Optional splitting formula in a progressively enlarged filtration

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Abstract

Let $\mathbb{F}$ be a filtration and τ be a random time. Let $\mathbb{G}$ be the progressive enlargement of $\mathbb{F}$ with τ. We study the validity of the following formula, called optional splitting formula : For any $\mathbb{G}$-optional process Y, there exist a $\mathbb{F}$-optional process Y' and a function Y'' defined on $[0,\infty]\times(\mathbb{R}_+\times\Omega)$ being $\mathcal{B}[0,\infty]\otimes\mathcal{O}(\mathbb{F})$ measurable, such that

$$Y = Y'\mathbb{1}_{[0,\tau)} + Y''(\tau)\mathbb{1}_{[\tau,\infty)}$$

We are interested in this formula, because it has been taken for granted in number of recent works in credit risk modeling, whilst such a formula can not be true in general. Sufficient conditions will be given for the validity of the above formula as well as of its extension in the case of multiple random times.

Keywords : optional process, progressive enlargement of filtrations, credit risk modeling, conditional density hypothesis

MSC Classification : 60G07, 60G44, 91G40, 97M30.

1 Intoduction

In credit risk modeling with progressive enlargement of filtration, one may be concerned with measurability problems. For example, given a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ on some space Ω and a random time $\tau \geq 0$ which is not a $\mathbb{F}$ stopping time, one may attempt to apply the monotone class theorem to prove results on $\cap_{s>t}(\sigma(\tau\wedge s)\vee\mathcal{F}_s)$ (for a $t\geq 0$), and therefore, hope to have the following equality

$$\cap_{s>t}(\sigma(\tau\wedge s)\vee\mathcal{F}_s) = \sigma(\tau\wedge t)\vee\mathcal{F}_t.$$

While it can not be valid in general as explained in [31], such an equality takes place in some models [4, 10, 32] (See also Subsection 3.2 below).

To give another example, we recall the formula in [24, Lemme(4.4)] : Let $\mathbb{G}$ denotes the filtration of $\cap_{s>t}(\sigma(\tau \wedge s) \vee \mathcal{F}_s), t \geq 0$. Let $\mathcal{P}(\mathbb{G})$, $\mathcal{O}(\mathbb{G})$, $\mathcal{P}(\mathbb{F})$ and $\mathcal{O}(\mathbb{F})$ be the usual notations denoting the predictable or optional σ-algebras with respect to different filtrations. Then, for any $\mathbb{G}$-predictable process Y, there exist a $\mathbb{F}$-predictable process Y' and a function Y'' defined on $[0,\infty] \times (\mathbb{R}_+ \times \Omega)$ being $\mathcal{B}[0,\infty] \otimes \mathcal{P}(\mathbb{F})$ measurable, such that

$$Y = Y' \mathbb{1}_{[0,\tau]} + Y''(\tau)\mathbb{1}_{(\tau,\infty)} \tag{1}$$

This formula has been very useful in [24]. In recent works on credit risk modeling, an optional version of the above formula slips into computations : For any $\mathbb{G}$-optional process Y, there exist a $\mathbb{F}$-optional process Y' and a function Y'' defined on $[0,\infty] \times (\mathbb{R}_+ \times \Omega)$ being $\mathcal{B}[0,\infty] \otimes \mathcal{O}(\mathbb{F})$ measurable, such that

$$Y = Y' \mathbb{1}_{[0,\tau)} + Y''(\tau)\mathbb{1}_{[\tau,\infty)} \tag{2}$$

Taken for granted, this optional version formula has been used in number of papers. One might believe that the optional version formula (2) can be proved by the monotone class theorem in the same way as formula (1) has been proved, or that (2) follows from (1) in taking the right limit in (1). Unfortunately, this idea, as well as the optional version formula, can not be valid in general as pointed out by [1]. See also [12, 13] for complementary analysis.

Number of works on credit risk modeling are concerned by the above mentioned measurability problems directly or indirectly (see, for example, [4, 5, 9, 15, 25, 27, 26, 32]). These measurability problems lie in the differentiation between the filtrations $\mathbb{G}$ and $(\sigma(\tau \wedge t) \vee \mathcal{F}_t)_{t\geq 0}$ (or $(\sigma(\tau \mathbb{1}_{\{\tau \leq t\}} + \infty \mathbb{1}_{\{t<\tau\}}) \vee \mathcal{F}_t)_{t\geq 0}$). They are linked to the interchangeability problem between intersection and supremum of σ-algebras. See [8, 16, 17, 18, 31, 33] to have some idea on this subtle interchangeability problem. Other related topics are, for example, the equality between $\mathcal{G}_{\tau-}$ and $\mathcal{G}_\tau$ ([24, 21]), or the notion of multiplicity of σ-algebras ([3, 2, 11, 30]). In this paper we study the consequences of the optional version formula (2) and we find sufficient conditions for it to be valid. We study also its extension in the case of multiple random times.

2 Set up

In this paper we work on a probability space $(\Omega, \mathcal{A}, \mathbb{Q})$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ of sub-σ-algebras in $\mathcal{A}$. We assume that $\mathbb{F}$ is right continuous and $\mathcal{F}_0$ contains $\mathcal{N}^{\mathcal{F}_\infty}$, where, for a σ-algebra $\mathcal{T}$ contained in $\mathcal{A}$, $\mathcal{N}^{\mathcal{T}}$ denotes the σ-algebra generated by $\{A \subset \Omega : \exists B \in \mathcal{T}, A \subset B, \mathbb{Q}[B] = 0\}$.

Let τ be a random time (i.e. a random variable taking values in $\mathbb{R}_+$) on $(\Omega, \mathcal{A})$. We introduce $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ the progressive enlargement of the filtration $\mathbb{F}$ with the random time τ, defined by

$$\mathcal{G}_t = \mathcal{N}^{\sigma(\tau)\vee\mathcal{F}_\infty} \vee (\cap_{s>t}(\mathcal{F}_s \vee \sigma(\tau \wedge s))),\ t \geq 0.$$

(Using the result in [31], we check that

$$\mathcal{N}^{\sigma(\tau)\vee\mathcal{F}_\infty} \vee (\cap_{s>t}(\mathcal{F}_s \vee \sigma(\tau \wedge s))) = \cap_{s>t}(\mathcal{N}^{\sigma(\tau)\vee\mathcal{F}_\infty} \vee (\mathcal{F}_s \vee \sigma(\tau \wedge s)))$$

so that $\mathbb{G}$ is a right continuous filtration.) We denote by $\mathcal{O}(\mathbb{F})$ (resp. $\mathcal{O}(\mathbb{G})$) the $\mathbb{F}$-optional (resp. $\mathbb{G}$-optional) σ-algebra. We define in a similar way the predictable σ-algebra $\mathcal{P}(\mathbb{F})$ and $\mathcal{P}(\mathbb{G})$.

Other notational conventions

For two elements a, b in $[0, \infty]$ we denote

$$a \nmid b = \begin{cases} a & \text{si } a \leq b \\ \\ \infty & \text{si } a > b \end{cases}$$

Let T be a $\mathbb{G}$ stopping time. Let $A \in \mathcal{G}_T$. We denote by $T_A = T\mathbb{1}_A + \infty\mathbb{1}_A$ (called the restrction of T on A). T_A is again a $\mathbb{G}$ stopping time (cf. [19]).

For a random time R, the graph $[R]$ is defined as $[R] = \{(t, \omega) : t \in \mathbb{R}_+, t = R(\omega)\}$. Note that, if $R = \infty$, $[R] = \emptyset$.

If a random variable ξ is measurable with respect to a σ-algebra $\mathcal{T}$, we will write $\xi \in \mathcal{T}$ and say that "ξ is in $\mathcal{T}$".

For any random time R on Ω, we denote (cf. [24])

$$\mathcal{F}_R := \sigma\{X_R \mathbb{1}_{\{R<\infty\}} : \ X \text{ a } \mathbb{F}\text{-optional process}\}$$
$$\mathcal{F}_{R+} := \sigma\{X_R \mathbb{1}_{\{R<\infty\}} : \ X \text{ a } \mathbb{F}\text{-progressively measurable process}\}$$

For an $\mathbb{Q}$-integrable random variable $\xi \in \mathcal{A}$, we denote by $\xi\!\!\!|^{\mathbb{Q}-\mathbb{G}}$ the $(\mathbb{Q}, \mathbb{G})$-martingale $\mathbb{Q}[\xi|\mathcal{G}_t], t \geq 0$. If the probability $\mathbb{Q}$ is clearly determined, we denote it by $\xi\!\!\!|^{\mathbb{G}}$. If we replace the filtration $\mathbb{G}$ by $\mathbb{F}$, we define similarly $\xi\!\!\!|^{\mathbb{Q}-\mathbb{F}}$ or $\xi\!\!\!|^{\mathbb{F}}$.

In this paper, an (in)equation between two measurable functions (resp. two random processes) is to be understood as an almost sure relation (resp. an indistinguishable relation).

Let D be a subset of Ω and $\mathcal{T}$ be a σ-algebra on Ω. We denote by $D \cap \mathcal{T}$ the family of all subsets $D \cap A$ with A running through $\mathcal{T}$. If D itself is an element in $\mathcal{T}$, $D \cap \mathcal{T}$ coincides with $\{A \in \mathcal{T} : A \subset D\}$. We will use the symbol "+" to present the union of two disjoint subsets. For two disjoint sets D_1, D_2 in Ω, for two families $\mathcal{T}_1, \mathcal{T}_2$ of sets in Ω, we denote by $D_1 \cap \mathcal{T}_1 + D_2 \cap \mathcal{T}_2$ the family of sets $D_1 \cap B_1 + D_2 \cap B_2$ where $B_1 \in \mathcal{T}_1, B_2 \in \mathcal{T}_2$. For two sets D and D', we have

$$D \cap \mathcal{T} \subset D' \cap D \cap \mathcal{T} + D'^c \cap D \cap \mathcal{T}$$

If $D' \in \mathcal{T}$, we have

$$D \cap \mathcal{T} = D' \cap D \cap \mathcal{T} + D'^c \cap D \cap \mathcal{T}$$

Lemma 2.1 *Let $\mathcal{T}$ and $\mathcal{T}'$ be two σ-algebras. Let D be a set. Let $(A_i)_{i\geq 1}$ be a sequence of sets. Suppose that $D \cap A_i \cap \mathcal{T} \subset D \cap A_i \cap \mathcal{T}'$ for all $i \geq 1$. If $A_i \in \mathcal{T}'$ for all $i \geq 1$, we have also*

$$D \cap (\cup_{i\geq 1} A_i) \cap \mathcal{T} \subset D \cap (\cup_{i\geq 1} A_i) \cap \mathcal{T}'$$

Proof. For any $C \in \mathcal{T}$, for any $i \geq 1$, there exists $C_i \in \mathcal{T}'$ such that $D \cap A_i \cap C = D \cap A_i \cap C_i$. Let

$$B_1 = A_1, \ B_k = A_k \setminus (\cup_{i=1}^{k-1} A_i)$$

Then, $B_k \in \mathcal{T}'$ and $D \cap C \cap B_k = D \cap C_k \cap B_k$. Therefore,

$$\begin{aligned} D \cap (\cup_{i\geq 1} A_i) \cap C &= (\cup_{i\geq 1} A_i) \cap D \cap (\cup_{i\geq 1} B_i) \cap C \\ &= (\cup_{i\geq 1} A_i) \cap D \cap (\cup_{k\geq 1} (B_k \cap C_k)) \end{aligned}$$

This proves the result. ∎

3 $\mathbb{G}$-Optional splitting formula at a random time τ with respect to $\mathbb{F}$

3.1 Definition

Notation. Let E be a space. Let $Y(\theta,t,\omega)$ be a map defined on $(\theta,t,\omega) \in E \times (\mathbb{R}_+ \times \Omega)$. For $\theta \in E$, we will denote by $Y(\theta)$ (resp. by $Y_t(\theta)$ for $t \in \mathbb{R}_+$) the map $(s,\omega) \to Y(\theta,s,\omega)$ (resp. $\omega \to Y(\theta,t,\omega)$). For a map Υ defined on Ω into E, $Y(\Upsilon)$ denotes the map $(s,\omega) \to Y(\Upsilon(\omega),s,\omega)$.

