Comparing Brownian stochastic integrals for the convex order

Francis HIRSCH[(1)] and Marc YOR[(2)(3)]

[(1)] Laboratoire d'Analyse et Probabilités,
Université d'Évry - Val d'Essonne, 23 Boulevard de France,
F-91037 Évry Cedex
e-mail: `francis.hirsch@univ-evry.fr`

[(2)] Laboratoire de Probabilités et Modèles Aléatoires,
Université Paris VI et VII, 4 Place Jussieu - Case 188,
F-75252 Paris Cedex 05
e-mail: `deaproba@proba.jussieu.fr`

[(3)] Institut Universitaire de France

Abstract: We show that, in general, inequalities between integrands with respect to Brownian motion do not lead to majorization in the convex order for the corresponding stochastic integrals. Particular examples and counter-examples are discussed.

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1 Introduction

In this paper, we are interested in the following general question. Let X and Y be square integrable Brownian centered random variables given by their predictable representations:

$$X = \int_0^\infty H_t \, \mathrm{d}B_t \qquad Y = \int_0^\infty K_t \, \mathrm{d}B_t \tag{1.1}$$

with

$$\int_0^\infty \mathbb{E}[H_t^2] \, \mathrm{d}t < \infty \ \text{ and } \ \int_0^\infty \mathbb{E}[K_t^2] \, \mathrm{d}t < \infty. \tag{1.2}$$

Is it possible to give conditions on H and K ensuring that $X \leq Y$ in the convex order? We recall that two integrable random variables X and Y are said to satisfy $X \leq Y$ in the convex order, which will be denoted in the sequel by $X \overset{(c)}{\leq} Y$, if, for every convex $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$,

$$-\infty < \mathbb{E}[\varphi(X)] \leq \mathbb{E}[\varphi(Y)] \leq +\infty.$$

There is an obvious necessary condition:

$$X \overset{(c)}{\leq} Y \Longrightarrow \int_0^\infty \mathbb{E}[H_t^2]\, \mathrm{d}t \leq \int_0^\infty \mathbb{E}[K_t^2]\, \mathrm{d}t. \tag{1.3}$$

This condition is far from being sufficient. In next Section 2 we present an easy counter-example.
Then, in Section 3, we consider the elementary case where (H_t) or (K_t) is deterministic. In this case, there is a simple sufficient condition.
In the three following sections, we study particular families $(X^f(a); a \geq 0)$ defined by:

$$X^f(a) = \int_0^\infty f(a, s, B_s)\, \mathrm{d}B_s,$$

where f denotes a non negative Borel function on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$.
In Section 4, $f(a, s, x) = 1_{(0,1)}(s)\, 1_{(a,+\infty)}(x)$. Then f is decreasing with respect to a, and the map: $a \longrightarrow X^f(a)$ also is decreasing in the convex order.
In Section 5, $f(a, s, x) = 1_{(0,1)}(s)\, 1_{(-\infty,a)}(x)$. Then f is increasing with respect to a, and the map: $a \longrightarrow X^f(a)$ also is increasing in the convex order.
In Section 6, $f(a, s, x) = a\, 1_{(0,1)}(s) + 1_{(1,2)}(s)\, 1_{(-\infty,0)}(x)$. Then f is increasing with respect to a, and the map: $a \longrightarrow X^f(a)$ is not monotone in the convex order. More precisely, $a \longrightarrow \mathbb{E}[(X^f(a))^2]$ is obviously increasing, but $a \longrightarrow \mathbb{E}[\exp(X^f(a))]$ is strictly decreasing on $[0, a_0]$ for some $a_0 > 0$.

2 A simple example

In this section, we show that, in general, the converse of (1.3) does not hold, even if H is deterministic.

Proposition 2.1 *We set, for $\lambda \geq 0$,*

$$X_\lambda = \int_0^1 \exp(\lambda\, B_s)\, \mathrm{d}B_s.$$

Obviously, for any $\lambda \geq 0$,

$$\int_0^1 \mathbb{E}[\exp(2\lambda\, B_s)]\, \mathrm{d}s = \int_0^1 \exp(2\lambda^2 s)\, \mathrm{d}s \geq 1.$$

However, there exists $\lambda > 0$ such that $B_1 \overset{(c)}{\leq} X_\lambda$ does not hold.

Proof For every C^1-function φ with bounded first derivative, we have

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathbb{E}[\varphi(X_\lambda)]\Big|_{\lambda=0} = \frac{1}{2}\,\mathbb{E}[\varphi'(B_1)\,(B_1^2-1)].$$

Suppose that for every $\lambda > 0$, one has: $B_1 \overset{(c)}{\leq} X_\lambda$. Then, for every convex C^1-function φ with bounded first derivative, $\mathbb{E}[\varphi'(B_1)\,(B_1^2-1)] \geq 0$. In particular, we obtain for the convex function

$$\varphi(x) = (x+1)^2\,1_{(-1,0)}(x) + (1+2x)\,1_{(0,+\infty)},$$

$2\,\mathbb{E}[((B_1+1)^+ \wedge 1)\,(B_1^2-1)] \geq 0$. Now,

$$\mathbb{E}[((B_1+1)^+ \wedge 1)\,(B_1^2-1)] = \frac{1}{\sqrt{2\pi}}\,(\mathrm{e}^{-1/2}-1) < 0,$$

which yields a contradiction. □

3 Case where H or K is deterministic

The following proposition partially extends a result of Pagès ([P, Proposition 2.4]) with a different method.

