Counterparty Risk Modeling: Beyond immersion

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Abstract

We use marked stopping times to model the defaults of two parties involved in OTC derivative transactions. The role of the mark is to convey some information about the defaults, in order to account for various possible counterparty risk and wrong-way risk scenarios and features. In a bilateral counterparty risk setup also accounting for the funding costs incurred, the pricing equations for counterparty risk and funding are implicit and nonlinear. Moreover they are posed over the random time interval ended by the first default time of a party. To cope with these technicalities we extend to marked default times the standard reduced-form credit risk modeling approach, obtaining a pre-default Markovian pricing BSDE/PDE with a fixed time horizon (the maturity of the underlying portfolio). The case of counterparty risk on credit derivatives, a major wrong-way and gap risks concern, poses specific dependence modeling and dimensionality challenges. To address these, we resort to dynamic copula models of portfolio credit risk and we develop the above-mentioned reduced-form modeling approach in these setups.

1 Introduction

Counterparty risk is the risk of default of a party in an OTC derivative transaction. This is a topical issue since the 2007-09 credit crisis, in its genuine form as in the derived one of the volatility of the related CVA, i.e. credit valuation adjustment (see e.g. the papers by (Prisco and Rosen 2005), Brigo and Pallavicini (2008), Brigo and Capponi (2010), Crépey (2011,2012b), Pallavicini, Perini, and Brigo (2011, 2012), Burgard and Kjaer (2011a,2011b), Hull and White (2013a, 2013b, 2013c)); see also the books by Gregory (2012), Cesari, Aquilina, Charpillon, Filipovic, Lee, and Manda (2010), Brigo, Morini, and Pallavicini (2013) and Crépey, Bielecki, and Brigo (2013)). The basic CVA mitigation tool is collateralization through margins which must be posted by the two parties. However, accounting for various frictions and delays, a collateralized position still has CVA through gap risk,

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which is the risk of a residual gap between the collateral and the debt of the defaulted counterparty. As the size of the residue is typically quite volatile, one can even claim that collateralization leverages rather than eliminates counterparty risk. Moreover, gap risk is magnified in the presence of wrong-way risk, i.e. a positive dependence between the size of the counterparty risk exposure and the default riskiness of the counterparty. This is a special case of concern regarding counterparty risk on credit derivatives, given the strong dependence effects (frailty and contagion) between the credit risks of the two parties and the ones of names underlying credit derivative positions between them.

In this paper we use marked stopping times for modeling the defaults of the two parties. The role of the mark is to convey some information about the default, in order to account for various possible wrong-way risk scenarios and features. In a bilateral counterparty risk perspective also accounting for the nonlinear funding costs involved, the pricing equations for the corresponding valuation adjustments (TVA equations, where TVA stands for total valuation adjustment) are implicit and nonlinear (see Crépey (2011,2012b), Pallavicini, Perini, and Brigo (2011, 2012)). Moreover, they are posed over the random time interval ended by the first default time of a party. To cope with these technicalities, in this paper, we extend to marked default times the credit risk reduced-form modeling approach of Crépey (2011,2012b), obtaining a pre-default TVA pricing equation with a fixed time horizon (the maturity of the underlying portfolio). The case of counterparty risk on credit derivatives, a central wrong-way and gap risks issue, poses specific dependence modeling and dimensionality challenges. To address these we resort to suitably dynamic copula models of portfolio credit risk, in which, using the preliminary results of Crépey (2013b), we illustrate the above-mentioned reduced-form pre-default modeling approach.

All the cash-flows that appear in the paper are assumed to be integrable under the prevailing pricing measure. "Martingale" is meant everywhere as local martingale, but true martingality is assumed whenever necessary.

2 Marked Default Time Reduced-Form Modeling

We consider a netted portfolio of OTC derivatives with maturity T between two defaultable counterparties. This portfolio is generically referred to as the "contract between the bank and its counterparty". To fix the framework we adopt the perspective of the bank. After having bought the contract, with promised dividends dD_t, from the counterparty at time 0, the bank sets-up a hedging, margining (i.e. collateralization) and funding portfolio. We call an "external funder" of the bank (or funder for short) a generic third-party, possibly composed in practice of several entities or devices, insuring funding of the position of the bank. The funder, assumed default-free for simplicity, plays the role of lender/borrower of last resort after exhaustion of the internal sources of funding provided to the bank by repo bases on its hedge and through the collateral (which, as opposed to margins in future contracts, is typically remunerated). By prices we mean the cost of hedging, margining and funding. For reasons explained Sect. 1 in Crépey (2012b), the price of the contract is computed by the bank as the difference between, on one hand, the "clean" price provided by the relevant business unit ("mark-to-market" ignoring counterparty risk and risky funding costs) and, on the other hand, an adjustment computed by a central CVA (or TVA) unit.

Let $(\Omega, \mathcal{G}_T, \mathbb{G}, \mathbb{Q})$ represent a prevailing pricing stochastic basis. The meaning of a risk-neutral pricing measure in this context, with different funding rates in particular (see Crépey (2012a)), will be specified by suitable martingale conditions introduced below in the form

of pricing BSDEs, i.e. backward stochastic differential equations (see El Karoui, Peng, and Quenez (1997) for a seminal reference in finance and Crépey (2013a) for a recent reference in book form). In particular, in our framework, a pricing measure must be such that the cumulative gains made on the hedging assets follow martingales (see Crépey (2012a)). We denote by r_t an OIS rate (overnight indexed swap rate, the best market proxy for a risk-free rate) and by $\beta_t = e^{-\int_0^t r_s ds}$ the corresponding discount factor. Letting $\mathbb{E}_t$ stand for the conditional expectation given $\mathcal{G}_t$, the clean value process P_t of the contract is defined, for $t \in [0, \bar{\tau}]$, by:

$$\beta_t P_t = \mathbb{E}_t \left(\int_t^T \beta_s dD_s \right), \tag{2.1}$$

by the tower law.

However, the two parties are defaultable, with respective default times denoted by τ_b and τ_c. Writing $\tau = \tau_b \wedge \tau_c$, so that the effective time horizon of the problem is $\bar{\tau} = \tau \wedge T$ (as there are no cash-flows beyond this), this results in an effective dividend stream $J_t dD_t$, where $J_t = \mathbb{1}_{t<\tau}$. To mitigate counterparty risk, the basic tool is a credit support annex (CSA), i.e. a legal agreement between the parties which specifies a valuation scheme Q_t by the liquidator for the contract in case of a default at time t (specifying, in particular, the netting rules which apply in the portfolio) and a collateralization scheme Γ_t (similar as with futures contracts, except that a CSA margin account stays the ownership of the collateral's poster, which is therefore remunerated by the receiver). At time τ (if $< T$), a terminal cash-flow $\mathfrak{R}$ paid to the bank closes out its position, where $\mathfrak{R} = \mathfrak{R}(\pi)$ can also depend on the wealth π of the bank right before time τ (see Crépey (2012a)). The $\mathcal{G}_\tau$-measurable exposure at default for the bank is defined by

$$\xi(\pi) = P_\tau + \Delta_\tau - \mathfrak{R}(\pi), \tag{2.2}$$

where $\Delta = D - D_-$ is the jump process of D.

Moreover, the bank needs to fund its position (contract, hedge, and collateral). The OIS rate r_t is used as a reference for all the other funding rates, which are defined in terms of the corresponding bases to r_t. We denote by $g_t = g_t(\pi)$ a random function such that $(r_t \pi + g_t(\pi))dt$ represents the bank's funding cost over $(t, t + dt)$, depending on the bank's wealth π.

Remark 2.1 For simplicity in this paper, we only consider the most common situation where the hedge is self-funded, as entirely made of positions either swapped or traded via repo markets. Otherwise ξ and g_t would additionally depend on a hedge argument ς (see Crépey (2012a,2012b)).

Concrete specifications for $\mathfrak{R}$, hence ξ, and for g_t will be given in Sect. 3. Given the data ξ and g_t, the TVA process Θ can be (implicitly) defined on $[0, \bar{\tau}]$ as the solution to the following BSDE, written in integral form over the random time interval $[0, \bar{\tau}]$:

$$\beta_t \Theta_t = \mathbb{E}_t \Big[\beta_{\bar{\tau}} \mathbb{1}_{\tau<T} \xi (P_{\tau-} - \Theta_{\tau-}) + \int_t^{\bar{\tau}} \beta_s g_s (P_s - \Theta_s) ds \Big], \ \ t \in [0, \bar{\tau}]. \tag{2.3}$$

The reader is referred to Proposition 2.1 in (Crépey 2012b) for more details and for the derivation of the TVA process (2.3) as the output of a computation based on a TVA primarily defined over $[0, \bar{\tau}]$ as

$$\Theta_t = P_t - \Pi_t + \mathbb{1}_{t=\tau} \Delta_\tau, \tag{2.4}$$

where Π represents the overall price process of the contract (cost of the hedge, inclusive of counterparty risk and funding costs considerations). Instead, in this paper, we take (2.3) as a definition, for simplicity of presentation, and then we define Π via P and Θ through (2.4). Of course this approach implicitly assumes the existence (at the very least) of a solution to (2.3), which is quite a nonstandard BSDE. The object of this paper is precisely to propose a suitable reduced-form approach for the "full" TVA BSDE (2.3), beyond the standard immersion case of Crépey (2012b).

Remark 2.2 In the marked setup of this paper, there can be a non-zero promised dividend Δ_τ at τ, which explains the presence of the additional Δ_τ-term in (2.2) with respect to Crépey (2012b). See Bielecki and Crépey (2011), Crépey, Bielecki, and Brigo (2013) for the details.

