# A Note on BSDEs with singular coefficients

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**Abstract**

In this Note we study a class of BSDEs which admits a particular singularity in their driver. More precisely, we assume that the driver is not integrable and degenerates when approaching to the terminal time of the equation.

## 1 Introduction

Since the seminal works of Bismut [2] and of Pardoux and Peng [13], a lot of attention has been given to the study of Backward Stochastic Differential Equations (BSDEs) as this object naturally arises in stochastic control problems and was found to be an ad hoc tool for many situations in mathematical finance as illustrated in the famous guideline paper [9]. Recall that a BSDE takes the following form:

$$Y_t = Y_T - \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \quad t \in [0, T],$$

where $W$ is a multi-dimensional Brownian motion. The historical natural assumption for providing existence and uniqueness (in the appropriate spaces) is to assume the driver $f$ to be Lipschitz plus some integrability conditions on the terminal condition. However, in applications one may deal with drivers which are not Lipschitz continuous, and which exhibit e.g. a quadratic growth in $z$ (in incomplete markets situations in Finance), or only some monotonicity in the $y$-variable. One way of relaxing the Lipschitz growth condition in $y$ is the so-called stochastic Lipschitz assumption which basically consists in replacing the usual Lipschitz constant by a stochastic process satisfying appropriate integrability conditions. As noted in Section 2.1.2 "Pathology" in [8], even in the stochastic linear framework, one has to be very careful when relaxing the integrability conditions on the driver of the equation. As an illustration, consider the following example presented in [8] (cf. [8, (2.9)]):

$$Y_t = 0 + \int_t^T [rY_s + \sigma Z_s + \gamma Y_s (e^{\gamma(T-s)} - 1)^{-1}]ds + \int_t^T Z_s dW_s, \quad t \in [0, T], \tag{1.1}$$

where $W$ is a one-dimensional Brownian motion, $r, \sigma, \gamma > 0$ and $T$ is a fixed positive real number. It is proved in [8] that the BSDE (1.1) has an infinite number of solutions. Note

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that here the driver is not Lipschitz continuous in $y$ due to the exploding term $(e^{\gamma(T-t)}-1)^{-1}$ as $t$ goes to $T$, and completely escape the existing results of the literature.

The aim of this note is to elaborate on the pathology mentioned in [8] and to try to understand better what kind of behavior can appear as soon as the usual integrability conditions are relaxed. In light of Example (1.1), multiple solutions is one of the behavior which can be observed, however is it the only type of problems that can occur ? For instance is it clear that existence is guaranteed? This note is an attempt in this direction and is motivated by the work in preparation [11] where equations with this specific pathology appear naturally in the financial application under interest in [11].

We proceed as follows. First we make precise the context of our study and we explain what is the notion of solution we use for dealing with non-integrable drivers. Then we deal with the particular case of affine equations in Section 3. These equations already allow us to present several type of pathologic behaviour. We then study in Section 4 a class of non-linear drivers which will be of interest for a specific problem in Finance presented in [11]. In particular, in our main result Theorem 4.4 we provide an existence and uniqueness result under a monotonicty assumption on the the mapping $f$ in (2.1) defined below.

## 2 Preliminaries

In this note $T$ denotes a fixed positive real number and $d$ a given positive integer. We set $(W_t)_{t\in[0,T]} := (W_t^1,\dots,W_t^d)_{t\in[0,T]}$ a $d$-dimensional standard Brownian motion defined on a filtered probability space $(\Omega,\mathcal{F},\mathbb{F}:=(\mathcal{F}_t)_{t\in[0,T]},\mathbb{P})$ where $\mathbb{F}$ denotes the natural filtration of $W$ (completed and right-continuous) and $\mathcal{F}=\mathcal{F}_T$. Throughout this paper "$\mathbb{F}$-predictable" (rep. $\mathbb{F}$-adapted) processes will be referred as predictable processes (resp. adapted). For later use we set for $p\geq 1$:

$$\mathbb{S}^p := \left\{ (Y_t)_{t\in[0,T]} \text{ continuous adapted one dimensional process }, \mathbb{E}\left[\sup_{t\in[0,T]}|Y_t|^p\right] < +\infty \right\},$$

$$\mathbb{H}^p(\mathbb{R}^m) := \left\{ (Z_t)_{t\in[0,T]} \text{ predictable } m\text{-dimensional process }, \mathbb{E}\left[\left(\int_0^T \|Z_t\|^2 dt\right)^{p/2}\right] < +\infty \right\},$$

where $\|\cdot\|$ denotes the Euclidian norm on $\mathbb{R}^m$ ($m\geq 1$). For any element $Z$ of $\mathbb{H}^1(\mathbb{R}^d)$, we set $\int_0^\cdot Z_s dW_s := \sum_{i=1}^d \int_0^\cdot Z_s^i dW_s^i$.

Let $\lambda : (\lambda_t)_{t\in[0,T]}$ a one-dimensional non-negative adapted process. For convenience we set $\Lambda_t := \int_0^t \lambda_s ds,\ t\in[0,T]$. We assume that

**Standing assumption on $\lambda$:**

$$\Lambda_t < +\infty,\ \forall t<T, \text{ and } \Lambda_T = +\infty,\ \mathbb{P}-a.s..$$