Proposition 3.1 *Let (H_t) be an adapted process and k be a deterministic Borel function such that:*

$$\int_0^\infty \mathbb{E}[H_t^2]\,\mathrm{d}t < \infty \quad \textit{and} \quad \int_0^\infty k^2(t)\,\mathrm{d}t < \infty.$$

We set:

$$X = \int_0^\infty H_t\,\mathrm{d}B_t \qquad Y = \int_0^\infty k(t)\,\mathrm{d}B_t.$$

One has:

1. *if $\displaystyle\int_0^\infty H_t^2\,\mathrm{d}t \leq \int_0^\infty k(t)^2\,\mathrm{d}t$ a.e., then $X \overset{(c)}{\leq} Y$;*
2. *if $\displaystyle\int_0^\infty k(t)^2\,\mathrm{d}t \leq \int_0^\infty H_t^2\,\mathrm{d}t$ a.e., then $Y \overset{(c)}{\leq} X$.*

Proof By the Dubins-Schwarz theorem (see, e.g. [RY, Chapter V]), there exist a filtration $(\mathcal{G}_u)$ and a $\mathcal{G}$-Brownian motion (β_u) such that: $X = \beta_T$, with $T := \int_0^\infty H_t^2 \, \mathrm{d}t$ a $(\mathcal{G}_u)$-stopping time. By hypothesis, T is integrable. Denote by σ the deterministic time $\int_0^\infty k(t)^2 \, \mathrm{d}t$. Let also $\widetilde{Y} = \beta_\sigma$. Since k is deterministic, we have: $Y \overset{(law)}{=} \widetilde{Y}$. Now, if $T \leq \sigma$, then $X = \mathbb{E}[\widetilde{Y} \,|\, \mathcal{G}_T]$, and if $\sigma \leq T$, then $\widetilde{Y} = \mathbb{E}[X \,|\, \mathcal{G}_\sigma]$. The desired result then follows from Jensen's inequality. □

Remark 3.1 We shall show in Section 6 that in the above proposition, the hypothesis that k is deterministic cannot be deleted. Likewise, the hypothesis: $\int_0^\infty \mathbb{E}[H_t^2] \, \mathrm{d}t < \infty$ cannot be deleted as shown by the following example. Suppose that $k(t) = 1_{(0,1)}(t)$ and $H_t = 1_{(0,d_1)}(t)$ with $d_1 = \inf\{t \geq 1; B_t = 0\}$. Then, since $1 \leq d_1$, one has $0 \leq k(t) \leq H_t$. But, $X = B_{d_1} = 0$, $Y = B_1$ and, obviously, $B_1 \overset{(c)}{\leq} 0$ does not hold. On the contrary, we have $0 \overset{(c)}{\leq} B_1$, which shows that, in general, the implication (1.3) does not hold if the condition (1.2) is not fulfilled.

4 A decreasing family

We set, for $t \geq 0$ and $a \geq 0$,

$$X_t(a) = \int_0^t 1_{(B_s > a)} \, \mathrm{d}B_s$$

and we denote $X_1(a)$ simply by $X(a)$. We also denote as usual by T_a the entrance time of (B_t) in $[a, +\infty[$.

Proposition 4.1 *The map: $a \geq 0 \longrightarrow X(a)$ is decreasing in the convex order.*

Proof Denote by $B^{(a)}$ the Brownian motion defined by: $B_t^{(a)} = B_{t+T_a} - a$. It is independent of $\mathcal{F}_{T_a}^B$ (where $(\mathcal{F}_t^B)$ denotes the natural filtration of B) and, in particular, it is independent of T_a. We set, for $t \geq 0$,

$$X_t^{(a)} = \int_0^t 1_{(B_s^{(a)} > 0)} \, \mathrm{d}B_s^{(a)}.$$

We clearly have:

$$X(a) = X_{(1-T_a)^+}^{(a)}. \tag{4.1}$$

Let $\widetilde{B}$ be an independent copy of B. We deduce from (4.1) that

$$X(a) \overset{(law)}{=} \widetilde{X}_{(1-T_a)^+}(0) \tag{4.2}$$

where $\widetilde{X}_t(a)$ is defined as $X_t(a)$ from $\widetilde{B}$ in place of B. We set, for $a \geq 0$,

$$\widehat{\mathcal{F}}_a = \sigma\{B_s, s \geq 0; \widetilde{B}_{s\wedge(1-T_a)^+}, s \geq 0\}.$$

Then, the above family of σ-algebras is decreasing with respect to a. We have, if $a \leq b$,

$$\widetilde{X}_{(1-T_b)^+}(0) = \mathbb{E}[\widetilde{X}_{(1-T_a)^+}(0) \,|\, \widehat{\mathcal{F}}_b].$$

The desired result follows from (4.2) and Jensen's inequality. □

Remark 4.1 Proposition 4.1 says, in the terminology of Hirsch et al. [HPRY], that $(X(a); a \geq 0)$ is an *inverse peacock*. By the general theorem of Kellerer (see [K, HR] and also [HPRY, Exercise 1.6]) there exists an inverse martingale which is *associated* to $(X(a); a \geq 0)$, which means that both processes have the same 1-marginals (we also say that $(X(a); a \geq 0)$ is a *1-inverse martingale*). In the previous proof, we showed that we may take as associated inverse martingale: $(\widetilde{X}_{(1-T_a)^+}(0), a \geq 0)$.

