BSDEs with singular terminal condition and control problems with constraints

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We provide a probabilistic solution of a not necessarily Markovian control problem with a state constraint by means of a Backward Stochastic Differential Equation (BSDE). The novelty of our solution approach is that the BSDE possesses a *singular terminal condition*. We prove that a solution of the BSDE exists, thus partly generalizing existence results obtained by Popier in [9] and [10]. We perform a verification and discuss special cases for which the control problem has explicit solutions.

Introduction

In these notes we provide a pure probabilistic solution of the control problem that consists in minimizing the functional

$$J(x) = E\left[\int_0^T (\eta_t |\dot{x}_t|^p + \gamma_t |x_t|^p) dt\right] \tag{1}$$

over all absolutely continuous paths $(x_t)_{t\in[0,T]}$ starting in $\xi \in \mathbb{R}$ and ending in 0 at time T. Here $p > 1$ and (η, γ) are two non-negative stochastic processes that are progressively measurable with respect to the natural filtration $(\mathcal{F}_t)$ generated by a Brownian motion. We choose the control strategies x to be adapted to $(\mathcal{F}_t)$.

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Such a control problem arises for example when economic agents have to close a position of ξ asset shares in a market with a stochastic price impact (see e.g. [1] and the references therein). The first term $\int_0^T \eta_t |\dot{x}_t|^p dt$ in (1) can be interpreted as the liquidity costs entailed by closing the position, where η is a stochastic price impact factor. The second term can be seen as a measure of the risk associated to the open position.

Our method for solving the control problem (1) draws on the notion of backward stochastic differential equations (BSDEs). BSDEs have turned out to be a powerful tool for analyzing stochastic control problems, and for providing pure probabilistic solutions. We refer to the survey article [2] and the book by Pham [8] for examples of control problems solved with BSDEs. The control problem (1) considered here imposes a constraint on the terminal value of the control process x, namely $x_T = 0$. In the following we characterize its solution with the BSDE

$$dY_t = \left((p-1) \frac{Y_t^q}{\eta_t^{q-1}} - \gamma_t \right) dt + Z_t dW_t \tag{2}$$

(where $q = 1/(1 - \frac{1}{p})$) possessing the *singular* terminal condition

$$\lim_{t \nearrow T} Y_t = \infty. \tag{3}$$

We show that if η and γ satisfy some nice integrability condition, then there exists a minimal solution (Y, Z) of the BSDE (2) with terminal condition (3). We subsequently prove, without any further assumptions, that there exists an optimal control of the problem (1) and that it is given by $x_t^* = \xi e^{-\int_0^t \left(\frac{Y_s}{\eta_s}\right)^{q-1} ds}$. Note that the terminal condition (3) is necessary for the constraint $x_T^* = 0$ to be satisfied.

One can also derive the singularity (3) by considering the value function associated to the control problem: as t converges to T, the value function converges to infinity, provided the position $x \neq 0$. We will show that the value function is a power function of the position variable, multiplied with the solution of the BSDE (2). The singularity of the value function translates into the BSDE's singularity at the terminal condition.

BSDEs with singular terminal conditions have so far been studied only in Popier [9] and [10]. One of the present paper's goal is to reveal their power for solving the stochastic control problem (1). BSDEs with singular terminal conditions have not been detected as an efficient tool for solving stochastic control problems yet.

The control problem (1), more precisely some versions of it, have been already studied in the literature. In [11] a similar class of control problems is solved by means of so-called superprocesses. The functional of the control problem considered in [11] is slightly more general, but the pair (η, γ) is assumed to be Markovian. The BSDE approach we present here is *not* bound to a Markovian model set-up.

Ji and Zhou [3] consider a very general control problem with terminal state constraints. They assume that the state process is disturbed by some white noise with a volatility that is invertible in the control. Notice that in our setting the state process x is not disturbed.

In [1] the authors consider the special case of the control problem (1) where $p = 2$, η is a constant and γ is a function of a homogeneous Brownian martingale (in particular γ is a Markov process). They solve the control problem with analytical techniques, characterizing the optimal control and the value function with a solution of a PDE in the viscosity sense.

A probabilistic solution of a related control problem is given in [5] (and the preceding paper [6]), also by means of BSDEs: the authors consider the problem of how to optimally follow a trading target in an illiquid market with a non-temporary price impact depending on order sizes. Optimal controls, however, are *singular* and are verified with BSDEs that have non-singular terminal conditions.

The paper is organized as follows. In Section 1 we precisely describe the modeling set-up and present the main results. Moreover, we give a heuristic derivation of why the BSDE (2) with singular terminal condition provides a solution of the control problem.

In Section 2 we prove, given some nice integrability conditions on η and γ, that there exists a solution of the BSDE (2).

Section 3 turns to a verification: we show that the optimal control and value function can indeed be characterized by the BSDE solution constructed in Section 2.

Finally, in Section 4 we study in detail the special case where γ is zero and η has uncorrelated multiplicative increments. We show that in this case the optimal control is deterministic.

1 Main results

We fix a deterministic, finite time horizon $T > 0$ and a probability space $(\Omega, \mathcal{F}, P)$ which supports a d-dimensional Brownian motion $(W_t)_{0 \le t \le T}$, where $d \in \mathbb{N}$. Let $(\mathcal{F}_t)_{t \in [0,T]}$ denote the completed filtration generated $(W_t)_{0 \le t \le T}$. Throughout we assume that $(\eta_t)_{t \in [0,T]}$ and $(\gamma_t)_{t \in [0,T]}$ are nonnegative, progressively measurable stochastic processes. We assume $p > 1$ and denote by $q = 1/(1 - \frac{1}{p})$ its Hölder conjugate. We consider the stochastic control problem to minimize the functional

$$J(x) = E\left[\int_0^T (\eta_t |\dot{x}_t|^p + \gamma_t |x_t|^p) dt\right] \tag{4}$$

over all progressively measurable processes $x : \Omega \times [0,T] \to \mathbb{R}$ that possess absolutely continuous sample paths and satisfy the constraints $x_0 = \xi \in \mathbb{R}$ and $x_T = 0$ a.s. We denote the set of all these controls by $\mathcal{A}_0$, and define

$$v = \inf_{x \in \mathcal{A}_0} J(x). \tag{5}$$

We show that under some nice integrability conditions on η and γ there exists an optimal control $x^* \in \mathcal{A}_0$; i.e. $J(x^*) = v$. Moreover we characterize the optimal control by means of a BSDEs with a singular terminal condition. We define the notion of a solution in the style of [9].

Definition 1.1. We say that a pair of progressively measurable processes (Y, Z) with values in $\mathbb{R} \times \mathbb{R}^d$ solves the BSDE (2) with singular terminal condition $Y_T = \infty$ if it satisfies

(i) for all $0 \le s \le t < T$: $Y_s = Y_t - \int_s^t \left((p-1) \frac{Y_r^q}{\eta_r^{q-1}} - \gamma_r \right) dr - \int_s^t Z_r dW_r$;

(ii) for all $0 \le t < T$: $E\left[\sup_{0 \le s \le t} |Y_s|^2 + \int_0^t |Z_r|^2 dr \right] < \infty$;

(iii) $\liminf_{t \nearrow T} Y_t = \infty$, a.s.

We introduce the following spaces of processes. For $i = 1, 2$ and $t \le T$ let

$$\mathcal{M}^i(0,t) = L^i(\Omega \times [0,t], \mathcal{P}, P \otimes \lambda)$$

where λ is the Lebesgue measure and $\mathcal{P}$ denotes the σ-algebra of $(\mathcal{F}_t)$-progressively measurable subsets of $\Omega \times [0,T]$. Throughout we assume that η and γ satisfy the integrability conditions

$$\textbf{(I1)} \qquad \eta \in \mathcal{M}^2(0,T) \text{ and } 1/\eta^{q-1} \in \mathcal{M}^1(0,T),$$

$$\textbf{(I2)} \qquad E \int_0^T (T-s)^p \gamma_s ds < \infty \text{ and } \gamma \in \mathcal{M}^2(0,t) \text{ for all } t < T.$$

In our first main result we prove existence of a minimal solution of the BSDE (2).

Theorem 1.2. *Assume that Conditions* **(I1)** *and* **(I2)** *are satisfied. Then there exists a minimal solution* (Y, Z) *of the BSDE* (2) *with singular terminal condition* $Y_T = \infty$.

In the second main result we characterize the value function and the optimal control in terms of the minimal solution.

