# A unified approach to pricing and risk management of equity and credit risk

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**Abstract**

We propose a unified framework for equity and credit risk modeling, where the default time is a doubly stochastic random time with intensity driven by an underlying affine factor process. This approach allows for flexible interactions between the defaultable stock price, its stochastic volatility and the default intensity, while maintaining full analytical tractability. We characterise all risk-neutral measures which preserve the affine structure of the model and show that risk management as well as pricing problems can be dealt with efficiently by shifting to suitable survival measures. As an example, we consider a jump-to-default extension of the Heston stochastic volatility model.

**Key words:** default risk, affine processes, stochastic volatility, market price of risk, change of measure, jump-to-default

## 1 Introduction

The last few years have witnessed an increasing popularity of hybrid equity/credit risk models, see e.g. the recent papers [2, 5, 6, 8, 9, 10, 14, 15]. One of the most appealing features of such models is their capability to link the stochastic behavior of the stock price (and of its volatility) with the randomness of the default event and, hence, with the level of credit spreads. The relation between equity and credit risk is supported by strong empirical evidence (we refer the reader to [6, 15] for good overviews of the related literature) and several studies document significant relationships between stock volatility and credit spreads of corporate bonds and Credit Default Swaps ([4, 17]).

In this paper, we propose a general framework for the joint modeling of equity and credit risk which allows for a flexible dependence between stock price, stochastic volatility, default intensity and interest rate. The proposed framework is fully analytically tractable, since it relies on the powerful technology of affine processes, and nests several stochastic volatility models proposed in the literature, thereby extending their scope to a defaultable setting. Furthermore, a distinguishing feature of our approach is that, unlike the models proposed in [2, 6, 8, 9, 14, 15], we jointly consider both physical and risk-neutral probability measures, ensuring that the analytical tractability is preserved under a change of measure, while at the same time avoiding unnecessarily restrictive specifications of the risk

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premia. This enables us to efficiently compute several quantities which are of interest in view of both risk management and pricing applications.

The paper is structured as follows. Sect. 2 introduces the modeling framework, while Sect. 3 gives a characterisation of the family of risk-neutral measures which preserve the affine structure of the model. In Sect.s 4 and 5, we show how most quantities of interest for risk management and pricing applications, respectively, can be efficiently computed (we refer the reader to Sect. 2.5 of [29] for more detailed proofs of the results of Sect.s 4-5). Sect. 6 illustrates the main features of the proposed approach within a simple example, which corresponds to a defaultable extension of the Heston [30] model. Finally, Sect. 7 concludes.

# 2 The modeling framework

This section presents the mathematical structure of the modeling framework. Let $(\Omega, \mathcal{G}, P)$ be a reference probability space, with $P$ denoting the physical/statistical probability measure (we want to emphasise that our framework will be entirely formulated with respect to the physical measure $P$). Let $T \in (0, \infty)$ be a fixed time horizon and $W = (W_t)_{0 \le t \le T}$ an $\mathbb{R}^d$-valued Brownian motion on $(\Omega, \mathcal{G}, P)$, with $d \geq 2$, and denote by $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$ its $P$-augmented natural filtration.

We focus our attention on a single defaultable firm[1], whose *default time* $\tau : \Omega \to [0, T] \cup \{\infty\}$ is supposed to be a $(P, \mathbb{F})$-*doubly stochastic* random time, in the sense of Def. 9.11 of [32]. This means that there exists a strictly positive process $\lambda^P = (\lambda^P_t)_{0 \le t \le T}$ such that, for all $t \in [0, T]$, we have $P(\tau > t \,|\, \mathcal{F}_T) = P(\tau > t \,|\, \mathcal{F}_t) = \exp\big(-\int_0^t \lambda^P_u du\big)$. We call the process $\lambda^P$ the $P$-*intensity* of $\tau$, thus emphasising the role of the reference measure $P$. Let the filtration $\mathbb{G} = (\mathcal{G}_t)_{0 \le t \le T}$ be the *progressive enlargement*[2] of $\mathbb{F}$ with respect to $\tau$, i.e., $\mathcal{G}_t := \bigcap_{s>t} \big\{\mathcal{F}_s \vee \sigma(\tau \wedge s)\big\}$, for all $t \in [0, T]$, and let $\mathcal{G} = \mathcal{G}_T$. It is well-known that $\mathbb{G}$ is the smallest filtration (satisfying the usual conditions) which makes $\tau$ a $\mathbb{G}$-stopping time and contains $\mathbb{F}$, in the sense that $\mathcal{F}_t \subset \mathcal{G}_t$ for all $t \in [0, T]$.

The price at time $t \in [0, T]$ of one share issued by the defaultable firm is denoted by $S_t$. We assume that the $\mathbb{G}$-adapted process $S = (S_t)_{0 \le t \le T}$ is continuous and strictly positive on the stochastic interval $[\![0, \tau[\![$ and satisfies $S\mathbf{1}_{[\![\tau, T]\!]} = 0$. This means that $S$ drops to zero as soon as the default event occurs and remains thereafter frozen at that level. By relying on the Sect. 5.1 of [3] together with the fact that all $\mathbb{F}$-martingales are continuous, it can be proved that there exists a continuous strictly positive $\mathbb{F}$-adapted process $\widetilde{S} = (\widetilde{S}_t)_{0 \le t \le T}$ such that $S_t = \mathbf{1}_{\{\tau > t\}} \widetilde{S}_t$ holds for all $t \in [0, T]$. We shall refer to the process $\widetilde{S}$ as the *pre-default* value of $S$.

The pre-default value $\widetilde{S}$ is assumed to be influenced by the $\mathbb{F}$-adapted *stochastic volatility* process $v = (v_t)_{0 \le t \le T}$ and by an $\mathbb{R}^{d-2}$-valued $\mathbb{F}$-adapted *factor process* $X = (X_t)_{0 \le t \le T}$. The process $X$ can include macro-economic covariates describing the state of the economy as well as firm-specific and latent variables, as considered for instance in [28]. Let us define the process $L = (L_t)_{0 \le t \le T}$ by $L_t := \log \widetilde{S}_t$ and the $\mathbb{R}^d$-valued $\mathbb{F}$-adapted process $V = (V_t)_{0 \le t \le T}$ by $V_t := (v_t, X_t^\top, L_t)^\top$, with $^\top$ denoting transposition.

The processes $v$, $X$ and $L$ are jointly specified through the following square-root-type SDE for the process $V$ on the state space $\mathbb{R}^m_{++} \times \mathbb{R}^{d-m}$, where we let $\mathbb{R}^m_{++} := \{x \in \mathbb{R}^m : x_i > 0, \forall i = 1, \ldots, m\}$,

[1] The present modeling framework can be easily extended to the case of $M > 1$ defaultable firms if we suppose that the random default times $\{\tau_1, \ldots, \tau_M\}$ are $\mathbb{F}$-conditionally independent (see e.g. [32], Sect. 9.6).