The typical example we have in mind is a coefficient $\lambda$ of the form $\lambda_t := (e^{\gamma(T-t)}-1)^{-1}$ as in the introducing example (1.1), or when $\lambda$ is the intensity process related to a prescribed

random time $\tau$ in the context of enlargement of filtration as presented in [11]. In this Note, we aim in studying BSDEs of the form:

$$Y_t = A - \int_t^T [\varphi_s + \lambda_s f(Y_s)]ds - \int_t^T Z_s dW_s, \quad t \in [0,T], \tag{2.1}$$

where $A$ is a regular enough $\mathcal{F}_T$-measurable random variable, $f : \mathbb{R} \to \mathbb{R}$ is a deterministic map and $\varphi$ is a predictable processes with some integrability conditions to be specified. Before going further, we would like to stress that in contradistinction to the classical case where $\lambda$ is bounded (and $A$ and $\varphi$ are square-integrable), the space $\mathbb{S}^2 \times \mathbb{H}^2(\mathbb{R}^d)$ is no more the natural space for solutions of our BSDEs. For instance if $f(x) := x$, the fact that $(Y, Z)$ belongs to $\mathbb{S}^2 \times \mathbb{H}^2(\mathbb{R}^d)$ does not guarantee that

$$\mathbb{E}\left[\int_0^T |\lambda_s Y_s|^p ds\right] < +\infty$$

for some $p \geq 1$ (which would be immediately satisfied with $p = 2$ if $\lambda$ were bounded) leading to a possible definition problem for the term $\int_0^t \lambda_s Y_s ds$ in equation (2.1). For this reason we make very precise the notion of solution in our context.

**Definition 2.1** (Solution)**.** *Let $A$ be an element of $L^1$ and $f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ such that for any $(y, z)$ in $\mathbb{R} \times \mathbb{R}^d$ the stochastic process $(t, \omega) \mapsto f(t, y, z)$ (where as usual we omit the $\omega$-variable in the expression of $f$) is progressively measurable. We say that a pair of predictable processes $(Y, Z)$ with values in $\mathbb{R} \times \mathbb{R}^d$ is solution to the BSDE*

$$Y_t = A - \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \quad t \in [0,T], \tag{2.2}$$

*if*

$$\mathbb{E}\left[\int_0^T |f(t, Y_t, Z_t)|dt + \left(\int_0^T \|Z_t\|^2 dt\right)^{1/2}\right] < +\infty, \tag{2.3}$$

*and Relation* (2.2) *is satisfied for any $t$ in $[0,T]$, $\mathbb{P}$-a.s..*

**Remark 2.2.** *This notion of solution is related to the theory of $L^1$-solution (see e.g. [3, Definition 2.1] or [5, 4]) where in Relation* (2.3) *the expectation is replaced by a $\mathbb{P}$-a.s. criterion. The fact that $Z$ is an element of $\mathbb{H}^1(\mathbb{R}^d)$ implies that the martingale $\int_0^\cdot Z_s dW_s$ is uniformly integrable. Combining this property with the $(\Omega \times [0,T], \mathbb{P} \otimes dt)$-integrability of $f(\cdot, Y, Z)$, it immediately follows that the solution process $Y$ is of class (D) (which then finds similarities with the notion of solution used in [3]).*

**Remark 2.3.** *We would like to stress that even in the case where the terminal condition $A$ is in $L^2$ we do not require $Y$ to be an element of $\mathbb{S}^2$. This fact bears some similarities with the papers [5, 4] and with [3, Section 6].*

**Remark 2.4** (Classical $L^2$ setting)**.** *If $f$ is uniformly (in time) Lipschitz in $(y, z)$ and if $\mathbb{E}\left[|A|^2 + \int_0^T |f(s, 0, 0)|^2 ds\right] < +\infty$ then the fact that there exists $(Y, Z)$ in $\mathbb{S}^2 \times \mathbb{H}^2(\mathbb{R}^d)$ satisfying* (2.2) *implies that the process $f(\cdot, Y_\cdot, Z_\cdot)$ is in $\mathbb{H}^2(\mathbb{R}^d)$ and thus Relation* (2.3) *is satisfied.*

Another important issue in our context is uniqueness. The uniqueness for the $Z$ component will be understood in the $\mathbb{H}^1(\mathbb{R}^d)$ sense. Concerning the $Y$ component, since we do not impose $Y$ to belong to $\mathbb{S}^1$ we will say that $Y^1 = Y^2$ if the processes are indistinguishable (by definition of a solution, both processes are continuous so uniqueness boils down to require $Y^1$ to be a modification of $Y^2$). This definition for uniqueness in our very special setting coincides with the notion of uniqueness with respect to a particular norm. More precisely, according to Remark 2.2 a solution process $Y$ is of class (D). This space can be naturally equipped with the norm $\|\cdot\|_{(D)}$ defined as[1]:

$$\|X\|_{(D)} := \sup_{\tau \in \mathcal{T}} \mathbb{E}[|X_\tau|], \quad X \text{ of class } (D),$$

where $\mathcal{T}$ denotes the set of stopping time smaller than $T$. By [6, Theorem IV.86], uniqueness with respect to the norm $\|\cdot\|_{(D)}$ is equivalent to indistinguashility.

From now on, by solution to a BSDE we mean a solution in the sense of Definition 2.1. For any pair of ($\mathcal{F}_T$-measurable) random variables $(A, B)$, we write $A \not\equiv B$ if $\mathbb{P}[A \neq B] > 0$. Similarly, $A = B$, $\mathbb{P}$-a.s. will be denoted as $A \equiv B$. Throughout this paper $C$ will denote a generic constant which can differ from line to line. Finally we set $L^p := L^p(\Omega, \mathcal{F}_T, \mathbb{P})$ for any $p \geq 1$.

# 3 Affine equations with exploding coefficients

As the reader will figure out later, it seems pretty complicated to define a general theory since many situations (non-existence, non-uniqueness) can be found under our assumption on $\lambda$ for BSDEs of the form (2.1). These very different behaviours can be clearly illustrated by studying affine equations, that is when $f$ in (2.1) stands for the identity (or minus the identity). In some sense, our results find immediate counterparts in the deterministic realm while considering the corresponding ODEs when all the coefficients of the equation are deterministic. However, for this latter case, techniques of time-reversion can be employed to provide immediate results which unfortunately can not be applied in the stochastic framework due to the measurability feature of the solution to a BSDE calling for different techniques.