In the sequel of this section, we shall give another proof, more analytic, of Proposition 4.1, from a computation of the law of $X(a)$.

Proposition 4.2 *The law of $X(a)$ is:*

$$\sqrt{\frac{2}{\pi}} \left(\left(\int_0^a \mathrm{e}^{-u^2/2} \, \mathrm{d}u \right) \delta_0 + h_a(z) \, \mathrm{d}z \right)$$

where δ_0 denotes the Dirac measure at 0, and

$$h_a(z) = \frac{4}{3} \exp\left(-\frac{(a-2z)^2}{2} \right) 1_{(-\infty,0)}(z) + \frac{1}{3} \exp\left(-\frac{(a+z)^2}{2} \right) 1_{(0,+\infty)}(z).$$

Proof Denote by μ_a the law of T_a. One has:

$$\mu_a(\mathrm{d}y) = \frac{a}{\sqrt{2\pi}} \, 1_{(0,+\infty)}(y) \, y^{-3/2} \exp\left(-\frac{a^2}{2y} \right) \, \mathrm{d}y. \tag{4.3}$$

By (4.2), we have for every non negative φ,

$$\mathbb{E}[\varphi(X(a))] = \varphi(0) \int_1^{+\infty} \mu_a(\mathrm{d}y) + \int_0^1 \mathbb{E}[\varphi(X_{1-y}(0))] \, \mu_a(\mathrm{d}y). \tag{4.4}$$

By (4.3) and the change of variable $y = a^2\, u^{-2}$, we obtain:

$$\int_1^{+\infty} \mu_a(\mathrm{d}y) = \sqrt{\frac{2}{\pi}} \int_0^a \mathrm{e}^{-u^2/2}\, \mathrm{d}u. \tag{4.5}$$

By Tanaka's formula, $X_t(0) = B_t^+ - \frac{1}{2} L_t$ where L_t denotes the local time at 0 of the Brownian B. According to [RY, Exercise 2.18], the joint law of (B_t, L_t) has a density given by:

$$\frac{1}{\sqrt{2\pi t^3}} \left[(a+b) \exp\left(-\frac{(a+b)^2}{2t} \right) \right] \quad \text{for} \quad a, b \geq 0 \quad \text{and}$$

$$\frac{1}{\sqrt{2\pi t^3}} \left[(-a+b) \exp\left(-\frac{(-a+b)^2}{2t} \right) \right] \quad \text{for} \quad a \leq 0, b \geq 0.$$

Consequently, the density of the law of $X_t(0)$ is:

$$\sqrt{\frac{2}{\pi\, t}} \left[\frac{4}{3} \exp\left(-\frac{2u^2}{t} \right) 1_{(u<0)} + \frac{1}{3} \exp\left(-\frac{u^2}{2t} \right) 1_{(u>0)} \right]. \tag{4.6}$$

Thus, by (4.3) and (4.6),

$$\int_0^1 \mathbb{E}[\varphi(X_{1-y}(0))]\, \mu_a(\mathrm{d}y) = \frac{a}{3\pi} \int_0^\infty I_a(z)\, [2\varphi(-z/2) + \varphi(z)]\, \mathrm{d}z \tag{4.7}$$

with

$$I_a(z) = \int_0^1 y^{-3/2} (1-y)^{-1/2} \exp\left(-\frac{a^2}{2y} \right) \exp\left(-\frac{z^2}{2(1-y)} \right)\, \mathrm{d}y.$$

The change of variable: $y = a^2 u (1 + a^2 u)^{-1}$ yields:

$$\begin{aligned} I_a(z) &= a^{-1} \exp\left(-\frac{(a^2+z^2)}{2} \right) \int_0^\infty u^{-3/2} \exp\left(-\frac{1}{2u} \right) \exp\left(-\frac{a^2 z^2 u}{2} \right)\, \mathrm{d}u \\ &= \sqrt{2\pi}\, a^{-1} \exp\left(-\frac{(a^2+z^2)}{2} \right) \mathbb{E}\left[\exp\left(-\frac{(az)^2 T_1}{2} \right) \right] \end{aligned}$$

and hence

$$I_a(z) = \sqrt{2\pi}\, a^{-1} \exp\left(-\frac{(a+z)^2}{2} \right). \tag{4.8}$$

Finally, gathering (4.4), (4.5), (4.7) and (4.8), we obtain the announced result, after an obvious change of variable. □

The next corollary follows easily from Proposition 4.2.

Corollary 4.1 *Let φ be a suitably integrable function. Then, for any $a \geq 0$,*

$$\frac{\mathrm{d}}{\mathrm{d}a}\mathbb{E}[\varphi(X(a))] =$$

$$\sqrt{\frac{2}{\pi}} \int_0^\infty (a+z) \exp\left(-\frac{(a+z)^2}{2}\right) \left(\varphi(0) - \frac{2}{3}\varphi(-z/2) - \frac{1}{3}\varphi(z)\right) \mathrm{d}z.$$

In particular, if moreover φ is convex, then

$$\frac{\mathrm{d}}{\mathrm{d}a}\mathbb{E}[\varphi(X(a))] \leq 0$$

and the inequality is strict if and only if φ is not an affine function.

Clearly, the above corollary entails Proposition 4.1.