Theorem 1.3. *Suppose Conditions* **(I1)** *and* **(I2)**, *and let* (Y, Z) *be the minimal solution of* (2). *Then*

$$v = Y_0 |\xi|^p$$

and the optimal control is given by

$$x_t^* = \exp\left(- \int_0^t \left(\frac{Y_s}{\eta_s} \right)^{q-1} ds \right),$$

for all $t \in [0,T]$.

The following deterministic example illustrates that a violation of the integrability condition $1/\eta^{q-1} \in \mathcal{M}^1(0,T)$ may lead to a minimization problem where no optimal control exists.

Example 1.4. *Let $T = 1$, $\eta_t = (1-t)^\beta$ for some $\beta \geq 0$, $\gamma_t = 0$ and $p = q = 2$. Then we have $1/\eta^{q-1} \in L^1([0,T])$ if and only if $\beta < 1$. In this case Theorem 1.3 yields that $x_t = (1-t)^{1-\beta}$ is an optimal control. In the case $\beta \geq 1$ consider the control $x_t = (1-t)^\alpha$ for some $\alpha > 0$. We compute*

$$J(x) = \int_0^1 \eta_t \dot{x}_t^2 dt = \alpha^2 \int_0^1 (1-t)^{2\alpha+\beta-2} dt.$$

Since $\beta \geq 1 > 1 - 2\alpha$ the integral is finite and has the value

$$J(x) = \frac{\alpha^2}{2\alpha + \beta - 1}.$$

Taking the limit $\alpha \searrow 0$ yields $v = 0$, but there exists no control in $\mathcal{A}_0$ attaining this value.

Remark 1.5. If $p = 1$, then the control problem also does not possess an optimal control in $\mathcal{A}_0$ (except for some simple cases). For $p = 1$ the right formulation of the problem would be to allow for singular controls; and consequently the description of optimal controls would require different methods.

We prove Theorem 1.2 in Section 2 (see Theorem 2.2) and Theorem 1.3 in Section 3 (see Theorem 3.2). Before tackling the proofs we provide a heuristic derivation of the BSDE (2).

Heuristic derivation of the BSDE

Throughout this section we assume $\xi > 0$. First we show that in this case we can restrict attention to non-increasing non-negative controls. To this end we denote the set of controls in $\mathcal{A}_0$ with non-increasing sample paths by $\mathcal{D}_0$.

Lemma 1.6. *Every control $x \in \mathcal{A}_0$ can be modified to a control $\underline{x} \in \mathcal{D}_0$ such that $J(x) \geq J(\underline{x})$. In particular, we have $v = \inf_{x \in \mathcal{D}_0} J(x)$.*

Proof. Let $x \in \mathcal{A}_0$ and define its running minimum cut off at zero by $\underline{x}_t = \min_{0 \leq s \leq t} x_s \vee 0$. Notice that $\underline{x}$ is absolutely continuous since $\underline{x}_t = \int_0^t \dot{x}_s 1_{\{x_s = \underline{x}_s\}} ds$. Hence $\underline{x} \in \mathcal{D}_0$. Observe that $|\dot{\underline{x}}_t| \leq |\dot{x}_t|$, and therefore we have $E\left[\int_0^T \eta_t |\dot{x}_t|^p dt\right] \geq E\left[\int_0^T \eta_t |\dot{\underline{x}}_t|^p dt\right]$. Since $\underline{x} \leq x$ on $\Omega \times [0,T]$ it follows that $E\left[\int_0^T \gamma_t |x_t|^p dt\right] \geq E\left[\int_0^T \gamma_t |\underline{x}_t|^p dt\right]$. Thus, we obtain $J(x) \geq J(\underline{x})$. □

The next result, a maximum principle, provides a sufficient condition for optimality in (5). We remark that we use it only for the heuristic derivation of the BSDE (2). The rigorous verification in Section 3 will be performed via a penalization.

Proposition 1.7 (Maximum principle)**.** *Assume that $x \in \mathcal{D}_0$ and that $M_t = p\eta_t |\dot{x}_t|^{p-1} + p\int_0^t \gamma_s x_s^{p-1} ds$ is a martingale with $E[M_T^2] < \infty$. Then x is optimal in* (5)*.*

Proof. Let $g(z) = |z|^p$ and $x \in \mathcal{D}_0$ such that $M_t = p\eta_t|\dot{x}_t|^{p-1} + p\int_0^t \gamma_s x_s^{p-1} ds$ is a martingale with $E[M_T^2] < \infty$. Let $y \in \mathcal{D}_0$ and introduce $\theta_t = x_t - y_t$. Then θ satisfies $\theta_0 = \theta_T = 0$ a.s. Furthermore, since x and y are non-increasing it follows that θ is bounded: $|\theta_t| \leq 2|\xi|$. Since $\dot{x} \leq 0$ on $\Omega \times [0,T]$ we have $g'(\dot{x}_t) = -p|\dot{x}_t|^{p-1}$. The convexity of g implies for all $t \in [0,T]$

$$g(\dot{x}_t) - g(\dot{y}_t) \leq g'(\dot{x}_t)(\dot{x}_t - \dot{y}_t).$$

Thus, by integration by parts we obtain

$$\begin{aligned}\int_0^T \eta_t(g(\dot{x}_t) - g(\dot{y}_t))dt &\leq \int_0^T \eta_t g'(\dot{x}_t)d\theta_t = \int_0^T \left(\int_0^t p\gamma_s x_s^{p-1} ds - M_t\right) d\theta_t \\ &= \int_0^T \theta_t dM_t - \int_0^T \gamma_t g'(x_t)\theta_t dt.\end{aligned}$$

Since θ is bounded and M is a martingale with $E[M_T^2] < \infty$ it follows that the integral process $\int_0^\cdot \theta_t dM_t$ is a martingale starting in 0. In particular, it vanishes in expectation. Using again the convexity of g yields for $t \in [0,T]$

$$g(x_t) - g(y_t) \leq g'(x_t)(x_t - y_t).$$

Taking expectations implies optimality of x:

$$E\left[\int_0^T \eta_t(g(\dot{x}_t) - g(\dot{y}_t))dt\right] \leq -E\left[\int_0^T \gamma_t g'(x_t)\theta_t dt\right] \leq -E\left[\int_0^T \gamma_t(g(x_t) - g(y_t))dt\right].$$

□

Remark 1.8. Observe that the previous two results hold in a more general framework than the one under consideration here. We can replace $y \mapsto |y|^p$ by any convex function which attains its minimum at $y = 0$.

We next observe that the *relative* control rate $r_t = \frac{\dot{x}_t}{x_t}$ of an optimal control $x \in \mathcal{A}_0$ is independent of the current state x_t. To this end fix $t < T$ and $\xi_2 > \xi_1 > 0$. Assume that $(x_s^1)_{t \leq s \leq T}$ is an optimal control to close the position ξ_1 in the period $[t,T]$. Then the homogeneity of $y \mapsto |y|^p$ implies that the control $x_s^2 = \frac{\xi_2}{\xi_1} x_s^1$, $s \in [t,T]$, is optimal to close the position ξ_2 in the period $[t,T]$. In particular the relative control rates at time t coincide $\frac{\dot{x}_t^1}{\xi_1} = \frac{\dot{x}_t^2}{\xi_2}$. Hence, an optimal control can be represented in feedback form $\dot{x}_t = r_t x_t$, where r_t is the relative control rate, which does not depend on x_t. We denote by q the Hölder conjugate of p and rewrite r_t as $r_t = -\left(\frac{Y_t}{\eta_t}\right)^{q-1}$ for some semi-martingale Y and make the ansatz that an optimal control x is of the form

$$\dot{x}_t = -\left(\frac{Y_t}{\eta_t}\right)^{q-1} x_t \tag{6}$$

with $x_0 = 1$. The solution of this pathwise ordinary differential equation is given by

$$x_t = e^{-\int_0^t \left(\frac{Y_s}{\eta_s}\right)^{q-1} ds}. \tag{7}$$

Proposition 1.7 shows that $x \in \mathcal{A}_0$ is optimal if the process $p\eta|\dot{x}|^{p-1} + p\int_0^{\cdot} \gamma_s x_s^{p-1} ds$ is a martingale. Since $(\mathcal{F}_t)$ is a Brownian filtration this is equivalent to the existence of a predictable process ϕ such that

$$d(\eta|\dot{x}|^{p-1})_t + \gamma_t x_t^{p-1} dt = \phi_t dW_t.$$

Using the equality $\eta_t|\dot{x}_t|^{p-1} = Y_t x_t^{p-1}$ and applying the integration by parts formula to the product Yx^{p-1} we obtain

$$\begin{aligned} d(\eta|\dot{x}|^{p-1})_t + \gamma_t x_t^{p-1} dt &= x_t^{p-1} dY_t + (p-1) Y_t x_t^{p-2} dx_t + \gamma_t x_t^{p-1} dt \\ &= x_t^{p-1} \left(dY_t - \left((p-1) \frac{Y_t^q}{\eta_t^{q-1}} - \gamma_t \right) dt \right). \end{aligned}$$

Setting $Z_t = \phi_t / x_t^{p-1}$ we see that Y satisfies the BSDE

$$dY_t = \left((p-1) \frac{Y_t^q}{\eta_t^{q-1}} - \gamma_t \right) dt + Z_t dW_t. \tag{8}$$

In view of Equation (7) the singular terminal condition $Y_T = \infty$ is necessary to ensure that $x_T = 0$. In Theorem 3.2 we show that this condition is indeed sufficient.