2.1 Pre-default Setup

We assume that τ is endowed with a mark e in a finite set E, i.e. $\tau = \min_{e\in E} \tau_e$, where τ_e has a pre-default intensity γ_t^e. We also postulate that for every mark e, there exists a predictable random function $\widetilde{\xi}_t^e(\pi)$ such that

$$\xi(\pi) = \widetilde{\xi}_\tau^e(\pi) \tag{2.5}$$

on the event $\{\tau = \tau_e\}$. For every semimartingale Y, we have the compensated martingale defined, for $t \in [0, \bar{\tau}]$, as:

$$\xi(Y_{t-})dJ_t + \gamma_t \cdot \widetilde{\xi}_t(Y_t)dt, \tag{2.6}$$

where we denote $\gamma_t \cdot h_t = \sum_E \gamma_t^e h_t^e$, for every random function h_t^e. For every real $t \in [0, \bar{\tau}]$ and ϑ, we let

$$f_t(P_t - \vartheta) = g_t(P_t - \vartheta) + \gamma_t \cdot \widetilde{\xi}_t(P_t - \vartheta) - \widetilde{r}_t\vartheta, \tag{2.7}$$

where $\widetilde{r}_t = r_t + \widetilde{\gamma}_t$, in which $\widetilde{\gamma}_t = \sum_{e\in E} \gamma_t^e = \gamma_t \cdot \mathbb{1}_E$ is a pre-default intensity of τ. Let there be given, assumed to exist (a set of assumptions that we denote collectively by **(A)**):

(A.1) a filtration $\mathbb{F}$ over $[0, T]$ such that $\mathcal{F}_t \subseteq \mathcal{G}_t$ on $[0, \bar{\tau}]$ and $\mathbb{F}$-semimartingales stopped at τ are $\mathbb{G}$-semimartingales,

(A.2) a probability measure $\mathbb{P}$ equivalent to $\mathbb{Q}$ on $\mathcal{F}_{\bar{\tau}}$ such that an $(\mathbb{F}, \mathbb{P})$-martingale stopped at $\tau-$ is a $(\mathbb{G}, \mathbb{Q})$-martingale,

(A.3) an $\mathbb{F}$-progressive random function $\widetilde{f}_t(\vartheta)$ such that $\widetilde{f}_t(\vartheta)dt = f_t(P_t - \vartheta)dt$ on $[0, \bar{\tau}]$.

Remark 2.3 We refer the reader to Sect. 4 and 5 for two concrete examples, using an equivalent measure $\mathbb{P} \neq \mathbb{Q}$ in the first one and $\mathbb{P} = \mathbb{Q}$ (for a well chosen subfiltration $\mathbb{F}$) in the second one. The assumption (A.1) relates to the $(\mathcal{H}')$-hypothesis between $\mathbb{F}$ and $\mathbb{G}$ (see Bielecki, Jeanblanc, and Rutkowski (2009)). By a process stopped at $\tau-$ in (A.2), we mean the process deprived from its jump at τ. In particular, the assumption (A.2) holds if an $(\mathbb{F}, \mathbb{P})$-martingale does not jump at τ and an $(\mathbb{F}, \mathbb{P})$-martingale stopped at τ is a $(\mathbb{G}, \mathbb{Q})$-martingale. In the "immersion " case with $\mathbb{P} = \mathbb{Q}$, these properties are related to the notion of an $\mathbb{F}$-pseudo-stopping time τ (see Nikeghbali and Yor (2005)). However, even in

this immersion case, via the mark e the present setup is richer than a standard reduced-form intensity model of credit risk, where the full model filtration $\mathbb{G}$ is simply given as the reference filtration $\mathbb{F}$ progressively enlarged by τ. In particular, the assumption (A) as such does not a exclude jump at τ of an $\mathbb{F}$-adapted càdlàg process, whereas this can not happen in a standard reduced-form setup (see Subsection 4.3.2 in Crépey (2013a)). See also Section 3 of Kchia, Larsson, and Protter (2013) for a theoretical study of progressive filtration expansions with multiple random times.

Let $\widetilde{\mathbb{E}}_t$ stand for the conditional expectation given $\mathcal{F}_t$ under $\mathbb{P}$. Note that the equation (2.8) in the next proposition defines a BSDE for the process $\widetilde{\Theta}_t$ (in integral form, the differential formulation appearing in (2.12)). In the sequel this equation is referred to as the pre-default TVA BSDE.

Theorem 2.1 (Reduced-Form TVA Modeling) *Assume that an $(\mathbb{F},\mathbb{P})$-semimartingale $\widetilde{\Theta}$ satisfies the following equation over $[0,T]$:*

$$\widetilde{\Theta}_t = \widetilde{\mathbb{E}}_t \int_t^T \widetilde{f}_s(\widetilde{\Theta}_s)ds, \;\; t\in[0,T]. \tag{2.8}$$

Let $\Theta = \widetilde{\Theta}$ on $[0,\bar{\tau})$ and $\Theta_{\bar{\tau}} = \mathbb{1}_{\tau<T}\xi(P_{\tau-} - \widetilde{\Theta}_{\tau-})$. Then the process Θ satisfies the full TVA equation (2.3) on $[0,\bar{\tau}]$. Moreover, the $(\mathbb{G},\mathbb{Q})$-martingale component

$$d\mu_t = d\Theta_t + (g_t(P_t-\Theta_t) - r_t\Theta_t)dt$$

of Θ satisfies, for $t\in[0,\bar{\tau}]$:

$$d\mu_t = d\widetilde{\mu}_{t\wedge\tau-} - \Big((\xi - \widetilde{\Theta}_{t-})dJ_t + (\gamma_t\cdot\widetilde{\xi}^\star_t - \widetilde{\gamma}_t\Theta_t)dt\Big), \tag{2.9}$$

where

$$d\widetilde{\mu}_t = d\widetilde{\Theta}_t + \widetilde{f}_t(\widetilde{\Theta}_t)dt \tag{2.10}$$

is the $(\mathbb{F},\mathbb{P})$-martingale component of $\widetilde{\Theta}$ and where ξ and $\widetilde{\xi}^\star_t = \widetilde{\xi}^{\star,e}_t$ are shorthands for $\xi(P_{\tau-} - \widetilde{\Theta}_{\tau-})$ and $\widetilde{\xi}^e_t(P_t - \widetilde{\Theta}_t)$.

Proof. By definition of Θ in the proposition, where Θ thus defined is a $(\mathbb{G},\mathbb{Q})$-semimartingale in virtue of our assumption (A.1), we have, for $t\in[0,\bar{\tau}]$:

$$d(\beta_t\Theta_t) = d(J_t\beta_t\widetilde{\Theta}_t) + \beta_t\xi\boldsymbol{\delta}_\tau(dt) = d\big(\beta_{t\wedge\tau-}\widetilde{\Theta}_{t\wedge\tau-}\big) + \beta_t\widetilde{\Theta}_{t-}\,dJ_t - \beta_t\xi dJ_t, \tag{2.11}$$

where by (2.10):

$$-d\widetilde{\Theta}_t = \widetilde{f}_t(\widetilde{\Theta}_t)dt - d\widetilde{\mu}_t. \tag{2.12}$$

Therefore, for $t\in[0,\bar{\tau}]$:

$$\begin{aligned}-\beta_t^{-1}d(\beta_t\Theta_t) &= \big(\widetilde{f}_t(\widetilde{\Theta}_t) + r_t\widetilde{\Theta}_t\big)dt - d\widetilde{\mu}_{t\wedge\tau-} + (\xi-\widetilde{\Theta}_{t-})dJ_t\\ &= g_t(P_t-\widetilde{\Theta}_t)dt - d\widetilde{\mu}_{t\wedge\tau-} + \Big((\xi-\widetilde{\Theta}_{t-})dJ_t + (\gamma_t\cdot\widetilde{\xi}^\star_t - \widetilde{\gamma}_t\Theta_t)dt\Big),\end{aligned}$$

by the assumption (A.3) on $\widetilde{f}_t$. Moreover, by the assumption (A.2), $\widetilde{\mu}_{t\wedge\tau-}$ is a $(\mathbb{G},\mathbb{Q})$-martingale, whereas the process

$$(\xi-\widetilde{\Theta}_{t-})dJ_t + (\gamma_t\cdot\widetilde{\xi}^\star_t - \widetilde{\gamma}_t\widetilde{\Theta}_t)dt$$

is a $(\mathbb{G},\mathbb{Q})$-compensated martingale of the form (2.6). This yields the decomposition (2.9) of μ, which also implies (2.3), for $t\in[0,\bar\tau]$. □

Henceforth, we model a TVA process Θ as in the above proposition in terms of a solution $\widetilde\Theta$, assumed to exist (for which we refer the reader to Crépey (2012b,2013a) and the references there), to (2.8).

2.2 Markovian Case

Assume that the pre-default TVA BSDE (2.8) is Markovian, i.e. there exists an $(\mathbb{F},\mathbb{P})$-Markov factor process X such that, over $[0,T]$,

$$\widetilde f_t(\vartheta)=\widetilde f(t,X_t,\vartheta) \tag{2.13}$$

for some deterministic function $\widetilde f(t,x,\vartheta)$. Let there be given, for some integer q, an $\mathbb{R}^q$-valued $(\mathbb{F},\mathbb{P})$-Brownian motion W and an $(\mathbb{F},\mathbb{P})$-compensated jump counting measure m on $[0,T]\times\mathbb{R}^q$ (see for instance Crépey (2013a) and the references there). Given suitable coefficients $b(t,x)$, $\sigma(t,x)$ and $j(t,x,y)$, we assume that X_t satisfies the following Markovian (forward) SDE in $\mathbb{R}^q$: $X_0=x$ given as an observable or calibratable constant and, for $t\in[0,T]$:

$$dX_t=b(t,X_t)\,dt+\sigma(t,X_t)\,dW_t+j(t,X_{t-})\cdot dm_t, \tag{2.14}$$

with an underlying jump intensity measure of the form $c(t,X_t,dx)$ (possibly random via X_t). Note that for every vector-valued function $f=f(t,x,y)$ on $[0,T]\times\mathbb{R}^q\times\mathbb{R}^q$ (like $f=j$ in (2.14)), we write:

$$f(t,X_{t-})\cdot dm_t=\int_{\mathbb{R}^q}f(t,X_{t-},x)N(dt,dx),\ \ \big(f\cdot c\big)(t,x)\ \ =\int_{\mathbb{R}^q}f(t,x,y)c(t,x,dy).$$

The integrals are performed entry by entry of f, so that one ends up with vectors of the same dimension as f.