Definition 3.1 *We say that the $\mathbb{G}$-optional splitting formula holds at τ with respect to $\mathbb{F}$, if, for any $\mathbb{G}$-optional process Y, there exists a process $Y' \in \mathcal{O}(\mathbb{F})$ and a function Y'' defined on $[0,\infty] \times (\mathbb{R}_+ \times \Omega)$ being $\mathcal{B}[0,\infty] \otimes \mathcal{O}(\mathbb{F})$-measurable, such that*

$$Y = Y' \mathbb{1}_{[0,\tau)} + Y''(\tau)\mathbb{1}_{[\tau,\infty)}$$

We will denote $\mathfrak{p}^{[0,\tau)}Y = Y'$ and $\mathfrak{p}^{[\tau,\infty)}Y = Y''$.

Remark 3.2 The term "splitting" is of two folds. It invokes obviously that the formula is splitted at the random time τ. But, more importantly, it invokes also that the measurability of $Y''(\tau)$ is "splitted" into two components $\sigma(\tau)$ and $\mathcal{O}(\mathbb{F})$ ($Y'' \in \mathcal{B}[0,\infty] \otimes \mathcal{O}(\mathbb{F})$). Theorem 3.4 and 3.6 below compared to [24, 31] show that these splitting properties constitute a really strong condition.

3.2 Some consequences

Lemma 3.3 *Let $F \in \mathcal{B}[0,\infty] \otimes \mathcal{O}(\mathbb{F})$. Then, the map $\omega \in \Omega \to F_{\tau(\omega)}(\tau(\omega),\omega)$ is in $\mathcal{F}_\tau$*

Proof By the monotone class theorem, we need only to check the stated measurability for F in the particular form $F_t(s,\omega) = h(s)f_t(\omega)$ where $h \in \mathcal{B}[0,\infty]$ and $f \in \mathcal{O}(\mathbb{F})$. But, then, the assertion is an evidence. ∎

Theorem 3.4 *Assuming the optional splitting formula. We have necessarily $\mathcal{G}_\tau = \mathcal{F}_\tau$. In particular, $\mathcal{F}_{\tau+} = \mathcal{F}_\tau$.*

Proof We know that $\mathcal{G}_\tau$ is generated by X_τ (cf. [19, Corollary 3.23]), where X runs throught the family $\mathcal{O}(\mathbb{G})$. By the assumption of the lemma, there exists a $F \in \mathcal{B}[0,\infty] \otimes \mathcal{O}(\mathbb{F})$, such that

$$X_{\tau(\omega)}(\omega) = F_{\tau(\omega)}(\tau(\omega),\omega)$$

According to the Lemma 3.3, then, $X_\tau \in \mathcal{F}_\tau$. This proves that $\mathcal{G}_\tau \subset \mathcal{F}_\tau$. We have in fact an equality, because the inverse inclusion is an evidence.

As for the second assertion of the theorem, it is the consequnence of the relations $\mathcal{F}_\tau \subset \mathcal{F}_{\tau+} \subset \mathcal{G}_\tau = \mathcal{F}_\tau$. ∎

Corollary 3.5 *The optional splitting formula can not hold in general.*

Proof This is because, according to [24, Proposition (5.6)], $\mathcal{F}_\tau \neq \mathcal{F}_{\tau+}$ for some $\mathbb{F}$ and τ. ∎

Theorem 3.6 *If the optional splitting formula holds, for any $t \geq 0$, for any $A \in \mathcal{G}_t$, there exists $B \in \sigma(\tau \nmid t) \vee \mathcal{F}_t$ such that $\mathbb{1}_A = \mathbb{1}_B$. This means, in particular, that $\mathcal{N}^{\sigma(\tau)\vee\mathcal{F}_\infty} \vee \sigma(\tau \nmid t) \vee \mathcal{F}_t$, $t \geq 0$, form a right continuous filtration.*

Proof Let $0 \leq t < \infty$ and $Y \in \mathcal{O}(\mathbb{G})$. We write the optional splitting formula

$$Y = Y' \mathbb{1}_{[0,\tau)} + Y''(\tau) \mathbb{1}_{[\tau,\infty)}$$

where $Y' = \mathfrak{p}^{[0,\tau)}Y$ and $Y'' = \mathfrak{p}^{[\tau,\infty)}Y$. Since $Y'_t \in \mathcal{F}_t, Y''_t(\tau) \in \sigma(\tau) \vee \mathcal{F}_t$, we have

$$\begin{array}{rcl} Y_t & = & Y'_t \mathbb{1}_{\{t<\tau\}} + Y''_t(\tau) \mathbb{1}_{\{\tau\leq t\}} \\ & \in & \{t < \tau\} \cap \mathcal{F}_t + \{\tau \leq t\} \cap (\sigma(\tau) \vee \mathcal{F}_t) \\ & = & \{t < \tau\} \cap (\sigma(\tau \nmid t) \vee \mathcal{F}_t) + \{\tau \leq t\} \cap (\sigma(\tau \nmid t) \vee \mathcal{F}_t) \\ & = & \sigma(\tau \nmid t) \vee \mathcal{F}_t \text{ (Lemma 2.1)} \end{array}$$

This being true for any $Y \in \mathcal{O}(\mathbb{G})$ and $t \geq 0$, we conclude that $\mathcal{N}^{\mathcal{F}_\infty \vee \sigma(\tau)} \vee \sigma(\tau \nmid t) \vee \mathcal{F}_t$, $t \geq 0$, form a right continuous filtration. The theorem is proved. ∎

Remark 3.7 As a matter of fact, we can not replace in the above lemma $\tau \nmid t$ by $\tau \wedge t$. In general,

$$\{t < \tau\} \cap (\sigma(\tau \wedge t) \vee \mathcal{F}_t) + \{\tau \leq t\} \cap (\sigma(\tau \wedge t) \vee \mathcal{F}_t) \neq \sigma(\tau \wedge t) \vee \mathcal{F}_t$$

because $\{\tau \leq t\}$ (or more precisely $\{\tau = t\}$) is not necessarily in $\sigma(\tau \wedge t) \vee \mathcal{F}_t$. It is a tricky trap. See [12, Chapitre IV, paragraph 104] which comments on [11]. (The problem no longer arises if $\{\tau = t\}$ is negligible and if $\mathbb{F}$ is complete.) See also [28, Chapter VI.3].

3.3 Predictable processes

Lemma 3.8 *The $\mathbb{G}$-predictable processes satisfy the optional splitting formula.*

Proof The $\mathbb{G}$-predictable processes are generated (in the sense of monotone class) by the processes of the form

$$g(\tau \wedge a) \mathbb{1}_A \mathbb{1}_{]a,b]} + \mathbb{1}_B \mathbb{1}_{[0]}$$

where g is a bounded Borel function, $0 \leq a < b$ are real numbers, $A \in \mathcal{F}_a$ and $B \in \mathcal{G}_0$. We check directly that the process $g(\tau \wedge a) \mathbb{1}_A \mathbb{1}_{]a,b]}$ satisfies the optional splitting formula. As for $\mathbb{1}_B \mathbb{1}_{[0]}$, by [24, Lemme(4.4)], there exist $B', B'' \in \mathcal{F}_0$ such that

$$\mathbb{1}_B \mathbb{1}_{[0]} = \mathbb{1}_{B'} \mathbb{1}_{\{0<\tau\}} \mathbb{1}_{[0]} + \mathbb{1}_{B''} \mathbb{1}_{\{\tau=0\}} \mathbb{1}_{[0]} = \mathbb{1}_{B'} \mathbb{1}_{[0]} \mathbb{1}_{[0,\tau)} + \mathbb{1}_{B''} \mathbb{1}_{[0]} \mathbb{1}_{[\tau,\infty)}$$

i.e., $\mathbb{1}_B \mathbb{1}_{[0]}$ also satisfies the optional splitting formula. By monotone class theorem, we prove the lemma. ∎

4 Local optional splitting formula

4.1 Definition and basic properties

Definition 4.1 *Let A be a $\mathbb{G}$-optional set. We say that the local $\mathbb{G}$-optional splitting formula holds on A (at τ with respect to $\mathbb{F}$), if, for any $\mathbb{G}$-optional process Y, there exists a $Y' \in \mathcal{O}(\mathbb{F})$ and a function Y'' defined on $[0,\infty] \times (\mathbb{R}_+ \times \Omega)$ being $\mathcal{B}[0,\infty] \otimes \mathcal{O}(\mathbb{F})$ measurable, such that*

$$Y \mathbb{1}_A = (Y' \mathbb{1}_{[0,\tau)} + Y''(\tau) \mathbb{1}_{[\tau,\infty)}) \mathbb{1}_A$$

We will denote $\mathfrak{p}_A^{[0,\tau)} Y = Y'$ and $\mathfrak{p}_A^{[\tau,\infty)} Y = Y''$.

If the above formula holds for a specific process X, we will say that X satisfies the local optional splitting formula on A.

Obviously, this definition coincides with Definition 3.1 when $A = \mathbb{R}_+$. We have immediately the following properties.

Lemma 4.2 *Let A be a $\mathbb{G}$-optional set. Let $\mathcal{S}_A$ be the family of all $\mathbb{G}$-optional processes which satisfy the local optional splitting formula on A. Then, $\mathcal{S}_A$ is a linear space, closed by simple limit, by* $\inf, \max$*, by product operations.*

Lemma 4.3 *Let A, B be two $\mathbb{G}$-optional sets. If the local optional splitting formula holds on A and $B \subset A$. Then, the local optional splitting formula holds on B.*

4.2 Local optional splitting formula on predictable sets

Lemma 4.4 *Let A be a $\mathbb{G}$-predictable set. Then, the local optional splitting formula holds on A if and only if, for any $\mathbb{G}$-optional process Y, $Y \mathbb{1}_A$ satisfies the optional splitting formula on the whole time space $\mathbb{R}_+$.*

Proof Let Y be a $\mathbb{G}$-optional process. Suppose that $Y \mathbb{1}_A$ satisfies the optional splitting formula on the whole time space $\mathbb{R}_+$. Let respectively $Y' = \mathfrak{p}^{[0,\tau)}(Y \mathbb{1}_A)$ and $Y'' = \mathfrak{p}^{[\tau,\infty)}(Y \mathbb{1}_A)$. We have

$$Y \mathbb{1}_A = Y' \mathbb{1}_{[0,\tau)} + Y''(\tau) \mathbb{1}_{[\tau,\infty)}$$

Since $\mathbb{1}_A = \mathbb{1}_A^2$, we have also

$$Y \mathbb{1}_A = Y \mathbb{1}_A^2 = (Y' \mathbb{1}_{[0,\tau)} + Y''(\tau) \mathbb{1}_{[\tau,\infty)}) \mathbb{1}_A$$

i.e. the local optional splitting formula on A.