[2] Due to Lemma 6.1.1 and Lemma 6.1.2 of [3], the fact that $P(\tau > t \,|\, \mathcal{F}_T) = P(\tau > t \,|\, \mathcal{F}_t)$, for all $t \in [0, T]$, implies that all $(P, \mathbb{F})$-martingales are also $(P, \mathbb{G})$-martingales. In particular, $W = (W_t)_{0 \le t \le T}$ is a Brownian motion with respect to both $\mathbb{F}$ and $\mathbb{G}$. This important fact will be used in the following without further mention.

for some fixed $m \in \{1, \dots, d-1\}$:

$$dV_t = (AV_t + b)\, dt + \Sigma\sqrt{R_t}\, dW_t \qquad V_0 = \left(v_0, X_0^\top, \log S_0\right)^\top = \bar{v} \in \mathbb{R}^m_{++} \times \mathbb{R}^{d-m} \tag{2.1}$$

where $(A, b, \Sigma) \in \mathbb{R}^{d\times d} \times \mathbb{R}^d \times \mathbb{R}^{d\times d}$ and $R_t$ is a diagonal $(d \times d)$-matrix with elements given by $R_t^{i,i} = \alpha_i + \beta_i^\top V_t$, for all $t \in [0, T]$, with $\alpha := (\alpha_1, \dots, \alpha_d)^\top \in \mathbb{R}^d_+$ and $\beta := (\beta_1, \dots, \beta_d) \in \mathbb{R}^{d\times d}_+$. In order to guarantee the existence of a strong solution to the SDE (2.1), we introduce the following Assumption.

**Assumption 2.1.** *Let $m \in \{1, \dots, d-1\}$ and define $I := \{1, \dots, m\}$, $J := \{m+1, \dots, d\}$ and $D := I \cup J = \{1, \dots, d\}$. The parameters $A, b, \Sigma, \alpha, \beta$ satisfy the following conditions:*

*(i)* $b_i \geq (\Sigma_{i,i})^2 \beta_{i,i}/2$ *for all* $i \in I$;

*(ii)* $A_{i,j} = 0$ *for all* $i \in I$ *and* $j \in J$ *and* $A_{i,j} \geq 0$ *for all* $i, j \in I$ *with* $i \neq j$;

*(iii)* $\Sigma_{i,j} = 0$ *for all* $i \in I$ *and* $j \in D$ *with* $j \neq i$;

*(iv)* $\beta_{j,i} = 0$ *for all* $i \in D$ *and* $j \in J$, $\beta_{i,i} > 0$ *for all* $i \in I$ *and* $\beta_{i,j} = 0$ *for all* $i, j \in I$ *with* $i \neq j$;

*(v)* $\alpha_i = 0$ *for all* $i \in I$ *and* $\alpha_j > -\sum_{i=1}^m \beta_{i,j}$ *for all* $j \in J$.

For any $\bar{v} \in \mathbb{R}^m_{++} \times \mathbb{R}^{d-m}$, Assumption 2.1 ensures the existence of a unique strong solution $V = (V_t)_{0\leq t\leq T}$ to the SDE (2.1) on the filtered probability space $(\Omega, \mathcal{G}, \mathbb{F}, P)$ such that $V_0 = \bar{v}$ and $V_t \in \mathbb{R}^m_{++} \times \mathbb{R}^{d-m}$ $P$-a.s. for all $t \in [0, T]$. Indeed, the same arguments used in the proof of Lemma 10.6 of [27] give the existence of a unique strong solution $V = (V_t)_{0\leq t\leq T}$ on $\mathbb{R}^m_+ \times \mathbb{R}^{d-m}$, while Lemma A.3 of [24] together with Ex. 10.12 of [27] implies that $V$ actually takes values in $\mathbb{R}^m_{++} \times \mathbb{R}^{d-m}$. Due to conditions (iv)-(v) of Assumption 2.1, this also implies that the matrix $R_t$ is positive definite for all $t \in [0, T]$. We shall always assume that Assumption 2.1 is satisfied without further mention.

**Remark 2.2.** The parameter restrictions imposed by Assumption 2.1 bear resemblance to the canonical representation of [19]. However, we do not require the matrix $\Sigma$ to be diagonal, since this may lead to unnecessary restrictions on the model if $2 \leq m \leq d-2$, as pointed out in [12].

The following Proposition describes the dynamics of the defaultable stock price process $S$.

**Proposition 2.3.** *The process $S = (S_t)_{0\leq t\leq T}$ satisfies the following SDE on $(\Omega, \mathcal{G}, \mathbb{G}, P)$:*

$$\begin{aligned} dS_t = S_{t-} \left( \bar{s} + \mu_1 \log S_{t-} + \mu_2 v_t + \sum_{i=1}^{d-2} \eta_i X_t^i + \sum_{i=1}^{m-1} \bar{\eta}_i X_t^i \right) dt \\ + S_{t-}\, \sigma \sqrt{v_t}\, dW_t^1 + S_{t-} \sum_{i=2}^{d} \Sigma_{d,i} \sqrt{R_t^{i,i}} dW_t^i - S_{t-}\, d\mathbf{1}_{\{\tau \leq t\}} \end{aligned} \tag{2.2}$$

*with the convention $S_{t-} \log S_{t-} = 0$ on $\{\tau \leq t\}$ and where*

$$\bar{s} := b_d + \frac{1}{2} \sum_{k=m+1}^{d} (\Sigma_{d,k})^2\, \alpha_k, \qquad \mu_1 := A_{d,d}, \quad \mu_2 := A_{d,1} + \frac{1}{2} (\Sigma_{d,1})^2\, \beta_{11} + \frac{1}{2} \sum_{k=m+1}^{d} (\Sigma_{d,k})^2\, \beta_{1,k},$$

$$\eta_i := A_{d,i+1}, \qquad \sigma := \Sigma_{d,1} \sqrt{\beta_{1,1}}, \quad \bar{\eta}_i := \frac{1}{2} (\Sigma_{d,i+1})^2\, \beta_{i+1,i+1} + \frac{1}{2} \sum_{k=m+1}^{d} (\Sigma_{d,k})^2\, \beta_{i+1,k}$$

*Proof.* Observe first that $dS_t = \mathbf{1}_{\{\tau>t-\}}\widetilde{S}_{t-}\left(dL_t + d\langle L\rangle_t/2\right) - \widetilde{S}_{t-}\, d\mathbf{1}_{\{\tau\le t\}}$, due to Itô's formula and integration by parts. Equation (2.2) then follows from (2.1) together with Assumption 2.1 by means of simple computations. □

**Remark 2.4.** The defaultable price process $S$ has a rich structure, influenced by the factor process $X$ in both the drift and diffusion terms. Furthermore, there are three levels of dependence between $S$ and the stochastic volatility $v$: *(1)* a direct interaction, since $v$ explicitly appears in the dynamics of $S$; *(2)* a "semi-direct" interaction, since the Brownian motion $W^1$ driving the process $v$ is also one of the drivers of $S$; *(3)* an indirect interaction, since $S$ and $v$ both depend on the factor process $X$.

