DISPERSION FOR THE SCHRÖDINGER EQUATION ON THE LINE WITH MULTIPLE DIRAC DELTA POTENTIALS AND ON DELTA TREES

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ABSTRACT. In this paper we consider the time dependent one-dimensional Schrödinger equation with multiple Dirac delta potentials of different positive strengths. We prove that the classical dispersion property holds. The result is obtained in a more general setting of a Laplace operator on a tree with δ-coupling conditions at the vertices. The proof relies on a careful analysis of the properties of the resolvent of the associated Hamiltonian. With respect to the analysis done for Kirchoff conditions, here the resolvent is no longer in the framework of Wiener algebra of almost periodic functions, and its expression is harder to analyze.

1. INTRODUCTION

Let us first consider the linear Schrödinger equation on $\mathbb{R}$:

$$\begin{cases} iu_t + u_{xx} = 0,\ x \in \mathbb{R}, t \in \mathbb{R}, \\ u(0,x) = u_0(x),\ x \in \mathbb{R}. \end{cases} \tag{1}$$

The linear semigroup $e^{it\Delta}$ has two important properties, the conservation of the L^2-norm:

$$\|e^{it\Delta}u_0\|_{L^2(\mathbb{R})} = \|u_0\|_{L^2(\mathbb{R})}, \tag{2}$$

and a dispersive estimate, obtained by Fourier analysis, of the form:

$$\|e^{it\Delta}u_0\|_{L^\infty(\mathbb{R})} \leq \frac{C}{\sqrt{|t|}}\|u_0\|_{L^1(\mathbb{R})}, \quad t \neq 0. \tag{3}$$

It is now well-known that from these two inequalities space-time estimates follow, known as Strichartz estimates ([34],[19]):

$$\|e^{it\Delta}u_0\|_{L^q_t(\mathbb{R},\, L^r_x(\mathbb{R}))} \leq C\|u_0\|_{L^2(\mathbb{R})}, \tag{4}$$

where (q,r) are so-called admissible pairs:

$$\frac{2}{q} + \frac{1}{r} = \frac{1}{2}, \quad 2 \leq q, r \leq \infty. \tag{5}$$

These dispersive estimates have been successfully applied to obtain well-posedness results for the nonlinear Schrödinger equation (see [13], [35] and the reference therein).

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In the present paper we analyze the above dispersion property in two particular settings. The first one concerns the semigroup $\exp(-itH_\alpha)$ where H_α is a perturbation of the Laplace operator with n Dirac delta potentials with positive strengths $\{\alpha_k\}_{k=1}^n$

$$H_\alpha = -\Delta + \sum_{j=1}^{n} \alpha_j \delta(x - x_j). \tag{6}$$

The spectral properties of the Laplacian with multiple Dirac delta potentials on $\mathbb{R}^n$ have been extensively studied. We only refer to [9] and to the references within. The time dependent propagator of the linear Schrödinger equation has also been considered in the case of one Dirac delta potential [17], [33], [7], or one point interactions [8], [6], or two symmatric Dirac delta potentials [16], [28]. Concerning the nonlinear Schrödinger equation with a Dirac delta potential, standing wave and bound states have been analyzed [15], [16], [32], as well as the time dynamics of solitons [21], [23], [22].

Our main result in the case of H_α is the following one.

Theorem 1.1. *The solution of the linear Schrödinger equation on the line with multiple delta interactions with positive strengths satisfies the dispersion inequality*

$$\|e^{-itH_\alpha} u_0\|_{L^\infty(\mathbb{R})} \leq \frac{C}{\sqrt{|t|}} \|u_0\|_{L^1(\mathbb{R})}, \ \ t \neq 0. \tag{7}$$

We are considering here the case when all α_k, $k = 1, \dots, n$ are positive. This guarantees that operator H_α is positive definite and its discrete spectrum is empty. The case with one single delta interaction, without sign condition on the streght, has been analyzed in [7] where similar dispersive estimates has been proved but for $e^{-itH_\alpha} P_c$, P_c begin the projection of H_α on the continuous spectrum. The case of two point interactions has been considered in [28]. In both previous works, given the particular structure of the operator H_α, the authors obtain explicit representations of the resolvent and then of e^{-itH_α}. However in the general case of multiple delta interactions an explicit representation is not easy to obtain; even in [10], [9, Ch. II.2] the resolvent is obtained in terms of the inverse of some matrix that depends on $\{\alpha_k\}_{k=1}^n$ and on $\{x_k\}_{k=1}^n$.

The setting might be seen as the special case of the equation posed on a simple graph with n vertices, with only two edges starting from any vertex and with delta connection conditions at each vertex ($x_0 = -\infty$, $x_{n+1} = \infty$)

$$\begin{cases} iu_t(t,x) + u_{xx}(t,x) = 0, & x \in (x_{k-1}, x_k), k = 1, \dots, n, \\ u_x(t, x_k+) - u_x(t, x_k-) = \alpha_k u(x_k), & t > 0, \ k = 1, \dots, n. \end{cases} \tag{8}$$

Our second framework refers to the Dirac's delta Hamiltonian H_α^Γ on tree Γ with a finite number of vertices, with the external edges (thats that have only one internal vertex as an endpoint) formed by infinite strips. We consider the linear Schrödinger equation in the case of a tree Γ, with positive

delta conditions of non necessarily equal strength at the vertices

$$\begin{cases} i\mathbf{u}_t = H_\alpha^\Gamma \mathbf{u},\ x \in \Gamma, t \in \mathbb{R}, \\ \mathbf{u}(0,x) = \mathbf{u}_0(x),\ x \in \Gamma. \end{cases} \tag{9}$$

The presentation of operator H_α^Γ will be given in full details in Sec. 2. Let us just say here that H_α^Γ acts as $-\partial_{xx}$ on each edge of the tree and that at vertices the δ-coupling must be fulfilled: continuity condition for the functions on the graph and a δ transmission condition at the level of their first derivative at all internal vertices v:

$$\sum_{e \in E_v} \partial_n \mathbf{u}(v) = \alpha(v)\mathbf{u}(v).$$

The following questions have been considered for equation (9). The study of bound states on a star shaped tree with delta conditions have been analyzed in a series of papers [1], [5], [3], [2]. Dispersion for the linear Schrödinger equation on a star shaped tree with delta conditions has been proved in [4], where the main result concerns the time evolution of a fast soliton, in the spirit of [4]. The main result of this paper is the following.

Theorem 1.2. *The solution of the linear Schrödinger equation on a tree with delta connection conditions with positive strengths satisfies the dispersion inequality*

$$\|e^{-itH_\alpha^\Gamma}\mathbf{u}_0\|_{L^\infty(\Gamma)} \leq \frac{C}{\sqrt{|t|}}\|\mathbf{u}_0\|_{L^1(\Gamma)},\ \ t \neq 0. \tag{10}$$

We mention here that dispersion was previously proved in [12] (see also [24]) for the case of Kirchoff's connection condition on trees, i.e. $\alpha(v) = 0$ for all internal vertices of the tree. The case of δ and δ' coupling on a star shaped tree has been considered in [4]. The extension of the results presented here to the case of δ'- coupling or to general strengths remains to be analyzed. More coupling conditions are discussed in Section 2.

The proof of Theorem 1.2 uses elements from [11], [12], [18] in an appropriate way related to the delta connection conditions on the tree. The starting point consists in writing the solution in terms of the resolvent of the Laplacian, which in turn is determined by recursion on the number of vertices. With respect to the previous works with Kirchoff conditions, the novelty in here is that we are not any longer in the framework of the almost periodic Wiener algebra of functions, and that the expression of the resolvent is harder to analyse.

The linear solution $e^{-itH_\alpha^\Gamma}\mathbf{u}_0$ will be shown to be a combination of oscillatory integrals, that becomes more and more involved as the number of vertices of the tree grows. We do not have any more that $e^{-itH_\alpha^\Gamma}\mathbf{u}_0$ is a summable superposition of solutions of the linear Schrödinger equation on the line, as for Kirchoff conditions in [12].

Theorem 1.1 follow from Theorem 1.2 by considering the particular case of a tree Γ with all the internal vertices having degree two.

Denoting by H either H_α or H_α^Γ the arguments used in the context of NSE on $\mathbb{R}$ can also be used here to obtain the following as a typical result.