Conversely, suppose that the local optional splitting formula on A holds. Let respectively $C' = \mathfrak{p}_A^{[0,\tau)} Y$ and $C'' = \mathfrak{p}_A^{[\tau,\infty)} Y$. We write then

$$Y \mathbb{1}_A = (C' \mathbb{1}_{[0,\tau)} + C''(\tau) \mathbb{1}_{[\tau,\infty)}) \mathbb{1}_A$$

Note that A is $\mathbb{G}$-predictable. According to Lemma 3.8, $\mathbb{1}_A$ satisfies the global optional splitting formula. Let $B' = \mathfrak{p}^{[0,\tau)} \mathbb{1}_A$ and $B'' = \mathfrak{p}^{[\tau,\infty)} \mathbb{1}_A$. We have then

$$\begin{aligned} Y \mathbb{1}_A &= (C' \mathbb{1}_{[0,\tau)} + C''(\tau) \mathbb{1}_{[\tau,\infty)}) \mathbb{1}_A \\ &= C' B' \mathbb{1}_{[0,\tau)} + C''(\tau) B''(\tau) \mathbb{1}_{[\tau,\infty)} \end{aligned}$$

This is a global optional splitting formula for $Y \mathbb{1}_A$. ∎

Lemma 4.5 *Let $(A_i)_{i=1}^{\infty}$ be a sequence of $\mathbb{G}$-predictable sets. Suppose that the local optional splitting formula holds on each $A_i, i \geq 1$. Then, the local optional splitting formula holds on the union $\cup_{i=1}^{\infty} A_i$.*

Proof Let Y be any $\mathbb{G}$-optional process. We apply Lemma 4.4 in this proof. It is then enough to prove that $Y\mathbb{1}_{\cup_{i=1}^{\infty} A_i}$ satisfies the global optional splitting formula.

By induction, we see that $Y\mathbb{1}_{\cup_{i=1}^{k} A_i}$ satisfies the global optional splitting formula for any interger k. Actually, for $k = 1$, it is the case. Suppose that for an integer $k = n$, $Y\mathbb{1}_{\cup_{i=1}^{n} A_i}$ satisfies the global optional splitting formula. Let us prove the same for $k = n + 1$.

We write the identity :

$$Y\mathbb{1}_{\cup_{i=1}^{n+1} A_i} = Y\mathbb{1}_{\cup_{i=1}^{n} A_i} + Y\mathbb{1}_{A_{n+1}} - Y\mathbb{1}_{A_{n+1}\cap(\cup_{i=1}^{n} A_i)}$$

By assumption, $Y\mathbb{1}_{\cup_{i=1}^{n} A_i}$ and $Y\mathbb{1}_{A_{n+1}}$ satisfy the global optional splitting formula. For the term $Y\mathbb{1}_{A_{n+1}\cap(\cup_{i=1}^{n} A_i)}$, we write it in the form

$$Y\mathbb{1}_{A_{n+1}\cap(\cup_{i=1}^{n} A_i)} \quad = \quad (Y\mathbb{1}_{\cup_{i=1}^{n} A_i})(\mathbb{1}_{A_{n+1}})$$

$\mathbb{1}_{A_{n+1}}$ satisfies the global optional splitting formula, because A_{n+1} is $\mathbb{G}$-predictable (Lemma 3.8). $Y\mathbb{1}_{\cup_{i=1}^{n} A_i}$ satisfies the global optional splitting formula by assumption. Applying Lemma 4.2, we conclude that $Y\mathbb{1}_{\cup_{i=1}^{n+1} A_i}$ satisfies also the global optional splitting formula.

Now, taking the limit on $Y\mathbb{1}_{\cup_{i=1}^{k} A_i}$ when $k \to \infty$, we conclude that $Y\mathbb{1}_{\cup_{i=1}^{\infty} A_i}$ satisfies the global optional splitting formula (cf. Lemma 4.2). This proves the lemma. ∎

4.3 Local optional splitting formula on a left-closed right-open interval $[S, T)$

Lemma 4.6 *Let S, T be two $\mathbb{G}$-stopping times. To have the local optional splitting formula on $[S, T)$, it is necessary and sufficient that, for all bounded $(\mathbb{Q}, \mathbb{G})$-martingale X such that $X_T \in \mathcal{G}_{T-}$, X satisfies the optional splitting formula on $[S, T)$.*

Proof The necessity of the condition is an obvious fact. Let us consider the sufficiency. We follow the argument in [13, Chapter XX, section 22]. Let $\xi \in \mathcal{G}_\infty$ be a bounded random variable. Let us abuse the notation ξ to denote also $\xi|^{\mathbb{G}}$. Let

$$Y = \Delta_T \xi \mathbb{1}_{[T,\infty)} - (\Delta_T \xi \mathbb{1}_{[T,\infty)})^{\mathbb{G}-(p)}$$

and $X = \xi^T - Y$, where $\bullet^{\mathbb{G}-(p)}$ denotes the $(\mathbb{Q}, \mathbb{G})$-predictable dual projection. Note that X, Y are $(\mathbb{Q}, \mathbb{G})$-martingales. Because $(\Delta_T \xi \mathbb{1}_{[T,\infty)})^{\mathbb{G}-(p)}$ is a $\mathbb{G}$-predictable process, we have

$$\Delta_T X = \Delta_T (\Delta_T \xi \mathbb{1}_{[T,\infty)})^{\mathbb{G}-(p)} \in \mathcal{G}_{T-}$$

(cf. [19]) so that $X_T \in \mathcal{G}_{T-}$. We write now

$$\xi = (\xi - \xi^T) + X + \Delta_T \xi \mathbb{1}_{[T,\infty)} - (\Delta_T \xi \mathbb{1}_{[T,\infty)})^{\mathbb{G}-(p)}.$$

For the terms on the right hand of the above identity, $(\xi - \xi^T) + \Delta_T \xi \mathbb{1}_{[T,\infty)}$ satisfy clearly the optional splitting formula on $[S, T)$; by assumption X satisfies the

optional splitting formula on $[S,T)$; $(\Delta_T \xi \mathbb{1}_{[T,\infty)})^{\mathbb{G}-(p)}$ being $\mathbb{G}$-predictable, satisfies the optional splitting formula thanks to Lemma 3.8. Consequently, ξ satisfies the optional splitting formula on $[S,T)$ ((cf. Lemma 4.2)).

Introduce $\mathfrak{A}$ the family of all bounded $Y \in \mathcal{B}[0,\infty) \otimes \mathcal{G}_\infty$ such that ${}^{\mathbb{G}-(o)}Y$ satisfies the optional splitting formula on $[S,T)$, where ${}^{\mathbb{G}-(o)}\centerdot$ denotes the $(\mathbb{Q},\mathbb{G})$-optional projection. We check that $\mathfrak{A}$ is a functional monotone class (cf. [29, 19]) and, according to the preceding result, $\mathfrak{A}$ contains all the random variables $\mathbb{1}_{(a,b]}\xi$, where $a,b \in \mathbb{R}_+$ and ξ is a bounded random variable in $\mathcal{G}_\infty$. By monotone class theorem, $\mathfrak{A}$ contians all bounded $\mathcal{B}[0,\infty) \otimes \mathcal{G}_\infty$-measurable random variables. This implies that all bounded $\mathbb{G}$-optional process satisfies the local optional splitting formula on $[S,T)$. ∎

4.4 Local optional splitting formula on the graph of a stopping time

By the monotone class theorem we obtain the following lemma.

Lemma 4.7 *Let R be a $\mathbb{G}$-stopping time. For any random variable $\xi \in \mathcal{F}_R$, there exists a $\mathbb{F}$-optional process Y such that $\mathbb{1}_{\{R<\infty\}}\xi = \mathbb{1}_{\{R<\infty\}}Y_R$.*

In the same vein we have :

Lemma 4.8 *Let R be a $\mathbb{G}$-stopping time. Let $\zeta \in \mathcal{N}^{\sigma(\tau)\vee\mathcal{F}_\infty} \vee \sigma(\tau) \vee \mathcal{F}_\infty$ be a random variable. Then, $\zeta \in \mathcal{N}^{\sigma(\tau)\vee\mathcal{F}_\infty} \vee \sigma(\tau) \vee \mathcal{F}_R$, if and only if there exists a $Y \in \mathcal{B}[0,\infty] \otimes \mathcal{O}(\mathbb{F})$ such that*

$$\mathbb{1}_{\{R<\infty\}}Y_R(\tau) = \mathbb{1}_{\{R<\infty\}}\zeta$$

Proof "If" part Let $\hat{\mathcal{C}}$ be the family of all function Y defined on $[0,\infty] \times (\mathbb{R}_+ \times \Omega)$ such that $Y_R(\tau) \in \sigma(\tau) \vee \mathcal{F}_R$. $\hat{\mathcal{C}}$ is a functional monotone class, containing the functions $g(t)Z_s(\omega)$, where g is a Borel function on $[0,\infty]$ and $Z \in \mathcal{O}(\mathbb{F})$. By monotone class theorem, $\hat{\mathcal{C}}$ contains any function Y in $\mathcal{B}[0,\infty] \otimes \mathcal{O}(\mathbb{F})$.

If $\mathbb{1}_{\{R<\infty\}}Y_R(\tau) = \mathbb{1}_{\{R<\infty\}}\zeta$, we have

$$\begin{array}{rcl} \zeta & = & \mathbb{1}_{\{R<\infty\}}Y_R(\tau) + \mathbb{1}_{\{R=\infty\}}\zeta \\ & \in & \{R<\infty\} \cap (\sigma(\tau) \vee \mathcal{F}_R) + \{R=\infty\} \cap (\mathcal{N}^{\sigma(\tau)\vee\mathcal{F}_\infty} \vee \sigma(\tau) \vee \mathcal{F}_R) \\ & \subset & \mathcal{N}^{\sigma(\tau)\vee\mathcal{F}_\infty} \vee \sigma(\tau) \vee \mathcal{F}_R \end{array}$$

"Only if" part Let $\mathcal{C}$ be the family of all functions on Ω which satisfy the stated condition. Then, $\mathcal{C}$ is a functional monotone class and contains the random variables of form $\mathbb{1}_B g(\tau)\xi$, where $B \in \mathcal{N}^{\sigma(\tau)\vee\mathcal{F}_\infty}$, g is a bounded Borel function and $\xi \in \mathcal{F}_R$ (see Lemma 4.7). Applying the monotone class theorem, we conclude that $\mathcal{C}$ contians all $\mathcal{N}^{\sigma(\tau)\vee\mathcal{F}_\infty} \vee \sigma(\tau) \vee \mathcal{F}_R$-measurable random variables. ∎

Theorem 4.9 *Let us denote $\mathcal{N}^{\sigma(\tau)\vee\mathcal{F}_\infty}$ by $\mathcal{N}$. Let R be a $\mathbb{G}$-stopping time. Then, the local optional splitting formula holds on the graph $[R]$, if and only if*

$$\{R<\infty\} \cap \mathcal{G}_R = \{R<\infty\} \cap (\mathcal{N} \vee \sigma(\tau \nmid R) \vee \mathcal{F}_R)$$

Proof "Only if" part Suppose that the local optional splitting formula holds on the graph $[R]$. Let Y be any $\mathbb{G}$-optional process. Let $Y' = \mathfrak{p}^{[0,\tau)}_{[R]}Y$ and $Y'' = \mathfrak{p}^{[\tau,\infty)}_{[R]}Y$. Let $\mathcal{T}^0$ be the trivial σ-algebra : $\mathcal{T}^0 = \{\emptyset, \Omega\}$. Note that $R \in \mathcal{F}_R, \{\tau \leq R < \infty\} \in \{R < \infty\} \cap (\sigma(\tau \nmid R) \vee \mathcal{F}_R)$, we have

$$\begin{array}{rcl} & Y_R \mathbb{1}_{\{R<\infty\}} & \\ = & Y'_R \mathbb{1}_{\{0\leq R<\tau\}} + Y''_R(\tau)\mathbb{1}_{\{\tau\leq R<\infty\}} & \\ \in & \{0 \leq R < \tau\} \cap \mathcal{F}_R + \{\tau \leq R < \infty\} \cap (\sigma(\tau) \vee \mathcal{F}_R) + \{R = \infty\} \cap \mathcal{T}^0 & \\ = & \{0 \leq R < \tau\} \cap (\sigma(\tau \nmid R) \vee \mathcal{F}_R) + \{\tau \leq R < \infty\} \cap (\sigma(\tau \nmid R) \vee \mathcal{F}_R) + \{R = \infty\} \cap \mathcal{T}^0 & \\ = & \{R < \infty\} \cap (\sigma(\tau \nmid R) \vee \mathcal{F}_R) + \{R = \infty\} \cap \mathcal{T}^0 & \end{array}$$

This measurability relation yields

$$\{R < \infty\} \cap \mathcal{G}_R \subset \{R < \infty\} \cap (\mathcal{N} \vee \sigma(\tau \nmid R) \vee \mathcal{F}_R)$$

It is actually an equality, because the inverse inclusion is obvious.