To complete the description of the modeling framework, we specify as follows the $P$-intensity process $\lambda^P = (\lambda^P_t)_{0\le t\le T}$ and the risk-free interest rate process $r = (r_t)_{0\le t\le T}$:

$$\lambda^P_t := \bar{\lambda}^P + (\Lambda^P)^\top V_t \qquad r_t := \bar{r} + \Upsilon^\top V_t \qquad \text{for all } t\in[0,T] \tag{2.3}$$

where $\bar{\lambda}^P, \bar{r} \in \mathbb{R}_+$ and $\Lambda^P, \Upsilon \in \mathbb{R}^m_+ \times \{0\}^{d-m}$ satisfying the conditions $\bar{\lambda}^P + \sum_{i=1}^m \Lambda^P_i > 0$ and $\bar{r} + \sum_{i=1}^m \Upsilon_i > 0$. This ensures that the $P$-intensity and the risk-free rate are correlated and strictly positive, since 0 is an unattainable boundary for $V^i$, $\forall i \in I$. Furthermore, the linear structure (2.3) permits to obtain analytically tractable formulae for several quantities of interest, as shown in Sect.s 4-5. The specification (2.3) allows for a direct dependence of $\lambda^P$ on the stochastic volatility $v$, this feature being consistent with several empirical observations (see e.g. [4, 17]). Observe also that the defaultable price process $S$ and the $P$-intensity $\lambda^P$ are linked through the common factor process $X$. Finally, we want to note that the proposed modeling framework generalises to a defaultable setting several stochastic volatility models considered in the literature. For instance, defaultable versions of the models considered in [1, 9] and Sect. 4.3 of [25] can be easily recovered within our general setting.

# 3 Equivalent changes of measure which preserve the affine structure

The modeling framework introduced in Sect. 2 has been formulated entirely with respect to the physical probability measure $P$. However, since we aim at dealing with pricing as well as risk management applications, we need to study the structure of the model under a suitable *risk-neutral* probability measure, formally defined as a probability measure $Q \sim P$ on $(\Omega, \mathcal{G})$ such that the discounted defaultable price process $\exp\left(-\int_0^\cdot r_u du\right) S$ is a $(Q, \mathbb{G})$-local martingale[3].

It is important to be aware of the fact that most of the appealing features of the framework described in Sect. 2 may be lost after a change of measure. Aiming at a model which is analytically tractable under both the physical and a risk-neutral measure, we shall consider all risk-neutral measures $Q$ which *preserve the affine structure of* $(V,\tau)$, in the sense of the following Definition.

**Definition 3.1.** *Let $Q$ be a probability measure on $(\Omega,\mathcal{G})$ with $Q\sim P$. We say that $Q$* preserves the affine structure of $(V,\tau)$ *if the following hold:*

*(i) the process $V=(V_t)_{0\le t\le T}$ satisfies an SDE of the type* (2.1) *on $(\Omega,\mathcal{G},\mathbb{F},Q)$ with respect to an $\mathbb{R}^d$-valued $(Q,\mathbb{F})$-Brownian motion $W^Q=(W^Q_t)_{0\le t\le T}$ and for some parameters $A^Q, b^Q, \Sigma, \alpha$, $\beta$ satisfying Assumption 2.1;*

[3] Due to the fundamental result of [20], this is equivalent to the validity of *No Free Lunch with Vanishing Risk* (NFLVR) condition for the financial market $(S,\mathbb{G})$, being the process $\exp\left(-\int_0^\cdot r_u du\right) S$ locally bounded.

*(ii) the default time $\tau$ is a $(Q,\mathbb{F})$-doubly stochastic random time with $Q$-intensity $\lambda^Q=(\lambda^Q_t)_{0\le t\le T}$ of the form $\lambda^Q_t=\bar\lambda^Q+(\Lambda^Q)^\top V_t$, for $\bar\lambda^Q\in\mathbb{R}_+$ and $\Lambda^Q\in\mathbb{R}^m_+\times\{0\}^{d-m}$ with $\bar\lambda^Q+\sum_{i=1}^m\Lambda^Q_i>0$.*

We denote by $\mathcal{Q}$ the family of all risk-neutral measures which preserve the affine structure of $(V,\tau)$, in the sense of Def. 3.1. The following Theorem gives a complete characterisation of the family $\mathcal{Q}$. This result follows from a more general one in Chapt. 2 of [29], but we outline a self-contained proof for the convenience of the reader. We denote by $\mathcal{E}$ the stochastic exponential and by $M=(M_t)_{0\le t\le T}$ the $(P,\mathbb{G})$-martingale defined by $M_t:=\mathbf{1}_{\{\tau\le t\}}-\int_0^{t\wedge\tau}\lambda^P_u du$ (see e.g. [3], Prop. 5.1.3).

**Theorem 3.2.** *Let $Q$ be a probability measure on $(\Omega,\mathcal{G})$. Then we have $Q\in\mathcal{Q}$ if and only if*

$$\begin{aligned}\frac{dQ}{dP}&=\mathcal{E}\left(\int\theta\,dW+\int\gamma\,dM\right)_T\\&=\exp\left(\sum_{i=1}^d\int_0^T\theta^i_t\,dW^i_t-\frac{1}{2}\sum_{i=1}^d\int_0^T(\theta^i_t)^2dt-\int_0^{\tau\wedge T}\gamma_t\,\lambda^P_t dt\right)\Big(1+\mathbf{1}_{\{\tau\le T\}}\gamma_\tau\Big)\end{aligned}\tag{3.1}$$

*where $\theta=(\theta_t)_{0\le t\le T}$ and $\gamma=(\gamma_t)_{0\le t\le T}$ are $\mathbb{F}$-adapted processes of the following form:*

$$\theta_t=\theta(V_t):=R_t^{-1/2}\big(\hat\theta+\Theta V_t\big)\qquad\qquad\gamma_t=\gamma(V_t):=\frac{\big(\bar\lambda^Q-\bar\lambda^P\big)+\big(\Lambda^Q-\Lambda^P\big)^\top V_t}{\bar\lambda^P+\big(\Lambda^P\big)^\top V_t}\tag{3.2}$$

*for some $\hat\theta\in\mathbb{R}^d$ and $\Theta\in\mathbb{R}^{d\times d}$ such that:*

*(i) $\sum_{k=1}^d\Sigma_{i,k}\hat\theta_k\ge(\Sigma_{i,i})^2\beta_{i,i}/2-b_i$ for all $i\in I$;*

*(ii) $\sum_{k=1}^d\Sigma_{i,k}\Theta_{k,j}=0$, for all $i\in I$ and $j\in J$, and $\sum_{k=1}^d\Sigma_{i,k}\Theta_{k,j}\ge-A_{i,j}$, for all $i,j\in I$ with $i\ne j$;*

*for some $\bar\lambda^Q\in\mathbb{R}_+$ and $\Lambda^Q\in\mathbb{R}^m_+\times\{0\}^{d-m}$ with $\bar\lambda^Q+\sum_{i=1}^m\Lambda^Q_i>0$ and if the following equality holds $P$-a.s. on $\{\tau>t\}$, using the notations introduced in Prop. 2.3:*

$$\bar s+\mu_1\log S_{t-}+\left(\mu_2+\sigma\frac{\theta^1_t}{\sqrt{v_t}}\right)v_t+\sum_{i=1}^{d-2}\eta_iX^i_t+\sum_{i=1}^{m-1}\bar\eta_iX^i_t+\sum_{i=2}^d\Sigma_{d,i}\sqrt{R^{i,i}_t}\,\theta^i_t=r_t+\lambda^P_t(1+\gamma_t)\tag{3.3}$$

