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THE OBSTACLE PROBLEM FOR QUASILINEAR STOCHASTIC PDES: ANALYTICAL APPROACH

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Abstract: We prove an existence and uniqueness result for quasilinear Stochastic PDEs with obstacle (OSPDE in short). Our method is based on analytical technics coming from the parabolic potential theory. The solution is expressed as a pair (u, ν) where u is a predictable continuous process which takes values in a proper Sobolev space and ν is a random regular measure satisfying minimal Skohorod condition.

1. Introduction. The starting point of this work is the following parabolic stochastic partial differential equation (in short SPDE)

$$\begin{aligned} du_t(x) &= \partial_i \left(a_{i,j}(x)\partial_j u_t(x) + g_i(t,x,u_t(x),\nabla u_t(x))\right) dt + f(t,x,u_t(x),\nabla u_t(x))dt \\ &\quad + \sum_{j=1}^{+\infty} h_j(t,x,u_t(x),\nabla u_t(x))dB_t^j, \end{aligned} \tag{1}$$

where a is a symmetric bounded measurable matrix which defines a second order operator on an open domain $\mathcal{O} \subset \mathbb{R}^d$, with Dirichlet boundary condition. The initial condition is given as $u_0 = \xi$, a $L^2(\mathcal{O})-$valued random variable, and f, $g = (g_1, ..., g_d)$ and $h = (h_1, ...h_i, ...)$ are non-linear random functions. Given an obstacle $S : \Omega \times [0,T] \times \mathcal{O} \to \mathbb{R}$, we study the obstacle problem for SPDE (1), i.e. we want to find a solution of (1) which satisfies "$u \geq S$" where the obstacle S is regular in some sense and controlled by the solution of a SPDE.

Nualart and Pardoux [19] have studied the obstacle problem for a nonlinear heat equation on the spatial interval $[0,1]$ with Dirichlet boundary conditions, driven by an additive space-time white noise. They proved the existence and uniqueness of the solution and their method relied heavily on the results for a deterministic variational inequality. Donati-Martin and Pardoux [12] generalized the model of Nualart and Pardoux. The nonlinearity appears both in the drift and in the diffusion coefficients. They proved the existence of the solution by penalization method but they didn't obtain the uniqueness result. And then in 2009, Xu and Zhang solved the problem of the uniqueness, see [25]. However, in all their models, there isn't the term of divergence and they do not consider the case where the coefficients depend on ∇u.

The work of El Karoui et al [13] treats the obstacle problem for deterministic semilinear PDE's within the framework of backward stochastic differential equations (BSDE in short).

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Namely the equation (1) is considered with f depending on u and ∇u, while the function g is null (as well h) and the obstacle v is continuous. They considered the viscosity solution of the obstacle problem for the equation (1) , they represented this solution stochastically as a process and the main new object of this BSDE framework is a continuous increasing process that controls the set $\{u = v\}$. Bally et al [3] (see also [16]) point out that the continuity of this process allows one to extend the classical notion of strong variational solution (see Theorem 2.2 of [5] p.238) and express the solution to the obstacle as a pair (u, ν) where ν is supported by the set $\{u = v\}$.
Matoussi and Stoica [17] have proved an existence and uniqueness result for the obstacle problem of backward quasilinear stochastic PDE on the whole space $\mathbb{R}^d$ and driven by a finite dimensional Brownian motion. The method is based on the probabilistic interpretation of the solution by using the backward doubly stochastic differential equation (BDSDE in short). They have also proved that the solution is a pair (u, ν) where u is a predictable continuous process which takes values in a proper Sobolev space and ν is a random regular measure satisfying minimal Skohorod condition. In particular they gave for the regular measure ν a probabilistic interpretation in term of the continuous increasing process K where (Y, Z, K) is the solution of a reflected generalized BDSDE.
Michel Pierre [20, 21] has studied the parabolic PDE with obstacle using the parabolic potential as a tool. He proved that the solution uniquely exists and is quasi-continuous. With the help of Pierre's result, under suitable assumptions on f, g and h, our aim is to prove existence and uniqueness for the following SPDE with given obstacle S that we write formally as:

$$
(2)\quad \begin{cases} du_t(x) = \partial_i \left(a_{i,j}(x)\partial_j u_t(x) + g_i(t,x,u_t(x),\nabla u_t(x))\right) dt + f(t,x,u_t(x),\nabla u_t(x))dt \\ \qquad\qquad + \sum_{j=1}^{+\infty} h_j(t,x,u_t(x),\nabla u_t(x))dB_t^j, \\ u_t(x) \geq S_t(x)\,, \forall (t,x) \in \mathbb{R}^+ \times \mathcal{O}, \\ u_0(x) = \xi(x)\,,\ \forall x \in \mathcal{O}, \\ u_t(x) = 0,\ \forall (t,x) \in \mathbb{R}^+ \times \partial\mathcal{O}\,. \end{cases}
$$

To give a rigorous definition to the notion of solution to this equation, we will use the technics of parabolic potential theory developed by M. Pierre in the stochastic framework. We first prove a quasi-continuity result for the solution of the SPDE (1) with null Dirichlet condition on given domain $\mathcal{O}$ and driven by an infinite dimensional Brownian motion. This result is not obvious and is based on a mixing pathwise arguments and Mignot and Puel [18] existence result of the obstacle problem for some deterministic PDEs. Moreover, we prove in our context that the reflected measure ν is a regular random measure and we give the analytical representation of such measure in term of parabolic potential in the sense given by M. Pierre in [20]. The main theorem we obtain is:

THEOREM 1. *Assume that f, g and h satisfy some Lipschitz continuity and integrability hypotheses, $\xi \in L^2(\Omega \times \mathcal{O})$, S is quasi-continuous and $S_t \leq S'_t$ where S' is the solution of the linear SPDE with null boundary condition:*

$$
\begin{cases} dS'_t &= LS'_t dt + f'_t dt + \sum_{i=1}^d \partial_i g'_{i,t} dt + \sum_{j=1}^{+\infty} h'_{j,t} dB_t^j \\ S'(0) &= S'_0, \end{cases}
$$

where $S_0' \in L^2(\Omega \times \mathcal{O})$, f', g' and h' are square integrable adapted processes.
Then there exists a unique solution (u, ν) of the obstacle problem for the SPDE (2) associated to (ξ, f, g, h, S) i.e. u is a predictable continuous process which takes values in a proper Sobolev space, $u \geq S$ and ν is a random regular measure such that:

1. *the following relation holds almost surely, for all $t \in [0,T]$ and $\forall \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^+) \otimes \mathcal{C}_c^2(\mathcal{O})$,*

$$(u_t, \varphi_t) - (\xi, \varphi_0) - \int_0^t (u_s, \partial_s \varphi_s) ds + \int_0^t \mathcal{E}(u_s, \varphi_s) ds + \sum_{i=1}^d \int_0^t (g_s^i(u_s, \nabla u_s), \partial_i \varphi_s) ds$$
$$= \int_0^t (f_s(u_s, \nabla u_s), \varphi_s) ds + \sum_{j=1}^{+\infty} \int_0^t (h_s^j(u_s, \nabla u_s), \varphi_s) dB_s^j + \int_0^t \int_{\mathcal{O}} \varphi_s(x) \nu(dx, ds)$$

2. *u admits a quasi-continuous version, $\tilde{u}$, and we have the mininal Skohorod condition*

$$\int_0^T \int_{\mathcal{O}} (\tilde{u}(s,x) - S(s,x)) \nu(dx, ds) = 0 \quad a.s.$$

This paper is divided as follows: in the second section, we set the assumptions then we introduce in the third section the notion of regular measure associated to parabolic potentials. The fourth section is devoted to prove the quasi-continuity of the solution of SPDE without obstacle. The fifth section is the main part of the paper in which we prove existence and uniqueness of the solution, to do that we begin with the linear case, and then by Picard iteration we get the result in the nonlinear case, we also establish an Itô formula. Finally, in the sixth section, we prove a comparison theorem for the solution of SPDE with obstacle.

2. Preliminaries. We consider a sequence $((B^i(t))_{t \geq 0})_{i \in \mathbb{N}^*}$ of independent Brownian motions defined on a standard filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ satisfying the usual conditions.
Let $\mathcal{O} \subset \mathbb{R}^d$ be an open domain and $L^2(\mathcal{O})$ the set of square integrable functions with respect to the Lebesgue measure on $\mathcal{O}$, it is an Hilbert space equipped with the usual scalar product and norm as follows

$$(u, v) = \int_{\mathcal{O}} u(x) v(x) dx, \qquad \| u \| = (\int_{\mathcal{O}} u^2(x) dx)^{1/2}.$$

Let A be a symmetric second order differential operator, with domain $\mathcal{D}(A)$, given by

$$A := -L = -\sum_{i,j=1}^d \partial_i (a^{i,j}(x) \partial_j).$$

We assume that $a(x) = (a^{i,j}(x))_{i,j}$ is a measurable symmetric matrix defined on $\mathcal{O}$ which satisfies the uniform ellipticity condition

$$\lambda |\xi|^2 \leq \sum_{i,j=1}^d a^{i,j}(x) \xi^i \xi^j \leq \Lambda |\xi|^2, \ \forall x \in \mathcal{O}, \ \xi \in \mathbb{R}^d,$$

where λ and Λ are positive constants.
Let $(F, \mathcal{E})$ be the associated Dirichlet form given by $F := \mathcal{D}(A^{1/2}) = H_0^1(\mathcal{O})$ and

$$\mathcal{E}(u,v) := (A^{1/2}u, A^{1/2}v) \text{ and } \mathcal{E}(u) = \| A^{1/2}u \|^2, \ \forall u, v \in F,$$

where $H_0^1(\mathcal{O})$ is the first order Sobolev space of functions vanishing at the boundary. As usual we shall denote $H^{-1}(\mathcal{O})$ its dual space.
We consider the quasilinear stochastic partial differential equation (1) with initial condition $u(0,\cdot) = \xi(\cdot)$ and Dirichlet boundary condition $u(t,x) = 0, \ \forall \ (t,x) \in \mathbb{R}^+ \times \partial\mathcal{O}$.
We assume that we have predictable random functions

$$\begin{aligned} &f : \mathbb{R}^+ \times \Omega \times \mathcal{O} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}, \\ &g = (g_1, ..., g_d) : \mathbb{R}^+ \times \Omega \times \mathcal{O} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d, \\ &h = (h_1, ..., h_i, ...) : \mathbb{R}^+ \times \Omega \times \mathcal{O} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^{\mathbb{N}^*}, \end{aligned}$$

In the sequel, $|\cdot|$ will always denote the underlying Euclidean or l^2-norm. For example

$$|h(t,\omega,x,y,z)|^2 = \sum_{i=1}^{+\infty} |h_i(t,\omega,x,y,z)|^2.$$

Assumption (H): There exist non-negative constants $C, \ \alpha, \ \beta$ such that for almost all ω, the following inequalities hold for all $(t,x,y,z) \in \mathbb{R}^+ \times \mathcal{O} \times \mathbb{R} \times \mathbb{R}^d$:

1. $|f(t,\omega,x,y,z) - f(t,\omega,x,y',z')| \leq C(|y-y'| + |z-z'|)$,
2. $(\sum_{i=1}^d |g_i(t,\omega,x,y,z) - g_i(t,\omega,x,y',z')|^2)^{\frac{1}{2}} \leq C|y-y'| + \alpha|z-z'|$,
3. $(|h(t,\omega,x,y,z) - h(t,\omega,x,y',z')|^2)^{\frac{1}{2}} \leq C|y-y'| + \beta|z-z'|$,
4. the contraction property: $2\alpha + \beta^2 < 2\lambda$.

Remark 1. *This last contraction property ensures existence and uniqueness for the solution of the SPDE without obstacle (see [8]).*

With the uniform ellipticity condition we have the following equivalent conditions:

$$\| f(u,\nabla u) - f(v,\nabla v) \| \leq C \| u - v \| + C\lambda^{-1/2}\mathcal{E}^{1/2}(u-v)$$

$$\| g(u,\nabla u) - g(v,\nabla v) \|_{L^2(\mathcal{O};\mathbb{R}^d)} \leq C \| u - v \| + \alpha\lambda^{-1/2}\mathcal{E}^{1/2}(u-v)$$

$$\| h(u,\nabla u) - h(v,\nabla v) \|_{L^2(\mathcal{O};\mathbb{R}^{\mathbb{N}^*})} \leq C \| u - v \| + \beta\lambda^{-1/2}\mathcal{E}^{1/2}(u-v)$$

Moreover for simplicity, we fix a terminal time $T > 0$, we assume that:

Assumption (I):

$$\xi \in L^2(\Omega \times \mathcal{O}) \text{ is an } \mathcal{F}_0 - \text{measurable random variable}$$

$$f(\cdot,\cdot,\cdot,0,0) := f^0 \in L^2([0,T] \times \Omega \times \mathcal{O}; \mathbb{R})$$

$$g(\cdot,\cdot,\cdot,0,0) := g^0 = (g_1^0, ..., g_d^0) \in L^2([0,T] \times \Omega \times \mathcal{O}; \mathbb{R}^d)$$

$$h(\cdot,\cdot,\cdot,0,0) := h^0 = (h_1^0, ..., h_i^0, ...) \in L^2([0,T] \times \Omega \times \mathcal{O}; \mathbb{R}^{\mathbb{N}^*}).$$

Now we introduce the notion of weak solution.
We denote by $\mathcal{H}_T$ the space of $H_0^1(\mathcal{O})$-valued predictable $L^2(\mathcal{O})$-continuous processes $(u_t)_{t\geq 0}$ which satisfy

$$E \sup_{t\in[0,T]} \| u_t \|^2 + E\int_0^T \mathcal{E}(u_t)dt < +\infty.$$

It is the natural space for solutions.
The space of test functions is denote by $\mathcal{D} = \mathcal{C}_c^\infty(\mathbb{R}^+) \otimes \mathcal{C}_c^2(\mathcal{O})$, where $\mathcal{C}_c^\infty(\mathbb{R}^+)$ is the space of all real valued infinitely differentiable functions with compact support in $\mathbb{R}^+$ and $\mathcal{C}_c^2(\mathcal{O})$ is the set of C^2-functions with compact support in $\mathcal{O}$.
Heuristically, a pair (u, ν) is a solution of the obstacle problem for (1) with Dirichlet boundary condition if we have the followings:

1. $u \in \mathcal{H}_T$ and $u(t,x) \geq S(t,x),\ dP\otimes dt\otimes dx - a.e.$ and $u_0(x) = \xi,\ dP\otimes dx - a.e.$;
2. ν is a random measure defined on $[0,T)\times\mathcal{O}$;
3. the following relation holds almost surely, for all $t \in [0,T]$ and $\forall\varphi \in \mathcal{D}$,

$$(u_t,\varphi_t) - (\xi,\varphi_0) - \int_0^t (u_s,\partial_s\varphi_s)ds + \int_0^t \mathcal{E}(u_s,\varphi_s)ds + \sum_{i=1}^d \int_0^t (g_s^i(u_s,\nabla u_s),\partial_i\varphi_s)ds$$
$$= \int_0^t (f_s(u_s,\nabla u_s),\varphi_s)ds + \sum_{j=1}^{+\infty}\int_0^t (h_s^j(u_s,\nabla u_s),\varphi_s)dB_s^j + \int_0^t\int_{\mathcal{O}} \varphi_s(x)\nu(dx,ds);$$

4. $$\int_0^T\int_{\mathcal{O}} (u(s,x) - S(s,x))\nu(dx,ds) = 0, \quad a.s..$$

But, the random measure which in some sense obliges the solution to stay above the barrier is a local time so, in general, it is not absolutely continuous w.r.t Lebesgue measure. As a consequence, for example, the condition

$$\int_0^T\int_{\mathcal{O}} (u(s,x) - S(s,x))\nu(dxds) = 0$$

makes no sense. Hence we need to consider precise version of u and S defined $\nu-$almost surely.
In order to tackle this difficulty, we introduce in the next section the notions of parabolic capacity on $[0,T]\times\mathcal{O}$ and quasi-continuous version of functions introduced by Michel Pierre in several works (see for example [20, 21]). Let us remark that these tools were also used by Klimsiak ([14]) to get a probabilistic interpretation to semilinear PDE's with obstacle.