"If" part Suppose now $\{R < \infty\} \cap \mathcal{G}_R = \{R < \infty\} \cap (\mathcal{N} \vee \sigma(\tau \nmid R) \vee \mathcal{F}_R)$. Let Y be any $\mathbb{G}$-optional process. We have

$$\begin{array}{rcl} \mathbb{1}_{\{R<\infty\}}Y_R & \in & \{R < \infty\} \cap (\mathcal{N} \vee \sigma(\tau \nmid R) \vee \mathcal{F}_R) + \{R = \infty\} \cap \mathcal{T}^0 \\ & \subset & \{R < \tau\} \cap (\mathcal{N} \vee \mathcal{F}_R) + \{\tau \leq R < \infty\} \cap (\mathcal{N} \vee \sigma(\tau) \vee \mathcal{F}_R) + \{R = \infty\} \cap \mathcal{T}^0 \end{array}$$

and, therefore, there exist $\zeta' \in \mathcal{F}_R$ and $\zeta'' \in \sigma(\tau) \vee \mathcal{F}_R$ such that

$$\mathbb{1}_{\{R<\infty\}}Y_R = \zeta' \mathbb{1}_{\{R<\tau\}} + \zeta'' \mathbb{1}_{\{\tau\leq R<\infty\}}$$

Let $Y' \in \mathcal{O}(\mathbb{F})$ and $Y'' \in \mathcal{B}[0,\infty] \otimes \mathcal{O}(\mathbb{F})$ such that $\mathbb{1}_{\{R<\infty\}}Y'_R = \mathbb{1}_{\{R<\infty\}}\zeta'$ and $\mathbb{1}_{\{R<\infty\}}Y''_R(\tau) = \mathbb{1}_{\{R<\infty\}}\zeta''$ (see Lemma 4.7 and Lemma 4.8). We get

$$Y\mathbb{1}_{[R]} = Y'\mathbb{1}_{[0,\tau)}\mathbb{1}_{[R]} + Y''(\tau)\mathbb{1}_{[\tau,\infty)}\mathbb{1}_{[R]} \quad \blacksquare$$

4.5 Local optional splitting formula on intervals such as $[S,T]$ or $(S,T]$

Lemma 4.10 *Let S, T be $\mathbb{G}$-stopping times. Suppose that the local optional splitting formula holds on $[S,T)$ as well as on $[T_{\{S\leq T<\infty\}}]$. Suppose that $\mathbb{1}_{[T_{\{S\leq T<\infty\}}]}$ satisfies the local optional splitting formula on $[S,T]$. Then, the local optional splitting formula holds on $[S,T]$.*

Proof Let Y be a $\mathbb{G}$-optional process. Let $A' = \mathfrak{p}^{[0,\tau)}_{[S,T)}Y$ and $A'' = \mathfrak{p}^{[\tau,\infty)}_{[S,T)}Y$. Let $B' = \mathfrak{p}^{[0,\tau)}_{[T_{\{S\leq T<\infty\}}]}Y$ and $B'' = \mathfrak{p}^{[\tau,\infty)}_{[T_{\{S\leq T<\infty\}}]}Y$. Let $C' = \mathfrak{p}^{[0,\tau)}_{[S,T]}\mathbb{1}_{[T_{\{S\leq T<\infty\}}]}$ and $C'' = \mathfrak{p}^{[\tau,\infty)}_{[S,T]}\mathbb{1}_{[T_{\{S\leq T<\infty\}}]}$. Note that $\mathbb{1}_{[T_{\{S\leq T<\infty\}}]} = \mathbb{1}_{[T]}\mathbb{1}_{\{S\leq T<\infty\}}$. We can write

$$\begin{array}{rcl} Y\mathbb{1}_{[S,T]} & = & Y\mathbb{1}_{[S,T)} + Y\mathbb{1}_{[T]}\mathbb{1}_{\{S\leq T<\infty\}} \\ & = & (A'\mathbb{1}_{[0,\tau)} + A''\mathbb{1}_{[\tau,\infty)})\mathbb{1}_{[S,T)} + (B'\mathbb{1}_{[0,\tau)} + B''\mathbb{1}_{[\tau,\infty)})\mathbb{1}_{[T_{\{S\leq T<\infty\}}]} \\ & = & (A'\mathbb{1}_{[0,\tau)} + A''\mathbb{1}_{[\tau,\infty)})\mathbb{1}_{[S,T]}(1 - \mathbb{1}_{[T_{\{S\leq T<\infty\}}]}) + (B'\mathbb{1}_{[0,\tau)} + B''\mathbb{1}_{[\tau,\infty)})\mathbb{1}_{[T_{\{S\leq T<\infty\}}]}\mathbb{1}_{[S,T]} \\ & = & ((A' + (B' - A')C')\mathbb{1}_{[0,\tau)} + (A'' + (B'' - A'')C'')\mathbb{1}_{[\tau,\infty)})\mathbb{1}_{[S,T]} \quad \blacksquare \end{array}$$

In the same way we can prove

Lemma 4.11 *Let S, T be $\mathbb{G}$-stopping times. Suppose that the local optional splitting formula holds on (S,T) as well as on $[T_{\{S<T<\infty\}}]$. Suppose that $\mathbb{1}_{[T_{\{S<T<\infty\}}]}$ satisfies the local optional splitting formula on $(S,T]$. Then, the local optional splitting formula holds on $(S,T]$.*

5 Sufficient conditions to have optional splitting formulas at τ with respect to $\mathbb{F}$

5.1 Optional splitting formula on $[0,\tau)$

Theorem 5.1 *Local optional splitting formula holds on the time interval $[0,\tau)$.*

Proof Let $\xi \in \mathcal{G}_\infty$ be a bounded random variable. We write the identity (cf. [13])

$$\xi^{\mathbb{G}}_t \mathbb{1}_{\{t<\tau\}} = \mathbb{1}_{\{t<\tau\}} \frac{\mathbb{Q}[\xi \mathbb{1}_{\{t<\tau\}}|\mathcal{F}_t]}{\mathbb{Q}[t<\tau|\mathcal{F}_t]} \mathbb{1}_{\{\mathbb{Q}[t<\tau|\mathcal{F}_t]>0\}}, t \geq 0.$$

It is an optional splitting formula of $\xi^{\mathbb{G}}$ on $[0,\tau)$. Now, applying Lemma 4.6, we conclude the lemma. ∎

From this thoerem we deduce once again the following known result.

Corollary 5.2 *Let R be a $\mathbb{G}$-stopping time. We have*

$$\{R<\tau\} \cap \mathcal{G}_R = \{R<\tau\} \cap (\mathcal{N} \vee \mathcal{F}_R)$$

Proof. We have the identity

$$\mathbb{1}_{[R]} \mathbb{1}_{[0,\tau)} = \mathbb{1}_{[R]} \mathbb{1}_{\{R<\tau\}} = \mathbb{1}_{[R_{\{R<\tau\}}]}$$

Since the optional splitting formula holds on $[0,\tau)$ by Theorem 5.1, it holds on $[R_{\{R<\tau\}}]$ (Lemma 4.3). According to Theorem 4.9,

$$\{R_{\{R<\tau\}} < \infty\} \cap \mathcal{G}_{R_{\{R<\tau\}}} = \{R_{\{R<\tau\}} < \infty\} \cap (\mathcal{N} \vee \sigma(\tau \nmid R_{\{R<\tau\}}) \vee \mathcal{F}_{R_{\{R<\tau\}}})$$

which is equivalent to $\{R<\tau\} \cap \mathcal{G}_R = \{R<\tau\} \cap (\mathcal{N} \vee \mathcal{F}_R)$. ∎

5.2 $\mathcal{SH}$-measure and optional splitting formula on $[\tau,\infty)$

Definition 5.3 *Let S, T be $\mathbb{G}$-stopping times. A probability measure $\mathbb{Q}'$ defined on $\mathcal{G}_\infty$ is called a $\mathcal{SH}$-measure over the random time interval $(S,T]$ (with respect to $(\mathbb{Q},\mathbb{F},\mathbb{G})$), if $\mathbb{Q}'$ is equivalent to $\mathbb{Q}$ on $\mathcal{G}_\infty$, and if, for any $(\mathbb{Q},\mathbb{F})$ local martingale X, $X^{(S,T]}$ is a $(\mathbb{Q}',\mathbb{G})$ local martingale, where $X^{(S,T]}_t = X^{S\vee T}_t - X^S_t, t \geq 0$.*

Remark 5.4 The notion of $\mathcal{SH}$-measure is introduced in [22] to study the martingale representation property. The above Definition 5.3 is a different but equivalent version of that used in [22] (see Lemma A.5 and Lemma A.6 in [22]).

Note also that, if $\mathbb{Q}'$ is a $\mathcal{SH}$-measure on $(S,T]$, if $(S',T'] \subset (S,T]$, then, $\mathbb{Q}'$ is a $\mathcal{SH}$-measure on $(S',T']$.

Remark 5.5 Note that the property of local optional splitting formula on a $\mathbb{G}$-optional set is invariant by the equivalent change of probability measure on $\mathcal{G}_\infty$.

Theorem 5.6 *For any $\mathbb{F}$-stopping time T, for any $\mathbb{G}$-stopping time S such that $S \geq \tau$, if a $\mathcal{SH}$-measure $\mathbb{Q}'$ over $(S,T]$ exists, the local optional splitting formula holds on $[S,T)$.*

Proof. Let $\mathbb{Q}'$ be a $\mathcal{SH}$-measure over $(S,T]$. Let ζ be a $\mathcal{F}_T$-measurable bounded random variable. We introduce the martingale $X = \zeta\big|^{\mathbb{Q}-\mathbb{F}}$. We note that $\zeta = X_t$ for all $t \geq T$.

Since X is bounded, $X^{(S,T]}$ is a bounded $(\mathbb{Q}', \mathbb{G})$ martingale. Hence,

$$\mathbb{Q}'[X_\infty^{(S,T]}|\mathcal{G}_t] = X_t^{(S,T]} = X_t^{S\vee T} - X_t^S, t \geq 0.$$

The relation $\tau \leq S$ (which implies $\sigma(\tau) \vee \mathcal{F}_S \subset \mathcal{G}_S$) held under $\mathbb{Q}$ remains valid under $\mathbb{Q}'$. Let g be a bounded Borel function. Noting that $g(\tau)\mathbb{1}_{\{S<t\}} \in \mathcal{G}_t$, we have

$$\begin{aligned}\mathbb{Q}'[g(\tau)X_{S\vee T} - g(\tau)X_S|\mathcal{G}_t] &= \mathbb{Q}'[g(\tau)X_\infty^{(S,T]}|\mathcal{G}_{S\vee t}|\mathcal{G}_t] \\ &= \mathbb{Q}'[g(\tau)X_t^{(S,T]}\mathbb{1}_{\{S<t\}}|\mathcal{G}_t] \\ &= g(\tau)X_t^{S\vee T} - g(\tau)X_t^S, t \geq 0.\end{aligned}$$

Consider this identity on the set $\{S \leq t < T\}$. We obtain

$$\mathbb{1}_{\{S\leq t<T\}}g(\tau)X_t = \mathbb{1}_{\{S\leq t<T\}}\mathbb{Q}'[g(\tau)\zeta|\mathcal{G}_t]$$

This identity means that the $(\mathbb{Q}', \mathbb{G})$-martingale $(g(\tau)\zeta)\big|_t^{\mathbb{Q}'-\mathbb{G}} = \mathbb{Q}'[g(\tau)\zeta|\mathcal{G}_t], t \geq 0$, satisfies the local optional splitting formula on $[S,T)$.