Under this specification of a factor process X_t, the pre-default reduced-form TVA representation (2.8) boils down to the following Markovian BSDE in $\widetilde\Theta_t=\widetilde\Theta(t,X_t)$:

$$\begin{cases}\widetilde\Theta(T,X_T)=0 \text{ and, for } t\in[0,T]:\\ -d\widetilde\Theta(t,X_t)=\widetilde f\left(t,X_t,\widetilde\Theta(t,X_t)\right)dt-(\nabla\widetilde\Theta\sigma)(t,X_t)dW_t-\delta\widetilde\Theta(t,X_{t-})\cdot dm_t,\end{cases} \tag{2.15}$$

where ∇u is the row-gradient with respect to x of a function $u=u(t,x)$ and where $\delta u(t,x,y)=u(t,x+\delta(t,x,y))-u(t,x)$. In particular, we have in (2.9), for $t\in[0,\bar\tau]$, that:

$$d\widetilde\mu_{t\wedge\tau-}=(\nabla\widetilde\Theta\sigma)(t,X_t)dW_t+\delta\widetilde\Theta(t,X_{t-})\cdot dm_{t\wedge\tau-}. \tag{2.16}$$

The semilinear PDE associated with (2.15) is written as:

$$\begin{cases}\widetilde\Theta(T,x)=0\,,\ x\in\mathbb{R}^q\\ \left(\partial_t+\mathcal{A}\right)\widetilde\Theta(t,x)+\widetilde f(t,x,\widetilde\Theta(t,x))=0 \text{ on } [0,T)\times\mathbb{R}^q,\end{cases} \tag{2.17}$$

where $\mathcal{A}$ is the $(\mathbb{F},\mathbb{P})$-generator of the Markov process X. Any solution scheme for (2.15) and/or (2.17) can be used for computing the TVA $\widetilde\Theta$.

3 Wrong Way Risk

The standard reduced-form counterparty risk modeling approach of Crépey (2012b) hinges on weak dependence assumptions between the contract and the default times of the two parties:

1. Immersion of the reference filtration $\mathbb{F}$ into the full model filtration $\mathbb{G}$,
2. Continuity of some of the data at the default time of a party.

This is suitable for routine applications such as counterparty risk on interest-rate derivatives, but it is insufficient to deal with the strong wrong-way risk effects which may occur in other applications like counterparty risk on credit derivatives, or through a collateral posted in another currency strongly dependent on the default of a party. In view of a proper wrong-way risk modeling we extend in this section the reduced-form counterparty risk modeling approach of Crépey (2012b) to the marked default time setup of this paper. For this purpose we first need to specify the close-out cash-flow R (paid by the bank at time τ if $< T$) in ξ (see (2.2)) and the funding coefficient g in f.

Let semimartingales Q and Γ represent the CSA value process of the contract and the value process of a CSA collateralization scheme, respectively (see the explanations following 2.1). The bank's close-out cash-flow $\mathfrak{R} = \mathfrak{R}(\pi)$ is in fact twofold, decomposing into a close-out cash-flow $\mathfrak{R}^c$ from the bank to the counterparty, minus, in case $\tau = \tau_b$ (default of the bank), a cash-flow from the funder to the bank, $\mathfrak{R}^f(\pi)$, depending on the wealth π of the bank right before time τ. These two cash-flows are respectively derived from the debts of the bank to its counterparty and to its funder at time τ, modeled as

$$\chi = Q_\tau + \Delta_\tau - \Gamma_\tau, \;\; \mathfrak{X}(\pi) = (\pi - \Gamma_{\tau-}). \tag{3.1}$$

The close-out cash-flow is finally given as $\mathfrak{R}(\pi) = \mathfrak{R}^c + \mathbb{1}_{\tau=\tau_b}\mathfrak{R}^f(\pi)$, where

$$\begin{aligned} \mathfrak{R}^c &= \Gamma_\tau + \mathbb{1}_{\tau=\tau_c}\big(R_c\chi^+ - \chi^-\big) - \mathbb{1}_{\tau=\tau_b}\big(R_b\chi^- - \chi^+\big) - \mathbb{1}_{\tau_b=\tau_c}\chi \\ \mathfrak{R}^f(\pi) &= (1-\bar{R}_b)\mathfrak{X}^+(\pi). \end{aligned} \tag{3.2}$$

Here R_b and R_c stand for the recovery rates between the two parties and $\bar{R}_b$ stands for the recovery rate of the bank to its funder. The ensuing exposure at default results from (2.2) as

$$\begin{aligned} \xi(\pi) &= P_\tau + \Delta_\tau - \mathfrak{R}(\pi) \\ &= P_\tau - Q_\tau \qquad\qquad + \mathbb{1}_{\tau=\tau_c}(1-R_c)\chi^+ - \mathbb{1}_{\tau=\tau_b}\big((1-R_b)\chi^- + (1-\bar{R}_b)\mathfrak{X}^+(\pi)\big), \end{aligned} \tag{3.3}$$

by an elementary computation.

We now consider the funding cashflows. Given bases b_t and $\bar{b}_t$ for the remuneration of the collateral posted and received by the bank, as well as bases λ_t and $\bar{\lambda}_t$ for the external lending and borrowing funding positions of the bank , the funding coefficient of the bank is defined by

$$g_t(\pi) = (b_t\Gamma_t^+ - \bar{b}_t\Gamma_t^-) + \lambda_t\,(\pi - \Gamma_t)^+ - \bar{\lambda}_t\,(\pi - \Gamma_t)^-\,. \tag{3.4}$$

(see Sect. 2 and Crépey, Gerboud, Grbac, and Ngor (2013)).

To proceed, we assume that τ_b and τ_c can be represented in terms of underlying $\mathbb{G}$-stopping times $\tau_e, e \in E$ as

$$\tau_b = \min_{e\in E^b}\tau_e, \;\; \tau_c = \min_{e\in E^c}\tau_e$$

with $E^b \cup E^c = E$ (not necessarily a disjoint union, as will be illustrated in Sect. 5). Moreover, we assume that

$$\begin{aligned}&\text{For every process } U = P, \Delta, Q, \Gamma \text{ and for every } e \in E,\\ &\text{there exists a predictable process } \widetilde{U}^e_t \text{ such that } U_\tau = \widetilde{U}^e_\tau \text{ on the event } \{\tau = \tau_e\}.\end{aligned} \tag{3.5}$$

Therefore, in view of (3.1) through (3.3), the condition (2.5) holds with

$$\begin{aligned}\widetilde{\xi}^e_t(\pi) = \widetilde{P}^e_t - \widetilde{Q}^e_t + \mathbb{1}_{e\in E^c}(1-R_c)\widetilde{\chi}^{e,+}_t \\ - \mathbb{1}_{e\in E^b}\big((1-R_b)\widetilde{\chi}^{e,-}_t + (1-\bar{R}_b)(\pi - \Gamma_{t-})^+\big),\end{aligned} \tag{3.6}$$

where $\widetilde{\chi}^{e,\pm}_t$ denote the positive and negative parts of

$$\chi^e_t = \widetilde{Q}^e_t + \widetilde{\Delta}^e_t - \widetilde{\Gamma}^e_t. \tag{3.7}$$

The resulting coefficient f in (2.7) is written as

$$\begin{aligned}f_t(P_t - \vartheta) + r_t\vartheta = \quad & (1-R_c)\gamma_t \cdot \Big(\mathbb{1}_{E^c}\big(\widetilde{Q}_t + \widetilde{\Delta}_t - \widetilde{\Gamma}_t\big)^+\Big)\\ & - (1-R_b)\gamma_t \cdot \Big(\mathbb{1}_{E^b}\big(\widetilde{Q}_t + \widetilde{\Delta}_t - \widetilde{\Gamma}_t\big)^-\Big)\\ & + \bar{b}_t\Gamma^+_t - b_t\Gamma^-_t \; + \; \widetilde{\lambda}_t\,(P_t - \vartheta - \Gamma_t)^+ - \lambda_t\,(P_t - \vartheta - \Gamma_t)^-\\ & + \gamma_t \cdot \Big(\widetilde{P}_t - \mathbb{1}_E\vartheta - \widetilde{Q}_t\Big),\end{aligned} \tag{3.8}$$

where $\widetilde{\lambda}_t = \bar{\lambda}_t - (1-\bar{R}_b)\gamma_t \cdot \mathbb{1}_{E^b}$. The coefficient $\widetilde{\lambda}_t$ of $(P_t - \vartheta - \Gamma_t)^+$ can be interpreted as an external borrowing basis for the bank, net of the bank's credit spread. The basis $\widetilde{\lambda}_t$ corresponds to the liquidity component of $\bar{\lambda}$. From the perspective of the bank, the four terms in the decomposition (3.8) of the TVA coefficient $f_t(P_t - \vartheta)$ (up to the $r_t\vartheta$-discount term at the reference OIS rate r_t) can respectively be interpreted as a costly credit valuation adjustment (CVA), a beneficial debt valuation adjustment (DVA), a funding liquidity valuation adjustment (LVA) and a replacement cost/benefit (RC). The positive (resp. negative) TVA terms can be considered as "deal adverse" (resp. "deal friendly") as they increase the TVA and therefore decrease the price (cost of the hedge) for the bank (with, depending on the sign of Π, a "less positive" Π interpreted as a lower buying price by the bank or a "more negative" Π interpreted as a higher selling price by the bank). See Crépey, Gerboud, Grbac, and Ngor (2013) or Crépey, Grbac, Ngor, and Skovmand (2013) for numerical illustrations on interest rate derivatives.

Remark 3.1 The materiality of the DVA (through the coefficient proportianal to $(1-R_b)$ in (3.8)) or of a funding benefit at own default (through the $(1-\bar{R}_b)\gamma_t \cdot \mathbb{1}_{E^b}$ coefficient) is clearly subject to caution, unless a corresponding hedge allows the bank to monetize this benefit before its default. Otherwise, the bank should better set its recovery rates R_b and $\bar{R}_b$ equal to one, in order to avoid reckoning such a "fake benefit".

