Optimal Trading in a Two-Sided Limit Order Book

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Abstract

This paper studies four trading algorithms of a professional trader, in a realistic two-sided limit order book whose dynamics are driven by the order book events. The identity of the trader can be either privileged or regular, either a hedge fund or a brokery agency. The speed and cost of trading can be balanced by properly choosing active strategies on the displayed orders in the book and passive strategies on the hidden orders within the spread. We shall show that the price switching algorithms provide lower and upper bounds of the singular trading algorithms. For both a privileged trader and a regular trader, the optimal price switching strategy exists and is expressed in terms of the value function. A parallelizable algorithm to numerically compute the value function and optimal price switching strategy for the discretized state process is provided.

Keywords and Phrases: Limit order book, algorithmic trading, switching control.

AMS 2000 Subject Classifications: 60G40, 93E20.

1 Introduction

1.1 Literature and overview

Market microstructure is an interdisciplinary field involving economics, finance, probability and optimization, statistics, and even psychology, that studies the order-driven price formulation processes in equity markets like those of stocks, futures and foreign exchanges. Due to the complexity of the phenomena, the research works on market microstructure usually focus on individual aspects of the problem. Interesting questions studied so far include econometrics of the order books and of the market maker's inventory levels, optimal market making, a buyer or seller's optimal order execution, and limiting behaviors of the queueing system of limit orders and bid and ask prices.

The study of market microstructure dates back to at least four decades ago, and persists up till present time. It is hard to enumerate all the literature on this field. The books O'Hara (1997) and Hasbrouck (2007) provide an overview of quantitative analysis of market microstructure. One significant development in recent years is the prevalence of electronic trading platforms as an alternative to markets where prices are determined via a market maker's auction and the traders' bidding; the other is the popularity of applying stochastic control to solving optimal execution and optimal market making problems. Readers are welcome to Lehalle and Laruelle (2013) for latest updates in the field of market microstructure and algorithmic trading.

Stochastic control provides the theory and methodologies to find actions that optimize an objective, while the actions can influence the evolution of some random processes to which the objective is associated. It naturally facilitates the study of financial markets where participants, the assembly of whose activities contribute to the price evolution, seek to maximize profits and minimize losses. The application of stochastic control to optimizing activities in an order book traces back to early works like Ho and Stoll (1981).

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There have been many frameworks to study trading and order execution in limit order books. Among them are the equilibrium models surveyed in Parlour and Seppi (2008), the model with stochastic bid and ask prices and deterministic order book shape as in Alfonsi, Schied and coauthors (2009, 2010, 2012) and in Predoiu, Shaikhet and Shreve (2011), the model with stochastic mid price and deterministic or stochastic spread as in Avellaneda and Stoikov (2008) and Guilbaud and Pham (2013), the Almgren-Chriss model used by many in the industry, as in Almgren (2003), Almgren and Chriss (2000), Bouchard, Dang and Lehalle (2011) and Gatheral and Schied (2011), and maximizing the utility by choosing an optimal posting distance that determines the intensity of the execution process as in Guéant, Lehalle and Fernandez-Tapia (2012a), Guéant, Lehalle and Fernandez-Tapia (2012b) and Laruelle, Lehalle and Pagès (2011).

1.2 Contribution of this paper

This paper studies four trading algorithms based on the observations of a realistic limit order book of a stock, namely a regular trader's price switching strategy, a regular trader's singular strategy, a privileged trader's singular strategy and a privileged trader's price switching strategy. We shall first show that the value functions of the four algorithms are in ascending order, with the last two being identical. Afterwards, we shall solve the two price switching algorithms, whose value functions can be viewed as the lower and upper bounds of the value functions of the two singular algorithms, by drawing on the stochastic control theory. The optimal price switching strategies are expressed in terms of the value function. Numerical techniques on how to implement the price switching algorithms in an industrial environment will be provided.

The limit order book is described as what it is seen at the trading venues by a two-sided model. The movements in the order book are physically driven by the events of arrival, cancelation and execution, hence the only parameters to be estimated are the volatilities of the queue lengths at the bid ask prices and the intensities of the order arrivals within the spread. The model is genuinely two-sided, because the exact mutual influence between the dynamics on the two-sides is specified. The model also specifies the exact interactions among the standing liquidity, the trader's transaction and the price impact and recovery as they actually are. The specifications are achieved without adding to any complexity in model fitting. Another benefit of our way of describing the order book dynamics is not having to assume equilibrium in the market or the existence of an uncontrollable martingale mid price.

By choosing specific forms of the reward function, the algorithm can serve different trading goals, like order execution by a brokerage agency and speculation by a hedge fund or a proprietary trading firm.

A trading algorithm would look for an optimal balance between the speed and cost of trading by using both active strategies for the orders displayed in the book and passive strategies for orders yet to arrive at a better price. The kind of passive strategies incorporated in this paper is hidden orders within the spread, similar to the dark pool mechanism studied in the concurrent works Horst and Naujokat (2011) and Kratz and Schöneborn (2012a,b). There is another kind of strategies more passive than hidden orders, which is orders queuing up at the best available prices. Because even the optimal queuing problem itself has been a wide field of research, it would be too ambitious to include it into this paper. Interested readers are invited to Lachapelle, Lasry, Lehalle and Lions (2013) for a trading algorithm that balances between active strategies for displayed orders and passive orders in the queue at the best available prices.

In terms of the theoretical background, the singular algorithms are problems of singular control type. We cite Bather and Chernoff (1966), Karatzas (1983), Taksar (1985), Karatzas and Wang (2001) and Fukushima and Taksar (2002) as references on singular controls. The price switching algorithms are problems of optimal switching type with price impact, if viewing the prices as an index; they are impulse control problems if viewing the prices as part of the state process. Also as references, we cite works on optimal switching problems with no price impact – via the backward stochastic differential equation approach by Hu and Tang (2010) and Romuald and Kharroubi (2011), via the PDE approach by Tang and Yong (1993), Pham (2007), Pham, Ly Vath and Zhou (2007) and Ly Vath and Pham (2007), and via the probabilistic method by Aïd, Campi, Langrené and Pham (2012) and Gassiat, Kharroubi and Pham (2012). Øksendal and Sulem (2007) is

a good reference where the readers can find the verification approach to impulse control of Markov processes.

2 The two-sided order book dynamics

The trading activities take place in a trading platform of one stock, where all the orders are submitted electronically to the processing center and matched directly with each other. There is no intermediary who conducts auctions. The participants in the market are one professional trader and many noise traders. The professional trader (thereafter "the trader") could be an agency broker, a hedge fund or a proprietary trading firm. He observes the entire limit order book and can place either displayed or hidden orders. The noise traders trade for exogenous liquidity reasons. They do not monitor the limit order book, tracking at most the bid and ask prices. All the orders from the noise traders are displayed in the book. This section will setup the order book dynamics resulted from the collective activities of the noise traders, when the trader does not trade.

2.1 Illustration of the order book

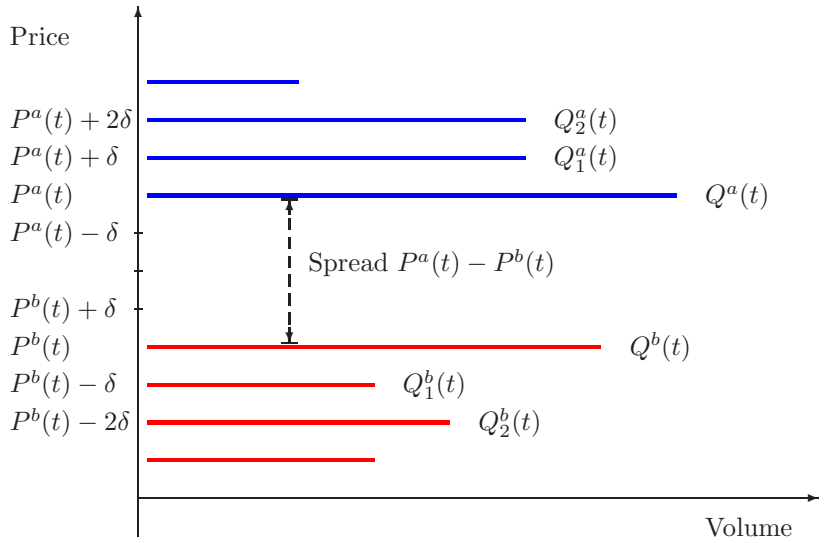
In preparation, let us present a few terminologies that appear frequently in discussions about a limit order book. For every stock in the market, there are several types of orders, the most commonly used types being the *limit order* and the *market order*. A market buy (sell) order only specifies the number of shares and is executed immediately at the lowest ask (highest bid) price available in the market. A limit buy (sell) order specifies the number of shares and the highest (lowest) price at which the trader is willing to buy (sell). According to the rules of best price first and FIFO at the same price level, limit orders are executed when there are matching sell (buy) orders at their specified prices. The records of all limit orders waiting to be executed are maintained. The set of the records is called a *limit order book*. A limit order book is a "reservoir" of limit orders. It records the number of shares, the price and the time of order arrival or cancelation for every limit order. Once a limit order is submitted, if it is not executed immediately, then this order is "stored" in the limit order book until being "released" and disappearing from the book for one of the three reasons – execution, cancelation, or expiration. The total of limit orders at each price level is called one *limit*. The lowest ask price (highest bid price) in the book is called the *ask price* (*bid price*) for short. The difference between the ask price and the bid price is called the *spread*. The distance between two adjacent price levels at which limit orders can be submitted is called the *tick size*.

Figure 2.1 illustrates a snapshot of a typical limit order book at some time t. The vertical axis represents the different price levels in the book, where $P^a(t)$ is the current ask price, $P^b(t)$ is the current bid price and δ is the tick size. The horizontal axis represents the volume, in other words the number of shares, of limit orders at each price level. The sell side of the book is shown in blue and the buy side in red. For example, the volume of limit sell orders at the ask price is denoted as $Q^a(t)$, which equals the length of the blue horizontal line at the price level $P^a(t)$; the volume of limit buy orders at the price level $P^b(t) - 2\delta$ is denoted as $Q^b_2(t)$, which equals the length of the red horizontal line at that level. The spread is defined as $P^a(t) - P^b(t)$. Without loss of generality, the tick size is set as $\delta = 1$.

We shall first describe narratively the model to be used in this paper with the help of Figure 2.1 and then provide the mathematical formulation in the next subsection.

All the limit sell (buy) orders at and higher (lower) than the ask (bid) price are displayed to the market participants (even though the noise traders do not look). The volumes of limit orders beyond the ask and bid prices are constants. In the notations illustrated in Figure 2.1, this means that $Q^a_1(t) = Q^a_2(t) = \cdots = \Delta^a$ and $Q^b_1(t) = Q^b_2(t) = \cdots = \Delta^b$, for all $0 \le t \le T$, where Δ^a and Δ^b are two positive constants. The number Q^a of limit sell orders at the ask price and the number Q^b of limit buy orders at the bid price are two stochastic processes. When the spread $P^a - P^b$ is more than one tick, limit sell orders can arrive one tick below the ask price P^a, and limit buy orders can arrive one tick above the bid price P^b. The ask (bid) price remains constant, until either all the sell (buy) orders at the current price get depleted or new limit sell

Figure 2.1: A Limit Order Book



(buy) orders arrive at the price one tick lower (higher). If the number of all the sell (buy) orders at the ask (bid) price reaches zero, then the ask (bid) price increases (decreases) by one tick, i.e.

$$P^a(t) = P^a(t-) + 1 \text{ or } P^b(t) = P^b(t-) - 1, \tag{2.1}$$

and the volume at the new ask (bid) price is given by

$$Q^a(t) = \Delta^a \text{ or } Q^b(t) = \Delta^b. \tag{2.2}$$

If limit sell (buy) orders arrive at time t at one tick below the ask price $P^a(t-)$ (above the bid price $P^b(t-)$), the ask (bid) price decreases (increases) by one tick, i.e.

$$P^a(t) = P^a(t-) - 1 \text{ or } P^b(t) = P^b(t-) + 1, \tag{2.3}$$

and each arrival contains Δ^a (Δ^b) shares, i.e. the expression (2.2) holds; the number of limit sell (buy) orders at the old ask price $P^a(t-)$ (the old bid price $P^b(t-)$) remains $Q^a(t-)$ ($Q^b(t-)$) at time t and resets to Δ^a (Δ^b) at time $t+$. We could make $Q_1^a(t), Q_2^a(t), \cdots$ and $Q_1^b(t), Q_2^b(t), \cdots$ Markov processes with independent increments. The assumption that they are constants will significantly reduce the dimensionality of the control problem while making decisions based on the major driving forces of the order book dynamics.

Besides all the displayed orders that form the limit order book in Figure 2.1, there is a "dark pool" mechanism within the spread. When the spread is greater than one tick, the trader can post limit sell (buy) orders at one tick below (above) the current ask (bid) price to passively wait for execution by upcoming liquidity events. If the trader chooses to do so, the occurrence of each liquidity event will fill Δ^a (Δ^b) shares in his limit sell (buy) orders. The trader's passive orders and the liquidity events are not displayed in this book, this is why they are said to be "hidden".

Unless specified otherwise, a "limit order" thereafter refers to a "limit order which is displayed to the market participants and is not hidden".

2.2 Mathematical formulation of the order book dynamics

This subsection will formulate rigorously the dynamics of the limit order book over a deterministic finite time horizon $[0, T]$ in a probability space $(\Omega, \mathbb{F}, \mathbb{P})$. The volumes at the ask and bid prices are driven by

the independent standard Brownian motions $\sigma^a W^a$ and $\sigma^b W^b$, where the volatilities σ^a and σ^b are two positive constants. The active limit sell (buy) orders arrive at one tick below (above) the ask (bid) price according to an inhomogeneous Poisson process N^a (respectively N^b) with the intensity $\theta^a(P^a(t-)-P^b(t-))$ (respectively $\theta^b\left(P^a(t-)-P^b(t-)\right)$) at time t. The occurrence of the liquidity events that fill the trader's hidden limit sell (buy) orders is an inhomogeneous Poisson process H^a (respectively H^b) with the intensity $\lambda^a\left(P^a(t-)-P^b(t-)\right)$ (respectively $\lambda^b\left(P^a(t-)-P^b(t-)\right)$). The known measurable functions θ^i and $\lambda^i:\mathbb{N}\to[0,\infty)$ satisfy $\theta^i(1)=\lambda^i(1)=0$, for $i=a,b$. This means that the displayed limit orders and hidden liquidity events arrive within the spread only if the spread is greater than one tick. Conditioning on the spread, the next arrival times of N^a, N^b, H^a and H^b are independent of each other and independent of the future increment of W^a and W^b. The filtration generated by the processes W^a, W^b, N^a, N^b, H^a and H^b are denoted as $\mathscr{F}=\{\mathscr{F}_t\}_{0\le t\le T}$.

Consistent with existing works, the dynamics setup in this section indeed captures the main features of a limit order book. Empirical studies (Cont, Stoikov and Talreja (2010) and Hasbrouck and Saar (2010)) observe that inhomogeneous Poisson processes are proper to model the order arrivals and cancelations at different prices, and that the orders in the neighborhoods closest to the bid and ask prices being the most influential to the stock price dynamics. An explanation for the latter observation is that the limit orders whose execution prices are far away from the bid and ask prices are more likely to be placed by speculators to profit from sudden dramatic price changes. Hence, if tracking only the volumes at the bid and ask prices, it makes a reasonable approximation to the real limit order books. Cont, Kukanov and Stoikov (2013) proposed an *order flow imbalance* model to describe the stylized features of an order book, where the number of shares at each price level beyond the best prices is constant and limit order arrivals and cancelations occur only at the best bid and ask prices. Further, Cont and de Larrard (2011, 2013) discovered that the volumes at the best bid and ask prices are approximately Brownian motions.

To prove the well-posedness of the control problem, the intensities of the order arrival processes within the spread are assumed uniformly bounded.

Assumption 2.1 *The intensity functions θ^a, θ^b, λ^a and λ^b of the inhomogeneous Poisson processes N^a, N^b, H^a and H^b satisfy*

$$\theta^{i*}:=\sup_{p\in\mathbb{R}}\left\{\theta^i(p)\right\}<\infty \text{ and } \lambda^{i*}:=\sup_{p\in\mathbb{R}}\left\{\lambda^i(p)\right\}<\infty,\ i=a,b. \tag{2.4}$$

Any change to the limit order book, either in the bid and ask prices, or in the available shares at each price level, is caused by one of the four types of events – limit order arrival, limit order cancelation or expiration, limit order execution, and market order arrival and immediate execution. Depending on the price level at which these events happen, the two sources of movements are the changes in the volumes (Q^a,Q^b) at the current best prices and the arrivals within the spread according to (N^a,N^b), which in turn result in all the changes in the prices.

The number of times over $[0,t]$ that all the orders at the current ask and bid prices are depleted is

$$L^i(t)=\sum_{0\le s\le t}\mathbf{1}_{\{Q^i(s-)\le 0\}},\ 0\le t\le T,\ i=a,b. \tag{2.5}$$

At every time the volume at the ask (bid) price is depleted, meaning that $L^i(t)-L^i(t-)=1$, the Δ^a (respectively Δ^b) shares at the higher (lower) price level are exposed and the ask (bid) price increases (decreases) by one tick. At every arrival of limit sell (buy) orders within the spread, meaning that $N^i(t)-N^i(t-)=1$, the new limit at the lower (higher) price level contains Δ^a (respectively Δ^b) shares and the ask (bid) price decreases (increases) by one tick. At any other time, the volumes move according to the Brownian motions and the prices remain constants.