Theorem 1.3. *Let $p \in (0,4)$. For any $\mathbf{u}_0 \in L^2(\Gamma)$ there exists a unique solution*

$$\mathbf{u} \in C(\mathbb{R}, L^2(\Gamma)) \cap \bigcap_{(q,r)\, admissible} L^q_{loc}(\mathbb{R}, L^r(\Gamma)),$$

of the nonlinear Schrödinger equation

$$\begin{cases} i\mathbf{u}_t + H\mathbf{u} \pm |\mathbf{u}|^p\mathbf{u} = 0, & t \neq 0, \\ \mathbf{u}(0) = \mathbf{u}_0, & t = 0. \end{cases} \tag{11}$$

Moreover, the $L^2(\Gamma)$-norm of $\mathbf{u}$ is conserved along the time

$$\|\mathbf{u}(t)\|_{L^2(\Gamma)} = \|\mathbf{u}_0\|_{L^2(\Gamma)}.$$

The proof is standard once the dispersion property is obtained and it follows as in [13], p. 109, Theorem 4.6.1.

The paper is organized as follows. In the next section we introduce the framework of the Laplacian analysis on a graph. In §3 we give the proof of Theorem 1.2.

2. Preliminaries on graphs and δ-coupling

In this section we present some generalities about metric graphs and introduce the Dirac's delta Hamiltonian H^Γ_α on such structure. More general type of self-adjoint operators, $\Delta(A,B)$, have been considered in [27], [26]. We collect here some basic facts on metric graphs and on some operators that could be defined on such structure [31], [29], [30], [27], [20], [14].

Let $\Gamma = (V,E)$ be a graph where V is a set of vertices and E the set of edges. For each $v \in V$ we denote by $E_v = \{e \in E : v \in e\}$ the set of edges branching from v. We assume that V is connex and the degree of each vertex v of Γ is finite: $d(v) = |E_v| < \infty$. The edges could be of finite length and then their ends are vertices of V or they have infinite length and then we assume that each infinite edge is a ray with a single vertex belonging to V (see [31] for more details on graphs with infinite edges). The vertices are called internal if $d(v) \geq 2$ or external if $d(v) = 1$. In this paper we will assume that there are not external vertices.

We fix an orientation of Γ and for each oriented edge e, we denote by $I(e)$ the initial vertex and by $T(e)$ the terminal one. Of course in the case of infinite edges we have only initial vertices.

We identify every edge e of Γ with an interval I_e, where $I_e = [0, l_e]$ if the edge is finite and $I_e = [0,\infty)$ if the edge is infinite. This identification introduces a coordinate x_e along the edge e. In this way Γ is a metric space and is often named metric graph [31].

Let v be a vertex of V and e be an edge in E_v. We set for finite edges e

$$j(v,e) = \begin{cases} 0 & \text{if} \quad v = I(e), \\ l_e & \text{if} \quad v = T(e) \end{cases}$$

and

$$j(v,e) = 0, \text{ if } v = I(e)$$

for infinite edges.

We identify any function $\mathbf{u}$ on Γ with a collection $\{u^e\}_{e\in E}$ of functions u^e defined on the edges e of Γ. Each u^e can be considered as a function on the interval I_e. In fact, we use the same notation u^e for both the function on the edge e and the function on the interval I_e identified with e. For a function $\mathbf{u}:\Gamma\to\mathbb{C}$, $\mathbf{u}=\{u^e\}_{e\in E}$, we denote by $f(\mathbf{u}):\Gamma\to\mathbb{C}$ the family $\{f(u^e)\}_{e\in E}$, where $f(u^e):e\to\mathbb{C}$.

A function $\mathbf{u}=\{u^e\}_{e\in E}$ is continuous if and only if u^e is continuous on I_e for every $e\in E$, and moreover, is continuous at the vertices of Γ:

$$u^e(j(v,e))=u^{e'}(j(v,e')),\quad \forall\ e,e'\in E_v,\quad \forall\ v\in V.$$

The space $L^p(\Gamma)$, $1\le p<\infty$ consists of all functions $\mathbf{u}=\{u_e\}_{e\in E}$ on Γ that belong to $L^p(I_e)$ for each edge $e\in E$ and

$$\|\mathbf{u}\|^p_{L^p(\Gamma)}=\sum_{e\in E}\|u^e\|^p_{L^p(I_e)}<\infty.$$

Similarly, the space $L^\infty(\Gamma)$ consists of all functions that belong to $L^\infty(I_e)$ for each edge $e\in E$ and

$$\|\mathbf{u}\|_{L^\infty(\Gamma)}=\sup_{e\in E}\|u^e\|_{L^\infty(I_e)}<\infty.$$

The Sobolev space $H^m(\Gamma)$, $m\ge 1$ an integer, consists in all continuous functions on Γ that belong to $H^m(I_e)$ for each $e\in E$ and

$$\|\mathbf{u}\|^2_{H^m(\Gamma)}=\sum_{e\in E}\|u^e\|^2_{H^m(e)}<\infty.$$

The above spaces are Hilbert spaces with the inner products

$$(\mathbf{u},\mathbf{v})_{L^2(\Gamma)}=\sum_{e\in E}(u^e,v^e)_{L^2(I_e)}=\sum_{e\in E}\int_{I_e}u^e(x)\overline{v^e}(x)dx$$

and

$$(\mathbf{u},\mathbf{v})_{H^m(\Gamma)}=\sum_{e\in E}(u^e,v^e)_{H^m(I_e)}=\sum_{e\in E}\sum_{k=0}^{m}\int_{I_e}\frac{d^ku^e}{dx^k}\overline{\frac{d^kv^e}{dx^k}}dx.$$

We now define the normal exterior derivative of a function $\mathbf{u}=\{u^e\}_{e\in E}$ at the endpoints of the edges. For each $e\in E$ and v an endpoint of e we consider the normal derivative of the restriction of $\mathbf{u}$ to the edge e of E_v evaluated at $j(v,e)$ to be defined by:

$$\frac{\partial u^e}{\partial n_e}(j(v,e))=\begin{cases}-u^e_x(0+) & \text{if}\quad j(v,e)=0,\\ u^e_x(l_e-) & \text{if}\quad j(v,e)=l_e.\end{cases}$$

We now introduce H^Γ_α. It generalizes the classical Dirac's delta interactions with strength parameters (6). The Dirac's delta Hamiltonian is defined on the domain

$$D(H^\Gamma_\alpha)=\Big\{\mathbf{u}\in H^2(\Gamma),\ \sum_{e\in E_v}\frac{\partial u^e}{\partial n_e}(j(v,e))=\alpha(v)\mathbf{u}(v),\quad \forall v\in V\Big\}. \tag{12}$$

Operator H^Γ_α acts as following, for any $\mathbf{u}=\{u^e\}_{e\in E}$

$$(H^\Gamma_\alpha\mathbf{u})(x)=-u^e_{xx}(x),\quad x\in I_e,\ e\in E.$$

The quadratic form associated to H_α^Γ is defined on $H^1(\Gamma)$ and it is given by

$$\mathcal{E}_\alpha^\Gamma(u)=\sum_{e\in E}\int_{I_e}|u_x^e(x)|^2dx+\sum_{v\in V}\alpha(v)|\mathbf{u}(v)|^2.$$

Observe that in our case, since all the strength parameters are nonnegative, the quadratic form $\mathcal{E}_\alpha^\Gamma$ is nonnegative. The case when all strengths vanish corresponds to the Kirchhoff coupling analyzed in [12].

Finally, let us mention that there are other coupling conditions (see [26]) which allow to define a "Laplace" operator on a metric graph. To be more precise, let us consider an operator that acts on functions on the graph Γ as the second derivative $\frac{d^2}{dx^2}$, and its domain consists in all functions $\mathbf{u}$ that belong to the Sobolev space $H^2(e)$ on each edge e of Γ and satisfy the following boundary condition at the vertices:

$$A(v)\mathbf{u}(v)+B(v)\mathbf{u}'(v)=0\quad\text{for each vertex } v. \tag{13}$$

Here $\mathbf{u}(v)$ and $\mathbf{u}'(v)$ are correspondingly the vector of values of $\mathbf{u}$ at v attained from directions of different edges converging at v and the vector of derivatives at v in the outgoing directions. For each vertex v of the tree we assume that matrices $A(v)$ and $B(v)$ are of size $d(v)$ and satisfy the following two conditions

(1) the joint matrix $(A(v),B(v))$ has maximal rank, i.e. $d(v)$,

(2) $A(v)B(v)^T=B(v)A(v)^T$.