3.1 Gap Risk

The basic CVA mitigation tool is collateralization, through margins, with cumulative value Γ_t, which must be posted by the two parties according to the rules prescribed by the CSA. The extreme case of continuous collateralization would correspond to $\Gamma = Q$. However,

accounting for various frictions and delays regarding formation and delivery of the collateral, continuous collateralization is not possible and even the most safely collateralized position still has CVA through gap risk, which is the risk of a residual gap between the collateral and the debt of a defaulting party at time τ. As illustrated by AIG's bailout on 16 September 2008, which was largely triggered by AIG's inability to face increasing margin calls on sell-protection CDS positions on the distressed Lehman, it is very important to use an accurate model of the collateral. In this subsection we introduce realistic collateralization features.

3.1.1 Collateral Slippage, Thresholds and Minimal Transfer Amounts

In practice, margin calls are executed according to a discrete time schedule $(t_l)_{l\geq 0}$ (with $t_0 = 0$, starting from $\Gamma_{0-} = 0$). The collateral process also depends on the following CSA data:

- the thresholds ("free credit lines") of the bank and the counterparty : $\eta^b \leq 0$ and $\eta^c \geq 0$
- the minimum transfer amounts (MTA) of the bank and the counterparty: $\epsilon^b \leq 0$ and $\epsilon^c \geq 0$.

In a realistic collateralization scheme (conform to ISDA requirements in particular), the collateral Γ_t "tracks" the thresholded exposure,

$$\widehat{Q}_t \quad = \quad (Q_t - \eta^b)^+ - (Q_t - \eta^c)^-, \tag{3.9}$$

through a càdlàg and piecewise-constant process reset at every $t_l < \tau$, unless the corresponding margin adjustment would be less than the minimum transfer amount of the concerned party. More precisely, Γ_t is reset at every $t_l < \tau$ following

$$\Gamma_{t_l} - \Gamma_{t_l-} = \mathbb{1}_{\widehat{Q}_{t_l-} - \Gamma_{t_l-} \notin [\epsilon^c, \epsilon^b]} (\widehat{Q}_{t_l-} - \Gamma_{t_l-}); \tag{3.10}$$

In particular,

$$\Gamma_{t_l} \in [Q_{t_l-} - (\eta^b + \epsilon^b), Q_{t_l-} - (\eta^c + \epsilon^c)]. \tag{3.11}$$

Note that a discrete-time collateralization scheme induces a path dependence, which, from a computational perspective, implies to add one more dimension Γ_t to the factor process X_t.

3.1.2 Cure Period

An additional important feature is a time lag $\delta > 0$, called the cure period, to be taken no less than $\delta =$ two weeks for Basel III capital charges CVA computations, between the first default time of a party and the time of delivery of the close-out cash-flow R. The time interval between the last margin call preceding the first default time of a party and the time of delivery of the close-out cash-flow is known as the margin period of risk. The margin period of risk thus consists of the time interval between the last margin call preceding the first default time of a party and this first default time, plus the cure period.

In order to account for the cure period, a possible fix (see however the remark 3.4 for an in-depth analysis) is to think everywhere above of τ_b, τ_c and τ as the corresponding default times plus δ (so that τ represents the time of delivery of the close-out cash-flow, i.e.

the end of the margin period of risk) and of T as the nominal maturity of the contract plus δ, also changing the definition of the exposure χ in (3.1) into

$$\chi = Q_\tau + \Delta_\tau - \Gamma_{\widehat{\tau}}, \tag{3.12}$$

where, for every t:

- Δ_t is now defined as

$$\Delta_t = \int_{[t-\delta,t]} \beta_t^{-1}\beta_s dD_s = \int_{[t-\delta,t)} \beta_t^{-1}\beta_s dD_s \, + \, (D_t - D_{t-}), \tag{3.13}$$

- $\widehat{t}$ denotes the greatest t_l less or equal than $(t-\delta)$.

Remark 3.2 With this understanding, we have $\tau \geq \delta$, so that the intensity $\widetilde{\gamma}$ has to be set to zero before δ, and there are no promised dividends dD_t of the contract after $T-\delta$.

The equation (3.12), with $\Gamma_{\widehat{\tau}}$ instead of Γ_τ in (3.1), reflects the fact that the collateral is frozen during the margin period of risk $(\widehat{\tau}, \tau]$. Put differently, we have:

$$\chi = (Q_{\widehat{\tau}-} - \Gamma_{\widehat{\tau}}) + \Delta_\tau + (Q_\tau - Q_{\widehat{\tau}-}), \tag{3.14}$$

where $Q_{\widehat{\tau}-} - \Gamma_{\widehat{\tau}} \in [\eta^c + \epsilon^c, \eta^b + \epsilon^b]$ (by (3.11)) is controlled by the thresholds and the MTAs, but where the two residual "gap risk" terms Δ_τ (which includes $(D_\tau - D_{\tau-})$) and $(Q_\tau - Q_{\widehat{\tau}-})$ can be quite substantial.

Moreover, accounting for the cure period in the context of a discrete-time collateralization scheme leads to modify the condition on Γ in (3.5) into

$$\begin{array}{l} \text{For every } e \in E, \text{ one can find a predictable} \\ \text{process } \widetilde{\Gamma}^e \text{ such that } \Gamma_{\widehat{\tau}} = \widetilde{\Gamma}^e_\tau \text{ on } \{\tau = \tau_e\}. \end{array} \tag{3.15}$$

Then, as is classical with such rolling-window path-dependence features (see e.g. Chassagneux and Crépey (2013) or Chapter 10 in Crépey (2013a)), for computational (Markov) purposes, one must augment the factor process X_t not only by one additional dimension Γ_t as in the end of Sect. 3.1.1, but also by the past values of Γ at all the margining time-grid points between the current time t and $(t-\delta)$; see Subsection 5.3 for a detailed example. For instance, with daily collateralization and δ = two weeks, where we recall that two weeks is a regulatory minimum for Basel III CVA capital charges computations, this results in ten extra dimensions (ten, i.e. the number of working days within two weeks) in the TVA pricing equations.

Remark 3.3 In the above approach, the treatment of funding costs during the cure period is only an approximation, since on $(\tau-\delta, \tau]$ we still use funding costs given as the next-to-last line of (3.8), with Γ_t there evolving according to (3.10), whereas the reality is more involved (depending, in particular, on which party defaults first and, in case it is the counterparty, on whether the bank survives until the close-out or not, so that three different cases need be considered). It would be possible to eliminate the corresponding bias by adding approriate corrective terms in the exposures $\widetilde{\xi}^e_t$ in (3.6). However, the exact formulation would lead to a time-delayed BSDE (see Delong and Imkeller (2010)). This raises a mathematical difficulty since, as shown in the just-mentioned paper, even for a Lipschitz coeffficient, a time-delayed

BSDE may only have a solution for T small enough, depending on the Lipschitz constant of the coefficient (not to mention numerical issues). But, beyond these technicalities, the fix used in this section to cope with a positive cure period is questionable from a financial point of view, as it models the close-out time by a totally inaccessible stopping time (endowed with an intensity). In reality the unpredictable time is rather the time of default, which in fact announces the close-out an amount of time δ later. However, for simplicity in this paper, we live with this approach regarding the cure period, in the hope that it should be fine at a first level of approximation.

Remark 3.4 The specification of Γ in Sect. 3.1.1, coupled with a positive cure period as described above, finely renders the path-dependence of the collateral. It does not include other possible gap risk features, such as a possible jump of the collateral at the default of a party, in case of a collateral posted in a currency strongly dependent on this party's default (see the beginning of Sect. 3 and Ehlers and Schönbucher (2006)). However, the general framework (3.5), (3.15) does allow for such features if need be, through a $\widetilde{\Gamma}^e$ in (3.15) that would effectively depend on e – whereas it does in fact not in the above-described specification (see Subsection 5.3 for a concrete illustration).

In the remaining sections, we apply the above framework to the modeling of the TVA on credit derivatives in the dynamic copula models of Crépey, Jeanblanc, and Wu (2013) (dynamic Gaussian copula) and Bielecki, Cousin, Crépey, and Herbertsson (2013d), Bielecki and Crépey (2011) (dynamic Marshall-Olkin copula or common shock model), building on the auxiliary results already proven in Crépey (2013b). Let $N = \{-1, 0, 1, \ldots, n\}$, $N^\star = \{1, \ldots, n\}$. In both cases, we dynamize a copula model for the $(\tau_i)_{i\in N}$ by introduction of a suitable filtration, where $\tau_{-1} = \tau_b$ and $\tau_0 = \tau_c$ are used to model the default times of the the bank and its counterparty , whereas the $(\tau_i)_{i\in N^\star}$ represent the default times of a pool of reference entities underlying a credit portfolio between the two parties.

For any Euclidean vector $\mathbf{k}$, we denote

$$\mathrm{supp}(\mathbf{k}) = \{i;\, k_i \neq 0\},\ \ \mathrm{supp}^c(\mathbf{k}) = \{i;\, k_i = 0\}. \tag{3.16}$$

In the financial interpretation, $\mathrm{supp}(\mathbf{k})$ denotes the obligors who have defaulted in the portfolio-state $\mathbf{k}$ and therefore $\mathrm{supp}^c(\mathbf{k})$ corresponds to the survivors.