*Proof.* Let $\theta=(\theta_t)_{0\le t\le T}$ and $\gamma=(\gamma_t)_{0\le t\le T}$ be two $\mathbb{F}$-adapted processes satisfying (3.2). Since $\theta$ and $\gamma$ are continuous functions of $V$ and the process $V$ is continuous, hence locally bounded, the process $Z:=\mathcal{E}\Big(\int\theta\,dW+\int\gamma\,dM\Big)$ is well-defined as a strictly positive $(P,\mathbb{G})$-local martingale and, as a consequence of Fatou's lemma, it is also a $(P,\mathbb{G})$-supermartingale. Moreover, Thm. 2.4 and Remark 2.5 of [13] allow to conclude that $E[Z_T]=1$, thus implying that $Z$ is a uniformly integrable $(P,\mathbb{G})$-martingale. So, we can define a probability measure $Q$ on $(\Omega,\mathcal{G})$ via (3.1). Part (i) of Def. 3.1 then follows from Girsanov's theorem together with (3.2), while part (ii) follows from Thm. 6.3 of [16], Girsanov's theorem together with (3.2) and Prop. 6.2.2 of [3]. Finally, the $(Q,\mathbb{G})$-local martingale property of $\exp\big(-\int_0^\cdot r_u du\big)\,S$ easily follows from Girsanov's theorem together with Prop. 2.3 and (3.3). Conversely, suppose that $Q\in\mathcal{Q}$. The existence of a representation of the form (3.1) follows from Corollary 5.2.4 of [3], while (3.2) and (3.3) follow from Girsanov's theorem together with Def. 3.1 and Prop. 2.3, respectively. □

**Remark 3.3.** The processes $\theta=(\theta_t)_{0\le t\le T}$ and $\gamma=(\gamma_t)_{0\le t\le T}$ admit the financial interpretation of *risk premia* (or *market prices of risk*) associated to the randomness generated by the Brownian motion $W$ and by the random default time $\tau$, respectively. More specifically:

(a) The process $\theta = (\theta_t)_{0\le t\le T}$ represents the risk premium associated to the diffusive risk generated by the Brownian motion $W$. Since the stock price, its stochastic volatility, the default intensity and the interest rate all depend on $W$ through $V$, the risk premium $\theta$ can be considered as a market-wide non-diversifiable risk premium[4].

(b) The process $\gamma = (\gamma_t)_{0\le t\le T}$ represents the risk premium associated to the default event or, more precisely, the risk premium associated to the idiosyncratic component of the risk generated by the occurrence of the default event (to this effect, see also [5, 26] and Sect. 9.3 of [32]).

The importance of explicitly distinguishing between $\theta$ and $\gamma$ has been demonstrated in [21]. Assuming $\gamma \equiv 0$ means that the idiosyncratic component of default risk can be diversified away in the market, as explained in [31], and, therefore, market participants do not require a compensation for it. However, the jump-type risk premium can be significant when it is difficult to hedge the risk associated with the timing of the default event of a given firm. Note that, as can be seen from (3.2), the risk premia $\theta$ and $\gamma$ both depend on the common driving process $V$.

Due to Thm. 3.2, our modeling framework enjoys full analytical tractability under both the physical measure $P$ and any risk-neutral measure $Q \in \mathcal{Q}$, thus enabling us to efficiently solve risk management as well as a pricing problems, as we are going to show in Sect.s 4-5. We close this Section with the following fundamental result, which follows from Thm. 10.4 of [27] together with part (i) of Definition 3.1, (2.1) and Assumption 2.1. For $z \in \mathbb{C}^d$ we denote by $\Re(z)$ and $\Im(z)$ the real and imaginary parts of $z$, respectively, and $\mathbb{C}^m_- := \left\{z \in \mathbb{C}^m : \Re(z) \in \mathbb{R}^m_-\right\}$. For $Q \in \mathcal{Q} \cup \{P\}$, we denote by $E^Q$ the (conditional) expectation operator under the measure $Q$.

**Proposition 3.4.** *For every $Q \in \mathcal{Q} \cup \{P\}$ and for all $z \in \mathbb{C}^m_- \times i\mathbb{R}^{d-m}$, there exists a unique solution $\left(\Phi^Q(\cdot, z), \Psi^Q(\cdot, z)\right) : [0, T] \to \mathbb{C} \times \mathbb{C}^d$ to the following system of Riccati ODEs:*

$$
\begin{aligned}
\partial_t \Phi^Q(t,z) &= (b^Q)^\top \Psi^Q(t,z) + \frac{1}{2}\sum_{k=m+1}^{d} [\Sigma^\top \Psi^Q(t,z)]_k^2\, \alpha_k - \bar{\lambda}^Q - \bar{r}\, \mathbf{1}_{Q\neq P} \\
\Phi^Q(0,z) &= 0 \\
\partial_t \Psi_i^Q(t,z) &= \sum_{k=1}^{d} A_{k,i}^Q\, \Psi_k^Q(t,z) + \frac{1}{2}[\Sigma^\top \Psi^Q(t,z)]_i^2 \beta_{i,i} + \frac{1}{2}\sum_{k=m+1}^{d} [\Sigma^\top \Psi^Q(t,z)]_k^2\, \beta_{i,k} - \Lambda_i^Q - \Upsilon_i \mathbf{1}_{Q\neq P} \qquad \forall i \in I \\
\partial_t \Psi_j^Q(t,z) &= \sum_{k=m+1}^{d} A_{k,j}^Q\, \Psi_k^Q(t,z) \qquad \forall j \in J \\
\Psi^Q(0,z) &= z
\end{aligned}
\tag{3.4}
$$

*Furthermore, for any $Q \in \mathcal{Q} \cup \{P\}$, the following holds for all $0 \le t \le u \le T$ and for all $z \in \mathbb{C}^m_- \times i\mathbb{R}^{d-m}$:*

$$
E^Q\left[\exp\left(-\int_t^u (\lambda_s^Q + r_s \mathbf{1}_{Q\neq P})\, ds + z^\top V_u\right)\middle|\, \mathcal{F}_t\right] = \exp\left(\Phi^Q(u-t,z) + \Psi^Q(u-t,z)^\top V_t\right) \tag{3.5}
$$

[4] In the context of default-free term structure modeling, in [11] the authors demonstrate that the specification (3.2) has a considerably better fit to market data than the simpler market price of risk specifications traditionally considered in the literature (see e.g. [10, 19, 22, 23, 30]).

# 4 Risk management applications

Many quantities of interest in view of risk management applications can be computed as conditional expectations under the physical measure $P$. As a first and basic application, let us compute the $\mathcal{G}_t$-conditional survival probability of the defaultable firm up to the final horizon $T$. We denote by $\Phi^P(\cdot,\cdot)$ and $\Psi^P(\cdot,\cdot)$ the solutions to the Riccati ODEs (3.4) with $Q = P$.