Let $\mathcal{C}$ be the class of all bounded random variables $\xi \in \mathcal{G}_\infty$ such that $\xi\big|^{\mathbb{Q}'-\mathbb{G}}$ satisfies the local optional splitting formula on $[S,T)$. The preceding result together with monotone class theorem implies that $\mathcal{C}$ contains all bounded $\sigma(\tau) \vee \mathcal{F}_T$ measurable random variables. By [24, Lemme(4.4)], $\mathcal{G}_{T-} \subset (\mathcal{N} \vee \sigma(\tau) \vee \mathcal{F}_T)$ (noting that we have the same family of negligible sets under $\mathbb{Q}$ and under $\mathbb{Q}'$). Lemma 4.6 is applicable to conclude the local optional splitting formula on (S,T) under the probability measure $\mathbb{Q}'$. Finally, Remark 5.5 achieves the proof. ∎

5.3 Structure of $\mathcal{G}_R$ under a $\mathcal{SH}$-measure

Lemma 5.7 *Let R be a $\mathbb{G}$-stopping time. For any $\mathbb{F}$-stopping time T, for any $\mathbb{G}$-stopping time S, if a $\mathcal{SH}$-measure $\mathbb{Q}'$ over $(S,T]$exists, we have*

$$\{\tau \leq R\} \cap \{S \leq R < T\} \cap \mathcal{G}_R = \{\tau \leq R\} \cap \{S \leq R < T\} \cap (\mathcal{N} \vee \sigma(\tau) \vee \mathcal{F}_R)$$

Proof Let $\mathbb{Q}'$ be a $\mathcal{SH}$-measure over $(S,T]$. Let ζ be a $\mathcal{F}_T$-measurable bounded random variable. We introduce the martingale $X = \zeta\big|^{\mathbb{Q}-\mathbb{F}}$. Let g be a bounded Borel function. As in the preceding lemma, we prove

$$\mathbb{1}_{\{\tau\leq R\}}\mathbb{1}_{\{S\leq R<T\}}g(\tau)X_R = \mathbb{1}_{\{\tau\leq R\}}\mathbb{1}_{\{S\leq R<T\}}\mathbb{Q}'[g(\tau)\zeta|\mathcal{G}_R]$$

We can then write

$$\begin{aligned}&\mathbb{1}_{\{\tau\leq R\}}\mathbb{1}_{\{S\leq R<T\}}\mathbb{Q}'[g(\tau)\zeta|\mathcal{G}_R] \\ \in\ &\{\tau \leq R\} \cap \{S \leq R < T\} \cap (\sigma(\tau) \vee \mathcal{F}_R) + (\{\tau \leq R\} \cap \{S \leq R < T\})^c \cap \mathcal{T}^0\end{aligned}$$

By the monotone class theorem, this relation is extended to any bounded $\xi \in \sigma(\tau) \vee \mathcal{F}_T$:

$$\begin{array}{rl} & \mathbb{1}_{\{\tau \leq R\}} \mathbb{1}_{\{S \leq R < T\}} \mathbb{Q}'[\xi | \mathcal{G}_R] \\ \in & \{\tau \leq R\} \cap \{S \leq R < T\} \cap (\sigma(\tau) \vee \mathcal{F}_R) \\ & \qquad + (\{\tau \leq R\} \cap \{S \leq R < T\})^c \cap \mathcal{T}^0 \end{array} \tag{3}$$

We note that $\{\tau \leq R\} \cap \{S \leq R < T\} \in \mathcal{G}_{T-}$ and

$$\begin{array}{rcl} \{\tau \leq R\} \cap \{S \leq R < T\} \cap \mathcal{G}_R & \subset & \{\tau \leq R\} \cap \{S \leq R < T\} \cap \mathcal{G}_{T-} \\ & \subset & \mathcal{N} \vee \sigma(\tau) \vee \mathcal{F}_{T-} \end{array}$$

where the last inclusion comes from [24, Lemme(4.4)]. Applying the equation (3) to all the $\xi \in \mathcal{G}_R$, we conclude

$$\{\tau \leq R\} \cap \{S \leq R < T\} \cap \mathcal{G}_R \subset \{\tau \leq R\} \cap \{S \leq R < T\} \cap (\mathcal{N} \vee \sigma(\tau) \vee \mathcal{F}_R)$$

But the inverse inclusion relation being obvious, we have actually an equality. ∎

5.4 $s\mathcal{H}$-measures with covering condition

Theorem 5.8 *Suppose that there exists a countable family of couples of $\mathbb{G}$ stopping times $\{S_j, T_j\}, j \in \mathbb{N}$, such that*

(1) T_j are $\mathbb{F}$-stopping times.

(2) $(\tau, \infty) \subset \cup_{i \in \mathbb{N}} (S_j, T_j)$ (covering condition)

Suppose that, for any $j \in \mathbb{N}$, there exists a $s\mathcal{H}$-measure $\mathbb{Q}_j$ over the time interval $(S_j, T_j]$. Then, the local optional splitting formula holds on (τ, ∞).

If we replace the condition (2) *by the condition :*

(2)' $[\tau, \infty) \cap (0, \infty) \subset \cup_{i \in \mathbb{N}} (S_j, T_j)$ (covering condition)

Then, the global optional splitting formula holds.

Proof Suppose firstly the covering condition (2). Let $k \in \mathbb{N}$ and consider the couple $\{S_k, T_k\}$. We state that Lemme 4.11 is applicable so that the local splitting formula holds on $(\tau \vee S_k, T_k]$.

Let us check the conditions in Lemma 4.11. Firstly, according to Theorem 5.6, the local optional splitting formula holds on $[\tau \vee S_k, T_k)$, and *à fortiorie* on $(\tau \vee S_k, T_k)$. Apply then Lemma 5.7 to write the equality : $j \geq 1$,

$$\{\tau \leq T_k\} \cap \{S_j \leq T_k < T_j\} \cap \mathcal{G}_{T_k} = \{\tau \leq T_k\} \cap \{S_j \leq T_k < T_j\} \cap (\mathcal{N} \vee \sigma(\tau) \vee \mathcal{F}_{T_k})$$

From this, we get

$$\{\tau < T_k\} \cap \{S_j < T_k < T_j\} \cap \mathcal{G}_{T_k} = \{\tau < T_k\} \cap \{S_j < T_k < T_j\} \cap (\mathcal{N} \vee \sigma(\tau) \vee \mathcal{F}_{T_k})$$

Because

$$\begin{array}{l} \{S_j < T_k < T_j\} \in \mathcal{G}_{T_k} \\ \{T_k < T_j\} \in \mathcal{F}_{T_k} \subset \sigma(\tau) \vee \mathcal{F}_{T_k} \\ \{S_j < T_k\} \in \mathcal{G}_{T_k-} \subset \mathcal{N} \vee \sigma(\tau) \vee \mathcal{F}_{T_k} \end{array}$$

Lemma 2.1 is applicable. Taking then the union on $j \geq 0$, using the covering condition (2), we obtain

$$\{\tau < T_k < \infty\} \cap \mathcal{G}_{T_k} = \{\tau < T_k < \infty\} \cap (\mathcal{N} \vee \sigma(\tau) \vee \mathcal{F}_{T_k})$$

By Theorem 4.9, the local splitting formula holds on $[(T_k)_{\{\tau<T_k\}}]$, and *à fortiorie* on $[(T_k)_{\{\tau\vee S_k<T_k<\infty\}}]$. Finally, we write

$$\mathbb{1}_{[(T_k)_{\{\tau\vee S_k<T_k<\infty\}}]} = \mathbb{1}_{[T_k]}\mathbb{1}_{\{\tau\vee S_k<T_k<\infty\}} = \mathbb{1}_{[T_k]}\mathbb{1}_{(\tau\vee S_k,T_k]}$$

Since $\mathbb{1}_{[T_k]}$ is a $\mathbb{F}$ optional process, the above formula is an optional splitting formula for $\mathbb{1}_{[(T_k)_{\{\tau\vee S_k<T_k<\infty\}}]}$ on $(\tau\vee S_k, T_k]$.

$(\tau\vee S_k, T_k]$ being $\mathbb{G}$-predictable sets, applying Lemma 4.5, the local splitting formula holds on $\cup_{k\in\mathbb{N}}(\tau\vee S_k, T_k] = (\tau,\infty)$, because of the covering condition (2). The first part of the theorem is proved.

Now suppose the covering condition $(2)'$. Applying Lemme 5.7 to the random time τ, we write

$$\{S_j<\tau<T_j\}\cap\mathcal{G}_\tau = \{S_j<\tau<T_j\}\cap(\mathcal{N}\vee\sigma(\tau)\vee\mathcal{F}_\tau)$$

Since $\{S_j<\tau<T_j\}\in\mathcal{G}_\tau$, $\{S_j<\tau\}\in\mathcal{G}_{\tau-}\subset\sigma(\tau)\vee\mathcal{F}_\tau$, and $\{\tau<T_j\} = \{\mathbb{1}_{[0,T_k)}(\tau)=1\}\in\mathcal{F}_\tau$, Lemma 2.1 is applicable. Taking the union on $j\geq 0$, using the covering condition $(2)'$, we obtain

$$\{0<\tau<\infty\}\cap\mathcal{G}_\tau = \{0<\tau<\infty\}\cap(\mathcal{N}\vee\sigma(\tau)\vee\mathcal{F}_\tau) = \{0<\tau<\infty\}\cap(\mathcal{N}\vee\mathcal{F}_\tau)$$

On the other side, according to [24, Lemme 4.4],

$$\{\tau=0\}\cap\mathcal{G}_\tau = \{\tau=0\}\cap\mathcal{G}_0 = \{\tau=0\}\cap(\mathcal{N}\vee\mathcal{F}_0) = \{\tau=0\}\cap(\mathcal{N}\vee\mathcal{F}_\tau)$$

Taking the union of these two identities, we conclude

$$\{\tau<\infty\}\cap\mathcal{G}_\tau = \{\tau<\infty\}\cap(\mathcal{N}\vee\mathcal{F}_\tau)$$

Theorem 4.9 is now applicable so that the local splitting formula holds on $[\tau]$.

Let Y be a $\mathbb{G}$ optional process. Let $A' = \mathfrak{p}^{[0,\tau)}_{(\tau,\infty)}Y$ and $A'' = \mathfrak{p}^{[\tau,\infty)}_{(\tau,\infty)}Y$. Let $B' = \mathfrak{p}^{[0,\tau)}_{[\tau]}Y$ and $B'' = \mathfrak{p}^{[\tau,\infty)}_{[\tau]}Y$. We check immediately

$$Y\mathbb{1}_{[\tau,\infty)} \quad = \quad (A''\mathbb{1}_{(\tau,\infty)} + B''\mathbb{1}_{[\tau]})\mathbb{1}_{[\tau,\infty)}$$

This proves the local splitting formula on $[\tau,\infty)$. Since the local splitting formula always holds on $[0,\tau)$ (Theorem 5.1), the second part of the theorem is proved. ∎

6 Examples

In this short section we show that Theorem 5.8 is applicable to a wide number of models that we encounter in credit risk study.

Suppose that $(\mathbb{F},\mathbb{G})$ satisfies the $(\mathcal{H})$ hypothesis under $\mathbb{Q}$ (see for example [7, 23]), i.e., any $(\mathbb{Q},\mathbb{F})$ local martingale is a $(\mathbb{Q},\mathbb{G})$ local martingale. Then, if we consider the couple $T=\infty$ and $S=0$, the covering condition $(2)'$ is satisfied and the probability measure $\mathbb{Q}$ itself is an $s\mathcal{H}$-measure on $(S,T]$. Theorem 5.8 is applicable and the global optional splitting formula holds. Cox model is a typical example where the $(\mathcal{H})$ hypothesis is satisfied. A model where τ is independent of $\mathcal{F}_\infty$ gives rise to another example.