Finally and to end this section, we give an important example of stochastic noise which is covered by our framework:

Example 1. *Let W be a noise white in time and colored in space, defined on a standard filtered probability space $\big(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},P\big)$ whose covariance function is given by:*

$$\forall s,t\in\mathbb{R}^+,\ \forall x,y\in\mathcal{O},\quad E[\dot{W}(x,s)\dot{W}(y,t)] = \delta(t-s)k(x,y),$$

where $k : \mathcal{O} \times \mathcal{O} \mapsto \mathbb{R}^+$ is a symmetric and measurable function.
Consider the following SPDE driven by W:

$$du_t(x) = \big(\sum_{i,j=1}^{d} \partial_i a_{i,j}(x)\partial_j u_t(x) + f(t,x,u_t(x),\nabla u_t(x)) + \sum_{i=1}^{d} \partial_i g_i(t,x,u_t(x),\nabla u_t(x))\big)dt + \tilde{h}(t,x,u_t(x),\nabla u_t(x))\, W(dt,x), \tag{3}$$

where f and g are as above and $\tilde{h}$ is a random real valued function.
We assume that the covariance function k defines a trace class operator denoted by K in $L^2(\mathcal{O})$. It is well known (see [22]) that there exists an orthogonal basis $(e_i)_{i\in\mathbb{N}^*}$ of $L^2(\mathcal{O})$ consisting of eigenfunctions of K with corresponding eigenvalues $(\lambda_i)_{i\in\mathbb{N}^*}$ such that

$$\sum_{i=1}^{+\infty} \lambda_i < +\infty,$$

and

$$k(x,y) = \sum_{i=1}^{+\infty} \lambda_i e_i(x) e_i(y).$$

It is also well known that there exists a sequence $((B^i(t))_{t\geq 0})_{i\in\mathbb{N}^*}$ of independent standard Brownian motions such that

$$W(dt,\cdot) = \sum_{i=1}^{+\infty} \lambda_i^{1/2} e_i B^i(dt).$$

So that equation (3) is equivalent to (1) with $h = (h_i)_{i\in\mathbb{N}^*}$ where

$$\forall i \in \mathbb{N}^*,\ h_i(s,x,y,z) = \sqrt{\lambda_i}\tilde{h}(s,x,y,z)e_i(x).$$

Assume as in [23] that for all $i \in \mathbb{N}^*$, $\|e_i\|_\infty < +\infty$ and

$$\sum_{i=1}^{+\infty} \lambda_i \|e_i\|_\infty^2 < +\infty.$$

Since

$$\Big(|h(t,\omega,x,y,z) - h(t,\omega,x,y',z')|^2\Big)^{\frac{1}{2}} \leq \left(\sum_{i=1}^{+\infty} \lambda_i \|e_i\|_\infty^2\right) \left|\tilde{h}(t,x,y,z) - \tilde{h}(t,x,y',z')\right|^2,$$

h satisfies the Lipschitz hypothesis **(H)-(ii)** *if $\tilde{h}$ satisfies a similar Lipschitz hypothesis.*

3. Parabolic potential analysis.

3.1. *Parabolic capacity and potentials.* In this section we will recall some important definitions and results concerning the obstacle problem for parabolic PDE in [20] and [21]. $\mathcal{K}$ denotes $L^\infty([0,T];L^2(\mathcal{O})) \cap L^2([0,T];H_0^1(\mathcal{O}))$ equipped with the norm:

$$\begin{aligned} \| v \|_{\mathcal{K}}^2 &= \| v \|_{L^\infty([0,T];L^2(\mathcal{O}))}^2 + \| v \|_{L^2([0,T];H_0^1(\mathcal{O}))}^2 \\ &= \sup_{t\in[0,T[} \| v_t \|^2 + \int_0^T \big(\| v_t \|^2 + \mathcal{E}(v_t)\big)\, dt. \end{aligned}$$

$\mathcal{C}$ denotes the space of continuous functions on compact support in $[0,T[\times\mathcal{O}$ and finally:

$$\mathcal{W}=\{\varphi\in L^2([0,T];H_0^1(\mathcal{O}));\ \frac{\partial\varphi}{\partial t}\in L^2([0,T];H^{-1}(\mathcal{O}))\},$$

endowed with the norm$\| \varphi \|^2_{\mathcal{W}}=\| \varphi \|^2_{L^2([0,T];H_0^1(\mathcal{O}))}+\| \frac{\partial\varphi}{\partial t} \|^2_{L^2([0,T];H^{-1}(\mathcal{O}))}$.
It is known (see [15]) that $\mathcal{W}$ is continuously embedded in $C([0,T];L^2(\mathcal{O}))$, the set of $L^2(\mathcal{O})$-valued continuous functions on $[0,T]$. So without ambiguity, we will also consider $\mathcal{W}_T=\{\varphi\in\mathcal{W};\varphi(T)=0\}$, $\mathcal{W}^+=\{\varphi\in\mathcal{W};\varphi\geq 0\}$, $\mathcal{W}_T^+=\mathcal{W}_T\cap\mathcal{W}^+$.
We now introduce the notion of parabolic potentials and regular measures which permit to define the parabolic capacity.

Definition 1. *An element $v\in\mathcal{K}$ is said to be a* **parabolic potential** *if it satisfies:*

$$\forall\varphi\in\mathcal{W}_T^+,\ \int_0^T-(\frac{\partial\varphi_t}{\partial t},v_t)dt+\int_0^T\mathcal{E}(\varphi_t,v_t)dt\geq 0.$$

We denote by $\mathcal{P}$ the set of all parabolic potentials.

The next representation property is crucial:

Proposition 1. *(Proposition 1.1 in [21]) Let $v\in\mathcal{P}$, then there exists a unique positive Radon measure on $[0,T[\times\mathcal{O}$, denoted by ν^v, such that:*

$$\forall\varphi\in\mathcal{W}_T\cap\mathcal{C},\ \int_0^T(-\frac{\partial\varphi_t}{\partial t},v_t)dt+\int_0^T\mathcal{E}(\varphi_t,v_t)dt=\int_0^T\int_{\mathcal{O}}\varphi(t,x)d\nu^v.$$

Moreover, v admits a right-continuous (resp. left-continuous) version $\hat{v}$ (resp. $\bar{v}$) : $[0,T]\mapsto L^2(\mathcal{O})$.
Such a Radon measure, ν^v is called **a regular measure** *and we write:*

$$\nu^v=\frac{\partial v}{\partial t}+Av.$$

Remark 2. *As a consequence, we can also define for all $v\in\mathcal{P}$:*

$$v_T=\lim_{t\uparrow T}\bar{v}_t\ \in L^2(\mathcal{O}).$$

Definition 2. *Let $K\subset[0,T[\times\mathcal{O}$ be compact, $v\in\mathcal{P}$ is said to be $\nu-$superior than 1 on K, if there exists a sequence $v_n\in\mathcal{P}$ with $v_n\geq 1$ a.e. on a neighborhood of K converging to v in $L^2([0,T];H_0^1(\mathcal{O}))$.*

We denote:

$$\mathscr{S}_K=\{v\in\mathcal{P};\ v\ is\ \nu-superior\ to\ 1\ on\ K\}.$$

Proposition 2. *(Proposition 2.1 in [21]) Let $K\subset[0,T[\times\mathcal{O}$ compact, then $\mathscr{S}_K$ admits a smallest $v_K\in\mathcal{P}$ and the measure ν_K^v whose support is in K satisfies*

$$\int_0^T\int_{\mathcal{O}}d\nu_K^v=\inf_{v\in\mathcal{P}}\{\int_0^T\int_{\mathcal{O}}d\nu^v;\ v\in\mathscr{S}_K\}.$$

DEFINITION 3. *(Parabolic Capacity)*

- *Let $K \subset [0,T[\times\mathcal{O}$ be compact, we define $cap(K) = \int_0^T \int_{\mathcal{O}} d\nu_K^v$;*
- *let $O \subset [0,T[\times\mathcal{O}$ be open, we define $cap(O) = \sup\{cap(K);\ K \subset O \text{ compact}\}$;*
- *for any borelian $E \subset [0,T[\times\mathcal{O}$, we define $cap(E) = \inf\{cap(O);\ O \supset E \text{ open}\}$.*

DEFINITION 4. *A property is said to hold quasi-everywhere (in short q.e.) if it holds outside a set of null capacity.*

DEFINITION 5. *(Quasi-continuity)*
A function $u : [0,T[\times\mathcal{O} \to \mathbb{R}$ is called quasi-continuous, if there exists a decreasing sequence of open subsets O_n of $[0,T[\times\mathcal{O}$ with:

1. *for all n, the restriction of u_n to the complement of O_n is continuous;*
2. *$\lim_{n\to+\infty} cap\,(O_n) = 0$.*

We say that u admits a quasi-continuous version, if there exists $\tilde{u}$ quasi-continuous such that $\tilde{u} = u$ a.e..

The next proposition, whose proof may be found in [20] or [21] shall play an important role in the sequel:

PROPOSITION 3. *Let $K \subset \mathcal{O}$ a compact set, then $\forall t \in [0,T[$*

$$cap(\{t\} \times K) = \lambda_d(K),$$

where λ_d is the Lebesgue measure on $\mathcal{O}$.
As a consequence, if $u : [0,T[\times\mathcal{O} \to \mathbb{R}$ is a map defined quasi-everywhere then it defines uniquely a map from $[0,T[$ into $L^2(\mathcal{O})$. In other words, for any $t \in [0,T[$, u_t is defined without any ambiguity as an element in $L^2(\mathcal{O})$. Moreover, if $u \in \mathcal{P}$, it admits version $\bar{u}$ which is left continuous on $[0,T]$ with values in $L^2(\mathcal{O})$ so that $u_T = \bar{u}_{T^-}$ is also defined without ambiguity.

REMARK 3. *The previous proposition applies if for example u is quasi-continuous.*

PROPOSITION 4. *(Theorem III.1 in [21]) If $\varphi \in \mathcal{W}$, then it admits a unique quasi-continuous version that we denote by $\tilde{\varphi}$. Moreover, for all $v \in \mathcal{P}$, the following relation holds:*

$$\int_{[0,T[\times\mathcal{O}} \tilde{\varphi} d\nu^v = \int_0^T (-\partial_t\varphi, v) + \mathcal{E}(\varphi, v)\, dt + (\varphi_T, v_T)\,.$$

3.2. *Applications to PDE's with obstacle.* For any function $\psi : [0,T[\times\mathcal{O} \to \mathbb{R}$ and $u_0 \in L^2(\mathcal{O})$, following M. Pierre [20, 21], F. Mignot and J.P. Puel [18], we define

$$\kappa(\psi, u_0) = ess\inf\{u \in \mathcal{P};\ u \geq \psi \text{ a.e.},\ u(0) \geq u_0\}. \tag{4}$$

This lower bound exists and is an element in $\mathcal{P}$. Moreover, when ψ is quasi-continuous, this potential is the solution of the following reflected problem:

$$\kappa \in \mathcal{P},\ \kappa \geq \psi,\ \frac{\partial\kappa}{\partial t} + A\kappa = 0 \text{ on } \{u > \psi\},\ \kappa(0) = u_0.$$

Mignot and Puel have proved in [18] that $\kappa(\psi, u_0)$ is the limit (increasingly and weakly in $L^2([0,T]; H_0^1(\mathcal{O}))$) when ϵ tends to 0 of the solution of the following penalized equation

$$u_\epsilon \in \mathcal{W},\ u_\epsilon(0) = u_0,\ \frac{\partial u_\epsilon}{\partial t} + Au_\epsilon - \frac{(u_\epsilon - \psi)^-}{\epsilon} = 0.$$

Let us point out that they obtain this result in the more general case where ψ is only measurable from $[0,T[$ into $L^2(\mathcal{O})$.
For given $f \in L^2([0,T]; H^{-1}(\mathcal{O}))$, we denote by $\kappa_{u_0}^f$ the solution of the following problem:

$$\kappa \in \mathcal{W},\ \kappa(0) = u_0,\ \frac{\partial \kappa}{\partial t} + A\kappa = f.$$

The next theorem ensures existence and uniqueness of the solution of parabolic PDE with obstacle, it is proved in [20], Theorem 1.1. The proof is based on a regularization argument of the obstacle, using the results of [6].

THEOREM 2. *Let $\psi : [0,T[\times\mathcal{O} \to \mathbb{R}$ be quasi-continuous, suppose that there exists $\zeta \in \mathcal{P}$ with $|\psi| \leq \zeta$ a.e., $f \in L^2([0,T]; H^{-1}(\mathcal{O}))$, and the initial value $u_0 \in L^2(\mathcal{O})$ with $u_0 \geq \psi(0)$, then there exists a unique $u \in \kappa_{u_0}^f + \mathcal{P}$ quasi-continuous such that:*

$$u(0) = u_0,\ \tilde{u} \geq \psi,\ q.e.; \quad \int_0^T \int_{\mathcal{O}} (\tilde{u} - \tilde{\psi}) d\nu^{u - \kappa_{u_0}^f} = 0$$

We end this section by a convergence lemma which plays an important role in our approach (Lemma 3.8 in [21]):

LEMMA 1. *If $(v^n)_n \in \mathcal{P}$ is a bounded sequence in $\mathcal{K}$ and converges weakly to v in $L^2([0,T]; H_0^1(\mathcal{O}))$; if u is a quasi-continuous function and $|u|$ is bounded by a element in $\mathcal{P}$. Then*

$$\lim_{n \to +\infty} \int_0^T \int_{\mathcal{O}} u d\nu^{v^n} = \int_0^T \int_{\mathcal{O}} u d\nu^v.$$

REMARK 4. *For the more general case one can see [21] Lemma 3.8.*

4. Quasi–continuity of the solution of SPDE without obstacle. As a consequence of well-known results (see for example [8], Theorem 8), we know that under assumptions **(H)** and **(I)**, SPDE (1) with zero Dirichlet boundary condition, admits a unique solution in $\mathcal{H}_T$, (for the definition of solution see for example Definition 1 in [8]) we denote it by $\mathcal{U}(\xi, f, g, h)$.
The main theorem of this section is the following:

THEOREM 3. *Under assumptions* **(H)** *and* **(I)**, *$u = \mathcal{U}(\xi, f, g, h)$ the solution of SPDE (1) admits a quasi-continuous version denoted by $\tilde{u}$ i.e. $u = \tilde{u}$ $dP \otimes dt \otimes dx-$a.e. and for almost all $w \in \Omega$, $(t,x) \to \tilde{u}_t(w,x)$ is quasi-continuous.*

Before giving the proof of this theorem, we need the following lemmas. The first one is proved in [21], Lemma 3.3:

LEMMA 2. *There exists $C > 0$ such that, for all open set $\vartheta \subset [0,T[\times\mathcal{O}$ and $v \in \mathcal{P}$ with $v \geq 1$ a.e. on ϑ:*

$$cap\vartheta \leq C \parallel v \parallel^2_{\mathcal{K}} .$$

Let $\kappa = \kappa(u, u^+(0))$ be defined by relation (4). One has to note that κ is a random function. From now on, we always take for κ the following measurable version

$$\kappa = \sup_n v^n,$$

where $(v^n)_n$ is the non-decreasing sequence of random functions given by

$$\begin{cases} \dfrac{\partial v^n_t}{\partial t} = Lv^n_t + n(v^n_t - u_t)^- \\ v^n_0 = u^+(0). \end{cases} \tag{5}$$

Using the results recalled in Subsection 3, we know that for almost all $w \in \Omega$, $v^n(w)$ converges weakly to $v(w) = \kappa(u(w), u^+(0)(w))$ in $L^2([0,T]; H^1_0(\mathcal{O}))$ and that $v \geq u$.