Following the above reasoning, the dynamics of the order book can be described by the four-dimensional process (Q^a,Q^b,P^a,P^b). The volumes Q^a and Q^b move according to

$$Q^i(t)=Q_0^i+\sigma^i W^i(t)+\int_0^t\left(\Delta^i-Q^i(t-)\right)d\left(L^i+N^i\right)(t),\ 0\le t\le T,\ i=a,b. \tag{2.6}$$

The prices move according to

$$P^a(t) = P^a(0) + L^a(t) - N^a(t), \text{ and } P^b(t) = P^b(0) - L^b(t) + N^b(t),\ 0 \le t \le T. \tag{2.7}$$

The process (Q^a, Q^b, P^a, P^b) defined in (2.7) and (2.6) is Markovian.

Figure 2.2: Simulation of the Bid and Ask Prices

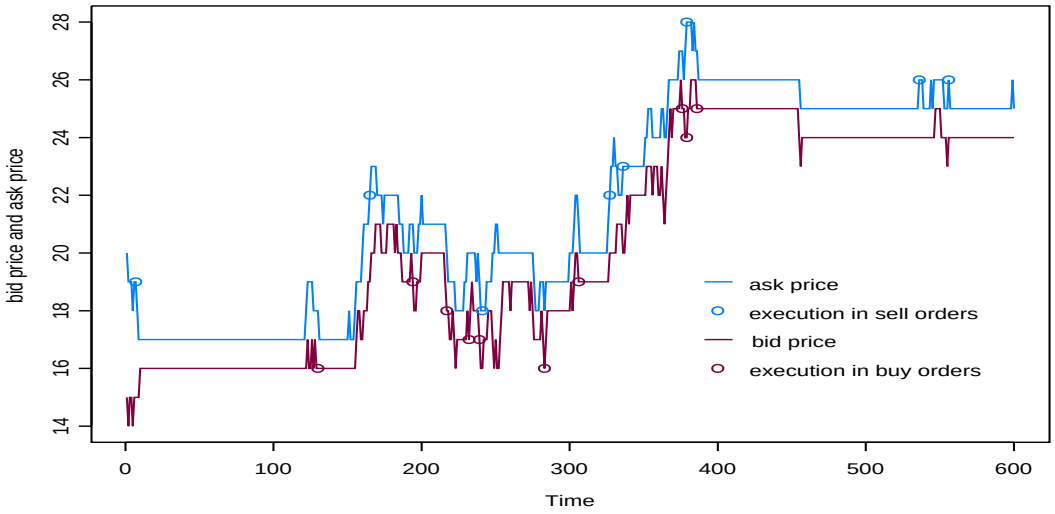
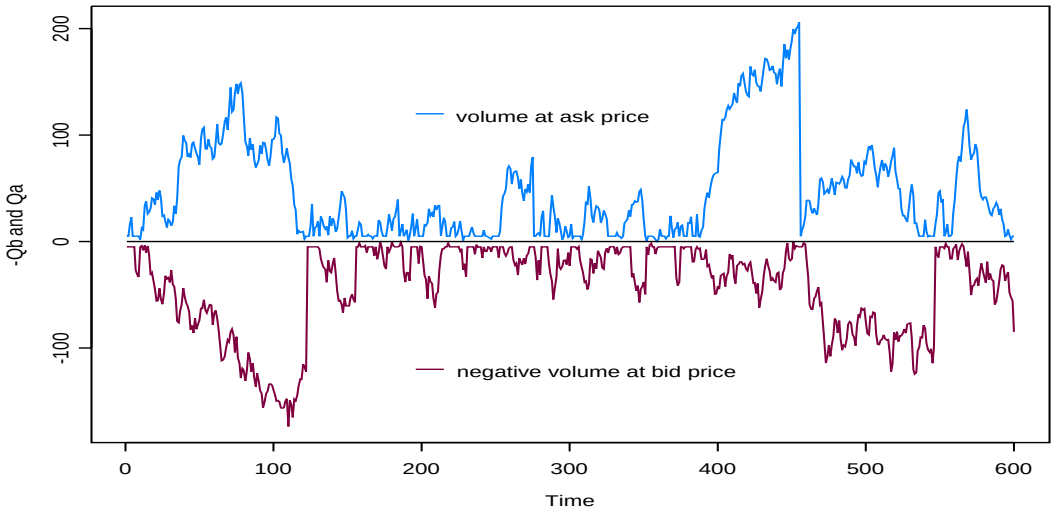


Figure 2.3: Simulation of the Volumes



Figures 2.2 and 2.3 plot a simulated path of the two-sided order book dynamics (2.7) and (2.6). Figure 2.2 shows the ask (top blue line) and bid (bottom red line) prices. Each time of price change due to order depletion is assigned a 10% probability that it is an execution (indicated by circles) and a 90% probability that it is a cancelation. Figure 2.3 shows the volumes at the ask (value of the top blue line) and bid (absolute value of the bottom red line) prices respectively in the positive and negative axis. The parameters are $T = 600$, $P^a(0) = 20$, $P^b(0) = 15$, $Q^a(0) = Q^b(0) = \Delta^a = \Delta^b = 5$, $\sigma^a = \sigma^b = 10$ and $\theta^a(P^a(t) - P^b(t)) = \theta^b(P^a(t) - P^b(t)) = 0.5(P^a(t) - P^b(t))$.

3 The optimal trading problem

Suppose the collective activities of the noise traders form the order book dynamics are described in the previous section. Besides the noise traders, the *trader* - a hedge fund or a brokery agency - now trades in this limit order book. He places a combination of active and passive orders. The active orders immediately fill the limit orders displayed in the book at the best prices available. The passive orders within the spread may or may not be executed at the next moment, but once they are executed the trader receives a price one tick lower or higher than the current ask or bid price. The flexibility to choose between active and passive orders enables finding an optimal balance between the speed and cost of trading.

Depending on his informational advantage, the trader is identified as either *regular* or *privileged*. Most traders in today's markets, including all the traders in Europe, are regular. They observe and only observe the current records in the order book. A privileged trader has the priority of observing incoming orders and acting upon them immediately before the orders are displayed to other market participants. The several official market makers at the NYSE used to have such privilege; the privilege was also a choice at trading platforms like Direct Edge. After some discussions between 2008 and 2011, currently the US Securities and Exchange Commission (SEC) does not document any attitude towards the privilege, while the stock exchanges display the order book to all the market participants simultaneously. Neither do the authors in this paper assume any attitude towards the privilege. The privileged trader's algorithms in this paper are intended to provide a numerically computable theoretical upper bound of the regular trader's algorithms.

3.1 Actively filling displayed orders

An active trading strategy places orders that will be fully and immediately executed at the best prices available. It should be the very role of market orders, but in practice one would prefer limit orders whose execution prices equal those of the market orders, so as to prevent losses in case of unexpected drastic increase or decrease in the ask or bid prices.

The total shares that the trader has bought and sold up till time t by actively filling the limit orders displayed in the book consist of those filled at the best available price, denoted by the processes $\{Z^a(t)\}_{0\le t\le T}$ and $\{Z^b(t)\}_{0\le t\le T}$, and of those filled at the old prices when new limit orders arrive within the spread, denoted by the processes $\{\beta^a(t)\}_{0\le t\le T}$ and $\{\beta^b(t)\}_{0\le t\le T}$ as the proportion of shares filled at the old price with respect to the total number of existing shares at the old price right before the transaction. The active trading strategy $Z = (Z^a, Z^b, \beta^a, \beta^b)$ is called

- *continuously re-balanced*, if Z^a and Z^b are absolutely continuous with respect to time t;
- *discretely re-balanced*, if Z^a and Z^b are a pure jump process;
- *singular*, if Z^a and Z^b are a mixture of the above two.

This subsection formulates singular active trading strategies ("singular trading strategies" for short) respectively for a privileged trader and a regular trader.

Definition 3.1 *(singular trading strategy)*
(1) The set of admissible singular trading strategies $\mathscr{Z}^p$ of a privileged trader is the collection of all trading strategies $Z = (Z^a, Z^b, \beta^a, \beta^b)$ satisfying the following two criteria.
(1.1) The $\mathscr{F}$-adapted càdlàg processes $\{Z^a(t)\}_{0\le t\le T}$ and $\{Z^b(t)\}_{0\le t\le T}$ are non-negative and non-decreasing

over the time interval $[0,T]$. *The* $\mathscr{F}$*-adapted processes* $\{\beta^a(t)\}_{0\le t\le T}$ *and* $\{\beta^b(t)\}_{0\le t\le T}$ *take values within the interval* $[0,1]$.

(1.2) For two given positive integers $\underline{p}^b < \bar{p}^a$, *the process* Z^a *is flat on*

$$\{(t,\omega)\in[0,T]\times\Omega | P^a(t)\ge \bar{p}^a\}, \tag{3.1}$$

and Z^b *is flat on*

$$\{(t,\omega)\in[0,T]\times\Omega | P^b(t)\le \underline{p}^b\}; \tag{3.2}$$

the process β^a *is non-zero only on*

$$\{(t,\omega)\in[0,T]\times\Omega | P^a(t)<\bar{p}^a,\ N^a(t)-N^a(t-)=1 \text{ and } Z^a(t)-Z^a(t-)<\Delta^a\}, \tag{3.3}$$

and the process β^b *is non-zero only on*

$$\{(t,\omega)\in[0,T]\times\Omega | P^b(t)>\underline{p}^b,\ N^b(t)-N^b(t-)=1 \text{ and } Z^b(t)-Z^b(t-)<\Delta^b\}. \tag{3.4}$$

(2) The set of admissible singular trading strategies $\mathscr{Z}^r$ *of a regular trader is defined as*

$$\mathscr{Z}^r := \left\{Z=(Z^a,Z^b,\beta^a,\beta^b)\in\mathscr{Z}^p \,\middle|\, \beta^a(t)\equiv\beta^b(t)\equiv 0, \text{ for all } t\in[0,T]\right\}. \tag{3.5}$$

Criterion *(1.1)* in Definition 3.1 defines a privileged trader's singular trading strategy. The bounded prices in Criterion *(1.2)* requires that the trader only buy below the price $\bar{p}^a$ and sell above the price $\underline{p}^b$. In practice, when the prices goes outside of their normal range, no trading reflects a psyche of not to take up risk or push the prices further towards the extreme; technically, it will be used to prove the well-posedness of the control problem, or in other words that the value function is finite. The other requirement about β^a and β^b in Criterion *(1.2)* means that it is only necessary to consider filling a fractional column at the old price when new limit orders arrive within the spread. The privilege to optimize over this fraction allows for a pathwise replication of the effect of every singular trading strategy by a price switching strategy, as will be shown in section 5. A regular trader's singular trading strategy defined in Criterion *(2)* is the same as that of a privileged trader, except that at the time of order arrival within the spread, the regular trader can no longer fill the orders at the old price, which is formulated as $\beta^a\equiv\beta^b\equiv 0$.

When a privileged trader trades according to a generic singular strategy in the order book, all the situations that could trigger a change in the bid or ask price at time t are enumerated by the cases (3.a-f) in subsection 7.1 of the Appendix. Details, together with the computation of changes in the prices and volumes for each case, are provided there. The illustrations are provided in subsection 7.3 of the Appendix. The rest of the current subsection will write, in the case of a privileged trader, the order book dynamics and the trader's stock inventory and cash amount in all the six cases into compact formulae. To adjust to the case of a regular trader is only a matter of setting $\beta^a\equiv\beta^b\equiv 0$.

The order book dynamics from equations (2.5), (2.7) and (2.6), is now controlled by the trader using an admissible singular trading strategy Z. The number of times over $[0,t]$ that all the orders at the current ask and bid prices are depleted can now be expressed as

$$L^i(t)=\sum_{0\le s\le t}\mathbf{1}_{\{Q^i(s-)-(Z^i(s)-Z^i(s-))\le 0\}},\ i=a,b. \tag{3.6}$$

For $i=a,b$, the changes

$$\mu^a(t)=P^a(t)-P^a(t-) \text{ and } \mu^b(t)=-\left(P^b(t)-P^b(t-)\right) \tag{3.7}$$

in the controlled ask and bid price processes at time t can be computed by (see subsection 7.1 of the Appendix for details)

$$\mu^i(t)=\begin{cases} -1, \text{ if } N^i(t)-N^i(t-)=1 \text{ and } Z^i(t)-Z^i(t-)<\Delta^i;\\ 0, \text{ if } N^i(t)-N^i(t-)=1 \text{ and } \Delta^i\le Z^i(t)-Z^i(t-)<Q^i(t-)+\Delta^i,\\ \quad \text{or if } t=T \text{ and } Z^i(T)-Z^i(T-)<Q^i(T-);\\ \left\lfloor\left(Z^i(t)-Z^i(t-)-Q^i(t-)\right)/\Delta^i\right\rfloor+1-(N^i(t)-N^i(t-)), \text{ else.}\end{cases} \tag{3.8}$$

Summarizing up the analysis of the cases (3.a-f) in subsection 7.1 of the Appendix, we are able to write down the dynamics of the controlled order book. The **volumes** Q^a and Q^b at the ask and bid prices move according to

$$\begin{aligned} Q^i(t) =& Q_0^i + \sigma^i W^i(t) - Z^i(t) + \Delta^i \int_0^t (\mu^i(s))^+ dL^i(s) \\ &+ \int_0^t \mathbf{1}_{\{\mu^i(s)\le 0\}} \left(\Delta^i - (\mu^i(s))^- Q^i(s-)\right) dN^i(s),\ 0 \le t \le T,\ i = a, b. \end{aligned} \tag{3.9}$$

The ask and bid **prices** P^a and P^b move according to

$$\begin{aligned} P^a(t) =& P^a(0) + \int_0^t (\mu^a(s))^+ dL^a(s) - \int_0^t (\mu^a(s))^- dN^a(s); \\ P^b(t) =& P^b(0) - \int_0^t (\mu^b(s))^+ dL^b(s) + \int_0^t (\mu^b(s))^- dN^b(s),\ 0 \le t \le T. \end{aligned} \tag{3.10}$$

At every time $t \in [0, T]$, a singular trading strategy Z gives the trader an **inventory** of stock shares

$$I^Z(t) = Z^a(t) - Z^b(t) + \int_0^t \beta^a(s) Q^a(s-) dN^a(s) - \int_0^t \beta^b(s) Q^b(s-) dN^b(s) \tag{3.11}$$

and a **cash** amount

$$\begin{aligned} C^Z(t) =& \int_0^t P^b(s-) dZ^b(s) - \int_0^t P^a(s-) dZ^a(s) \\ &- \int_0^t \sum_{i=a,b} \left(\frac{1}{2} \Delta^i d(P^i(s))^2 + \left(\frac{1}{2} \Delta^i - Q^i(s) \right) dP^i(s) - \Delta^i dN^i(s) \right) \\ &+ \int_0^t \beta^b(s) Q^b(s-) P^b(s-) dN^b(s) - \int_0^t \beta^a(s) Q^a(s-) P^a(s-) dN^a(s). \end{aligned} \tag{3.12}$$

In the equation (3.11), the two integrals count the the total number of shares bought and sold at the old prices when limit orders arrive within the spread. In the equation (3.12), the first line on the right hand side of the identity is the amount of cash paid and received if all the orders placed at the best available prices were executed at the bid and ask prices observed right before each transaction, as they are when the prices do not change and there is no arrival within the spread. The expression in the second line on the right hand side of the equation collects the additional cost paid for trading deeper into the book and the savings from filling the arriving limit orders within the spread. The third line counts the cash amount paid and received from filling the limit orders at the old prices when new limit orders arrive within the spread.

3.2 Passively executed hidden orders

Our formulation of the hidden orders strategies are similar to that by Kratz and Schöneborn (2012a,b) and Horst and Naujokat (2011). In their papers, the limit order book and the dark pool are located at two different venues, so the exact dependence between prices at the two venues does not matter much. In our paper, the hidden activities inside the spread can be viewed as a "dark pool" mechanism. The limit orders displayed in the order book and those hidden within the spread co-exist at the same trading platform for the same stock. This gives the advantage, as well as the necessity, of specifying the influence of the displayed orders on the hidden liquidity events H^a and H^b, in the sense that the intensities λ^a and λ^b are functions of the spread.

The trader's decision on whether to accept an upcoming liquidity event is indicated by the set of admissible hidden order strategies defined below. Be it a privileged trader or a regular trader, the admissible set of hidden order strategies is the same.

Definition 3.2 *(hidden order strategy) The trader's decisions on whether to accept the hidden orders are indicated by the $\mathscr{F}$-adapted, right-continuous, $\{0,1\}$-valued processes h^a and $h^b : [0,T]\times\Omega \to \{0,1\}$. The process h^a equals zero on the set*

$$\big\{(t,\omega)\in[0,T]\times\Omega | P^a(t) - P^b(t) = 1\big\} \cap \{(t,\omega)\in[0,T]\times\Omega | P^b(t) \geq \bar{p}^a\}, \tag{3.13}$$

and the process h^b equals zero on the set

$$\big\{(t,\omega)\in[0,T]\times\Omega | P^a(t) - P^b(t) = 1\big\} \cap \{(t,\omega)\in[0,T]\times\Omega | P^a(t) \leq \underline{p}^b\}. \tag{3.14}$$

The collection of all such processes $h = (h^a, h^b)$ are denoted as $\mathscr{H}$. The subset of $\mathscr{H}$ restricted on $[t_1,t_2]\times\Omega$ is denoted as $\mathscr{H}_{t_1,t_2}$, and $\mathscr{H}_{t,T}$ is denoted as $\mathscr{H}_t$ for short.