2 Construction of a BSDE solution with singular terminal condition

In this section we construct a solution of the BSDE (2) with singular terminal condition. To this end we first show existence of solutions to BSDEs with cut off drivers and finite deterministic terminal condition $L > 0$. In a second step we let L tend to infinity and obtain a solution with a singular terminal condition. We show that this particular solution is the minimal solution of (8). We remark that the second step of our construction bears similarities with the existence proof conducted by Popier in [9] resp. [10].

Let us clarify some terminology concerning BSDEs. The pair consisting of the driver and the terminal condition of a BSDE will be referred to as its *parameters*. Given a solution (Y, Z) of a BSDE, we call the first component Y the *solution process* and the second component Z the *martingale component*.

2.1 Approximation

Consider the BSDE

$$dY_t^L = \left((p-1) \frac{(Y_t^L)^q}{\eta_t^{q-1}} - (\gamma_t \wedge L) \right) dt + Z_t^L dW_t, \tag{9}$$

with terminal condition $Y_T^L = L$.

Proposition 2.1. *Assume that $\eta \in \mathcal{M}^2(0,T)$ and $\frac{1}{\eta^{q-1}} \in \mathcal{M}^1(0,T)$. Then there exists a solution (Y^L, Z^L) to (9) with $Z^L \in \mathcal{M}^2(0,T)$. For every $t \in [0,T]$ the random variable Y_t^L is bounded from below and above as follows*

$$\frac{1}{\left(\frac{1}{L^{q-1}} + E\left[\int_t^T \frac{1}{\eta_s^{q-1}} ds \middle| \mathcal{F}_t\right]\right)^{p-1}} \leq Y_t^L \leq (1+T)L \wedge \frac{1}{(T-t)^p} E\left[\int_t^T (\eta_s + (T-s)^p \gamma_s) ds \middle| \mathcal{F}_t\right]. \quad (10)$$

Proof. Let $f(t,y) = -(p-1)\frac{y^q}{\eta_t^{q-1}} + (\gamma_t \wedge L)$ denote the driver of the BSDE (9). Define $f^\delta(t,y) = -(p-1)\frac{y^q}{(\eta_t \vee \delta)^{q-1}} + (\gamma_t \wedge L)$ for $\delta > 0$. Being decreasing in y, bounded in ω, the driver $(\omega, t, y) \mapsto f^\delta(t, y \vee 0)$ - which does not depend on z - satisfies all conditions of Theorem 2.2. in [7]. Hence, for every $L > 0$ there exists a solution $(Y^{\delta,L}, Z^{\delta,L})$ to the BSDE with parameters $(f^\delta(t, y \vee 0), L)$. Moreover, any such solution satisfies

$$E\left[\sup_{0 \leq t \leq T} |Y_t^{\delta,L}|^2 + \int_0^T (Z_t^{\delta,L})^2 dt\right] < \infty. \quad (11)$$

For $L = 0$ the solution is given by $(Y^{\delta,0}, Z^{\delta,0}) = (0,0)$. The comparison theorem [7, Theorem 2.4] implies that $Y^{\delta,L}$ is nonnegative and, hence, $Y^{\delta,L}$ is also a solution to the BSDE with parameters (f^δ, L).

We can also derive an upper bound for $Y^{\delta,L}$ by appealing to the comparison theorem. Note that we have $f^\delta(t,y) \leq L$ for $y \geq 0$. This implies

$$Y_t^{\delta,L} \leq (1+T)L \quad (12)$$

for all $t \in [0,T]$.

We obtain a solution of the BSDE (9) by letting δ converge to zero. Indeed, the mapping $\delta \mapsto f^\delta$ is increasing, which implies that $Y^{\delta_1,L} \leq Y^{\delta_2,L}$ if $\delta_1 \leq \delta_2$. In particular we can define Y^L as the decreasing limit of $Y^{\delta,L}$ as $\delta \searrow 0$. For the convergence of the control process $Z^{\delta,L}$, let $(\delta_n)_{n \geq 0}$ be a sequence with $\delta_n \searrow 0$ as $n \to \infty$. Fix $n \geq m$. Then we have $Y^{\delta_n,L} \leq Y^{\delta_m,L}$. For all $0 \leq t \leq T$ Itô's formula leads to

$$\begin{aligned}\int_0^T (Z_s^{\delta_n,L} - Z_s^{\delta_m,L})^2 ds = &- (Y_0^{\delta_n,L} - Y_0^{\delta_m,L})^2 \\ &- 2\int_0^T (Y_s^{\delta_n,L} - Y_s^{\delta_m,L})(Z_s^{\delta_n,L} - Z_s^{\delta_m,L}) dW_s \\ &+ 2\int_0^T (Y_s^{\delta_n,L} - Y_s^{\delta_m,L})(f^{\delta_n}(s, Y_s^{\delta_n,L}) - f^{\delta_m}(s, Y_s^{\delta_m,L})) ds \end{aligned} \quad (13)$$

Estimates (11) and (12) imply

$$E\left[\int_0^T (Y_s^{\delta_n,L} - Y_s^{\delta_m,L})(Z_s^{\delta_n,L} - Z_s^{\delta_m,L}) dW_s\right] = 0.$$

By monotonicity of f^{δ_m} and estimate (12) we have

$$\begin{aligned}(Y_s^{\delta_n,L} - Y_s^{\delta_m,L})(f^{\delta_n}(s,Y_s^{\delta_n,L}) - f^{\delta_m}(s,Y_s^{\delta_m,L})) \\ &\leq (Y_s^{\delta_n,L} - Y_s^{\delta_m,L})(f^{\delta_n}(s,Y_s^{\delta_n,L}) - f^{\delta_m}(s,Y_s^{\delta_n,L})) \\ &= (p-1)(Y_s^{\delta_m,L} - Y_s^{\delta_n,L})(Y_s^{\delta_n,L})^q \left(\frac{1}{(\eta_s \vee \delta_n)^{q-1}} - \frac{1}{(\eta_s \vee \delta_m)^{q-1}}\right) \\ &\leq C\left(\frac{1}{(\eta_s \vee \delta_n)^{q-1}} - \frac{1}{(\eta_s \vee \delta_m)^{q-1}}\right)\end{aligned}$$

for all $s \in [0,T]$ and a constant $C > 0$. Taking expectations in Equation (13) yields

$$E\left[\int_0^T (Z_s^{\delta_n,L} - Z_s^{\delta_m,L})^2 ds\right] \leq 2CE\left[\int_0^T \left(\frac{1}{(\eta_s \vee \delta_n)^{q-1}} - \frac{1}{(\eta_s \vee \delta_m)^{q-1}}\right) ds\right].$$

The sequence $\left(\frac{1}{(\eta \vee \delta_n)^{q-1}}\right)_{n\geq 0}$ converges in $\mathcal{M}^1(0,T)$ to $\frac{1}{\eta^{q-1}}$ as $n \to \infty$. This implies that $(Z^{\delta_n,L})_{n\geq 0}$ is a Cauchy sequence in $\mathcal{M}^2(0,T)$ and converges to $Z^L \in \mathcal{M}^2(0,T)$. In particular the random variable $\int_t^T Z_r^{\delta_n,L} dW_r$ converges to $\int_t^T Z_r^L dW_r$ in $L^2(\Omega)$ as $n \to \infty$. We obtain almost sure convergence by passing to a subsequence. Taking the limit $n \to \infty$ in

$$Y_t^{\delta_n,L} = L - (p-1)\int_t^T \frac{(Y_r^{\delta_n,L})^q}{(\eta_r \vee \delta_n)^{q-1}} dr - \int_t^T Z_r^{\delta_n,L} dW_r,$$

and using estimate (12) yields that (Y^L, Z^L) satisfies almost surely the BSDE

$$Y_t^L = L - (p-1)\int_t^T \frac{(Y_r^L)^q}{\eta_r^{q-1}} dr - \int_t^T Z_r^L dW_r.$$

