Noname manuscript No.
(will be inserted by the editor)

Non-Arbitrage up to Random Horizon and after Honest Times for Semimartingale Models

Tahir Choulli · Anna Aksamit · Jun Deng · Monique Jeanblanc

Version of October 7, 2013

Abstract This paper addresses the question of how an arbitrage-free semimartingale model is affected when stopped at a random horizon or when an honest time is incorporated. Precisely, we focus on No-Unbounded-Profit-with-Bounded-Risk (called NUPBR hereafter) concept, which is also known in the literature as the first kind of non-arbitrage. Herein, we prove that any quasi-left-continuous process satisfying NUPBR, will preserve the NUPBR when one stops it at any random horizon or when one incorporates a *specific* honest time. For the general case of semimartingale, we provide necessary and sufficient conditions for the NUPBR to be preserved. Precisely, we elaborate two types of results. For a fixed semimartingale and a random time, we provide necessary and sufficient conditions on the process and the random time for which the non-arbitrage concept remains valid. The second type of results consists of giving necessary and sufficient conditions on the random time for which the non-arbitrage is preserved for any process. Our class of honest times —that we consider in the paper— is much larger than the class of all stopping times, and plays an important rôle in classifying random times. The crucial stochastic tool that drives our analysis lies in the *optional stochastic integral* (or *compensated stochastic integral*) that was introduced in early eighties.

1 Introduction

Since the seminal work of Arrow-Debreu for discrete markets (see [3], [15], [11] and [19]) where the authors stated what is know in the financial literature as the fundamental theorem of asset pricing (FTAP hereafter), arbitrage becomes one of the fundamental concepts in modern finance. This importance was

Tahir Choulli (corresponding author) · Jun Deng
Mathematical and Statistical Sciences Depart., University of Alberta, Edmonton, Canada
E-mail: tchoulli@ualberta.ca
Anna Aksamit, Monique Jeanblanc
Laboratoire Analyse et Probabilités, Université d'Evry Val d'Essonne, Evry, France

strengthen by establishing the link between the two theories of arbitrage and asset pricing through the Arrow-Debreu model (see [11] Chapter 7), the Black and Scholes formula (see [5]), and the Cox and Ross linear pricing model (see [10]). These works has been formalized in general framework by many scholars such as Duffie, Harrison, Huang, Kreps and Pliska (see [22], [23] and [12]). Other links between arbitrage theory and other financial/economical concepts have been discovered, elaborated, and explored further such as stochastic dominance, market's equilibrium, market's viability, portfolio analysis, numéraire portfolio...,etcoetera. Arbitrage plays crucial role in the analysis of securities markets, because its effect is to bring prices to fundamental values and to keep markets efficient.

In both mathematical finance and financial economics, arbitrage (or equivalently absence of arbitrage) has numerous definitions such as no-free-lunch (NFL), no-free-lunch-with-vanishing-risk (NFLVR), cheap thrill, free snack, No-Unbounded-Profit-with-Bounded-Risk (NUPBR), ..., etcoetera (see [36],[15], [18], [16], [29], [34], and the references therein to cite a few). It is worthwhile to mention that the majority of these arbitrage concepts coincides for discrete markets (i.e. markets with finite number of scenarios and finite number of trading times), and at some extent in discrete-time markets with finite horizon. Philosophically, an arbitrage opportunity is a transaction with no cash outlay that results in a sure profit. The mathematical formulation of this philosophy varies substantially from model to another.

Recently there has been an upsurge interest in studying the effect of additional information on the NFLVR concept as well as on the utility maximization problem. For these topics we refer the reader to [35], [4], [40] and others. Herein, we focus on the NUPBR concept instead for two main reasons. Firstly, the NUPBR is the non-arbitrage concept that is intimately related to the weakest forms of markets' viability (see [9] and [32] for details about this issue, and [1] for many examples of market models violating NFLVR and fulfilling NUPBR). Secondly, due to [41] and again [9], the NUPBR is mathematically very attractive and possesses the 'dynamic/localization' feature that the NFLVR and others arbitrage concepts lack to possess. By localization feature, we mean that if the property holds locally (i.e. the property holds for the stopped models with a sequence of stopping times that increases to infinity), then it holds globally.
Precisely, in this paper we address the NUPBR in market models with two groups of agents, where one group receives extra information dynamically over time. This extra information is principally generated by a random time that can model retirement time, death time, or any time of event that affects the market and the agents' behaviors. This random time is not adapted to the flow of "public information" that the other group of investors get through time. We suppose that the market model endowed with public information is arbitrage free, and we investigate which models that preserve the NUPBR condition. Within this framework of informational markets, we consider two main situ-

ations. The first situation consists of the random stopping/horizon context, where we study how the NUPBR is preserved under this random stopping. The second situation lies when one completely incorporates a *precise random time* called "honest" time. This precise choice of random time is due to the fact that honest times are among the random times that preserve the semimartingale structure when incorporated totally into the model.

This paper is organized as follows. The next section (Section 2) introduces the mathematical model as well preliminary results, most of them being established in the literature except the results of Subsection 2.4 which sounds new to us. Section 3 focuses on the effect of random horizon on the NUPBR, while Section 4 deals with a class of honest times.

2 Mathematical Framework and Preliminary Results

This section presents the framework that we will work on. We start with a given filtered probability space on which we consider a market model that will be mathematically represented by a d-dimensional semimartingale. This describes the basic market model. Then, our goal is to incorporate a new information, modeled as a random time that will be fixed for the entire paper. This random time can be a default time, a retirement date, the occurrence of a political or economic event, or a change of regime date. The knowledge of that information will change the mathematical structures of the processes in the basic market. We will detail this framework and all its stochastic tools in the first subsection of this section (Subsection 2.1). Then Subsection 2.2 will present some preliminary results as well the existing results on progressive enlargement.

2.1 The Mathematical Framework

We consider a stochastic basis $(\Omega, \mathcal{G}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, P)$, where $\mathbb{F}$ is a filtration satisfying the usual hypotheses (i.e., right continuity and completeness), and $\mathcal{F}_\infty \subseteq \mathcal{G}$. Financially speaking, the filtration $\mathbb{F}$ represents the flow of public information through time. Throughout the paper, a financial market will be represented by a d-dimensional semimartingale S, defined on the above filtered probability space, with decomposition $S = S_0 + M^S + A^S$ where M^S is a local martingale and A^S a predictable process with finite variation. The riskless asset is supposed to have a constant price.