The same conclusion remains true, if there exists a probability $\mathbb{Q}'$ on $\mathcal{G}_\infty$, equivalent to $\mathbb{Q}$, such that the couple $(\mathbb{F},\mathbb{G})$ satisfies the $(\mathcal{H})$ hypothesis under $\mathbb{Q}'$. This remark

concerns, for example, the models satisfying the conditional density hypothesis in [15] with a strictly positive density function, or the proportionality model in [20].

There exists models which are not covered by the above discussion on the $(\mathcal{H})$ hypothesis, but they can, nevertheless, be studied using Theorem 5.8. It is the case of honest time model. Suppose that τ is an $\mathbb{F}$ honest time (cf. [24]). Suppose that all $(\mathbb{Q},\mathbb{F})$ local martingales are continuous, and τ does not intersect $\mathbb{F}$-stopping time. In [22] it is shown that the covering condition (2) with $\mathcal{sH}$-measures is satisfied. Consequently, the local optional splitting formula holds on (τ,∞). Recall that the local optional splitting formula always holds on $[0,\tau)$ (Theorem 5.1). Note also that, in general, $\mathcal{G}_\tau$ differ from $\mathcal{F}_\tau$ (cf. [24]) which means, according to Theorem 4.9, that the local optional splitting formula may not hold on $[\tau]$.

Another example is the $(\natural)$-model with conditions **Hy**(C) and **Hy**(Mc) studied in [21]. It is shown in [22] that the $(\natural)$-model satisfies the covering condition $(2)'$ with $\mathcal{sH}$-measures. Consequently, the global optional splitting formula holds in this model.

7 Splitting formula at multiple random times

In this section we extend the preceding results to the case of multiple random times.

7.1 Ordering of functions on $\{1,\dots,k\}$

Let $\mathfrak{a}$ be a function defined on $\{1,\dots,k\}$ (where $k>0$ is an integer) taking values in $[0,\infty]$. Let $\{a_1,\dots,a_k\}$ denote the values of $\mathfrak{a}$. Let

$$R^{\mathfrak{a}}(i)=R^{\{a_1,\dots,a_k\}}(i)=\sum_{j=1}^{k}\mathbb{1}_{\{a_j<a_i\}}+\sum_{j=1}^{k}\mathbb{1}_{\{j<i,a_j=a_i\}}+1$$

The map $i\in\{1,\dots,k\}\to R^{\mathfrak{a}}(i)\in\{1,\dots,k\}$ is a bijection. Let $\rho^{\mathfrak{a}}$ be its inverse. Define $\uparrow\mathfrak{a}=\mathfrak{a}(\rho^{\mathfrak{a}})$. We check that $\uparrow\mathfrak{a}$ is an non decreasing function on $\{1,\dots,k\}$ taking the same values of $\mathfrak{a}$.

Let $1\le j\le k$, $i=\rho^{\mathfrak{a}}(j)$ and $b\in\mathbb{R}_+$. Let $\mathfrak{a}\nmid b=\{a_1\nmid b,\dots,a_k\nmid b\}$. Then, if $b\ge\uparrow\mathfrak{a}(j)=a_i$, it can be checked that $a_h\nmid b<a_i\nmid b$ (resp. $a_h\nmid b=a_i\nmid b$) is equivalent to $a_h<a_i$ (resp. $a_h=a_i$). Therefore, $R^{\mathfrak{a}\nmid b}(i)=R^{\mathfrak{a}}(i)=j$ and

$$\uparrow(\mathfrak{a}\nmid b)(j)=\ a_i\nmid b=a_i=\uparrow\mathfrak{a}(j).$$

7.2 The enlargement of filtration with multiple random times

Let $m>0$ be an integer and $\tau_1,\dots,\tau_m$ be m random times. For a $1\le k\le m$, consider the function $\mathfrak{t}_k$ on $\{1,\dots,k\}$ taking respectively the values $\{\tau_1,\dots,\tau_k\}$. We define $\sigma_{k,i}=\uparrow\mathfrak{t}_k(i), 1\le i\le k$. Note that, if the τ_i are stopping times with respect to some filtration, the $\sigma_{k,i}$ are stopping times with respect to the same filtration, because

$$\{\sigma_{k,i}\le t\}=\cup_{I\subset\{1,\dots,k\},\sharp I=i}\{\tau_j\le t,\forall j\in I\},\ t\ge 0.$$

This same equation shows that there exists a Borel function $\mathfrak{s}_{k,i}$ on $[0,\infty]^k$ such that $\sigma_{k,i}=\mathfrak{s}_{k,i}(\tau_1,\dots,\tau_k)$.

Let $\mathbb{G}^0 = \mathbb{F}$. For $1 \leq k \leq m$, let $\mathbb{G}^k = (\mathcal{G}_t^k)_{t\geq 0}$ where

$$\mathcal{G}_t^k = \mathcal{N}^k \vee (\cap_{s>t}(\mathcal{G}_s^{k-1} \vee \sigma(\tau_k \wedge s))),\ t \geq 0,$$

and

$$\mathcal{N}^k = \mathcal{N}^{\sigma(\tau_1)\ldots\vee\sigma(\tau_k)\vee\mathcal{F}_\infty}$$

By induction, we can prove that $\mathbb{G}^k$ is the smallest right continuous filtration containing $\mathbb{F}$ and $\mathcal{N}^k$, making the $\tau_1, \ldots, \tau_k$ stopping times.

Remark 7.1 Let us denote temporarily $\mathbb{G}^k$ by $\mathbb{G}^{(\tau_1,\ldots,\tau_k)}$ in reference to the dependance of $\mathbb{G}^k$ on the random times $(\tau_1, \ldots, \tau_k)$. We have the relation

$$\mathbb{G}^{(\tau_1,\ldots,\tau_k)} \supset \mathbb{G}^{(\sigma_{k,1},\ldots,\sigma_{k,k})}$$

because the $\sigma_{k,i}, 1 \leq i \leq k$, are $\mathbb{G}^{(\tau_1,\ldots,\tau_k)}$-stopping times. In general, there is no equality between $\mathbb{G}^{(\sigma_{k,1},\ldots,\sigma_{k,k})}$ and $\mathbb{G}^{(\tau_1,\ldots,\tau_k)}$. For example, let $\{A, B, C\}$ be a partition of Ω. If

$$\begin{aligned}\tau_1 &= 1\mathbb{1}_A + 2\mathbb{1}_B + 3\mathbb{1}_C\\ \tau_2 &= 2\mathbb{1}_A + 3\mathbb{1}_B + 1\mathbb{1}_C\\ \tau_3 &= 3\mathbb{1}_A + 1\mathbb{1}_B + 2\mathbb{1}_C,\end{aligned}$$

we get $\sigma_{3,1} \equiv 1, \sigma_{3,2} \equiv 2, \sigma_{3,3} \equiv 3$.

7.3 The splitting formulas

Definition 7.2 *We say that the $\mathbb{G}^m$-optional splitting formula holds at the times $\tau_1, \ldots, \tau_m$ with respect to $\mathbb{F}$, if, for any $\mathbb{G}^m$-optional process Y, there exist functions $Y^{(0)}, Y^{(1)}, \ldots, Y^{(m)}$ defined on $[0,\infty]^m \times (\mathbb{R}_+ \times \Omega)$ being $\mathcal{B}[0,\infty]^m \otimes \mathcal{O}(\mathbb{F})$-measurable such that*

$$Y = \sum_{i=0}^{m} Y^{(i)}(\tau_1 \nmid \sigma_{m,i}, \ldots, \tau_m \nmid \sigma_{m,i})\mathbb{1}_{[\sigma_{m,i},\sigma_{m,i+1})}$$

where $\sigma_{m,0} \equiv 0$ and $\sigma_{m,m+1} \equiv \infty$ by definition.

Note that this definition is in coherence with Definition 3.1.

Lemma 7.3 *Let $(E, \mathcal{E})$ be a measurable space. Let $1 \leq k \leq m$. Suppose that the $\mathbb{G}^k$-optional splitting formula holds at the times $\tau_1, \ldots, \tau_k$ with respect to $\mathbb{F}$. Then, for any $\mathcal{E} \otimes \mathcal{O}(\mathbb{G}^k)$-measurable function $Y(\theta, s, \omega)$, there exist functions $Y^{(0)}, Y^{(1)}, \ldots, Y^{(k)}$ defined on $E \times [0,\infty]^k \times (\mathbb{R}_+ \times \Omega)$ being $\mathcal{E} \otimes \mathcal{B}[0,\infty]^k \otimes \mathcal{O}(\mathbb{F})$-measurable such that*

$$Y(\theta) = \sum_{i=0}^{k} Y^{(i)}(\theta, \tau_1 \nmid \sigma_{k,i}, \ldots, \tau_k \nmid \sigma_{k,i})\mathbb{1}_{[\sigma_{k,i},\sigma_{k,i+1})}$$

Proof We need only to check the lemma upon the functions of the form $Y(\theta, s, \omega) = g(\theta)F_s(\omega), g \in \mathcal{E}, F \in \mathcal{O}(\mathbb{G}^k)$, and apply the monotone class theorem. ∎

Theorem 7.4 *Suppose $m > 1$. Supose that $\mathbb{G}^{m-1}$-optional splitting formula holds at the times $\tau_1, \ldots, \tau_{m-1}$ with respect to $\mathbb{F}$. Suppose $\mathbb{G}^m$-optional splitting formula holds at the time τ_m with respect to $\mathbb{G}^{m-1}$. Then, $\mathbb{G}^m$-optional splitting formula holds at the times $\tau_1, \ldots, \tau_m$ with respect to $\mathbb{F}$.*

Proof Let Y be a $\mathbb{G}^m$-optional process. By assumption, there exist Y', Y'' such that $Y' \in \mathcal{O}(\mathbb{G}^{m-1})$ and $Y'' \in \mathcal{B}[0,\infty] \otimes \mathcal{O}(\mathbb{G}^{m-1})$ and

$$Y = Y' \mathbb{1}_{[0,\tau_m)} + Y''(\tau_m)\mathbb{1}_{[\tau_m,\infty)}$$

Now according to the $\mathbb{G}^{m-1}$-optional splitting formula at the times $\tau_1, \ldots, \tau_{m-1}$ with respect to $\mathbb{F}$, there exist functions $Y'^{(0)}, Y'^{(1)}, \ldots, Y'^{(m-1)}$ defined on $[0,\infty]^{m-1} \times (\mathbb{R}_+ \times \Omega)$ being $\mathcal{B}[0,\infty]^{m-1} \otimes \mathcal{O}(\mathbb{F})$-measurable such that

$$Y' = \sum_{i=0}^{m-1} Y'^{(i)}(\tau_1 \restriction \sigma_{m-1,i}, \ldots, \tau_{m-1} \restriction \sigma_{m-1,i})\mathbb{1}_{[\sigma_{m-1,i},\sigma_{m-1,i+1})}$$

According to Lemma 7.3, there exist functions $Y''^{(0)}, Y''^{(1)}, \ldots, Y''^{(m-1)}$ defined on $[0,\infty] \times [0,\infty]^{m-1} \times (\mathbb{R}_+ \times \Omega)$ being $\mathcal{B}[0,\infty] \otimes \mathcal{B}[0,\infty]^{m-1} \otimes \mathcal{O}(\mathbb{F})$-measurable such that

$$Y''(\theta) = \sum_{i=0}^{m-1} Y''^{(i)}(\theta, \tau_1 \restriction \sigma_{m-1,i}, \ldots, \tau_{m-1} \restriction \sigma_{m-1,i})\mathbb{1}_{[\sigma_{m-1,i},\sigma_{m-1,i+1})}$$