We proceed by deriving the upper and lower bound in (10). We first estimate $Y^{\delta,L}$ against a linear BSDE with driver

$$g(t,y) = -p\frac{y}{T-t} + \frac{\eta_t \vee \delta}{(T-t)^p} + \gamma_t.$$

By using the inequality

$$(p-1)y^q - pa^{q-1}y + a^q \geq 0,$$

which holds for all $y \geq 0, a \geq 0$, one can show that $f^\delta(t,y) \leq g(t,y)$ (take $a = (\eta_t \vee \delta)(T-t)^{-p/q}$). Let $\epsilon > 0$ and denote by Ψ^ϵ the solution process of the BSDE on $[0, T-\epsilon]$ with parameters $(g, Y_{T-\epsilon}^{\delta,L})$. By the solution formula for linear BSDEs we have

$$\Psi_t^\epsilon = E\left[\Gamma_{T-\epsilon} Y_{T-\epsilon}^{\delta,L} + \int_t^{T-\epsilon} \Gamma_s \left(\frac{\eta_s \vee \delta}{(T-s)^p} + \gamma_s\right) ds | \mathcal{F}_t\right],$$

where

$$\Gamma_t \;=\; \exp\left(-\int_0^t \frac{p}{T-s}ds\right) = \left(\frac{T-t}{T}\right)^p.$$

The comparison theorem implies

$$Y_t^{\delta,L} \le \Psi_t^\epsilon = \frac{1}{(T-t)^p} E\left[\epsilon^p Y_{T-\epsilon}^{\delta,L} + \int_t^{T-\epsilon} \left((\eta_s \vee \delta) + (T-s)^p \gamma_s\right) ds | \mathcal{F}_t\right] \tag{14}$$

for all $t \in [0,T]$ and $\epsilon > 0$. By letting $\epsilon \downarrow 0$ we obtain with dominated convergence

$$Y_t^{\delta,L} \le \frac{1}{(T-t)^p} E\left[\int_t^{T} \left((\eta_s \vee \delta) + (T-s)^p \gamma_s\right) ds | \mathcal{F}_t\right].$$

By letting $\delta \downarrow 0$ we obtain the upper bound in (10).

In order to derive the lower estimate, let $V_t = \frac{1}{L^{q-1}} + E\left[\int_t^T \frac{1}{(\eta_s \vee \delta)^{q-1}} ds \big| \mathcal{F}_t\right]$, and observe that there exists a process $Z \in \mathcal{M}^2(0,T)$ such that

$$dV_t = -\frac{1}{(\eta_t \vee \delta)^{q-1}} dt + Z_t dW_t.$$

Notice that $\frac{1}{L^{q-1}} \le V_t \le \kappa := \frac{1}{L^{q-1}} + \frac{T}{\delta^{q-1}}$. Next let $U_t = \frac{1}{V_t^{p-1}}$. With Ito's formula one can show that there exists $\tilde{Z} \in \mathcal{M}^2(0,T)$ such that

$$dU_t = -h(t, U_t, \tilde{Z}_t)dt + \tilde{Z}_t dW_t,$$

where

$$h(t,u,z) = -(p-1)\frac{(u \wedge L)^q}{(\eta_s \vee \delta)^{q-1}} - \frac{1}{2}\frac{p}{p-1}\frac{z^2}{u \vee (1/\kappa^{p-1})}.$$

Note that $h(t,u,z) \le f^\delta(t,u)$. Since $U_T = L = Y_T^{\delta,L}$, the comparison theorem for quadratic BSDEs (see e.g. Theorem 2.6 in [4]) implies that $U_t \le Y_t^{\delta,L}$. Finally, by letting $\delta \downarrow 0$, we obtain the lower estimate in (10).

□

2.2 Existence of solutions for BSDEs with singular terminal condition

First we establish the convergence of (Y^L, Z^L) from Proposition 2.1 to a pair (Y,Z) which is a solution to the BSDE (8) with singular terminal condition $Y_T = \infty$ in the sense of Definition 1.1.

Theorem 2.2. *Assume* (**I1**) *and* (**I2**) *hold true. Let* (Y^L, Z^L) *be the solution to* (9) *from Proposition 2.1. Then there exists a process* (Y, Z) *such that for every* $0 \leq t < T$ *the random variable* Y_t^L *converges a.s. to* Y_t *and* Z^L *converges in* $\mathcal{M}^2(0,t)$ *to* Z *as* $L \to \infty$. *The limit process* (Y, Z) *is a solution to the BSDE* (8) *with singular terminal condition* $Y_T = \infty$. *Moreover, for every* $t \in [0,T]$ *the random variable* Y_t *is almost surely positive:*

$$Y_t \geq \frac{1}{\left(E\left[\int_t^T \frac{1}{\eta_s^{q-1}} ds \middle| \mathcal{F}_t\right]\right)^{p-1}}. \tag{15}$$

Proof. The proof is partly a generalization of the arguments in [9] to our setting. Appealing to the comparison theorem [7, Theorem 2.4] yields that $Y^L \leq Y^N$ if $N > L$ (Observe that although assumption (ii) of [7, Theorem 2.4] is not satisfied here, the comparison holds, since the process α_t from the proof is non-positive here as well). By Equation (10) for fixed $t < T$ the family of random variables $(Y_t^L, L \geq 0)$ is bounded from above as follows

$$Y_t^L \leq \frac{1}{(T-t)^p} E\left[\int_t^T (\eta_s + (T-s)^p \gamma_s) ds | \mathcal{F}_t\right]. \tag{16}$$

Hence, for all $t < T$ we can define Y_t as the increasing limit of Y_t^L as $L \to \infty$. Notice that by Conditions (**I1**) and (**I2**) the random variable on the RHS of (16) is square integrable. By dominated convergence, therefore, Y_t^L converges to Y_t in $L^2(\Omega)$.

Taking the limit $L \nearrow \infty$ in the lower bound for Y^L from Inequality (10) yields that Y satisfies (15). We write $E\left[\int_t^T \frac{1}{\eta_s^{q-1}} ds | \mathcal{F}_t\right] = M_t - A_t$ with $M_t = E\left[\int_0^T \frac{1}{\eta_s^{q-1}} ds | \mathcal{F}_t\right]$ and $A_t = \int_0^t \frac{1}{\eta_s^{q-1}} ds$. Since $(\mathcal{F}_t)$ is a Brownian filtration the martingale M is continuous. This implies

$$\lim_{t \nearrow T} E\left[\int_t^T \frac{1}{\eta_s^{q-1}} ds \middle| \mathcal{F}_t\right] = \lim_{t \nearrow T} (M_t - A_t) = M_T - A_T = 0.$$

Hence, it follows from (15) that Y satisfies the singular terminal condition $\liminf_{t \nearrow T} Y_t = \infty$.

For the convergence of (Z^L) let $0 \leq s \leq t < T$. Then Itô's formula implies, for $N, L \geq 0$,

$$\begin{aligned}(Y_s^N - Y_s^L)^2 + \int_s^t |Z_r^N - Z_r^L|^2 dr = (Y_t^N - Y_t^L)^2 - 2\int_s^t (Y_r^N - Y_r^L)(Z_r^N - Z_r^L) dW_r \\ + 2\int_s^t (Y_r^N - Y_r^L)(f^N(r, Y_r^N) - f^L(r, Y_r^L)) dr.\end{aligned} \tag{17}$$

The monotonicity of the driver $f^L(r,y) = -(p-1)\frac{y^q}{\eta_t^{q-1}} + (\gamma_r \wedge L)$ in y yields for $y, y' \geq 0$

$$(y-y')(f^N(r,y) - f^L(r,y')) \leq (y-y')(f^N(r,y) - f^L(r,y)) = (y-y')(\gamma_r \wedge N - \gamma_r \wedge L),$$

and hence

$$(Y_s^N - Y_s^L)^2 + \int_s^t |Z_r^N - Z_r^L|^2 dr \leq (Y_t^N - Y_t^L)^2 - 2\int_s^t (Y_r^N - Y_r^L)(Z_r^N - Z_r^L) dW_r + 2\int_s^t (Y_r^N - Y_r^L)(\gamma_r \wedge N - \gamma_r \wedge L) dr. \tag{18}$$