In what follows, $\mathbb{H}$ is a filtration satisfying the usual hypotheses. The set of martingales for the filtration $\mathbb{H}$ will be denoted by $\mathcal{M}(\mathbb{H})$. As usual, $\mathcal{A}^+(\mathbb{H})$ denotes the set of increasing, right-continuous, $\mathbb{H}$-adapted and integrable processes.
If $\mathcal{C}(\mathbb{H})$ is a class of $\mathbb{H}$ adapted processes, we denote by $\mathcal{C}_0(\mathbb{H})$ the set of processes $X \in \mathcal{C}(\mathbb{H})$ with $X_0 = 0$, and by $\mathcal{C}_{loc}$ the set of processes X such that

there exists a sequence $(T_n)_{n\geq 1}$ of $\mathbb{H}$-stopping times that increases to $+\infty$ and the stopped process X^{T_n} belongs to $\mathcal{C}(\mathbb{H})$. We put $\mathcal{C}_{0,loc} = \mathcal{C}_0 \cap \mathcal{C}_{loc}$.
For a process K with $\mathbb{H}$-locally integrable variation, we denote by $K^{o,\mathbb{H}}$ its dual optional projection. The dual predictable projection of K (also called the $\mathbb{H}$-predictable dual projection) is denoted $K^{p,\mathbb{H}}$. For a process X, we denote ${}^{o,\mathbb{H}}X$ (resp. ${}^{p,\mathbb{H}}X$) its optional (resp. predictable) projection with respect to $\mathbb{H}$. For an $\mathbb{H}$- semi-martingale Y, the set $L(Y,\mathbb{H})$ is the set of $\mathbb{H}$ predictable processes integrable w.r.t. Y and for $H \in L(Y,\mathbb{H})$, we denote $H \bullet Y_t := \int_0^t H_s dY_s$.

We recall the notion of non-arbitrage that will be addressed in this paper.

Definition 2.1 An $\mathbb{H}$-semimartingale S satisfies the *No-Unbounded-Profit-with-Bounded-Risk* condition under $(\mathbb{H}, Q)$ (hereafter called NUPBR$(\mathbb{H}, Q)$) if the set

$$\mathcal{K} := \Big\{ (H \bullet S)_\infty \ \Big| \quad H \in L(S,\mathbb{H}), \text{ and } H \bullet S \geq -1 \Big\}$$

is bounded in probability under Q. When $Q \sim P$, we simply write NUPBR$(\mathbb{H})$).

For more details about this type of arbitrage and other related non-arbitrage concepts we refer the reader to Delbaen and Schachermayer [20], Kabanov [29], Takaoka [41], and Karatzas and Kardaras[31]. It is obvious that the property NUPBR is stable under an equivalent change of probability, however its stability under stopping localization is not immediate. The NUPBR property is intimately related to the existence of σ-martingale density for the process S. Below, we recall the definition of σ-martingale and σ-martingale density for a process.

Definition 2.2 An $\mathbb{H}$-adapted process X is called an $(\mathbb{H},\sigma)$-martingale if there exists an $\mathbb{H}$-predictable process ϕ such that

$$0 < \phi \leq 1, \quad \text{and } \phi \bullet X \text{ is an } \mathbb{H}\text{-martingale.}$$

If S is $\mathbb{H}$-adapted, we call $(\mathbb{H},\sigma)$-martingale density for S, any positive $\mathbb{H}$-local martingale L such that SL is an $(\mathbb{H},\sigma)$-martingale. The set of all $(\mathbb{H},\sigma)$-martingale densities for S is denoted by

$$\mathcal{L}_\sigma(\mathbb{H}, S) := \big\{ L \in \mathcal{M}_{loc}(\mathbb{H}) \big|\ L > 0, \quad LS \ \text{ is an } (\mathbb{H},\sigma)\text{-martingale } \big\}.$$

The relationship between NUPBR$(\mathbb{H})$ for a process S and the set $\mathcal{L}_\sigma(\mathbb{H}, S)$ is established in full generality by Takaoka[41]. Below we recall this important result.

Theorem 2.3 *[41] Let S be an $\mathbb{H}$-adapted process. Then, S satisfies NUPBR $(\mathbb{H})$ if and only if $\mathcal{L}_\sigma(\mathbb{H}, S) \neq \emptyset$.*

It will be useful to note that $\mathcal{L}_\sigma(\mathbb{H}, S) \neq \emptyset$ if and only if there exist $L \in \mathcal{M}_{loc}(\mathbb{H})$, $L > 0$ and $0 < \phi \leq 1$ such that $L(\phi \bullet S)$ is an $\mathbb{H}$-local martingale.

Beside any basic model that is represented by $(\Omega, \mathbb{F}, P, S)$, we consider a finite random time τ (arbitrary but fixed throughout the paper). To this random time we associate the process D and the filtration $\mathbb{G}$ given by

$$D := I_{[\![\tau,+\infty[\![}, \quad \mathbb{G} = (\mathcal{G}_t)_{t\geq 0}\,, \quad \mathcal{G}_t = \bigcap_{s>t}\Big(\mathcal{F}_s \vee \sigma(D_u, u \leq s)\Big).$$

The filtration $\mathbb{G}$ is the smallest right-continuous filtration which contains $\mathbb{F}$ and makes τ a stopping time. In the probabilistic literature, $\mathbb{G}$ is called the progressive enlargement of $\mathbb{F}$ with τ. In addition to $\mathbb{G}$ and D, we associate to τ two important $\mathbb{F}$-supermartingales given by

$$Z_t := P\left(\tau > t \mid \mathcal{F}_t\right) \text{ and } \quad \widetilde{Z}_t := P\left(\tau \geq t \;\middle|\; \mathcal{F}_t\right). \tag{2.1}$$