LEMMA 3. *We have the following estimate:*

$$E \parallel \kappa \parallel^2_{\mathcal{K}} \leq C \left(E \parallel u^+_0 \parallel^2 + E \parallel u_0 \parallel^2 + E \int_0^T \parallel f^0_t \parallel^2 + \parallel |g^0_t| \parallel^2 + \parallel |h^0_t| \parallel^2 dt \right),$$

where C is a constant depending only on the structure constants of the equation.

PROOF. All along this proof, we shall denote by C or C_ϵ some constant which may change from line to line.
The following estimate for the solution of the SPDE we consider is well-known:

$$E \sup_{t\in[0,T]} \parallel u_t \parallel^2 + E \int_0^T \mathcal{E}(u_t)dt \leq CE(\parallel u_0 \parallel^2 + \int_0^T (\parallel f^0_t \parallel^2 + \parallel |g^0_t| \parallel^2 + \parallel |h^0_t| \parallel^2)dt) \tag{6}$$

where C is a constant depending only on the structure constants of the equation.
Consider the approximation $(v^n)_n$ defined by (5), P-almost surely, it converges weakly to $v = \kappa(u, u^+(0))$ in $L^2([0,T]; H^1_0(\mathcal{O}))$.
We remark that $v^n - u$ satisfies the following equation:

$$d(v^n_t - u_t) + A(v^n_t - u_t)dt = -f_t(u_t, \nabla u_t)dt - \sum_{i=1}^d \partial_i g^i_t(u_t, \nabla u_t)dt - \sum_{j=1}^{+\infty} h^j_t(u_t, \nabla u_t)dB^j_t + n(v^n_t - u_t)^- dt,$$

applying Itô's formula to $(v^n - u)^2$, see Lemma 7 in [9], we have almost surely, for all $t \in [0,T]$,

$$\begin{aligned} \parallel v^n_t - u_t \parallel^2 + 2\int_0^t \mathcal{E}(v^n_s - u_s)ds &= \parallel u^+_0 - u_0 \parallel^2 - 2\int_0^t (v^n_s - u_s, f_s(u_s, \nabla u_s))ds \\ &+ 2\sum_{i=1}^d \int_0^t (\partial_i(v^n_s - u_s), g^i_s(u_s, \nabla u_s))ds + \int_0^t \parallel |h_s(u_s, \nabla u_s)| \parallel^2 ds \\ &- 2\sum_{j=1}^{+\infty} \int_0^t (v^n_s - u_s, h^j_s(u_s, \nabla u_s))dB^j_s + 2\int_0^t (n(v^n_s - u_s)^-, v^n_s - u_s)ds. \end{aligned} \tag{7}$$

The last term in the right member of (7) is obviously non-positive so

$$\| v_t^n - u_t \|^2 + 2\int_0^t \mathcal{E}(v_s^n - u_s)ds \leq \| u_0^+ - u_0 \|^2 - 2\int_0^t (v_s^n - u_s, f_s(u_s, \nabla u_s))ds$$
$$+ \int_0^t \| |h_s(u_s, \nabla u_s)| \|^2 ds + 2\sum_{i=1}^d \int_0^t (\partial_i(v_s^n - u_s), g_s^i(u_s, \nabla u_s))ds$$
$$-2\sum_{j=1}^{+\infty} \int_0^t (v_s^n - u_s, h_s^j(u_s, \nabla u_s))dB_s^j, \quad a.s. \tag{8}$$

Then taking expectation and using Cauchy-Schwarz's inequality, we get

$$E \| v_t^n - u_t \|^2 + (2 - \frac{\epsilon}{\lambda})E\int_0^t \mathcal{E}(v_s^n - u_s)ds \leq E \| u_0^+ - u_0 \|^2 + E\int_0^t \| v_s^n - u_s \|^2 ds$$
$$+E\int_0^t \| f_s(u_s, \nabla u_s) \|^2 ds + C_\epsilon E\int_0^t \| |g_s(u_s, \nabla u_s)| \|^2 ds + E\int_0^t \| |h_s(u_s, \nabla u_s)| \|^2 ds.$$

Therefore, by using the Lipschitz conditions on the coefficients we have:

$$E \| v_t^n - u_t \|^2 + (2 - \frac{\epsilon}{\lambda})E\int_0^t \mathcal{E}(v_s^n - u_s)ds \leq E \| u_0^+ - u_0 \|^2 + E\int_0^t \| v_s^n - u_s \|^2 ds$$
$$+CE\int_0^t (\| f_s^0 \|^2 + \| |g_s^0| \|^2 + \| |h_s^0| \|^2)ds + CE\int_0^t \| u_s \|^2 ds + (\frac{C}{\lambda} + \frac{\alpha}{\lambda} + \frac{\beta^2}{\lambda})E\int_0^t \mathcal{E}(u_s)ds.$$

Combining with (6), this yields

$$E \| v_t^n - u_t \|^2 + (2 - \frac{\epsilon}{\lambda})E\int_0^t \mathcal{E}(v_s^n - u_s)ds \leq E \| u_0^+ - u_0 \|^2 + E\int_0^t \| v_s^n - u_s \|^2 ds$$
$$+ CE(\| u_0 \|^2 + \int_0^T (\| f_t^0 \|^2 + \| |g_t^0| \|^2 + \| |h_t^0| \|^2)dt).$$

We take now ϵ small enough such that $(2 - \frac{\epsilon}{\lambda}) > 0$, then, with Gronwall's lemma, we obtain for each $t \in [0, T]$

$$E \| v_t^n - u_t \|^2 \leq Ce^{c'T}(E \| u_0^+ - u_0 \|^2 + E \| u_0 \|^2 + E\int_0^T \| f_t^0 \|^2 + \| |g_t^0| \|^2 + \| |h_t^0| \|^2 dt).$$

As we a priori know that P-almost surely, $(v^n)_n$ tends to κ strongly in $L^2([0, T] \times \mathcal{O})$, the previous estimate yields, thanks to the dominated convergence theorem, that $(v^n)_n$ converges to κ strongly in $L^2(\Omega \times [0, T] \times \mathcal{O})$ and

$$\sup_{t\in[0,T]} E \| \kappa_t - u_t \|^2 \leq Ce^{c'T}(E \| u_0^+ - u_0 \|^2 + E \| u_0 \|^2 + E\int_0^T \| f_t^0 \|^2 + \| |g_t^0| \|^2 + \| |h_t^0| \|^2 dt).$$

Moreover, as $(v^n)_n$ tends to κ weakly in $L^2([0, T]; H_0^1(\mathcal{O}))$ P-almost-surely, we have for all $t \in [0, T]$:

$$E\int_0^T \mathcal{E}(\kappa_s - u_s)ds \leq \liminf_n E\int_0^T \mathcal{E}(v_s^n - u_s)ds$$
$$\leq TCe^{c'T}(E \| u_0^+ - u_0 \|^2 + E \| u_0 \|^2 + E\int_0^T \| f_t^0 \|^2 + \| |g_t^0| \|^2 + \| |h_t^0| \|^2 dt).$$

Let us now study the stochastic term in (8). Let define the martingales

$$M_t^n = \sum_{j=1}^{+\infty} \int_0^t (v_s^n - u_s, h_s^j) dB_s^j \text{ and } M_t = \sum_{j=1}^{+\infty} \int_0^t (\kappa_s - u_s, h_s^j) dB_s^j.$$

Then

$$E[|M_T^n - M_T|^2] = E \int_0^T \sum_{j=1}^{+\infty} (\kappa_s - v_s^n, h_s)^2 \, ds \leq E \int_0^T \| \kappa_s - v_s^n \|^2 \| |h_s| \|^2 \, ds.$$

Using the strong convergence of $(v^n)_n$ to κ we conclude that $(M^n)_n$ tends to M in L^2 sense. Passing to the limit in (8), we get, almost surely, for all $t \in [0,T]$,

$$\begin{aligned} \| \kappa_t - u_t \|^2 + 2 \int_0^t \mathcal{E}(\kappa_s - u_s) ds \leq & \| u_0^+ - u_0 \|^2 - 2 \int_0^t (\kappa_s - u_s, f_s(u_s, \nabla u_s)) ds \\ & + 2 \sum_{i=1}^d \int_0^t (\partial_i (\kappa_s - u_s), g_s^i(u_s, \nabla u_s)) ds \\ & - 2 \sum_{j=1}^{+\infty} \int_0^t (\kappa_s - u_s, h_s^j(u_s, \nabla u_s)) dB_s^j + \int_0^t \| |h_s(u_s, \nabla u_s)| \|^2 \, ds. \end{aligned}$$

As a consequence of Burkholder-Davies-Gundy's inequalities, we get

$$\begin{aligned} E \sup_{t \in [0,T]} | \sum_{j=1}^{+\infty} \int_0^t (\kappa_s - u_s, h_s^j(u_s, \nabla u_s)) dB_s^j | &\leq CE[\int_0^T \sum_{j=1}^{+\infty} (\kappa_s - u_s, h_s^j(u_s, \nabla u_s))^2 ds]^{1/2} \\ &\leq CE[\int_0^T \sum_{j=1}^{+\infty} \sup_{t \in [0,T]} \| \kappa_t - u_t \|^2 \| h_s^j(u_s, \nabla u_s) \|^2 \, ds]^{1/2} \\ &\leq CE[\sup_{t \in [0,T]} \| \kappa_t - u_t \| (\int_0^T \| |h_t(u_t, \nabla u_t)| \|^2 \, dt)^{1/2}] \\ &\leq \epsilon E \sup_{t \in [0,T]} \| \kappa_t - u_t \|^2 + C_\epsilon E \int_0^T \| |h_t(u_t, \nabla u_t)| \|^2 \, dt. \end{aligned}$$

By Lipschitz conditions on h and (6) this yields

$$\begin{aligned} E \sup_{t \in [0,T]} | \sum_{j=1}^{+\infty} \int_0^t (\kappa_s - u_s, h_s(u_s, \nabla u_s)) dB_s | &\leq \epsilon E \sup_{t \in [0,T]} \| \kappa_t - u_t \|^2 + C(E \| u_0 \|^2 \\ &+ E \int_0^T (\| f_t^0 \|^2 + \| |g_t^0| \|^2 + \| |h_t^0| \|^2) dt) \end{aligned}$$

Hence,

$$\begin{aligned} (1-\epsilon) E \sup_{t \in [0,T]} \| \kappa_t - u_t \|^2 + (2 - \frac{\epsilon}{\lambda}) E \int_0^T \mathcal{E}(\kappa_t - u_t) dt \leq & C(E \| u_0^+ - u_0 \|^2 + E \| u_0 \|^2 \\ & + E \int_0^T \| f_t^0 \|^2 + \| |g_t^0| \|^2 + \| |h_t^0| \|^2 \, dt). \end{aligned}$$

We can take ϵ small enough such that $1-\epsilon>0$ and $2-\frac{\epsilon}{\lambda}>0$, hence,

$$\begin{aligned} E\sup_{t\in[0,T]}\parallel\kappa_t-u_t\parallel^2 &+ E\int_0^T\mathcal{E}(\kappa_t-u_t)dt\leq C(E\parallel u_0^+-u_0\parallel^2+E\parallel u_0\parallel^2 \\ &+ E\int_0^T\parallel f_t^0\parallel^2+\parallel|g_t^0|\parallel^2+\parallel|h_t^0|\parallel^2\,dt). \end{aligned}$$

Then, combining with (6), we get the desired estimate:

$$E\sup_{t\in[0,T]}\parallel\kappa_t\parallel^2+E\int_0^T\mathcal{E}(\kappa_t)dt\leq C(E\parallel u_0^+\parallel^2+E\parallel u_0\parallel^2+E\int_0^T\parallel f_t^0\parallel^2+\parallel|g_t^0|\parallel^2+\parallel|h_t^0|\parallel^2\,dt).$$

□

Proof of Theorem 3: For simplicity, we put

$f_t(x)=f(t,x,u_t(x),\nabla u_t(x))$, $g_t(x)=g(t,x,u_t(x),\nabla u_t(x))$ and $h_t(x)=h(t,x,u_t(x),\nabla u_t(x))$.

We introduce (P_t) the semi-group associated to operator A and put for each $n\in\mathbb{N}^*$, $i\in\{1,\cdots,d\}$ and each $j\in\mathbb{N}^*$:

$$u_0^n=P_{\frac{1}{n}}u_0,\quad f^n=P_{\frac{1}{n}}f,\quad g_i^n=P_{\frac{1}{n}}g_i,\quad \textit{and } h_j^n=P_{\frac{1}{n}}h_j.$$

Then $(u_0^n)_n$ converges to u_0 in $L^2(\Omega;L^2(\mathcal{O}))$, $(f^n)_n$, $(g^n)_n$ and $(h^n)_n$ are sequences of elements in $L^2(\Omega\times[0,T];\mathcal{D}(A))$ which converge respectively to f, g and h in $L^2(\Omega\times[0,T];L^2(\mathcal{O}))$. For all $n\in\mathbb{N}^*$ we define

$$\begin{aligned} u_t^n &= P_tu_0^n+\int_0^tP_{t-s}f_s^nds+\sum_{i=1}^d\int_0^tP_{t-s}\partial_ig_{i,s}^nds+\sum_{j=1}^{+\infty}\int_0^tP_{t-s}h_{j,s}^ndB_s^j \\ &= P_{t+\frac{1}{n}}u_0+\int_0^tP_{t+\frac{1}{n}-s}f_sds+\sum_{i=1}^d\int_0^tP_{t+\frac{1}{n}-s}\partial_ig_{i,s}ds+\sum_{j=1}^{+\infty}\int_0^tP_{t+\frac{1}{n}-s}h_{j,s}dB_s^j. \end{aligned}$$

We denote by $G(t,x,s,y)$ the kernel associated to P_t, then

$$\begin{aligned} u^n(t,x) &= \int_{\mathcal{O}}G(t+\frac{1}{n},x,0,y)u_0(y)dy+\int_0^t\int_{\mathcal{O}}G(t+\frac{1}{n},x,s,y)f(s,y)dyds \\ &+ \sum_{i=1}^d\int_0^t\int_{\mathcal{O}}G(t+\frac{1}{n},x,s,y)\partial_ig_s^i(y)dyds+\sum_{j=1}^{+\infty}\int_0^t\int_{\mathcal{O}}G(t+\frac{1}{n},x,s,y)h_s^j(y)dydB_s^i. \end{aligned}$$