**Proposition 4.1.** *For any $t \in [0,T]$, the following holds:*

$$P\left(\tau > T \,|\, \mathcal{G}_t\right) = \mathbf{1}_{\{\tau>t\}} \exp\left(\Phi^P(T-t,0) + \Psi^P(T-t,0)^\top V_t\right) \tag{4.1}$$

*Proof.* Corollary 5.1.1 of [3] implies that $P(\tau > T \,|\, \mathcal{G}_t) = \mathbf{1}_{\{\tau>t\}} E[\exp(-\int_t^T \lambda_s^P ds) \,|\, \mathcal{F}_t]$. The result then follows by applying (3.5) with $Q = P$, $z = 0$ and $u = T$. □

As can be easily checked from (3.4), the right-hand side of (4.1) only depends on $\{V^i : i \in I\}$, i.e., on the components of the process $V$ on which the $P$-intensity $\lambda^P$ depends. For computing conditional expectations (under the measure $P$) of more general quantities needed for risk management purposes, it turns out to be convenient to introduce the *$T$-survival measure* $P^T \sim P$ on $(\Omega, \mathcal{G})$ defined by $dP^T/dP := \exp\big(-\int_0^T \lambda_t^P dt\big)/E\big[\exp\big(-\int_0^T \lambda_t^P dt\big)\big]$.

**Lemma 4.2.** *For any random variable $F \in L^1(P, \mathcal{F}_T)$ and for any $t \in [0,T]$ the following holds:*

$$E\left[F\,\mathbf{1}_{\{\tau>T\}} \,|\, \mathcal{G}_t\right] = P(\tau > T \,|\, \mathcal{G}_t) E^{P^T}[F \,|\, \mathcal{F}_t] \tag{4.2}$$

*Proof.* Corollary 5.1.1 of [3] implies that $E\left[F\,\mathbf{1}_{\{\tau>T\}} \,|\, \mathcal{G}_t\right] = \mathbf{1}_{\{\tau>t\}} E\big[F \exp\big(-\int_t^T \lambda_s^P ds\big) \,|\, \mathcal{F}_t\big]$. Equation (4.2) then follows by using the definition of the measure $P^T$ together with the conditional version of Bayes' formula (see e.g. [27], Ex. 4.9). □

Lemma 4.2 shows that the computation of the $\mathcal{G}_t$-conditional expectation of an $\mathcal{F}_T$-measurable random variable $F$ in the case of survival until time $T$ reduces to the computation of the $\mathcal{F}_t$-conditional expectation of $F$ under the $T$-survival measure $P^T$, the term $P(\tau > T\,|\mathcal{G}_t)$ being computed as in (4.1). Furthermore, the $\mathcal{F}_t$-conditional characteristic function of the vector $V_T$ under the $T$-survival measure $P^T$ can be computed in closed form, as shown in the following Lemma.

**Lemma 4.3.** *For any $z \in i\mathbb{R}^d$ and for any $t \in [0,T]$ the following holds:*

$$\varphi_t^{P^T}(z) := E^{P^T}\left[e^{z^\top V_T} \,|\, \mathcal{F}_t\right] = \exp\Big(\Phi^P(T-t,z) - \Phi^P(T-t,0) + \big(\Psi^P(T-t,z) - \Psi^P(T-t,0)\big)^\top V_t\Big) \tag{4.3}$$

*Proof.* The definition of the measure $P^T$ together with the conditional version of Bayes' formula gives $E^{P^T}\left[e^{z^\top V_T} \,|\, \mathcal{F}_t\right] = E\big[\exp\big(-\int_t^T \lambda_s^P ds + z^\top V_T\big) \,|\, \mathcal{F}_t\big]/E\big[\exp\big(-\int_t^T \lambda_s^P ds\big) \,|\, \mathcal{F}_t\big]$. By applying (3.5) with $Q = P$, $u = T$ and $z \in i\mathbb{R}^d$ ($z = 0$, resp.) to the numerator (to the denominator, resp.), we then obtain equation (4.3). □

Due to Lemma 4.2 and Lemma 4.3, we can compute the $\mathcal{G}_t$-conditional expectation (under the physical probability measure $P$) of arbitrary functions of the random vector $V_T$ in the case of survival by relying on well-known Fourier inversion techniques. As an example, we can explicitly compute quantiles of the $\mathcal{G}_t$-conditional distribution of the defaultable price $S_T$ in the case of survival. This is crucial for the computation of *Value-at-Risk* and related risk measures.

**Proposition 4.4.** *For any $x \in (0,\infty)$ and for any $t \in [0,T]$ the following holds:*

$$P(S_T \le x, \tau > T \,|\, \mathcal{G}_t) = P(\tau > T \,|\, \mathcal{G}_t) \left( \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \frac{\Im\big(e^{-iy\log x}\, \varphi_t^{P^T}(0,\dots,0,iy)\big)}{y}\, dy \right) \tag{4.4}$$

*where $P(\tau > T \,|\, \mathcal{G}_t)$ and $\varphi_t^{P^T}(\cdot)$ are explicitly given in* (4.1) *and Lemma 4.3, respectively.*

*Proof.* Note that:

$$P(S_T \le x, \tau > T|\mathcal{G}_t) = P(L_T \le \log x, \tau > T|\mathcal{G}_t) = P(\tau > T|\mathcal{G}_t)\, P^T(L_T \le \log x|\mathcal{F}_t)$$

where the second equality follows from Lemma 4.2. Equation (4.4) then follows from standard Fourier inversion techniques (see e.g. [29], Prop. 2.5.12, and [33], Sect. 1.2.6). □

# 5 Valuation of default-sensitive payoffs and defaultable options

Throughout this section, we fix an element $Q \in \mathcal{Q}$. For the purpose of valuing default-sensitive payoffs, the *$u$-survival risk-neutral measure* $Q^u$, for $u \in [0,T]$, turns out to be quite useful. The measure $Q^u$ is defined by $dQ^u/dQ = \exp\big(-\int_0^u (r_s + \lambda_s^Q)\, ds\big)/E^Q\big[\exp\big(-\int_0^u (r_s + \lambda_s^Q)\, ds\big)\big]$. For $u = T$, the measure $Q^T$ bears resemblance to the *$T$-survival measure* $P^T$ introduced in Sect. 4, except that $Q^T$ is defined with respect to some $Q \in \mathcal{Q}$ and the density $dQ^T/dQ$ also involves the risk-free interest rate besides the $Q$-intensity $\lambda^Q$ (compare also with [3], Def. 15.2.2). Following the same logic of Sect. 4, we show that many pricing problems can be simplified by shifting to the measure $Q^u$, for some $u \in [0,T]$. As a preliminary, let us compute the arbitrage-free price $\Pi(t,T)$ of a *zero-coupon defaultable bond*. We denote by $\Phi^Q(\cdot,\cdot)$ and $\Psi^Q(\cdot,\cdot)$ the solutions to the Riccati ODE system (3.4).

**Lemma 5.1.** *For any $t \in [0,T]$ the following holds:*

$$\Pi(t,T) = \mathbf{1}_{\{\tau > t\}} \exp\big(\Phi^Q(T-t,0) + \Psi^Q(T-t,0)^\top V_t\big) \tag{5.1}$$

*Proof.* Note first that:

$$\Pi(t,T) = E^Q\big[\exp(-\int_t^T r_s ds)\mathbf{1}_{\{\tau > T\}}|\mathcal{G}_t\big] = \mathbf{1}_{\{\tau > t\}} E^Q\big[\exp(-\int_t^T (r_s + \lambda_s^Q)\, ds)|\mathcal{F}_t\big]$$

where the second equality follows from Thm. 9.23 of [32]. Equation (5.1) then follows from Prop. 3.4 with $u = T$ and $z = 0$. □

Of course, coupon-bearing corporate bonds can be valued as linear combinations of zero-coupon defaultable bonds (see [3], Sect. 1.1.5). More generally, most default-sensitive payoffs can be decomposed into linear combinations of *zero-recovery* and *pure recovery* payments, the latter being paid only in the case of default, see e.g. Sect. 9.4 of [32]. The next Proposition provides general valuation formulas for zero-recovery and pure recovery payments.