The supermartingale Z is right-continuous with left limits and coincides with the optional projection of $I_{[\![0,\tau[\![}$, while $\widetilde{Z}$ admits right limits and left limits only and is the optional projection of $I_{[\![0,\tau]\!]}$. The decomposition of Z leads to an important martingale that we denote by m, given by

$$m := Z + D^{o,\mathbb{F}}, \tag{2.2}$$

where $D^{o,\mathbb{F}}$ is the $\mathbb{F}$-dual optional projection of $D = I_{[\![\tau,\infty[\![}$.
It is proved in Yor [42, Lemme 0] that Z and Z_- are strictly positive on $[\![0,\tau[\![$. We define, for future use, the quasi-left-continuous $\mathbb{F}$-martingale $m^{(qc)}$ obtained by taking away all the predictable jumps (σ_n) of m and the set $\Gamma := \bigcup_{n=1}^{+\infty}[\![\sigma_n]\!]$, so that

$$m^{(qc)} = m - I_\Gamma \bullet m\,. \tag{2.3}$$

Remark 2.4 The decomposition $Z = m - D^{o,\mathbb{F}}$ is, in general, different from the Doob-Meyer decomposition $Z = M^Z - D^{p,\mathbb{F}}$ where M^Z is a martingale and $D^{p,\mathbb{F}}$ is with finite variation and predictable. If τ avoids the $\mathbb{F}$-stopping times, or if all $\mathbb{F}$ martingales are continuous, then $D^{p,\mathbb{F}} = D^{o,\mathbb{F}}$ and $Z = m - D^{o,\mathbb{F}}$ is the Doob-Meyer decomposition of Z. In this paper, we are interested with general filtration, and our class of honest times (see the definition below if needed) do not avoid $\mathbb{F}$ stopping times. Here is a simple example: let N be a standard Poisson process with intensity equal to one, and $\mu > 1$. Then, for $a > 0$, the time $\tau = \sup\{t : \mu t - N_t \geq a\}$ is a finite honest time; moreover $P(\tau = S_n) \neq 0$ where $(S_n, n \geq 1)$ is the sequence of successive hitting times of the level a by the process $(\mu t - N_t, t \geq 0)$.

In the second part of this paper, we shall deal with a particular class of random times, called honest times.

Definition 2.5 A random time τ is honest, if for any t, there exists an $\mathcal{F}_t$ measurable r.v. τ_t such that $\tau I_{\{\tau<t\}} = \tau_t I_{\{\tau<t\}}$.

We refer to Jeulin [28, Chapter 4] for more information on honest times. It will be useful to remember that, if an honest time τ avoids $\mathbb{F}$-stopping times, then $Z_\tau = 1$ [39].

2.2 Progressive Enlargement

To distinguish the effect of filtration, we will denote $\langle .,. \rangle^{\mathbb{F}}$, or $\langle .,. \rangle^{\mathbb{G}}$ to specify the sharp bracket (predictable covariation process) calculated in the filtration $\mathbb{F}$ or $\mathbb{G}$, if confusion may rise.

We recall that, for general semi-martingales X and Y, the sharp bracket is (if it exists) the dual predictable projection of the covariation process $[X,Y]$.

To any $\mathbb{F}$-local martingale M, we associate the stopped process

$$\widehat{M^\tau_t} := M^\tau_t - \int_0^{t\wedge\tau} \frac{d\langle M,m\rangle^{\mathbb{F}}_s}{Z_{s-}}, \tag{2.4}$$

and if τ is honest, we define

$$\widehat{M_t} := M_t - \int_0^{t\wedge\tau} \frac{d\langle M,m\rangle^{\mathbb{F}}_s}{Z_{s-}} + \int_\tau^t \frac{d\langle M,m\rangle^{\mathbb{F}}_s}{1-Z_{s-}}. \tag{2.5}$$

We recall standard results on progressive enlargement of filtration (we refer the reader to [6,14,28] for proofs).

Lemma 2.6 *Let M be an $\mathbb{F}$-local martingale.*
(a) *For any random time τ, the process $\widehat{M^\tau}$ is a $\mathbb{G}$-local martingale.*
(b) *If τ is honest, the process $\widehat{M}$ is a $\mathbb{G}$-local martingale. In particular, the $\mathbb{G}$-canonical representation of S is given by*

$$S = S_0 + \widehat{M^S} + \widehat{A^S},$$

where

$$\widehat{M^S} := M^S - \int_0^{\cdot\wedge\tau} \frac{d\langle M^S,m\rangle^{\mathbb{F}}_s}{Z_{s-}} + \int_\tau^{\cdot} \frac{d\langle M^S,m\rangle^{\mathbb{F}}_s}{1-Z_{s-}},$$
$$\widehat{A^S} := A^S + \int_0^{\cdot\wedge\tau} \frac{d\langle M^S,m\rangle^{\mathbb{F}}_s}{Z_{s-}} - \int_\tau^{\cdot} \frac{d\langle M^S,m\rangle^{\mathbb{F}}_s}{1-Z_{s-}}.$$

2.3 Optional Stochastic Integral

In the general case addressed herein, the notion of compensated stochastic integral (or optional stochastic integral), defined in [26] (Chapter III.4.b p. 106-109) and also studied in [13] (Chapter VIII.2 sections 32-35 p. 356-361) is crucial.

Definition 2.7 (see [26], Definition (3.80)) Let N be an $\mathbb{H}$-local martingale with continuous martingale part N^c, and let H be an $\mathbb{H}$-optional process.
i) The process H is said to be integrable with respect to N if ${}^{p,\mathbb{H}}H$ is N^c integrable and the process

$$\left(\sum_{s\le t} \left(H_s \Delta N_s - {}^{p,\mathbb{H}}(H\Delta N)_s \right)^2 \right)^{1/2}$$

is locally integrable. The set of integrable processes with respect to N is denoted by ${}^{o}L^{1}_{loc}(N,\mathbb{H})$.
ii) For $H \in {}^{o}L^{1}_{loc}(N,\mathbb{H})$, the compensated stochastic integral of H with respect to N, denoted by $H \odot N$, is the unique local martingale M satisfies

$$M^c = {}^{p,\mathbb{H}}H \bullet N^c \quad \text{and} \quad \Delta M = H\Delta N - {}^{p,\mathbb{H}}(H\Delta N). \tag{2.6}$$

We recall some important properties of the compensated stochastic integral:

Proposition 2.8 *(see [13])* (a) *The compensated stochastic integral $M = H \odot N$ is the unique $\mathbb{H}$-local martingale such that, for any $\mathbb{H}$-local martingale Y,*

$$\mathbb{E}\left([M,Y]_\infty\right) = \mathbb{E}\left(\int_0^\infty H_s d[N,Y]_s\right).$$

(b) *The process $[M,Y] - H \bullet [N,Y]$ is an $\mathbb{H}$-local martingale. As a result $[M,Y] \in \mathcal{A}_{loc}(\mathbb{H})$ if and only if $H \bullet [N,Y] \in \mathcal{A}_{loc}(\mathbb{H})$ and in this case we have*

$$\langle M,Y\rangle^{\mathbb{H}} = (H \bullet [N,Y])^{p,\mathbb{H}}.$$

2.4 Some New Preliminary Results

In this subsection, we develop easy results that play important roles in simplifying proofs throughout the paper. Most of these new results deal with the NUPBR condition.