Let $k = R^{\{\tau_1,\ldots,\tau_m\}}(m)$. We have $\sigma_{m-1,i} = \sigma_{m,i}$ for $i \leq k-1$ and $\sigma_{m-1,i} = \sigma_{m,i+1}$ for $k \leq i < m$. Moreover, if $k < m$, $\sigma_{m,k+1} = \sigma_{m-1,k} > \tau_m$. Now we write

$$\begin{array}{rcl} Y'\mathbb{1}_{[0,\tau_m)} & = & \sum_{i=0}^{k-1} Y'^{(i)}(\tau_1 \restriction \sigma_{m,i}, \ldots, \tau_{m-1} \restriction \sigma_{m,i})\mathbb{1}_{[\sigma_{m,i},\sigma_{m,i+1})} \\ & = & \sum_{i=0}^{m-1} Y'^{(i)}(\tau_1 \restriction \sigma_{m,i}, \ldots, \tau_{m-1} \restriction \sigma_{m,i})\mathbb{1}_{[\sigma_{m,i},\sigma_{m,i+1})}\mathbb{1}_{\{\sigma_{m,i}<\tau_m\}} \end{array}$$

where the last equality comes from the fact that, if $i < k$ and $\sigma_{m,i} = \tau_m$, $[\sigma_{m,i}, \sigma_{m,i+1}) = \emptyset$. Because $\mathbb{1}_{[\sigma_{m,i},\sigma_{m,i+1})} = \mathbb{1}_{[\sigma_{m,i},\sigma_{m,i+1})}\mathbb{1}_{\{\sigma_{m,i}<\infty\}}$ so that

$$\begin{array}{rcl} \mathbb{1}_{[\sigma_{m,i},\sigma_{m,i+1})}\mathbb{1}_{\{\sigma_{m,i}<\tau_m\}} & = & \mathbb{1}_{[\sigma_{m,i},\sigma_{m,i+1})}\mathbb{1}_{\{\sigma_{m,i}<\tau_m,\sigma_{m,i}<\infty\}} \\ & = & \mathbb{1}_{[\sigma_{m,i},\sigma_{m,i+1})}\mathbb{1}_{\{\tau_m \restriction \sigma_{m,i}=\infty,\sigma_{m,i}<\infty\}} \\ & = & \mathbb{1}_{[\sigma_{m,i},\sigma_{m,i+1})}\mathbb{1}_{\{\tau_m \restriction \sigma_{m,i}=\infty\}} \end{array}$$

the above expression of $Y'\mathbb{1}_{[0,\tau_m)}$ gives it a $\mathbb{G}^m$-optional splitting formula at $\tau_1, \ldots, \tau_m$ with respect to $\mathbb{F}$.

We write nextly

$$Y''(\tau_m)\mathbb{1}_{[\tau_m,\infty)} = \sum_{j=1}^{m} Y''^{(j-1)}(\tau_m, \tau_1 \restriction \sigma_{m-1,j-1}, \ldots, \tau_{m-1} \restriction \sigma_{m-1,j-1})\mathbb{1}_{[\sigma_{m,j},\sigma_{m,j+1})}\mathbb{1}_{\{\tau_m \leq \sigma_{m,j}\}}$$

where we use the fact that, for any $j < k$, $\sigma_{m,j} \geq \tau_m$ implies $\sigma_{m,j} = \sigma_{m,j+1} = \tau_m$, i.e., $[\sigma_{m,j}, \sigma_{m,j+1}) = \emptyset$. According to Subsection 7.1,

$$\sigma_{m-1,i} = \Upsilon(\mathfrak{t}_{m-1} \restriction \sigma_{m,i+1})(i) = \mathfrak{s}_{m-1,i}(\tau_1 \restriction \sigma_{m,i+1}, \ldots, \tau_{m-1} \restriction \sigma_{m,i+1}),\ 1 \leq i \leq m-1$$

Notice that $\tau_m = \tau_m \restriction \sigma_{m,i+1}$ for $i \geq k-1$ and

$$\begin{array}{rcl} \tau_a \restriction \sigma_{m-1,i} & = & \begin{cases} \tau_a & \text{if } \tau_a \leq \sigma_{m-1,i} \\ \infty & \text{if } \tau_a > \sigma_{m-1,i} \end{cases} \\ & = & \begin{cases} \tau_a \restriction \sigma_{m,i+1} & \text{if } \tau_a \restriction \sigma_{m,i+1} \leq \sigma_{m-1,i} \\ \infty & \text{if } \tau_a \restriction \sigma_{m,i+1} > \sigma_{m-1,i} \end{cases} \\ & = & (\tau_a \restriction \sigma_{m,i+1}) \restriction \mathfrak{s}_{m-1,i}(\tau_1 \restriction \sigma_{m,i+1}, \ldots, \tau_{m-1} \restriction \sigma_{m,i+1}), \end{array}$$

for $1 \leq a \leq m-1, 1 \leq i \leq m-1$. Notice also that

$$\mathbb{1}_{[\sigma_{m,j},\sigma_{m,j+1})}\mathbb{1}_{\{\tau_m \leq \sigma_{m,j}\}} = \mathbb{1}_{[\sigma_{m,j},\sigma_{m,j+1})}\mathbb{1}_{\{\tau_m \restriction \sigma_{m,j}<\infty\}}.$$

We deduce from these facts that there exist Borel functions $Z''^{(i)}$ on $[0,\infty]^m \times (\mathbb{R}_+ \times \Omega)$ being $\mathcal{B}[0,\infty]^m \otimes \mathcal{O}(\mathbb{F})$-measurable such that

$$Y''(\tau_m)\mathbb{1}_{[\tau_m,\infty)} \quad = \quad \sum_{j=1}^m Z''^{(j)}(\tau_1 \nmid \sigma_{m,j}, \ldots, \tau_{m-1} \nmid \sigma_{m,j}, \tau_m \nmid \sigma_{m,j})\mathbb{1}_{[\sigma_{m,j},\sigma_{m,j+1})}$$

This is a $\mathbb{G}^m$-optional splitting formula for $Y''(\tau_m)\mathbb{1}_{[\tau_m,\infty)}$ at the times $\tau_1, \ldots, \tau_m$ with respect to $\mathbb{F}$. The theorem is proved. ∎

7.4 Conditional density hypothesis

Definition 7.5 *We say that $(\tau_1, \ldots, \tau_m)$ satisfied the (conditional) density hypothesis (with a strictly positive density function) with respect to $\mathcal{F}_\infty$, if there exists a Borel probability measure μ on $[0,\infty]$ and a stictly positive function γ on $[0,\infty]^m \times \Omega$ being $\mathcal{B}[0,\infty]^m \otimes \mathcal{F}_\infty$ measurable such that*

$$\mathbb{Q}[(\tau_1, \ldots, \tau_m) \in \mathtt{A} \mid \mathcal{F}_\infty] = \int_{\mathtt{A}} \gamma(t_1, \ldots, t_m)\mu^{\otimes m}(\mathsf{d}t_1, \ldots, \mathsf{d}t_m)$$

for any $\mathtt{A} \in \mathcal{B}[0,\infty]^m$.

We have the following results.

Lemma 7.6 *If $(\tau_1, \ldots, \tau_m)$ satisfies the density hypothesis with respect to $\mathcal{F}_\infty$, τ_m satisfies the density hypothesis with respect to $\mathcal{G}_\infty^{m-1}$,*

The proof of the lemma is straightforward.

Lemma 7.7 *If $m = 1$ and the density hypothesis holds with respect to $\mathcal{F}_\infty$, the global $\mathbb{G}$-optional splitting formula holds at τ_1 with respect to $\mathbb{F}$*

Proof Let

$$\mathbb{Q}' = \frac{1}{\gamma(\tau_1)} \cdot \mathbb{Q}$$

It is an probability equivalent to $\mathbb{Q}$. By Remark 5.5, we need only to prove the lemma under this new probability. But under $\mathbb{Q}'$, τ_1 is independent of $\mathcal{F}_\infty$ so that $(\mathcal{H})$ hypothesis is satisfied. According to Theorem 5.8, the global $\mathbb{G}$-optional splitting formula holds at τ_1 with respect to $\mathbb{F}$. ∎

Now look at Lemma 7.7, Theorem 7.4, and Lemma 7.6. They constitute a perfect scheme of mathematic induction. We obtain the main result of this section :

Theorem 7.8 *If the times $(\tau_1, \ldots, \tau_m)$ satisfy the density hypothesis with respect to $\mathcal{F}_\infty$, then, $\mathbb{G}^m$-optional splitting formula holds at the times $\tau_1, \ldots, \tau_m$ with respect to $\mathbb{F}$.*

8 Splitting formula at random times with marks

The preceeding results can be extended to the case of random times with marks. The proofs will follow the same idea with some notational complication.

Let $(E,\mathcal{E})$ be a separable complete metric space with its Borel σ-algebra. Let $\triangle\in E$ and $E^\circ = E\setminus\{\triangle\}$. Let $(\xi_1,\dots,\xi_m)$ be m random variables taking values in E°. Define, for $1\le i\le m$,

$$H_i(t) = \begin{cases} \triangle & \text{if } t<\tau_i, \\ & \qquad\qquad t\ge 0, \\ \xi_i & \text{if } \tau_i\le t, \end{cases}$$

and $\mathcal{H}_i(t) = \sigma(H_i(u) : 0\le u\le t)$. Note that $(\mathcal{H}_i(t))_{t\ge0}$ is a right continuous filtration. In fact, for any $0\le t<\infty$, we have

$$\begin{aligned} \{\tau_i\le t\}\cap(\cap_{s>t}\mathcal{H}_i(s)) &= \{\tau_i\le t\}\cap\sigma(\xi_i,\tau_i) = \{\tau_i\le t\}\cap\mathcal{H}_i(t) \\ \{\tau_i> t\}\cap(\cap_{s>t}\mathcal{H}_i(s)) &= \{\tau_i> t\}\cap\{\emptyset,\Omega\} = \{\tau_i> t\}\cap\mathcal{H}_i(t) \end{aligned}$$

Let $\overline{\mathcal{H}}_i(t) = \mathcal{N}^{\mathcal{H}_i(\infty)}\vee\mathcal{H}_i(t)$ $(1\le i\le m)$ and $\mathbb{H}_i = (\overline{\mathcal{H}}_i(t))_{t\ge0}$ which is a right continuous filtration. From the definition of $\mathcal{H}_i(t)$ we deduce a simple formula to compute the $\mathbb{H}_i$ martingales.

Lemma 8.1 *For any integrable randon variable $X\in\mathcal{N}^{\mathcal{H}_i(\infty)}\vee\sigma(\xi_i,\tau_i)$,*

$$\mathbb{Q}[X|\overline{\mathcal{H}}_i(t)] = \mathbb{1}_{\{t<\tau_i\}}\frac{\mathbb{Q}[X\mathbb{1}_{\{t<\tau_i\}}]}{\mathbb{Q}[t<\tau_i]} + \mathbb{1}_{\{\tau_i\le t\}}X$$

Let $\mathcal{H}_t^{\{1,\dots,m\}} = \sigma(H_i(s) : 1\le i\le m,\ 0\le s\le t)$ and

$$\mathcal{G}_t^{*m} = \mathcal{N}^{*m}\vee\cap_{s>t}(\mathcal{F}_s\vee\mathcal{H}_s^{\{1,\dots,m\}})$$

where $\mathcal{N}^{*m}$ denotes $\mathcal{N}^{\mathcal{H}_\infty^{\{1,\dots,m\}}\vee\mathcal{F}_\infty}$. Let $\mathbb{G}^{*m}$ be the filtration of $\mathcal{G}_t^{*m}, t\ge0$.

Let $\mathfrak{D}(E)$ be the space of all càdlàg functions taking values in E equipped with the Skorokhod topology (cf. [6]) and its Borel σ-algebra $\mathcal{D}$.