**Proposition 5.2.** *For any $t \in [0,T]$ and for any measurable function $G : \mathbb{R}^m_{++} \times \mathbb{R}^{d-m} \to \mathbb{R}_+$ the following hold:*

$$E^Q\Big[e^{-\int_t^T r_s ds} G(V_T)\mathbf{1}_{\{\tau > T\}}\Big|\, \mathcal{G}_t\Big] = \Pi(t,T) E^{Q^T}[G(V_T) \,|\, \mathcal{F}_t] \tag{5.2}$$

$$E^Q\Big[e^{-\int_t^\tau r_s ds} G(V_\tau)\mathbf{1}_{\{\tau \le T\}}\Big|\, \mathcal{G}_t\Big] = \int_t^T \Pi(u,T) E^{Q^u}[\lambda_u^Q\, G(V_u) \,|\, \mathcal{F}_t]\, du \tag{5.3}$$

*Proof.* Note first that, due to Thm. 9.23 of [32], we can write:

$$E^Q\Big[e^{-\int_t^T r_s ds}G(V_T)\mathbf{1}_{\{\tau>T\}}\big|\,\mathcal{G}_t\Big] = \mathbf{1}_{\{\tau>t\}}E^Q\Big[e^{-\int_t^T (r_s+\lambda_s^Q)\,ds}G(V_T)\,\big|\,\mathcal{F}_t\Big]$$
$$E^Q\Big[e^{-\int_t^\tau r_s ds}G(V_\tau)\mathbf{1}_{\{\tau\le T\}}\big|\,\mathcal{G}_t\Big] = \mathbf{1}_{\{\tau>t\}}E^Q\left[\int_t^T e^{-\int_t^u (r_s+\lambda_s^Q)\,ds}\lambda_u^Q\,G(V_u)\,du\,\Big|\,\mathcal{F}_t\right]$$

Equations (5.2)-(5.3) then follow by using the definition of the measure $Q^u$, for $u\in[t,T]$, together with the conditional version of Bayes' formula and also, for (5.3), with Tonelli's theorem. □

For $0\le t\le u\le T$, the following Lemma gives an explicit expression for the $\mathcal{F}_t$-conditional characteristic function $\varphi_t^{Q^u}$ of the random vector $V_u$ under the $u$-survival risk-neutral measure $Q^u$. Its proof follows from (3.5) and, being analogous to that of Lemma 4.3, is omitted.

**Lemma 5.3.** *For any $0\le t\le u\le T$ and for any $z\in i\mathbb{R}^d$ the following holds:*

$$\varphi_t^{Q^u}(z) := E^{Q^u}\big[e^{z^\top V_u}\big|\,\mathcal{F}_t\big] = \exp\Big(\Phi^Q(u-t,z)-\Phi^Q(u-t,0)+\big(\Psi^Q(u-t,z)-\Psi^Q(u-t,0)\big)^\top V_t\Big) \tag{5.4}$$

By combining Prop. 5.2 with Lemma 5.1 and Lemma 5.3 and using well-known Fourier inversion techniques, we can obtain semi-explicit formulas for a wide range of default-sensitive payoffs. In particular, we now derive valuation formulas for *Call* and *Put* options (issued by a default-free third party) written on the defaultable stock $S$. We denote by $\Pi_{rf}(t,T) := E^Q\big[\exp(-\int_t^T r_s ds)\,|\,\mathcal{G}_t\big] = E^Q\big[\exp(-\int_t^T r_s ds)\,|\,\mathcal{G}_t\big]$ the arbitrage-free price at time $t\in[0,T]$ of a zero-coupon default-free bond.

**Corollary 5.4.** *For any $t\in[0,T]$ and for any strike price $K>0$ the following hold:*

$$C_K(t,T) := E^Q\Big[e^{-\int_t^T r_s ds}(S_T-K)^+\big|\,\mathcal{G}_t\Big] \tag{5.5}$$
$$= \frac{\Pi(t,T)}{2\pi}\int_{-\infty}^{+\infty}\varphi_t^{Q^T}(0,\dots,0,w+iu)\frac{K^{-(w-1+iu)}}{(w+iu)(w-1+iu)}\,du \tag{5.6}$$
$$P_K(t,T) := E^Q\Big[e^{-\int_t^T r_s ds}(K-S_T)^+\big|\,\mathcal{G}_t\Big]$$
$$= K\left(\Pi_{rf}(t,T)-\Pi(t,T)\right)+\frac{\Pi(t,T)}{2\pi}\int_{-\infty}^{+\infty}\varphi_t^{Q^T}(0,\dots,0,y+iu)\frac{K^{-(y-1+iu)}}{(y+iu)(y-1+iu)}\,du \tag{5.7}$$

*for some $w>1$ and $y<0$ such that the system of Riccati ODEs* (3.4) *has a unique solution for the initial conditions $z=(0,\dots,0,w)^\top$ and $z=(0,\dots,0,y)^\top$.*

*Proof.* Observe first that:

$$E^Q\Big[e^{-\int_t^T r_s ds}(S_T-K)^+\big|\,\mathcal{G}_t\Big] = E^Q\Big[e^{-\int_t^T r_s ds}\big(\widetilde{S}_T-K\big)^+\mathbf{1}_{\{\tau>T\}}\big|\,\mathcal{G}_t\Big] = \Pi(t,T)E^{Q^T}\Big[\big(e^{L_T}-K\big)^+\,\big|\,\mathcal{F}_t\Big]$$

where the second equality follows from (5.2). As in [7] and [27], Lemma 10.2, it can be shown that:

$$(e^x-K)^+ = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{(w+iu)x}\frac{K^{-(w-1+iu)}}{(w+iu)(w-1+iu)}\,du$$

for some $w>1$. Equation (5.6) then follows by Fubini's theorem (see Cor. 2.5.21 of [29] for more details). Equation (5.7) follows by an analogous computation once we observe that:

$$E^Q\Big[e^{-\int_t^T r_s ds}(K-S_T)^+\big|\,\mathcal{G}_t\Big] = E^Q\Big[e^{-\int_t^T r_s ds}\big(K-\widetilde{S}_T\big)^+\mathbf{1}_{\{\tau>T\}}\big|\,\mathcal{G}_t\Big]+KE^Q\Big[e^{-\int_t^T r_s ds}\left(1-\mathbf{1}_{\{\tau>T\}}\right)\big|\,\mathcal{G}_t\Big]$$

□

# 6 An example: the Heston with jump-to-default model

In this section, we illustrate some of the essential features of the proposed modeling framework within a simple example, which corresponds to a generalisation of the stochastic volatility model introduced by Heston [30], here extended by allowing the stock price process to be killed by a *jump-to-default* event, in the spirit of [9].