By Remark 4.1, $(X(a); a \geq 0)$ is a 1-inverse martingale. One may wonder whether it also is a *2-inverse martingale*, that is whether it has the same 2-marginals as an inverse martingale. We answer this question in the next proposition.

Proposition 4.3 *For every $a > 0$, $\mathbb{E}[X(0)\, X(a)^2] > \mathbb{E}[X(a)^3]$. Consequently, $(X(a); a \geq 0)$ is not a 2-inverse martingale.*

Proof Set:

$$E(a) = \mathbb{E}[X(0)\, X(a)^2] - \mathbb{E}[X(a)^3]. \tag{4.9}$$

By Itô's formula:

$$E(a) = \int_0^1 \mathbb{E}\left[\left(\int_0^t 1_{(0<B_s<a)}\, \mathrm{d}B_s\right) 1_{(B_t>a)}\right] \mathrm{d}t.$$

We set, for $t > 0$,

$$U(t) = \mathbb{E}\left[\left(\int_0^t 1_{(0<B_s<1)}\, \mathrm{d}B_s\right) 1_{(B_t>1)}\right] \tag{4.10}$$

By scaling,

$$\mathbb{E}\left[\left(\int_0^t 1_{(0<B_s<a)}\, \mathrm{d}B_s\right) 1_{(B_t>a)}\right] = a\, U(a^{-2}t).$$

Hence,

$$E(a) = a \int_0^1 U(a^{-2}t)\, \mathrm{d}t = a^3 \int_0^{a^{-2}} U(t)\, \mathrm{d}t. \tag{4.11}$$

The result will then follow from the next lemma.

Lemma 4.1 *For every $t > 0$,*

$$U(t) = \frac{1}{2\sqrt{2\pi}} \int_{t^{-1}}^{\infty} \mathrm{e}^{-u/2}\, u^{-1/2}\, \mathrm{d}u$$

where $U(t)$ is defined in (4.10).

Proof By Tanaka's formula,

$$U(t) = \mathbb{E}\left[\left(1 + \frac{1}{2}\, L_t^1 - \frac{1}{2}\, L_t^0\right) 1_{(B_t>1)}\right]$$

where L_t^1 (resp. L_t^0) denotes the local time of the Brownian motion at time t in 1 (resp. 0). Using the classical property:

$$L_t^x = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t 1_{(x<B_s<x+\varepsilon)}\, \mathrm{d}s$$

we obtain:

$$\mathbb{E}[L_t^1\, 1_{(B_t>1)}] = \frac{1}{2\sqrt{2\pi}} \int_0^t \mathrm{e}^{-1/2s}\, s^{-1/2}\, \mathrm{d}s, \tag{4.12}$$

$$\mathbb{E}[L_t^0\, 1_{(B_t>1)}] = \frac{1}{2\pi} \int_0^t \mathrm{e}^{-1/2s}\, (t-s)^{1/2}\, s^{-3/2}\, \mathrm{d}s. \tag{4.13}$$

By (4.13),

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[L_t^0\, 1_{(B_t>1)}] = \frac{1}{4\pi} \int_0^t \mathrm{e}^{-1/2s}\, (t-s)^{-1/2}\, s^{-3/2}\, \mathrm{d}s$$

and the change of variable: $s = t\,(2vt+1)^{-1}$ yields:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[L_t^0\, 1_{(B_t>1)}] = \frac{1}{2\sqrt{2\pi}}\, \mathrm{e}^{-1/2t}\, t^{-1/2},$$

which is equal to $\dfrac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[L_t^1\, 1_{(B_t>1)}]$ by (4.12). Thus, $\mathbb{E}[L_t^0\, 1_{(B_t>1)}] = \mathbb{E}[L_t^1\, 1_{(B_t>1)}]$, and consequently $U(t) = \mathbb{P}[B_t > 1]$, which is the announced result. □

The proposition follows then from the above lemma and (4.11). □

Remark 4.2 By Lemma 4.1, $\lim_{t\to\infty} U(t) = \frac{1}{2}$ and $U(t)$ is equivalent to $\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-1/2t}t^{1/2}$ when t tends to 0. Therefore, by (4.11), $E(a)$ is equivalent to $\frac{a}{2}$ when a tends to 0, and is equivalent to $\sqrt{\frac{2}{\pi}}\mathrm{e}^{-a^2/2}a^{-2}$ when a tends to ∞.

5 An increasing family

We set, for $t \geq 0$ and $a \geq 0$,

$$Y_t(a) = \int_0^t 1_{(B_s<a)}\, \mathrm{d}B_s$$

and we denote $Y_1(a)$ simply by $Y(a)$.

Proposition 5.1 *The law of $Y(a)$ is:*

$$\sqrt{\frac{2}{\pi}} \left(A_a(z)\, 1_{(-\infty,a)}(z) + D_a(z)\, 1_{(a,+\infty)}(z) \right)\, \mathrm{d}z$$

with

$$A_a(z) = \left[\frac{1}{2} \exp\left(-\frac{z^2}{2} \right) - \frac{1}{6} \exp\left(-\frac{(2a-z)^2}{2} \right) \right]$$

$$\textit{and} \quad D_a(z) = \frac{4}{3} \exp\left(-\frac{(2z-a)^2}{2} \right).$$