Proposition 2.9 *Let S be an $\mathbb{H}$ adapted process. Then, the following assertions are equivalent.*
(a) *There exists a sequence of $\mathbb{H}$-stopping times that increases to $+\infty$, $(T_n)_{n\geq 1}$, such that for each $n \geq 1$, there exists a probability Q_n on (Ω, F_{T_n}) such that $Q_n \sim P$ and S^{T_n} satisfies the NUPBR($\mathbb{H}$) under Q_n.*
(b) *S satisfies the NUPBR($\mathbb{H}$).*
(c) *There exists a $\mathbb{H}$-predictable process ϕ, such that $0 < \phi \leq 1$ and $(\phi \bullet S)$ satisfies the NUPBR($\mathbb{H}$).*

Proof The proof for (a)$\Longleftrightarrow$(b) follows from the stability of NUPBR under localisation which is due to [41] (see also Choulli/Deng/Ma [8] for further discussion about this issue), and the fact that the NUPBR is stable under any equivalent probability change.
In order to prove (b)$\Longrightarrow$(c), we suppose that assertion (b) holds. Then Theorem 2.3 implies the existence of an $\mathbb{H}$-predictable process ψ and a positive $\mathbb{H}$-local martingale $Z = \mathcal{E}(N)$ such that $Z(\psi \bullet S)$ is a local martingale. This latter fact is equivalent to $\psi \bullet S + [\psi \bullet S, N]$ is a local martingale. Then for any bounded and predictable process ϕ, the process $\phi\psi \bullet S + [\phi\psi \bullet S, N]$ is a local martingale. Hence we deduce that $Z(\phi\psi \bullet S)$ is a local martingale and Z is a σ-martingale density for $\phi \bullet S$ for any predictable and bounded process ϕ. This proves (b)$\Longrightarrow$(c). To prove the reverse, we assume that (c) holds. Then again

Theorem 2.3 implies the existence of a real-valued and $\mathbb{H}$-predictable process ψ such that $0 < \psi \leq 1$ and a positive $\mathbb{H}$-local martingale $Z = \mathcal{E}(N)$ such that $Z(\psi\phi \bullet S)$ is a local martingale. Since $\psi\phi$ is predictable and $0 < \psi\phi \leq 1$, we deduce that S satisfies the NUPBR. This ends the proof of the proposition.

Lemma 2.10 *Let S be an $\mathbb{H}$-predictable process with finite variation. Then S satisfies NUPBR($\mathbb{H}$) if and only if $S \equiv S_0$ (i.e. the process S is constant).*

Proof It is obvious that if $S \equiv S_0$, then S satisfies NUPBR($\mathbb{H}$). Suppose that S satisfies NUPBR($\mathbb{H}$). Consider $Y \in \mathcal{L}_\sigma(\mathbb{H}, S)$, and an $\mathbb{H}$-predictable process ϕ such that $0 < \phi \leq 1$ and $Y(\phi \bullet S)$ is a local martingale. Let $(T_n)_{n\geq 1}$ be a sequence of $\mathbb{H}$-stopping times that increases to $+\infty$ such that Y^{T_n} and $Y^{T_n}(\phi \bullet S)^{T_n}$ are true martingales. Then, for each $n \geq 1$, put

$$Q_n := \frac{Y_{T_n}}{Y_0} \cdot P.$$

Since S is predictable, then $(\phi \bullet S)^{T_n}$ is also predictable with finite variation and is a Q_n-martingale. Thus, we deduce that $(\phi \bullet S)^{T_n} \equiv 0$ for each $n \geq 1$. Therefore, we deduce that S is constant (since $S^{T_n} - S_0 = \phi^{-1} \bullet (\phi \bullet S)^{T_n} \equiv 0$). This ends the proof of the lemma. □

Definition 2.11 Consider an $\mathbb{H}$-semimartingale S. Then, S is said to admit an $\mathbb{H}$-deflator if there exists a positive $\mathbb{H}$-supermartingale L such that $L(\theta \bullet S)$ is a supermartingale, for any $\theta \in L(S, \mathbb{H})$ such that $\theta \bullet S \geq -1$.

Proposition 2.12 *Let S be an $\mathbb{H}$-semimartingale. Then, the following assertions are equivalent.*
(a) *S admits an $\mathbb{H}$-deflator.*
(b) *S satisfies NUPBR($\mathbb{H}$).*

Proof Thanks to Theorem 2.3, it is obvious that (b)$\Longrightarrow$(a). To prove the reverse, using Proposition 2.9, it is enough to prove it on each finite time interval $[0, T]$. Suppose that assertion (a) holds, and consider an $\mathbb{H}$-deflator L. Thus for any $\epsilon > 0$ any $c > 0$, and any $\theta \in \Theta := \{\theta \in L(S) \mid 1 + \theta \bullet S \geq 0\}$, we calculate

$$\begin{aligned} P(\theta \bullet S_T > c) &\leq P(L_T \leq \epsilon) + P(\theta \bullet S_T > c \ \& \ L_T > \epsilon) \\ &\leq P(L_T \leq \epsilon) + \frac{1}{\epsilon(1+c)} E\left(L_T(1 + (\theta \bullet S)_T)\right) \\ &\leq P(L_T \leq \epsilon) + \frac{1}{\epsilon(1+c)}. \end{aligned}$$

Then by taking the supremum over all such θ and letting c go to infinity, we get for any $\epsilon > 0$

$$\lim_{c \longrightarrow +\infty} \sup_{\theta \in \Theta} P(\theta \bullet S_T > c) \leq P(L_T \leq \epsilon).$$

This proves that $\{(\theta \bullet S)_T \mid \theta \in \Theta\}$ is bounded in probability. This ends the proof of the proposition. □

Lemma 2.13 *Define* $\widetilde{\mathcal{P}}(\mathbb{G}) := \mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(\mathbb{R}^d)$ *and* $\widetilde{\mathcal{P}}(\mathbb{F}) := \mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^d)$ *where* $\mathcal{P}(\mathbb{H})$ *is the* $\mathbb{H}$*-predictable* σ*-algebra for* $\mathbb{H} \in \{\mathbb{F}, \mathbb{G}\}$ *and* $\mathcal{B}(\mathbb{R}^d)$ *is the Borel* σ*-field on* $\mathbb{R}^d$*. Then the following two properties hold:*
(a) *For any predictable functional* $H^{\mathbb{G}} \in \widetilde{\mathcal{P}}(\mathbb{G})$*, there exist an* $\widetilde{\mathcal{P}}(\mathbb{F})$*-measurable functional* $H^{\mathbb{F}}$ *and a* $\mathcal{B}(\mathbb{R}_+) \otimes \widetilde{\mathcal{P}}(\mathbb{F})$*-measurable functionals* $K^{\mathbb{F}} : \mathbb{R}_+ \times \mathbb{R}_+ \times \Omega \times \mathbb{R}^d \to \mathbb{R}$ *such that*

$$H^{\mathbb{G}}(\omega, t, x) = H^{\mathbb{F}}(\omega, t, x) I_{[\![0,\tau]\!]} + K^{\mathbb{F}}(\tau(\omega), t, \omega, x) I_{[\![\tau,+\infty]\!]}. \tag{2.7}$$