## 6.1 The model

Using the notations introduced in Sect. 2, we let $d = 3$ and consider the following specification:

$$A = \begin{pmatrix} -k & 0 & 0 \\ 0 & -k_0 & 0 \\ -1/2 & 0 & 0 \end{pmatrix} \qquad b = \begin{pmatrix} k\hat{v} \\ k_0\hat{x} \\ \mu \end{pmatrix} \qquad \Sigma = \begin{pmatrix} \bar{\sigma} & 0 & 0 \\ 0 & \sigma_0 & 0 \\ \rho & 0 & \sqrt{1-\rho^2} \end{pmatrix} \qquad R_t = \begin{pmatrix} v_t & 0 & 0 \\ 0 & X_t & 0 \\ 0 & 0 & v_t \end{pmatrix} \tag{6.1}$$

with $k\hat{v} \geq \bar{\sigma}^2/2$, $k_0\hat{x} \geq \sigma_0^2/2$ and $\rho \in [-1, 1]$. The $P$-intensity $(\lambda_t^P)_{0\leq t\leq T}$ is specified as in (2.3), i.e., we have $\lambda_t^P = \bar{\lambda}^P + \Lambda_1^P v_t + \Lambda_2^P X_t$, for some $\bar{\lambda}, \Lambda_1^P, \Lambda_2^P \in \mathbb{R}_+$ with $\bar{\lambda} + \Lambda_1^P + \Lambda_2^P > 0$. For simplicity, we assume that $r_t = \bar{r} \in \mathbb{R}_+$ for all $t \in [0, T]$. Note that this specification extends the Heston jump-to-default model considered in [9] by allowing $\lambda_t^P$ to depend on $v_t$ and on the additional stochastic factor $X_t$. It can be easily checked that the specification (6.1) satisfies Assumption 2.1 and, due to Prop. 2.3, the defaultable stock price process $S = (S_t)_{0\leq t\leq T}$ has the following dynamics:

$$dS_t = S_{t-}\left(\mu - \lambda_t^P\right)dt + S_{t-}\sqrt{v_t}\left(\rho\, dW_t^1 + \sqrt{1-\rho^2}\, dW_t^3\right) - S_{t-}\, dM_t \tag{6.2}$$

where $M = (M_t)_{0\leq t\leq T}$ is the $(P, \mathbb{G})$-martingale defined by $M_t := \mathbf{1}_{\{\tau\leq t\}} - \int_0^{t\wedge\tau} \lambda_u^P du$. We also have:

$$\begin{aligned} dv_t &= k(\hat{v} - v_t)dt + \bar{\sigma}\sqrt{v_t}\, dW_t^1 \\ dX_t &= k_0(\hat{x} - X_t)dt + \sigma_0\sqrt{X_t}\, dW_t^2 \end{aligned} \tag{6.3}$$

## 6.2 Risk-neutral measures which preserve the Heston with jump-to-default structure

By relying on Thm. 3.2, we now characterise the family of all risk-neutral measures $Q \in \mathcal{Q}$ which *preserve the Heston with jump-to-default structure*, namely all risk-neutral measures $Q \in \mathcal{Q}$ which leave unchanged the structure of the SDEs (6.2)-(6.3) (compare also with [29], Sect. 2.4.1).

**Lemma 6.1.** *A risk-neutral measure $Q \in \mathcal{Q}$ preserves the Heston with jump-to-default structure if and only if $dQ/dP$ admits the representation* (3.1) *for some $\mathbb{F}$-adapted processes $\theta = (\theta_t)_{0\leq t\leq T}$ and $\gamma = (\gamma_t)_{0\leq t\leq T}$ of the form* (3.2) *with $\bar{\theta} \in \mathbb{R}^3$ and $\Theta \in \mathbb{R}^{3\times 3}$ satisfying the following restrictions:*

$$\hat{\theta} = \begin{pmatrix} \hat{\theta}_1 \\ \hat{\theta}_2 \\ \frac{\bar{r}+\bar{\lambda}^Q-\mu-\rho\,\hat{\theta}_1}{\sqrt{1-\rho^2}} \end{pmatrix} \qquad \Theta = \begin{pmatrix} \Theta_{1,1} & 0 & 0 \\ 0 & \Theta_{2,2} & 0 \\ \frac{\Lambda_1^Q-\rho\,\Theta_{1,1}}{\sqrt{1-\rho^2}} & \frac{\Lambda_2^Q}{\sqrt{1-\rho^2}} & 0 \end{pmatrix} \tag{6.4}$$

*with $\hat{\theta}_1 \geq \bar{\sigma}/2 - k\hat{v}/\bar{\sigma}$ and $\hat{\theta}_2 \geq \sigma_0/2 - k_0\hat{x}/\sigma_0$.*

*Proof.* The result follows from conditions (3.2)-(3.3) of Thm. 3.2, noting that the preservation of the Heston with jump-to-default structure consists in the additional restriction $\Theta_{1,2} = \Theta_{2,1} = 0$. □

The main benefit of working with risk-neutral measures which preserve the Heston with jump-to-default structure consists in the possibility of obtaining closed-form solutions to the system of Riccati ODEs (3.4), as shown in the next Lemma, which follows from Lemma 10.12 of [27] by means of simple (but tedious and, hence, omitted) computations.

**Lemma 6.2.** *Let $Q \in \mathcal{Q}$ be a risk-neutral measure which preserves the Heston with jump-to-default structure. Then the system of Riccati ODEs* (3.4) *admits the following solution, for all $z \in \mathbb{C}_-^2 \times i\mathbb{R}$:*

$$\Psi_1^Q(t,z) = -\frac{\left(z_3 - z_3^2 + 2\Lambda_1^Q(1-z_3)\right)\left(e^{\sqrt{\Delta_1}t}-1\right) - \left(\sqrt{\Delta_1}\left(e^{\sqrt{\Delta_1}t}+1\right) + \left(\bar\sigma\left(\Theta_{1,1}+\rho z_3\right)-k\right)\left(e^{\sqrt{\Delta_1}t}-1\right)\right) z_1}{\sqrt{\Delta_1}\left(e^{\sqrt{\Delta_1}t}+1\right) - \left(\bar\sigma\left(\Theta_{1,1}+\rho z_3\right)-k\right)\left(e^{\sqrt{\Delta_1}t}-1\right) - \bar\sigma^2\left(e^{\sqrt{\Delta_1}t}-1\right) z_1}$$

$$\Psi_2^Q(t,z) = -\frac{2\Lambda_2^Q(1-z_3)\left(e^{\sqrt{\Delta_2}t}-1\right) - \left(\sqrt{\Delta_2}\left(e^{\sqrt{\Delta_2}t}+1\right) + \left(\sigma_0\,\Theta_{2,2}-k_0\right)\left(e^{\sqrt{\Delta_2}t}-1\right)\right) z_2}{\sqrt{\Delta_2}\left(e^{\sqrt{\Delta_2}t}+1\right) - \left(\sigma_0\Theta_{2,2}-k_0\right)\left(e^{\sqrt{\Delta_2}t}-1\right) - \sigma_0^2\left(e^{\sqrt{\Delta_2}t}-1\right) z_2}$$