Proof We shall follow the same lines as in the proof of Proposition 4.2, of which we keep the notation.
We set, for $t \geq 0$,

$$Y_t^{(a)} = \int_0^t 1_{(B_s^{(a)}<0)}\, \mathrm{d}B_s^{(a)}.$$

We clearly have:

$$Y(a) = 1_{(T_a>1)}\, B_1 + 1_{(T_a<1)}\, \left(a + Y^{(a)}_{(1-T_a)} \right) \tag{5.1}$$

and

$$1_{(T_a<1)}\, B_1 = 1_{(T_a<1)}\, \left(a + B^{(a)}_{(1-T_a)} \right). \tag{5.2}$$

Since $Y^{(a)}$ and $B^{(a)}$ are independent of T_a, by (5.1) and(5.2) we have for every non negative φ,

$$\begin{aligned} &\mathbb{E}[\varphi(Y(a))] = \\ &\mathbb{E}[\varphi(B_1)] + \int_0^1 \left(\mathbb{E}[\varphi(a + Y_{1-y}(0))] - \mathbb{E}[\varphi(a + B_{1-y})] \right)\, \mu_a(\mathrm{d}y). \end{aligned} \tag{5.3}$$

Clearly, $Y_t(0) \overset{(law)}{=} -X_t(0)$. Consequently, by (4.6) the density of the law of $Y_t(0)$ is:

$$\sqrt{\frac{2}{\pi\, t}}\, \left[\frac{1}{3} \exp\left(-\frac{u^2}{2t} \right) 1_{(u<0)} + \frac{4}{3} \exp\left(-\frac{2u^2}{t} \right) 1_{(u>0)} \right]. \tag{5.4}$$

By (5.4) and (4.3),

$$\int_0^1 \mathbb{E}[\varphi(a + Y_{1-y}(0))]\ \mu_a(\mathrm{d}y) =$$

$$\frac{a}{3\pi} \int_0^\infty I_a(z)\,[2\varphi(a + z/2) + \varphi(a - z)]\ \mathrm{d}z \tag{5.5}$$

where $I_a(z)$ is given in (4.8). Likewise,

$$\int_0^1 \mathbb{E}[\varphi(a + B_{1-y})]\ \mu_a(\mathrm{d}y) =$$

$$\frac{a}{2\pi} \int_0^\infty I_a(z)\,[\varphi(a + z) + \varphi(a - z)]\ \mathrm{d}z. \tag{5.6}$$

Finally, gathering (5.3), (5.5), (5.6) and (4.8), we obtain the announced result, after obvious changes of variable. □

The next corollary follows easily from Proposition 5.1.

Corollary 5.1 *Let φ be a suitably integrable function. Then, for any $a \geq 0$,*

$$\frac{\mathrm{d}}{\mathrm{d}a}\mathbb{E}[\varphi(Y(a))] =$$

$$\sqrt{\frac{2}{\pi}} \int_0^\infty (a + z) \exp\left(-\frac{(a+z)^2}{2}\right) \left(\frac{2}{3}\varphi(a + z/2) + \frac{1}{3}\varphi(a - z) - \varphi(a)\right)\ \mathrm{d}z.$$

In particular, if moreover φ is convex, then

$$\frac{\mathrm{d}}{\mathrm{d}a}\mathbb{E}[\varphi(Y(a))] \geq 0$$

and the inequality is strict if and only if φ is not an affine function.

Clearly, the above corollary entails the following proposition.

Proposition 5.2 *The map: $a \geq 0 \longrightarrow Y(a)$ is increasing in the convex order.*

Proposition 5.2 says, in the terminology of Hirsch et al. [HPRY], that $(Y(a); a \geq 0)$ is a *peacock*. By the general forementioned theorem of Kellerer, there exists a martingale which is associated to $(Y(a); a \geq 0)$ (in the sense that both processes have the same 1-marginals). We shall exhibit such a martingale, using *the stochastic differential equation method* (see [HPRY,

Chapter 6] and [HR]). We first introduce some further notation.
We set, for $(a,x) \in \mathbb{R}_+ \times \mathbb{R}$,

$$\begin{aligned} p(a,x) &= \frac{1}{\sqrt{2\pi}} \left[\exp\left(-\frac{x^2}{2}\right) - \frac{1}{3} \exp\left(-\frac{(2a-x)^2}{2}\right) \right] \quad \text{if } x < a \\ \text{and } \ p(a,x) &= \frac{1}{\sqrt{2\pi}} \frac{8}{3} \exp\left(-\frac{(2x-a)^2}{2}\right) \quad \text{if } x \geq a. \end{aligned}$$

Thus, by Proposition 5.1, the law of $Y(a)$ is $p(a,x)\,\mathrm{d}x$. We then define the *call function* C by:

$$\forall (a,x) \in \mathbb{R}_+ \times \mathbb{R}, \quad C(a,x) = \mathbb{E}[(Y(a)-x)^+].$$

We also set, for $x \in \mathbb{R}$,

$$\mathcal{N}(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\left(-\frac{u^2}{2}\right) \mathrm{d}u.$$

Lemma 5.1 *1. For every $(a,x) \in \mathbb{R}_+ \times \mathbb{R}$,*

$$p(a,x) \geq \frac{2}{3\sqrt{2\pi}} \exp\left(-\frac{[(2x-a) \vee (2a-x)]^2}{2}\right) \quad \textit{and}$$

$$p(a,x) \leq \frac{8}{3\sqrt{2\pi}} \exp\left(-\frac{[(2x-a) \vee x]^2}{2}\right).$$