In this section, we consider stochastic affine BSDEs of one of the following forms:

$$Y_t = A - \int_t^T (\varphi_s - \lambda_s Y_s) ds - \int_t^T Z_s dW_s; \quad t \in [0, T], \tag{3.1}$$

$$Y_t = A - \int_t^T (\varphi_s + \lambda_s Y_s) ds - \int_t^T Z_s dW_s; \quad t \in [0, T], \tag{3.2}$$

We start with Equation (3.1).

**Proposition 3.1.** *Let $A$ be in $L^1$ and $\varphi := (\varphi_t)_{t\in[0,T]}$ be an element of $\mathbb{H}^1(\mathbb{R})$. The Brownian BSDE*

$$dY_t = (\varphi_t - \lambda_t Y_t) dt + Z_t dW_t; \quad Y_T = A. \tag{3.3}$$

*admits no solution if $A \not\equiv 0$. If $A \equiv 0$, the BSDE (3.3) may admit infinitely many solutions.*

[1] This norm is referred as $\|\cdot\|_1$ in [7, Definition VI.20], we do not use this notation here to avoid any confusion.

*Proof.* **Step 1: non-existence of solution if $A \not\equiv 0$**

Let $(Y, Z)$ be a solution to (3.3). Assume there exists a set $\mathcal{A}$ in $\mathcal{F}_T$ such that $A > 0$ on $\mathcal{A}$. By definition of a solution, it holds that

$$\int_0^T |\lambda_s Y_s| ds < \infty, \quad \mathbb{P}\text{-a.s.} \tag{3.4}$$

since

$$\int_0^T |\lambda_s Y_s| ds \leq \int_0^T |\varphi_s + \lambda_s Y_s| ds + \int_0^T |\varphi_s| ds.$$

For $\omega$ in $\mathcal{A}$, let $t_0(\omega) := \sup\{t \in [0,T],\ Y_t(\omega) < A/2\}$. By continuity of $Y$ and the fact that $Y_T = A$, for $\mathbb{P}$-almost all $\omega$ in $\mathcal{A}$, $t_0(\omega) < T$ and $Y_t(\omega)\mathbf{1}_{[t_0(\omega),T]}(t) \geq A/2$. Note that $t_0$ is not a stopping time but only a random $\mathcal{F}_T$-measurable variable. As a consequence, on $\mathcal{A}$, it holds that

$$\int_0^T |\lambda_s Y_s| ds \geq \int_{t_0}^T \lambda_s Y_s ds \geq A/2 \underbrace{\int_{t_0}^T \lambda_s ds}_{=+\infty},$$

which contradicts (3.4). As a consequence, $A \leq 0$, $\mathbb{P}$-a.s.. Similarly, one proves that $A \geq 0$, $\mathbb{P}$-a.s..

**Step 2: Multiplicity of solutions if $A \equiv 0$**

If $A \equiv 0$, we will provide examples of non-uniqueness of solution. Remark that if $(\mathcal{Y}, \mathcal{Z})$ is a particular solution to the BSDE and that $(Y, Z)$ is a solution to the (fundamental) BSDE:

$$dY_t = -\lambda_t Y_t dt + Z_t dW_t, \quad Y_T = 0, \tag{3.5}$$

then as for ODEs, the sum of any of these fundamental solutions and $\mathcal{Y}$ is solution to (3.3) (together with the sum of the associated $Z$ processes). In addition, Equation (3.5) admits an infinite number of solutions (like $Y_t = Y_0 e^{-\Lambda_t}$ and $Z \equiv 0$ which is an adapted continuous solution to the BSDE satisfying $\mathbb{E}\left[\int_0^T \lambda_s |Y_s| ds\right] = |Y_0|$ for any chosen real number $Y_0$). An example of particular solution can be given by the process $\mathcal{Y}_t := -\mathbb{E}\left[\int_t^T e^{\int_t^s \lambda_u du} \varphi_s ds | \mathcal{F}_t\right]$ if it is well-defined, such that Relation (2.3) is satisfied. In that case, the existence of $\mathbb{E}\left[\int_t^T \varphi_s e^{\Lambda_s} ds | \mathcal{F}_t\right]$ entails that it converges to 0 as $t$ goes to $T$. In that case $\mathcal{Y}_T = 0$ and one can check that $\mathcal{Y}$ together with the process $\mathcal{Z} := \tilde{Z} e^{-\Lambda}$ is solution to (3.3), where $\tilde{Z}$ is such that $\mathbb{E}\left[\int_0^T \varphi_s e^{\Lambda_s} ds | \mathcal{F}_t\right] = \mathbb{E}\left[\int_0^T \varphi_s e^{\Lambda_s} ds\right] - \int_0^t \tilde{Z}_s dW_s$ ($t \in [0,T]$). We conclude the proof with an example: set $\varphi_t := e^{-\Lambda_t}$. With this choice, the process $\mathcal{Y}$ satisfies all the requirements above providing infinitely many solutions to (3.3). □

Note that in the previous proof the non-existence when $A \not\equiv 0$ relies on the assumption that $\int_0^T |\varphi_s| ds < +\infty$, $\mathbb{P}-a.s.$. If the latter is not satisfied, one may find existence of solutions for $A \not\equiv 0$ as the following proposition illustrates in the deterministic case.