Under those assumptions it has been proved in [26] that the considered operator, denoted by $\Delta(A,B)$, is self-adjoint. The case considered in this paper, the δ-coupling, corresponds to the matrices

$$A(v)=\begin{pmatrix}1&-1&0&\dots&0&0\\0&1&-1&\dots&0&0\\0&0&1&\dots&0&0\\\vdots&\vdots&\vdots&&\vdots&\vdots\\0&0&0&\vdots&1&-1\\0&0&0&\vdots&0&-\alpha(v)\end{pmatrix},\ B(v)=\begin{pmatrix}0&0&0&\dots&0&0\\0&0&0&\dots&0&0\\0&0&0&\dots&0&0\\\vdots&\vdots&\vdots&&\vdots&\vdots\\0&0&0&\dots&0&0\\1&1&1&\dots&1&1\end{pmatrix}.$$

More examples of matrices satisfying the above conditions are given in [26, 25].

3. Proof of Theorem 1.2

We shall use a description of the solution of the linear Schrödinger equation in terms of the resolvent. For $\omega>0$ let R_ω be the resolvent of the Laplacian on a tree

$$R_\omega\mathbf{u}_0=(H_\alpha^\Gamma+\omega^2I)^{-1}\mathbf{u}_0.$$

Before starting let us choose an orientation on tree Γ. Let us choose an internal vertex $\mathcal{O}$. This will be the root of the tree and the initial vertex for all the edges that branch from it. This procedure introduces an orientation for all the edges staring from $\mathcal{O}$. For the other endpoints of the edges belonging to $E_\mathcal{O}$ we repeat the above procedure and inductively we construct an orientation on Γ.

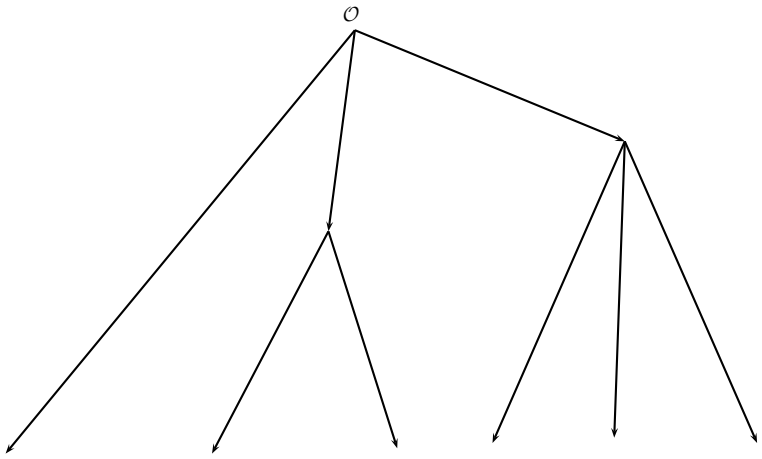


FIGURE 1. The orientation on Γ

3.1. **The structure of the resolvent.** In order to obtain the expression of the resolvent second-order equations

$$(R_\omega \mathbf{u}_0)'' = \omega^2 R_\omega \mathbf{u}_0 - \mathbf{u}_0$$

must be solved on each edge of the tree together with coupling conditions at each vertex. Then, on each edge parametrized by I_e, for $x \in I_e$,

$$R_\omega \mathbf{u}_0(x) = c_e\, e^{\omega x} + \tilde{c}_e\, e^{-\omega x} + \frac{t_e(x,\omega)}{\omega}, \tag{14}$$

with

$$t_e(x,\omega) = \frac{1}{2}\int_{I_e} \mathbf{u}_0(y)\, e^{-\omega|x-y|} dy.$$

Since $R_\omega \mathbf{u}_0$ belongs to $L^2(\Gamma)$ the coefficients c's are zero on the infinite edges $e \in \mathcal{E}$, parametrized by $[0,\infty)$. If we denote by $\mathcal{I}$ the set of internal edges, we have $2|\mathcal{I}| + |\mathcal{E}|$ coefficients. The delta conditions of continuity of $R_\omega \mathbf{u}_0$ and of transmission of $(R_\omega \mathbf{u}_0)'$ at the vertices of the tree give the system of equations on the coefficients. We have the same number of equations as the number of unknowns. We denote $D_{\Gamma_p}(\omega)$ the matrix of the system, where p stands for the number of vertices of the tree, and by $T_{\Gamma_p}(\omega)$ the column of the free terms in the system.

Therefore the resolvent $R_\omega \mathbf{u}_0(x)$ on an edge I_e is

$$R_\omega \mathbf{u}_0(x) = \frac{\det M^{c_e}_{\Gamma_p}(\omega)}{\det D_{\Gamma_p}(\omega)}\, e^{\omega x} + \frac{\det M^{\tilde{c}_e}_{\Gamma_p}(\omega)}{\det D_{\Gamma_p}(\omega)}\, e^{-\omega x} + \frac{t_e(x,\omega)}{\omega}, \tag{15}$$

where $M^{c_e}_{\Gamma_p}(\omega)$ and $M^{\tilde{c}_e}_{\Gamma_p}(\omega)$ are obtained from $D_{\Gamma_p}(\omega)$ by replacing the column corresponding to the unknown c_e, and respectively $\tilde{c}_e$ by the column of the free terms $T_{\Gamma_p}(\omega)$.

3.2. **The expression of** $\det D_{\Gamma_p}(\omega)$**.** In view of the form (14) of the resolvent, we obtain on an edge I_e

$$\begin{aligned} R_\omega \mathbf{u}_0(0) &= c_e + \tilde{c}_e + \frac{t_e(0,\omega)}{\omega}, \\ (R_\omega \mathbf{u}_0)'(0) &= c_e\, \omega - \tilde{c}_e\, \omega + t_e(0,\omega), \end{aligned} \tag{16}$$

and in case I_e is parametrized by $[0,a]$ with $a<\infty$,

$$R_\omega \mathbf{u}_0(a) = c_e\, e^{\omega a} + \tilde{c}_e\, e^{-\omega a} + \frac{t_e(0,\omega)}{\omega}, \tag{17}$$
$$(R_\omega \mathbf{u}_0)'(a) = c_e\, \omega\, e^{\omega a} - \tilde{c}_e\, \omega\, e^{-\omega a} - t_e(a,\omega).$$

3.2.1. *The star-shaped tree case.* In the case of a single vertex and $n_1 \geq 2$ edges I_j, $1 \leq j \leq n_1$, parametrized by $[0,\infty)$ we have only the coefficient $\tilde{c}_j$ on each edge I_j since all c_j vanish. The delta conditions are continuity of the resolvent at the vertex, together with the fact that the sum of the first derivatives must be equal to α times the value of the resolvent at the vertex

$$(R_\omega \mathbf{u}_0)_j(0) = (R_\omega \mathbf{u}_0)_1(0), \quad \sum_{1\leq j\leq n_1} (R_\omega \mathbf{u}_0)_j'(0) = \alpha_1\, (R_\omega \mathbf{u}_0)_1(0).$$

From (16) we obtain as matrix for the system of $\tilde{c}$'s

$$D_{\Gamma_1}(\omega) = \begin{pmatrix} 1 & -1 & & & & & & \\ & 1 & -1 & & & & & \\ & & \cdot & \cdot & & & & \\ & & & \cdot & \cdot & & & \\ & & & & \cdot & \cdot & & \\ & & & & & 1 & -1 & \\ & & & & & & 1 & -1 \\ 1 & \frac{\omega}{\omega+\alpha_1} & \frac{\omega}{\omega+\alpha_1} & \cdot & \cdot & \frac{\omega}{\omega+\alpha_1} & \frac{\omega}{\omega+\alpha_1} & \frac{\omega}{\omega+\alpha_1} \end{pmatrix},$$

and as a free term column

$$T_{\Gamma_1}(\omega) = \begin{pmatrix} \frac{t_2(0,\omega)-t_1(0,\omega)}{\omega} \\ \cdots \\ \frac{t_{n_1}(0,\omega)-t_{n_1-1}(0,\omega)}{\omega} \\ \\ \frac{\omega-\alpha_1}{\omega+\alpha_1}\frac{t_1(0,\omega)}{\omega} + \frac{\omega}{\omega+\alpha_1}\sum_{2\leq j\leq n_1}\frac{t_j(0,\omega)}{\omega} \end{pmatrix}.$$

By developing $\det D_{\Gamma_1}(\omega)$ with respect to its last column, we obtain by recursion that

$$\det D_{\Gamma_1}(\omega) = \frac{n_1\,\omega + \alpha_1}{\omega+\alpha_1}.$$

Thus $\det D_{\Gamma_1}$ does no vanish on the imaginary axis and $\omega R_\omega \mathbf{u}_0$ can be analytically continued in a region containing the imaginary axis.