In Definition 3.2, the right-continuity of h^a and h^b guarantees that, for each scenario $\omega\in\Omega$, the hidden orders are revised finitely many times over the time horizon $[0,T]$. The value $h^a(t) = 1$ ($h^b(t) = 1$) means that the trades places a hidden limit buy (sell) order of Δ^a (Δ^b) shares with the execution the price $P^b(t)+1$ (respectively $P^a(t)-1$); the value $h^a(t) = 0$ ($h^b(t) = 0$) means that he does not place the hidden order. The processes h^a and h^b are required to be zero on the first set in (3.13) and (3.14), because there is no space for the hidden events when the spread is one tick. The second sets in (3.13) and (3.14) require that the trader's hidden orders would only buy below the price $\bar{p}^a$ and sell above the price $\underline{p}^b$.

Suppose an liquidity sell (buy) event occurs at time t, meaning that $H^a(t) - H^a(t-) = 1$ (respectively $H^a(t) - H^a(t-) = 1$), if the trader placed a hidden limit buy (sell) order right before time t with the execution price $P^b(t-)+1$ (respectively $P^a(t-)-1$), then he successfully buys Δ^a shares (sells Δ^b shares) at time t and pays (receives) a cash amount of $\Delta^a(P^b(t-)+1)$ (respectively $\Delta^b(P^a(t-)-1)$). Using a generic hidden order strategy $h = (h^a, h^b)\in\mathscr{H}$, the trader's stock **inventory** and **cash** amount from trading hidden orders are

$$\begin{aligned} I^h(t) =& \Delta^a\int_0^t h^a(s-)dH^a(s) - \Delta^b\int_0^t h^b(s-)dH^b(s);\\ C^h(t) =& -\Delta^a\int_0^t h^a(s-)(P^b(s-)+1)dH^a(s) + \Delta^b\int_0^t h^b(s-)(P^a(s-)-1)dH^b(s),\ 0\leq t\leq T. \end{aligned} \tag{3.15}$$

3.3 The goal of trading

The trader's total stock inventory and cash amount are sums of those from trading hidden and displayed orders, being

$$I^{h,Z}(t) = I_0 + I^h(t) + I^Z(t) \text{ and } C^{h,Z}(t) = C_0 + C^h(t) + C^Z(t),\ 0\leq t\leq T, \tag{3.16}$$

where the real numbers I_0 and C_0 are the initial stock inventory and the initial cash amount, and the processes I^h, C^h, I^Z and C^Z are defined in the equations (3.15), (3.11) and (3.12). When there is no ambiguity on which trading strategies are used, the superscripts in the stock inventory $I^{h,Z}$ and the cash amount $C^{h,Z}$ are omitted.

Let r^I and r^C be two real numbers and

$$F : \mathbb{R}\times\mathbb{N}\times\mathbb{N}\to\mathbb{R};\ (z,p^a,p^b)\mapsto F(z,p^a,p^b), \tag{3.17}$$

be a measurable function with quadratic growth.

Assumption 3.1 *There exists a constant $r^F > 0$, such that for all $z\in\mathbb{R}$ and all $p^b\leq p^a$, we have*

$$\big|F(z,p^a,p^b)\big| \leq r^F\left((z)^2 + (p^a)^2 + (p^b)^2 + 1\right) \tag{3.18}$$

and

$$\big|F(z_1,p^a,p^b) - F(z_2,p^a,p^b)\big| \leq r^F\left(|p^a| + |p^b| + 1\right)\left(|z_1| + |z_2| + 1\right)|z_1 - z_2|. \tag{3.19}$$

The trading activities are measured by the reward

$$\xi\left(I(T),C(T)\right)=r^{C}C(T)+r^{I}F\left(I(T),P^{a}(T),P^{b}(T)\right). \tag{3.20}$$

The trader's objective of maximizing the reward in expectation is formulated as a stochastic control problem.

Problem 3.1 *(1) A privileged trader looks for an optimal trading strategy* $Z^{*}=(Z^{a*},Z^{b*},\beta^{a*},\beta^{b*})\in\mathscr{Z}^{p}$ *and* $h^{*}=(h^{a*},h^{b*})\in\mathscr{H}$ *to achieve the maximum expected reward*

$$\sup_{(Z^{a},Z^{b},\beta^{a},\beta^{b})\in\mathscr{Z}^{p},(h^{a},h^{b})\in\mathscr{H}}\mathbb{E}\left[\xi\left(I^{h,Z}(T),C^{h,Z}(T)\right)\right]. \tag{3.21}$$

(2) A regular trader looks for an optimal trading strategy $Z^{*}=(Z^{a*},Z^{b*},0,0)\in\mathscr{Z}^{r}$ *and* $h^{*}=(h^{a*},h^{b*})\in\mathscr{H}$ *to achieve the maximum expected reward*

$$\sup_{(Z^{a},Z^{b},0,0)\in\mathscr{Z}^{r},(h^{a},h^{b})\in\mathscr{H}}\mathbb{E}\left[\xi\left(I^{h,Z}(T),C^{h,Z}(T)\right)\right]. \tag{3.22}$$

Control problems with reward functions of the form (3.20) have a five dimensional state process $(Q^{a},Q^{b},I,P^{a},P^{b})$. Problem 3.1 is still solvable for reward functions that are not linear in $C(T)$, but in that case the state process will have the cash amount C a sixth dimension. Assumption 3.1 is a technical assumption under which the control problem is well proposed.

Several common situations where the reward criteria satisfy Assumption 3.1 are linear combination of the cash and inventory, liquidating or filling a certain number of stock shares, and holding cash only at the terminal time, corresponding to the following forms of $\xi(I(T),C(T))$ defined in (3.20):

$$\begin{aligned}
&\textit{(1) } \xi(I(T),C(T))=r^{C}C(T)+r^{I}I(T);\\
&\textit{(2) } \xi(I(T),C(T))=r^{C}C(T)+r^{I}|I(T)-z_{0}| \text{ or } \xi(I(T),C(T))=r^{C}C(T)+r^{I}(I(T)-z_{0})^{2};\\
&\textit{(3) } \xi(I(T),C(T))=r^{C}C(T)+r^{I}\left(\mathbf{1}_{\{I(T)>0\}}(P^{b}(T)-U^{b})+\mathbf{1}_{\{I(T)<0\}}(P^{a}(T)+U^{a})\right)I(T).
\end{aligned} \tag{3.23}$$

The criterion (3.23)*(1)* means that the trader has a utility function linear in cash and inventory. In (3.23)*(2)*, the coefficient r^{I} is negative, and the constant z_{0} is the number of shares that the trader would like to hold at the terminal time T. In (3.23)*(3)*, the coefficient r^{I} is positive and U^{a} and U^{b} are two positive integers; if the terminal inventory $I(T)$ is positive, the trader sells all his stocks at the price $(P^{b}(T)-U^{b})$ per share; if $I(T)$ is negative, he pays the price $(P^{a}(T)+U^{a})$ for each share.

Remark 3.1 *In application, a trader in a hedge fund or a proprietary trading firm is entitled to both buying and selling during any trading period, hence* Z^{a}, Z^{b}, h^{a} *and* h^{b} *can be all positive. For a trader in a brokerage agency, if he trades during the time* $[0,T]$ *to fill a buy (sell) order for the customer, a simplest way to comply with the regulations is not to sell (buy) throughout the same time period, hence, when using the results in this paper, he should set* $Z^{b}(t)\equiv 0$, $\beta^{b}(t)\equiv 0$ *and* $h^{b}(t)\equiv 0$ *(respectively* $Z^{a}(t)\equiv 0$, $\beta^{a}(t)\equiv 0$ *and* $h^{a}(t)\equiv 0$*) for all* $0\le t\le T$. *Especially, a regular trader sets* $\beta^{a}\equiv\beta^{b}\equiv 0$. *The brokerage agency's admissible set of equivalent price switching strategies is a modification of Definition 4.1 by setting* $u^{b}_{n}\equiv -(N^{b}(S_{n})-N^{b}(S_{n}-))$ *and* $h^{b}(t)\equiv 0$ *(respectively* $u^{a}_{n}\equiv -(N^{a}(S_{n})-N^{a}(S_{n}-))$ *and* $h^{a}(t)\equiv 0$*). Optimizing over the buying (selling) strategies only is a special case of the algorithm in subsection 6.2.*

4 The optimal switching problem

This section presents the problem of how the trader could optimally switch the bid and ask prices and shows the well-posedness of the switching problem. Compared to the optimal trading Problem 3.1, price switching considers a smaller and simpler set of active trading strategies, which are discretely re-balanced strategies in the form of the times to trade and the number of limits to fill at each time.

4.1 Switching prices

Let $[u]$ and $\{u\}$ respectively denote the integer part and the fractional part of a real number u. The identity $u = [u] + \{u\}$ holds. Switching the prices means that the trader chooses a sequence of times $\{S_n\}_{n=1}^{\infty}$ and two sequences of positive numbers $\{u_n^a\}_{n=1}^{\infty}$ and $\{u_n^b\}_{n=1}^{\infty}$, such that the ask and bid prices are pushed from $P^a(S_{n-1})$ to $P^a(S_n) = P^a(S_{n-1}) + [u_n^a]$ and from $P^b(S_{n-1})$ to $P^b(S_n) = P^b(S_{n-1}) - [u_n^b]$. The ask and bid prices stay constants over every time interval (S_{n-1}, S_n).

In the limit order book where the trader and the noise traders all act, a price change could occur due to two possible reasons.

(1) Limit orders at the ask or bid price is depleted by the noise trader, or limit orders arrive within the spread. Suppose the previous time of price switching is S_{n-1}, then the next time T_n when either event happens can be expressed as

$$\begin{aligned} T_n := \inf\{S_{n-1} < t \le T | Q^a(S_{n-1}) + \sigma^a(W^a(t) - W^a(S_{n-1})) = 0,\ N^a(t) - N^a(S_{n-1}) = 1, \\ Q^b(S_{n-1}) + \sigma^b(W^b(t) - W^b(S_{n-1})) = 0, \text{ or } N^b(t) - N^b(S_{n-1}) = 1\} \wedge T. \end{aligned} \tag{4.1}$$

(2) The trader fills all the shares at the ask or bid price and, at his choice, some limits beyond the best prices. If he trades at some time $S_n \in (S_{n-1}, T_n)$, then he has to fill all the shares at either the ask price or the bid price to trigger a price change.

After the $(n-1)$th price switching at the time S_{n-1}, if he waits until the time T_n, the trader may choose to fill some shares at time T_n; even if he does not trade, the noise traders will switch the prices at time T_n any way, in which case the time of the nth price switching is set as $S_n = T_n$. Requiring that the trader has to "trade" at time T_n, though possibly zero share, makes sure that the prices are constants between two switching times and gives a neater expression of the order book dynamics.

Furthermore, formulating with the help of T_n's reduces the price switching problem from seven-dimensional path-dependent to five-dimensional Markovian. The actual controlled state process is

$$\{(Q^a(t-), Q^b(t-), I^\alpha(t-) + I^h(t), P^a(t-), P^b(t-), N^a(t), N^b(t))\}_{0 \le t \le T}. \tag{4.2}$$

Since what matter in $(N^a(t), N^b(t))$ is their exponentially distributed arrival times, the trader only need to monitor whether or not there is an arrival of limit sell or buy orders within the spread. Viewing the controlled arrival times T_n's as a sequence of exit times when decision marking has to take place, the state process is simplified to $\{(Q^a(t-), Q^b(t-), I^\alpha(t-) + I^h(t), P^a(t-), P^b(t-))\}_{0 \le t \le T}$.

The admissible price switching strategies are switching controls defined below.

Definition 4.1 *(switching control) The admissible set of switching controls of a privileged trader and that of a regular trader are denoted respectively as $\mathscr{A}^p$ and $\mathscr{A}^r$. For $j = p, r$, the admissible set $\mathscr{A}^j$ consists of switching controls $\alpha := \{(S_n, u_n^a, u_n^b)\}_{n=0}^{\infty}$ satisfying, for every $n = 1, 2, \cdots$, the three criteria below, with the convention that $S_0 = u_0^a = u_0^b = 0$.*
(1) The switching time S_n is an $\mathscr{F}$-stopping time such that $0 \le S_1 < \cdots < S_{n-1} < S_n < \cdots$. If $S_{n-1} \ge T$, then $S_n = T + 1$; if $S_{n-1} < T$, then $S_n \in (S_{n-1}, T_n]$, where T_n defined in (4.1) is the next time of price change if the trader does not trade.
(2) For the same positive integers $\bar{p}^a$ and $\underline{p}^b$ as in Definition 3.1, we have $(u_n^a, u_n^b) \in \mathscr{U}^j(S_n, P^a(S_{n-1}), P^b(S_{n-1}))$, where

$$\mathscr{U}^j(t, p^a, p^b) := \begin{cases} \{0, 1, \cdots, (\bar{p}^a - p^a)^+\} \times \{0, 1, \cdots, (p^b - \underline{p}^b)^+\} \setminus \{(0,0)\}, \text{ if } t < T_n, \text{ or } Q^i(t-) = 0,\ i = a \text{ or } b; \\ D^j(\bar{p}^a - p^a) \cup \{2, \cdots, (\bar{p}^a - p^a)^+\} \times \{0, 1, \cdots, (p^b - \underline{p}^b)^+\}, \text{ if } N^a(t) - N^a(t-) = 1; \\ \{0, 1, \cdots, (\bar{p}^a - p^a)^+\} \times D^j(p^b - \underline{p}^b) \cup \{2, \cdots, (p^b - \underline{p}^b)^+\}, \text{ if } N^b(t) - N^b(t-) = 1; \\ [0, (\bar{p}^a - p^a)^+] \times [0, (p^b - \underline{p}^b)^+], \text{ if } t = T; \\ \{(0,0)\}, \text{ if } t = T + 1. \end{cases} \tag{4.3}$$

In expression (4.3), the sets $D^j(\bar{p}^a - p^a)$ and $D^j(p^b - \underline{p}^b)$ of real numbers are defined as

$$D^p(x) := \begin{cases} [-1,1], \text{ if } x \geq 2; \\ [-1,1), \text{ if } x = 1; \\ \{-1,0\}, \text{ if } x = 0; \\ \{-1\}, \text{ if } x \leq -1 \end{cases} \tag{4.4}$$

for a privileged trader, and as

$$D^r(x) := \begin{cases} \{-1,0,1\}, \text{ if } x \geq 1; \\ \{-1,0\}, \text{ if } x = 0; \\ \{-1\}, \text{ if } x \leq -1. \end{cases} \tag{4.5}$$

for a regular trader. For $j = p, r$ and $t \in [0,T]$, the set $\mathscr{A}_t^j$ is defined the same as $\mathscr{A}^j$ except that $S_0 = t$. Also, we denote as

$$\mathscr{A}_{t,1}^j := \left\{ (S_1, u_1^a, u_1^b) \middle| \, \alpha = \{(S_n, u_n^a, u_n^b)\}_{n=0}^{\infty} \in \mathscr{A}_t^j \right\} \tag{4.6}$$

the set of the first elements of all switching controls in $\mathscr{A}_t^j$.

In Definition 4.1, Criterion *(1)* specifies the trading times S_n. Criterion *(2)* specifies the possible numbers of limits to buy and sell at each transaction. Setting $S_n = T + 1$ means the trader no longer trades, hence there has to be $u_n^a = u_n^b = 0$.

The admissible switching control sets of a privileged trader and a regular trader are different only up to the sets D^p and D^r defined in (4.4) and (4.5). Throughout this paper, when a claim is valid for a trader regardless of whether he is privileged or regular, the admissible switching control sets will be generically denoted as $\mathscr{A}$, $\mathscr{A}_t$ and $\mathscr{A}_{t,1}$.

For $n = 1, 2, \cdots$, at every time S_n a switching control $\alpha \in \mathscr{A}$ causes the ask price to increase from $P^a(S_{n-1})$ to $P^a(S_n) = P^a(S_{n-1}) + [u_n^a]$ and the bid price to decrease from $P^b(S_{n-1})$ to $P^b(S_n) = P^b(S_{n-1}) - [u_n^b]$. All the possible situations of a price change cause by a privileged trader are enumerated as cases (4.a-f) in subsection 7.2 with illustrations in 7.3 in the Appendix. A regular trader can only cause cases (4.a,b,c,e,f), but not case (4.d). Based on the analysis in each of the six cases, the order book dynamics and the changes in the stock inventory and cash amount can be summarized in compact formulae.