**Proposition 3.2.** *Let $A$ be a given constant and $\varphi := (\varphi_t)_{t\in[0,T]}$ be a deterministic map. We assume that $\lambda$ is a deterministic function which is such that $\Lambda_t = \int_0^t \lambda_s ds < +\infty$, for $t < T$ and $\int_0^T \lambda_s ds = +\infty$. Then*

*(i) If* $e^{-\Lambda_t}\int_0^t e^{\Lambda_s}\varphi_s ds$ *converges to* $C \neq 0$ *when* $t$ *goes to* $T$*, then the ODE*

$$dY_t = (\varphi_t - \lambda_t Y_t)dt; \quad Y_T = A. \tag{3.6}$$

*admits no solution if* $A \neq C$*. If* $A = C$*, the ODE* (3.6) *admits a unique solution* $Y_t = e^{-\Lambda_t}\int_0^t e^{\Lambda_s}\varphi_s ds$ *provided that* $\int_0^T |\varphi_t + \lambda_t Y_t|dt < \infty$*.*

*(ii) If* $e^{-\Lambda_t}\int_0^t e^{\Lambda_s}\varphi_s ds$ *converges to* $0$*, the ODE* (3.6) *has no solution if* $A \neq 0$*, and a infinite number of solutions given by* $Y_t = e^{-\Lambda_t} - Y_0 + \int_0^t e^{\Lambda_s}\varphi_s ds$ *in the case* $A = 0$*.*

*(iii) If* $e^{-\Lambda_t}\int_0^t e^{\Lambda_s}\varphi_s ds$ *does not converge, the ODE* (3.6) *has no solution.*

**Remark 3.3.** *Note that the assumption in (i) of Proposition 3.2 can be met only if* $\int_0^T |\varphi_s|ds = +\infty$*. Indeed, assume that* $\int_0^T |\varphi_s|ds < \infty$*. Let* $\varepsilon > 0$ *and* $t < T$*. We have:*

$$\begin{aligned} e^{-\Lambda_t}\left|\int_0^t e^{\Lambda_s}\varphi_s ds\right| &\leq e^{-\Lambda_t}\int_0^{t-\varepsilon} e^{\Lambda_s}|\varphi_s|ds + e^{-\Lambda_t}\int_{t-\varepsilon}^t e^{\Lambda_s}|\varphi_s|ds \\ &\leq e^{-\Lambda_t}e^{\Lambda_{t-\varepsilon}}\int_0^T |\varphi_s|ds + \int_{t-\varepsilon}^t |\varphi_s|ds. \end{aligned}$$

*Hence as* $t$ *goes to* $T$*, we have that* $\lim_{t\to T} e^{-\Lambda_t}\left|\int_0^t e^{\Lambda_s}\varphi_s ds\right| \leq \int_{T-\varepsilon}^T |\varphi_s|ds$*, which implies that*

$$\lim_{t\to T} e^{-\Lambda_t}\left|\int_0^t e^{\Lambda_s}\varphi_s ds\right| = 0,$$

*which contradicts the assumption of (i).*

**Remark 3.4.** *Since* $\lambda$ *is unbounded, assuming* $A, \lambda$ *and* $\varphi$ *to be deterministic in Equation* (3.1) *does not lead to deterministic solutions (and so differs from the ODE framework of Proposition 3.2) as the following example illustrates. Assume* $A \equiv 0$*,* $\varphi \equiv 0$ *and* $\lambda$ *is a deterministic mapping. Then for any element* $\beta := (\beta_t)_{t\in[0,T]}$ *in* $\mathbb{H}^1(\mathbb{R}^d)$*, the pair of adapted processes* $(Y, Z)$ *defined as:*

$$Y_t := Y_0 e^{-\Lambda_t} + e^{-\Lambda_t}\int_0^t \beta_s dW_s, \quad Z_t := e^{-\Lambda_t}\beta_t, \quad t \in [0,T]$$

*is solution to* (3.1)*. This provides in turn, a generalization of the fundamental solution to Equation* (3.5)*.*

We continue with the BSDE:

$$dY_t = (\varphi_t + \lambda_t Y_t)dt + Z_t dW_t; \quad Y_T = A.$$

**Proposition 3.5.** *Let* $A$ *be in* $L^1$ *and* $\varphi := (\varphi_t)_{t\in[0,T]}$ *be a bounded predictable process. The Brownian BSDE*

$$dY_t = (\varphi_t + \lambda_t Y_t)dt + Z_t dW_t; \quad Y_T = A. \tag{3.7}$$

*admits no solution unless* $A \equiv 0$*. In that case the BSDE admits a unique solution.*

*Proof.* Let $(Y, Z)$ be a solution and set $\tilde{Y} := Ye^{-\Lambda} - \int_0^{\cdot} e^{-\Lambda_s}\varphi_s ds$. We have that

$$d\tilde{Y}_t = e^{-\Lambda_t} Z_t dW_t, \quad \text{and } \tilde{Y}_T = -\int_0^T e^{-\Lambda_s}\varphi_s ds.$$

Hence $\tilde{Y}$ is a $L^1$-martingale and

$$\tilde{Y}_t = -\mathbb{E}\left[\int_0^T e^{-\Lambda_s}\varphi_s ds|\mathcal{F}_t\right], \quad t \in [0,T],$$

leading to

$$Y_t = -\mathbb{E}\left[\int_t^T e^{-\int_t^s \lambda_u du}\varphi_s ds|\mathcal{F}_t\right], \quad t \in [0,T]. \tag{3.8}$$

In particular, $Y_T = 0$. Indeed, since $\varphi$ is bounded

$$e^{\Lambda_t}\left|\int_t^T e^{-\Lambda_s}\varphi_s ds\right| \leq e^{\Lambda_t}e^{-\Lambda_t}\int_t^T |\varphi_s| ds \leq \|\varphi\|_\infty (T-t).$$