If furthermore $H^{\mathbb{G}} > 0$*, then we can choose* $H^{\mathbb{F}} > 0$ *such that*

$$H^{\mathbb{G}}(\omega, t, x) I_{[\![0,\tau]\!]} = H^{\mathbb{F}}(\omega, t, x) I_{[\![0,\tau]\!]}.$$

(b) *If* τ *is an* $\mathbb{F}$*-honest time, for any predictable functional* $H^{\mathbb{G}} \in \widetilde{\mathcal{P}}(\mathbb{G})$*, there exist two* $\widetilde{\mathcal{P}}(\mathbb{F})$*-measurable functionals* $H^{\mathbb{F}}$ *and* $K^{\mathbb{F}}$ *such that*

$$H^{\mathbb{G}} = H^{\mathbb{F}} I_{[\![0,\tau]\!]} + K^{\mathbb{F}} I_{[\![\tau,+\infty]\!]}. \tag{2.8}$$

If furthermore $H^{\mathbb{G}} > 0$*, then we can choose both* $H^{\mathbb{F}}$ *and* $K^{\mathbb{F}}$ *positive such that (2.8) holds.*

Proof We are mimicking the approach of Jeulin[28]. Notice that $\widetilde{\mathcal{P}}(\mathbb{G})$ is generated by the process of the form

$$h^{\mathbb{G}}(\omega, t, x) := \alpha\left(s \wedge \tau(\omega)\right) f_s(\omega) g(x) I_{\{s<t\}},$$

where α and g are Borel measurable functions and $f_s \in \mathcal{F}_s$. Now, by taking

$$\begin{aligned} h^{\mathbb{F}}(\omega, t, x) &:= \alpha\left(s\right) f_s(\omega) g(x) I_{\{s<t\}} \in \widetilde{\mathcal{P}}(\mathbb{F}) \quad \text{and} \\ k^{\mathbb{F}}(y, t, \omega, x) &:= \alpha\left(y \wedge s\right) f_s(\omega) g(x) I_{\{s<t\}} \in \mathcal{B}(R_+) \otimes \widetilde{\mathcal{P}}(\mathbb{F}) \end{aligned}$$

we have

$$h^{\mathbb{G}}(\omega, t, x) = h^{\mathbb{F}} I_{[\![0,\tau]\!]} + k^{\mathbb{F}}(\tau(\omega), t, \omega, x) I_{[\![\tau,+\infty[\![}$$

Therefore, (2.7) follows immediately. To prove the positivity of $H^{\mathbb{F}}$ when $H^{\mathbb{G}} > 0$, we consider

$$\overline{H}^{\mathbb{F}} := (H^{\mathbb{F}})^+ + I_{\{H^{\mathbb{F}}=0\}} > 0,$$

and we remark that due to (2.7), we have $]\!]0, \tau]\!] \subset \{H^{\mathbb{G}} = H^{\mathbb{F}}\} \subset \{H^{\mathbb{F}} > 0\}$. Thus, we get

$$H^{\mathbb{G}} I_{[\![0,\tau]\!]} = \overline{H}^{\mathbb{F}} I_{[\![0,\tau]\!]}.$$

This ends the proof of assertion (a). The proof of assertion (b) follows the same arguments (see Jeulin [28, p.73]). This ends the proof of the proposition. □

3 Non-Arbitrage under Random Horizon

This section addresses the NUPBR concept under random horizon, and contains four subsections. The first subsection outlines some new stochastic developments which are summarized in two lemmas (Lemmas 3.2-3.3) and two propositions (Propositions 3.4-3.5). The main results of this section rely deeply on these results. The second subsection proves the stability of the NUPBR under any random stopping for processes that are quasi-left-continuous. The third subsection focuses on pure jump processes that jump at predictable times only, while the last subsection combines these two cases and elaborates a summary results for the general framework. This separation of cases sounds important to us for two reasons. The first reason lies in the sake of simplicity of exposition. Indeed, the optional stochastic integral for quasi-left-continuous processes becomes much simpler and less technical. Our second reason resides in the sake of elaborating more precise results and highlights the precise source of difficulties when one deals with random horizon/stopping.

3.1 The Dual Predictable Projections under $\mathbb{G}$ and $\mathbb{F}$ before τ

In this subsection, we derive some innovative results that establish the relationship between the $\mathbb{G}$–(dual) predictable projections and the $\mathbb{F}$-(dual) predictable projections. To this end, we start with a lemma that recalls some results and defines some notations related to the random time τ. Most of the results of this lemma can be found in the literature [42].

Lemma 3.1 *Let Z and $\widetilde{Z}$ be the two supermartingales given by (2.1).*
(a) *The three sets $\{\widetilde{Z}=0\}$, $\{Z=0\}$ and $\{Z_-=0\}$ have the same début which is an $\mathbb{F}$-stopping time that we denote by*

$$\widehat{R} := \inf\{t \geq 0 \big|\ Z_{t-}=0\}. \tag{3.1}$$

(b) *The following stopping times*

$$\widehat{R}_0 := \begin{cases} \widehat{R} & \text{on } \{Z_{\widehat{R}-}=0\} \\ +\infty & \text{otherwise} \end{cases} \qquad \text{and} \qquad \widetilde{R}_0 := \begin{cases} \widehat{R} & \text{on } \{\widetilde{Z}_{\widehat{R}}=0\} \\ +\infty & \text{otherwise} \end{cases}$$

are such that $\widehat{R}_0$ is a predictable stopping time, and

$$\tau \leq \widehat{R}, \qquad \tau < \widetilde{R}_0, \qquad \widetilde{Z}_\tau > 0, \quad Z_{\tau-} > 0, \quad P-a.s. \tag{3.2}$$