The order book dynamics can be written in terms of the switching control $\alpha = \{(S_n, u_n^a, u_n^b)\}_{n=0}^{\infty} \in \mathscr{A}$: the controlled prices move according to

$$P^a(t) = P^a(0) + \sum_{n: S_n \leq t} [u_n^a], \text{ and } P^b(t) = P^b(0) - \sum_{n: S_n \leq t} [u_n^b], \; 0 \leq t \leq T; \tag{4.7}$$

the controlled volumes move according to

$$Q^i(t) = \begin{cases} Q^i(0) + \sigma^i W^i(t) \\ \quad + \sum\limits_{n: S_n \leq t} \left(\mathbf{1}_{\{[u_n^i] \neq 0\}} \left(\Delta^i - Q^i(S_n-) \right) - \mathbf{1}_{\{[u_n^i]=0\}} \{u_n^i\} Q^i(S_n-) \right), \; 0 \leq t < T; \\ \sum\limits_{n: S_n = T} (1 - \{u_n^i\}) \left(\mathbf{1}_{\{u_n^i \geq 1\}} \Delta^i + \mathbf{1}_{\{[u_n^i]=0\}} Q^i(T-) \right), \; t = T, \; i = a, b. \end{cases} \tag{4.8}$$

Defining the functions g^a and $g^b : \Omega \times [0,T] \times [0,\infty) \times \mathbb{Z} \to \mathbb{R}$ as

$$g^i(t,q,u) = \begin{cases} \mathbf{1}_{\{u \geq 1\}} \left(q + (u-1)\Delta^i \right) + (N^i(t) - N^i(t-)) \left(\mathbf{1}_{\{u \geq 0\}} \Delta^i + \mathbf{1}_{\{[u] \leq 0\}} \{u\} q \right), \; t < T; \\ \mathbf{1}_{\{u \geq 1\}} (q + (u-1)\Delta^i) + \mathbf{1}_{\{[u]=0\}} u \cdot q, \; t = T, \; i = a, b, \end{cases} \tag{4.9}$$

then $g^a\left(S_n, Q^a(S_n-), u_n^a\right)$ and $g^b\left(S_n, Q^b(S_n-), u_n^b\right)$ are respectively the number of shares that the trader buys and sells at the time S_n. The quantity

$$g_\alpha\left(S_n, Q^a(S_n-), Q^b(S_n-), u_n^a, u_n^b\right) := g^a\left(S_n, Q^a(S_n-), u_n^a\right) - g^b\left(S_n, Q^b(S_n-), u_n^b\right) \tag{4.10}$$

is the change in the trader's stock inventory from transactions at the time S_n.

Defining the functions f^a and $f^b : \Omega \times [0,T] \times [0,\infty) \times \mathbb{Z} \times \mathbb{N} \to \mathbb{R}$ as

$$f^a(t,q,p,u) = \begin{cases} \mathbf{1}_{\{u\geq 1\}}\left(p\left(q+(u-1)\Delta^a\right) + \frac{1}{2}u(u-1)\Delta^a\right) \\ \quad + \left(N^a(t) - N^a(t-)\right)\left(\mathbf{1}_{\{u\geq 0\}}(p-1)\Delta^a + \mathbf{1}_{\{[u]\leq 0\}}p\{u\}q\right),\ t<T; \\ \mathbf{1}_{\{u\geq 1\}}\left(p\left(q+(u-1)\Delta^a\right) + \left(\frac{1}{2}[u]([u]-1) + [u]\cdot\{u\}\right)\Delta^a\right) + \mathbf{1}_{\{[u]=0\}}p\cdot u\cdot q,\ t=T, \end{cases} \tag{4.11}$$

and

$$f^b(t,q,p,u) = \begin{cases} \mathbf{1}_{\{u\geq 1\}}\left(p\left(q+(u-1)\Delta^b\right) + \frac{1}{2}u(u-1)\Delta^b\right) \\ \quad + \left(N^b(t) - N^b(t-)\right)\left(\mathbf{1}_{\{u\geq 0\}}(p+1)\Delta^b + \mathbf{1}_{\{[u]\leq 0\}}p\{u\}q\right),\ t<T; \\ \mathbf{1}_{\{u\geq 1\}}\left(p\left(q+(u-1)\Delta^b\right) + \left(\frac{1}{2}[u]([u]-1) + [u]\cdot\{u\}\right)\Delta^b\right) + \mathbf{1}_{\{[u]=0\}}p\cdot u\cdot q,\ t=T, \end{cases} \tag{4.12}$$

then $f^a\left(S_n, Q^a(S_n-), P^a(S_n-), u_n^a\right)$ and $f^b\left(S_n, Q^b(S_n-), P^b(S_n-), u_n^b\right)$ are respectively the cash amount that the trader pays for and receives from transactions at the time S_n. The quantity

$$\begin{aligned} &f_\alpha\left(S_n, Q^a(S_n-), Q^b(S_n-), P^a(S_n-), P^b(S_n-), u_n^a, u_n^b\right) \\ :=& -f^a\left(S_n, Q^a(S_n-), P^a(S_n-), u_n^a\right) + f^b\left(S_n, Q^b(S_n-), P^b(S_n-), u_n^b\right) \end{aligned} \tag{4.13}$$

is the change in the trader's cash amount from transactions at the time S_n.

Actively trading according to a generic switching control $\alpha \in \mathscr{A}$ defined by Definition 4.1, the trader's inventory and cash amount from the displayed orders are respectively

$$\begin{aligned} I^\alpha(t) &= \sum_{n:S_n\leq t} g_\alpha\left(S_n, Q^a(S_n-), Q^b(S_n-), u_n^a, u_n^b\right) \text{ and} \\ C^\alpha(t) &= \sum_{n:S_n\leq t} f_\alpha\left(S_n, Q^a(S_n-), Q^b(S_n-), P^a(S_n-), P^b(S_n-), u_n^a, u_n^b\right), \text{ for } 0\leq t\leq T. \end{aligned} \tag{4.14}$$

Taking into account the hidden orders, the trader's total terminal stock inventory and cash amount are respectively

$$I(T)^{h,\alpha} = I_0 + I^h(T) + I^\alpha(T) \text{ and } C^{h,\alpha}(T) = C_0 + C^h(T) + C^\alpha(T), \tag{4.15}$$

where the quantities $I^h(T)$ and $C^h(T)$ are defined in the equations (3.15). When there is no ambiguity on which trading strategies are used, the superscripts in $I^{h,\alpha}(T)$ and $C^{h,\alpha}(T)$ are omitted.

Since each transaction using active orders causes a change in the prices, the trader's trading activities are a matter of choosing when and to what level to "switch" the ask and bid prices. The profit or cost from switching at time S_n is the change in his cash amount expressed as the quantity in (4.13). This is why we give the name "price switching" to the set of discretely re-balances trading strategies introduced in this section. Optimizing the trading algorithm over all the price switching strategies is a problem of switching control with impact on the state process.

Problem 4.1 *(1) A privileged trader looks for an admissible switching control $\alpha^* \in \mathscr{A}^p$ and an optimal hidden order strategy $h^* \in \mathscr{H}$ that achieve the supremum*

$$\sup_{\alpha\in\mathscr{A}^p,\, h\in\mathscr{H}} \mathbb{E}[\xi(I(T)^{h,\alpha}, C^{h,\alpha}(T))]. \tag{4.16}$$

(2) A regular trader looks for an admissible switching control $\alpha^* \in \mathscr{A}^r$ and an optimal hidden order strategy $h^* \in \mathscr{H}$ that achieve the supremum

$$\sup_{\alpha\in\mathscr{A}^r,\, h\in\mathscr{H}} \mathbb{E}[\xi(I(T)^{h,\alpha}, C^{h,\alpha}(T))]. \tag{4.17}$$

Generically for either a privileged trader or a regular trader, Problem 4.1 requires finding an admissible switching control $\alpha^* \in \mathscr{A}$ and an optimal hidden order strategy $h^* \in \mathscr{H}$ that achieve the supremum

$$\begin{aligned}
&\sup_{\alpha\in\mathscr{A},\, h\in\mathscr{H}} \mathbb{E}[\xi(I(T)^{h,\alpha}, C^{h,\alpha}(T))] \\
=&\sup_{\alpha\in\mathscr{A},\, h\in\mathscr{H}} \mathbb{E}\Bigg[r^C \sum_{n=1}^{\infty} f_\alpha\left(S_n, Q^a(S_n-), Q^b(S_n-), P^a(S_n-), P^b(S_n-), u_n^a, u_n^b\right) \\
&\qquad + r^C C^h(T) + r^I F(I^{h,\alpha}(T), P^a(T), P^b(T))\Bigg].
\end{aligned} \tag{4.18}$$

The **value process** V of (4.18) is defined as

$$V(t) := \sup_{\alpha\in\mathscr{A}_t,\, h\in\mathscr{H}_t} \mathbb{E}\left[\,\xi(I(T), C(T))|\,\mathscr{F}(t)\right] - r^C C(t). \tag{4.19}$$

Then the best expected reward (4.18) can be written as

$$\sup_{\alpha\in\mathscr{A},\, h\in\mathscr{H}} \mathbb{E}[\xi(I(T), C(T))] = r^C C_0 + V(0). \tag{4.20}$$

4.2 Well-posedness of the switching problem

Before looking for an optimal trading strategy that achieves the supremum in (4.18), it is necessary to verify that the problem is well-defined. In other words, is the value process V in (4.19) finite? This subsection will prove that the answer is yes.

The well-posedness is stated as Theorem 4.1 at the end of this subsection. To prepare for the proof of the theorem, Lemma 4.1 and Lemma 4.2 will respectively provide uniform $\mathbb{L}^2$ or $\mathbb{L}^1$ bounds of the prices $(P^a(T), P^b(T))$, the stock inventory $I^{h,\alpha}(T)$ and the cash amount $C^{h,\alpha}(T)$ for all admissible switching controls. Because the reward criterion ξ defined in (3.20) is a function of $(P^a(T),\, P^b(T))$, $I^{h,\alpha}(T)$ and $C^{h,\alpha}(T)$, the two Lemmas lead to Theorem 4.1. In addition, an interpretation worth noting of Lemma 4.1 is that the trader's trading activities will not push the prices towards explosion.

Lemma 4.1 *There exists a constant $c_1 > 0$, such that for any admissible switching control $\alpha \in \mathscr{A}$ defined by Definition 4.1, the controlled ask and bid prices at any time $t \in [0, T]$ has the $\mathbb{L}^2$ bounds*

$$\begin{aligned}
\mathbb{E}\left[\left(P^a(t)\right)^2\right] \leq& (P^a(0))^2 + (\bar{p}^a)^2 + c_1 T\left(((\theta^{a*})^2+1)T + \theta^{a*} + 1\right); \\
\mathbb{E}\left[\left(P^b(t)\right)^2\right] \leq& (P^b(0))^2 + (\underline{p}^b)^2 + c_1 T\left(((\theta^{b*})^2+1)T + \theta^{b*} + 1\right),
\end{aligned} \tag{4.21}$$

where the constants θ^{a*} and θ^{b*} defined in (2.4) are the maximum arrival rates of limit orders within the spread.

Proof. The total number of downward (upward) movements in the ask (bid) price equals the number of times N^i that limit sell (buy) orders arrive within the spread. The total number of upward (downward) movements in the ask (bid) price equals the number of limits that the trader has filled plus the number of

times when the volume at the ask (bid) price is depleted by the noise traders. The total number of depletions at the ask (bid) price by the noise traders does not exceed a renewal process R^i with independent interarrival times identically distributed as the leverage hitting time

$$\inf\left\{0 \le t \le T \,\middle|\, \Delta^i + \sigma^i W^i(t) \le 0\right\} \wedge T, \text{ for } i = a, b. \tag{4.22}$$

The highest ask (lowest bid) price happens when no limit sell (buy) orders arrive within the spread and the trader fills all the limit sell (buy) order below the price $\bar{p}^a$ (above the price $\underline{p}^b$). The lowest ask (highest bid) price happens when there is no depletion by the trader or the noise trader, and limit sell (buy) orders arrive according to the largest possible intensity. The prices have the upper and lower bounds

$$\begin{aligned} &P^a(0) - N^a(T) \le P^a(0) - N^a(t) \le P^a(t) \le \bar{p}^a \vee P^a(0) + R^a(t) \le \bar{p}^a \vee P^a(0) + R^a(T); \\ &\underline{p}^b \wedge P^b(0) - R^b(T) \le \underline{p}^b \wedge P^b(0) - R^b(t) \le P^b(t) \le P^b(0) + N^b(t) \le P^b(0) + N^b(T). \end{aligned} \tag{4.23}$$

Using the inequalities in (4.23) and the bounds for variances of stationary renewal processes derived in [14] Daley (1978), we get the inequalities in (4.21). □

Lemma 4.2 *There exist positive constants c_2 and c_3, such that for any admissible switching control $\alpha \in \mathscr{A}$ defined by Definition 4.1 and any hidden order strategy $h \in \mathscr{H}$ defined by Definition 3.2, the trader's total cash amount and stock inventory from the strategies α and h have the bounds*

$$\mathbb{E}[|C^h(T) + C^\alpha(T)|] \le c_2\left((Q^a(0))^2 + (Q^b(0))^2 + (P^a(0))^2 + (P^b(0))^2 + T^2 + 1\right) \tag{4.24}$$

and

$$\mathbb{E}[|I_0 + I^h(T) + I^\alpha(T)|^2] \le c_3\left((Q^a(0))^4 + (Q^b(0))^4 + (I_0)^2 + (P^a(0))^4 + (P^b(0))^4 + T^4 + 1\right). \tag{4.25}$$

Proof. Throughout the time interval $[0, T]$, the ask price P^a (bid price P^b) moves downward $N^a(T)$ times (upward $N^b(T)$ times). At least one limit at a time, the trader could take at most all the initially existing displayed limit sell (buy) orders and all the displayed limit sell (buy) orders ever arriving within the spread, as long as the ask (bid) price is below $\bar{p}^a$ (above $\underline{p}^b$). Then the total number of limits that the trader would ever buy (sell) until time T does not exceed $(\bar{p}^a - P^a(0))^+ + N^a(T)$ (respectively $(P^b(0) - \underline{p}^b)^+ + N^b(T)$). At any time $t \in [0, T]$, each limit of limit sell (buy) orders contains no more than $\Delta^a + Q^a(0) + 2\sigma^a \sup_{0\le t\le T} |W^a(t)|$ (respectively $\Delta^b + Q^b(0) + 2\sigma^b \sup_{0\le t\le T} |W^b(t)|$) shares. Each time buying (selling), it is impossible to take more than the number of all the currently existing limit sell (buy) orders below (above) the price $\bar{p}^a$ (respectively $\underline{p}^b$). We may bound the stock inventory and cash amount from displayed orders by

$$\begin{aligned} |I^\alpha(T)| \le{}& \left((\bar{p}^a - P^a(0))^+ + N^a(T)\right)\left(\Delta^a + Q^a(0) + 2\sigma^a \sup_{0\le t\le T} |W^a(t)|\right) \\ &+ \left((P^b(0) - \underline{p}^b)^+ + N^b(T)\right)\left(\Delta^b + Q^b(0) + 2\sigma^b \sup_{0\le t\le T} |W^b(t)|\right); \\ |C^\alpha(T)| \le{}& \bar{p}^a\left((\bar{p}^a - P^a(0))^+ + N^a(T)\right)\left(\Delta^a + Q^a(0) + 2\sigma^a \sup_{0\le t\le T} |W^a(t)|\right) \\ &+ \underline{p}^b\left((P^b(0) - \underline{p}^b)^+ + N^b(T)\right)\left(\Delta^b + Q^b(0) + 2\sigma^b \sup_{0\le t\le T} |W^b(t)|\right). \end{aligned} \tag{4.26}$$

Considering the hidden orders, the trader could receive at most $\Delta^a H^a(T)$ shares of hidden limit sell orders and $\Delta^b H^b(T)$ shares of hidden limit buy orders. The greatest possible price for each share does not exceed the bounds in (4.23). The stock inventory and cash amount from hidden orders can be bounded by

$$\begin{aligned} |I^h(T)| \le{}& \Delta^a H^a(T) + \Delta^b H^b(T); \\ |C^h(T)| \le{}& \Delta^a H^a(T)\left(P^a(0) + \bar{p}^a + N^a(T) + R^a(T) + 1\right) \\ &+ \Delta^b H^b(T)\left(P^b(0) + \underline{p}^b + N^b(T) + R^b(T) + 1\right). \end{aligned} \tag{4.27}$$

By the inequations (4.26) and (4.27), by Lemma 4.1, and by applying Burkholder-Gundy-Davis inequality to $\sup_{0\le t\le T} |W^a(t)|$ and $\sup_{0\le t\le T} |W^b(t)|$, the claim in this lemma can be justified. □

Theorem 4.1 *There exists a constant $c_4 > 0$, such that for any switching control $\alpha \in \mathscr{A}$ and any hidden order strategy $h \in \mathscr{H}$, we have*

$$\mathbb{E}\left[\left|\xi\left(C(T), I(T)\right) - r^C C_0\right|\right] \leq c_4\left((Q^a(0))^4 + (Q^b(0))^4 + (I_0)^2 + (P^a(0))^4 + (P^b(0))^4 + T^4 + 1\right) < \infty. \tag{4.28}$$

Furthermore, for all $t \in [0,T]$, the value process $V(t)$ defined in (4.19) has the growth rate

$$|V(t)| \leq c_4\left((Q^a(t))^4 + (Q^b(t))^4 + (I(t))^2 + (P^a(t))^4 + (P^b(t))^4 + (T-t)^4 + 1\right). \tag{4.29}$$

Proof. By Assumption 3.1 (1), equation (3.20) and equation (4.15), we know that

$$|\xi(C(T), I(T)) - r^C C_0| \leq |r^C| \cdot |C^h(T) + C^\alpha(T)| + |r^I| \cdot r^F(|z + I^h(T) + I^\alpha(T)|^2 + 1). \tag{4.30}$$

Substituting the inequalities (4.24) and (4.25) into inequality (4.30) gives the inequality (4.28). Because of the Markov property of the order book dynamics, the inequality in (4.29) can be derived in the same way as (4.28). □

Theorem 4.1 implies that the value process V defined in (4.19) indeed exists and is finite.