Definition 8.2 *We say that the $\mathbb{G}^{*m}$-optional splitting formula holds at the times $\tau_1,\dots,\tau_m$ with respect to $\mathbb{F}$, if, for any $\mathbb{G}^{*m}$-optional process Y, there exist functions $Y^{(0)},Y^{(1)},\dots,Y^{(m)}$ defined on $\mathfrak{D}(E)^m\times(\mathbb{R}_+\times\Omega)$ being $\mathcal{D}^m\otimes\mathcal{O}(\mathbb{F})$-measurable such that*

$$Y = \sum_{i=0}^m Y^{(i)}(H_1^{\sigma_{m,i}},\dots,H_m^{\sigma_{m,i}})\mathbb{1}_{[\sigma_{m,i},\sigma_{m,i+1})}$$

where $H_i^{\sigma_{m,i}}$ denotes the process H_i stopped at $\sigma_{m,i}$.

Note that, if ξ_i are constant random variables, the above Definition 8.2 coincides with Definition 7.2.

Theorem 8.3 *Suppose $m>1$. Supose that $\mathbb{G}^{*m-1}$-optional splitting formula holds at the times $\tau_1,\dots,\tau_{m-1}$ with respect to $\mathbb{F}$. Suppose $\mathbb{G}^{*m}$-optional splitting formula holds at the time τ_m with respect to $\mathbb{G}^{*m-1}$. Then, $\mathbb{G}^{*m}$-optional splitting formula holds at the times $\tau_1,\dots,\tau_m$ with respect to $\mathbb{F}$.*

Proof Let Y be a $\mathbb{G}^{*m}$-optional process. By assumption, there exist Y', Y'' such that $Y' \in \mathcal{O}(\mathbb{G}^{m-1})$ and $Y'' \in \mathcal{D} \otimes \mathcal{O}(\mathbb{G}^{m-1})$ and

$$Y = Y' \mathbb{1}_{[0,\tau_m)} + Y''(H_m)\mathbb{1}_{[\tau_m,\infty)}$$

Now according to the $\mathbb{G}^{*m-1}$-optional splitting formula at the times $\tau_1, \ldots, \tau_{m-1}$ with respect to $\mathbb{F}$, there exist functions $Y'^{(0)}, Y'^{(1)}, \ldots, Y'^{(m-1)}$ defined on $\mathfrak{D}(E)^{m-1} \times (\mathbb{R}_+ \times \Omega)$ being $\mathcal{D}^{m-1} \otimes \mathcal{O}(\mathbb{F})$-measurable such that

$$Y' = \sum_{i=0}^{m-1} Y'^{(i)}(H_1^{\sigma_{m-1,i}}, \ldots, H_{m-1}^{\sigma_{m-1,i}})\mathbb{1}_{[\sigma_{m-1,i},\sigma_{m-1,i+1})}$$

Also, there exist functions $Y''^{(0)}, Y''^{(1)}, \ldots, Y''^{(m-1)}$ defined on $\mathfrak{D}(E) \times \mathfrak{D}(E)^{m-1} \times (\mathbb{R}_+ \times \Omega)$ being $\mathcal{D} \otimes \mathcal{D}^{m-1} \otimes \mathcal{O}(\mathbb{F})$-measurable such that

$$Y''(H_m) = \sum_{i=0}^{m-1} Y''^{(i)}(H_m, H_1^{\sigma_{m-1,i}}, \ldots, H_{m-1}^{\sigma_{m-1,i}})\mathbb{1}_{[\sigma_{m-1,i},\sigma_{m-1,i+1})}$$

We have (cf. the proof of Theorem 7.4)

$$Y'\mathbb{1}_{[0,\tau_m)} \;=\; \textstyle\sum_{i=0}^{m-1} Y'^{(i)}(H_1^{\sigma_{m,i}}, \ldots, H_{m-1}^{\sigma_{m,i}})\mathbb{1}_{[\sigma_{m,i},\sigma_{m,i+1})}\mathbb{1}_{\{\sigma_{m,i}<\tau_m\}}$$

Since $\mathbb{1}_{\{\sigma_{m,i}<\tau_m\}} = \mathbb{1}_{\{H_m(\sigma_{m,i})=\triangle\}}$, the above expression is simply a $\mathbb{G}^{*m}$-optional splitting formula for $Y'\mathbb{1}_{[0,\tau_m)}$ at $\tau_1, \ldots, \tau_m$ with respect to $\mathbb{F}$.

We write nextly

$$\begin{array}{rl} & Y''(H_m)\mathbb{1}_{[\tau_m,\infty)} \\ = & \sum_{j=1}^{m} Y''^{(j-1)}(H_m, H_1^{\sigma_{m-1,j-1}}, \ldots, H_{m-1}^{\sigma_{m-1,j-1}})\mathbb{1}_{[\sigma_{m,j},\sigma_{m,j+1})}\mathbb{1}_{\{\tau_m \le \sigma_{m,j}\}} \\ = & \sum_{j=1}^{m} Y''^{(j-1)}(H_m^{\sigma_{m,j}}, H_1^{\sigma_{m-1,j-1}}, \ldots, H_{m-1}^{\sigma_{m-1,j-1}})\mathbb{1}_{[\sigma_{m,j},\sigma_{m,j+1})}\mathbb{1}_{\{H_m(\sigma_{m,j})\neq\triangle\}} \end{array}$$

Recall that

$$\sigma_{m-1,i} = \Lambda(\mathfrak{t}_{m-1} \nmid \sigma_{m,i+1})(i) = \mathfrak{s}_{m-1,i}(\tau_1 \nmid \sigma_{m,i+1}, \ldots, \tau_{m-1} \nmid \sigma_{m,i+1}),\ 1 \le i \le m-1$$

and also, for $1 \le a \le m-1$ and $u \ge 0$,

$$\begin{array}{rcl} H_a^{\sigma_{m-1,i}}(u) & = & (H_a^{\sigma_{m,i+1}})^{\sigma_{m-1,i}}(u) \\ & = & \begin{cases} H_a^{\sigma_{m,i+1}}(u) & \text{if } u \le \sigma_{m-1,i} \\ H_a^{\sigma_{m,i+1}}(\sigma_{m-1,i}) & \text{if } u > \sigma_{m-1,i} \end{cases} \end{array}$$

Consider the measurability of the above objects. First of all,

$$\forall 0 \le t < \infty,\ \{\tau_a \nmid \sigma_{m,i+1} \le t\} = \{\tau_a \le t, \tau_a \le \sigma_{m,i+1}\} = \{H_a^{\sigma_{m,i+1}}(t) \in E^\circ\} \in \sigma(H_a^{\sigma_{m,i+1}})$$

This implies that

$$\sigma_{m-1,i} \in \sigma(H_1^{\sigma_{m,i+1}}, \ldots, H_{m-1}^{\sigma_{m,i+1}})$$

Note that the map $(t,\omega) \longrightarrow H_a^{\sigma_{m,i+1}}(t)$ is $\mathcal{B}[0,\infty] \otimes \sigma(H_a^{\sigma_{m,i+1}})$ measurable. Composing this map with that one $\omega \longrightarrow (\sigma_{m-1,i}(\omega), \omega)$, we see that $H_a^{\sigma_{m,i+1}}(\sigma_{m-1,i})$ is in $\sigma(H_1^{\sigma_{m,i+1}}, \ldots, H_{m-1}^{\sigma_{m,i+1}})$. Putting together the above results we conclude that

$$\forall u \ge 0,\ H_a^{\sigma_{m-1,i}}(u) \in \sigma(H_1^{\sigma_{m,i+1}}, \ldots, H_{m-1}^{\sigma_{m,i+1}})$$

Consequently, there exist Borel functions $Z''^{(i)}$ on $\mathfrak{D}(E)^m \times (\mathbb{R}_+ \times \Omega)$ being $\mathcal{D}^m \otimes \mathcal{O}(\mathbb{F})$-measurable such that

$$\begin{array}{rl} & Y''(H_m)\mathbb{1}_{[\tau_m,\infty)} \\ = & \sum_{j=1}^{m} Y''^{(j-1)}(H_m^{\sigma_{m,j}}, H_1^{\sigma_{m-1,j-1}}, \ldots, H_{m-1}^{\sigma_{m-1,j-1}})\mathbb{1}_{[\sigma_{m,j},\sigma_{m,j+1})}\mathbb{1}_{\{H_m(\sigma_{m,j})\neq\triangle\}} \\ = & \sum_{j=1}^{m} Z''^{(j)}(H_1^{\sigma_{m,j}}, \ldots, H_{m-1}^{\sigma_{m,j}}, H_m^{\sigma_{m,j}})\mathbb{1}_{[\sigma_{m,j},\sigma_{m,j+1})} \end{array}$$

This is a $\mathbb{G}^{*m}$-optional splitting formula for $Y''(\tau_m)\mathbb{1}_{[\tau_m,\infty)}$ at the times $\tau_1, \ldots, \tau_m$ with respect to $\mathbb{F}$. The theorem is proved. ∎

Definition 8.4 *We say that* $((\xi_1,\tau_1),\dots,(\xi_m,\tau_m))$ *satisfied the (conditional) density hypothesis (with a strictly positive density function) with respect to* $\mathcal{F}_\infty$*, if there exists a Borel probability measure* ν *on* $E\times[0,\infty]$ *and a stictly positive function* γ^* *on* $(E\times[0,\infty])^m\times\Omega$ *being* $(\mathcal{E}\otimes\mathcal{B}[0,\infty])^m\otimes\mathcal{F}_\infty$ *measurable such that*

$$\mathbb{Q}[((\xi_1,\tau_1),\dots,(\xi_m,\tau_m))\in \mathtt{A}\,|\mathcal{F}_\infty]=\int_{\mathtt{A}}\gamma^*((x_1,t_1),\dots,(x_m,t_m))\nu^{\otimes m}(\mathsf{d}(x_1,t_1),\dots,\mathsf{d}(x_m,t_m))$$

for any $\mathtt{A}\in(\mathcal{E}\times\mathcal{B}[0,\infty])^m$*.*

Lemma 8.5 *If* $m=1$*, if* (ξ_1,τ_1) *satisfies the density hypothesis with respect to* $\mathcal{F}_\infty$*, the* $\mathbb{G}^{*1}$*-optional splitting formula holds at* τ_1 *with respect to* $\mathbb{F}$

Proof There exists a probability measure $\mathbb{Q}'$ equivalent to $\mathbb{Q}$, under which (ξ_1,τ_1) is independent of $\mathcal{F}_\infty$. We need only to prove the lemma under $\mathbb{Q}'$.

Let X be a bounded random variable in $\sigma(\xi_1,\tau_1)$. Let Y be a bounded random variable in $\mathcal{F}_\infty$. With independence, we have the following identity

$$\mathbb{Q}'[XY|\mathcal{G}_t^*]=\mathbb{Q}'[Y|\mathcal{F}_t]\;\mathbb{Q}'[X|\overline{\mathcal{H}}_1(t)],\;t\geq 0.$$

Applying Lemma 8.1 under $\mathbb{Q}'$, we see that $\mathbb{Q}'[XY|\mathcal{G}_t^*], t\geq 0$, satisfies the $\mathbb{G}^{*1}$-optional splitting formula at τ_1 with respect to $\mathbb{F}$.

Now, to prove the lemma, we need only to repeat the argument in [13, Chapter XX, section 22], as we did in Lemma 4.6. ∎

Reproducing the argument in the proof of Theorem 7.8, we obtain also :

Theorem 8.6 *If the times* $((\xi_1,\tau_1),\dots,(\xi_m,\tau_m))$ *satisfy the density hypothesis with respect to* $\mathcal{F}_\infty$*, then,* $\mathbb{G}^{*m}$*-optional splitting formula holds at the times* $\tau_1,\dots,\tau_m$ *with respect to* $\mathbb{F}$*.*

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