4 Dynamic Gaussian Copula TVA Model

4.1 Model of Default Times

We consider a multivariate Brownian motion $\mathbf{B} = (B^i)_{i\in N}$ with pairwise correlation ϱ in its own completed filtration $\mathbb{F}^{\mathbf{B}}$, under a pricing probability measure $\mathbb{Q}$. For any $i \in N$, let h_i be a differentiable increasing function from $\mathbb{R}_+$ to $\mathbb{R}$ with $\lim_0 h_i(s) = -\infty$ and $\lim_{+\infty} h_i(s) = +\infty$. For every $i \in N$, we define a random time

$$\tau_i = h_i^{-1}\Big(\int_0^{+\infty} \varsigma(u) dB_u^i\Big), \tag{4.1}$$

where $\varsigma(\cdot)$ is a square integrable function with unit L^2-norm. So the τ_i jointly follow the standard (static) Gaussian copula model of Li (2000), with correlation parameter ϱ and with

marginal distribution function $\Phi \circ h_i$ of τ_i, where Φ is the the standard normal distribution function. In order to make the model dynamic as required by counterparty risk applications, we introduce a model filtration $\mathbb{G}$ given as $\mathbb{F}^{\mathbf{B}}$ progressively enlarged by the τ_i, so, for every t,

$$\mathcal{G}_t = \mathcal{F}_t^{\mathbf{B}} \vee \big(\bigvee_{i \in N} \sigma(\tau_i \wedge t) \big).$$

Let also, for every $i \in N$,

$$m_t^i = \int_0^t \varsigma(u) dB_u^i, \;\; k_t^i = \tau_i \mathbb{1}_{\{\tau_i \leq t\}}$$

and, in vector form,

$$\mathbf{m}_t = (m_t^i)_{i \in N}, \;\; \mathbf{k}_t = (k_t^i)_{i \in N}, \;\; \widetilde{\mathbf{k}}_t = (k_t^i)_{i \in N^\star}, \;\; X_t = (\mathbf{m}_t, \widetilde{\mathbf{k}}_t).$$

The above "informational dynamization" of the Gaussian copula (DGC, i.e. dynamic Gaussian copula) was introduced in Crépey, Jeanblanc, and Wu (2013) (see also Fermanian and Vigneron (2010, 2013) for a structural perspective), where one can find that for explicit processes μ_t^i and λ_t^i of the form

$$\mu_t^i, \gamma_t^i = \mu_i, \gamma_i \left(t, \mathbf{m}_t, \mathbf{k}_t\right), \tag{4.2}$$

the $\mathbb{G}$-Brownian motions $dW_t^i = dB_t^i - \mu_t^i dt$ and the $\mathbb{G}$-compensated default indicator processes $dM_t^i = d\mathbb{1}_{\tau_i \leq t} - \gamma_t^i dt$ have the $\mathbb{G}$-martingale representation property.

4.2 TVA Model

A DGC setup can be used as a TVA credit derivatives model, for E^b and E^c in Sect. 3 respectively given as

$$E^b = \{-1\}, \;\; E^c = \{0\}.$$

Since there are no joint defaults in this model, we can assume that the contract promises no cash-flow at τ (see Crépey, Jeanblanc, and Wu (2013)), i.e. $\Delta_\tau = 0$ and χ in (3.1) reduces to $\chi = Q_{\tau} - \Gamma_\tau$. Next, we assume that for every process $U = P$, Q and Γ, there exists a function $\widetilde{U}_i$ such that

$$U_\tau = \widetilde{U}_i(\tau, \mathbf{m}_\tau, \mathbf{k}_{\tau-}) = \widetilde{U}_\tau^i \tag{4.3}$$

on every event of the form $\{\tau = \tau_i\}$ for $i = -1, 0$. The results of Crépey, Jeanblanc, and Wu (2013) show that the condition (4.3) holds regarding $U = P$ (for every $i \in N$) for standard credit derivatives, including CDS and CDO; moreover, for CDS and CDO, P can be computed explicitly. The condition (4.3) regarding $U = Q$ and Γ may be satisfied or not depending on the CSA. We postulate them at this stage, referring the reader to the remark 4.1 for more in this regard. Then, setting for $i = -1, 0$

$$\chi_t^e = \widetilde{Q}_t^e - \widetilde{\Gamma}_t^e, \tag{4.4}$$

the coefficients $\widetilde{\xi}$ and f of equations (3.6) and (3.8) are given as

$$\begin{aligned} \widetilde{\xi}_t^e(\pi) = \widetilde{P}_t^e - \widetilde{Q}_t^e + \mathbb{1}_{i=0}(1 - R_c)\widetilde{\chi}_t^{e,+} \\ - \mathbb{1}_{i=-1}\big((1 - R_b)\widetilde{\chi}_t^{e,-} + (1 - \bar{R}_b)(\pi - \Gamma_{t-})^+\big) = \widetilde{\xi}_i(t, \mathbf{m}_t, \mathbf{k}_{t-}; \pi) \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} f_t(P_t-\vartheta)+r_t\vartheta = \quad & (1-R_c)\gamma_t^0\big(\widetilde{Q}_t^0-\widetilde{\Gamma}_t^0\big)^+ \\ & -(1-R_b)\gamma_t^{-1}\big(\widetilde{Q}_t^{-1}-\widetilde{\Gamma}_t^{-1}\big)^- \\ & +\bar{b}_t\Gamma_t^+-b_t\Gamma_t^- \;+\widetilde{\lambda}_t\left(P_t-\vartheta-\Gamma_t\right)^+-\lambda_t\left(P_t-\vartheta-\Gamma_t\right)^- \\ & +\sum_{i=-1,0}\gamma_t^i\left(\widetilde{P}_t^i-\vartheta-\widetilde{Q}_t^i\right), \end{aligned} \tag{4.6}$$

where $\widetilde{\lambda}_t=\bar{\lambda}_t-(1-\bar{R}_b)\gamma_t^{-1}$.

Consequently, in view of the results of Sect. 3 in Crépey (2013b), provided the processes $r,b,\bar{b},\lambda,\bar{\lambda},P$ and Γ are given before τ as continuous functions of (t,X_t) (functions denoted below by the same letters as the related processes), the assumption (A) holds in a predefault DGC TVA model, for:

(DGC.1) a reference filtration $\mathbb{F}=(\mathcal{F}_t)$ in (A.1) generated by the Brownian motion $\mathbf{B}$ and the default times of references names $i\in N^\star$, i.e.

$$\mathcal{F}_t=\mathcal{F}_t^{\mathbf{B}}\vee\big(\bigvee_{i\in N^\star}\sigma(\tau_i\wedge t)\big),$$

(DGC.2) a changed measure $\mathbb{P}$ ($\neq\mathbb{Q}$, so that this is a case of no immersion in the sense of the remark 2.3) such that the "$(\mathbb{F},\mathbb{P})$-intensities" of the $B_t^i, i\in N$ and of the $\mathbb{1}_{\tau_i\le t}, i\in N^\star$ are respectively equal, writing $\mathbf{k}=(0,0,\widetilde{\mathbf{k}})$ for every $\widetilde{\mathbf{k}}\in\mathbb{R}_+^n$, to:

$$\begin{aligned} \widetilde{\mu}_t^i&=\widetilde{\mu}_i(t,\mathbf{m}_t,\widetilde{\mathbf{k}}_t):=\mu_i(t,\mathbf{m}_t,\mathbf{k}_t)\\ \widetilde{\lambda}_t^i&=\widetilde{\gamma}_i(t,\mathbf{m}_t,\widetilde{\mathbf{k}}_t):=\gamma_i(t,\mathbf{m}_t,\mathbf{k}_t), \end{aligned} \tag{4.7}$$

(DGC.3) a Markovian specification $\widetilde{f}_t(\vartheta):=\widetilde{f}(t,X_t,\vartheta)$ in (A.3), for the function $\widetilde{f}=\widetilde{f}(s,\mathbf{m},\widetilde{\mathbf{k}},\vartheta)$ given, writing $\mathbf{k}=(0,0,\widetilde{\mathbf{k}})$ for every $\widetilde{\mathbf{k}}\in\mathbb{R}_+^n$, through[1]

$$\begin{aligned} \widetilde{f}(s,\mathbf{m},\widetilde{\mathbf{k}},\vartheta)+r(s,\mathbf{m},\mathbf{k})\vartheta = \quad & (1-R_c)\gamma_0\left(\widetilde{Q}_0-\widetilde{\Gamma}_0\right)^+(s,\mathbf{m},\mathbf{k}) \\ & -(1-R_b)\gamma_{-1}\left(\widetilde{Q}_{-1}-\widetilde{\Gamma}_{-1}\right)^-(s,\mathbf{m},\mathbf{k}) \\ & +\left(\bar{b}\Gamma^+-b\Gamma^-+\widetilde{\lambda}\left(P-\vartheta-\Gamma\right)^+-\lambda\left(P-\vartheta-\Gamma\right)^-\right)(s,\mathbf{m},\mathbf{k}) \\ & +\sum_{i=-1,0}\gamma_i\left(\widetilde{P}_i-\vartheta-\widetilde{Q}_i\right)(s,\mathbf{m},\mathbf{k}), \end{aligned} \tag{4.8}$$

where $\widetilde{\lambda}=\bar{\lambda}-(1-\bar{R}_b)\gamma_{-1}$, and for the pre-default factor process $(t,X_t)=(t,\mathbf{m}_t,\widetilde{\mathbf{k}}_t)$, an $(\mathbb{F},\mathbb{P})$-Markov process with generator acting on every function $u=u(s,\mathbf{m},\widetilde{\mathbf{k}})$ as

$$\mathcal{A}u=\partial_s u+\mathcal{A}_{\mathbf{m}}u+\sum_{i\in N^\star}\widetilde{\gamma}_i\delta_i u, \tag{4.9}$$

where

$$\mathcal{A}_{\mathbf{m}}u=\varsigma\sum_{i\in N}\widetilde{\mu}_i\partial_{m_i}u+\frac{\varsigma^2}{2}\left(\sum_{i\in N}\partial^2_{m_i^2}u+\varrho\sum_{i\neq j\in N}\partial^2_{m_i,m_j}u\right) \tag{4.10}$$

[1] Compare (4.6).

and

$$\delta_i u(s,m,\mathbf{k}) = u(s,m,\mathbf{k}^{i,s}) - u(s,m,\mathbf{k}),$$

in which $\mathbf{k}^{i,s}$ stands for $\mathbf{k}$ with component i replaced by s.

Note that time t has to be included in the state variables in $(t,\mathbf{m}_t,\widetilde{\mathbf{k}}_t) = (t,X_t)$, because X_t is not Markov unless considered with time jointly (see Crépey (2013b)). Moreover, a family of fundamental $(\mathbb{F},\mathbb{P})$-martingales is written as

$$d\widetilde{W}^i_t = dB^i_t - \widetilde{\mu}^i_t dt,\ i\in N;\ d\widetilde{M}^i_t = d\mathbb{1}_{\tau_i\le t} - \widetilde{\lambda}^i_t dt,\ i\in N^\star. \tag{4.11}$$

Let

$$\widetilde{J}_t = \{j\in N^\star;\tau_j > t\} = \mathrm{supp}^c(\widetilde{\mathbf{k}}_{t-})$$

represent the set of survivors right before time t (see (3.16)). The following proposition follows of the above by application of Theorem 2.1 (Markovian specification in the sense of Sect. 2.2).