This proves that there is no solution to the equation unless $A \equiv 0$. We now assume that $A \equiv 0$. In that case, we prove that the process given by (3.8) together with a suitable process $Z$ is a solution to the BSDE. We begin with the integrability condition

$$\mathbb{E}\left[\int_0^T |\lambda_s Y_s| ds\right] < +\infty.$$

We have that:

$$\begin{aligned}
\mathbb{E}\left[\int_0^T |\lambda_s Y_s| ds\right] &= \mathbb{E}\left[\int_0^T \left|\lambda_s \mathbb{E}\left[\int_s^T e^{-\int_s^u \lambda_r dr}\varphi_u du|\mathcal{F}_s\right]\right| ds\right] \\
&\leq \mathbb{E}\left[\int_0^T \lambda_s e^{\Lambda_s}\int_s^T e^{-\Lambda_u}|\varphi_u| du ds\right] \\
&= \mathbb{E}\left[\lim_{s\to T, s<T}[e^{\Lambda_s}\int_s^T e^{-\Lambda_u}|\varphi_u| du] - \int_0^T e^{-\Lambda_u}|\varphi_u| du + \int_0^T e^{\Lambda_s}e^{-\Lambda_s}|\varphi_s| ds\right] \\
&< +\infty
\end{aligned}$$

where we have used the estimate $e^{-\Lambda_u} \leq e^{-\Lambda_s}$ for $u \geq s$. We now turn to the definition of the $Z$ process in the equation. Consider the $L^2$ martingale $\hat{L}$ defined as:

$$\hat{L}_t := \mathbb{E}\left[\int_0^T e^{-\Lambda_s}\varphi_s ds|\mathcal{F}_t\right], \quad t \in [0,T].$$

By martingale representation Theorem, there exists a process $\hat{Z}$ in $\mathbb{H}^2(\mathbb{R}^d)$ such that $\hat{L}_t = \hat{L}_0 + \int_0^t \hat{Z}_s dW_s$. Now let $Z_t := -e^{\Lambda_t}\hat{Z}_t$ and $L_t := \int_0^t Z_s dW_s$ which is a local martingale. With this definition, it is clear that the pair $(Y, Z)$ has the dynamics:

$$dY_t = (\varphi_t + \lambda_t Y_t)dt + Z_t dW_t, \quad t \in [0,T].$$

Note that a priori $\int_0^{\cdot} Z_s dW_s$ is only a local martingale. From the equation, there exists a constant $C > 0$ such that

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t Z_s dW_s\right|\right] \leq C\left(2\mathbb{E}[\sup_{t\in[0,T]}|Y_t|] + T\|\varphi\|_\infty + \mathbb{E}\left[\int_0^T |\lambda_s Y_s| ds\right]\right) < \infty,$$

since by definition $Y$ is bounded. Hence $Z$ is an element of $\mathbb{H}^1(\mathbb{R}^d)$ by Burkholder-Davis-Gundy's inequality. Finally note that this argument provides uniqueness of the solution since we have characterized any solution via the process $\tilde{Y}$. □

**Remark 3.6.** *Up to a Girsanov transformation (as in the proof of Corollary 4.6 below), the previous result can be generalized to equations of the form:*

$$Y_t = A - \int_t^T (\varphi_s + \sigma_t Z_t - \lambda_s Y_s)ds - \int_t^T Z_s dW_s; \quad t \in [0,T],$$

$$Y_t = A - \int_t^T (\varphi_s + \sigma_t Z_t + \lambda_s Y_s)ds - \int_t^T Z_s dW_s; \quad t \in [0,T],$$

*where $\sigma := (\sigma_t)_{t\in[0,T]}$ is any bounded predictable process. In particular, our results contain the motivating example* (2.1) *from [8].*

# 4 A class of non-linear equations

From the results of Section 3 it appears clearly that there is no hope to provide a general theory for equations of the form (2.1) with a non-integrable coefficient $\lambda$. However, motivated by a problem in Finance (which will be presented in [11]), we need to prove that the particular equation (2.1) with $f(x) := \alpha^{-1}(1 - e^{-\alpha x})$ admits a unique solution if and only if $Y_T = 0$. In addition, in that case in order to solve the financial problem associated to this equation, we need to prove that the process $Y$ is bounded and that the martingale $\int_0^{\cdot} Z_s dW_s$ is a BMO-martingale (whose definition will be recalled below). This section is devoted to the study of a class of equations which generalizes this particular case. We start with a generalization of Proposition 3.1.

**Proposition 4.1.** *Let $\varphi$ be an element of $\mathbb{H}^1(\mathbb{R})$ and $A$ in $L^1$. Let $f : \mathbb{R} \to \mathbb{R}$ be a increasing (respectively decreasing) map with $f(0) = 0$. The BSDE*

$$Y_t = A - \int_t^T [\varphi_s + \lambda_s f(Y_s)]ds - \int_t^T Z_s dW_s, \quad t \in [0,T] \tag{4.1}$$

*admits no solution if $A \not\equiv 0$.*

*Proof.* The proof follows the lines of the one of Proposition 3.1. □

**Remark 4.2.** *Note that the previous result does not contradict the conclusion of Proposition 3.2 in the deterministic setting, since according to Remark 3.3 the assumption of (i) in Proposition 3.2 on $\lambda$ is not compatible with the $\mathbb{H}^1(\mathbb{R})$-requirement of Proposition 4.1.*

The following lemma will be of interest for proving the main result of this section.