5 Relation between trading and price switching

The candidate active strategies in Problem 3.1 are a set of singular trading controls, while those in Problem 4.1 are a smaller set of strategies that only trade at a sequence of times. This section will prove Theorem 5.1 that states the relation between the two problems – the two price switching algorithms provide lower and upper bounds of value functions of the two singular trading algorithms.

Definition 5.1 *(step trading strategy) The admissible set of step trading strategy of a privileged trader and that of a regular trader are denoted respectively as $\mathscr{S}^p$ and $\mathscr{S}^r$. For $j = p, r$, let $\alpha = \{(S_n, u_n^a, u_n^b)\}_{n=0}^\infty \in \mathscr{A}^j$ be an arbitrary admissible switching control. The processes Z_α^a and Z_α^b, being the total shares that the trader has bought and sold according to the switching control α, are computed from*

$$Z_\alpha^i(t) = \sum_{n:S_n \leq t} g^i\left(S_n, Q^i(S_n-), u_n^i\right)\mathbf{1}_{\{u_n^i \geq 0\}},\ 0 \leq t \leq T,\ i = a, b, \tag{5.1}$$

where the mappings g^a and g^b are defined in (4.9). When limit orders arrive within the spread, the proportion of shares that the trader fills at the old price $P^i(S_n-)$ is computed from

$$\beta_\alpha^i(t) = \begin{cases} \{u_n^i\}\mathbf{1}_{\{u_n^i \in [-1,0)\}},\ \text{if } t = S_n \text{ and } N^i(S_n) - N^i(S_n-) = 1; \\ 0,\ \text{otherwise; } 0 \leq t \leq T,\ i = a, b. \end{cases} \tag{5.2}$$

The set $\mathscr{S}^j$ of admissible step trading strategies is defined as the collection of all the trading strategies $Z_\alpha = (Z_\alpha^a, Z_\alpha^b, \beta_\alpha^a, \beta_\alpha^b)$ satisfying (5.1) and (5.2) for some switching control $\alpha \in \mathscr{A}^j$. Namely, $\mathscr{S}^p = \{Z_\alpha | \alpha \in \mathscr{A}^p\}$ and $\mathscr{S}^r = \{Z_\alpha | \alpha \in \mathscr{A}^r\}$.

Seen from Definition 4.1 and Definition 5.1, each step trading strategy of a privileged or regular trader is his price switching strategy denoted in terms of the total numbers of shares bought and sold, so they are the same active trading strategy under different names. The two definitions further imply that

$$\mathscr{S}^r = \left\{Z_\alpha = (Z_\alpha^a, Z_\alpha^b, \beta_\alpha^a, \beta_\alpha^b) \in \mathscr{S}^p | \beta_\alpha^a(t) \equiv \beta_\alpha^b(t) \equiv 0, \text{ for all } t \in [0,T]\right\}. \tag{5.3}$$

Theorem 5.1 *The optimal trading problem is equivalent to an optimal switching problem in the sense that*

$$\begin{aligned} &\sup_{\alpha \in \mathscr{A}^r,\, h \in \mathscr{H}} \mathbb{E}\left[\xi(I^{h,Z}(T), C^{h,Z}(T))\right] = \sup_{Z_\alpha \in \mathscr{S}^r,\, h \in \mathscr{H}} \mathbb{E}\left[\xi\left(I^{h,Z_\alpha}(T), C^{h,Z_\alpha}(T)\right)\right] \\ \leq &\sup_{Z \in \mathscr{Z}^r,\, h \in \mathscr{H}} \mathbb{E}\left[\xi\left(I^{h,\alpha}(T), C^{h,\alpha}(T)\right)\right] \leq \sup_{Z \in \mathscr{Z}^p,\, h \in \mathscr{H}} \mathbb{E}\left[\xi(I^{h,Z}(T), C^{h,Z}(T))\right] \\ = &\sup_{Z_\alpha \in \mathscr{S}^p,\, h \in \mathscr{H}} \mathbb{E}\left[\xi\left(I^{h,Z_\alpha}(T), C^{h,Z_\alpha}(T)\right)\right] = \sup_{\alpha \in \mathscr{A}^p,\, h \in \mathscr{H}} \mathbb{E}\left[\xi\left(I^{h,\alpha}(T), C^{h,\alpha}(T)\right)\right], \end{aligned} \tag{5.4}$$

where the superscripts of $I(T)$ and $C(T)$ indicate the trading strategies used to derive the three pairs of terminal stock inventories and cash amounts.

Theorem 5.1 can be proved by the results from the next subsections 5.1 and 5.2. In (5.4), the second identity comes from Lemma 5.2, Lemma 5.1(1) and Proposition 5.1; the first and last identities are due to Definition 5.1; the first and second inequalities are deduced from Lemma 5.1(2) and Lemma 5.1(3).

The main idea of the proof of the second identity in Theorem 5.1 is to construct, in Lemma 5.2, a step trading strategy $Z_\alpha \in \mathscr{S}^p$ that pathwisely replicates the stock inventory and the cash amount produced by the singular trading strategy $Z \in \mathscr{Z}^p$. By Lemma 5.1(1), every step trading strategy is a singular trading strategy. Furthermore, as will be shown by Proposition 5.1, a privileged trader's two active trading strategies Z and Z_α pathwisely result in the same bid ask prices, replacing the former with the latter does not change the stock inventory and cash amount produced by a passive strategy on the hidden orders. This is where the second identity in (5.4) comes.

By their definitions, step trading strategies (Definition 5.1) and price switching strategies (Definition 4.1) have one-to-one correspondence between each other, because the two sets in fact consist of the same active trading strategies denoted in different terms. A price switching strategy denotes the times of transactions and the numbers of limits to buy and sell at each time; a step strategy denotes the total numbers of shares bought and sold up-to-date. This explains the first and last identities in Theorem 5.1.

The detailed results needed to prove Theorem 5.1 will be provided in subsections 5.1 and 5.2.

5.1 Analysis of active strategies

Every choice of admissible switching control defines an admissible step trading strategy. Every admissible step trading strategy is an admissible singular trading strategy. For every singular trading strategy, this subsection will construct a step trading strategy that pathwisely replicates the terminal stock inventory, the terminal cash amount and the entire price evolution.

It can be verified that the processes Z^a_α, Z^b_α, β^a_α and β^b_α defined in (5.1) and (5.2) satisfy Definition 3.1, so every step trading strategy $Z_\alpha = (Z^a_\alpha, Z^b_\alpha, \beta^a_\alpha, \beta^b_\alpha) \in \mathscr{S}^j$ is a singular trading strategy in the admissible set $\mathscr{Z}^j$, for $j = p, r$. However, the contrary is not true, because a singular trading strategy can be continuous over some time interval, but a step trading strategy is a pure jump process. By Definition 3.1(2), a regular trader's admissible set of singular trading strategies is the subset of a privileged trader's singular trading strategies that do not fill orders at the old price at the time of order arrival within the spread.

Lemma 5.1 *The admissible sets $\mathscr{S}^p$, $\mathscr{S}^r$, $\mathscr{Z}^p$ and $\mathscr{Z}^r$ of trading strategies defined in Definition 4.1 and Definition 5.1 have the inclusion relations*
(1) $\mathscr{S}^p \subsetneqq \mathscr{Z}^p$; (2) $\mathscr{S}^r \subsetneqq \mathscr{Z}^r$; (3) $\mathscr{Z}^r \subsetneqq \mathscr{Z}^p$.

Though it is a much smaller set as stated in Lemma 5.1(1), the set of his admissible step trading strategies performs equally well as a privileged trader's set of admissible step trading strategies. Whatever stock inventory and cash amount a singular trading strategy can produce at the terminal time, a privileged trader can always find a step trading strategy that pathwisely does the same. Hence a privileged trader's best expected reward can be achieved over a smaller and simpler set of admissible trading strategies. This is the role of Lemma 5.2.

Lemma 5.2 will be proved by constructing a price switching strategy $\alpha' \in \mathscr{A}^p$ (in other words, a step trading strategy $Z_{\alpha'} \in \mathscr{S}^p$) that trades entire limits at the times of price changes resulted from the singular trading strategy Z, unless when limit sell (buy) orders arrive within the spread or at the terminal time. It should be more obvious that the terminal stock inventory and cash amount of a singular trading strategy Z can be pathwisely replicated by a discretely re-balanced trading strategy, which trades only at the times of price changes resulted from the strategy Z and each time fills all the shares that Z would fill since the previous price change. The step trading strategy $Z_{\alpha'}$ trades at the same times of price changes, but it only

fills entire limits unless at the current ask (bid) price when limit sell (buy) orders arrive within the spread. The fractional limit at the new price that the other discretely re-balanced trading strategy would fill will be filled at the next time of price change. Sticking to entire limits when possible requires considering fewer candidate strategies than the other discretely re-balanced trading strategy does.

Lemma 5.2 *For any admissible singular trading strategy $Z = (Z^a, Z^b, \beta^a, \beta^b) \in \mathscr{Z}^p$, there exists an admissible switching control $\alpha = \{(S_n, u_n^a, u_n^b)\}_{n=0}^{\infty} \in \mathscr{A}^p$ such that the step trading strategy $Z_\alpha = (Z_\alpha^a, Z_\alpha^b, \beta_\alpha^a, \beta_\alpha^b) \in \mathscr{S}^p$ defined by (5.1) and (5.2) for this α almost surely satisfies*

$$I^Z(T) = I^{Z_\alpha}(T) \text{ and } C^Z(T) = C^{Z_\alpha}(T). \tag{5.5}$$

Proof. It suffices to find a specific $\alpha' = \{(S'_n, u_n^{a'}, u_n^{b'})\}_{n=0}^{\infty} \in \mathscr{A}^p$ such that the step trading strategy $Z_{\alpha'} = (Z_{\alpha'}^a, Z_{\alpha'}^b, \beta_{\alpha'}^a, \beta_{\alpha'}^b)$ defined in (5.1) and (5.2) for this α' satisfies the identities in (5.5).

Let L^a and L^b be the numbers of order depletion defined in (3.6), and N^a and N^b be the inhomogeneous Poisson processes representing order arrivals within the spread. The times of price changes are the sequence of $\mathscr{F}$-stopping times $\{S'_n\}_{n=0}^{\infty}$ defined as $S_0 = 0$ and, for $n = 1, 2, \cdots$, as

$$S'_n := \begin{cases} \inf\Big\{ S'_{n-1} < t \le T \Big| (L^a(t) - L^a(t-)) \vee (L^b(t) - L^b(t-)) \vee (N^a(t) - N^a(t-)) \\ \qquad\qquad \vee (N^b(t) - N^b(t-)) = 1 \Big\} \wedge T, \text{ if } S'_{n-1} < T; \\ T+1, \text{ if } S'_{n-1} \ge T. \end{cases} \tag{5.6}$$

Among $\{S'_n\}_{n=0}^{\infty}$, the time of the nth depletion of limit sell (buy) order of arrival of limit sell (buy) order within the spread is

$$S'_{n_k^i} := \begin{cases} \inf\Big\{ S'_{n_{k-1}^i} < t \le T \Big| (L^{i'}(t) - L^{i'}(t-)) \vee (N^{i'}(t) - N^{i'}(t-)) = 1 \Big\} \wedge T, \text{ if } S'_{n_{k-1}^i} < T; \\ T+1, \text{ if } S'_{n_{k-1}^i} \ge T, \end{cases} \tag{5.7}$$

with the convention that $S'_{n_0^i} = 0$, for $i = a, b$. Let μ^i be defined in (3.8), and the volume Q^i be the queue length controlled by the singular trading strategy Z as in (3.9). Then, the number of entire limits that the price switching strategy α' buys (sells) is

$$[u_n^{i'}] := \begin{cases} \mu^i\left(S'_{n_k^i}\right), \text{ if } S'_n = S'_{n_k^i} \le T \text{ for some } k; \\ 0, \text{ otherwise.} \end{cases} \tag{5.8}$$

For every $k = 1, 2, \cdots$, the singular trading strategy Z fills

$$\Delta z_k^i := \mathbf{1}_{\left\{[u_n^{i'}] \neq 0\right\}} \left(\Delta^i - Q^i\left(S'_{n_k^i}\right)\right) \tag{5.9}$$

shares in the fractional limit at the price $P^i\left(S'_{n_k^i}\right)$, while the price switching strategy α' postpones filling these Δz_k^i shares until the next time $S'_{n_{k+1}^i}$ of price change in the ask (bid) price. The singular trading strategy Z fills

$$\Delta Z_k^i := Z^i\left(S'_{n_k^i}\right) - Z^i\left(S'_{n_{k-1}^i}\right), \text{ for } i = a, b. \tag{5.10}$$

shares of limit sell (buy) orders over the time interval $\left(S'_{n_{k-1}^i}, S'_{n_k^i}\right]$.

Suppose $S'_n = S'_{n_k^i} \le T$ for some k, then the price switching strategy α' fills a fractional limit and the number of shares that α' fills is

$$\pi_n^{i'} := \begin{cases} \Delta Z_k^i - \Delta^i + \Delta z_{k-1}^i, \text{ if in case (3.c), } [u_n^{i'}] = 0 \text{ and } N^i(S'_n) - N^i(S'_n-) = 1; \\ Z^i\left(S'_{n_k^i}-\right) - Z^i\left(S'_{n_{k-1}^i}\right) + \beta^i\left(S'_{n_k^i}\right) Q^i\left(S'_{n_k^i}-\right) + \Delta z_{k-1}^i, \\ \qquad \text{if in case (3.d), } [u_n^{i'}] = -1 \text{ and } N^i(S'_n) - N^i(S'_n-) = 1; \\ \left(\Delta^i - Q^i(T)\right) + \Delta z_{k-1}^i, \text{ if in case (3.e), } [u_n^{i'}] \ge 1 \text{ and } S'_n = T; \\ \Delta Z_k^i + \Delta z_{k-1}^i, \text{ if in case (3.f), } [u_n^{i'}] = 0 \text{ and } S'_n = T. \end{cases} \tag{5.11}$$

The fractional limit that α' fills is at the price level $P^i\left(S'_n\right)$ in cases (3.c,d), at the price level $P^i(T)$ in cases (3.e,f). Using the singular strategy Z, the number of shares at those price levels at the time S'_n- is

$$q_n^{i'} := \begin{cases} Q^i\left(S'_{n^i_{k-1}}\right) + \sigma^i\left(W^i\left(S'_{n^i_k}\right) - W^i\left(S'_{n^i_{k-1}}\right)\right) + \Delta z^i_{k-1}, \text{ if in case (3.c,d,f), } [u_n^{i'}] \le 0; \\ \Delta^i, \text{ if in case (3.e), } [u_n^{i'}] \ge 1 \text{ and } S'_n = T. \end{cases} \tag{5.12}$$

When $S'_n = S'_{n^i_k} \le T$ for some k, the fractional part of $u_n^{i'}$ is

$$\{u_n^{i'}\} = \pi_n^{i'} / \, q_n^{i'}, \text{ if in case (3.c,d,f) } [u_n^{i'}] \le 0, \text{ or in case (3.e) } [u_n^{i'}] \ge 1 \text{ and } S'_n = T; \tag{5.13}$$

in any other situation $\{u_n^{i'}\}$ is set to be zero. From equations (5.8) and (5.13), the total number of limits that the price switching strategy α' buys (sells) at the time S'_n is

$$u_n^{i'} = [u_n^{i'}] + \{u_n^{i'}\}, \, i = a, b. \tag{5.14}$$

The singular strategy Z and the price switching strategy α' (equivalently, the step trading strategy $Z_{\alpha'} = (Z^a_{\alpha'}, Z^b_{\alpha'}, \beta^a_{\alpha'}, \beta^b_{\alpha'})$ defined according to (5.1) and (5.2)) buys and sells the same amount of shares at the same price levels, hence they produce the same terminal inventory and cash amount. This can also be verified by direct computation via the expressions (3.11), (3.12) and (6.3). □

Remark 5.1 *Some traders in practice would avoid triggering a price change by filling a fraction of a limit, which is necessary in two possible extensions of our framework. The first possibility is that there is more than one privileged trader in addition to the noise traders, like in the era before 2008-2011. The several privileged traders face a question of competing and/or collaborating with each other while maximizing their individual trading objectives. In this case, using the method in this section, one can still construct a discretely re-balanced trading strategy to replicate every privileged trader's terminal stock inventory and cash amount resulted from a singular trading strategy, except that the traders may have to fill fractional limits at the times of price change to achieve their optimal reward. The other possibility is that there is a unique regular trader in the market who does not have the privilege that entitles him to act upon the arriving limit orders in the spread. His admissible set of singular trading strategies becomes the subset of $\mathscr{Z}^p$ which satisfies $\beta^a \equiv \beta^b \equiv 0$. In this case, we do not see how to find a pathwise replication by price switching strategies. Instead, the conjecture is that a regular trader's best expected reward over all the singular trading strategies can be achieved within a set of discretely re-balanced trading strategies that allows for filling fractional limits at any time. Since fractional limits bring a much larger admissible control set which can be implemented on very few computer clusters, we shall stop at the optimal choice among entire limits.*

5.2 Effect on a privileged trader's hidden orders

Given an arbitrary admissible singular trading strategy Z from the privileged trader, the proof of Lemma 5.2 constructed a step trading strategy $Z_{\alpha'}$ for some $\alpha' = \{(S'_n, u_n^{a'}, u_n^{b'})\}_{n=0}^{\infty} \in \mathscr{A}^p$ to replicate the terminal stock inventory and cash amount. Let P^a_Z and P^b_Z denote the price processes (3.10) controlled by the singular trading strategy Z, and $P^a_{\alpha'}$ and $P^b_{\alpha'}$ denote the price processes (4.7) controlled by the switching control α'. Then by the construction of $(S'_n, u_n^{a'}, u_n^{b'})$ in equations (5.6), (5.8) and (5.14), and by the expressions (3.7) and (4.7) of the price dynamics, we know that the two strategies also produce the same times and amounts of price change, meaning that

$$P^a_Z(t) = P^a_{\alpha'}(t) \text{ and } P^b_Z(t) = P^b_{\alpha'}(t), \text{ for all } (t, \omega) \in [0, T] \times \Omega. \tag{5.15}$$

Let us recall that the intensities of the liquidity event processes H^a and H^b are functions of the spread only. The equations (3.15) and (5.15) further imply that, both the inventory I^h and cash amount C^h from an arbitrary hidden order strategy $h = (h^a, h^b) \in \mathscr{H}$ remain the same regardless of whether the singular trading strategy Z or the step trading strategy $Z_{\alpha'}$ is used. The analysis in this paragraph has verified a reinforcement of Lemma 5.2, stated as the proposition below.