(c) *The $\mathbb{G}$-predictable process*

$$H_t := (Z_{t-})^{-1}\, I_{[\![0,\tau]\!]}(t), \tag{3.3}$$

is $\mathbb{G}$-locally bounded.
(d) *We have $Z_- + \Delta m = \widetilde{Z}$.*

Proof From [14, paragraph 14, Chapter XX], for any random time τ, the sets $\{\widetilde{Z} = 0\}$ and $\{Z_- = 0\}$ are disjoint from $]\!]0,\tau]\!]$ and have the same lower bound $\widehat{R}$, the smallest $\mathbb{F}$-stopping time greater than τ. Thus, we also conclude that $\{Z = 0\}$ is disjoint from $]\!]0,\tau[\![$. This leads to assertion (a). The process $X := Z^{-1} I_{]\!]0,\tau[\![}$ being a càdlàg $\mathbb{G}$-supermartingale [42], its left limit is locally bounded. Then, due to

$$(Z_-)^{-1} I_{]\!]0,\tau]\!]} = X_-,$$

the local boundedness of H follows. The last assertion (d) can be derived from

$$\Delta D^{o,\mathbb{F}} = {}^{o,\mathbb{F}}(\Delta D) = \widetilde{Z} - Z \text{ and } \Delta m = \Delta Z + \Delta D^{o,\mathbb{F}}.$$

This ends the proof of the lemma. □

In the following lemma, we will express the $\mathbb{G}$–dual predictable projection in terms of the $\mathbb{F}$–dual predictable projection.

Lemma 3.2 *The following assertions hold.*
(a) *For any $\mathbb{F}$-adapted process V with locally integrable variation, we have*

$$(V^\tau)^{p,\mathbb{G}} = (Z_-)^{-1} I_{]\!]0,\tau]\!]} \cdot \left(\widetilde{Z} \cdot V\right)^{p,\mathbb{F}}. \tag{3.4}$$

(b) *For any $\mathbb{F}$-local martingale M, we have, on $[\![0,\tau]\!]$*

$${}^{p,\mathbb{G}}\left(\frac{\Delta M}{\widetilde{Z}}\right) = \frac{{}^{p,\mathbb{F}}\left(\Delta M I_{\{\widetilde{Z}>0\}}\right)}{Z_-}, \quad \textit{and} \quad {}^{p,\mathbb{G}}\left(\frac{1}{\widetilde{Z}}\right) = \frac{{}^{p,\mathbb{F}}\left(I_{\{\widetilde{Z}>0\}}\right)}{Z_-}. \tag{3.5}$$

(c) *For any quasi-left-continuous $\mathbb{F}$-local martingale M, we have, on $[\![0,\tau]\!]$*

$${}^{p,\mathbb{G}}\left(\frac{\Delta M}{\widetilde{Z}}\right) = 0, \quad \textit{and} \quad {}^{p,\mathbb{G}}\left(\frac{1}{Z_- + \Delta m^{(qc)}}\right) = \frac{1}{Z_-}, \tag{3.6}$$

where $m^{(qc)}$ is the quasi-left-continuous $\mathbb{F}$-martingale defined in (2.3).

Proof (a) In terms of (3.3), the equality (2.4) takes the form

$$M^\tau = \widehat{M^\tau} + H I_{]\!]0,\tau]\!]} \cdot \langle M, m\rangle^{\mathbb{F}}.$$

By taking $M = V - V^{p,\mathbb{F}}$, we obtain

$$V^\tau = I_{]\!]0,\tau]\!]} \cdot V^{p,\mathbb{F}} + \widehat{M^\tau} + H I_{]\!]0,\tau]\!]} \cdot \langle V, m\rangle^{\mathbb{F}} = \widehat{M^\tau} + I_{]\!]0,\tau]\!]} \cdot V^{p,\mathbb{F}} + \frac{1}{Z_-} I_{]\!]0,\tau]\!]} \cdot (\Delta m \cdot V)^{p,\mathbb{F}},$$

which proves assertion (a).
(b) Let M be an $\mathbb{F}$-local martingale, then, for any positive integers (n,k) the process $V^{(n,k)} := \sum \frac{\Delta M}{\widetilde{Z}} I_{\{|\Delta M| \geq k^{-1},\ \widetilde{Z} \geq n^{-1}\}}$ has a locally integrable variation. Then, by using the known equality ${}^{p,\mathbb{G}}(\Delta V) = \Delta(V^{p,\mathbb{G}})$ (see Theorem 76 in

pages 149–150 of [13] or Theorem 5.27 in page 150 of [24]), and applying assertion (a) to the process $V^{(n,k)}$, we get, on $]\!]0,\tau]\!]$

$$ {}^{p,\mathbb{G}}\Big(\Delta V^{(n,k)}\Big) = {}^{p,\mathbb{G}}\Big(\frac{\Delta M}{\widetilde{Z}} I_{\{|\Delta M|\geq k^{-1},\ \widetilde{Z}\geq n^{-1}\}}\Big) = \frac{1}{Z_-}\ {}^{p,\mathbb{F}}\Big(\Delta M I_{\{|\Delta M|\geq k^{-1},\ \widetilde{Z}\geq n^{-1}\}}\Big).$$

Since M is a local martingale, then by stopping we can exchange limits with projections in both sides. Then by letting n and k go to infinity, and using the fact that $\widetilde{Z} > 0$ on $]\!]0,\tau]\!]$, we deduce that

$$ {}^{p,\mathbb{G}}\Big(\frac{\Delta M}{\widetilde{Z}}\Big) = \frac{1}{Z_-}\ {}^{p,\mathbb{F}}\Big(\Delta M I_{\{\widetilde{Z}>0\}}\Big).$$

This proves the first equality in (3.5), while the second equality follows from $\widetilde{Z} = \Delta m + Z_-$:

$$\begin{aligned} Z_-\ {}^{p,\mathbb{G}}\Big(\widetilde{Z}^{-1}\Big) &= {}^{p,\mathbb{G}}\Big((\widetilde{Z}-\Delta m)/\widetilde{Z}\Big) = 1 - {}^{p,\mathbb{G}}\Big(\Delta m/\widetilde{Z}\Big) \\ &= 1 - (Z_-)^{-1}\ {}^{p,\mathbb{F}}\Big(\Delta m I_{\{\widetilde{Z}>0\}}\Big) = 1 - {}^{p,\mathbb{F}}\Big(I_{\{\widetilde{Z}=0\}}\Big) = {}^{p,\mathbb{F}}\Big(I_{\{\widetilde{Z}>0\}}\Big). \end{aligned}$$

In the above string of equalities, the third equality follows from the first equality of (3.5), while the fourth equality is due to ${}^{p,\mathbb{F}}(\Delta m) = 0$ and $\Delta m I_{\{\widetilde{Z}=0\}} = -Z_- I_{\{\widetilde{Z}=0\}}$. This ends the proof of assertion (b).