$$\Psi_3^Q(t,z) = z_3$$

$$\begin{aligned}\Phi^Q(t,z) = {} & \frac{2(k\hat v + \bar\sigma\hat\theta_1)}{\bar\sigma^2}\log\left(\frac{2\sqrt{\Delta_1}\exp\left(\frac{\sqrt{\Delta_1}-(\bar\sigma(\Theta_{1,1}+\rho z_3)-k)}{2}t\right)}{\sqrt{\Delta_1}(e^{\sqrt{\Delta_1}t}+1) - (\bar\sigma(\Theta_{1,1}+\rho z_3)-k)(e^{\sqrt{\Delta_1}t}-1) - \bar\sigma^2(e^{\sqrt{\Delta_1}t}-1)z_1}\right)\\ & + \frac{2(k_0\hat x + \sigma_0\hat\theta_2)}{\sigma_0^2}\log\left(\frac{2\sqrt{\Delta_2}\exp\left(\frac{\sqrt{\Delta_2}-(\sigma_0\Theta_{2,2}-k_0)}{2}t\right)}{\sqrt{\Delta_2}(e^{\sqrt{\Delta_2}t}+1) - (\sigma_0\Theta_{2,2}-k_0)(e^{\sqrt{\Delta_2}t}-1) - \sigma_0^2(e^{\sqrt{\Delta_2}t}-1)z_2}\right)\\ & + \left(r+\bar\lambda^Q\right)(z_3-1)t\end{aligned}$$

*where:*

$$\Delta_1 := \left(\bar\sigma\left(\Theta_{1,1}+\rho z_3\right)-k\right)^2 + \bar\sigma^2\left(z_3 - z_3^2 + 2\Lambda_1^Q(1-z_3)\right)$$
$$\Delta_2 := \left(\sigma_0\,\Theta_{2,2}-k_0\right)^2 + 2\,\sigma_0^2\,\Lambda_2^Q(1-z_3)$$

By combining the above Lemma with the results of Sect.s 4-5, we can efficiently solve risk management problems and compute arbitrage-free prices of general default-sensitive payoffs.

## 6.3 Numerical results

This Section reports the results of some numerical computations for the Heston with jump-to-default model (6.1)-(6.3). We adopt the following parameters' specification: $k = 0.565$, $\hat v = 0.07$, $\bar\sigma = 0.281$, $k_0 = 0.325$, $\hat x = 0.003$, $\sigma_0 = 0.036$, $\mu = 0.1$, $\rho = -0.558$. These values have been obtained in [10] by calibrating a jump-to-default model to equity options and CDS spreads on the Citigroup company (period: 5/2002 - 5/2006). The remaining parameters appearing in (6.4) are specified as $\bar r = 0$, $\Theta_{1,1} = \Theta_{2,2} = 0.002$, $\hat\theta_1 = \hat\theta_2 = 0.001$ and $\Lambda_1^P = \Lambda_2^P = \bar\lambda^P = 0.1225$.

By relying on Proposition 4.4 and Lemma 6.2, we compute the distribution function of the defaultable stock price $S_T$ in the case of survival, i.e., we compute the map $(T,x) \mapsto P\left(S_T \le x, \tau > T\right)$, for $T \in [0.5, 3.0]$ and $x \in [0.7, 1.3]$, for $S_0 = 1$. In particular, we can observe from Fig. 1 that $\lim_{x\to+\infty} P\left(S_T \le x, \tau > T\right) = P(\tau > T) < 1$, with a more pronounced effect as $T$ increases.

As a second application, we construct the implied volatility surface generated by the model (6.1)-(6.3). To this effect, we first compute a matrix of prices $P_K(0,T)$ of Put options on the defaultable stock $S_T$, issued by a default-free third party, with maturity $T \in [0.5, 3.0]$ and moneyness $K/S_0 \in [0.7, 1.3]$, letting $\bar\lambda^Q = 0.001$ and $\Lambda_i^Q = \Lambda_i^P$, for $i = 1, 2$. The computation is performed via the Fast Fourier Transform method of [7], by relying on Corollary 5.4 and Lemma 6.2. The corresponding implied volatilities are then computed by using the *blsimpv* function in Matlab© (R2012a 64-bit version). Fig. 2 compares the implied volatility surface generated by the model (6.1)-(6.3) with the

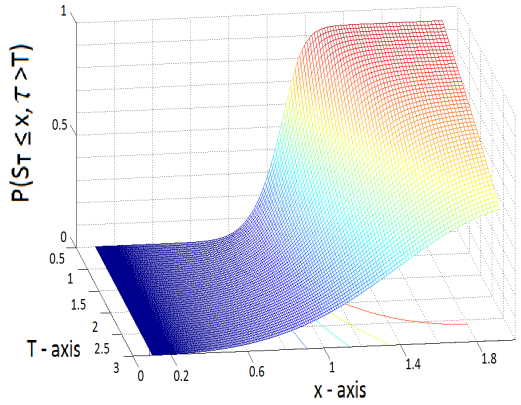



Figure 1: Surface $(T, x) \mapsto P(S_T \leq x, \tau > T)$ for the Heston with Jump-to-Default model.

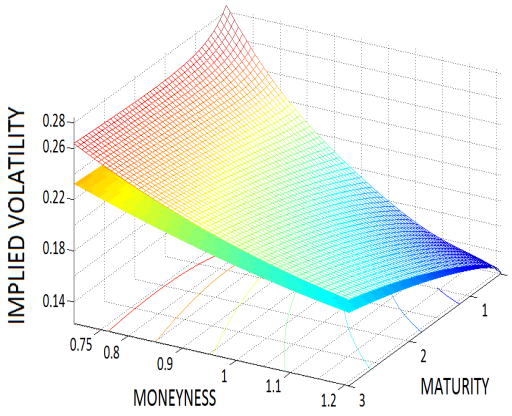



Figure 2: Implied volatility surfaces: standard Heston (solid) and Heston + Jump-to-Default (mesh).

implied volatility surface obtained from a standard (default-free) Heston [30] model, i.e., by letting $\bar{\lambda}^Q = \Lambda_1^Q = \Lambda_2^Q = 0$. It is evident that the introduction of a jump-to-default increases the implied volatility along all maturities and strikes. The increase is more pronounced for deep out-of-the-money options, due to the possibility of obtaining $K$ in the case of default (compare also with eqn. (5.7)). There is also a strong skew effect, which tends to flatten as the maturity increases but is always more significant than in the default-free case.

# 7 Conclusions and further developments

We have proposed a general framework based on an affine process $V$ and on a doubly stochastic random time $\tau$ for the modeling of a defaultable stock. This approach allows to jointly model equity and credit risk, together with stochastic volatility and stochastic interest rate. Moreover, analytical tractability is ensured under both the physical and a risk-neutral probability measure, thanks to a characterisation of all risk-neutral measures which preserve the affine structure of $(V, \tau)$. In the present paper, the driving process $V$ has been specified as an affine diffusion on $\mathbb{R}^m_{++} \times \mathbb{R}^{d-m}$. However, our approach can be easily adapted to the more general case where $V$ is a continuous matrix-valued affine process (e.g., a Wishart process), as recently considered in [18].

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