But, as A is strictly elliptic, G is uniformly continuous in space-time variables on any compact away from the diagonal in time (see Theorem 6 in [1]) and satisfies Gaussian estimates (see Aronson [2]), this ensures that for all $n\in\mathbb{N}^*$, u^n is P-almost surely continuous in (t,x).
We consider a sequence of random open sets

$$\vartheta_n=\{|u^{n+1}-u^n|>\epsilon_n\},\quad \Theta_p=\bigcup_{n=p}^{+\infty}\vartheta_n.$$

Let $\kappa_n = \kappa(\frac{1}{\epsilon_n}(u^{n+1}-u^n), \frac{1}{\epsilon_n}(u^{n+1}-u^n)^+(0)) + \kappa(-\frac{1}{\epsilon_n}(u^{n+1}-u^n), \frac{1}{\epsilon_n}(u^{n+1}-u^n)^-(0))$, from the definition of κ and the relation (see [21])

$$\kappa(|v|) \leq \kappa(v, v^+(0)) + \kappa(-v, v^-(0))$$

we know that κ_n satisfy the conditions of Lemma 2, i.e. $\kappa_n \in \mathcal{P}$ et $\kappa_n \geq 1$ *a.e.* on ϑ_n, thus we get the following relation

$$cap\,(\Theta_p) \leq \sum_{n=p}^{+\infty} cap\,(\vartheta_n) \leq \sum_{n=p}^{+\infty} \parallel \kappa_n \parallel^2_{\mathcal{K}}.$$

Thus, remarking that $u^{n+1}-u^n = \mathcal{U}(u_0^{n+1}-u_0^n, f^{n+1}-f^n, g^{n+1}-g^n, h^{n+1}-h^n)$, we apply Lemma 3 to $\kappa(\frac{1}{\epsilon_n}(u^{n+1}-u^n), \frac{1}{\epsilon_n}(u^{n+1}-u^n)^+(0))$ and $\kappa(-\frac{1}{\epsilon_n}(u^{n+1}-u^n), \frac{1}{\epsilon_n}(u^{n+1}-u^n)^-(0))$ and obtain:

$$\begin{aligned} E[cap\,(\Theta_p)] \leq \sum_{n=p}^{+\infty} E \parallel \kappa_n \parallel^2_{\mathcal{K}} &\leq 2C \sum_{n=p}^{+\infty} \frac{1}{\epsilon_n^2} (E \parallel u_0^{n+1}-u_0^n \parallel^2 + E\int_0^T \parallel f_t^{n+1}-f_t^n \parallel^2 \\ &+ \parallel |g_t^{n+1}-g_t^n| \parallel^2 + \parallel |h_t^{n+1}-h_t^n| \parallel^2 dt) \end{aligned}$$

Then, by extracting a subsequence, we can consider that

$$E \parallel u_0^{n+1}-u_0^n \parallel^2 + E\int_0^T \parallel f_t^{n+1}-f_t^n \parallel^2 + \parallel |g_t^{n+1}-g_t^n| \parallel^2 + \parallel |h_t^{n+1}-h_t^n| \parallel^2 dt \leq \frac{1}{2^n}$$

Then we take $\epsilon_n = \frac{1}{n^2}$ to get

$$E[cap\,(\Theta_p)] \leq \sum_{n=p}^{+\infty} \frac{2Cn^4}{2^n}$$

Therefore

$$\lim_{p\to+\infty} E[cap\,(\Theta_p)] = 0.$$

For almost all $\omega \in \Omega$, $u^n(\omega)$ is continuous in (t,x) on $(\Theta_p(w))^c$ and $(u^n(\omega))_n$ converges uniformly to u on $(\Theta_p(w))^c$ for all p, hence, $u(\omega)$ is continuous in (t,x) on $(\Theta_p(w))^c$, then from the definition of quasi-continuous, we know that $u(\omega)$ admits a quasi-continuous version since $cap\,(\Theta_p)$ tends to 0 almost surely as p tends to $+\infty$. □

5. Existence and uniqueness of the solution of the obstacle problem.

5.1. *Weak solution.* **Assumption (O):** The obstacle S is assumed to be an adapted process, quasi-continuous, such that $S_0 \leq \xi$ P-almost surely and controlled by the solution of a SPDE, i.e. $\forall t \in [0,T]$,

$$S_t \leq S'_t \tag{9}$$

where S' is the solution of the linear SPDE with Dirichlet boundary condition:

$$\left\{ \begin{array}{rcl} dS'_t &=& LS'_t dt + f'_t dt + \sum_{i=1}^d \partial_i g'_{i,t} dt + \sum_{j=1}^{+\infty} h'_{j,t} dB_t^j \\ S'(0) &=& S'_0, \end{array} \right. \tag{10}$$

where $S'_0 \in L^2(\Omega \times \mathcal{O})$ is $\mathcal{F}_0$-measurable, f', g' and h' are adapted processes respectively in $L^2([0,T]\times\Omega\times\mathcal{O};\mathbb{R})$, $L^2([0,T]\times\Omega\times\mathcal{O};\mathbb{R}^d)$ and $L^2([0,T]\times\Omega\times\mathcal{O};\mathbb{R}^{\mathbb{N}^*})$.

REMARK 5. *Here again, we know that S' uniquely exists and satisfies the following estimate:*

(11)
$$E \sup_{t\in[0,T]} \| S'_t \|^2 + E\int_0^T \mathcal{E}(S'_t)dt \leq CE\left[\| S'_0 \|^2 + \int_0^T (\| f'_t \|^2 + \| |g'_t| \|^2 + \| |h'_t| \|^2)dt\right]$$

Moreover, from Theorem 3, S' admits a quasi-continuous version.
Let us also remark that even if this assumption seems restrictive since S' is driven by the same operator and Brownian motions as u, it encompasses a large class of examples.

We now are able to define rigorously the notion of solution to the problem with obstacle we consider.

DEFINITION 6. *A pair (u, ν) is said to be a solution of the obstacle problem for (1) with Dirichlet boundary condition if*

1. *$u \in \mathcal{H}_T$ and $u(t,x) \geq S(t,x)$, $dP \otimes dt \otimes dx - a.e.$ and $u_0(x) = \xi$, $dP \otimes dx - a.e.$;*
2. *ν is a random regular measure defined on $[0,T) \times \mathcal{O}$;*
3. *the following relation holds almost surely, for all $t \in [0,T]$ and $\forall \varphi \in \mathcal{D}$,*

$$(u_t, \varphi_t) - (\xi, \varphi_0) - \int_0^t (u_s, \partial_s \varphi_s)ds + \int_0^t \mathcal{E}(u_s, \varphi_s)ds + \sum_{i=1}^d \int_0^t (g_s^i(u_s, \nabla u_s), \partial_i \varphi_s)ds$$
$$(12) \quad \int_0^t (f_s(u_s, \nabla u_s), \varphi_s)ds + \sum_{j=1}^{+\infty} \int_0^t (h_s^j(u_s, \nabla u_s), \varphi_s)dB_s^j + \int_0^t \int_{\mathcal{O}} \varphi_s(x)\nu(dx, ds)$$

4. *u admits a quasi-continuous version, $\tilde{u}$, and we have*

$$\int_0^T \int_{\mathcal{O}} (\tilde{u}(s,x) - S(s,x))\nu(dx, ds) = 0, \quad a.s..$$

The main result of this paper is the following:

THEOREM 4. *Under assumptions* **(H)**, **(I)** *and* **(O)**, *there exists a unique weak solution of the obstacle problem for the SPDE (1) associated to $(\xi,\ f,\ g,\ h,\ S)$.*
We denote by $\mathcal{R}(\xi, f, g, h, S)$ the solution of SPDE (1) with obstacle when it exists and is unique.

As the proof of this theorem is quite long, we split it in several steps: first we prove existence and uniqueness in the linear case then establish an Itô formula and finally prove the Theorem thanks to a fixed point argument.

5.2. *Proof of Theorem 4 in the linear case .* All along this subsection, we assume that f, g and h do not depend on u and ∇u, so we consider that f, g and h are adapted processes respectively in $L^2([0,T] \times \Omega \times \mathcal{O}; \mathbb{R})$, $L^2([0,T] \times \Omega \times \mathcal{O}; \mathbb{R}^d)$ and $L^2([0,T] \times \Omega \times \mathcal{O}; \mathbb{R}^{\mathbb{N}^*})$. For $n \in \mathbb{N}^*$, let u^n be the solution of the following SPDE

$$(13) \qquad du_t^n = Lu_t^n dt + f_t dt + \sum_{i=1}^d \partial_i g_{i,t} dt + \sum_{j=1}^{+\infty} h_{j,t} dB_t^j + n(u_t^n - S_t)^- dt$$

with initial condition $u_0^n = \xi$ and null Dirichlet boundary condition. We know from Theorem 8 in [8] that this equation admits a unique solution in $\mathcal{H}_T$ and that the solution admits $L^2(\mathcal{O})-$continuous trajectories.

LEMMA 4. *For all $n \in \mathbb{N}^*$, u^n satisfies the following estimate:*

$$E \sup_{t\in[0,T]} \parallel u_t^n \parallel^2 + E\int_0^T \mathcal{E}(u_t^n)dt + E\int_0^T n \parallel (u_t^n - S_t)^- \parallel^2 dt \leq C,$$

where C is a constant depending only on the structure constants of the SPDE.

PROOF. From (13) and (10), we know that $u^n - S'$ satisfies the following equation:

$$d(u_t^n - S_t') = L(u_t^n - S_t')dt + \tilde{f}_t dt + \sum_{i=1}^d \partial_i \tilde{g}_t^i dt + \sum_{j=1}^{+\infty} \tilde{h}_t^j dB_t^j + n(u_t^n - S_t)^- dt$$

where $\tilde{f} = f - f'$, $\tilde{g} = g - g'$and $\tilde{h} = h - h'$. Applying Itô's formula to $(u^n - S')^2$, we have:

$$\parallel u_t^n - S_t' \parallel^2 + 2\int_0^t \mathcal{E}(u_s^n - S_s')ds = 2\int_0^t ((u_s^n - S_s'), \tilde{f}_s)ds + 2\sum_{j=1}^{+\infty}\int_0^t ((u_s^n - S_s'), \tilde{h}_s^j)dB_s^j$$
$$-2\sum_{i=1}^d \int_0^t (\partial_i(u_s^n - \tilde{S}_s), \tilde{g}_s^i)ds + 2\int_0^t \int_{\mathcal{O}} (u_s^n - S_s')n(u_s^n - S_s)^- ds + \int_0^t \parallel |\tilde{h}_s| \parallel^2 ds, \quad a.s.$$

We remark first:

$$\begin{aligned}\int_0^t \int_{\mathcal{O}} (u_s^n - S_s')n(u_s^n - S_s)^- ds &= \int_0^t \int_{\mathcal{O}} (u_s^n - S_s + S_s - S_s')n(u_s^n - S_s)^- ds \\ &= -\int_0^t \int_{\mathcal{O}} n((u_s^n - S_s)^-)^2 ds + \int_0^t \int_{\mathcal{O}} (S_s - S_s')n(u_s^n - S_s)^- dxds\end{aligned}$$

the last term in the right member is non-positive because $S_t \leq S_t'$, thus,

$$\parallel u_t^n - S_t' \parallel^2 + 2\int_0^t \mathcal{E}(u_s^n - S_s')ds + 2\int_0^t n \parallel (u_s^n - S)^- \parallel^2 ds \leq 2\int_0^t (u_s^n - S_s', \tilde{f}_s)ds$$
$$-2\sum_{i=1}^d \int_0^t (\partial_i(u_s^n - S_s'), \tilde{g}_s^i)ds + 2\sum_{j=1}^{+\infty}\int_0^t (u_s^n - S_s', \tilde{h}_s^j)dB_s^j + \int_0^t \parallel |\tilde{h}_s| \parallel^2 ds, \quad a.s.$$

Then using Cauchy-Schwarz's inequality, we have $\forall t \in [0, T]$,

$$2|\int_0^t (u_s^n - S_s', \tilde{f}_s)ds| \leq \epsilon \int_0^T \parallel u_s^n - S_s' \parallel^2 ds + \frac{1}{\epsilon}\int_0^T \parallel \tilde{f}_s \parallel^2 ds$$

and

$$2|\sum_{i=1}^d \int_0^t (\partial_i(u_s^n - S_s'), \tilde{g}_s^i)ds| \leq \epsilon \int_0^T \parallel \nabla(u_s^n - S_s') \parallel^2 ds + \frac{1}{\epsilon}\int_0^T \parallel |\tilde{g}| \parallel^2 ds.$$

Moreover, thanks to Burkholder-Davies-Gundy's inequality, we get

$$\begin{aligned} E\sup_{t\in[0,T]}|\sum_{j=1}^{+\infty}\int_0^t(u_s^n-S'_s,\tilde{h}_s^j)dB_s^j| &\leq c_1E[\int_0^T\sum_{j=1}^{+\infty}(u_s^n-S'_s,\tilde{h}_s^j)^2ds]^{1/2} \\ &\leq c_1E[\int_0^T\sum_{j=1}^{+\infty}\sup_{s\in[0,T]}\parallel u_s^n-S'_s\parallel^2\parallel\tilde{h}_s^j\parallel^2\,ds]^{1/2} \\ &\leq c_1E[\sup_{s\in[0,T]}\parallel u_s^n-S'_s\parallel(\int_0^T\parallel|\tilde{h}_s|\parallel^2\,ds)^{1/2}] \\ &\leq \epsilon E\sup_{s\in[0,T]}\parallel u_s^n-S'_s\parallel^2+\frac{c_1}{4\epsilon}E\int_0^T\parallel|\tilde{h}_s|\parallel^2\,ds. \end{aligned}$$

Then using the strict ellipticity assumption and the inequalities above, we get

$$\begin{aligned} &(1-2\epsilon(T+1))E\sup_{t\in[0,T]}\parallel u_t^n-S'_t\parallel^2+(2\lambda-\epsilon)E\int_0^T\mathcal{E}(u_s^n-S'_s)ds+2E\int_0^T n\parallel(u_s^n-S_s)^-\parallel^2\,ds \\ &\leq C(E\parallel\xi\parallel^2+\frac{2}{\epsilon}E\int_0^T\parallel\tilde{f}_s\parallel^2+\frac{2}{\epsilon}\parallel|\tilde{g}_s|\parallel^2+(\frac{c_1}{2\epsilon}+1)\parallel|\tilde{h}_s|\parallel^2\,ds). \end{aligned}$$

We take ϵ small enough such that $(1-2\epsilon(T+1))>0$, this yields $(2\lambda-\epsilon)>0$

$$E\sup_{t\in[0,T]}\parallel u_t^n-S'_t\parallel^2+E\int_0^T\mathcal{E}(u_t^n-S'_t)dt+E\int_0^T n\parallel(u_t^n-S_t)^-\parallel^2\,dt\leq C.$$

Then with (11), we obtain the desired estimate. □

End of the proof of Theorem 4. We now introduce z, the solution of the corresponding SPDE without obstacle:

$$dz_t+Az_tdt=f_tdt+\sum_{i=1}^d\partial_ig_{i,t}dt+\sum_{j=1}^{+\infty}h_{j,t}dB_t^j,$$

starting from $z_0=\xi$, with null Dirichlet condition on the boundary. As a consequence of Theorem 3, we can take for z a quasi-continuous version.
For each $n\in\mathbb{N}^*$, we put $v^n=u^n-z$. Clearly, v^n satisfies

$$dv_t^n+Av_t^ndt=n(v_t^n-(S_t-z_t))^-dt=n(u_t^n-S_t)^-dt.$$

Since $S-z$ is quasi-continuous almost-surely, by the results established by Mignot and Puel in [18], we know that P-almost surely, the sequence $(v^n)_n$ is increasing and converges in $L^2([0,T]\times\mathcal{O})$ P-almost surely to v and that the sequence of random measures $\nu^{v^n}=n(u_t^n-S_t)^-dtdx$ converges vaguely to a measure associated to v: $\nu=\nu^v$. As a consequence of the previous lemma, $(u^n)_n$ and $(v^n)_n$ are bounded sequences in $L^2(\Omega\times[0,T];H_0^1(\mathcal{O}))$ which is an Hilbert space (equipped the norm $(E\int_0^T\parallel u_t\parallel^2_{H_0^1(\mathcal{O})}\,dt)^{1/2}$), by a double extraction argument, we can construct subsequences $(u^{n_k})_k$ and $(v^{n_k})_k$ such that the first one converges weakly in $L^2(\Omega\times[0,T];H_0^1(\mathcal{O}))$ to an element that we denote u and the

second one to an element which necessary is equal to v since $(v^n)_n$ is increasing. Moreover, we can construct sequences $(\hat{u}^n)_n$ and $(\hat{v}^n)_n$ of convex combinations of elements of the form

$$\hat{u}^n = \sum_{k=1}^{N_n} \alpha_k^n u^{n_k} \text{ and } \hat{v}^n = \sum_{k=1}^{N_n} \alpha_k^n v^{n_k}$$

converging strongly to u an v respectively in $L^2(\Omega \times [0,T]; H_0^1(\mathcal{O}))$.