Since Y^L and Y^N are bounded and $Z^L, Z^N \in \mathcal{M}^2(0,T)$, we have

$$E\left[\int_s^t (Y_r^N - Y_r^L)(Z_r^N - Z_r^L) dW_r\right] = 0.$$

Then estimate (18) implies

$$E\left[\int_0^t |Z_r^N - Z_r^L|^2 dr\right] \leq E\left[(Y_t^N - Y_t^L)^2] + 2\int_s^t (Y_r^N - Y_r^L)(\gamma_r \wedge N - \gamma_r \wedge L) dr\right], \tag{19}$$

and for a constant C_1

$$\begin{aligned} E\left[\sup_{0\leq s\leq t} (Y_s^N - Y_s^L)^2\right] &\leq E[(Y_t^N - Y_t^L)^2] + C_1 E\left[\sqrt{\int_0^t (Y_r^N - Y_r^L)^2 |Z_r^N - Z_r^L|^2 dr}\right] \\ &\quad + 2E\left[\int_0^t (Y_r^N - Y_r^L)(\gamma_r \wedge N - \gamma_r \wedge L) dr\right], \end{aligned} \tag{20}$$

where we used the Burkholder-Davis-Gundy inequality. From Young's inequality we derive

$$\begin{aligned} E\left[\sqrt{\int_0^t (Y_r^N - Y_r^L)^2 |Z_r^N - Z_r^L|^2 dr}\right] &\leq E\left[\sup_{0\leq s\leq t} |Y_s^N - Y_s^L| \sqrt{\int_0^t |Z_r^N - Z_r^L|^2 dr}\right] \\ &\leq \frac{1}{4C_1} E\left[\sup_{0\leq s\leq t} (Y_s^N - Y_s^L)^2\right] + C_1 E\left[\int_0^t |Z_r^N - Z_r^L|^2 dr\right], \end{aligned}$$

which implies, together with (20) and (19),

$$\begin{aligned} \frac{3}{4} E\left[\sup_{0\leq s\leq t} (Y_s^N - Y_s^L)^2\right] &\leq C_2 E[(Y_t^N - Y_t^L)^2] \\ &\quad + 2C_2 E\left[\int_0^t (Y_r^N - Y_r^L)(\gamma_r \wedge N - \gamma_r \wedge L) dr\right], \end{aligned}$$

where $C_2 = 1 + C_1^2$. Again with Young's inequality we get

$$\begin{aligned} & E\left[\int_0^t (Y_r^N - Y_r^L)(\gamma_r \wedge N - \gamma_r \wedge L) dr\right] \\ \leq\ & \frac{1}{4C_2} E\left[\sup_{0\leq s\leq t} (Y_s^N - Y_s^L)^2\right] + C_2 E\left[\left(\int_0^t |\gamma_r \wedge N - \gamma_r \wedge L| dr\right)^2\right]. \end{aligned}$$

Finally we arrive at

$$E\left[\sup_{0\le s\le t}(Y_s^N - Y_s^L)^2\right] \;\le\; C_3 E\left[(Y_t^N - Y_t^L)^2 + \int_0^t (\gamma_r \wedge N - \gamma_r \wedge L)^2 dr\right], \quad (21)$$

for a constant $C_3 \ge 0$. The RHS of (21) converges to zero as N, $L \to \infty$. In particular, Inequality (19) implies that (Z^L) is a Cauchy sequence in $\mathcal{M}^2(0,t)$ and converges to $Z \in \mathcal{M}^2(0,t)$ for every $t < T$. Moreover, Inequality (21) yields that $E\left[\sup_{0\le s\le t} Y_s^2\right] < \infty$. Finally, taking the limit $L \nearrow \infty$ in

$$Y_s^L = Y_t^L - \int_s^t \left((p-1)\frac{(Y_r^L)^q}{\eta_r^{q-1}} - \gamma_r\right) dr - \int_s^t Z_r^L dW_r$$

implies that Y satisfies (8) for every $0 \le s \le t < T$. □

Proposition 2.3. *The solution obtained in Theorem 2.2 is minimal: If (Y', Z') is another nonnegative solution of (8) with singular terminal condition $Y_T = \infty$, then $Y_t' \ge Y_t$ a.s. for all $t \in [0,T]$.*

Proof. The proof is an adaptation of [10, Theorem 7] to our setting.

Fix $L > 0$ and let (Y^L, Z^L) denote the solution of (9) with terminal condition $Y_T^L = L$. Let (Y', Z') be a nonnegative solution of (8) in the sense of Definition 1.1. Set $\Delta_t = Y_t' - Y_t^L$, $\Gamma_t = Z_t' - Z_t^L$ and

$$\alpha_t = \begin{cases} \dfrac{p-1}{\eta_t^{q-1}} \dfrac{(Y_t')^q - (Y_t^L)^q}{Y_t' - Y_t^L}, & \text{if } \eta_t^{q-1}(Y_t' - Y_t^L) \ne 0 \\ 0, & \text{else.} \end{cases}$$

Note that α is nonnegative. For every $t < T$ the process (Δ, Γ) solves the linear BSDE

$$d\Delta_s = [\alpha_s \Delta_s - (\gamma_t - L)^+] ds + \Gamma_s dW_s$$

on $[0,t]$ with terminal condition $\Delta_t = Y_t' - Y_t^L$. Hence, by Lemma 4.10 in the Appendix the solution Δ admits the explicit representation

$$\Delta_s = E\left[\Delta_t e^{-\int_s^t \alpha_r dr} + \int_s^t e^{-\int_s^u \alpha_r dr}(\gamma_u - L)^+ du | \mathcal{F}_s\right].$$

Since Y' is nonnegative and $Y^L \le (1+T)L$ by Proposition 2.1, we have $\Delta_t \ge -(1+T)L$. Thus $\Delta_t e^{-\int_s^t \alpha_r dr}$ is bounded from below by $-(1+T)L$ and we can apply Fatou's lemma to obtain

$$\begin{aligned} Y_s' - Y_s^L &= \Delta_s = \liminf_{t \nearrow T} E\left[\Delta_t e^{-\int_s^t \alpha_r dr} + \int_s^t e^{-\int_s^u \alpha_r dr}(\gamma_u - L)^+ du | \mathcal{F}_s\right] \\ &\ge E\left[\liminf_{t \nearrow T} \Delta_t e^{-\int_s^t \alpha_r dr} | \mathcal{F}_s\right] \ge 0. \end{aligned}$$

Finally, taking the limit $L \nearrow \infty$ yields the claim. □

3 Optimal Controls

In this section we first consider a variant of the minimization problem (5), where we omit the constraint $x_T = 0$ in the set of admissible controls but penalize any nonzero terminal state by $L|x_T|^p$. We show that optimal controls for this unconstrained minimization problem admit a representation in terms of the solutions Y^L from Proposition 2.1. We then use this result to derive an optimal control for (5).

Throughout this section we assume (**I1**) and (**I2**) without further mentioning it.

3.1 Penalization

In this section we consider the unconstrained minimization problem

$$v^L = \inf_{x\in\mathcal{A}} J^L(x) = \inf_{x\in\mathcal{A}} E\left[\int_0^T \big(\eta_t|\dot{x}_t|^p + (\gamma_t \wedge L)|x_t|^p\big)dt + L|x_T|^p\right] \tag{22}$$

for some $L > 0$, where we take the infimum over $\mathcal{A}$, the set of all progressively measurable processes $x : \Omega \times [0,T] \to \mathbb{R}$ with absolutely continuous sample paths starting in $x_0 = \xi$. Next, we show how to obtain a minimizing control for (22) from the solution Y^L to (9).

Proposition 3.1. *Let (Y^L, Z^L) be the solution to (9) from Proposition 2.1. Then*

$$x_t^L = \xi e^{-\int_0^t \left(\frac{Y_s^L}{\eta_s}\right)^{q-1} ds}$$

is optimal in (22) and we have $v^L = Y_0^L|\xi|^p$.