**Lemma 4.3.** *Let $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(0) = 0$, $f$ is non-decreasing and $f(x) - x \leq 0$, $\forall x \in \mathbb{R}$. Then the equation*

$$Y_t = 0 - \int_t^T \lambda_s f(Y_s)ds - \int_t^T Z_s dW_s, \quad t \in [0,T] \tag{4.2}$$

*admits $(0,0)$ as unique solution.*

*Proof.* It is clear that $(0,0)$ solves (4.2). Let $(Y,Z)$ be any solution and $\tilde{Y} := e^{-\Lambda}Y$. It holds that $\tilde{Y}_T = 0$ and that

$$d\tilde{Y}_t = \lambda_t e^{-\Lambda_t}(f(Y_t) - Y_t)dt + e^{-\Lambda_t} Z_t dW_t.$$

Since $f(x) - x \leq 0$, for all $x \in \mathbb{R}$, $\tilde{Y}_t \geq 0$. Hence by definition, $Y_t \geq 0$, $\forall t \in [0,T]$, $\mathbb{P}$-a.s.. From Equation (4.2), since $Y \geq 0$ we have that $f(Y_t) \geq 0$ which implies that

$$Y_t = 0 - \mathbb{E}\left[\int_t^T \lambda_s f(Y_s)ds|\mathcal{F}_t\right] \leq 0, \quad \forall t \in [0,T],\ \mathbb{P}-a.s..$$

As a consequence $Y_t = 0$ for all $t$, $\mathbb{P}$-a.s. which in turn gives $Z = 0$ (in $\mathbb{H}^1(\mathbb{R}^d)$), which concludes the proof. □

We now consider a class of BSDE that will be of interest in [11].

**Theorem 4.4.** *Let $\varphi$ be a non-negative bounded predictable process and $f : \mathbb{R} \to \mathbb{R}$ a continuously differentiable map satisfying: $f(0) = 0$, $f$ is non-decreasing, there exists $\delta > 0$ such that*

$$f(x) - x \leq 0,\ \forall x \in \mathbb{R} \text{ and } f'(x) \geq \delta,\ \forall x \leq 0.$$

*Then the BSDE*

$$Y_t = A - \int_t^T [\varphi_s + \lambda_s f(Y_s)]ds - \int_t^T Z_s dW_s, \quad t \in [0,T]. \tag{4.3}$$

*admits a solution if and only if $A \equiv 0$. In that case, the solution is unique, $Y$ is bounded and $\int_0^\cdot Z_s dW_s$ is a BMO-martingale that is:*

$$\text{esssup}_{\tau \in \mathcal{T}} \mathbb{E}\left[\int_\tau^T \|Z_s\|^2 ds|\mathcal{F}_\tau\right] < \infty,$$

*where we recall that $\mathcal{T}$ denotes the set of stopping time smaller than $T$.*

*Proof.* We have seen in Proposition 4.1 that the only possible value for $A$ to admit a solution is 0. From now on, we assume that $A \equiv 0$.

**Step 1: some estimates**
We start with some estimates on the (possible) solution to the BSDE. Assume that there exists a solution $(Y,Z)$ to Equation (4.3). Since $\varphi$ is non-negative, $(Y,Z)$ is a sub-solution to the BSDE:

$$\mathcal{Y}_t = 0 - \int_t^T \lambda_s f(\mathcal{Y}_s)ds - \int_t^T \mathcal{Z}_s dW_s, \quad t \in [0,T],$$

which admits $(0,0)$ as unique solution by Lemma 4.3. Indeed, this sub-solution property is classical for ($L^2$) Lipschitz BSDE and follows from comparison theorem, however here the BSDE (4.2) is not Lipschitz due to the unboundedness of $\lambda$. In our context the result can be proved explicitly. Since $f(0) = 0$ the BSDE (4.3) rewrites as[2]

$$Y_t = 0 - \int_t^T \tilde{\lambda}_s Y_s ds - \int_t^T Z_s dW_s - \int_t^T \varphi_s ds, \quad t \in [0,T], \tag{4.4}$$

[2]by: $f(x) - f(y) = (x-y)\int_0^1 f'(y + \theta(x-y))d\theta$, $(x,y) \in \mathbb{R}^2$

with $\tilde{\lambda}_t := \lambda_t \int_0^1 f'(\theta Y_t) d\theta$ which is non-negative. Following the lines of Proposition 3.5 with $\lambda$ replaced by $\tilde{\lambda}$, and using the non-negativity of $\varphi$, we get that

$$Y_t = -\mathbb{E}\left[\int_t^T e^{-\int_t^s \tilde{\lambda}_u du} \varphi_s ds | \mathcal{F}_t\right] \leq 0.$$

From the non-positivity of $Y$, we can deduce that $\tilde{\lambda} \geq \delta\lambda$ from which we get that

$$Y_t \geq -\mathbb{E}\left[\int_t^T e^{-\int_t^s \lambda_u du} \varphi_s ds | \mathcal{F}_t\right], \quad t \in [0,T].$$

To summarize, we have proven that

$$-T\|\varphi\|_\infty \leq -\mathbb{E}\left[\int_t^T e^{-\int_t^s \lambda_u du} \varphi_s ds | \mathcal{F}_t\right] \leq Y_t \leq 0, \quad \forall t \in [0,T],\ \mathbb{P}-a.s.. \tag{4.5}$$

We now prove that the process $\int_0^\cdot Z_s dW_s$ is a BMO-martingale. Let $\tau$ be any stopping time such that $\tau \leq T$. By Itô's formula, we have that

$$|Y_\tau|^2 = 0 - 2\int_\tau^T \varphi_s Y_s ds - 2\int_\tau^T Y_s Z_s dW_s - \int_\tau^T \|Z_s\|^2 ds - 2\int_\tau^T Y_s f(Y_s)\lambda_s ds.$$

Since $Y$ is bounded and $Z$ is an element of $\mathbb{H}^1(\mathbb{R}^d)$, the stochastic integral process is a true martingale, and since $Y$ is non-positive, the last term of the previous expression is non-positive. As a consequence, it holds that

$$\mathbb{E}\left[\int_\tau^T \|Z_s\|^2 ds | \mathcal{F}_\tau\right] \leq -2\mathbb{E}\left[\int_\tau^T \varphi_s Y_s ds | \mathcal{F}_\tau\right] \leq 2T^2\|\varphi\|_\infty^2,$$

So the claim is proved.