2. For $(a,x) \in \mathbb{R}_+ \times \mathbb{R}$,

$$\frac{\partial}{\partial a} C(a,x) = \frac{2}{3} \mathcal{N}((2x-a) \vee (2a-x)).$$

3. For $(a,x) \in \mathbb{R}_+ \times \mathbb{R}$,

$$\frac{\partial}{\partial a} C(a,x) \leq \frac{\sqrt{2\pi}}{2}\, p(a,x).$$

Proof The first point follows from the above definition of p.
The second point is a direct consequence of Corollary 5.1.
For the third point, we remark that: $(2x-a) \vee (2a-x) \geq a \geq 0$, and

$$\sup\{\exp(u^2/2)\,\mathcal{N}(u); u \geq 0\} = \mathcal{N}(0) = 1/2.$$

□

We now set, for $(a,x) \in \mathbb{R}_+ \times \mathbb{R}$,

$$\sigma(a,x) = \left(2 \, \frac{\frac{\partial}{\partial a} C(a,x)}{p(a,x)} \right)^{1/2}.$$

This definition of σ comes from Dupire [D].

Proposition 5.3 *The stochastic differential equation:*

$$M_t = M_0 + \int_0^t \sigma(s, M_s) \, \mathrm{d}B_s, \quad M_0 \overset{(law)}{=} Y(0) \tag{5.7}$$

admits a weak solution which is unique in law. Such a solution is a continuous, strong Markov martingale, which is associated to the peacock $(Y(a); a \geq 0)$.

Proof We shall first prove the existence of a weak solution to (5.7). We remark that, by Lemma 5.1, one has: $0 < \sigma \leq (2\pi)^{1/4}$. However, σ is not continuous on $\mathbb{R}_+ \times \mathbb{R}$, but only continuous on the complement of $\{(a,a); a \geq 0\}$. So, we need to approximate σ. We set, for $\varepsilon > 0$ and $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$,

$$p_\varepsilon(t,x) = \int_0^1 p(t, x + \varepsilon \, u) \, \mathrm{d}u.$$

Thus, $p_\varepsilon(t,x) \, \mathrm{d}x$ is the law of $Y_t^\varepsilon := Y(t) - \varepsilon \, U$, where U denotes a uniform variable on $[0,1]$, independent of $Y(t)$. Clearly, p_ε is continuous and > 0 on $\mathbb{R}_+ \times \mathbb{R}$. We set:

$$C_\varepsilon(t,x) = \mathbb{E}[(Y_t^\varepsilon - x)^+] = \int_0^1 C(t, x + \varepsilon \, u) \, \mathrm{d}u.$$

Consequently, by Lemma 5.1, for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$,

$$0 < \frac{\partial}{\partial t} C_\varepsilon(t,x) = \int_0^1 \frac{\partial}{\partial t} C(t, x + \varepsilon \, u) \, \mathrm{d}u \leq \frac{\sqrt{2\pi}}{2} \, p_\varepsilon(t,x).$$

We then set, for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$,

$$\sigma_\varepsilon(t,x) = \left(2 \, \frac{\frac{\partial}{\partial t} C_\varepsilon(t,x)}{p_\varepsilon(t,x)} \right)^{1/2}.$$

Thus, σ_ε is continuous and, for every $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$,

$$0 < \sigma_\varepsilon(t,x) \leq (2\pi)^{1/4}. \tag{5.8}$$

Therefore, the stochastic differential equation:

$$M_t = M_0 + \int_0^t \sigma_\varepsilon(s, M_s)\, \mathrm{d}B_s, \quad M_0 \overset{(law)}{=} Y_0^\varepsilon \tag{5.9}$$

admits a weak solution, and, using M. Pierre's uniqueness theorem ([HPRY, Theorem 6.1]), one sees as in the proof of Theorem 6.2 in [HPRY], that such a solution M^ε is unique in law, and M^ε is a continuous martingale, which is associated to the peacock $(Y_t^\varepsilon; t \geq 0)$. Besides, by (5.8) and BDG inequalities, for every $\gamma > 0$, there exists $c_\gamma > 0$ such that

$$\forall \varepsilon > 0,\ \forall s, t \geq 0, \quad \mathbb{E}[|M_t^\varepsilon - M_s^\varepsilon|^\gamma] \leq c_\gamma |t - s|^{\gamma/2}. \tag{5.10}$$

We denote by $\mathbb{P}^\varepsilon$ the law of M^ε on $C(\mathbb{R}_+; \mathbb{R})$. We deduce from (5.10) and Kolmogorov's criterion (see e.g. [RY, Theorem 1.8, Chapter XIII]) that the family of laws: $\{\mathbb{P}^\varepsilon; \varepsilon > 0\}$ is weakly relatively compact. Therefore, there exists a sequence (ε_n) tending to 0 and a probability $\mathbb{P}$ on $C(\mathbb{R}_+; \mathbb{R})$ such that $\mathbb{P}^{\varepsilon_n}$ weakly converges to $\mathbb{P}$ when n tends to infinity. We denote by M a continuous process with law $\mathbb{P}$. Obviously, the law of M_t is $p(t, x)\, \mathrm{d}x$. To show that M is a weak solution to (5.7), we shall prove that $\mathbb{P}$ is a solution to the corresponding martingale problem. Let f be a C^2-function with compact support, let $0 \leq s \leq t$ and g be a bounded continuous function on $C([0, s]; \mathbb{R})$. By (5.9), for every n,

$$\mathbb{E}^{\varepsilon_n}\left[\left(f(y_t) - f(y_s) - \frac{1}{2}\int_s^t \sigma_{\varepsilon_n}^2(u, y_u)\, f''(y_u)\, \mathrm{d}u\right) g(y|_{[0,s]})\right] = 0.$$