Proof. To simplify notation we assume $\xi = 1$ and set $\gamma_t^L = \gamma_t \wedge L$. Let $g(z) = |z|^p$ and $M_t = pY_t^L(x_t^L)^{p-1} + p\int_0^t \gamma_s^L (x_s^L)^{p-1}dt$. Applying the integration by parts formula to M results in

$$\begin{aligned} dM_t &= p(x_t^L)^{p-1}dY_t^L + p(p-1)Y_t^L(x_t^L)^{p-2}dx_t^L + p\gamma_t^L(x_t^L)^{p-1}dt \\ &= p(x_t^L)^{p-1}Z_t^L dW_t. \end{aligned}$$

Since x^L is bounded and $Z^L \in \mathcal{M}^2(0,T)$, the process M is a martingale. Let $x \in \mathcal{A}$ and introduce $\theta_t = x_t^L - x_t$. Then θ satisfies $\theta_0 = 0$. Similar considerations as in Lemma 1.6 imply that we can assume that x is pathwise non-increasing and hence $|\theta_t| \le 2$. Furthermore, we have $\eta_t g'(\dot{x}_t^L) = -p\eta_t|\dot{x}_t^L|^{p-1} = -pY_t^L(x_t^L)^{p-1}$. The convexity of g implies for all $t \in [0,T]$

$$g(\dot{x}_t^L) - g(\dot{x}_t) \le g'(\dot{x}_t^L)(\dot{x}_t^L - \dot{x}_t).$$

Thus, it follows from integration by parts

$$\begin{aligned} \int_0^T \eta_t(g(\dot{x}_t^L) - g(\dot{x}_t))dt &\le \int_0^T \eta_t g'(\dot{x}_t^L)d\theta_t = \int_0^T \left(\int_0^t p\gamma_s^L x_s^{p-1}ds - M_t\right)d\theta_t \\ &= -Lg'(x_T^L)\theta_T + \int_0^T \theta_t dM_t - \int_0^T \gamma_t^L g'(x_t)\theta_t dt \end{aligned}$$

Since M is a square integrable martingale, we obtain $E\left[\int_0^T \theta_t dM_t\right] = 0$. Using convexity of g once more, we obtain

$$g(x_t^L) - g(x_t) \leq g'(x_t^L)(x_t^L - x_t).$$

This implies optimality of x^L:

$$\begin{aligned} E\left[\int_0^T \eta_t(g(\dot{x}_t^L) - g(\dot{x}_t))dt\right] &\leq -E\left[Lg'(x_T^L)\theta_T + \int_0^T \gamma_t^L g'(x_t)\theta_t dt\right] \\ &\leq -E\left[L(g(x_T^L) - g(x_T)) + \int_0^T \gamma_t^L(g(x_t^L) - g(x_t))dt\right]. \end{aligned}$$

It remains to verify the identity $v^L = Y_0^L$. To this end we apply the integration by parts formula to the process $Y(x^L)^p$ to obtain

$$d(Y(x^L)^p)_t = -\left(\frac{(Y_t^L)^q}{\eta_t^{q-1}}(x_t^L)^p + \gamma_t^L(x_t^L)^p\right)dt + (x_t^L)^p Z_t^L dW_t.$$

Moreover we have

$$|\dot{x}_t^L|^p = \left(\left(\frac{Y_t^L}{\eta_t}\right)^{q-1} x_t^L\right)^p = \left(\frac{Y_t^L}{\eta_t}\right)^q (x_t^L)^p.$$

Thus we obtain

$$Y_0^L = E\left[\int_0^T \eta_t|\dot{x}_t^L|^p + \gamma_t^L|x_t^L|^p dt\right] = J^L(x^L) = v^L.$$

□

3.2 The constrained case

We now turn to the constrained case and prove Theorem 1.3. For the reader's convenience we restate the theorem here.

Theorem 3.2. *Let (Y, Z) be the minimal solution to (8) with singular terminal condition $Y_T = \infty$ from Theorem 2.2. Then $v = Y_0|\xi|^p$; moreover the control $x_t = \xi \exp\left(-\int_0^t \left(\frac{Y_s}{\eta_s}\right)^{q-1} ds\right)$ belongs to $\mathcal{A}_0$ and is optimal in (5).*

Proof. To simplify notation assume that $\xi = 1$. As in the proof of Proposition 3.1 we introduce $M_t = pY_t x_t^{p-1} + p\int_0^t \gamma_s x_s^{p-1} dt$. Performing integration by parts yields

$$dM_t = x_t^{p-1} Z_t dW_t.$$

Hence, M is a nonnegative local martingale on $[0,T)$ and in particular a nonnegative super-martingale. Thus it converges almost surely in $\mathbb{R}$ as $t \nearrow T$. Since Y satisfies the terminal condition $\liminf_{t\nearrow T} Y_t = \infty$ we have that

$$0 \leq x_t = \left(\frac{M_t - p\int_0^t \gamma_s x_s^{p-1} ds}{pY_t} \right)^{q-1} \leq \left(\frac{M_t}{pY_t} \right)^{q-1} \to 0$$

a.s. for $t \nearrow T$. It follows that $x \in \mathcal{A}_0$.

Next we apply the integration by parts formula to the process Yx^p to obtain

$$d(Yx^p)_t = -\left(\eta_t |\dot{x}_t|^p + \gamma_t x_t^p\right) dt + x_t^p Z_t dW_t.$$

Since $Z \in \mathcal{M}^2(0,t)$ and $|x_t| \leq 1$ we can deduce for $t < T$

$$Y_0 = E\left[\int_0^t \left(\eta_s |\dot{x}_s|^p + \gamma_s x_s^p\right) ds\right] + E\left[Y_t x_t^p\right] \geq E\left[\int_0^t \left(\eta_s |\dot{x}_s|^p + \gamma_s x_s^p\right) ds\right].$$

Taking the limit $t \nearrow T$ and appealing to monotone convergence theorem yields

$$Y_0 \geq E\left[\int_0^T \left(\eta_s |\dot{x}_s|^p + \gamma_s x_s^p\right) ds\right] = J(x). \tag{23}$$

Next, note that for every $\overline{x} \in \mathcal{A}_0$ we have $J(\overline{x}) \geq J^L(\overline{x})$. This implies $v \geq v^L$ for every $L > 0$. By Proposition 3.1 we have $Y_0^L = v^L$. Minimality of Y implies $Y_0 = \lim_{L\nearrow\infty} Y_0^L = \lim_{L\nearrow\infty} v^L \leq v$. Consequently we obtain with Equation (23)

$$Y_0 \geq J(x) \geq v \geq Y_0$$

and thus optimality of x. □

Remark 3.3. The solution Y from Theorem 2.2 does not only lead to optimal controls in the case where the liquidation period begins at time $t = 0$ and the initial position position is equal to $x_0 = 1$ but also for general initial states. Let $x \in \mathcal{A}_0$ denote the optimal control from Theorem 3.2. For a general initial position $\xi \in \mathbb{R}$ the homogeneity of $z \mapsto |z|^p$ implies that the process $t \mapsto \xi x_t$ minimizes the functional $E\left[\int_0^T \left(\eta_t |\dot{\tilde{x}}_t|^p + \gamma_t |\tilde{x}_t|^p\right) dt\right]$ over all progressively measurable processes $\tilde{x}$ with absolutely continuous paths starting in x_0 and ending in 0. The value of this minimization problem is then given by $Y_0 |x_0|^p$. If liquidation starts at an arbitrary time $t < T$ the minimization problem reads

$$V_t = \inf E\left[\int_t^T \left(\eta_s |\dot{\tilde{x}}_s|^p + \gamma_s |\tilde{x}_s|^p\right) ds \middle| \mathcal{F}_t\right],$$

where the infimum is taken over all progressively measurable processes $\tilde{x}$ starting in a $\mathcal{F}_t$-measurable random variable ξ and ending in 0. In this case the optimal control is given by

$$x_s = \xi \exp\left(-\int_t^s \left(\frac{Y_r}{\eta_r}\right)^{q-1} dr\right)$$

and the value is equal to $V_t = Y_t |\xi|^p$.

In the next proposition we state an integrability condition that allows to identify the minimal solution of (8).