We introduce here the matrix $\tilde{D}_{\Gamma_1}(\omega)$ which is the matrix of the coefficients of the resolvent, if on the last edge I_{n_1} we should have $c_{n_1}e^{\omega x}$ instead of $\tilde{c}_{n_1}e^{-\omega x}$. This changes only the (n_1,n_1)-entry of $D_{\Gamma_1}(\omega)$ in $-\frac{\omega}{\omega+\alpha_1}$ instead of $\frac{\omega}{\omega+\alpha_1}$,

$$\tilde{D}_{\Gamma_1}(\omega) = \begin{pmatrix} 1 & -1 & & & & & & \\ & 1 & -1 & & & & & \\ & & \cdot & \cdot & & & & \\ & & & \cdot & \cdot & & & \\ & & & & \cdot & \cdot & & \\ & & & & & 1 & -1 & \\ & & & & & & 1 & -1 \\ 1 & \frac{\omega}{\omega+\alpha_1} & \frac{\omega}{\omega+\alpha_1} & \cdot & \cdot & \frac{\omega}{\omega+\alpha_1} & \frac{\omega}{\omega+\alpha_1} & -\frac{\omega}{\omega+\alpha_1} \end{pmatrix}.$$

Moreover, the free term column remains the same for this new system. We have again by recursion

$$\det \tilde{D}_{\Gamma_1}(\omega) = \frac{(n_1-2)\,\omega+\alpha_1}{\omega+\alpha_1}.$$

3.2.2. *The general tree case.* Any tree Γ_p with p vertices, $p \geq 2$ can be seen as a tree Γ_{p-1} with $p-1$ vertices, to which we add a new vertex on one of its infinite edges, and $n_p - 1$ new infinite edges from it. Let us denote by N the number of edges of Γ_{p-1}. By this transformation I_N becomes an internal edge, parametrized by $[0, a_{p-1}]$, and we have in addition I_{N+j} as external edges, for $1 \leq j \leq n_p - 1$. We denote α_p the positive strength of the δ condition in the new p^{th} vertex. The matrix of the new system (unknowns of the Γ_{p-1} system, together with an extra-unknown on the new internal line I_N, as well as $n_p - 1$ unknowns on the new $n_p - 1$ external edges) is denoted by $D_{\Gamma_p}(\omega)$. Notice that if we write the system of unknowns of Γ_p by changing the order of the unknowns (i.e. permuting columns) or the order of the conditions at vertices (i.e. permuting lines), then the determinant remains unchanged or it changes sign, and the ratio $\frac{\det \tilde{D}_{\Gamma_p}(\omega)}{\det D_{\Gamma_p}(\omega)}$ remains unchanged. We shall prove the following Lemma.

Lemma 3.1. *We have the recursion formulae*

$$\det D_{\Gamma_1}(\omega) = \frac{n_1\,\omega+\alpha_1}{\omega+\alpha_1}, \quad \frac{\det \tilde{D}_{\Gamma_1}(\omega)}{\det D_{\Gamma_1}(\omega)} = \frac{(n_1-2)\,\omega+\alpha_1}{n_1\,\omega+\alpha_1},$$

$$\det D_{\Gamma_p}(\omega) = \frac{n_p\,\omega+\alpha_p}{\omega+\alpha_p} e^{\omega a_{p-1}} \det D_{\Gamma_{p-1}}(\omega) \left(1 - \frac{(n_p-2)\,\omega+\alpha_p}{n_p\,\omega+\alpha_p}\, e^{-2\omega a_{p-1}} \frac{\det \tilde{D}_{\Gamma_{p-1}}(\omega)}{\det D_{\Gamma_{p-1}}(\omega)}\right).$$

$$\frac{\det \tilde{D}_{\Gamma_p}(\omega)}{\det D_{\Gamma_p}(\omega)} = \frac{\frac{(n_p-2)\,\omega+\alpha_p}{n_p\,\omega+\alpha_p} - \frac{(n_p-4)\,\omega+\alpha_p}{n_p\,\omega+\alpha_p}\, e^{-2\omega a_{p-1}} \frac{\det \tilde{D}_{\Gamma_{p-1}}(\omega)}{\det D_{\Gamma_{p-1}}(\omega)}}{1 - \frac{(n_p-2)\,\omega+\alpha_p}{n_p\,\omega+\alpha_p}\, e^{-2\omega a_{p-1}} \frac{\det \tilde{D}_{\Gamma_{p-1}}(\omega)}{\det D_{\Gamma_{p-1}}(\omega)}}.$$

Proof. The part on Γ_1 was proved in subsection §3.2.1.

For Γ_p, by writing the delta conditions at the end of I_N, together with the two conditions involving the coefficients on I_N at the begining of I_N, we obtain the matrix $D_{\Gamma_p}(\omega)$ as

$$\left(\begin{array}{c|cccccccc}
D_{\Gamma_{p-1}}(\omega) & -\frac{1}{\frac{\omega}{\omega+\alpha_{p-1}}} & & & & & & & \\
\hline
e^{-\omega a_{p-1}} & e^{\omega a_{p-1}} & -1 & & & & & & \\
 & & 1 & -1 & & & & & \\
 & & & \cdot & \cdot & & & & \\
 & & & & \cdot & \cdot & & & \\
 & & & & & \cdot & \cdot & & \\
 & & & & & & 1 & -1 & \\
 & & & & & & & 1 & -1 \\
\frac{-\omega+\alpha_p}{\omega+\alpha_p} e^{-\omega a_{p-1}} & e^{\omega a_{p-1}} & \frac{\omega}{\omega+\alpha_p} & \frac{\omega}{\omega+\alpha_p} & . & . & \frac{\omega}{\omega+\alpha_p} & \frac{\omega}{\omega+\alpha_p} & \frac{\omega}{\omega+\alpha_p}
\end{array}\right)$$

and the free term column as

$$T_{\Gamma_p}(\omega)=\begin{pmatrix} T_{\Gamma_{p-1}}(\omega)\\ \frac{t_{N+1}(0,\omega)-t_N(a_{p-1},\omega)}{\omega}\\ \cdots\\ \frac{t_{N+n_p-1}(0,\omega)-t_{N+n_p-2}(0,\omega)}{\omega}\\ \frac{\omega-\alpha_p}{\omega+\alpha_p}\frac{t_N(a_{p-1},\omega)}{\omega}+\frac{\omega}{\omega+\alpha_p}\sum_{1\leq j\leq n_p-1}\frac{t_{N+j}(0,\omega)}{\omega}\end{pmatrix}.$$

By developing $\det D_{\Gamma_p}$ with respect to the last n_p lines we obtain an alternated sum of determinants of $n_p\times n_p$ minors composed from the last n_p lines of D_{Γ_p} times the determinant of the matrix D_{Γ_p} without the lines and columns the minor is made of. On the last n_p lines, there are only n_p+1 columns that does not identically vanish. The only possibility to obtain a $n_p\times n_p$ minor composed from the last n_p lines of D_{Γ_p} with determinant different from zero is to choose all last n_p-1 columns together with a previous one. This follows from the fact that if we eliminate from $\det D_{\Gamma_n}$ both previous columns together with n_p-2 columns among the last n_p columns, we obtain a block-diagonal type matrix, with first diagonal block $D_{\Gamma_{p-1}}$ with its last column replaced by zeros, so its determinant vanishes. Therefore

$$\det D_{\Gamma_p}=\det D_{\Gamma_{p-1}}\det A^{n_p}-\det\tilde{D}_{\Gamma_{p-1}}\det B^{n_p},$$

where for $m\geq 1$, A^m and B^m are the $m\times m$ matrices

$$A^m=\begin{pmatrix} e^{\omega a_{p-1}} & -1 & & & & & & \\ & 1 & -1 & & & & & \\ & & \cdot & \cdot & & & & \\ & & & \cdot & \cdot & & & \\ & & & & \cdot & \cdot & & \\ & & & & & 1 & -1 & \\ & & & & & & 1 & -1\\ e^{\omega a_{p-1}} & \frac{\omega}{\omega+\alpha_p} & \frac{\omega}{\omega+\alpha_p} & \cdot & \cdot & \frac{\omega}{\omega+\alpha_p} & \frac{\omega}{\omega+\alpha_p} & \frac{\omega}{\omega+\alpha_p}\end{pmatrix},$$

$$B^m=\begin{pmatrix} e^{-\omega a_{p-1}} & -1 & & & & & & \\ & 1 & -1 & & & & & \\ & & \cdot & \cdot & & & & \\ & & & \cdot & \cdot & & & \\ & & & & \cdot & \cdot & & \\ & & & & & 1 & -1 & \\ & & & & & & 1 & -1\\ \frac{-\omega+\alpha_p}{\omega+\alpha_p}e^{-\omega a_{p-1}} & \frac{\omega}{\omega+\alpha_p} & \frac{\omega}{\omega+\alpha_p} & \cdot & \cdot & \frac{\omega}{\omega+\alpha_p} & \frac{\omega}{\omega+\alpha_p} & \frac{\omega}{\omega+\alpha_p}\end{pmatrix}.$$