Proposition 4.1 *We obtain a solution Θ to the full TVA equation (2.3) by setting $\Theta = \widetilde{\Theta}$ on $[0,\bar\tau)$ and $\Theta_{\bar\tau} = \mathbb{1}_{\tau<T}\xi$, where $\widetilde{\Theta}_t = \widetilde{\Theta}(t,X_t)$ is such that $\widetilde{\Theta}_T = 0$ and, for $t\in[0,T]$:*

$$-d\widetilde{\Theta}_t = \widetilde{f}(t,\mathbf{m}_t,\widetilde{\mathbf{k}}_t,\widetilde{\Theta}_t)dt - d\widetilde{\mu}_t, \tag{4.12}$$

where $\widetilde{\mu}_t$ is an $(\mathbb{F},\mathbb{P})$-martingale starting from 0 such that for $t\in[0,T]$

$$d\widetilde{\mu}_t = \varsigma(t)\sum_{i\in N}\partial_{m_i}\widetilde{\Theta}(t,\mathbf{m}_t,\widetilde{\mathbf{k}}_t)d\widetilde{W}^i_t + \sum_{j\in\widetilde{J}_{t-}}\delta_j\widetilde{\Theta}(t,\mathbf{m}_t,\widetilde{\mathbf{k}}_{t-})d\widetilde{M}^j_t,$$

in which the pre-default TVA function $\widetilde{\Theta} = \widetilde{\Theta}(t,\mathbf{m},\widetilde{\mathbf{k}})$ solves the PDE (2.17) for the generator $\mathcal{A}$ of (4.9) and for the coefficient $\widetilde{f}$ of (4.8). Moreover, the $(\mathbb{G},\mathbb{Q})$-martingale component

$$d\mu_t = d\Theta_t + (g_t(P_t-\Theta_t) - r_t\Theta_t)dt$$

of Θ satisfies, for $t\in[0,\bar\tau]$:

$$d\mu_t = \varsigma(t)\sum_{i\in N}\partial_{m_i}\widetilde{\Theta}(t,\mathbf{m}_t,\widetilde{\mathbf{k}}_t)dW^i_t + \sum_{j\in\widetilde{J}_{t-}}\delta_j\widetilde{\Theta}(t,\mathbf{m}_t,\widetilde{\mathbf{k}}_{t-})dM^j_t - \left((\xi-\widetilde{\Theta}_{t-})dJ_t + \sum_{i=-1,0}\gamma^i_t(\widetilde{\xi}^{\star,i}_t - \widetilde{\Theta}_t)dt\right)$$

where ξ and $\widetilde{\xi}^{\star,i}_t$ are shorthands for $\xi(P_{\tau-}-\widetilde{\Theta}_{\tau-})$ and $\widetilde{\xi}^i_t(P_t-\widetilde{\Theta}_t)$.

This proposition can then be used for any TVA valuation and hedging purposes.

Remark 4.1 (Gap risk) Proposition 4.1 can be adapted to the path-dependent collateralization scheme of Sect. 3.1 (also with a positive cure period ϵ as in Sect. 3.1.2). We refer the reader to Sect. 5.3 for the corresponding development in a DMO setup. As in the basic case without path-dependence (compare this section with Sect. 5.2), the situation is in fact slightly more intricate in a DMO setup, which is why we reserve the detailed exposition for this case. But in a DGC setup very similar considerations apply, in terms of an augmented Markovian pre-default model $(t,X_t) = (t,\mathbf{m}_t,\widetilde{\mathbf{k}}_t,\mathbf{C}_t)$, where $\mathbf{m}_t$ and $\widetilde{\mathbf{k}}_t$ are as above and $\mathbf{C}_t$ is defined as in Sect. 5.3.

5 Dynamic Marshall-Olkin Copula TVA Model

The above dynamic Gaussian copula (DGC) model can be sufficient to deal with CVA on portfolios of CDSs. If CDOs are also present in the reference portfolio a Gaussian copula dependence structure is not rich enough. Instead one can use a dynamic Marshall-Olkin (DMO) copula model (common shock model of Bielecki, Cousin, Crépey, and Herbertsson (2013d), Bielecki and Crépey (2011)).

5.1 Model of Default Times

Let's briefly revisit the model. We define a certain number m (typically small: a few units) of groups $I_j \subseteq N$ of obligors who are likely to default simultaneously, for $l = 1, \ldots, m$. The idea is that at every time t, there will be a positive probability that the survivors of the group of obligors I_j (obligors of group I_j still alive at time t) default simultaneously. Let $\mathcal{I} = \{I_1, \ldots, I_m\}$, $\mathcal{Y} = \{\{-1\}, \{0\}, \{1\}, \ldots, \{n\}, I_1, \ldots, I_m\}$. Let shock intensity processes X^Y be given in the form of extended CIR processes as, for every $Y \in \mathcal{Y}$,

$$dX_t^Y = a(b_Y(t) - X_t^Y)dt + c\sqrt{X_t^Y}dW_t^Y \tag{5.1}$$

for non-negative constants a, c, non-negative functions $b_Y(t)$ and independent Brownian motions W^Y (in their own filtration $\mathcal{F}^{\mathbf{W}}$), under a pricing measure $\mathbb{Q}$..

Remark 5.1 The case of deterministic intensities $X_t^Y = b_Y(t)$ can be embedded in this framework as the limiting case of an "infinite mean-reversion speed" a where (see Bielecki, Cousin, Crépey, and Herbertsson (2013a)).

For every $Y \in \mathcal{Y}$, we define

$$\widehat{\tau}_Y = \inf\{t > 0; \int_0^t X_s^Y ds > \mathcal{E}_Y\},$$

where the $\mathcal{E}_Y$ are i.i.d. standard exponential random variables. Then, for every obligor $i \in N$, we set

$$\tau_i = \min_{\{Y \in \mathcal{Y}; i \in Y\}} \widehat{\tau}_Y, \;\; H_t^i = \mathbb{1}_{\tau_i \leq t}. \tag{5.2}$$

We finally consider the dynamic model $(\mathbf{X}, \mathbf{H}) = ((X^Y)_{Y \in \mathcal{Y}}, (H^i)_{i \in N})$ relatively to the filtration $\mathbb{G}$ such that for every t

$$\mathcal{G}_t = \mathcal{F}_t^{\mathbf{W}} \vee \big(\bigvee_{i \in N} \sigma(\tau_i \wedge t)\big),$$

For every $Z \subseteq N$, we have the following expression for the predictable intensity γ_t^Z of the indicator process of the event of a joint default of names in set Z and only in Z:

$$\gamma_t^Z = \gamma_Z(t, \mathbf{X}_t, \mathbf{H}_{t-}) = \sum_{Y \in \mathcal{Y}; Y_t = Z} X_t^Y, \tag{5.3}$$

where Y_t stands for the set of survivors of set Y "right before time t," for every $Y \in \mathcal{Y}$. So $Y_t = Y \cap \text{supp}^c(\mathbf{H}_{t-})$. One denotes by M^Z the corresponding compensated set-event martingale, so for $t \in [0, T]$,

$$dM_t^Z = d\mathbb{1}_{\tau_Z \leq t} - \gamma_t^Z dt \tag{5.4}$$

where $\mathbb{1}_{\tau_Z \leq t}$ stands for the time of a joint default of names in set Z and only in Z. The W^Y and the M^Z have the $(\mathbb{G}, \mathbb{Q})$-martingale representation property.

5.2 TVA Model

A DMO setup can also be used as a TVA wrong-way risk model on credit derivatives, with E^b and E^c respectively given as

$$E^b = \{Z \subseteq N;\, -1 \in Z\},\ \ E^c = \{Z \subseteq N;\, 0 \in Z\}.$$

We assume that for every process $U = P$, Δ, Q and Γ, there exists a function $\widetilde{U}$ such that

$$U_\tau = \widetilde{U}(\tau, \mathbf{X}_\tau, \mathbf{H}_{\tau-}, \mathbf{H}^Z_{\tau-}) = \widetilde{U}^Z_\tau \tag{5.5}$$

on every event of the form $\{\tau = \tau_e\}$ for some $Z \in E$. The results of Bielecki, Cousin, Crépey, and Herbertsson (2013c) show that the condition (5.5) holds on $U = P$ and Δ for standard credit derivatives, including CDS and CDOs; moreover, for CDS and CDOs, P can be computed explicitly (at least for piecewise-constant $b_Y(t)$, which is enough for applications); see Bielecki, Cousin, Crépey, and Herbertsson (2013b, 2013a). The conditions (5.5) on $U = Q$ and Γ may be satisfied or not depending on the CSA. We postulate them at this stage, postponing to Sect. 5.3 the discussion of various specifications. Then, letting

$$\chi^e_t = \widetilde{Q}^e_t + \widetilde{\Delta}^e_t - \widetilde{\Gamma}^e_t, \tag{5.6}$$

the coefficient $\widetilde{\xi}$ of (3.6) is given as

$$\begin{aligned}\widetilde{\xi}^e_t(\pi) = \widetilde{P}^e_t - \widetilde{Q}^e_t + \mathbb{1}_{Z\in\mathcal{Z}^c}(1-R_c)\widetilde{\chi}^{e,+}_t \\ - \mathbb{1}_{Z\in\mathcal{Z}^b}\big((1-R_b)\widetilde{\chi}^{e,-}_t + (1-\bar{R}_b)(\pi-\Gamma_{t-})^+\big) = \widetilde{\xi}(t, \mathbf{X}_t, \mathbf{H}_{t-}, \mathbf{H}^Z_{t-}; \pi),\end{aligned} \tag{5.7}$$

by (5.5) assumed for every $U = P$, Δ, Q and Γ. Then, letting

$$\begin{aligned}&\mathcal{Y}^b = \{Y \in \mathcal{Y};\, -1 \in Y\},\ \ \mathcal{Y}^c = \{Y \in \mathcal{Y};\, 0 \in Y\},\ \ \mathcal{Y}^\bullet = \mathcal{Y}^b \cup \mathcal{Y}^c \\ &\mathcal{Y}^b_t = \{Y \in \mathcal{Y};\, -1 \in Y_t\},\ \ \mathcal{Y}^c_t = \{Y \in \mathcal{Y};\, 0 \in Y_t\},\ \ \mathcal{Y}^\bullet_t = \mathcal{Y}^b_t \cup \mathcal{Y}^c_t,\end{aligned}$$

it follows from the results of Sect. 4.2 in (Crépey 2013b) that

$$\begin{aligned}f_t(P_t - \vartheta) + r_t\vartheta = \ & (1-R_c)\sum_{Y\in\mathcal{Y}^c_t} X^Y_t\big(\widetilde{Q}^Y_t + \widetilde{\Delta}^Y_t - \widetilde{\Gamma}^Y_t\big)^+ \\ & - (1-R_b)\sum_{Y\in\mathcal{Y}^b_t} X^Y_t\big(\widetilde{Q}^Y_t + \widetilde{\Delta}^Y_t - \widetilde{\Gamma}^Y_t\big)^- \\ & + \bar{b}_t\Gamma^+_t - b_t\Gamma^-_t + \widetilde{\lambda}_t\,(P_t - \vartheta - \Gamma_t)^+\big) - \lambda_t\,(P_t - \vartheta - \Gamma_t)^- \\ & + \sum_{Y\in\mathcal{Y}^\bullet_t} X^Y_t\left(\widetilde{P}^Y_t - \vartheta - \widetilde{Q}^Y_t\right),\end{aligned} \tag{5.8}$$

where $\widetilde{\lambda}_t = \bar{\lambda}_t - (1-\bar{R}_b)\sum_{Y\in\mathcal{Y}^b_t} X^Y_t$.