From the fact that u^n is the weak solution of (13), we get

$$\begin{aligned}
&(u_t^n, \varphi_t) - (\xi, \varphi_0) - \int_0^t (u_s^n, \partial_s \varphi_s) ds + \int_0^t \mathcal{E}(u_s^n, \varphi_s) ds + \sum_{i=1}^d \int_0^t (g_s^i, \partial_i \varphi_s) ds \\
&= \int_0^t (f_s, \varphi_s) ds + \sum_{j=1}^{+\infty} \int_0^t (h_s^j, \varphi_s) dB_s^j + \int_0^t \int_{\mathcal{O}} \varphi_s(x) n (u_s^n - S_s)^- dx ds, \quad a.s.
\end{aligned} \tag{14}$$

Hence

$$\begin{aligned}
&(\hat{u}_t^n, \varphi_t) - (\xi, \varphi_0) - \int_0^t (\hat{u}_s^n, \partial_s \varphi_s) ds + \int_0^t \mathcal{E}(\hat{u}_s^n, \varphi_s) ds + \sum_{i=1}^d \int_0^t (g_s^i, \partial_i \varphi_s) ds \\
&\int_0^t (f_s, \varphi_s) ds + \sum_{j=1}^{+\infty} \int_0^t (h_s^j, \varphi_s) dB_s^j + \int_0^t \int_{\mathcal{O}} \varphi_s(x) \left(\sum_{k=1}^{N_n} n_k (u_s^{n_k} - S_s)^- \right) dx ds, \ a.s.
\end{aligned} \tag{15}$$

We have

$$\int_0^t \int_{\mathcal{O}} \varphi_s(x) \left(\sum_{k=1}^{N_n} n_k (u_s^{n_k} - S_s)^- \right) dx ds = \int_0^T -(\frac{\partial \varphi_t}{\partial t}, \hat{v}_t^n) dt + \int_0^T \mathcal{E}(\varphi_t, \hat{v}_t^n) dt$$

so that we have almost-surely, at least for a subsequence:

$$\begin{aligned}
\lim_{n \to +\infty} \int_0^t \int_{\mathcal{O}} \varphi_s(x) \left(\sum_{k=1}^{N_n} n_k (u_s^{n_k} - S_s)^- \right) dx ds &= \int_0^T -(\frac{\partial \varphi_t}{\partial t}, v_t) dt + \int_0^T \mathcal{E}(\varphi_t, v_t) dt \\
&= \int_0^T \int_{\mathcal{O}} \varphi_t(x) \nu(dx, dt).
\end{aligned}$$

As $(\hat{u}^n)_n$ converges to u in $L^2(\Omega \times [0,T]; H_0^1(\mathcal{O}))$, by making n tend to $+\infty$ in (15), we obtain:

$$\begin{aligned}
&(u_t, \varphi_t) - (\xi, \varphi_0) - \int_0^t (u_s, \partial_s \varphi_s) ds + \int_0^t \mathcal{E}(u_s, \varphi_s) ds + \sum_{i=1}^d \int_0^t (g_s^i, \partial_i \varphi_s) ds \\
&= \int_0^t (f_s, \varphi_s) ds + \sum_{j=1}^{+\infty} \int_0^t (h_s^j, \varphi_s) dB_s^j + \int_0^t \int_{\mathcal{O}} \varphi_s(x) \nu(dx, ds), \quad a.s.
\end{aligned}$$

In the next subsection, we'll show that u satisfies an Itô formula, as a consequence by applying it to u_t^2, using standard arguments we get that $u \in \mathcal{H}_T$ so for almost all $\omega \in \Omega$,

$u(\omega) \in \mathcal{K}$. And from Theorem 9 in [8], we know that for almost all $\omega \in \Omega$, $z(\omega) \in \mathcal{K}$. Therefore, for almost all $\omega \in \Omega$, $v(\omega) = u(w) - z(w) \in \mathcal{K}$. Hence, $\nu = \partial_t v + Av$ is a regular measure by definition. Moreover, by [20, 21] we know that v admits a quasi continuous version $\tilde{v}$ which satisfies the minimality condition

$$\int\int (\tilde{v} - S + \tilde{z})\nu(dxdt) = 0. \tag{16}$$

z is quasi-continuous version hence $\tilde{u} = z + \tilde{v}$ is a quasi-continuous version of u and we can write (16) as

$$\int\int (\tilde{u} - S)\nu(dxdt) = 0.$$

The fact that $u \geq S$ comes from the fact that $v \geq u - z$, so at this stage we have proved that (u, ν) is a solution to the obstacle problem we consider.
Uniqueness comes from the fact that both z and v are unique, which ends the proof of Theorem 4.

□

5.3. *Itô's formula.* The following Itô formula for the solution of the obstacle problem is fundamental to get all the results in the nonlinear case. Let us also remark, that any solution of the nonlinear equation (1) may be viewed as the solution of a linear one so that it satisfies also the Itô formula.

THEOREM 5. *Under assumptions of the previous subsection 5.2, let u be the solution of SPDE (1) with obstacle and $\Phi : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ be a function of class $\mathcal{C}^{1,2}$. We denote by Φ' and Φ'' the derivatives of Φ with respect to the space variables and by $\frac{\partial \Phi}{\partial t}$ the partial derivative with respect to time. We assume that these derivatives are bounded and $\Phi'(t, 0) = 0$ for all $t \geq 0$. Then $P - a.s.$ for all $t \in [0, T]$,*

$$\int_{\mathcal{O}} \Phi(t, u_t(x))dx + \int_0^t \mathcal{E}(\Phi'(s, u_s), u_s)ds = \int_{\mathcal{O}} \Phi(0, \xi(x))dx + \int_0^t \int_{\mathcal{O}} \frac{\partial \Phi}{\partial s}(s, u_s(x))dxds$$
$$+ \int_0^t (\Phi'(s, u_s), f_s)ds - \sum_{i=1}^{d} \int_0^t \int_{\mathcal{O}} \Phi''(s, u_s(x))\partial_i u_s(x) g_i(x)dxds + \sum_{j=1}^{+\infty} \int_0^t (\Phi'(s, u_s), h_j)dB_s^j$$
$$+ \frac{1}{2} \sum_{j=1}^{+\infty} \int_0^t \int_{\mathcal{O}} \Phi''(s, u_s(x))(h_{j,s}(x))^2 dxds + \int_0^t \int_{\mathcal{O}} \Phi'(s, \tilde{u}_s(x))\nu(dxds).$$

PROOF. We keep the same notations as in the previous subsection and so consider the sequence $(u^n)_n$ approximating u and also $(\hat{u}^n)_n$ the sequence of convex combinations $\hat{u}^n = \sum_{k=1}^{N_n} \alpha_k^n u^{n_k}$ converging strongly to u in $L^2(\Omega \times [0, T]; H_0^1(\mathcal{O}))$.
Moreover, by standard arguments such as the Banach-Saks theorem, since $(u^n)_n$ is non-decreasing, we can choose the convex combinations such that $(\hat{u}^n)_n$ is also a non-decreasing sequence. We start by a key lemma:

LEMMA 5. *Let $t \in [0, T]$, then*

$$\lim_{n \to +\infty} E \int_0^t \int_{\mathcal{O}} (\hat{u}_s^n - S_s)^- \sum_{k=1}^{N_n} \alpha_k^n n_k (u_s^{n_k} - S_s)^- dxds = 0.$$

PROOF. We write as above $u^n = v^n + z$ and we denote $\hat{\nu}^n = \sum_{k=1}^{N_n} \alpha_k^n n_k (u_s^{n_k} - S_s)^-$ so that

$$\int_0^t \int_{\mathcal{O}} (\hat{u}_s^n - S_s)\hat{\nu}^n(dxds) = \int_0^t \int_{\mathcal{O}} \hat{v}_s^n \hat{\nu}^n(dxds) + \int_0^t \int_{\mathcal{O}} (z_s - S_s)\hat{\nu}^n(dxds)$$

From Lemma 1, we know that

$$\int_0^t \int_{\mathcal{O}} (z_s - S_s)\hat{\nu}^n(dxds) \to \int_0^t \int_{\mathcal{O}} (z_s - S_s)\nu(dxds).$$

Moreover, by Lemma II.6 in [20] we have for all n:

$$\frac{1}{2} \parallel \hat{v}_T^n \parallel^2 + \int_0^T \mathcal{E}(\hat{v}_s^n)ds = \int_0^T \int_{\mathcal{O}} \hat{v}_s^n \hat{\nu}^n(dxds),$$

and

$$\frac{1}{2} \parallel v_T \parallel^2 + \int_0^T \mathcal{E}(v_s)ds = \int_0^T \int_{\mathcal{O}} \tilde{v}_s \nu(dxds).$$

As $(\hat{v}^n)_n$ tends to v in $L^2([0,T], H_0^1(\mathcal{O}))$,

$$\lim_{n\to+\infty} \int_0^T \mathcal{E}(\hat{v}_s^n)ds = \int_0^T \mathcal{E}(v_s)ds.$$

Let us prove that $(\parallel \hat{v}_T^n \parallel)_n$ tends to $\parallel v_T \parallel$.
Since, $(\hat{v}_T^n)_n$ is non-decreasing and bounded in $L^2(\mathcal{O})$ it converges in $L^2(\mathcal{O})$ to $m = \sup_n \hat{v}_T^n$.
Let $\rho \in H_0^1(\mathcal{O})$ then the map defined by $\varphi(t,x) = \rho(x)$ belongs to $\mathcal{W}$ hence as a consequence of Proposition 4

$$\int_{[0,T[\times\mathcal{O}} \rho d\hat{\nu}^n = \int_0^T \mathcal{E}(\rho, \hat{v}_s^n)\, ds + (\rho, \hat{v}_T^n),$$

and

$$\int_{[0,T[\times\mathcal{O}} \tilde{\rho} d\nu = \int_0^T \mathcal{E}(\rho, v_s)\, ds + (\rho, v_T),$$

making n tend to $+\infty$ and using one more time Lemma 1, we get

$$\lim_{n\to+\infty} (\rho, \hat{v}_T^n) = (\rho, m) = (\rho, v_T).$$

Since ρ is arbitrary, we have $v_T = m$ and so $\lim_{n\to+\infty} \parallel \hat{v}_T^n \parallel = \parallel v_T \parallel$ and this yields

$$\lim_{n\to+\infty} \int_0^T \int_{\mathcal{O}} \hat{v}_s^n \hat{\nu}^n(dxds) = \int_0^T \int_{\mathcal{O}} \tilde{v}_s \nu(dxds) = \int_0^T \int_{\mathcal{O}} (S_s - z_s)\nu(dxds).$$

This proves that

$$\lim_{n\to+\infty} \int_0^t \int_{\mathcal{O}} (\hat{u}_s^n - S_s)\hat{\nu}^n(dxds) = 0,$$

we conclude by remarking that

$$\lim_{n\to+\infty} \int_0^t \int_{\mathcal{O}} (\hat{u}_s^n - S_s)^+ \hat{\nu}^n(dxds) \leq \lim_{n\to+\infty} \int_0^t \int_{\mathcal{O}} (u_s - S_s)\hat{\nu}^n(dxds) = \int_0^t \int_{\mathcal{O}} (\tilde{u}_s - S_s)\nu(dxds) = 0.$$

□

Proof of Theorem 5: We consider the penalized solution $(u^n)_n$, we know that its convex combination $(\hat{u}^n)_n$ converges strongly to u in $L^2(\Omega \times [0,T]; H_0^1(\mathcal{O}))$. And $\hat{u}^n$ satisfies the following SPDE

$$d\hat{u}_t^n + A\hat{u}_t^n dt = f_t dt + \sum_{i=1}^{d} \partial_i g_t^i dt + \sum_{j=1}^{+\infty} h_t^j dB_t^j + \sum_{k=1}^{N_n} \alpha_k^n n_k (u_s^{n_k} - S_s)^- dt$$

From the Itô formula for the solution of SPDE without obstacle (see Lemma 7 in [9]), we have, almost surely, for all $t \in [0,T]$,

$$\begin{aligned}
&\int_{\mathcal{O}} \Phi(t, \hat{u}_t^n(x))dx + \int_0^t \mathcal{E}(\Phi'(s,\hat{u}_s^n), \hat{u}_s^n)ds = \int_{\mathcal{O}} \Phi(0,\xi(x))dx + \int_0^t \int_{\mathcal{O}} \frac{\partial \Phi}{\partial s}(s,\hat{u}_s^n)dxds \\
&+ \int_0^t (\Phi'(s,\hat{u}_s^n), f_s)ds - \sum_{i=1}^{d} \int_0^t \int_{\mathcal{O}} \Phi''(s,\hat{u}_s^n(x))\partial_i \hat{u}_s^n(x) g_i(x) dxds + \sum_{j=1}^{+\infty} \int_0^t (\Phi'(s,\hat{u}_s^n), h_j) dB_s^j \\
&+ \frac{1}{2} \sum_{j=1}^{+\infty} \int_0^t \int_{\mathcal{O}} \Phi''(s,\hat{u}_s^n(x))(h_j(x))^2 dxds + \int_0^t \int_{\mathcal{O}} \Phi'(s,\hat{u}_s^n) \sum_{k=1}^{N_n} \alpha_k^n n_k (u_s^{n_k} - S_s)^- dxds.
\end{aligned}$$

Because of the strong convergence of $(\hat{u}^n)_n$, the convergence of all the terms except the last one are clear. To obtain the convergence of the last term, we do as follows:

$$\begin{aligned}
\int_0^t \int_{\mathcal{O}} \Phi'(s,\hat{u}_s^n) \sum_{k=1}^{N_n} \alpha_k^n n_k (u_s^{n_k} - S_s)^- dxds &= \int_0^t \int_{\mathcal{O}} (\Phi'(s,\hat{u}_s^n) - \Phi'(s,S_s)) \sum_{k=1}^{N_n} \alpha_k^n n_k (u_s^{n_k} - S_s)^- dxds \\
&+ \int_0^t \int_{\mathcal{O}} \Phi'(s,S_s) \sum_{k=1}^{N_n} \alpha_k^n n_k (u_s^{n_k} - S_s)^- dxds.
\end{aligned}$$