We have

$$\det A^2=\frac{2\omega+\alpha_p}{\omega+\alpha_p}e^{\omega a_{p-1}},$$

and by developing A^m with respect to the first last column we obtain the recursion formula $\det A^m=\frac{\omega}{\omega+\alpha_p}e^{\omega a_{p-1}}+\det A^{n_p-1}$, so

$$\det A^m=\frac{m\,\omega+\alpha_p}{\omega+\alpha_p}e^{\omega a_{p-1}}.$$

Similarly we obtain

$$\det B^m = \frac{(m-2)\,\omega+\alpha_p}{\omega+\alpha_p} e^{-\omega a_{p-1}}.$$

Therefore we find indeed

$$\det D_{\Gamma_p}(\omega) = \frac{n_p\,\omega+\alpha_p}{\omega+\alpha_p} e^{\omega a_{p-1}} \det D_{\Gamma_{p-1}}(\omega) \left(1 - \frac{(n_p-2)\,\omega+\alpha_p}{n_p\,\omega+\alpha_p}\, e^{-2\omega a_{p-1}} \frac{\det \tilde{D}_{\Gamma_{p-1}}(\omega)}{\det D_{\Gamma_{p-1}}(\omega)}\right).$$

In a similar way we get

$$\det \tilde{D}_{\Gamma_p}(\omega) = \frac{(n_p-2)\,\omega+\alpha_p}{\omega+\alpha_p} e^{\omega a_{p-1}} \det D_{\Gamma_{p-1}}(\omega) - \frac{(n_p-4)\,\omega+\alpha_p}{\omega+\alpha_p} e^{-\omega a_{p-1}} \det \tilde{D}_{\Gamma_{p-1}}(\omega),$$

so

$$\frac{\det \tilde{D}_{\Gamma_p}(\omega)}{\det D_{\Gamma_p}(\omega)} = \frac{\frac{(n_p-2)\,\omega+\alpha_p}{n_p\,\omega+\alpha_p} - \frac{(n_p-4)\,\omega+\alpha_p}{n_p\,\omega+\alpha_p}\, e^{-2\omega a_{p-1}} \frac{\det \tilde{D}_{\Gamma_{p-1}}(\omega)}{\det D_{\Gamma_{p-1}}(\omega)}}{1 - \frac{(n_p-2)\,\omega+\alpha_p}{n_p\,\omega+\alpha_p}\, e^{-2\omega a_{p-1}} \frac{\det \tilde{D}_{\Gamma_{p-1}}(\omega)}{\det D_{\Gamma_{p-1}}(\omega)}},$$

and the proof of the Lemma is complete. □

3.3. **A lower bound for** $\det D_{\Gamma_p}(i\tau)$ **away from** 0**.**

Lemma 3.2. *Function* $\det D_{\Gamma_p}(\omega)$ *is lower bounded by a positive constant on a strip containing the imaginary axis, away from zero:*

$$\forall \delta > 0,\, \exists c_{\Gamma_p}, \epsilon_{\Gamma_p} > 0, \exists 0 < r_{\Gamma_p} < 1 \; s.t. \, |\det D_{\Gamma_p}(\omega)| > c_{\Gamma_p},\; \left|\frac{\det \tilde{D}_{\Gamma_p}(\omega)}{\det D_{\Gamma_p}(\omega)}\right| < r_\Gamma,$$

for all $\omega \in \mathbb{C}$ *with* $|\Re\omega| < \epsilon_{\Gamma_p}$ *and* $|\Im\omega| > \delta$.

Proof. We shall prove this Lemma by recursion on p. For $p = 1$ Lemma 3.1 insures us that

$$\det D_{\Gamma_1}(\omega) = \frac{n_1\,\omega+\alpha_1}{\omega+\alpha_1}, \quad \frac{\det \tilde{D}_{\Gamma_1}(\omega)}{\det D_{\Gamma_1}(\omega)} = \frac{(n_1-2)\,\omega+\alpha_1}{n_1\,\omega+\alpha_1}.$$

We obtain a positive lower bound for $|\det D_{\Gamma_1}(\omega)|$ if we avoid that it approaches zero. Therefore the existence of $c_{\Gamma_p} > 0$ is obtained by considering $\epsilon_{\Gamma_1} \leq \frac{\alpha_1}{2n_1}$. Next, we have

$$\left|\frac{(n_1-2)\,\omega+\alpha_1}{n_1\,\omega+\alpha_1}\right| < 1 \iff 0 < \alpha_1 \Re\omega + (n_1-1)|\omega|^2,$$

so for any $\delta > 0$ we get an appropriate $0 < r_{\Gamma_1} < 1$ by choosing

$$\epsilon_{\Gamma_1} \leq \frac{(n_1-1)\delta^2}{2\alpha_1}.$$

Assume that we have proved this Lemma for $p-1$. We shall show now that it also holds for p. Now, from ratio information part in this Lemma for Γ_{p-1} we can choose ϵ_{Γ_p} small enough to have for $|\Re\omega| < \epsilon_{\Gamma_p}$ and $|\Im\omega| > \delta$

$$\left|1 - \frac{(n_p-2)\,\omega+\alpha_p}{n_p\,\omega+\alpha_p}\, e^{-2\omega a_{p-1}} \frac{\det \tilde{D}_{\Gamma_{p-1}}(\omega)}{\det D_{\Gamma_{p-1}}(\omega)}\right| > c_0 > 0.$$

Also from this Lemma for Γ_{p-1} we have the existence of two positive constants $c_{\Gamma_{p-1}}$ and $\epsilon_{\Gamma_{p-1}}$ such that $|\det D_{\Gamma_{p-1}}(\omega)| > c_{\Gamma_{p-1}}$, $\forall \omega \in \mathbb{C}, |\Re\omega| <$

$\epsilon_{\Gamma_{p-1}}$ and $|\Im\omega| > \delta$. Finally, $\frac{n_p\omega+\alpha_p}{\omega+\alpha_p}$ is lower bounded by a positive constant for $\Re\omega$ small enough, so eventually we get

$$\exists c_{\Gamma_p}, \epsilon_{\Gamma_p} > 0,\ |\det D_{\Gamma_p}(\omega)| > c_{\Gamma_p},\ \forall \omega \in \mathbb{C}, |\Re\omega| < \epsilon_{\Gamma_p}, |\Im\omega| > \delta.$$

We are left with showing that the ratio $\frac{\det \tilde{D}_{\Gamma_p}(\omega)}{\det D_{\Gamma_p}(\omega)}$ is of modulus less than one. In view of the recursion formula on the ratio from Lemma 3.1, we first impose as a condition on ϵ_{Γ_p} that

$$\tilde{r}_{\Gamma_{p-1}} := e^{2\epsilon_{\Gamma_p} a_{p-1}} r_{\Gamma_{p-1}} < 1,$$

and then we have to show that for $|z| < \tilde{r}_{\Gamma_{p-1}}$

$$\left| \frac{(n_p-2)\,\omega + \alpha_p - [(n_p-4)\,\omega+\alpha_p]z}{n_p\,\omega+\alpha_p - [(n_p-2)\,\omega+\alpha_p]z} \right| < r_{\Gamma_p},$$

for all complex ω with $|\Re\omega| < \epsilon_{\Gamma_p}$ and $|\Im\omega| > \delta$, for ϵ_{Γ_p} to be chosen and $r_{\Gamma_p} < 1$. Denoting $q = (n_p-2)\omega + \alpha_p$, the above inequality is written in as

$$|q-(q-2\omega)z| < |(q+2\omega)-qz| \iff |q(1-z)+2\omega z| < |q(1-z)+2\omega|.$$

Expanding this last inequality we find that we have to prove that

$$0 < |\omega|^2(1-|z|^2) + |1-z|^2\Big((n_p-2)|\omega|^2 + \alpha_p\Re(\omega)\Big).$$