**Step 2: existence**

Now, we prove the existence of a solution for the BSDE (4.3). For any positive integer $n$, we set $\lambda_\cdot^n := \lambda_\cdot \wedge n$, $\tilde{f}(x) := f(x)\mathbf{1}_{\{[-T\|\varphi\|_\infty, 0]\}}(x) + f(-T\|\varphi\|_\infty)\mathbf{1}_{\{(-\infty, -T\|\varphi\|_\infty]\}}(x)$, and $(Y^n, Z^n)$ the unique (classical) solution in $\mathbb{S}^2 \times \mathbb{H}^2(\mathbb{R}^d)$ to the BSDE

$$Y_t^n = 0 - \int_t^T [\varphi_s + \tilde{f}(Y_s^n)\lambda_s^n] ds - \int_t^T Z_s^n dW_s, \quad t \in [0,T]. \tag{4.6}$$

It is clear that this equation admits a unique solution since $\tilde{f}$ is Lipschitz continuous and $\lambda^n$ is bounded. In addition, following the same lines as above we can prove that $Y^n \leq 0$ and

$$|Y_t^n| \leq T\|\varphi\|_\infty, \quad \forall t \in [0,T],\ \mathbb{P}-a.s..$$

Hence we can re-write Equation (4.6) as:

$$Y_t^n = 0 - \int_t^T [\varphi_s + f(Y_s^n)\lambda_s^n] ds - \int_t^T Z_s^n dW_s, \quad t \in [0,T]. \tag{4.7}$$

Repeating the same argument used in the previous step we can prove that

$$\sup_n \mathbb{E}\left[\int_0^T |Z_t^n|^2 dt\right] < \infty \tag{4.8}$$

showing that the sequence $(Y^n, Z^n)_n$ is bounded in the Hilbert space $\mathbb{S}^2 \times \mathbb{H}^2(\mathbb{R}^d)$. As a consequence it converges weakly to an element $(Y, Z)$ in $\mathbb{S}^2 \times \mathbb{H}^2(\mathbb{R}^d)$. By Mazur's theorem for every $n$, there exists a sequence of real numbers $(\mu_k^n)_k$ in $[0,1]$ such that $\sum_{k\geq n}^{\infty} \mu_k^n = 1$ with $\#\{k \geq n,\ \mu_k^n \neq 0\} < \infty$ such that

$$(\tilde{Y}^n, \tilde{Z}^n) := \left( \sum_{k\geq n} \mu_k^n Y^k, \sum_{k\geq n} \mu_k^n Z^k \right)$$

converges strongly in $\mathbb{S}^2 \times \mathbb{H}^2(\mathbb{R}^d)$ to $(Y, Z)$. We would like to point out at this stage that by construction $Y$ take values in $[-T\|\varphi\|_\infty, 0]$. Now it remains to prove that $(Y, Z)$ is a solution to Equation (4.3). To this end we aim in applying [1, Theorem 1]. We have obtained already that $\lim_{n\to\infty} \mathbb{E}[\sup_{t\in[0,T]} |\tilde{Y}_t^n - Y_t|] = 0$. To satisfy the assumptions of [1, Theorem 1], we have to prove that for every $n$

$$\sup_n \mathbb{E}\left[ \left( \int_0^T |\tilde{Z}_s^n|^2 ds \right)^{1/2} \right] \leq C \tag{4.9}$$

(which by Burkholder-Davis-Gundy's inequality implies that $\mathbb{E}\left[\sup_{t\in[0,T]} \left|\int_0^t \tilde{Z}_s^n dW_s\right|\right] \leq C$ for every $n$) and that

$$\sup_n \mathbb{E}\left[ \int_0^T \sum_{k\geq n} \mu_k^n \lambda_s^k |f(Y_s^k)| ds \right] \leq C, \quad \forall n \geq 1. \tag{4.10}$$

Relation (4.9) is a direct consequence of (4.8). With this estimate at hand we can deduce Relation (4.10). Indeed, using Equation (4.6) and the uniform estimates on the $Y^k$ obtained above we deduce that

$$\begin{aligned} \mathbb{E}\left[ \int_0^T \sum_{k\geq n} \mu_k^n \lambda_s^k |\tilde{f}(Y_s^k)| ds \right] &= \mathbb{E}\left[ \left| \int_0^T \sum_{k\geq n} \mu_k^n \lambda_s^k f(Y_s^k) ds \right| \right] \\ &= \mathbb{E}\left[ \left| \tilde{Y}_0^n + \int_0^T \varphi_s ds + \int_0^T \tilde{Z}_s^n dW_s \right| \right] \leq C, \quad n \geq 1, \end{aligned}$$

where $C$ depends only on $T$ and $\|\varphi\|_\infty$ (and not on $n$). Thus, by [1, Theorem 1], $Y$ writes down as $Y_t = A_t + \int_0^t \varphi_s ds + \int_0^t Z_s dW_s$ and

$$\lim_{n\to\infty} \mathbb{E}[\sup_{t\in[0,T]} |A_t - \int_0^t \sum_{k\geq n} \mu_k^n \lambda_s^k f(Y_s^k) ds|] = 0. \tag{4.11}$$