Let further $\widetilde{\mathcal{Y}} = \{\{1\}, \ldots, \{n\}\} \cup \widetilde{\mathcal{I}}$, where $\widetilde{\mathcal{I}}$ consists of those I_j in $\mathcal{I}$ which do not contain -1 nor 0. Defining for every obligor $i \in N^\star$

$$\widetilde{\tau}_i = \min_{\{Y\in\widetilde{\mathcal{Y}};\, i\in Y\}} \widehat{\tau}_Y,\ \ \widetilde{H}^i_t = \mathbb{1}_{\widetilde{\tau}_i \le t},$$

we now set $X_t = (\mathbf{X}_t, \widetilde{\mathbf{H}}_t)$, where $\widetilde{\mathbf{H}} = (\widetilde{H}^i)_{i\in N^\star}$. In view of the above and of the developments of Sect. 4 in Crépey (2013b), provided the processes $r, b, \bar{b}, \lambda, \bar{\lambda}, P$ and Γ are given before τ as continuous functions of (t, X_t) (functions denoted below by the same letters as the related processes), the assumption (A) holds in a predefault DMO TVA model, for:

(DMO.1) a reference filtration $\mathbb{F} = (\mathcal{F}_t)$ in (A.1) generated by the Brownian motion $\mathbf{W}$ and the "reduced default times" $\widetilde{\tau}_i$ of references names $i \in N^\star$, i.e.

$$\mathcal{F}_t = \mathcal{F}_t^{\mathbf{W}} \vee \big(\bigvee_{i \in N^\star} \sigma(\widetilde{\tau}_i \wedge t) \big),$$

(DMO.2) $\mathbb{P} = \mathbb{Q}$ in (A.2) (immersion holds in the sense of the remark 2.3),

(DMO.3) a Markovian specification $\widetilde{f}_t(\vartheta) := \widetilde{f}(t, X_t, \vartheta)$ in (A.3), for the function $\widetilde{f} = \widetilde{f}(t, \mathbf{x}, \widetilde{\mathbf{k}}, \vartheta)$ given, letting $\mathbf{k} = (0, 0, \widetilde{\mathbf{k}})$ for every $\widetilde{\mathbf{k}} \in \{0,1\}^n$, through[2]

$$\begin{aligned}
\widetilde{f}(t, \mathbf{x}, \widetilde{\mathbf{k}}, \vartheta) + r(t, \mathbf{x}, \mathbf{k})\vartheta = {} & (1 - R_c) \sum_{Y \in \mathcal{Y}^c} x_Y \left(\widetilde{Q}_t^Y + \widetilde{\Delta}_t^Y - \widetilde{\Gamma}_t^Y \right)^+ (t, \mathbf{x}, \mathbf{k}, \mathbf{k}^Y) \\
& - (1 - R_b) \sum_{Y \in \mathcal{Y}^b} x_Y \left(\widetilde{Q}_t^Y + \widetilde{\Delta}_t^Y - \widetilde{\Gamma}_t^Y \right)^- (t, \mathbf{x}, \mathbf{k}, \mathbf{k}^Y) \\
& + \left(\bar{b}\Gamma^+ - b\Gamma^- + \widetilde{\lambda} \left(P - \vartheta - \Gamma\right)^+ - \lambda \left(P - \vartheta - \Gamma\right)^- \right) (t, \mathbf{x}, \mathbf{k}) \\
& + \sum_{Y \in \mathcal{Y}^\bullet} x_Y \left(\widetilde{P} - \vartheta - \widetilde{Q} \right) (t, \mathbf{x}, \mathbf{k}, \mathbf{k}^Y),
\end{aligned} \tag{5.9}$$

where $\widetilde{\lambda} = \bar{\lambda} - (1 - \bar{R}_b) \sum_{Y \in \mathcal{Y}^b} x_Y$, and for the pre-default factor process $X_t = (\mathbf{X}_t, \widetilde{\mathbf{H}}_t)$, an $(\mathbb{F}, \mathbb{Q})$- (in fact, even $(\mathbb{G}, \mathbb{Q})$-) Markov process with generator acting on every function $u = u(t, \mathbf{x}, \widetilde{\mathbf{k}})$ as:

$$\mathcal{A}u(t, \mathbf{x}, \widetilde{\mathbf{k}}) = \mathcal{A}_x u(t, \mathbf{x}, \widetilde{\mathbf{k}}) + \sum_{Y \in \widetilde{\mathcal{Y}}} x_Y \delta_Y u(t, \mathbf{x}, \widetilde{\mathbf{k}}). \tag{5.10}$$

with

$$\mathcal{A}_x u = \sum_{Y \in \mathcal{Y}} \Big(a(b_Y(t) - x_Y) \partial_{x_Y} u + \frac{1}{2} c^2 \, x_Y \partial^2_{x_Y^2} \Big) u$$

and

$$\delta_Y u(t, \mathbf{x}, \mathbf{k}) = u(t, \mathbf{x}, \mathbf{k}^Y) - u(t, \mathbf{x}, \mathbf{k})$$

in which $\mathbf{k}^Y$ denotes the vector obtained from $\mathbf{k} = (k_i)_{i \in N}$ by replacing the components k_i, $i \in Y$, by numbers one.

Moreover, let, for $Z \subseteq N^\star$,

$$d\widetilde{M}_t^Z = d\mathbb{1}_{\tau_Z \leq t} - \widetilde{\gamma}_t^Z dt \text{ with } \widetilde{\gamma}_t^Z = \sum_{Y \in \widetilde{\mathcal{Y}};\, Y_t = Z} X_t^Y. \tag{5.11}$$

The $W^Y, Y \in \mathcal{Y}$ and the $\widetilde{M}^Z, Z \subseteq N^\star$ have the $(\mathbb{F}, \mathbb{Q})$-martingale representation property. Also note that, for $Z \subseteq N^\star$, we have $\widetilde{M}_t^Z = M_t^Z$ on $\{t \leq \bar{\tau}\}$, since by (5.3) and (5.11):

$$\mathbb{1}_{\{t \leq \bar{\tau}\}} \gamma_t^Z dt = \mathbb{1}_{\{t \leq \bar{\tau}\}} \widetilde{\gamma}_t^Z dt.$$

The following proposition follows of the above by application of Theorem 2.1 (Markovian specification in the sense of Sect. 2.2).

[2] Compare with (5.8).

Proposition 5.1 *We obtain a solution Θ to the full TVA equation* (2.3) *by setting $\Theta = \widetilde{\Theta}$ on $[0, \bar{\tau})$ and $\Theta_{\bar{\tau}} = \mathbb{1}_{\tau<T}\xi$, where $\widetilde{\Theta}_t = \widetilde{\Theta}(t, X_t)$ is such that $\widetilde{\Theta}_T = 0$ and, for $t \in [0, T]$,*

$$-d\widetilde{\Theta}_t = \widetilde{f}(t, \mathbf{X}_t, \widetilde{\mathbf{H}}_t, \widetilde{\Theta}_t)dt - d\widetilde{\mu}_t$$

where $\widetilde{\mu}_t$ is an $(\mathbb{F}, \mathbb{Q})$-martingale starting from 0 and such that for $t \in [0, T]$

$$d\widetilde{\mu}_t = c \sum_{Y \in \widetilde{\mathcal{Y}}} \sqrt{x_Y}\partial_{x_Y}\widetilde{\Theta}(t, \mathbf{X}_t, \widetilde{\mathbf{H}}_t)dW_t^Y \; + \sum_{Z \subseteq N^\star} \delta_Z \widetilde{\Theta}(t, \mathbf{X}_t, \widetilde{\mathbf{H}}_{t-})d\widetilde{M}_t^Z,$$

in which the pre-default TVA function $\widetilde{\Theta} = \widetilde{\Theta}(t, \mathbf{x}, \widetilde{\mathbf{k}})$ solves the PDE (2.17) *for the generator $\mathcal{A} = \widetilde{\mathcal{A}}$ of* (5.10) *and for the coefficient $\widetilde{f}$ of* (5.9)*. Moreover, the $(\mathbb{G}, \mathbb{Q})$-martingale component*

$$d\mu_t = d\Theta_t + (g_t(P_t - \Theta_t) - r_t\Theta_t)dt$$

of Θ satisfies, for $t \in [0, \bar{\tau}]$:

$$d\mu_t = c \sum_{Y \in \widetilde{\mathcal{Y}}} \sqrt{x_Y}\partial_{x_Y}\widetilde{\Theta}(t, \mathbf{X}_t, \widetilde{\mathbf{H}}_t)dW_t^Y \; + \sum_{Z \subseteq N^\star} \delta_Z \widetilde{\Theta}(t, \mathbf{X}_t, \widetilde{\mathbf{H}}_{t-})dM_t^Z$$
$$- \left((\xi - \widetilde{\Theta}_{t-})dJ_t + \sum_{Y \in \mathcal{Y}^\bullet} X_t^Y (\widetilde{\xi}_t^{\star,Y} - \widetilde{\Theta}_t)dt \right),$$

where ξ and $\widetilde{\xi}_t^{\star,Y}$ ar shorthands for $\xi(P_{\tau-} - \widetilde{\Theta}_{\tau-})$ and $\widetilde{\xi}_t^Y(P_t - \widetilde{\Theta}_t)$.