Since $n_p \geq 2$ and $|z| < \tilde{r}_{\Gamma_{p-1}} < 1$, it is enough to have

$$0 < |\omega|^2(1-|z|^2) + |1-z|^2\alpha_p\Re(\omega).$$

Also, $\Re z < \tilde{r}_{\Gamma_{p-1}} < 1$, so by choosing

$$\epsilon_{\Gamma_p} \leq \frac{(1-r^2_{\Gamma_{p-1}})\delta^2}{2\alpha_p(1-r_{\Gamma_{p-1}})^2},$$

we get the existence of $r_{\Gamma_p} < 1$. □

3.4. **The multiplicity of the root $\omega = 0$ of $\det D_{\Gamma_p}(\omega)$.**

Lemma 3.3. *For all $p \geq 1$ we have the following informations*

$$(\mathcal{P}^1_p):\ \frac{\det \tilde{D}_{\Gamma_p}}{\det D_{\Gamma_p}}(0) = 1\ ,\ \ (\mathcal{P}^2_p):\ \partial_\omega\left(\frac{\det \tilde{D}_{\Gamma_p}}{\det D_{\Gamma_p}}\right)(0) < 0.$$

Proof. Lemma 3.1 insures us that $\frac{\det \tilde{D}_{\Gamma_1}}{\det D_{\Gamma_1}}(\omega) = \frac{(n_1-2)\,\omega+\alpha_1}{n_1\,\omega+\alpha_1}$, and in particular

$$\partial_\omega\left(\frac{\det \tilde{D}_{\Gamma_1}}{\det D_{\Gamma_1}}\right)(\omega) = -\frac{2\alpha_1}{(n_1\omega+\alpha_1)^2},$$

and the Lemma follows for $p = 1$, since $\alpha_1 > 0$. We shall show the general case by recursion. Let us denote by $P_p(\omega)$ and $Q_p(\omega)$ the numerator and respectively the denominator in recursion formula of the ratio from Lemma 3.1

$$P_p(\omega) = \frac{(n_p-2)\,\omega+\alpha_p}{n_p\,\omega+\alpha_p} - \frac{(n_p-4)\,\omega+\alpha_p}{n_p\,\omega+\alpha_p}\,e^{-2\omega a_{p-1}}\frac{\det \tilde{D}_{\Gamma_{p-1}}(\omega)}{\det D_{\Gamma_{p-1}}(\omega)},$$

$$Q_p(\omega) = 1 - \frac{(n_p-2)\,\omega+\alpha_p}{n_p\,\omega+\alpha_p}\,e^{-2\omega a_{p-1}}\frac{\det \tilde{D}_{\Gamma_{p-1}}(\omega)}{\det D_{\Gamma_{p-1}}(\omega)}.$$

We have $P_p(0) = Q_p(0) = 0$, and in view of $(\mathcal{P}^1_{p-1})$ we compute

$$\partial_\omega P_p(0) = \partial_\omega Q_p(0) = \frac{2}{\alpha_p} + 2a_{p-1} - \partial_\omega \left(\frac{\det \tilde{D}_{\Gamma_{p-1}}}{\det D_{\Gamma_{p-1}}} \right)(0).$$

Therefore $(\mathcal{P}^2_{p-1})$ insures us that $\partial_\omega P_p(0) = \partial_\omega Q_p(0) \neq 0$ and we apply l'Hôpital's rule to conclude $(\mathcal{P}^1_p)$.

Since

$$P_p(\omega) - Q_p(\omega) = -\frac{2\omega}{n_p\omega + \alpha_p} \left(1 - e^{-2a_{p-1}\omega} \frac{\det \tilde{D}_{\Gamma_{p-1}}}{\det D_{\Gamma_{p-1}}}(\omega) \right).$$

we define $\tilde{P}_p(\omega)$ and $\tilde{Q}_p(\omega)$ by

$$P_p(\omega) = \frac{2\omega}{n_p\omega + \alpha_p} \tilde{P}_p(\omega), \; Q_p(\omega) = \frac{2\omega}{n_p\omega + \alpha_p} \tilde{Q}_p(\omega).$$

In particular

$$\frac{\det \tilde{D}_{\Gamma_p}}{\det D_{\Gamma_p}}(\omega) = \frac{\tilde{P}_p}{\tilde{Q}_p}(\omega), \quad \tilde{P}_p(\omega) - \tilde{Q}_p(\omega) = -\left(1 - e^{-2a_{p-1}\omega} \frac{\det \tilde{D}_{\Gamma_{p-1}}}{\det D_{\Gamma_{p-1}}}(\omega) \right).$$

By using $(\mathcal{P}^1_{p-1})$ and $(\mathcal{P}^2_{p-1})$

$$\partial_\omega(\tilde{P}_p - \tilde{Q}_p)(0) = -2a_{p-1} + \partial_\omega \left(\frac{\det \tilde{D}_{\Gamma_{p-1}}}{\det D_{\Gamma_{p-1}}} \right)(0)$$

Moreover,

$$\tilde{P}_p(0) = \tilde{Q}_p(0) = \frac{\alpha_p}{2} \partial_\omega P_p(0) = \frac{\alpha_p}{2} \partial_\omega Q_p(0) \neq 0,$$

and we can compute

$$\partial_\omega \left(\frac{\det \tilde{D}_{\Gamma_p}}{\det D_{\Gamma_p}} \right)(0) = \frac{\partial_\omega \tilde{P}_p(0)\tilde{Q}_p(0) - \tilde{P}_p(0)\partial_\omega \tilde{Q}_p(0)}{(\tilde{Q}_p(0))^2} = \frac{\partial_\omega(\tilde{P}_p - \tilde{Q}_p)(0)}{\tilde{Q}_p(0)}$$

$$= -\frac{2a_{p-1} - \partial_\omega \left(\frac{\det \tilde{D}_{\Gamma_{p-1}}}{\det D_{\Gamma_{p-1}}} \right)(0)}{\frac{\alpha_p}{2} \left(\frac{2}{\alpha_p} + 2a_{p-1} - \partial_\omega \left(\frac{\det \tilde{D}_{\Gamma_{p-1}}}{\det D_{\Gamma_{p-1}}} \right)(0) \right)}.$$

By using again $(\mathcal{P}^2_{p-1})$ we obtain $(\mathcal{P}^2_p)$.

□

Lemma 3.4. *$\omega = 0$ is a root of order $p-1$ of $\det D_{\Gamma_p}(\omega)$.*

Proof. From Lemma 3.1 we have $\det D_{\Gamma_1}(\omega) = n_1\,\omega + \alpha$, so $\det D_{\Gamma_1}(0) \neq 0$. Lemma 3.1 provides us the expression

$$\det D_{\Gamma_p}(\omega) = \frac{n_p\,\omega + \alpha_p}{\omega + \alpha_p} e^{\omega a_{p-1}} \det D_{\Gamma_{p-1}}(\omega) \left(1 - \frac{(n_p - 2)\,\omega + \alpha_p}{n_p\,\omega + \alpha_p} e^{-2\omega a_{p-1}} \frac{\det \tilde{D}_{\Gamma_{p-1}}}{\det D_{\Gamma_{p-1}}}(\omega) \right),$$

so by recursion it is enough to show that $\omega = 0$ is a simple root for

$$1 - \frac{(n_p - 2)\,\omega + \alpha_p}{n_p\,\omega + \alpha_p} e^{-2\omega a_{p-1}} \frac{\det \tilde{D}_{\Gamma_{p-1}}}{\det D_{\Gamma_{p-1}}}(\omega).$$

This expression is precisely $Q_p(\omega)$ from the proof of Lemma 3.3, and it was proved there that $\partial_\omega Q_p(0) \neq 0$. □

3.5. **Vanishing of the numerator at $\tau = 0$.** Recall that we have denoted by $M^{c_e}_{\Gamma_p}(\omega)$ (respectively $\det M^{\tilde{c}_e}_{\Gamma_p}(\omega)$) the matrix $D_{\Gamma_p}(\omega)$ with the column corresponding to the unknown c_e (respectively $\tilde{c}_e$), $D^e_{\Gamma_p}(\omega)$ (respectively $D^{\tilde{e}}_{\Gamma_p}(\omega)$), replaced by the free terms column $T_{\Gamma_p}(\omega)$. In particular $\omega \det M^{c_e}_{\Gamma_p}(\omega)$ (respectively $\omega \det M^{\tilde{c}_e}_{\Gamma_p}(\omega)$) is the determinant of the matrix $D_{\Gamma_p}(\omega)$ with the column corresponding to the unknown c_e (respectively $\tilde{c}_e$) replaced by $\omega T_{\Gamma_p}(\omega)$.