We now identify the process $A$. We proceed in two steps: first we prove that $A_t = \int_0^t \tilde{f}(Y_s)\lambda_s ds$ for $t < T$ and then we prove the relation for $t = T$. Fix $t < T$. We have that

$$\begin{aligned} &\left| \int_0^t \sum_{k\geq n} \mu_k^n f(Y_s^k)(\lambda_s^k - \lambda_s) ds \right| \\ &\leq C \int_0^t |\lambda_s^n - \lambda_s| ds \to_{n\to\infty} 0, \quad \mathbb{P} - a.s. \end{aligned}$$

by monotone convergence Theorem, since the $Y^k$ are uniformly bounded and $\Lambda_t < \infty$, $\mathbb{P}$-a.s.. Hence up to a subsequence,

$$\lim_{n\to\infty} \left| A_t - \int_0^t \sum_{k\geq n} \mu_k^n f(Y_s^k)\lambda_s ds \right| = 0.$$

In addition by comparison theorem (for Lipschitz BSDEs), the sequence $(Y^n)_n$ is non-decreasing and uniformly bounded so it converges almost surely to some predictable process $\hat{Y}$ (which is then bounded). The strong convergence of $(\tilde{Y}^n)_n$ to $Y$ immediately implies that $\hat{Y} = Y$ so $(Y^n)_n$ converges a.s. to $Y$ and $Y^n \leq Y$. Thus

$$\begin{aligned} &\left| \int_0^t \sum_{k\geq n} \mu_k^n (f(Y_s^k) - f(Y_s))\lambda_s ds \right| \\ &\leq C \int_0^t |Y_s^n - Y_s|\lambda_s ds \to_{n\to\infty} 0 \end{aligned}$$

where once again we have used monotone convergence Theorem. This leads to

$$A_t = \int_0^t f(Y_s)\lambda_s ds, \quad \mathbb{P} - a.s.$$

for any $t < T$. The relation for $t = T$ follows from the continuity of $A$ by (4.11). Finally according to Definition 2.1 it remains to prove Relation (2.3). This is done as follows by combining the monotone convergence theorem together with (4.10) and (4.11):

$$\mathbb{E}\left[\int_0^T |f(Y_s)|\lambda_s ds\right] = \lim_{t\to T} \mathbb{E}\left[\int_0^t |f(Y_s)|\lambda_s ds\right] = \lim_{t\to T} \mathbb{E}[|A_t|] < \infty.$$

**Step 3: uniqueness**

Assume there exist two solutions $(Y^1, Z^1)$ and $(Y^2, Z^2)$ to the BSDE (4.3). Then, the difference processes $(\delta Y := Y^1 - Y^2, \delta Z := Z^1 - Z^2)$ satisfies

$$\delta Y_t = 0 - \int_t^T \lambda_s (f(Y_s^1) - f(Y_s^2)) ds - \int_t^T \delta Z_s dW_s, \quad t \in [0, T].$$

From the existence part we know that both processes $Y^1$ and $Y^2$ are uniformly bounded. As a consequence the mapping $f$ restricted to the set $[-T\|\varphi\|_\infty, 0]$ has a non-negative derivative. Hence the equation re-writes as:

$$\delta Y_t = 0 - \int_t^T \tilde{\lambda}_s \delta Y_s ds - \int_t^T \delta Z_s dW_s, \quad t \in [0, T],$$

where $\tilde{\lambda}_t := \lambda_t \int_0^1 f'(Y_t^2 + \theta(Y_t^1 - Y_t^2))d\theta$ is a non-negative process which satisfies $\int_0^t \tilde{\lambda}_s ds < \infty$ for $t < T$, $\mathbb{P}$-a.s. and $\int_0^T \tilde{\lambda}_s ds = \infty$, $\mathbb{P}$-a.s.. Similar to Proposition 3.5 with $\lambda$ replaced with $\tilde{\lambda}$, we deduce that $(\delta Y, \delta Z) = (0, 0)$ is the unique solution. □

**Remark 4.5.** *Note that our previous result is not contained in the theory of monotonic drivers for BSDEs (see e.g. [12, 3] or [10]) where conditions of the form [3, (H5) and (H1")] are not satisfied in our setting due to the non-integrability at $T$ of $\Lambda$.*

We conclude with an example which will be of importance in [11].

**Corollary 4.6.** *Let $\theta := (\theta_t)_{t\in[0,T]}$ be a bounded predictable process with values in $\mathbb{R}^d$ and $\alpha > 0$. The BSDE*

$$Y_t = A - \int_t^T [\frac{\|\theta_s\|^2}{\alpha} + Z_s\theta_s + \frac{\lambda_s}{\alpha}(1 - e^{-\alpha Y_s})]ds - \int_t^T Z_s dW_s, \quad t \in [0,T]. \tag{4.12}$$

*admits a solution if and only if $A \equiv 0$. In that case, the solution is unique, $Y$ is bounded and $\int_0^\cdot Z_s dW_s$ is a BMO-martingale*

*Proof.* Since $\theta$ is bounded the stochastic process $\mathfrak{E}_\cdot := \exp\left(-\sum_{i=1}^d \int_0^\cdot \theta_s^i dW_s^i - \frac{1}{2}\int_0^\cdot \|\theta_s\|^2 ds\right)$ is a martingale and thus defines a probability measure $\mathbb{Q}$ as $\frac{d\mathbb{Q}}{d\mathbb{P}} := \mathfrak{E}_T$. From Girsanov's theorem the process $W^\theta := (W^{\theta,1}, \dots, W^{\theta,d})$ with $W^{\theta^i} := W^i + \int_0^\cdot \theta_s^i ds$ is a $\mathbb{Q}$-Brownian motion and so the BSDE (4.12) re-writes as:

$$Y_t = A - \int_t^T [\frac{\|\theta_s\|^2}{\alpha} + \frac{\lambda_s}{\alpha}(1 - e^{-\alpha Y_s})]ds - \int_t^T Z_s dW_s^\theta, \quad t \in [0,T].$$

The proof then follows by Theorem 4.4 with $f(x) := \alpha^{-1}(1 - e^{-\alpha x})$. □

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