For the first term in the right member, we have:

$$\begin{aligned}
&\Big| \int_0^t \int_{\mathcal{O}} (\Phi'(s,\hat{u}_s^n) - \Phi'(s,S_s)) \sum_{k=1}^{N_n} \alpha_k^n n_k (u_s^{n_k} - S_s)^- dxds \Big| \\
&\leq C \int_0^t \int_{\mathcal{O}} |\hat{u}_s^n - S_s| \cdot \sum_{k=1}^{N_n} \alpha_k^n n_k (u_s^{n_k} - S_s)^- dxds \\
&= C \int_0^t \int_{\mathcal{O}} ((\hat{u}_s^n - S_s)^+ + (\hat{u}_s^n - S_s)^-) \sum_{k=1}^{N_n} \alpha_k^n n_k (u_s^{n_k} - S_s)^- dxds \\
&= C \int_0^t \int_{\mathcal{O}} (\hat{u}_s^n - S_s)^+ \sum_{k=1}^{N_n} \alpha_k^n n_k (u_s^{n_k} - S_s)^- dxds + C \int_0^t \int_{\mathcal{O}} (\hat{u}_s^n - S_s)^- \sum_{k=1}^{N_n} \alpha_k^n n_k (u_s^{n_k} - S_s)^- dxds
\end{aligned}$$

We have the following inequality because $(\hat{u}^n)_n$ converges to u increasingly:

$$\begin{aligned}
\int_0^t \int_{\mathcal{O}} (\hat{u}_s^n - S_s)^- \sum_{k=1}^{N_n} \alpha_k^n n_k (u_s^{n_k} - S_s)^- dxds &\leq \int_0^t \int_{\mathcal{O}} (u_s - S_s)^+ \sum_{k=1}^{N_n} \alpha_k^n n_k (u_s^{n_k} - S_s)^- dxds \\
&= \int_0^t \int_{\mathcal{O}} (u_s - S_s) \sum_{k=1}^{N_n} \alpha_k^n n_k (u_s^{n_k} - S_s)^- dxds
\end{aligned}$$

With Lemma 1, we know that

$$\lim_{n\to\infty}\int_0^t\int_{\mathcal{O}}(u_s-S_s)\sum_{k=1}^{N_n}\alpha_k^n n_k(u_s^{n_k}-S_s)^-dxds\to\int_0^t\int_{\mathcal{O}}(\tilde{u}_s-\tilde{S}_s)\nu(dxds)=0.$$

And from Lemma 5, we have

$$\int_0^t\int_{\mathcal{O}}(\hat{u}_s^n-S_s)^-\sum_{k=1}^{N_n}\alpha_k^n n_k(u_s^{n_k}-S_s)^-dxds\to 0.$$

Therefore,

$$\int_0^t\int_{\mathcal{O}}(\Phi'(s,\hat{u}_s^n)-\Phi'(s,S_s))\sum_{k=1}^{N_n}\alpha_k^n n_k(u_s^{n_k}-S_s)^-dxds\to 0.$$

Moreover, with Lemma 1, we have

$$\int_0^t\int_{\mathcal{O}}\Phi'(s,S_s)\sum_{k=1}^{N_n}\alpha_k^n n_k(u_s^{n_k}-S_s)^-dxds\to\int_0^t\int_{\mathcal{O}}\Phi'(s,S_s)\nu(dxds)$$

and

$$\begin{aligned}|\int_0^t\int_{\mathcal{O}}\Phi'(s,u_s)\nu(dxds)-\int_0^t\int_{\mathcal{O}}\Phi'(s,S_s)\nu(dxds)| &\leq C\int_0^t\int_{\mathcal{O}}|\tilde{u}_s-S_s|\nu(dxds)\\ &= C\int_0^t\int_{\mathcal{O}}(\tilde{u}_s-S_s)\nu(dxds)=0\end{aligned}$$

Therefore, taking limit, we get the desired Itô formula. □

5.4. *Itô's formula for the difference of the solutions of two OSPDEs.* We still consider (u,ν) solution of the linear equation as in Subsection 5.2

$$\left\{\begin{array}{rcl} du_t & + & Au_tdt=f_tdt+\sum_{i=1}^d\partial_i g_t^i dt+\sum_{j=1}^{+\infty}h_t^j dB_t^j+\nu(dt,x)\\ u & \geq & S\end{array}\right.$$

and consider another linear equation with adapted coefficients $\bar{f}$, $\bar{g}$, $\bar{h}$ respectively in $L^2([0,T]\times\Omega\times\mathcal{O};\mathbb{R})$, $L^2([0,T]\times\Omega\times\mathcal{O};\mathbb{R}^d)$ and $L^2([0,T]\times\Omega\times\mathcal{O};\mathbb{R}^{\mathbb{N}^*})$ and obstacle $\bar{S}$ which satisfies the same hypotheses (**O**) as S i.e; $\bar{S}_0\leq\xi$ and $\bar{S}$ is dominated by the solution of an SPDE (not necessarily the same as S). We denote by $(y,\bar{\nu})$ the unique solution to the associated SPDE with obstacle with initial condition $y_0=u_0=\xi$.

$$\left\{\begin{array}{rcl} dy_t & + & Ay_tdt=\bar{f}_tdt+\sum_{i=1}^d\partial_i\bar{g}_t^i dt+\sum_{j=1}^{+\infty}\bar{h}_t^j dB_t^j+\bar{\nu}(dt,x)\\ y & \geq & \bar{S}\end{array}\right.$$

THEOREM 6. *Let Φ as in Theorem 5, then the difference of the two solutions satisfy the following Itô formula for all $t \in [0,T]$:*

$$\int_{\mathcal{O}} \Phi(t, u_t(x) - y_t(x))dx + \int_0^t \mathcal{E}(\Phi'(s, u_s - y_s), u_s - y_s)ds = \int_0^t (\Phi'(s, u_s - y_s), f_s - \bar{f}_s)ds$$
$$- \sum_{i=1}^d \int_0^t \int_{\mathcal{O}} \Phi''(s, u_s - y_s)\partial_i(u_s - y_s)(g_s^i - \bar{g}_s^i)dxds + \sum_{j=1}^{+\infty} \int_0^t (\Phi'(s, u_s - y_s), h_s^j - \bar{h}_s^j)dB_s^j$$
$$+ \frac{1}{2}\sum_{j=1}^{+\infty} \int_0^t \int_{\mathcal{O}} \Phi''(s, u_s - y_s)(h_s^j - \bar{h}_s^j)^2 dxds + \int_0^t \int_{\mathcal{O}} \frac{\partial \Phi}{\partial s}(s, u_s - y_s)dxds$$
$$\int_0^t \int_{\mathcal{O}} \Phi'(s, \tilde{u}_s - \tilde{y}_s)(\nu - \bar{\nu})(dx, ds) \quad a.s. \tag{17}$$

PROOF. We begin with the penalized solutions. The corresponding penalization equations are

$$du_t^n + Au_t^n dt = f_t dt + \sum_{i=1}^d \partial_i g_t^i dt + \sum_{j=1}^{+\infty} h_t^j dB_t^j + n(u_t^n - S_t)^- dt$$

and

$$dy_t^m + Ay_t^m dt = \bar{f}_t dt + \sum_{i=1}^d \partial_i \bar{g}_t^i dt + \sum_{j=1}^{+\infty} \bar{h}_t^j dB_t^j + m(y_t^m - \bar{S}_t)^- dt$$

from the proofs above, we know that the penalized solution converges weakly to the solution and we can take convex combinations $\hat{u}^n = \sum_{i=1}^{N_n} \alpha_i^n u^{n_i}$ and $\hat{y}^n = \sum_{i=1}^{N'_n} \beta_i^n y^{n'_i}$ such that $(\hat{u}^n)_n$ and $(\hat{y}^n)_n$ are non-decreasing and converge strongly to u and y respectively in $L^2(\Omega \times [0,T], H_0^1(\mathcal{O}))$ as n tends to $+\infty$.
As in the proof of Theorem 5, we first establish a key lemma:

LEMMA 6. *For all $t \in [0,T]$,*

$$\lim_{n\to+\infty} E\int_0^t \int_{\mathcal{O}} \hat{u}_s^n \sum_{k=1}^{N'_n} \beta_k^n n'_k (y_s^{n'_k} - \bar{S}_s)^- dxds = E\int_0^t \int_{\mathcal{O}} \tilde{u}\bar{\nu}(ds, dx),$$

and

$$\lim_{n\to+\infty} E\int_0^t \int_{\mathcal{O}} \hat{y}_s^n \sum_{k=1}^{N_n} \alpha_k^n n_k (u_s^{n_k} - S_s)^- dxds = E\int_0^t \int_{\mathcal{O}} \tilde{y}\nu(ds, dx).$$

PROOF. We put for all n:

$$\nu^n(ds, dx) = \sum_{k=1}^{N_n} \alpha_k^n n_k (u_s^{n_k} - S_s)^- dxds \text{ and } \bar{\nu}^n(ds, dx) = \sum_{k=1}^{N'_n} \beta_k^n n'_k (y_s^{n'_k} - \bar{S}_s)^- dxds.$$

As in the proof of Lemma 5, we write for all $n \in \mathbb{N}^*$: $u^n = z + v^n$.
In the same spirit, we introduce $\bar{z}$ the solution of the linear SPDE:

$$d\bar{z}_t + A\bar{z}_t = \bar{f}_t dt + \sum_{i=1}^d \partial_i \bar{g}_t^i dt + \sum_{j=1}^{+\infty} \bar{h}_t^j dB_t^j,$$

with initial condition $\bar{z}_0 = \xi$ and put $\forall n \in \mathbb{N}^*$, $\bar{v}^n = y^n - \bar{z}$, $\hat{\bar{v}}^n = \hat{y}^n - \bar{z}$ and $\bar{v} = y - \bar{z}$. As a consequence of Lemma II.6 in [21], we have for all $n \in \mathbb{N}^*$, P-almost surely:

$$\frac{1}{2} \parallel \hat{v}_t^n - \hat{\bar{v}}_t^n \parallel^2 + \int_0^t \mathcal{E}(\hat{v}_s^n - \hat{\bar{v}}_s^n)ds = \int_0^t \int_{\mathcal{O}} (\hat{v}_s^n - \hat{\bar{v}}_s^n)(\nu^n - \bar{\nu}^n)(dx, ds),$$

and

$$\frac{1}{2} \parallel v_t - \bar{v}_t \parallel^2 + \int_0^t \mathcal{E}(v_s - \bar{v}_s)ds = \int_0^t \int_{\mathcal{O}} (\tilde{v}_s - \tilde{\bar{v}}_s)(\nu - \bar{\nu})(dx, ds).$$

But, as in the proof of Lemma 5, we get that $(\hat{v}_t^n - \hat{\bar{v}}_t^n)_n$ tends to $v_t - \bar{v}_t$ in $L^2(\mathcal{O})$ almost surely and

$$\lim_n \int_0^t \int_{\mathcal{O}} \hat{v}_s^n \nu^n(dx, ds) = \int_0^t \int_{\mathcal{O}} \tilde{v}_s \nu(dx, ds),$$
$$\lim_n \int_0^t \int_{\mathcal{O}} \hat{\bar{v}}_s^n \bar{\nu}^n(dx, ds) = \int_0^t \int_{\mathcal{O}} \tilde{\bar{v}}_s \bar{\nu}(dx, ds).$$

This yields:

$$\lim_n \left(\int_0^t \int_{\mathcal{O}} \hat{v}_s^n \bar{\nu}^n(dx, ds) + \int_0^t \int_{\mathcal{O}} \hat{\bar{v}}_s^n \nu^n(dx, ds) \right) = \int_0^t \int_{\mathcal{O}} \tilde{v}_s \bar{\nu}(dx, ds) + \int_0^t \int_{\mathcal{O}} \tilde{\bar{v}}_s \nu(dx, ds).$$

But, we have

$$\limsup_n \int_0^t \int_{\mathcal{O}} \hat{v}_s^n \bar{\nu}^n(dx, ds) \quad \leq \quad \limsup_n \int_0^t \int_{\mathcal{O}} v_s \bar{\nu}^n(dx, ds) = \int_0^t \int_{\mathcal{O}} \tilde{v}_s \bar{\nu}(dx, ds),$$

and in the same way:

$$\limsup_n \int_0^t \int_{\mathcal{O}} \hat{\bar{v}}_s^n \nu^n(dx, ds) \leq \int_0^t \int_{\mathcal{O}} \tilde{\bar{v}}_s \nu(dx, ds).$$

Let us remark that these inequalities also hold for any subsequence. From this, it is easy to deduce that necessarily:

$$\lim_n \int_0^t \int_{\mathcal{O}} \hat{v}_s^n \bar{\nu}^n(dx, ds) = \int_0^t \int_{\mathcal{O}} \tilde{v}_s \bar{\nu}(dx, ds),$$

and

$$\lim_n \int_0^t \int_{\mathcal{O}} \hat{\bar{v}}_s^n \nu^n(dx, ds) = \int_0^t \int_{\mathcal{O}} \tilde{\bar{v}}_s \nu(dx, ds).$$

We end the proof of this lemma by using similar arguments as in the proof of Lemma 5. □

End of the proof of Theorem 6: We begin with the equation which $\hat{u}^n - \hat{y}^n$ satisfies:

$$\begin{aligned} d(\hat{u}_t^n - \hat{y}_t^n) + A(\hat{u}_t^n - \hat{y}_t^n)dt &= (f_t - \bar{f}_t)dt + \sum_{i=1}^{d} \partial_i (g_t^i - \bar{g}_t^i)dt + \sum_{j=1}^{+\infty} (h_t^j - \bar{h}_t^j)dB_t^j \\ &+ (\nu^n - \bar{\nu}^n)(x, dt) \end{aligned}$$

Applying Itô's formula to $\Phi(\hat{u}^n - \hat{y}^n)$, we have

$$\int_{\mathcal{O}} \Phi(t, \hat{u}_t^n(x) - \hat{y}_t^n(x))dx + \int_0^t \mathcal{E}(\Phi'(s, \hat{u}_s^n - \hat{y}_s^n), \hat{u}_s^n - \hat{y}_s^n)ds = \int_0^t (\Phi'(s, \hat{u}_s^n - \hat{y}_s^n), f_s - \bar{f}_s)ds$$
$$- \sum_{i=1}^d \int_0^t \int_{\mathcal{O}} \Phi''(s, \hat{u}_s^n - \hat{y}_s^n)\partial_i(\hat{u}_s^n - \hat{y}_s^n)(g_s^i - \bar{g}_s^i)dxds + \sum_{j=1}^{+\infty} \int_0^t (\Phi'(s, \hat{u}_s^n - \hat{y}_s^n), h_s^j - \bar{h}_s^j)dB_s^j$$
$$+ \frac{1}{2}\sum_{j=1}^{+\infty} \int_0^t \int_{\mathcal{O}} \Phi''(s, \hat{u}_s^n - \hat{y}_s^n)(h_s^j - \bar{h}_s^j)^2 dxds + \int_0^t \int_{\mathcal{O}} \frac{\partial \Phi}{\partial s}(s, \hat{u}_s^n - \hat{y}_s^n)dxds$$
$$+ \int_0^t \int_{\mathcal{O}} \Phi'(s, \hat{u}_s^n - \hat{y}_s^n)(\nu^n - \bar{\nu}^n)(dx, dt) \quad a.s.$$