Lemma 3.5. *The following holds*

$$-(\omega T_{\Gamma_p}(\omega))(0) = \sum_{e\in\mathcal{E}} t_e(0,0) D^e_{\Gamma_p}(0) + \sum_{e\in\mathcal{I}} t_e(0,0) D^{\tilde{e}}_{\Gamma_p}(0) \tag{18}$$

Remark 3.6. *From the shape of $D_{\Gamma_p}(\omega)$ displayed in the proof of Lemma 3.1 we notice that the two junction columns with $D_{\Gamma_{p-1}}(\omega)$, corresponding to the coefficients of the resolvent on the connecting edge I_N, are*

$$D^{I_N}_{\Gamma_p}(\omega) = {}^t\left(0,\dots,0,-1,\frac{\omega}{\omega+\alpha_{p-1}},e^{-\omega a_{p-1}},0,\dots,0,\frac{-\omega+\alpha_p}{\omega+\alpha_p}e^{-\omega a_{p-1}}\right)$$

and

$$D^{\tilde{I}_N}_{\Gamma_p}(\omega) = {}^t\left(0,\dots,0,-1,-\frac{\omega}{\omega+\alpha_{p-1}},e^{\omega a_{p-1}},0,\dots,0,e^{\omega a_{p-1}}\right).$$

In particular, these two columns are the same at $\omega = 0$. Moreover, $D_{\Gamma_p}(\omega)$ contains $p-1$ such pair of columns, $D^e_{\Gamma_p}(0) = D^{\tilde{e}}_{\Gamma_p}(0)$ for all $e \in \mathcal{I}$. Thus, the last term in the right hand side of (18) could be $D^e_{\Gamma_p}(0)$ either $D^{\tilde{e}}_{\Gamma_p}(0)$, $e \in \mathcal{I}$.

Proof. We will prove this identity inductively. In the case $p = 1$ we use that (ωT_{Γ_1}) is given in Section 3.2.1. We choose $X_1 = (t_1(0,0), t_2(0,0), \dots, t_{n_1}(0,0))$ and then $D_{\Gamma_1}(0)X_1 = -(\omega T_{\Gamma_1})(0)$ which proves (18) when $p = 1$.

Given now X_{p-1} such that $D_{\Gamma_{p-1}}(0)X_{p-1} = -(\omega T_{\Gamma_{p-1}}(\omega))(0)$ we construct X_p as follows

$${}^t X_p = ({}^t X_{p-1}, 0, t_{N+1}(0,0), \dots, t_{N+n_p-1}(0,0)).$$

Using the recursion between D_{Γ_p} and $D_{\Gamma_{p-1}}$ used in the proof of Lemma 3.1, identity

$$\omega T_{\Gamma_p}(\omega) = \begin{pmatrix} \omega T_{\Gamma_{p-1}}(\omega) \\ t_{N+1}(0,\omega) - t_N(a_{p-1},\omega) \\ \dots \\ t_{N+n_p-1}(0,\omega) - t_{N+n_p-2}(0,\omega) \\ \frac{\omega-\alpha_p}{\omega+\alpha_p} t_N(a_{p-1},\omega) + \frac{\omega}{\omega+\alpha_p}\sum_{1\le j\le n_p-1} t_{N+j}(0,\omega) \end{pmatrix},$$

and the fact that $t_e(0,0) = t_e(a_e,0)$ for all $e \in \mathcal{I}$, we obtain that X_p satisfies the system $D_{\Gamma_p}(0)X_p = -(\omega T_{\Gamma_p}(\omega))(0)$. Writing this identity in terms of the columns of matrix $D_{\Gamma_p}(0)$ we obtain the desired identity. □

Lemma 3.7. *$\omega = 0$ is a root of order $p-1$ of $\omega \det M^{c_e}_{\Gamma_p}(\omega)$ and of $\omega \det M^{\tilde{c}_e}_{\Gamma_p}(\omega)$ for all edge e.*

Proof. We shall perform the proof for $\omega \det M^{c_e}_{\Gamma_p}(\omega)$; the result for $\omega \det M^{\tilde{c}_e}_{\Gamma_p}(\omega)$ will be the same. From the shape of $D_{\Gamma_p}(\omega)$ displayed in the proof of Lemma 3.1 and Remark 3.6 we have $p-1$ pairs of columns that are equal at $\omega = 0$. Moreover, by Lemma 3.5, $(\omega T_{\Gamma_p})(0)$ is a linear combination of these columns evaluated at $\omega = 0$.

The derivative of a determinant is the sum of the determinants of the matrices obtained by differentiating one column. When T_{Γ_p} does not replace any of these $2(p-1)$ columns it follows that the result of this Lemma holds since there are always two columns identically. Then by the above argument we have already

$$\partial^k_\omega(\omega \det M^{c_e}_{\Gamma_p})(0) = 0, \ \forall\, 0 \le k \le p-3. \tag{19}$$

Assume now that T_{Γ_p} replaces one of these $2(p-1)$ columns. For proving the Lemma we are left to show that

$$\partial^{p-2}_\omega(\omega \det M^{c_e}_{\Gamma_p})(0) = 0.$$

Using again the fact that $D_{\Gamma_p}(\omega)$ contains $p-1$ pairs of columns that are the same two by two at $\omega = 0$, we only need to show that $\det A_{\Gamma_p}(0) = 0$, where $A_{\Gamma_p}(\omega)$ is $D_{\Gamma_p}(\omega)$ with the column $\omega T_{\Gamma_p}(\omega)$ replacing one column of one pair, and one column of each remaining $p-2$ pairs of columns is differentiated. In particular $A_{\Gamma_p}(0)$ contains one column of each $p-1$ pairs unchanged. Since by Lemma 3.5 we know that $(\omega T_{\Gamma_p}(\omega))(0)$ is a linear combination of the columns corresponding to external edges and of the internal ones (each one from the $p-1$ pairs) the new determinant vanishes and the proof is finished. □

Lemma 3.8. *For all edges indices λ and e, $\omega = 0$ is a root of order $p-2$ for the coefficient $f_{\lambda,e}(\omega)$ of $t_\lambda(0,\omega)$ in $\omega \det M^{c_e}_{\Gamma_p}(\omega)$, and the same holds for the coefficient $\tilde{f}_{\lambda,e}(\omega)$ of $t_\lambda(0,\omega)$ in $\omega \det M^{\tilde{c}_e}_{\Gamma_p}(\omega)$.*

Proof. This result follows from the discussion that led to (19): the matrix $\omega M^{c_e}_{\Gamma_p}(\omega)$ has $p-2$ pairs of columns that are identical at $\omega = 0$. □

Lemma 3.9. *For all edge index e and all external edge index λ, $\omega = 0$ is a root of order $p-1$ for the coefficient $f_{\lambda,e}(\omega)$ of $t_\lambda(0,\omega)$ in $\omega \det M^{c_e}_{\Gamma_p}(\omega)$, and the same holds for the coefficient $\tilde{f}_{\lambda,e}(\omega)$ of $t_\lambda(0,\omega)$ in $\omega \det M^{\tilde{c}_e}_{\Gamma_p}(\omega)$.*

Proof. The statement corresponds to the particular case of Lemma 3.7 where all the components of T_{Γ_p} are taken to be zero except $t_\lambda(0,\omega)$ which is replaced by one. □

Lemma 3.10. *For all edge index e and all internal edge index λ, $\omega = 0$ is a root of order $p-1$ for $f^1_{\lambda,e}(\omega) + f^2_{\lambda,e}(\omega)$ where $f^1_{\lambda,e}(\omega)$ is the coefficient of $t_\lambda(0,\omega)$ in $\omega \det M^{c_e}_{\Gamma_p}(\omega)$ and $f^2_{\lambda,e}(\omega)$ is the coefficient of $t_\lambda(a_\lambda,\omega)$ in $\omega \det M^{c_e}_{\Gamma_p}(\omega)$. Also, the same holds for $\tilde{f}^1_{\lambda,e}(\omega) + \tilde{f}^2_{\lambda,e}(\omega)$, where $\tilde{f}^1_{\lambda,e}(\omega)$ is the coefficient of $t_\lambda(0,\omega)$ in $\omega \det M^{\tilde{c}_e}_{\Gamma_p}(\omega)$ and $\tilde{f}^2_{\lambda,e}(\omega)$ is the coefficient of $t_\lambda(a_\lambda,\omega)$ in $\omega \det M^{\tilde{c}_e}_{\Gamma_p}(\omega)$*

Proof. The proof goes the same as for Lemma 3.9. □

3.6. **The end of the proof.** From the previous subsections we conclude the following result.