Proposition 5.1 *Let $h = (h^a, h^b) \in \mathscr{H}$ be an arbitrary hidden order strategy. For any admissible singular trading strategy $Z = (Z^a, Z^b, \beta^a, \beta^b) \in \mathscr{Z}^p$, there exists an admissible switching control $\alpha = \{(S_n, u_n^a, u_n^b)\}_{n=0}^{\infty} \in \mathscr{A}^p$ such that the step trading strategy $Z_\alpha = (Z_\alpha^a, Z_\alpha^b, \beta_\alpha^a, \beta_\alpha^b) \in \mathscr{S}^p$ defined in (5.1) and (5.2) for this α almost surely satisfies*

$$I^{h,Z}(T) = I^{h,Z_\alpha}(T) \text{ and } C^{h,Z}(T) = C^{h,Z_\alpha}(T), \tag{5.16}$$

where $I^{h,Z}(T)$, $I^{h,Z_\alpha}(T)$, $C^{h,Z}(T)$ and $C^{h,Z_\alpha}(T)$ are defined in (3.16) and (4.15).

6 Solving the optimal switching problem

This section will provide the characterization of an optimal trading strategy and derive a trading algorithm for the optimal switching Problem 4.1. The solution is valid regardless of whether the trader is privileged or regular, hence the admissible set of switching controls is generically denoted as $\mathscr{A}$. Before getting down to the solution, a few notations are introduced.

The two dimensional quantities representing both sides of the order book are denoted as $Q(t) = (Q^a(t), Q^b(t))$, $P(t) = (P^a(t), P^b(t))$, $q = (q^a, q^b)$, $p = (p^a, p^b)$, $u = (u^a, u^b)$ and $h = (h^a, h^b)$ for short. As will be shown in Theorem 6.1, the decision making would only need to observe the state processes $\{(N^a, N^b)\}_{0\le t\le T}$ and $\{(Q(t-), I^\alpha(t-) + I^h(t), P(t-))\}_{0\le t\le T}$, which generate a smaller filtration than $\mathscr{F}(t)$. The domain of the process $\{(Q(t-), I^\alpha(t-) + I^h(t), P(t-))\}_{0\le t\le T}$ is denoted as

$$\mathscr{D} = [0,\infty)^2 \times \mathbb{R} \times \{(p^a, p^b) \in (P^a(0), P^b(0)) + \mathbb{N}^2 | p^a > p^b\}. \tag{6.1}$$

To express the change in the order book and in the inventory from the trader's transaction, the mapping $\gamma : \Omega \times [0,T] \times \mathscr{D} \times \mathbb{N}^2 \to [0,\infty)^2 \times \mathbb{R}$ is defined as

$$\gamma(t,q,z,u) = \begin{pmatrix} \mathbf{1}_{\{u^a \neq 0\}}\Delta^a + \mathbf{1}_{\{u^a=0\}}(1 - \{u^a\})q^a \\ \mathbf{1}_{\{u^b \neq 0\}}\Delta^b + \mathbf{1}_{\{u^b=0\}}(1 - \{u^b\})q^b \\ z + g_\alpha(t,q,u) \end{pmatrix}^{\text{transpose}}. \tag{6.2}$$

Immediately after applying the switching control u at time t, the volumes and inventory become

$$(Q(t), I(t)) = \gamma\left(t, Q(t-), I^\alpha(t-) + I^h(t), u\right). \tag{6.3}$$

The process $\{\int_0^t r(P(s-), h(s-))ds\}_{0\le t\le T}$ defined as

$$r(P(t), h(t)) = -\Delta^a h^a(t)(P^b(t) + 1)\lambda^a(P^a(t) - P^b(t)) + \Delta^b h^b(t)\lambda^b(P^a(t) - P^b(t)),\ 0 \le t \le T, \tag{6.4}$$

is the finite variation part in the Doob-Meyer decomposition of the semimartingale C^h defined in (3.15). By the bound in (4.27), the local martingale part of C^h is a martingale. Then the value process V defined in (4.19) can be written alternatively as

$$\begin{aligned} V(t) = \sup_{\alpha\in\mathscr{A}_t,\, h\in\mathscr{H}_t} \mathbb{E}\Bigg[& r^C \sum_{n=1}^{\infty} f_\alpha\left(S_n, Q(S_n-), P(S_n-), u_n\right) \\ & + r^C \int_t^T r(P(s-), h(s-))ds + r^I F(I^{h,\alpha}(T), P(T)) \Big| \mathscr{F}(t) \Bigg],\ 0 \le t \le T. \end{aligned} \tag{6.5}$$

For every $h \in \mathscr{H}_{0,t}$, the process $Y(\cdot\,; h)$ is defined as

$$Y(t; h) = \int_0^t r(P(s-), h(s-))ds + V(t),\ 0 \le t \le T. \tag{6.6}$$

For every measurable function $\phi : [0,T] \times \mathscr{D} \to \mathbb{R}$, the operator $\mathscr{M}$ is defined as

$$\mathscr{M}\phi(t,q,z,p) = \max_{u\in\mathscr{U}(t,p)} \left\{ f_\alpha(t,q,p,u) + \phi\left(t, \gamma(t,q,z,u), p^a + [u^a], p^b - [u^b]\right)\right\}. \tag{6.7}$$

6.1 Optimal trading strategy

It has been shown in subsection 4.2 that the value process of the optimal price switching problem is finite everywhere. This subsection will eventually derive in Proposition 6.1 expressions of an optimal price switching strategy in terms of the value process. The methodology is based on the principle that the value process of a control problem is a supermartingale, and becomes a martingale if and only if the control is optimal. It is called the "martingale method", first introduced for optimal stopping problems in Snell (1952) and for stochastic control problems in Davis (1979). The pivot of the arguments is the dynamic programming principle formulated in our setting as Theorem 6.1. A reference of the dynamic programming principle is Fleming and Soner (1993). Lemma 6.1 provides the right continuity of the value process, so that it is a qualified candidate for using the Snell envelop technique to sequentially determine each optimal time of trading. Lemma 6.2 is the characterization of the optimal trading strategy from the martingality of the value process. Because Theorem 4.1 has shown that the value process is finite, the expressions in Proposition 6.1 imply the existence of an optimal trading strategy.

Theorem 6.1 *(dynamic programming principle) Given $(Q(t-), I^\alpha(t-)+I^h(t), P(t-)) = (q, z, p)$, there exist deterministic measurable functions v^0, v^a and $v^b : [0,T] \times \mathscr{D} \to \mathbb{R}$, and a mapping $v : \Omega \times [0,T] \times \mathscr{D} \to \mathbb{R}$, such that the value process V defined by the equation (4.19) satisfies*

$$V(t) = v(t,q,z,p) = \begin{cases} v^0(t,q,z,p), \text{ if } N^i(t) - N^i(t-) = 0, \ i = a \text{ and } b; \\ v^i(t,q,z,p), \text{ if } N^i(t) - N^i(t-) = 1, \ i = a \text{ or } b. \end{cases} \tag{6.8}$$

The value functions v^0, v^a and v^b can be computed via the dynamic programming principle

$$\begin{aligned} & v^0\left(t, Q(t-), I^\alpha(t-) + I^h(t), P(t-)\right) \\ = & \sup_{(S_1,u_1)\in\mathscr{A}_{t,1},\, h\in\mathscr{H}_{t,S_1}} \mathbb{E}\Bigg[r^C \left(f_\alpha\left(S_1, Q(S_1-), P(S_1-), u_1\right) + \int_t^{S_1} r(P(t), h(s-))ds \right) \\ & + v\left(S_1, \gamma\left(S_1, Q(S_1-), I^\alpha(S_1-) + I^h(S_1), u_1\right), P^a(S_1-) + [u_1^a], P^b(S_1-) - [u_1^b]\right) \Big| \mathscr{F}(t) \Bigg], \end{aligned} \tag{6.9}$$

when $N^a(t) - N^a(t-) = 0$ and $N^b(t) - N^b(t-) = 0$, and

$$v^i\left(t, Q(t-), I^\alpha(t-) + I^h(t), P(t-)\right) = \mathscr{M} v^0\left(t, Q(t-), I^\alpha(t-) + I^h(t), P(t-)\right) \tag{6.10}$$

when $N^i(t) - N^i(t-) = 1$, $i = a, b$. Especially, at the terminal time T, the value function satisfies the terminal condition

$$v\left(T, Q(T-), I^\alpha(T-) + I^h(T), P(T-)\right) = \mathscr{M} F(I(T-), P(T-)). \tag{6.11}$$

Proof. The existence of the functions v^0, v^a and v^b comes from the Markovian structure of the processes (Q, I, P), and the memoryless property of the exponential interarrival times for the orders within the spread. The proof of the dynamic programming principle is routine. To wit, take arbitrary $\alpha \in \mathscr{A}_t$ and $h \in \mathscr{H}_t$ as defined in Definitions 4.1 and 3.2, and denote

$$\begin{aligned} \Sigma^1 =& r^C \left(f_\alpha\left(S_1, Q(S_1-), P(S_1-), u_1\right) + \int_t^{S_1} r(P(t), h(s-))ds \right), \text{ and} \\ \Sigma^2 =& r^C \sum_{n=2}^{\infty} \left(f_\alpha\left(S_n, Q(S_n-), P(S_n-), u_n\right) + \int_{S_{n-1}}^{S_n} r\left(P(S_{n-1}), h(s-)\right) ds \right) + r^I F(I(T), P(T)). \end{aligned} \tag{6.12}$$

Then

$$\mathbb{E}\left[\xi(I(T), C(T))\middle|\mathscr{F}(t)\right] - r^C C(t) = \mathbb{E}\left[\Sigma^1 + \mathbb{E}\left[\Sigma^2\middle|\mathscr{F}(S_1)\right]\middle|\mathscr{F}(t)\right]. \tag{6.13}$$

On one hand, taking supremum over $(S_1, u_1) \in \mathscr{A}_{t,1}$ on both sides of the inequality

$$\mathbb{E}\left[\Sigma^1 + \mathbb{E}\left[\Sigma^2\middle|\mathscr{F}(S_1)\right]\middle|\mathscr{F}(t)\right] \le \mathbb{E}\left[\Sigma^1 + v\left(S_1, Q(S_1-), I^\alpha(S_1-) + I^h(S_1), P(S_1-)\right)\middle|\mathscr{F}(t)\right] \tag{6.14}$$

shows that $V(t)$ is less than or equal to the right hand side of (6.9). On the other hand, the inequality

$$V(t) \geq \mathbb{E}\left[\Sigma^1 + \mathbb{E}\left[\Sigma^2 \,|\mathscr{F}(S_1)\right] \middle| \mathscr{F}(t)\right] \tag{6.15}$$

implies

$$V(t) \geq \mathbb{E}\left[\Sigma^1 + v\left(S_1, Q(S_1-), I^\alpha(S_1-) + I^h(S_1), P(S_1-)\right) \middle| \mathscr{F}(t)\right] \tag{6.16}$$

and thus $V(t)$ greater than or equal to the right hand side of (6.9). Both sides of the inequality hold, hence

$$\begin{aligned} V(t) = \sup_{(S_1,u_1)\in\mathscr{A}_{t,1},\, h\in\mathscr{H}_{t,S_1}} \mathbb{E}\Bigg[& r^C\left(f_\alpha\left(S_1, Q(S_1-), P(S_1-), u_1\right) + \int_t^{S_1} r(P(t), h(s-))ds\right) \\ & + v\left(S_1, \gamma\left(S_1, Q(S_1-), I^\alpha(S_1-) + I^h(S_1), u_1\right), P^a(S_1-) + [u_1^a], P^b(S_1-) - [u_1^b]\right) \Bigg| \mathscr{F}(t)\Bigg]. \end{aligned} \tag{6.17}$$

When $N^a(t) - N^a(t-) = 0$ and $N^b(t) - N^b(t-) = 0$, by (6.8) there is

$$V(t) = v^0\left(t, Q(t-), I^\alpha(t-) + I^h(t), P(t-)\right), \tag{6.18}$$

hence (6.17) takes the form (6.9). When $N^a(t) - N^a(t-) = 1$ or $N^b(t) - N^b(t-) = 1$ or $t = T$, the trader has to "trade" at time t, though possibly zero share, hence (6.17) takes the form (6.10) or (6.11). □

Lemma 6.1 *For every $0 \leq t < t+\Delta t \leq T$, suppose the trader does not trade over the time interval $[t, t+\Delta t]$. Then the processes $\left\{v^i\left(t, Q(t-), I^\alpha(t-) + I^h(t), P(t-)\right)\right\}_{0\leq t\leq T}$, $i = 0, a, b$, are right continuous at the time t.*

Proof. It suffices to prove the right continuity for v^0, then the right continuity for v^a and v^b follows from the expression (6.10).

Take an arbitrary time $t \in [0, T]$, an arbitrary price switching strategy $\alpha \in \mathscr{A}_t$ and an arbitrary hidden order strategy $h \in \mathscr{H}_t$. Suppose $N^a(t) - N^a(t-) = 0$ and $N^b(t) - N^b(t-) = 0$.

For any two sets of initial values (q, z, p) and $(q', z, p) \in \mathscr{D}$ of the state process, the difference in the trader's stock inventory does not exceed $\left|q^a - q^{a'}\right| + \left|q^b - q^{b'}\right|$, and that in the cash amount does not exceed $p^a\left|q^a - q^{a'}\right| + p^b\left|q^b - q^{b'}\right|$. By Assumption 3.1 (2) and from the bounds of the price, stock inventory and cash amount in Lemma 4.1 and Lemma 4.2, we know that

$$v^0(t, q, z, p) = \lim_{q'\to q} v^0(t, q', z, p). \tag{6.19}$$

For any initial values $(q, z, p) \in \mathscr{D}$ and any number $\Delta t \in [0, T-t]$, the expected reward from trading can be written as

$$\begin{aligned} & \mathbb{E}\left[\xi(I(T), C(T))|\,\mathscr{F}(t)\right] - r^C C(t) \\ =& \mathbb{E}\left[r^C\left(\sum_{n:\, t\leq S_n\leq T} f_n + \int_t^T r\left(P(t-), h(s-)\right) ds\right) + r^I F(I(T), P(T)) \middle| \mathscr{F}(t)\right] \\ =& \mathbb{E}\left[r^C\left(\sum_{n:\, t\leq S_n\leq T-\Delta t} f_n + \int_t^{T-\Delta t} r\left(P(t-), h(s-)\right) ds\right) + r^I F(I(T-\Delta t), P(T-\Delta t)) \middle| \mathscr{F}(t)\right] \\ & + \mathbb{E}\Bigg[r^C\Bigg(\sum_{n:\, T-\Delta t< S_n\leq T} f_n + \int_{T-\Delta t}^T r\left(P(t-), h(s-)\right) ds\Bigg) \\ & \qquad + r^I\left(F(I(T), P(T)) - F(I(T-\Delta t), P(T-\Delta t))\right) \Bigg| \mathscr{F}(t)\Bigg], \end{aligned} \tag{6.20}$$

where

$$f_n := f_\alpha\left(S_n, Q(S_n-), P(S_n-), u_n\right). \tag{6.21}$$

The random variable in the last expectation in (6.20) is, by Theorem 4.1, bounded by an integrable random variable for all $\alpha \in \mathscr{A}_t$, $h \in \mathscr{H}_t$ and $\Delta t \in [0, T-t]$, and converges almost surely to zero as $\Delta t \to 0+$. Taking supremum over the trading strategies $\alpha \in \mathscr{A}_t$ and $h \in \mathscr{H}_t$ in (6.20) gives

$$v^0(t,q,z,p) = \left(\sup_{\alpha \in \mathscr{A}_{t,T-\Delta t}, h \in \mathscr{H}_{t,T-\Delta t}} \mathbb{E}\left[\xi(I(T-\Delta t), C(T-\Delta t)) | \mathscr{F}(t) \right] - r^C C(t) \right) + O(\Delta t). \tag{6.22}$$

The term in the parenthesis in (6.22) equals $v^0(t+\Delta t, q, z, p)$, hence

$$v^0(t,q,z,p) = v^0(t+\Delta t, q, z, p) + O(\Delta t). \tag{6.23}$$

The the right continuity of the process $\left\{ v^0\left(t, Q(t-), I^\alpha(t-) + I^h(t), P(t-)\right) \right\}_{0 \le t \le T}$ can be concluded from the identities (6.19) and (6.23) and from the right continuity of the state process. □

The proofs of Lemma 6.2 and Proposition 6.1 follow the routine procedure on how to characterize the optimal control and optimal stopping time via the martingale method. Because they are very long, the proofs are not provided here. Interested readers could find the original idea in Davis (1979) and Snell (1952), and the arguments for a most similar result in section 2.2.2 of Li (2011).