Proposition 3.4. *Let (Y, Z) be a nonnegative solution of (8) with singular terminal condition $Y_T = \infty$. Let $x_t = \exp\left(-\int_0^t \left(\frac{Y_s}{\eta_s}\right)^{q-1} ds\right)$ denote the associated position path and assume that $x^{p-1}Z \in \mathcal{M}^2(0,T)$. Then Y is the minimal solution of (8).*

Proof. Let $Y^{\min}$ denote the minimal solution of (8). Without loss of generality we only consider the point in time $t = 0$ and show that $Y_0 = Y_0^{\min}$. For general $t < T$ we refer to Remark 3.3 which shows that $Y_t^{\min}$ is the value of the liquidation problem starting in time t. We proceed as in the proof of Theorem 3.2. Let $M_t = pY_t x_t^{p-1} + p\int_0^t \gamma_s x_s^{p-1} dt$. Then we obtain by integration by parts

$$dM_t = x_t^{p-1} Z_t dW_t.$$

Hence, M is a nonnegative true martingale with $E[M_T^2] < \infty$ and converges a.s. in $\mathbb{R}$ as $t \nearrow T$. Since Y satisfies the terminal condition $\liminf_{t\nearrow T} Y_t = \infty$ we have that $x_t \to 0$ as $t \nearrow T$. Consequently, $x \in \mathcal{A}_0$ and Lemma 1.7 implies optimality of x. Again an application of the integration by parts formula yields

$$d(Yx^p)_t = (\eta_t |\dot{x}_t|^p + \gamma_t x_t^p)dt + x_t^p Z_t dW_t.$$

By assumption the process $t \mapsto \int_0^t x_t^p Z_t dW_t$ is a true martingale. Moreover we have $\lim_{t\nearrow T} Y_t x_t^p = 0$ and hence Theorem 3.2 implies $Y_0 = J(x) = v = Y_0^{\min}$. □

4 Processes with uncorrelated multiplicative increments

In this section we study the special case of the control problem (5) where $\gamma = 0$ and η has uncorrelated multiplicative increments. We first give a rigorous definition of what the latter means.

We say that a positive, progressively measurable process η has uncorrelated multiplicative increments if $E\left[\frac{\eta_t}{\eta_s}|\mathcal{F}_s\right] = E\left[\frac{\eta_t}{\eta_s}\right]$ for all $s \le t < T$. We show that it is precisely this class of processes which leads to *deterministic* optimal controls for the minimization problem (5) (with $\gamma = 0$). Moreover we show that if η is a martingale, then it is optimal to close the position at a constant rate.

Observe that any process η where $\frac{\eta_t}{\eta_s}$ is independent of $\mathcal{F}_s$ for $s \le t < T$ has uncorrelated multiplicative increments. The converse does not hold true.

In the next lemma we give an equivalent characterization of processes with uncorrelated multiplicative increments.

Lemma 4.1. *A positive, progressively measurable process η has uncorrelated multiplicative increments if and only if the process $\left(\frac{\eta_t}{E[\eta_t]}\right)_{t<T}$ is a martingale. Any such process satisfies $E\left[\frac{\eta_t}{\eta_s}\right] = \frac{E[\eta_t]}{E[\eta_s]}$ for all $s \le t < T$.*

Proof. Let η have uncorrelated multiplicative increments. We first show that for $s \leq t < T$ any such η satisfies $E\left[\frac{\eta_t}{\eta_s}\right] = \frac{E[\eta_t]}{E[\eta_s]}$. Indeed, we have

$$E[\eta_t] = E\left[\eta_s E\left[\frac{\eta_t}{\eta_s}\middle|\mathcal{F}_s\right]\right] = E[\eta_s]E\left[\frac{\eta_t}{\eta_s}\right].$$

Next let $M_t = \frac{\eta_t}{E[\eta_t]}$ for $t < T$. For $s \leq t < T$ the process M satisfies

$$E[M_t|\mathcal{F}_s] = \frac{1}{E[\eta_t]}E[\eta_t|\mathcal{F}_s] = \frac{1}{E[\eta_t]}E\left[\frac{\eta_t}{\eta_s}\eta_s\middle|\mathcal{F}_s\right] = \frac{\eta_s}{E[\eta_t]}E\left[\frac{\eta_t}{\eta_s}\right] = M_s.$$

For the converse direction, let $M_t = \frac{\eta_t}{E[\eta_t]}$ be a martingale for $t < T$. Then we have for $s \leq t < T$

$$E[\eta_t|\mathcal{F}_s] = E[\eta_t]E[M_t|\mathcal{F}_s] = E[\eta_t]M_s = \frac{E[\eta_t]}{E[\eta_s]}\eta_s.$$

Thus the random variable $E\left[\frac{\eta_t}{\eta_s}|\mathcal{F}_s\right]$ is deterministic, which implies $E\left[\frac{\eta_t}{\eta_s}|\mathcal{F}_s\right] = E\left[\frac{\eta_t}{\eta_s}\right]$. □

Lemma 4.1 implies that any positive martingale has uncorrelated multiplicative increments. Further examples are provided by the following class of diffusions.

Example 4.2. *Let η be a diffusion with linear drift, i.e. η solves*

$$d\eta_t = \mu(t)\eta_t dt + \sigma(t,\eta_t)dW_t,$$

where the drift μ is a deterministic function of time and the stochastic volatility $\sigma : [0,T]\times\mathbb{R}\times\Omega \to \mathbb{R}_+$ is such that $t \mapsto \sigma(t,\eta_t) \in \mathcal{M}^2(0,T)$. Then the process $\eta\exp(-\int_0^\cdot \mu(r)dr)$ is a martingale, and hence we have $E[\eta_t|\mathcal{F}_s] = \eta_s\exp(\int_s^t \mu(r)dr)$. This implies that the random variable $E\left[\frac{\eta_t}{\eta_s}|\mathcal{F}_s\right]$ is deterministic. Therefore η has uncorrelated multiplicative increments.

We first show that if the optimal control from Theorem 3.2 is deterministic, then the process η has necessarily uncorrelated multiplicative increments.

Proposition 4.3. *Let η be positive, progressively measurable and such that $\eta \in \mathcal{M}^2(0,T)$, $1/\eta^{q-1} \in \mathcal{M}^1(0,T)$. Assume that the optimal control $x \in \mathcal{A}_0$ from Theorem 3.2 is deterministic. Then η has uncorrelated multiplicative increments.*

Proof. The optimal control from Theorem 3.2 satisfies $\dot{x}_t = -\left(\frac{Y_t}{\eta_t}\right)^{q-1} x_t$ where Y is the minimal solution of (8) with singular terminal condition $Y_T = \infty$. Since x is deterministic it follows that the nonnegative process $\alpha_t = \left(\frac{Y_t}{\eta_t}\right)^{q-1}$ is deterministic as well. Furthermore Y satisfies the linear BSDE

$$dY_t = (p-1)\alpha_t Y_t dt + Z_t dW_t$$

and hence Lemma 4.10 implies for $s \leq t < T$

$$\alpha_s^{p-1}\eta_s = Y_s = E\left[Y_t e^{-\int_s^t (p-1)\alpha_r dr}|\mathcal{F}_s\right] = \alpha_t^{p-1} e^{-\int_s^t (p-1)\alpha_r dr} E\left[\eta_t|\mathcal{F}_s\right].$$

Consequently, the random variable $E\left[\frac{\eta_t}{\eta_s}\middle|\mathcal{F}_s\right]$ is deterministic for all $s \leq t < T$ and hence η has uncorrelated multiplicative increments. □

We next show that the converse of Proposition 4.3 holds true as well: If η has uncorrelated multiplicative increments, then there exists an deterministic optimal control for (5).

Proposition 4.4. *Assume that η has uncorrelated multiplicative increments and satisfies the integrability assumptions* (**I1**) *and $\eta_T \in L^2(\Omega)$. Then*

$$Y_t = \frac{1}{\left(\int_t^T \frac{1}{E[\eta_s|\mathcal{F}_t]^{q-1}} ds\right)^{p-1}}$$

is the minimal solution to (8) *with singular terminal condition. The deterministic control*

$$x_t = \frac{1}{\int_0^T \frac{1}{E[\eta_s]^{q-1}} ds} \int_t^T \frac{1}{E[\eta_s]^{q-1}} ds$$

is optimal in (5). *In particular the optimal control rate is inversely proportional to $E[\eta_t]^{q-1}$.*

Proof. First note that we have by Jensen's inequality

$$\int_t^T \frac{1}{E[\eta_s|\mathcal{F}_t]^{q-1}} ds \geq (T-t)^q \frac{1}{\left(\int_t^T E[\eta_s|\mathcal{F}_t] ds\right)^{q-1}}.$$

This implies that Y is bounded from above as follows

$$Y_t \leq \frac{1}{(T-t)^p} E\left[\int_t^T \eta_s ds|\mathcal{F}_t\right]. \tag{24}$$