Because that $(\hat{u}^n)_n$ and $(\hat{y}^n)_n$ converge strongly to u and y respectively, the convergence of all the terms except the last term are clear. For the convergence of the last term, we do as follows:

$$\left| \int_0^t \int_{\mathcal{O}} [\Phi'(s, \hat{u}_s^n - \hat{y}_s^n) - \Phi'(s, u_s - \hat{y}_s^n)]\nu^n(dxds) + \int_0^t \int_{\mathcal{O}} [\Phi'(s, u_s - \hat{y}_s^n) - \Phi'(s, u_s - y_s)]\nu^n(dxds) \right|$$
$$\leq C \int_0^t \int_{\mathcal{O}} |\hat{u}_s^n - u_s|\nu^n(dxds) + \int_0^t \int_{\mathcal{O}} |\hat{y}_s^n - y_s|\nu^n(dx, ds)$$

As a consequence of Lemma 5 and using the fact that $\hat{u}^n \leq u$:

$$\lim_n \int_0^t \int_{\mathcal{O}} |\hat{u}_s^n - u_s|\nu^n(dxds) = \lim_n \int_0^t \int_{\mathcal{O}} (u_s - \hat{u}_s^n)\nu^n(dxds) = 0.$$

By Lemma 6 and the fact that $\hat{y}^n \leq y$:

$$\lim_n \int_0^t \int_{\mathcal{O}} |\hat{y}_s^n - y_s|\nu^n(dx, ds) = \lim_n \int_0^t \int_{\mathcal{O}} (y_s - \hat{y}_s^n)\nu^n(dx, ds) = 0,$$

this yields:

$$\lim_n \int_0^t \int_{\mathcal{O}} (\Phi'(s, \hat{u}_s^n - \hat{y}_s^n) - \Phi'(s, u_s - y_s)\nu^n(dx, dt) = 0,$$

but by Lemma 1, we know that

$$\lim_n \int_0^t \int_{\mathcal{O}} \Phi'(s, u_s - y_s)\nu^n(dx, dt) = \int_0^t \int_{\mathcal{O}} \Phi'(s, \tilde{u}_s - \tilde{y}_s)\bar{\nu}(dx, dt),$$

so

$$\lim_n \int_0^t \int_{\mathcal{O}} \Phi'(s, \hat{u}_s^n - \hat{y}_s^n)\nu^n(dx, dt) = \int_0^t \int_{\mathcal{O}} \Phi'(s, \tilde{u}_s - \tilde{y}_s)\nu(dx, dt).$$

In the same way, we prove:

$$\lim_n \int_0^t \int_{\mathcal{O}} \Phi'(s, \hat{u}_s^n - \hat{y}_s^n)\bar{\nu}^n(dx, dt) = \int_0^t \int_{\mathcal{O}} \Phi'(s, \tilde{u}_s - \tilde{y}_s)\bar{\nu}(dx, dt).$$

The proof is now complete. □

5.5. *Proof of Theorem 4 in the nonlinear case.* Let γ and δ be 2 positive constants. On $L^2(\Omega\times[0,T];H_0^1(\mathcal{O}))$, we introduce the norm

$$\| u \|_{\gamma,\delta}= E(\int_0^T e^{-\gamma s}(\delta \| u_s \|^2 + \| \nabla u_s \|^2)ds),$$

which clearly defines an equivalent norm on $L^2(\Omega\times[0,T];H_0^1(\mathcal{O}))$.
Let us consider the Picard sequence $(u^n)_n$ defined by $u^0=\xi$ and for all $n\in\mathbb{N}^*$ we denote by (u^{n+1},ν^{n+1}) the solution of the linear SPDE with obstacle

$$(u^{n+1},\nu^{n+1})=\mathcal{R}(\xi,f(u^n,\nabla u^n),g(u^n,\nabla u^n),h(u^n,\nabla u^n),S).$$

Then, by Itô's formula (17), we have almost surely

$$\begin{aligned}
&e^{-\gamma T}\| u_T^{n+1}-u_T^n\|^2 +2\int_0^T e^{-\gamma s}\mathcal{E}(u_s^{n+1}-u_s^n)ds=-\gamma\int_0^T e^{-\gamma s}\| u_s^{n+1}-u_s^n\|^2 ds\\
&+2\int_0^T e^{-\gamma s}(\hat f_s,u_s^{n+1}-u_s^n)ds-2\sum_{i=1}^d\int_0^T e^{-\gamma s}(\hat g_s^i,\partial_i(u_s^{n+1}-u_s^n))ds+2\sum_{j=1}^{+\infty}\int_0^T e^{-\gamma s}(\hat h_s^j,u_s^{n+1}-u_s^n)dB_s^j\\
&+\int_0^T e^{-\gamma s}\| |\hat h_s|\|^2 ds+2\int_0^T\int_{\mathcal{O}} e^{-\gamma s}(u_s^{n+1}-u_s^n)(\nu^{n+1}-\nu^n)(dxds)
\end{aligned}$$

where $\hat f = f(u^n,\nabla u^n)-f(u^{n-1},\nabla u^{n-1})$, $\hat g = g(u^n,\nabla u^n)-g(u^{n-1},\nabla u^{n-1})$ and $\hat h = h(u^n,\nabla u^n)-h(u^{n-1},\nabla u^{n-1})$. Clearly, the last term is non-positive so using Cauchy-Schwarz's inequality and the Lipschitz conditions on f, g and h, we have

$$\begin{aligned}
2\int_0^T e^{-\gamma s}(u_s^{n+1}-u_s^n,\hat f_s)ds &\le \frac{1}{\epsilon}\int_0^T e^{-\gamma s}\| u_s^{n+1}-u_s^n\|^2 ds+\epsilon\int_0^T\| \hat f_s\|^2 ds\\
&\le \frac{1}{\epsilon}\int_0^T e^{-\gamma s}\| u_s^{n+1}-u_s^n\|^2 ds+C\epsilon\int_0^T e^{-\gamma s}\| u_s^n-u_s^{n-1}\|^2 ds\\
&+ C\epsilon\int_0^T e^{-\gamma s}\| \nabla(u_s^n-u_s^{n-1})\|^2 ds
\end{aligned}$$

and

$$\begin{aligned}
&2\sum_{i=1}^d\int_0^T e^{-\gamma s}(\hat g_s^i,\partial_i(u_s^{n+1}-u_s^n))ds\le 2\int_0^T e^{-\gamma s}\| \nabla(u_s^{n+1}-u_s^n)(C\| u_s^n-u_s^{n-1}\|\\
&+\alpha\| \nabla(u_s^n-u_s^{n-1})\|)ds\le C\epsilon\int_0^T e^{-\gamma s}\| \nabla(u_s^{n+1}-u_s^n)\|^2 ds+\frac{C}{\epsilon}\int_0^T e^{-\gamma s}\| u_s^n-u_s^{n-1}\|^2 ds\\
&+\alpha\int_0^T e^{-\gamma s}\| \nabla(u_s^{n+1}-u_s^n)\|^2 ds+\alpha\int_0^T e^{-\gamma s}\| u_s^n-u_s^{n-1}\|^2 ds
\end{aligned}$$

and

$$\int_0^T e^{-\gamma s}\| |\hat h_s|\|^2 ds\le C(1+\frac{1}{\epsilon})\int_0^T e^{-\gamma s}\| u_s^n-u_s^{n-1}\|^2 ds+\beta^2(1+\epsilon)\int_0^T e^{-\gamma s}\| \nabla(u_s^n-u_s^{n-1})\|^2 ds$$

where C, α and β are the constants in the Lipschitz conditions. Using the elliptic condition and taking expectation, we get:

$$(\gamma - \frac{1}{\epsilon})E\int_0^T e^{-\gamma s} \parallel u_s^{n+1} - u_s^n \parallel^2 ds + (2\lambda - \alpha)E\int_0^T e^{-\gamma s} \parallel \nabla(u_s^{n+1} - u_s^n) \parallel^2 ds \leq$$
$$C(1+\epsilon+\frac{2}{\epsilon})\int_0^T e^{-\gamma s} \parallel u_s^n - u_s^{n-1} \parallel^2 ds + (C\epsilon + \alpha + \beta^2(1+\epsilon))E\int_0^T e^{-\gamma s} \parallel \nabla(u_s^n - u_s^{n-1}) \parallel^2 ds$$

We choose ϵ small enough and then γ such that

$$C\epsilon + \alpha + \beta^2(1+\epsilon) < 2\lambda - \alpha \; and \; \frac{\gamma - 1/\epsilon}{2\lambda - \alpha} = \frac{C(1+\epsilon+2/\epsilon)}{C\epsilon + \alpha + \beta^2(1+\epsilon)}$$

If we set $\delta = \frac{\gamma - 1/\epsilon}{2\lambda - \alpha}$, we have the following inequality:

$$\parallel u^{n+1} - u^n \parallel_{\gamma,\delta} \leq \frac{C\epsilon + \alpha + \beta^2(1+\epsilon)}{2\lambda - \alpha} \parallel u^n - u^{n-1} \parallel_{\gamma,\delta} \leq ... \leq (\frac{C\epsilon + \alpha + \beta^2(1+\epsilon)}{2\lambda - \alpha})^n \parallel u^1 \parallel_{\gamma,\delta}$$

when $n \to \infty$, $(\frac{C\epsilon+\alpha+\beta^2(1+\epsilon)}{2\lambda-\alpha})^n \to 0$, we deduce that $(u^n)_n$ converges strongly to u in $L^2(\Omega \times [0,T]; H_0^1(\mathcal{O}))$.
Moreover, as $(u^{n+1}, \nu^{n+1}) = \mathcal{R}(\xi, f(u^n, \nabla u^n), g(u^n, \nabla u^n), h(u^n, \nabla u^n), S)$, we have for any $\varphi \in \mathcal{D}$:

$$(u_t^{n+1}, \varphi_t) - (\xi, \varphi_0) - \int_0^t (u_s^n, \partial_s \varphi_s) ds + \int_0^t \mathcal{E}(u_s^{n+1}, \varphi_s) ds + \sum_{i=1}^d \int_0^t (g_s^i(u_s^n, \nabla u_s^n), \partial_i \varphi_s) ds$$
$$= \int_0^t (f_s(u_s^n, \nabla u_s^n), \varphi_s) ds + \sum_{j=1}^{+\infty} \int_0^t (h_s^j(u_s^n, \nabla u_s^n), \varphi_s) dB_s^j + \int_0^t \int_{\mathcal{O}} \varphi_s(x) \nu^{n+1}(dxds). \;\; a.s.$$

Let v^{n+1} the random parabolic potential associated to ν^{n+1}:

$$\nu^{n+1} = \partial_t v^{n+1} + A v^{n+1}.$$

We denote $z^{n+1} = u^{n+1} - v^{n+1}$, so

$$z^{n+1} = \mathcal{U}(\xi, f(u^n, \nabla u^n), g(u^n, \nabla u^n), h(u^n, \nabla u^n))$$

converges strongly to z in $L^2(\Omega \times [0,T]; H_0^1(\mathcal{O}))$. As a consequence of the strong convergence of $(u^{n+1})_n$, we deduce that $(v^{n+1})_n$ converges strongly to v in $L^2(\Omega \times [0,T]; H_0^1(\mathcal{O}))$. Therefore, for fixed ω,

$$\int_0^t (-\frac{\partial_s \varphi_s}{\partial s}, v_s) ds + \int_0^t \mathcal{E}(\varphi_s, v_s) ds = \lim \int_0^t (-\frac{\partial_s \varphi_s}{\partial s}, v_s^{n+1}) ds + \int_0^t \mathcal{E}(\varphi_s, v_s^{n+1}) ds \geq 0$$

i.e. $v(\omega) \in \mathcal{P}$. Then from Proposition 1, we obtain a regular measure associated with v, and $(\nu^{n+1})_n$ converges vaguely to ν.
Taking the limit, we obtain

$$(u_t, \varphi_t) - (\xi, \varphi_0) - \int_0^t (u_s, \partial_s \varphi_s) ds + \int_0^t \mathcal{E}(u_s, \varphi_s) ds + \sum_{i=1}^d \int_0^t (g_s^i(u_s, \nabla u_s), \partial_i \varphi_s) ds$$
$$= \int_0^t (f_s(u_s, \nabla u_s), \varphi_s) ds + \sum_{j=1}^{+\infty} \int_0^t (h_s^j(u_s, \nabla u_s), \varphi_s) dB_s^j + \int_0^t \int_{\mathcal{O}} \varphi_s(x) \nu(dx, ds), \quad a.s..$$

From the fact that u and z are in $\mathcal{H}_T$, we know that v is also in $\mathcal{H}_T$, by definition, ν is a random regular measure. □

6. Comparison theorem.

6.1. *A comparison Theorem in the linear case.* We first establish a comparison theorem for the solutions of linear SPDE with obstacle in the case where the obstacles are the same, this also gives a comparison between the regular measures.
So, for this part only, we consider the same hypotheses as in the Subsection 5.2. So we consider adapted processes f, g, h respectively in $L^2([0,T]\times\Omega\times\mathcal{O};\mathbb{R})$, $L^2([0,T]\times\Omega\times\mathcal{O};\mathbb{R}^d)$ and $L^2([0,T]\times\Omega\times\mathcal{O};\mathbb{R}^{\mathbb{N}^*})$, an obstacle S which satisfies assumption **(O)** and $\xi\in L^2(\Omega\times\mathcal{O})$ is an $\mathcal{F}_0$-measurable random variable such that $\xi\leq S_0$. We denote by (u,ν) be the solution of $\mathcal{R}(\xi,\ f,\ g,\ h,\ S)$.
We are given another $\xi'\in L^2(\Omega\times\mathcal{O})$ is $\mathcal{F}_0$-measurable and such that $\xi'\leq S_0$ and another adapted process f' in $L^2([0,T]\times\Omega\times\mathcal{O};\mathbb{R})$. We denote by (u',ν') the solution of $\mathcal{R}(\xi',\ f',\ g,\ h,\ S)$. We have the following comparison theorem:

THEOREM 7. *Assume that the following conditions hold*

1. $\xi\leq\xi',\ dx\otimes dP-a.e.$
2. $f\leq f',\ dt\otimes dx\otimes dP-a.e..$

Then for almost all $\omega\in\Omega$, $u\leq u',\ q.e.$ and $\nu\geq\nu'$ in the sense of distribution.