Lemma 3.11. *Function $\omega R_\omega \mathbf{f}(x)$ can be analytically continued in a region containing the imaginary axis.*

Proof. The proof is an immediate consequence of decomposition (15):

$$R_\omega \mathbf{u}_0(x) = \frac{\det M^{c_e}_{\Gamma_p}(\omega)}{\det D_{\Gamma_p}(\omega)}\, e^{\omega x} + \frac{\det M^{\tilde{c}_e}_{\Gamma_p}(\omega)}{\det D_{\Gamma_p}(\omega)}\, e^{-\omega x} + \frac{t_e(x,\omega)}{\omega}, \tag{20}$$

for $x \in I_e$, and Lemma 3.2, Lemma 3.3 and Lemma 3.7. □

Proof of Theorem 1.2. As a consequence of Lemma 3.11 we can use a spectral calculus argument to write the solution of the Schrödinger equation with initial data $\mathbf{u}_0$ as

$$e^{-itH^\Gamma_\alpha}\mathbf{u}_0(x) = \frac{1}{i\pi}\int_{-\infty}^{\infty} e^{-it\tau^2}\tau R_{i\tau}\mathbf{u}_0(x)d\tau. \tag{21}$$

In view of the definition of t_e and with the notations from Lemma 3.9 and Lemma 3.10 we can also write the decomposition (20) as

(22)

$$\begin{aligned}
\tau R_{i\tau}\mathbf{u}_0(x) = {} & \frac{1}{2}\int_{I_e}\mathbf{u}_0\, e^{-i\tau|x-y|}dy + \sum_{\lambda\in\mathcal{E}}\frac{f_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)}\int_{I_\lambda}\mathbf{u}_0(y)e^{i\tau y}dy\, e^{i\tau x} \\
& + \sum_{\lambda\in\mathcal{E}}\frac{\tilde{f}_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)}\int_{I_\lambda}\mathbf{u}_0(y)e^{i\tau y}dy\, e^{-i\tau x} \\
& + \sum_{\lambda\in\mathcal{I}}\int_{I_\lambda}\mathbf{u}_0(y)\left(e^{i\tau y}\frac{f^1_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)} + e^{i\tau(a_\lambda-y)}\frac{f^2_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)}\right)dy\, e^{i\tau x} \\
& + \sum_{\lambda\in\mathcal{I}}\int_{I_\lambda}\mathbf{u}_0(y)\left(e^{i\tau y}\frac{\tilde{f}^1_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)} + e^{i\tau(a_\lambda-y)}\frac{\tilde{f}^2_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)}\right)dy\, e^{-i\tau x}.
\end{aligned}$$

Moreover, in view of the results in Lemma 3.10 and Lemma 3.9 we gather the terms as follows

$$\begin{aligned}
\tau R_{i\tau}\mathbf{u}_0(x) = {} & \frac{1}{2}\int_{I_e}\mathbf{u}_0\, e^{-i\tau|x-y|}dy \\
& + \sum_{\lambda\in\mathcal{E}}\int_{I_\lambda}\mathbf{u}_0(y)\,\frac{f_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)}e^{i\tau(x+y)}\,dy + \sum_{\lambda\in\mathcal{E}}\int_{I_\lambda}\mathbf{u}_0(y)\,\frac{\tilde{f}_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)}e^{i\tau(y-x)}\,dy \\
& + \sum_{\lambda\in\mathcal{I}}\int_{I_\lambda}\mathbf{u}_0(y)\,\frac{f^1_{\lambda,e}(i\tau)+f^2_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)}\,e^{i\tau(x+y)}\,dy + \sum_{\lambda\in\mathcal{I}}\int_{I_\lambda}\mathbf{u}_0(y)\,\frac{\tilde{f}^1_{\lambda,e}(i\tau)+\tilde{f}^2_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)}\,e^{i\tau(y-x)}\,dy \\
& + \sum_{\lambda\in\mathcal{I}}\int_{I_\lambda}\mathbf{u}_0(y)\,\frac{\left(e^{i\tau(a_\lambda-y)}-e^{i\tau y}\right)f^2_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)}\,e^{i\tau x}\,dy \\
& + \sum_{\lambda\in\mathcal{I}}\int_{I_\lambda}\mathbf{u}_0(y)\,\frac{\left(e^{i\tau(a_\lambda-y)}-e^{i\tau y}\right)\tilde{f}^2_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)}\,e^{-i\tau x}\,dy.
\end{aligned} \tag{23}$$

Let e be an external edge. In view of Lemma 3.4 and Lemma 3.9 we obtain that the fraction $\frac{f_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)}$ is upper bounded near $\tau = 0$. Outside a neighbourhood of $\tau = 0$ we use Lemma 3.2 to infer that $|\det D_{\Gamma_p}(i\tau)|$ is positively lower bounded outside neighbourhoods of $\tau = 0$. Moreover, in view of the explicit entries of $M^{c_e}_{\Gamma_p}(i\tau)$, we see that $f_{\lambda,e}(i\tau)$ is upper bounded for any $\tau \in \mathbb{R}$ since all the entries of matrix $D_{\Gamma_p}(i\tau)$ as well as the coefficients of t_λ in $T_{\Gamma_p(i\tau)}$ have absolute value less than one. Summarizing, we have obtained that

$$\frac{f_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)} \in L^\infty(\mathbb{R}).$$

The derivative of this fraction is upper-bounded near $\tau = 0$ by limited development at $\tau = 0$. Outside neighbourhoods of $\tau = 0$ we have that $\partial_\tau f_{\lambda,e}(i\tau)$ and $\partial_\tau \det D_{\Gamma_p}(i\tau)$ have upper bounds of type $\frac{1}{\tau^2}$. This is because in each term of $\partial_\tau f_{\lambda,e}(i\tau)$ and $\partial_\tau \det D_{\Gamma_p}(i\tau)$ contains a derivative of an element of the line given by the δ-condition involving the derivatives in the root vertex $\mathcal{O}$. This vertex is the one which is an initial vertex for all n edges emerging from it: $I(e) = \mathcal{O}, \forall e \in E, \mathcal{O} \in e$. If α denotes the strength of the δ-condition in $\mathcal{O}$, then this line of the matrix $D_{\Gamma_p}(i\tau)$ is composed by $0, 1$ and $\pm\frac{i\tau}{i\tau+\alpha}$, where the minus sign appears only on the finite edges that stars from $\mathcal{O}$, and this line for the column matrix $i\tau T_{\Gamma_p}(i\tau)$ is

$$\left(\frac{i\tau-\alpha}{i\tau+\alpha}\, t_1(0,i\tau) + \frac{i\tau}{i\tau+\alpha} \sum_{2\le j\le n} t_j(0,i\tau)\right).$$

Finally, as above, $f_{\lambda,e}(i\tau)$ and $\det D_{\Gamma_p}(i\tau)$ are upper bounded and from Lemma 3.2 we have that $|\det D_{\Gamma_p}(i\tau)|$ is positively lower bounded outside neighbourhoods of $\tau = 0$. As a conclusion we infer that

$$\partial_\tau \frac{f_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)} \in L^1(\mathbb{R}).$$

The same argument using Lemma 3.9, Lemma 3.10 and Lemma 3.8 can be performed to obtain that

$$\frac{\tilde{f}_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)}, \quad \frac{f^1_{\lambda,e}(i\tau) + f^2_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)}, \quad \frac{\tilde{f}^1_{\lambda,e}(i\tau) + \tilde{f}^2_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)},$$

$$\frac{\left(e^{i\tau(a_\lambda - y)} - e^{i\tau y}\right) f^2_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)}, \quad \frac{\left(e^{i\tau(a_\lambda - y)} - e^{i\tau y}\right) \tilde{f}^2_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)}$$

are in L^∞ with derivative in L^1. Notice that when λ belongs to an internal edge I_λ it follows that the interval I_λ have finite length. Therefore for the last fractions we use that $\left(e^{i\tau(a_\lambda - y)} - e^{i\tau y}\right) f^2_{\lambda,e}(i\tau)$ vanishes or order $p-1$ at $\tau = 0$ and repeat the argument used above. The only difference with the previous cases is that we will obtain bounds in terms of parameter y. Since y is now on an internal edge I_λ of finite length we obtain uniform bounds. Therefore the dispersion estimate (10) of Theorem 1.2 follows from (21) by

using (23) and the classical oscillatory integral estimate

$$\left|\int_{-\infty}^{\infty} e^{-it\tau^2} e^{i\tau a} g(\tau) d\tau\right| \leq \frac{C}{\sqrt{|t|}} \left(\|g\|_{L^\infty} + \|g'\|_{L^\infty}\right). \tag{24}$$

□

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