We set

$$R_n = \mathbb{E}^{\varepsilon_n}\left[\int_s^t |\sigma_{\varepsilon_n}^2(u, y_u) - \sigma^2(u, y_u)|\, \mathrm{d}u\right].$$

Then,

$$R_n = \int_{\mathbb{R}} \mathrm{d}x \int_0^1 \mathrm{d}v \int_s^t \mathrm{d}u\, |\sigma_{\varepsilon_n}^2(u, x) - \sigma^2(u, x)|\, p(u, x + \varepsilon_n v).$$

Since, for every $u \geq 0$, $x \longrightarrow p(u, x)$ is right-continuous, we obtain by dominated convergence (see point 1 in Lemma 5.1): $\lim_{n\to\infty} R_n = 0$. We define the bounded function H on $C(\mathbb{R}_+; \mathbb{R})$ by:

$$H(y) = \left(f(y_t) - f(y_s) - \frac{1}{2}\int_s^t \sigma^2(u, y_u)\, f''(y_u)\, \mathrm{d}u\right) g(y|_{[0,s]}).$$

We obtain by what precedes: $\lim_{n\to\infty} \mathbb{E}^{\varepsilon_n}[H] = 0$. On the other hand, H is continuous at any y such that $u \neq y_u$ du-a.e. Now, since the law of y_u under $\mathbb{P}$ admits a density, namely $p(u,x)$, one has:

$$\int_0^\infty \int 1_{\{u=y_u\}}\, \mathbb{P}(\mathrm{d}y)\, \mathrm{d}u = 0.$$

Therefore, $\mathbb{P}$-a.s., $u \neq y_u$ du-a.e. Thus, H is continuous at every point of the complement of a $\mathbb{P}$-negligible set. By a classical result, this entails $\lim_{n\to\infty} \mathbb{E}^{\varepsilon_n}[H] = \mathbb{E}[H]$, and therefore, $\mathbb{E}[H] = 0$ which means:

$$\mathbb{E}\left[\left(f(y_t) - f(y_s) - \frac{1}{2}\int_s^t \sigma^2(u,y_u)\, f''(y_u)\, \mathrm{d}u\right)\, g(y|_{[0,s]})\right] = 0.$$

So, $\mathbb{P}$ is a solution to the martingale problem corresponding to $(Y(0), \sigma^2)$, which is equivalent to say that M is a weak solution of (5.7).

Now, by Lemma 5.1, σ satisfies the conditions allowing to apply M. Pierre's uniqueness theorem [HPRY, Theorem 6.1] with $a = \frac{1}{2}\sigma^2$. Consequently, we may show as in the proof of Theorem 6.2 in [HPRY], the uniqueness in law of the weak solution of (5.7), and the strong Markov property follows. □

Remark 5.1 By the results of Lowther [L1], the martingale M, weak solution of (5.7), is the only (in law) continuous, strong Markov martingale associated to $(Y(a); a \geq 0)$. It also follows from Lowther [L1] and [L2] (see also [HRY, Theorem 4.4]), that there exists a continuous inverse martingale associated to $(X(a); a \geq 0)$. This inverse martingale is therefore different, in law, of the one proposed in Remark 4.1, which is not continuous. Note that the above method does not apply to $(X(a); a \geq 0)$, since, as seen in Proposition 4.2, the law of $X(a)$ is not absolutely continuous.

Here again, one may wonder whether $(Y(a); a \geq 0)$ is a *2-martingale*, that is whether it has the same 2-marginals as a martingale. We answer this question in the next proposition.

Proposition 5.4 *For every $a > 0$, $\mathbb{E}[Y(0)^2\, Y(a)] < \mathbb{E}[Y(0)^3]$. Consequently, $(Y(a); a \geq 0)$ is not a 2-martingale.*

Proof We follow the same lines as in the proof of Proposition 4.3. Set:

$$F(a) = \mathbb{E}[Y(0)^2\, Y(a)] - \mathbb{E}[Y(0)^3]. \tag{5.11}$$

By Itô's formula:

$$F(a) = \int_0^1 \mathbb{E}\left[\left(\int_0^t 1_{(0<B_s<a)}\, \mathrm{d}B_s\right) 1_{(B_t<0)}\right]\, \mathrm{d}t.$$

We set, for $t > 0$,

$$V(t) = \mathbb{E}\left[\left(\int_0^t 1_{(0<B_s<1)}\, \mathrm{d}B_s\right) 1_{(B_t<0)}\right] \tag{5.12}$$

By scaling,

$$\mathbb{E}\left[\left(\int_0^t 1_{(0<B_s<a)}\, \mathrm{d}B_s\right) 1_{(B_t<0)}\right] = a\, V(a^{-2}t).$$

Hence,

$$F(a) = a \int_0^1 V(a^{-2}t)\, \mathrm{d}t = a^3 \int_0^{a^{-2}} V(t)\, \mathrm{d}t. \tag{5.13}$$

The result will then follow from the next lemma.

Lemma 5.2 *For every $t > 0$,*

$$V(t) = \frac{-1}{4\sqrt{2\pi}} \int_{t^{-1}}^{\infty} (1 - \mathrm{e}^{-2u})\, u^{-3/2}\, \mathrm{d}u$$

where $V(t)$ is defined in (5.12).