This proposition can then be used for any valuation and hedging purposes in a DMO TVA model. One can for instance extend to the present TVA setup the unilateral CVA dynamic hedging approach of Bielecki and Crépey (2011), the extension of the results being straightforward in an asymmetrical (but still bilateral) CVA approach where one sets $R_b = \bar{R}_b = 1$ (see Sect. 3.5 in Crépey, Gerboud, Grbac, and Ngor (2013)).

5.3 Gap Risk

We use the collateralization framework and notation of Sect. 3.1. Given a cure period $\delta > 0$, we assume for simplicity a uniform grid (t_l) of step h with δ and T multiples of h, i.e. $\delta = p\text{h}$ and $T = q \times (t_l - t_{l-1})$ (the adaptation of the results to less standard data is straightforward). Let $\mathbf{C}_t = (C_t^l)_{1 \le l \le p}$ denote the $\mathbb{R}^p$-valued process of the values of Γ at the last p margining dates, the components being ordered from the oldest value C^1 to the most recent C^p. So $\mathbf{C}_{0-} = \mathbf{0}$, $\mathbf{C}_t$ is constant over every interval $(t_{l-1}, t_l]$ and, at every $t = t_l$, we have:

$$\mathbf{C}_t = \left(C_{t-}^2, \ldots, C_{t-}^p, \Gamma_t\right). \tag{5.12}$$

In case of a credit derivative the process $\int_{(t-\delta,t)} \beta_t^{-1}\beta_s dD_s$ in Δ_t in (3.13) is a function of past default times. Using $K_t^i = \tau_i \mathbb{1}_{\tau_i \le t}$ (cf. the process k^i of Sect. 4) instead of $H_t^i = \mathbb{1}_{\tau_i \le t}$ above, and augmenting the factor process with $\mathbf{C}_t$, we accordingly amend the condition (5.5) into the existence of a function U such that

$$U_\tau = \widetilde{U}(\tau, \mathbf{X}_\tau, \mathbf{K}_{\tau-}, \mathbf{K}_{\tau-}^{Z,\tau}, \mathbf{C}_{\tau-}) \tag{5.13}$$

(also denoted for short by $\widetilde{U}_\tau^Z$) on every event $\{\tau = \tau_e\}$, for every process $U = P, \Delta, Q$ and C^1 (where $C_t^1 = \Gamma_{\widehat{t}}$; compare (3.15)). In (5.13), $\mathbf{K}_{\tau-}^{Z,\tau}$ stands for the vector obtained from $\mathbf{K}_{\tau-}$ by replacing all components with indices in Z by τ.

Remark 5.2 Under the collateralization specification of Sect. 3.1.1-3.1.2 with a positive cure period ϵ, C^1 only jumps at the constant times t_l whereas τ is a totally unpredictable stopping time, so that C^1 cannot jump at τ. This means that for $U = C^1$, $\widetilde{U}_\tau^Z$ does in fact not depend on Z in (5.13). As explained in the remark 3.4, the flexibility given by a potential dependence of $\widetilde{U}_\tau^Z$ in Z could be useful to deal with gap risk features like a jump of C^1 at τ in case of a collateral posted in another currency.

Letting $\widetilde{\mathbf{K}}_t = (K_t^i)_{i\in N^\star}$, we finally consider the augmented pre-default factor process $(t, X_t) = (t, \mathbf{X}_t, \widetilde{\mathbf{K}}_t, \mathbf{C}_t)$. Denoting $x = (\mathbf{x}, \widetilde{\mathbf{k}}, \mathbf{c})$, we write:

$$\delta_Z u(s,x) = u(s, \mathbf{x}, \widetilde{\mathbf{k}}^{Z,s}, \mathbf{c}) - u(s,x),$$

where $\widetilde{\mathbf{k}}^{Z,s}$ stands for the vector obtained from $\widetilde{\mathbf{k}}$ by replacing all components with indices in Z by s. This model (t, X_t) is a Markovian pre-default TVA model with generator

$$\mathcal{A}u(s,x) = \mathcal{A}_x u(s,x) + \sum_{Y\in\widetilde{\mathcal{Y}}} x_Y \delta_Y u(s,x) \tag{5.14}$$

between the t_l and with deterministic jumps of $\mathbf{C}_t$ as specified by (5.12) at the t_l. We also postulate that the processes $r, b, \bar{b}, \lambda, \bar{\lambda}, P$ and $\Gamma = C^p$ are given before τ as continuous functions of (t, X_t), functions that are denoted below by the same letters as the related processes. Replacing $\widetilde{\Gamma}_t^e$ by $\widetilde{C}_t^{1,Z}$ (obtained from (5.13)) in $\widetilde{\chi}_t^e$ in (5.6) and t and $(t, \mathbf{x}, \mathbf{k}, \mathbf{k}^Y)$ by s and $(s, \mathbf{x}, \mathbf{k}, \mathbf{k}^{Y,s}, \mathbf{c})$ everywhere in the equation (5.9) for $\widetilde{f}$ (with also $\widetilde{\mathbf{k}}$ now in $\mathbb{R}_+^n$), we can then introduce the following cascade of PDEs solved by a sequence of continuous functions $\widetilde{\Theta}_l = \widetilde{\Theta}_l(s,x)$ over the sets $[t_{l-1}, t_l]\times\mathbb{R}_+^{2+n+m}\times\mathbb{R}_+^n\times\mathbb{R}^p$ (recalling $\delta = p\times(t_l - t_{l-1})$ and $T = q\times(t_l - t_{l-1})$):

For l decreasing from q to 1:

- at $t = t_l$, for every $x = (\mathbf{x}, \widetilde{\mathbf{k}}, \mathbf{c})$,

$$\widetilde{\Theta}_l(t_l, x) = \widetilde{\Theta}_{l+1}(t_l, \mathbf{x}, \widetilde{\mathbf{k}}, \mathbf{c}') \tag{5.15}$$

with $\widetilde{\Theta}_{l+1}$ set equal to 0 in case $l = q$ and for

$$\mathbf{c}' = \left(c_2, \ldots, c_p, c_p + \mathbb{1}_{\widehat{Q}(s,\mathbf{x},\widetilde{\mathbf{k}}) - c_p \notin [\epsilon^c, \epsilon^b]}\left(\widehat{Q}(s,\mathbf{x},\widetilde{\mathbf{k}}) - c_p\right)\right);$$

- on the time interval $[t_{l-1}, t_l)$, the function $\widetilde{\Theta}$ solves the PDE (2.17) for the generator $\mathcal{A}$ given by (5.14) and for the coefficient $\widetilde{f}$ modified as described above.

We refer the reader to Subsection 14.1.2 in Crépey (2013a) for the technicalities related to cascades of PDEs in relation to Markovian BSDEs with deterministic jumps of a factor process. Finally, we obtain the following amended form of Proposition 5.1 with cure period.

Proposition 5.2 *We obtain a solution Θ to the full TVA equation (2.3) by setting $\Theta = \widetilde{\Theta}$ on $[0, \bar{\tau})$ and $\Theta_{\bar{\tau}} = \mathbb{1}_{\tau<T}\xi$, where $\widetilde{\Theta}_t = \widetilde{\Theta}(t, X_t)$ is such that $\widetilde{\Theta}_t = 0$ and, for $t \in [0,T]$,*

$$-d\widetilde{\Theta}_t = \widetilde{f}(t, \mathbf{X}_t, \widetilde{\mathbf{K}}_t, \mathbf{C}_t, \widetilde{\Theta}_t)dt - d\widetilde{\mu}_t \tag{5.16}$$

where

$$c\sum_{Y\in\widetilde{\mathcal{Y}}}\sqrt{x_Y}\partial_{x_Y}\widetilde{\Theta}(t,\mathbf{X}_t,\widetilde{\mathbf{K}}_t,\mathbf{C}_t)dW_t^Y \;+\; \sum_{Z\subseteq N^\star}\delta_Z\widetilde{\Theta}(t,\mathbf{X}_t,\widetilde{\mathbf{K}}_{t-},\mathbf{C}_t)d\widetilde{M}_t^Z$$

in which the function $\widetilde{\Theta}$ *is given by* $\widetilde{\Theta}_l$ *on every set* $[t_{l-1},t_l)\times\mathbb{R}_+^{2+n+m}\times\mathbb{R}_+^n\times\mathbb{R}^p$.

Moreover, the $\mathbb{G}$*-martingale component*

$$d\mu_t = d\Theta_t + (g_t(P_t-\Theta_t)-r_t\Theta_t)dt$$

of Θ *satisfies, for* $t\in[0,\bar{\tau}]$*,:*

$$d\mu_t = c\sum_{Y\in\widetilde{\mathcal{Y}}}\sqrt{x_Y}\partial_{x_Y}\widetilde{\Theta}(t,\mathbf{X}_t,\widetilde{\mathbf{K}}_t,\mathbf{C}_t)dW_t^Y \;+\; \sum_{Z\subseteq N^\star}\delta_Z\widetilde{\Theta}(t,\mathbf{X}_t,\widetilde{\mathbf{K}}_{t-},\mathbf{C}_t)dM_t^Z$$
$$-\left((\xi-\widetilde{\Theta}_{t-})dJ_t+\sum_{Y\in\mathcal{Y}^\bullet}X_t^Y(\widetilde{\xi}_t^{\star,Y}-\widetilde{\Theta}_t)dt\right)$$

where ξ *and* $\widetilde{\xi}_t^{\star,Y}$ *are shorthands for* $\xi(P_{\tau-}-\widetilde{\Theta}_{\tau-})$ *and* $\widetilde{\xi}_t^Y(P_t-\widetilde{\Theta}_t)$.

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