PROOF. We consider the following two penalized equations:

$$du_t^n = Au_t^n dt + f_t dt + \sum_{i=1}^{d}\partial_i g_t^i dt + \sum_{j=1}^{+\infty} h_t^j dB_t^j + n(u_t^n - S_t)^- dt$$

$$du_t'^n = Au_t'^n dt + f_t' dt + \sum_{i=1}^{d}\partial_i g_t^i dt + \sum_{j=1}^{+\infty} h_t^j dB_t^j + n(u_t'^n - S_t)^- dt$$

we denote

$$F_t(x,u_t^n) = f_t(x) + n(u_t^n - S_t)^-$$
$$F_t'(x,u_t^n) = f_t'(x) + n(u_t^n - S_t)^-$$

with assumption 2 we have that $F_t(x,u_t^n)\leq F_t'(x,u_t^n),\ dt\otimes dx\otimes dP-a.e.$, therefore, from the comparison theorem for SPDE (without obstacle, see [7]), we know that $\forall t\in[0,T]$, $u_t^n\leq u_t'^n,\ dx\otimes dP-a.e.$, thus $n(u_t^n-S_t)^-\geq n(u_t'^n-S_t)^-$.
The results are immediate consequence of the construction of (u,ν) and (u',ν') given in Subsection 5.2. □

6.2. *A comparison theorem in the general case.* We now come back to the general setting and consider $(u^1,\nu^1)=\mathcal{R}(\xi^1,f^1,g,h,S^1)$ the solution of the SPDE with obstacle with null

boundary condition:

$$\begin{cases} du_t^1(x) = Lu_t^1(x)dt + f^1(t,x,u_t^1(x),\nabla u_t^1(x))dt + \sum_{i=1}^{d} \partial_i g_i(t,x,u_t^1(x),\nabla u_t^1(x))dt \\ \qquad\qquad + \sum_{j=1}^{+\infty} h_j(t,x,u_t^1(x),\nabla u_t^1(x))dB_t^j + \nu^1(x,dt) \\ u^1 \geq S^1 \ , \ u_0^1 = \xi^1, \end{cases}$$

where we assume (ξ^1, f^1, g, h) satisfy hypotheses **(H)**, **(I)** and **(O)**.
We consider another coefficients f^2 which satisfies the same assumptions as f^1, another obstacle S^2 which satisfies **(O)** and another initial condition ξ^2 belonging to $L^2(\Omega \times \mathcal{O})$ and $\mathcal{F}_0$ adapted such that $\xi^2 \geq S_0^2$. We denote by $(u^2, \nu^2) = \mathcal{R}(\xi^2, f^2, g, h, S^2)$.

THEOREM 8. *Assume that the following conditions hold*

1. $\xi^1 \leq \xi^2,\ dx \otimes dP - a.e.$
2. $f^1(u^1, \nabla u^1) \leq f^2(u^1, \nabla u^1),\ dt \otimes dx \otimes dP - a.e.$
3. $S^1 \leq S^2,\ dt \otimes dx \otimes dP - a.e.$

Then for almost all $\omega \in \Omega$, $u^1(t,x) \leq u^2(t,x),\ q.e..$

We put $\hat{u} = u^1 - u^2$, $\hat{\xi} = \xi^1 - \xi^2$, $\hat{f}_t = f^1(t, u_t^1, \nabla u_t^1) - f^2(t, u_t^2, \nabla u_t^2)$, $\hat{g}_t = g(t, u_t^1, \nabla u_t^1) - g(t, u_t^2, \nabla u_t^2)$ and $\hat{h}_t = h(t, u_t^1, \nabla u_t) - h(t, u_t^2, \nabla u_t^2)$. The main idea is to evaluate $E \parallel \hat{u}_t^+ \parallel^2$, thanks to Itô's formula and then apply Gronwall's inequality. Therefore, we start by the following lemma

LEMMA 7. *For all* $t \in [0,T]$, *we have*

$$\begin{aligned} E \parallel \hat{u}_t^+ \parallel^2 + 2E \int_0^t \mathcal{E}(\hat{u}_s^+)ds &= E \parallel \hat{\xi}^+ \parallel^2 + 2E \int_0^t (\hat{u}_s^+, \hat{f}_s)ds - 2E \int_0^t (\nabla \hat{u}_s^+, \hat{g}_s)ds \\ + 2E \int_0^t \int_{\mathcal{O}} \hat{u}_s^+(x)(\nu - \nu')(dxds) &+ E \int_0^t \parallel I_{\{\hat{u}_s > 0\}} |\hat{h}_s| \parallel^2 ds. \end{aligned} \tag{18}$$

PROOF. We approximate the function $\psi : y \in R \to (y^+)^2$ by a sequence $(\psi_n)_{n \in \mathbb{N}^*}$ of regular functions: let φ be a C^∞ increasing function such that

$$\forall y \in]-\infty, 1],\ \varphi(y) = 0 \ and \ \forall y \in [2, +\infty[,\ \varphi(y) = 1.$$

We set for all $n \in \mathbb{N}^*$:

$$\forall y \in R,\ \psi_n(y) = y^2 \varphi(ny).$$

It is easy to verify that $(\psi_n)_n$ converges uniformly to the function ψ and that moreover we have the estimates:

$$\forall y \in R^+,\ \forall n,\ 0 \leq \psi_n(y) \leq \psi(y),\ 0 \leq \psi_n'(y) \leq Cy,\ |\psi_n''(y)| \leq C.$$

Thanks to Theorem 6, for all $n \in \mathbb{N}^*$ and $t \in [0,T]$ we have

$$E\int_{\mathcal{O}} \psi_n(\hat{u}_s)dx + E\int_0^t \mathcal{E}(\psi_n'(\hat{u}_s),\hat{u}_s)ds = E\int_{\mathcal{O}} \psi_n(\hat{\xi})dx + E\int_0^t (\psi_n'(\hat{u}_s),\hat{f}_s)ds \tag{19}$$
$$-E\int_0^t (\nabla\psi_n'(\hat{u}_s),\hat{g}_s)ds + E\int_0^t\int_{\mathcal{O}} \psi_n'(\hat{u}_s(x))\hat{\nu}(dxds) + \frac{1}{2}E\int_0^t\int_{\mathcal{O}} \psi_n''(\hat{u}_s(x))\hat{h}_s^2(x)dxds.$$

Taking the limit, thanks to the dominated convergence theorem, we obtain the convergences of all the terms except $E\int_0^t\int_{\mathcal{O}} \psi_n'(\hat{u}_s(x))\hat{\nu}(dxds)$.
From (19), we know that

$$-E\int_0^t\int_{\mathcal{O}} \psi_n'(\hat{u}_s(x))\hat{\nu}(dxds) \le C.$$

Moreover, we have the following relation:

$$\begin{aligned} & -E\int_0^t\int_{\mathcal{O}} \psi_n'(\hat{u}_s(x))\hat{\nu}(dxds) \\ = \; & -E\int_0^t\int_{\mathcal{O}} \psi_n'(S_s^1(x)-u_s^2(x))\nu^1(dxds) + E\int_0^t\int_{\mathcal{O}} \psi_n'(u_s^1(x)-S_s^2(x))\nu^2(dxds) \\ = \; & E\int_0^t\int_{\mathcal{O}} \psi_n'(u_s^2(x)-S_s^1(x))\nu^1(dxds) + E\int_0^t\int_{\mathcal{O}} \psi_n'(u_s^1(x)-S_s^2(x))\nu^2(dxds). \end{aligned}$$

By Fatou's lemma, we obtain

$$2E\int_0^t\int_{\mathcal{O}} (u_s^2(x)-S_s^1(x))^+\nu^1(dxds) + 2E\int_0^t\int_{\mathcal{O}} (u_s^1(x)-S_s^2(x))^+\nu^2(dxds) < +\infty.$$

Hence, the convergence of the term $E\int_0^t\int_{\mathcal{O}} \psi_n'(\hat{u}_s(x))\hat{\nu}(dxds)$ comes from the dominated convergence theorem. □

Proof of Theorem 8: Applying Itô's formula (18) to $(\hat{u}_t^+)^2$, we have

$$E\parallel \hat{u}_t^+ \parallel^2 + 2E\int_0^t I_{\{\hat{u}_s>0\}}\mathcal{E}(\hat{u}_s)ds = 2E\int_0^t (\hat{u}_s^+,\hat{f}_s)ds + 2E\int_0^t (\hat{u}_s^+,\hat{g}_s)ds$$
$$+E\int_0^t \parallel I_{\{\hat{u}_s>0\}}|\hat{h}_s| \parallel^2 ds + 2E\int_0^t\int_{\mathcal{O}} (u_s^1-u_s^2)^+(x)(\nu^1-\nu^2)(dx,ds).$$

As we assume that $f^1(u^1,\nabla u^1) \le f2(u^1,\nabla u^1)$,

$$\begin{aligned} \hat{u}_s^+\hat{f}_s &= \hat{u}_s^+\{f^1(s,u_s^1,\nabla u_s^1) - f^2(s,u_s^1,\nabla u_s^1)\} + \hat{u}_s^+\{f^2(s,u_s^1,\nabla u_s^1) - f^2(s,u_s^2,\nabla u_s^2)\} \\ &\le \hat{u}_s^+\{f^2(s,u_s^1,\nabla u_s^1) - f^2(s,u_s^2,\nabla u_s^2)\}. \end{aligned}$$

then with the Lipschitz condition, using Cauchy-Schwartz's inequality, we have the following relations:

$$E\int_0^t (\hat{u}_s^+,\hat{f}_s)ds \le (C+\frac{C}{\epsilon})E\int_0^t \parallel \hat{u}_s^+ \parallel^2 ds + \frac{C\epsilon}{\lambda}E\int_0^t \mathcal{E}(\hat{u}_s^+)ds.$$

$$E\int_0^t (\nabla \hat{u}_s^+, \hat{g}_s) \leq \frac{\epsilon+\alpha}{\lambda} E\int_0^t \mathcal{E}(\hat{u}_s^+)ds + \frac{C}{\epsilon} E\int_0^t \parallel \hat{u}_s^+ \parallel^2 ds$$

$$E\int_0^t \parallel I_{\{\hat{u}_s>0\}}|\hat{h}_s| \parallel^2 ds \leq CE\int_0^t \parallel \hat{u}_s^+ \parallel^2 ds + \frac{\beta^2+\epsilon}{\lambda} E\int_0^t \mathcal{E}(\hat{u}_s^+)ds.$$

The last term is equal to $-2E\int_0^t\int_{\mathcal{O}}(u_s^1-u_s^2)^+(x)\nu^2(dx,ds) \leq 0$, because that on $\{u^1 \leq u^2\}$, $(u^1-u^2)^+=0$ and on $\{u^1>u^2\}$, $\nu^1(dx,ds)=0$. Thus we have the following inequality

$$E \parallel \hat{u}_t^+ \parallel^2 + (2 - \frac{2\alpha+2\epsilon}{\lambda} - \frac{2C\epsilon}{\lambda} - \frac{\beta^2+\epsilon}{\lambda}) E\int_0^t \mathcal{E}(\hat{u}_s^+)ds \leq CE\int_0^t \parallel \hat{u}_s^+ \parallel^2 ds.$$

we can take ϵ small enough such that $2 - \frac{2\alpha+2\epsilon}{\lambda} - \frac{2C\epsilon}{\lambda} - \frac{\beta^2+\epsilon}{\lambda} > 0$, we have

$$E \parallel \hat{u}_t^+ \parallel^2 \leq CE\int_0^t \parallel \hat{u}_s^+ \parallel^2 ds,$$

then we deduce the result from Gronwall's lemma. □

REMARK 6. *Applying the comparison theorem to the same obstacle gives another proof of the uniqueness of the solution.*

REFERENCES

[1] Aronson D.G.: On the Green's function for second order parabolic differential equations with discontinuous coefficients. *Bulletin of the American Mathematical Society*, **69**, 841-847 (1963).

[2] Aronson, D.G. Non-negative solutions of linear parabolic equations, Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3, tome 22 (4), pp. 607-694 (1968):.

[3] Bally V., Caballero E., El-Karoui N. and Fernandez, B. : Reflected BSDE's PDE's and Variational Inequalities. preprint INRIA report (2004).

[4] Bally V. and Matoussi, A. : Weak solutions for SPDE's and Backward Doubly SDE's. *J. of Theoret.Probab.* **14**, 125-164 (2001).

[5] Bensoussan A. and Lions J.-L. Applications des Inéquations variationnelles en contrôle stochastique. *Dunod*, Paris (1978).

[6] Charrier P. and Troianiello G.M. Un résultat d'existence et de régularité pour les solutions fortes d'un problème unilatéral d'évolution avec obstacle dépendant du temps. *C.R.Acad. Sc. Paris*, **281**, série A, p. 621 (1975).

[7] Denis L.: Solutions of SPDE considered as Dirichlet Processes. *Bernoulli Journal of Probability*, **10**(5), 783-827 (2004).

[8] Denis L. and Stoïca L.: A general analytical result for non-linear s.p.d.e.'s and applications.Electronic *Journal of Probability*, **9**, p. 674-709 (2004).

[9] Denis L., Matoussi A. and Stoïca L.: L^p estimates for the uniform norm of solutions of quasilinear SPDE's. *Probability Theory Related Fields*, **133**, 437-463 (2005).

[10] Denis L., Matoussi A. and Stoïca L.: Maximum principle for parabolic SPDE's: a first approach. *Stochastic Partial Differential Equations and Applications VIII in the series "Quaderni di Matematica del Dipartimento di Matematica della Seconda Università di Napoli"* (2011).

[11] Denis L., Matoussi A. and Stoïca L.: Maximum Principle and Comparison Theorem for Quasi-linear Stochastic PDE's. *Electronic Journal of Probability*, **14**, p. 500-530 (2009).

[12] Donati-Martin C. and Pardoux E.: White noise driven SPDEs with reflection. *Probability Theory and Related Fields*, **95**, 1-24 (1993).

[13] El Karoui N., Kapoudjian C., Pardoux E., Peng S., and Quenez M.C.:*Reflected Solutions of Backward SDE and Related Obstacle Problems for PDEs. The Annals of Probability*, **25** (2), 702-737 (1997).

[14] Klimsiak T.: Reflected BSDEs and obstacle problem for semilinear PDEs in divergence form. *Stochastic Processes and their Applications*, **122** (1), 134-169 (2012).

[15] Lions J.L. and Magenes E.: Problèmes aux limites non homogènes et applications. **1** , *Dunod, Paris* (1968).

[16] Matoussi, A. Xu, M. : Sobolev solution for semilinear PDE with obstacle under monotonicity condition. *Electronic Journal of Probability* **13**, 1035-1067 (2008).

[17] Matoussi A. and Stoïca L.:*The Obstacle Problem for Quasilinear Stochastic PDE's*. The *Annals of Probability*, **38**, 3, 1143-1179 (2010).

[18] Mignot F. and Puel J.P. : Inéquations d'évolution paraboliques avec convexes dépendant du temps. Applications aux inéquations quasi-variationnelles d'évolution. *Arch. for Rat. Mech. and Ana.*, **64**, No.1, 59-91 (1977).

[19] Nualart D. and Pardoux E.: White noise driven quasilinear SPDEs with reflection. *Probability Theory and Related Fields*, **93**, 77-89 (1992).

[20] Pierre M.: Problèmes d'Evolution avec Contraintes Unilaterales et Potentiels Parabolique. *Comm. in Partial Differential Equations*, **4(10)**, 1149-1197 (1979).

[21] Pierre M. : Représentant Précis d'Un Potentiel Parabolique. *Séminaire de Théorie du Potentiel*, Paris, **No.5**, Lecture Notes in Math. 814, 186-228 (1980).

[22] Riesz, F. and Nagy, B. . Functional Analysis. *Dover, New York*, 1990.

[23] Sanz M. , Vuillermot P. (2003) : Equivalence and Hölder Sobolev regularity of solutions for a class of non-autonomous stochastic partial differential equations. *Ann. I. H. Poincaré* , **39** (4) 703-742.

[24] Walsh, J.B. : An introduction to stochastic partial differential equations. *Ecole d'Eté de St-Flour XIV, 1984, Lect. Notes in Math, Springer Verlag* , **1180** , 265-439 (1986).

[25] Xu T.G. and Zhang T.S.: White noise driven SPDEs with reflection: Existence, uniqueness and large deviation principles *Stochatic processes and their applications*, 119, 3453-3470 (2009).

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