(c) If M is a quasi-left-continuous $\mathbb{F}$-local martingale, then ${}^{p,\mathbb{F}}\Big(\Delta M I_{\{\widetilde{Z}>0\}}\Big) = 0$, and the first property of the assertion (c) follows. The second property of assertion (c) follows from the first property applied to $M = m^{(qc)}$ which is quasi-left-continuous. Indeed, on $]\!]0,\tau]\!]$, one has $\Delta m^{(qc)}\,(Z_- + \Delta m)^{-1} = \Delta m^{(qc)}\left(Z_- + \Delta m^{(qc)}\right)^{-1}$, and

$$\frac{1}{Z_-}\ {}^{p,\mathbb{G}}\Big(\frac{Z_-}{Z_- + \Delta m^{(qc)}}\Big) = \frac{1}{Z_-}\left(1 - {}^{p,\mathbb{G}}\Big(\frac{\Delta m^{(qc)}}{Z_- + \Delta m^{(qc)}}\Big)\right) = \frac{1}{Z_-}.$$

This proves assertion (c), and the proof of the lemma is achieved. □

The next lemma proves that $\widetilde{Z}^{-1} I_{]\!]0,\tau]\!]}$ is Lebesgue-Stieljes-integrable with respect to any process that is $\mathbb{F}$-adapted with $\mathbb{F}$-locally integrable variation. Using this fact, the lemma addresses the question of how an $\mathbb{F}$-compensator stopped at τ can be written in terms of a $\mathbb{G}$-compensator, and constitutes a *sort of* converse result to Lemma 3.2–(a).

Lemma 3.3 *Let V be an $\mathbb{F}$-adapted càdlàg process. Then the following properties hold.*
(a) *If V belongs to $\mathcal{A}^+_{loc}(\mathbb{F})$ (respectively $V \in \mathcal{A}^+(\mathbb{F})$), then the process*

$$U := \widetilde{Z}^{-1} I_{]\!]0,\tau]\!]} \bullet V, \tag{3.7}$$

belongs to $\mathcal{A}^+_{loc}(\mathbb{G})$ *(respectively to* $\mathcal{A}^+(\mathbb{G})$*).*
(b) *If* V *has* $\mathbb{F}$*-locally integrable variation, then the process* U *is well defined, its variation is* $\mathbb{G}$*-locally integrable, and its* $\mathbb{G}$*-dual predictable projection is given by*

$$U^{p,\mathbb{G}} = \left(\frac{1}{\widetilde{Z}} I_{]\!]0,\tau]\!]} \cdot V\right)^{p,\mathbb{G}} = \frac{1}{Z_-} I_{]\!]0,\tau]\!]} \cdot \left(I_{\{\widetilde{Z}>0\}} \cdot V\right)^{p,\mathbb{F}}. \tag{3.8}$$

Proof (a) Suppose that $V \in \mathcal{A}^+_{loc}(\mathbb{F})$. First, remark that, due to the fact that $\widetilde{Z}$ is positive on $]\!]0,\tau]\!]$, U is well defined. Let $(\vartheta_n)_{n\geq 1}$ be a sequence of $\mathbb{F}$-stopping times that increases to $+\infty$ such that $E\left(V_{\vartheta_n}\right) < +\infty$. Then, if $E\left(U_{\vartheta_n}\right) \leq E\left(V_{\vartheta_n}\right)$, assertion (a) follows. Thus, we calculate

$$\begin{aligned} E\left(U_{\vartheta_n}\right) &= E\left(\int_0^{\vartheta_n} I_{\{0<t\leq\tau\}} \frac{1}{\widetilde{Z}_t} dV_t\right) = E\left(\int_0^{\vartheta_n} \frac{P(\tau \geq t|\mathcal{F}_t)}{\widetilde{Z}_t} I_{\{\widetilde{Z}_t>0\}} dV_t\right) \\ &\leq E\left(V_{\vartheta_n}\right). \end{aligned}$$

The last inequality is obtained due to $\widetilde{Z}_t := P(\tau \geq t|\mathcal{F}_t)$. This ends the proof of assertion (a) of the lemma.
(b) Suppose that $V \in \mathcal{A}_{loc}(\mathbb{F})$, and denote by $W := V^+ + V^-$ its variation. Then $W \in \mathcal{A}^+_{loc}(\mathbb{F})$, and a direct application of the first assertion implies that

$$\left(\widetilde{Z}\right)^{-1} I_{]\!]0,\tau]\!]} \cdot W \in \mathcal{A}^+_{loc}(\mathbb{G}).$$

As a result, we deduce that U given by (3.7) for the case of $V = V^+ - V^-$ is well defined and has variation equal to $\left(\widetilde{Z}\right)^{-1} I_{]\!]0,\tau]\!]} \cdot W$ which is $\mathbb{G}$-locally integrable. By setting $U_n := I_{]\!]0,\tau]\!]} \cdot \left(\widetilde{Z}^{-1} I_{\{\widetilde{Z}\geq 1/n\}} \cdot V\right)$, we derive, due to (3.4),

$$(U_n)^{p,\mathbb{G}} = \frac{1}{Z_-} I_{]\!]0,\tau]\!]} \cdot \left(I_{\{\widetilde{Z}\geq 1/n\}} \cdot V\right)^{p,\mathbb{F}}.$$

Hence, since $U^{p,\mathbb{G}} = \lim_{n\longrightarrow+\infty} (U_n)^{p,\mathbb{G}}$, by taking the limit in the above equality, (3.8) follows immediately, and the proof of the lemma is completed. □

In the remaining part of this subsection, we will elaborate two propositions that are vital for the principal results of the forthcoming subsections.