Lemma 6.2 *The price switching strategy $\alpha^* = (S_1^*, u_1^*) \in \mathscr{A}_{t,1}$ and hidden order strategy $h^* \in \mathscr{H}_{t,S_1^*}$ achieve the supremum in (6.9), if and only if all of the four conditions below hold.*

(1) $\{Y(t;h)\}_{0 \le t \le T_1}$ is a supermartingale, for every $h \in \mathscr{H}_{t,T_1}$;
(2) $\{Y(t \wedge S_1^*; h^*)\}_{0 \le t \le T_1}$ is a martingale;
(3) either $\left(v^0 - \mathscr{M}v^0\right)\left(S_1^*, Q(S_1^*-), I^{\alpha^*}(S_1^*-) + I^{h^*}(S_1^*), P(S_1^*-)\right) = 0$, or $S_1^* = T_1$;
(4)

$$u_1^* = arg \max_{u \in \mathscr{U}(S_1^*, P(t))} \left\{ r^C \left(f_\alpha(S_1^*, Q(S_1^*-), P(t), u_1^*) + \int_t^{S_1^*} r(P(t), h^*(s-)) ds \right) \right.$$
$$\left. + v\left(S_1^*, \gamma(S_1^*, Q(S_1^*-), I^{\alpha^*}(S_1^*-) + I^{h^*}(S_1^*), u_1^*), P^a(t) + [u_1^{a*}], P^b(t) - [u_1^{b*}]\right) \right\}.$$

Proposition 6.1 *There exist an optimal switching control $\alpha^* = \{(S_n^*, u_n^{a*}, u_n^{b*})\}_{n=1}^\infty \in \mathscr{A}$ and an optimal hidden order strategy $h^* = (h^{a*}, h^{b*}) \in \mathscr{H}$, which are defined in the following way. Let $S_0^* = u_0^{a*} = u_0^{b*} = 0$. For $n = 1, 2, \cdots$, the optimal trading time S_n^* can be expressed as*

$$S_n^* = \begin{cases} \inf\left\{ S_{n-1}^* < t \le T \middle| \left(v^0 - \mathscr{M}v^0\right)\left(t, Q(t-), I^{\alpha^*}(t-) + I^{h^*}(t), P(t-)\right) = 0 \right\} \wedge T_n, \text{ if } S_{n-1}^* < T; \\ T+1, \text{ if } S_{n-1}^* = T. \end{cases} \tag{6.24}$$

If $S_n^* = T+1$, then $u_n^{a*} = u_n^{b*} = 0$; otherwise

$$\begin{aligned} u_n^* = arg \max_{u \in \mathscr{U}(S_n^*, P(S_{n-1}^*))} & \left\{ r^C f_\alpha(S_n^*, Q(S_n^*-), P(S_{n-1}^*), u) \right. \\ & \left. + v\left(S_n^*, \gamma(S_n^*, Q(S_n^*-), I^{\alpha^*}(S_n^*-) + I^{h^*}(S_n^*), u), P^a(S_{n-1}^*) + [u^a], P^b(S_{n-1}^*) - [u^b]\right) \right\}. \end{aligned} \tag{6.25}$$

Denoting as

$$\begin{aligned} & r^0\left(t, Q(t), I^{\alpha^*}(t), P(t), h(t)\right) \\ := & \left(v^0\left(t, Q(t), I^{\alpha^*}(t) + \Delta^a, P(t)\right) - v^0\left(t, Q(t), I^{\alpha^*}(t), P(t)\right) \right) h^a(t) \lambda^a \left(P^a(t) - P^b(t)\right) \\ & + \left(v^0\left(t, Q(t), I^{\alpha^*}(t) - \Delta^b, P(t)\right) - v^0\left(t, Q(t), I^{\alpha^*}(t), P(t)\right) \right) h^b(t) \lambda^b \left(P^a(t) - P^b(t)\right), \end{aligned} \tag{6.26}$$

for $0 \le t \le T$, the optimal hidden order strategy h^* can be expressed as

$$h^*(t) = arg \max_{h(t) \in \{0,1\}^2} \left\{ r\left(P(t), h(t)\right) + r^0\left(t, Q(t), I^{\alpha^*}(t), P(t), h(t)\right) \right\}. \tag{6.27}$$

6.2 Numerical algorithm

There are several types of numerical methods to compute the value function and the optimal strategy of a control problem. Here this price switching problem will be numerically solved by backward induction over a grid of discretized time and state space. This kind of method has been studied in great details in a sequel of works, notably by Pagès, Pham and Printems (2004) and Gassiat, Kharroubi and Pham (2012) for convergence, error rates and refinements to solve control problems very similar to ours. Since the calculation of the optimal grid of the space discretization, which will cost one entire paper, has not yet been done in our particular case, we shall use equi-partitions over the state space. Readers interested in further exploring the numerical method are invited to the works by Pham and Pagès.

This subsection will present the numerical algorithm to compute the value function and optimal trading strategy for the discretized version of the optimal price switching Problem 4.1. Like the numerical implementation of every optimal control problem on a discretized grid via the backward induction, the time complexity of the programme using serial computation on a PC is the number of time steps times the mesh size of the grid for the state space. However, it is feasible to compute all the nodes at each time step in parallel, because the computation at each node only need the results from one time step later and does not use result from the computation at any other node at the time step. By transferring time complexity to space complexity, the time complexity of the algorithm on a GPU cluster can be reduced to a minimum of the same order as the total number of time steps.

The time and the state process are discretized over a grid $\mathscr{T} \times \mathscr{X}$, where $\mathscr{T}$ as the grid for the time $t \in [0, T]$ is defined as

$$\mathscr{T} = \{0 = t_0, t_1, t_2, \cdots, t_K = T\} = \{0, \Delta t, 2\Delta t, \cdots, K\Delta t = T\} \tag{6.28}$$

and $\mathscr{X}$ as the grid for the state process (Q, I, P) is a bounded set in $\mathscr{D}$ with $|\mathscr{X}| < \infty$ elements. When the grid tends finer and finer, the limit

$$\lim_{K \to \infty, |\mathscr{X}| \to \infty} \mathscr{T} \times \mathscr{X} \tag{6.29}$$

is assumed to be a dense set in $[0, T] \times \mathscr{D}$.

Using the three steps to be specified below, Step **1.** computes the value function and the optimal trading strategy at the terminal time $t_K = T$ for all $x \in \mathscr{X}$; starting from $k = K - 1$ to $k = 0$ in descending order, the computations in Steps **2.** and **3.** are repeated for all $x \in \mathscr{X}$. Step **2.** simulates the state process in preparation of approximating the conditional expectation term in the dynamic programming principle. Step **3.** computes the value function and the optimal trading strategy at the time t_k, according to whether or not limit orders arrive within the spread over the time interval $(t_{k-1}, t_k]$. The outputs of this algorithm will be the value function and the optimal trading strategy $(\bar{v}^0(t_k, x), \bar{u}^{*0}(t_k, x), \bar{h}^*(t_k, x))$ when limit orders do not arrive within $(t_{k-1}, t_k]$, and $(\bar{v}^i(t_k, x), \bar{u}^{*i}(t_k, x))$ when limit sell (buy) orders arrive within $(t_{k-1}, t_k]$, for all $(t_k, x) \in \mathscr{T} \times \mathscr{X}$.

Step **1.** (at the terminal time) At the time $t_K = T$, the terminal condition is $\bar{v}(T, x) = \mathscr{M}F(z, p)$.

Step **1.1** Compute the reward from trading at the terminal time for every trading strategy $u \in \mathscr{U}(T, p)$ as

$$\bar{v}(T, x; u) = \bar{v}^0(T, x; u) = \bar{v}^i(T, x; u) = f_\alpha(T, q, p, u) + F(z + g_\alpha(T, q, u), p). \tag{6.30}$$

Step **1.2** The maximum reward from trading at the terminal time is

$$\bar{v}(T, x) = \bar{v}^0(T, x) = \bar{v}^i(T, x) = \max\{\bar{v}(T, x; u) | u \in \mathscr{U}(T, p)\}. \tag{6.31}$$

The optimal trading strategy is

$$\bar{u}^*(T, x) = \{u \in \mathscr{U}(T, p) \mid \text{such that } \bar{v}(T, x; u) = \bar{v}(T, x)\}. \tag{6.32}$$

□

Step **2.** (simulate the controlled state process) With the initial values

$$X_{t_k,x}(t_k;u) := (Q_{t_k,x}(t_k;u), I_{t_k,x}(t_k;u), P_{t_k,x}(t_k;u)) = \left(\gamma(t_k,q,z,p,u), p^a + [u^a], p^b - [u^b]\right), \tag{6.33}$$

simulate the state process

$$X_{t_k,x}(t_{k+1};u,h) = (Q_{t_k,x}(t_{k+1};u), I_{t_k,x}(t_{k+1};u,h), P_{t_k,x}(t_{k+1};u)) \tag{6.34}$$

according to

$$\begin{cases} Q^i_{t_k,x}(t_{k+1};u) = Q^i_{t_k,x}(t_k;u) + \sigma^i \times \text{ (a Normal r.v. with mean zero and variance } \Delta t); \\ I_{t_k,x}(t_{k+1};u,h) = I_{t_k,x}(t_k;u) + \Delta^a h^a \times \text{ (a Poisson r.v. with intensity } \lambda^a(p^a-p^b)\Delta t) \\ \qquad\qquad\qquad\qquad - \Delta^b h^b \times \text{ (a Poisson r.v. with intensity } \lambda^b(p^a-p^b)\Delta t); \\ P^a_{t_k,x}(t_{k+1};u) = p^a + [u^a]; \\ P^b_{t_k,x}(t_{k+1};u) = p^b - [u^b]. \end{cases} \tag{6.35}$$

So that the state process remains within the grid $\mathscr{X}$, the truncated value from each simulation $\bar{X}_{t_k,x}(t_{k+1})$ is obtained from

$$\bar{X}_{t_k,x}(t_{k+1};u,h) := arg\min\left\{ |X_{t_k,x}(t_{k+1};u,h) - y| \,\middle|\, y \in \mathscr{X} \right\}. \tag{6.36}$$

Run M simulations to get $\bar{X}_{t_k,x}(t_{k+1};u,h)$ according to the equations (6.34), (6.35) and (6.36). The M simulated values are denoted as $\left\{\bar{X}^m_{t_k,x}(t_{k+1};u,h)\right\}_{m=1}^M$. For $i=a,b$, simulate M Poisson random variables $\left\{N^i_m\right\}_{m=1}^M$ with the intensity $\theta^i(p^a-p^b)\Delta t$ to represent whether limit orders arrive within the spread during the time interval $(t_k, t_{k+1}]$. □

Step **3.** (value function and optimal trading strategy) This step conducts the optimization procedure by the dynamic programming principle

$$\begin{aligned} \bar{v}(t_k,x) = \max_{u\in\mathscr{U}(t_k,p),\, h\in\{0,1\}^2} \Big\{ & \left((p^a-1)h^b\lambda^b(p^a-p^b) - (p^b+1)h^a\lambda^a(p^a-p^b)\right)\Delta t \\ & + f_\alpha(t_k,q,p,u) + \mathbb{E}\left[\bar{v}(t_k, \bar{X}_{t_k,x}(t_{k+1};u,h))\right] \Big\}. \end{aligned} \tag{6.37}$$

Step **3.1** (approximating the expectation) The conditional expectation $\mathbb{E}\left[\bar{v}(t_k, \bar{X}_{t_k,x,}(t_{k+1}))\right]$ in (6.37) is approximated by computing

$$\hat{v}(t_k,x;u,h) := \frac{1}{M}\sum_{m=1}^M \left(\mathbf{1}_{\{N^a_m=0, N^b_m=0\}} \bar{v}^0\left(t_k, \bar{X}^m_{t_k,x}(t_{k+1};u,h)\right) + \sum_{i=a,b} \mathbf{1}_{\{N^i_m=0\}} \bar{v}^i\left(t_k, \bar{X}^m_{t_k,x}(t_{k+1};u,h)\right) \right). \tag{6.38}$$

Step **3.2** (value function and trading strategy when no arrival within the spread) This is the case when there is no limit order arrival within the spread throughout the time interval $(t_{k-1}, t_k]$, meaning that $N^i(t_k) - N^i(t_{k-1}) = 0$, for $i = a$ and b. The reward from using a generic trading strategy $u \in \mathscr{U}(t_k,p)$ and $h \in \{0,1\}^2$ is

$$\bar{v}^0(t_k,x;u,h) = \left((p^a-1)h^b\lambda^b(p^a-p^b) - (p^b+1)h^a\lambda^a(p^a-p^b)\right)\Delta t + f_\alpha(t_k,q,p,u) + \hat{v}(t_k,x;u,h) \tag{6.39}$$

The optimal value from trading is

$$\bar{v}^0(t_k,x) = \max\left\{\bar{v}^0(t_k,x;u,h) | u \in \mathscr{U}(t_k,p) \text{ and } h \in \{0,1\}^2\right\}. \tag{6.40}$$

The optimal trading strategy is

$$(\bar{u}^{0*}(t_k,x), \bar{h}^*(t_k,x)) = \left\{ u \in \mathscr{U}(t_k,p) \text{ and } h \in \{0,1\}^2 \,\middle|\, \text{ such that } \bar{v}^0(t_k,x;u,h) = \bar{v}^0(t_k,x) \right\}. \tag{6.41}$$

Step **3.3** (value function and trading strategy when there is arrival within the spread) This is the case when limit orders arrive within the spread at some point during the time interval $(t_{k-1}, t_k]$, meaning that $N^i(t_k) - N^i(t_{k-1}) = 1$, for $i = a$ or b. The reward from using a generic trading strategy $u \in \mathscr{U}(t_k, p)$ is

$$\bar{v}^i(t_k, x; u) = f_\alpha(t_k, q, p, u) + \bar{v}^0(t_k, x; u) \tag{6.42}$$

The optimal value from trading is

$$\bar{v}^i(t_k, x) = \max\left\{\bar{v}^i(t_k, x; u) | u \in \mathscr{U}(t_k, p)\right\}. \tag{6.43}$$

The optimal trading strategy is

$$\bar{u}^{i*}(t_k, x) = \left\{u \in \mathscr{U}(t_k, p) \mid \text{such that } \bar{v}^i(t_k, x; u) = \bar{v}^i(t_k, x)\right\}. \tag{6.44}$$

□

The backward induction algorithm can be implemented according to the pseudo codes. Due to the limited computing power of a PC, a separate on-going project is dedicated to implementing the algorithm over refined time and space grids on a cluster, and possibly testing it on real data. We shall leave detailed studies of the trading algorithm to that project.

---

beginning of the pseudo codes

```
for (x in 𝒳)
{
    for (u in 𝒰(T, p))
    {
     Step 1.1
    }
Step 1.2
print v̄(T, x) and u*(T, x) to file
}

for(k = K − 1, k − −, k ≥ 0)
{
    for (x in 𝒳)
    {
     for (u in 𝒰(t_k, p) and h in {0, 1}^2)
      {
       Step 2
       Step 3.1
       equation (6.39) in Step 3.2
      }
     equation (6.40) in Step 3.2
     equation (6.41) in Step 3.2
     print v̄^0(t_k, x), u^{0*}(t_k, x) and h*(t_k, x) to file
    }


    for (x in 𝒳)
    {
     for (u in 𝒰(t_k, p))
     {
```

```
      equation (6.42) in Step 3.3
    }
    equation (6.43) in Step 3.3
    equation (6.44) in Step 3.3
    print $\bar{v}^i(t_k, x)$ and $u^{i*}(t_k, x)$ to file
  }
}

## end of the pseudo codes
```

7 Appendix

The order book dynamics, as well as the stock inventory and cash amount from the active trading strategies are written in compact form, for singular trading strategies in subsection 3.1 and for price switching strategies in subsection 4.1. Those formulae come from summarizing the changes in all the possible situationsof a privileged trader, which are enumerated in subsections 7.1 and 7.2 and are illustrated in subsection 7.3. The case of a regular trader is a matter of setting $\beta^a \equiv \beta^b \equiv 0$.

7.1 Price changes using singular strategy

When the time t is between two price changes, the volumes evolve according to $dQ^i(t) = \sigma^i dW^i(t) - dZ^i(t)$, where (W^a, W^b) are the Brownian motions from the noise traders and (Z^a, Z^b) are the numbers of share bought and sold by the trader at the best available prices. When the prices change at the time t, the are six different situations, listed as Cases (3.a-f).