Next we use the fact from Lemma 4.1 that $E[\eta_s|\mathcal{F}_t] = \eta_t E\left[\frac{\eta_s}{\eta_t}\right] = \eta_t \frac{E[\eta_s]}{E[\eta_t]}$ for $s \geq t$ to rewrite Y as

$$Y_t = M_t \frac{1}{\left(\int_t^T \frac{1}{E[\eta_s]^{q-1}} ds\right)^{p-1}}$$

where the process M denotes the martingale $M_t = \frac{\eta_t}{E[\eta_t]}$. Moreover, we have by assumption $E[M_T^2] = E[\eta_T^2]/E[\eta_T]^2 < \infty$. Hence, M is a square integrable martingale. Let

$\phi \in \mathcal{M}^2(0,T)$ denote the integrand from its martingale representation. Then we obtain, by integration by parts,

$$\begin{aligned} dY_t &= (p-1)\frac{1}{E[\eta_t]^{q-1}}\frac{M_t}{\left(\int_t^T \frac{1}{E[\eta_s]^{q-1}}ds\right)^p}dt + \frac{\phi_t}{\left(\int_t^T \frac{1}{E[\eta_s]^{q-1}}ds\right)^{p-1}}dW_t \\ &= (p-1)\frac{Y_t^q}{\eta_t^{q-1}}dt + Z_t dW_t, \end{aligned}$$

with

$$Z_t = \frac{\phi_t}{\left(\int_t^T \frac{1}{E[\eta_s]^{q-1}}ds\right)^{p-1}}. \tag{25}$$

Hence, we have $Z \in \mathcal{M}^2(0,t)$ for every $t < T$. An application of the Burkholder-Davis-Gundy inequality as in the proof of Theorem 2.2 in combination with Inequality (24) yields $E[\sup_{0\le s\le t} Y_s^2] < \infty$ for all $t < T$. Hence, (Y, Z) is a solution to (8) with singular terminal condition $Y_T = \infty$.

The associated path x satisfies

$$\begin{aligned} x_t = \exp\left(-\int_0^t \left(\frac{Y_s}{\eta_s}\right)^{q-1} ds\right) &= \exp\left(-\int_0^t \frac{1}{E[\eta_s]^{q-1}\int_t^T \frac{1}{E[\eta_r]^{q-1}}dr}ds\right) \\ &= \frac{1}{\int_0^T \frac{1}{E[\eta_s]^{q-1}}ds}\int_t^T \frac{1}{E[\eta_s]^{q-1}}ds. \end{aligned}$$

In particular it follows from (25) that $x^{p-1}Z \in \mathcal{M}^2(0,T)$ and hence Proposition 3.4 yields that Y is the minimal solution of (8). Theorem 3.2 then implies optimality of x. □

If η is monotone in expectation, then we obtain the following result about the path of the optimal control.

Corollary 4.5. *Let η satisfy the assumptions of Proposition 4.4. If the mapping $t \mapsto E[\eta_t]$ is nondecreasing (nonincreasing), then the optimal control $x \in \mathcal{A}_0$ from Proposition 4.4 is a convex (concave) function of time.*

Proof. The optimal control rate from Proposition 4.4 is given by $\dot{x}_t = -\frac{1}{cE[\eta_t]^{q-1}}$ with $c = \int_0^T \frac{1}{E[\eta_s]^{q-1}}ds$. In particular $t \mapsto \dot{x}_t$ is nondecreasing (nonincreasing) if $t \mapsto E[\eta_t]$ is nondecreasing (nonincreasing). □

Proposition 4.4 includes the case where η is a martingale as a special case.

Corollary 4.6. *Let η be a positive martingale satisfying $1/\eta^{q-1} \in \mathcal{M}^1(0,T)$ and $\eta_T \in L^2(\Omega)$. Then $Y_t = \frac{\eta_t}{(T-t)^{p-1}}$ solves the BSDE (8) with singular terminal condition $Y_T = \infty$ and the control with constant control rate $x_t = 1 - \frac{t}{T}$ is optimal in (5).*

Proof. The process η^2 is a submartingale and hence $E[\eta_t^2] \leq E[\eta_T^2]$ for all $t \leq T$, which implies that $\eta \in \mathcal{M}^2(0,T)$. Moreover, Lemma 4.1 yields that η has uncorrelated multiplicative increments. Hence, all assumptions of Proposition 4.4 are satisfied which yields the claim. □

Another special case of Proposition 4.4 is the case where η is a deterministic function of time.

Corollary 4.7. *Assume that η is deterministic and satisfies $1/\eta^{q-1} \in L^1([0,T])$, $\eta \in L^2([0,T])$ and $\eta_T < \infty$. Then*

$$Y_t = \left(\frac{1}{\int_t^T \frac{1}{\eta_s^{q-1}} ds} \right)^{p-1}$$

solves (8) *with singular terminal condition $Y_T = \infty$ and the control*

$$x_t = \frac{\int_t^T \frac{1}{\eta_s^{q-1}} ds}{\int_0^T \frac{1}{\eta_s^{q-1}} ds} \tag{26}$$

is optimal in (5).

Remark 4.8. The results about the optimal control in Corollary 4.6 and Corollary 4.7 hold also true under weaker assumptions on the process η. In the martingale case it suffices to assume that η is a positive martingale with $E[\eta_T^2] < \infty$. Then Proposition 1.7 directly implies that the control with constant rate is optimal. In the deterministic case it is straightforward to show that under the integrability condition $1/\eta^{q-1} \in L^1([0,T])$ the function $\eta|\dot{x}|^{p-1}$ is constant for the control x from Equation (26). Then again Proposition 1.7 implies optimality of x.

A particular example for a process with uncorrelated multiplicative increments is the geometric Brownian motion.

Example 4.9. *Assume that η evolves according to a geometric Brownian motion*

$$d\eta_t = \mu \eta_t dt + \sigma \eta_t dW_t$$

with drift $\mu \in \mathbb{R}$, volatility $\sigma > 0$ and initial value $\eta_0 > 0$. In this case

$$\frac{\eta_t}{\eta_s} = e^{\left(\mu - \frac{\sigma^2}{2}\right)(t-s) + \sigma(W_t - W_s)}$$

for $s \leq t \leq T$ and hence η has uncorrelated multiplicative increments. Moreover we have $E[\eta_t|\mathcal{F}_s] = \eta_s e^{\mu(t-s)}$ and η satisfies the integrability conditions $\eta \in \mathcal{M}^2(0,T)$, $E[\eta_T^2 < \infty]$ and $\int_t^T \frac{1}{E[\eta_s]^{q-1}} ds < \infty$. In the case $\mu = 0$ the price impact process η is a martingale and

Corollary 4.6 yields that linear closure is optimal in (5)*. In the case* $\mu \neq 0$ *Proposition 4.4 implies that a solution of* (8) *is given by*

$$Y_t = \mu(q-1)^{p-1} \frac{\eta_t}{(1 - e^{-\mu(q-1)(T-t)})^{p-1}}$$

and that the optimal control for (5) *satisfies*

$$x_t = \frac{e^{-\mu(q-1)t} - e^{-\mu(q-1)T}}{1 - e^{-\mu(q-1)T}}.$$

Appendix

Here we provide a uniqueness result about linear BSDEs with a driver that is unbounded from below.

Lemma 4.10. *Let* $(\alpha_t)_{0\leq t\leq T}$ *and* $(\beta_t)_{0\leq t\leq T}$ *be progressively measurable processes and* ξ *a* $\mathcal{F}_T$*-measurable random variable. Assume that* α *is bounded from above. Any solution* (Y, Z) *with* $Z \in \mathcal{M}^2(0,T)$ *to the linear BSDE*

$$dY_t = (\alpha_t Y_t + \beta_t)\, dt + Z_t dW_t$$

with $Y_T = \xi$ *admits the representation*

$$Y_t = E\left[\xi e^{\int_t^T \alpha_s ds} + \int_t^T e^{\int_t^s \alpha_u du} \beta_s ds | \mathcal{F}_t\right].$$

Proof. Let (Y, Z) be a solution. Set

$$\varphi_t = Y_t e^{\int_0^t \alpha_s ds} + \int_0^t e^{\int_0^s \alpha_u du} \beta_s ds.$$

Then by integration by parts we obtain

$$d\varphi_t = e^{\int_0^t \alpha_s ds} Z_t dW_t.$$

Since α is bounded from above and $Z \in \mathcal{M}^2(0,T)$ the integrand belongs to $\mathcal{M}^2(0,T)$ as well. Therefore φ is a martingale and consequently

$$\varphi_t = E[\varphi_T | \mathcal{F}_t] = E\left[\xi e^{\int_0^T \alpha_s ds} + \int_0^T e^{\int_0^s \alpha_u du} \beta_s ds | \mathcal{F}_t\right],$$

which yields the claim. □

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