Proof By Tanaka's formula,

$$V(t) = \frac{1}{2}\, \mathbb{E}\left[\left(L_t^1 - L_t^0\right) 1_{(B_t<0)}\right]$$

where L_t^1 (resp. L_t^0) still denotes the local time of the Brownian motion at time t in 1 (resp. 0). Using again:

$$L_t^x = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t 1_{(x<B_s<x+\varepsilon)}\, \mathrm{d}s$$

we obtain:

$$\mathbb{E}[L_t^0\, 1_{(B_t<0)}] = \frac{t^{1/2}}{\sqrt{2\pi}} \tag{5.14}$$

$$\mathbb{E}[L_t^1\, 1_{(B_t<0)}] = \frac{1}{4\pi} \int_0^t \mathrm{e}^{-1/2s}\, s^{-1/2} \int_0^{t-s} \mathrm{e}^{-1/2u}\, u^{-3/2}\, \mathrm{d}u\, \mathrm{d}s. \tag{5.15}$$

By (5.15),

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[L_t^1\,1_{(B_t<0)}] = \frac{1}{4\pi}\,\int_0^t \exp\left(-\frac{t}{2}\,\frac{1}{s\,(t-s)}\right)\,(t-s)^{-3/2}\,s^{-1/2}\,\mathrm{d}s$$

and the change of variable: $s = t\,(v+1)^{-1}$ yields:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[L_t^1\,1_{(B_t<0)}] = \frac{1}{2\,\sqrt{2\pi}}\,\mathrm{e}^{-2/t}\,t^{-1/2}.$$

Thus, we obtain:

$$V(t) = \frac{-1}{4\,\sqrt{2\pi}}\int_0^t (1-\mathrm{e}^{-2/s})\,s^{-1/2}\,\mathrm{d}s$$

which yields the desired result. □

The proposition follows then from the above lemma and (5.13). □

Remark 5.2 By Lemma 5.2, $\lim_{t\to\infty} V(t) = -\frac{1}{2}$ and $V(t)$ is equivalent to $-\frac{1}{2\sqrt{2\pi}}t^{1/2}$ when t tends to 0. Therefore, by (5.13), $F(a)$ is equivalent to $-\frac{a}{2}$ when a tends to 0, and $\lim_{a\to\infty} F(a) = -\frac{1}{3\sqrt{2\pi}}$.

6 An example of non monotony

The following example (which is inspired from Pagès [P, Example 2.4.3]) shows that there exist functions f and g, satisfying

$$\forall s \geq 0, \forall x \in \mathbb{R}, \quad 0 \leq f(s,x) \leq g(s,x) \leq 1$$

and such that $\int f(s,B_s)\mathrm{d}B_s$ and $\int g(s,B_s)\mathrm{d}B_s$ are not comparable in the convex order. In particular, Proposition 3.1 does not hold in general, if neither H nor K are deterministic.

Proposition 6.1 *Set, for $a \geq 0$,*

$$Z(a) = a\,B_1 + \int_1^2 1_{(B_s<0)}\,\mathrm{d}B_s.$$

Then,there exists $a_0 > 0$ such that $a \longrightarrow \mathbb{E}[\exp(Z(a))]$ is strictly decreasing on $[0,a_0]$. In particular, if $0 \leq a_1 < a_2 \leq a_0$, then $Z(a_1)$ and $Z(a_2)$ are not comparable in the convex order.

Proof We begin with the following lemma.

Lemma 6.1 *We set, for* $x \in \mathbb{R}$,

$$\mathcal{E}(x) = \mathbb{E}\left[\exp\left(\int_0^1 1_{(B_s<x)}\, \mathrm{d}B_s\right)\right]$$

Then, for every $x \in \mathbb{R}$, $\mathcal{E}'(x) > 0$, *and hence* $\mathcal{E}$ *is strictly increasing on* $\mathbb{R}$.

Proof For $x \geq 0$, $\mathcal{E}(x) = \mathbb{E}[\exp(Y(x))]$, and, for $x \leq 0$, $\mathcal{E}(x) = \mathbb{E}[\exp(-X(-x))]$. The result follows then from Corollaries 5.1 and 4.1. □

We have:

$$Z(a) = a\, B_1 + \int_0^1 1_{(\widetilde{B}_s<-B_1)}\, \mathrm{d}\widetilde{B}_s$$

with $\widetilde{B}_s = B_{1+s} - B_1$. Since $(\widetilde{B}_s)_{0\leq s\leq 1}$ is independent of B_1, one obtains:

$$\mathbb{E}[\exp(Z(a))] = \mathbb{E}[\exp(aB_1)\, \mathcal{E}(-B_1)].$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}a}\mathbb{E}[\exp(Z(a))]\Big|_{a=0} = \mathbb{E}[B_1\, \mathcal{E}(-B_1)].$$

Since $x \longrightarrow \mathcal{E}(-x)$ is strictly decreasing by Lemma 6.1, and $x \longrightarrow x$ is strictly increasing,

$$\frac{\mathrm{d}}{\mathrm{d}a}\mathbb{E}[\exp(Z(a))]\Big|_{a=0} < \mathbb{E}[B_1]\, \mathbb{E}[\mathcal{E}(-B_1)] = 0.$$

This entails the desired result. □

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