(3.a) The shares at the ask or bid price are depleted, meaning that

$$Q^i(t-) - \left(Z^i(t) - Z^i(t-)\right) \leq 0,\ i = a \text{ or } b. \tag{7.1}$$

The number of shared filled deeper in the book is $Z^i(t) - Z^i(t-) - Q^i(t-)$. The new ask price becomes $P^a(t) = P^a(t-) + \lfloor (Z^a(t) - Z^a(t-) - Q^a(t-))/\Delta^a \rfloor + 1$, or the new bid price becomes $P^b(t) = P^b(t-) - \left\lfloor \left(Z^b(t) - Z^b(t-) - Q^b(t-)\right)/\Delta^b \right\rfloor - 1$. There are

$$Q^i(t) = Q^i(t-) + \left\lfloor \left(Z^i(t) - Z^i(t-) - Q^i(t-)\right)/\Delta^i \right\rfloor \cdot \Delta^i - \left(Z^i(t) - Z^i(t-)\right) \tag{7.2}$$

shares at the new ask or bid price.

(3.b) (3.c) and **(3.d)** Limit sell orders arrive within the spread at the price $P^a(t-) - 1$, or limit buy orders arrive within the spread at the price $P^b(t-) + 1$, meaning that

$$N^i(t) - N^i(t-) = 1,\ i = a \text{ or } b. \tag{7.3}$$

Case (3.b) The trading strategy Z fills all the shares at the prices $P^a(t-) - 1$ and $P^a(t-)$ (respectively $P^b(t-) + 1$ and $P^b(t-)$), meaning that

$$Z^i(t) - Z^i(t-) \geq Q^i(t-) + \Delta^i. \tag{7.4}$$

The number of shared filled deeper in the book is $Z^i(t) - Z^i(t-) - \left(Q^i(t-) + \Delta^i\right)$. The new ask price becomes $P^a(t) = P^a(t-) + \lfloor (Z^a(t) - Z^a(t-) - Q^a(t-))/\Delta^a \rfloor$, or the new bid price becomes $P^b(t) = P^b(t-) - \left\lfloor \left(Z^b(t) - Z^b(t-) - Q^b(t-)\right)/\Delta^i \right\rfloor$. The number of shares $Q^i(t-)$ at the new ask or bid price has the expression 7.2.

Case (3.c) The trading strategy Z fills all the shares at the price $P^a(t-)-1$ (respectively $P^b(t-)+1$) and at most part of the shares at the price $P^i(t-)$, meaning that

$$\Delta^i \le Z^i(t) - Z^i(t-) < Q^i(t-) + \Delta^i. \tag{7.5}$$

The ask (bid) price does not change. There are

$$Q^i(t) = Q^i(t-) + \Delta^i - \left(Z^i(t) - Z^i(t-)\right) \tag{7.6}$$

shares left at $P^i(t) = P^i(t-)$.

Case (3.d) The trading strategy Z fills at most part of the shares at the price $P^a(t-)-1$ (respectively $P^b(t-)+1$) and a proportion $\beta^i(t) \in [0,1]$ of the $Q^i(t-)$ shares at at the price $P^i(t-)$, meaning that

$$Z^i(t) - Z^i(t-) < \Delta^i. \tag{7.7}$$

The new ask price becomes $P^a(t) = P^a(t-)-1$, or the new bid price becomes $P^b(t) = P^b(t-)+1$. There are

$$Q^i(t) = \Delta^i - \left(Z^i(t) - Z^i(t-)\right) \tag{7.8}$$

shares at the new ask or bid price.

(3.e) At the terminal time T, all the shares at the price $P^i(T-)$ are filled. This is the same as case (3.a)

(3.f) At the terminal time T, not all the shares at the price $P^i(T-)$ are filled. The prices do not change.

7.2 Price changes using switching control

Using a price switching strategy $\alpha = \{S_n, u_n^a, u_n^b\}_{n=1}^{\infty}$, the trader only trades at the sequence of times $\{S_n\}_{n=1}^{\infty}$. At every time S_n, all the six different situations, in contrast to Cases (3.a-f), are listed as Cases (4.a-f).

(4.a) The shares at the ask or bid price are depleted by the trader or the noise trader. There is no arrival within the spread, meaning that

$$N^i(S_n) - N^i(S_n-) = 0,\ i = a \text{ or } b. \tag{7.9}$$

The trader can choose to push the ask or bid price to $P^a(S_n) = P^a(S_{n-1}) + u_n^a$ or $P^b(S_n) = P^b(S_{n-1}) - u_n^b$, for some $u_n^a \in \left\{1, 2, \cdots, (\bar{p}^a - P^a(S_{n-1}))^+\right\}$ or $u_n^b \in \left\{1, 2, \cdots, \left(P^b(S_{n-1}) - \underline{p}^b\right)^+\right\}$, by filling the $Q^i(S_n-)$ shares of limit sell (buy) orders at the old ask price $P^a(S_{n-1})$ (old bid price $P^b(S_{n-1})$) and the $u_n^i - 1$ limits deeper in the book. The total number of shared filled on the ask (bid) side is

$$Q^i(S_n-) + \left(u_n^i - 1\right)\Delta^i, \tag{7.10}$$

and the total cash amount paid for buying (received from selling) those shares is

$$P^i(S_{n-1})\left(Q^i(S_n-) + \left(u_n^i - 1\right)\Delta^i\right) + \frac{1}{2}u_n^i(u_n^i - 1)\Delta^i. \tag{7.11}$$

There are Δ^a (Δ^b) shares at the new ask (bid) price.

(4.b) (4.c) and **(4.d)** Limit sell orders arrive within the spread at the price $P^a(S_{n-1})-1$, or limit buy orders arrive within the spread at the price $P^b(S_{n-1})+1$, meaning that

$$N^i(S_n) - N^i(S_n-) = 1,\ i = a \text{ or } b. \tag{7.12}$$

Case (4.b) The trader can choose to push the ask or bid price to $P^a(S_n) = P^a(S_{n-1}) + u_n^a$ or $P^b(S_n) = P^b(S_{n-1}) - u_n^b$, for some $u_n^a \in \left\{1, 2, \cdots, (\bar{p}^a - P^a(S_{n-1}))^+\right\}$ or $u_n^b \in \left\{1, 2, \cdots, \left(P^b(S_{n-1}) - \underline{p}^b\right)^+\right\}$, by

filling all the shares at the prices $P^a(S_{n-1})$ and $P^a(S_{n-1})-1$ (respectively $P^b(S_{n-1})$ and $P^b(S_{n-1})+1$) and the u_n^i-1 limits deeper in the book. The total number of shared filled on the ask (bid) side is

$$Q^i(S_n-)+\Delta^i\left(u_n^i-1\right)+\Delta^i, \tag{7.13}$$

and the total cash amount paid for buying or received from selling those shares is

$$\begin{aligned} &P^a(S_{n-1})\left(Q^a(S_n-)+\left(u_n^a-1\right)\Delta^a+\Delta^a\right)+\frac{1}{2}u_n^a(u_n^a-1)\Delta^a-\Delta^a;\\ &P^b(S_{n-1})\left(Q^b(S_n-)+\left(u_n^b-1\right)\Delta^b+\Delta^b\right)+\frac{1}{2}u_n^b(u_n^b-1)\Delta^b+\Delta^b. \end{aligned} \tag{7.14}$$

There are Δ^a (Δ^b) shares at the new ask (bid) price.

Case (4.c) The trader chooses some $u_n^i\in[0,1)$. He fills all the Δ^i shares arriving at the price $P^a(S_{n-1})-1$ (respectively $P^b(S_{n-1})+1$) and $u_n^iQ^i(S_n-)$ shares at the price $P^i(S_{n-1})$. The total number of shared filled on the ask (bid) side is

$$\Delta^i+u_n^iQ^i(S_n-), \tag{7.15}$$

and the total cash amount paid for buying or received from selling those shares is

$$P^a(S_{n-1})\left(\Delta^a+u_n^aQ^a(S_n-)\right)-\Delta^a \text{ or } P^b(S_{n-1})\left(\Delta^b+u_n^bQ^b(S_n-)\right)+\Delta^b. \tag{7.16}$$

The ask (bid) price does not change. There are

$$Q^i(S_n)=Q^i(S_n-)-u_n^iQ^i(S_n-) \tag{7.17}$$

shares left at the price $P^i(S_n)=P^i(S_{n-1})$.

Case (4.d) The trader chooses some $u_n^i\in[-1,0)$. He only fills $\{u_n^i\}Q^i(S_n-)$ shares out of those $Q^i(S_n-)$ shares arriving at the price $P^i(S_{n-1})$, where $\{u_n^i\}=u_n^i-[u_n^i]=u_n^i+1\in[0,1)$. The total cash amount paid for buying or received from selling those shares is

$$P^i(S_{n-1})\left\{u_n^i\right\}Q^i(S_n-). \tag{7.18}$$

The ask (bid) price moves downward (upward) by one tick. There are Δ^i shares left at the new ask (bid) price $P^a(S_n)=P^a(S_{n-1})-1=P^a(S_{n-1})+[u_n^a]$ (respectively $P^b(S_n)=P^b(S_{n-1})+1=P^b(S_{n-1})-[u_n^b]$).

(4.e) At the terminal time $S_n=T$, the trader chooses some

$$u_n^a\in[1,(\bar{p}^a-P^a(T))^+] \text{ or } u_n^b\in[1,\left(P^b(S_{n-1})-\underline{p}^b\right)^+]. \tag{7.19}$$

He fills all the sell (buy) at the price $P^i(T-)$, $[u_n^i]$ limits deeper in the book and $\{u_n^i\}\cdot\Delta^i$ shares at the new ask (bid) price $P^a(T)=P^a(T-)+[u_n^i]$ (respectively $P^b(T)=P^b(T-)-[u_n^i]$). There are $Q^i(T)=\Delta^i-\{u_n^i\}\cdot\Delta^i$ shares left at $P^i(T)$. The total number of shared filled on the ask (bid) side is

$$Q^i(S_n-)+\left([u_n^i]-1\right)\Delta^i+\{u_n^i\}\cdot\Delta^i=Q^i(S_n-)+\left(u_n^i-1\right)\Delta^i, \tag{7.20}$$

and the total cash amount paid for buying (received from selling) those shares is

$$P^i(S_{n-1})\left(Q^i(S_n-)+\left(u_n^i-1\right)\Delta^i\right)+\left(\frac{1}{2}[u_n^i]([u_n^i]-1)+[u_n^i]\cdot\{u_n^i\}\right)\Delta^i. \tag{7.21}$$

(4.f) At the terminal time $S_n=T$, the trader chooses some $u_n^i\in[0,1)$. He fills $\{u_n^i\}\cdot Q^i(T-)$ shares of limit sell (buy) orders at the price $P^i(T-)$, and pays (receives) the cash amount $P^i(T-)\{u_n^i\}Q^i(T-)$. The ask (bid) price does not change. There are $Q^i(T)=Q^i(T-)-\{u_n^i\}\cdot Q^i(T-)$ shares of limit sell (buy orders left at the price $P^i(T)=P^i(T-)$.

7.3 Pictures

The Figures 7.3 to 8.6 illustrate all the possible situations enumerated in subsections 7.1 and 7.2, supposing that the bid price changes at the time t. The left plot in each figure is the case using a singular trading strategy $Z \in \mathscr{Z}$, and the right plot is the case using the price switching trading strategy $Z_\alpha \in \mathscr{S}^p$.

In each plot, all the horizontal colored lines represent the volumes of limit buy orders in the order book at the time $t-$; the red lines represent the volumes of limit buy orders left in the book at the time t; the blue and green lines represent the limit buy orders filled by the trader at the time t, which equals $Z^b(t) - Z^b(t-) + \beta^b(t)Q^b(t-)$ in a left plot and $Z^b_\alpha(t) - Z^b_\alpha(t-) + \beta^b_\alpha(t)Q^b(t-)$ in a right plot.

The two plots in each pictures are drawn in a way that the right one uses the price switching strategy $Z_{\alpha'}$ from Lemma 5.2 to replicate the singular trading strategy Z in the left one. In the left plots in Figures 8.1, 8.2 and 8.4, the strategy Z fills a fractional limit at the time t, while in the right plot, this fractional limit is postponed by the strategy $Z_{\alpha'}$ until the next time of price change. The green line in a right plot represents the number of limit buy orders in the fractional limit postponed from the previous time of price change plus the number of limit buy orders that the strategy Z has filled between the previous and current times of price change.

Figure 7.1: Case (3.a) on the left and Case (4.a) on the right

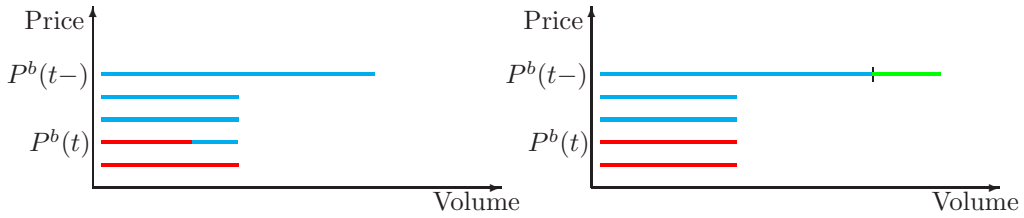


Figure 7.2: Case (3.b) on the left and Case (4.b) on the right

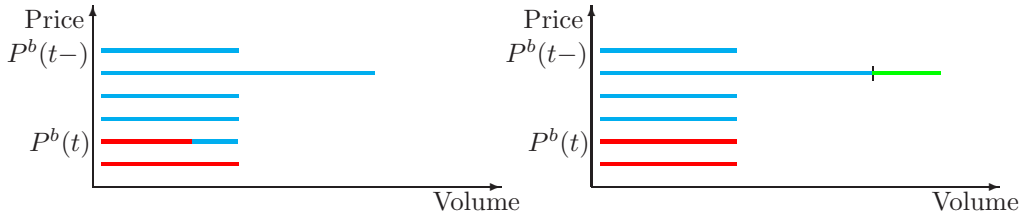


Figure 7.3: Case (3.c) on the left and Case (4.c) on the right

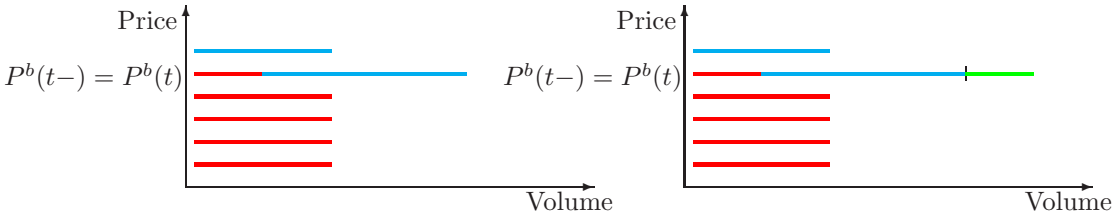


Figure 7.4: Case (3.d) on the left and Case (4.d) on the right

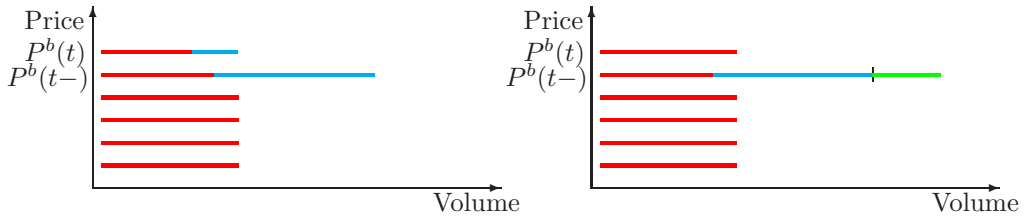


Figure 7.5: Case (3.e) on the left and Case (4.e) on the right

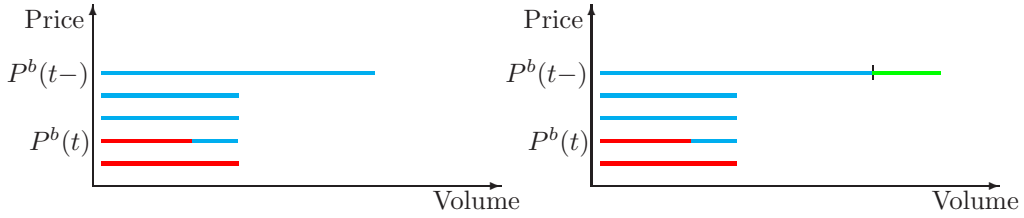
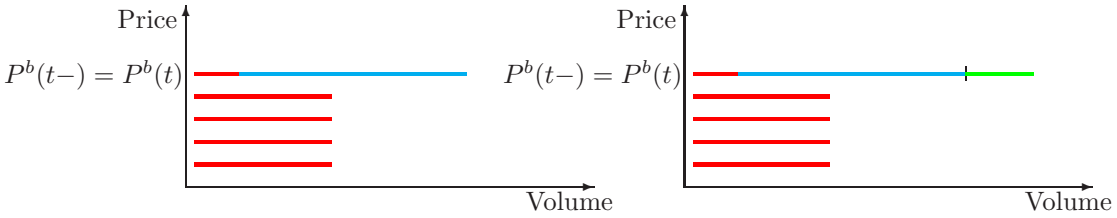


Figure 7.6: Case (3.f) on the left and Case (4.f) on the right



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