Proposition 3.4 *The following nondecreasing process*

$$V_t^{(b)} := \sum_{0\leq u\leq t} {}^{p,\mathbb{F}}\left(I_{\{\widetilde{Z}=0\}}\right)_u I_{\{u\leq\tau\}} \tag{3.9}$$

is $\mathbb{G}$*-predictable, càdlàg, and locally bounded.*

Proof The $\mathbb{G}$-predictability of $V^{(b)}$ is obvious. In the remaining part of the proof we will prove that this process is $\mathbb{G}$-locally bounded. Since $Z_-^{-1}I_{\rrbracket 0,\tau\rrbracket}$ is $\mathbb{G}$-locally bounded, then there exists a sequence of $\mathbb{G}$-stopping times $(\tau_n^{\mathbb{G}})_{n\geq 1}$ increasing to infinity such that

$$\left(\frac{1}{Z_-}I_{\rrbracket 0,\tau\rrbracket}\right)^{\tau_n^{\mathbb{G}}} \leq n+1.$$

Consider a sequence of $\mathbb{F}$-stopping times $(\sigma_n)_{n\geq 1}$ that increases to infinity such that $\langle m,m\rangle^{\mathbb{F}}_{\sigma_n} \leq n+1$. Then, for any nonnegative $\mathbb{F}$-predictable process H which is bounded by $C>0$, we calculate that

$$\begin{aligned}(H\bullet V^{(b)})_{\sigma_n\wedge\tau_n^{\mathbb{G}}} &= \sum_{0\leq u\leq \sigma_n\wedge\tau_n^{\mathbb{G}}} H_u\ {}^{p,\mathbb{F}}\left(I_{\{\widetilde{Z}=0\}}\right)_u I_{\{u\leq\tau\}}I_{\{Z_{u-}\geq\frac{1}{n+1}\}}\\ &\leq \sum_{0\leq u\leq \sigma_n} H_u\ {}^{p,\mathbb{F}}\left(I_{\{\Delta m\leq -\frac{1}{n+1}\}}\right)_u\\ &\leq (n+1)^2 H\bullet\langle m,m\rangle^{\mathbb{F}}_{\sigma_n} \leq C(n+1)^3.\end{aligned}$$

This ends the proof of the proposition. □

The next proposition proposes a $\mathbb{G}$-local martingale density candidate for a class of processes.

Proposition 3.5 *Consider the following $\mathbb{G}$-local martingale*

$$\widehat{m}^{(b)} := \frac{1}{Z_-}I_{\rrbracket 0,\tau\rrbracket}\bullet m - \frac{1}{Z_-^2}I_{\rrbracket 0,\tau\rrbracket}\bullet\langle m\rangle^{\mathbb{F}}, \tag{3.10}$$

and the space ${}^{o}L^1_{loc}(\widehat{m}^{(b)},\mathbb{G})$ introduced in Definition 2.7. Then, we have

$$K^{(b)} := \frac{Z_-^3}{Z_-^2+\Delta\langle m\rangle^{\mathbb{F}}}\frac{1}{\widetilde{Z}}I_{\rrbracket 0,\tau\rrbracket} \in\ {}^{o}L^1_{loc}(\widehat{m}^{(b)},\mathbb{G}). \tag{3.11}$$

Furthermore, the $\mathbb{G}$-local martingale

$$\widetilde{N}^{(b)} := -K^{(b)}\odot\widehat{m}^{(b)}, \tag{3.12}$$

satisfies the following
(a) $\mathcal{E}\left(\widetilde{N}^{(b)}\right)>0$ *(or equivalently $1+\Delta\widetilde{N}^{(b)}>0$).*
(b) *For any $M\in\mathcal{M}_{0,loc}(\mathbb{F})$, setting $M^{\mathbb{G}} := M^{\tau}-Z_-^{-1}I_{\rrbracket 0,\tau\rrbracket}\bullet\langle M,m\rangle^{\mathbb{F}}$, we have*

$$[\widetilde{N}^{(b)},M^{\mathbb{G}}]\in\mathcal{A}_{loc}(\mathbb{G}) \quad \left(i.e.\ \langle\widetilde{N}^{(b)},M^{\mathbb{G}}\rangle^{\mathbb{G}} \ \ exists\right). \tag{3.13}$$

Proof The proof of this proposition will be achieved in two steps. In the first step, which is the longest, we prove (3.11), while the second step will address the rest of the statements. For the sake of simplicity in notations, throughout this proof, we will use $\widetilde{N} := \widetilde{N}^{(b)}$, $K := K^{(b)}$ and $\kappa := Z_-^2 + \Delta\langle m\rangle^{\mathbb{F}}$.

Step 1: In this part, we prove $K \in {}^{o}L^1_{loc}(\widehat{m}^{(b)}, \mathbb{G})$. To this end, we start by calculating on $]\!]0,\tau]\!]$, making use of Lemma 3.2

$$
\begin{aligned}
K\Delta\widehat{m}^{(b)} - {}^{p,\mathbb{G}}\big(K\Delta\widehat{m}^{(b)}\big) &= \frac{I_{]\!]0,\tau]\!]} Z_-^3 \Delta\widehat{m}^{(b)}}{\kappa\widetilde{Z}} - {}^{p,\mathbb{G}}\left(\frac{I_{]\!]0,\tau]\!]} Z_-^3}{\kappa\widetilde{Z}}\,\Delta\widehat{m}^{(b)}\right) \\
&= \frac{(Z_-^2\Delta m - Z_-\Delta\langle m\rangle^{\mathbb{F}})}{\kappa\,\widetilde{Z}} + \frac{{}^{p,\mathbb{F}}(I_{\{\widetilde{Z}>0\}}\,\Delta\langle m\rangle^{\mathbb{F}})}{\kappa} - \frac{{}^{p,\mathbb{F}}(\Delta m I_{\{\widetilde{Z}>0\}})\,Z_-}{\kappa} \qquad (3.14)\\
&= \frac{\Delta m}{\widetilde{Z}} I_{]\!]0,\tau]\!]} - {}^{p,\mathbb{F}}\left(I_{\{\widetilde{Z}=0\}}\right) I_{]\!]0,\tau]\!]} =: \Delta V - \Delta V^{(b)}.
\end{aligned}
$$

Here, $V^{(b)}$ is defined in (3.9), which is nondecreasing, càdlàg and $\mathbb{G}$-locally bounded (see Proposition 3.4). Hence, we immediately deduce that $\sum(\Delta V^{(b)})^2 = \Delta V^{(b)} \bullet V^{(b)}$ is locally bounded, and in the rest of this part 1) we focus on proving $\sqrt{\sum(\Delta V)^2} \in \mathcal{A}^+_{loc}(\mathbb{G})$. To this end, we consider $\delta \in (0,1)$, and define $C := \{\Delta m < -\delta Z_-\}$ and C^c its complement in $\Omega \otimes [0,+\infty[$. Then we obtain

$$
\begin{aligned}
\sqrt{\sum(\Delta V)^2} &\le \left(\sum \frac{(\Delta m)^2}{\widetilde{Z}^2} I_C I_{]\!]0,\tau]\!]}\right)^{1/2} + \left(\sum \frac{(\Delta m)^2}{\widetilde{Z}^2} I_{C^c} I_{]\!]0,\tau]\!]}\right)^{1/2} \\
&\le \sum \frac{|\Delta m|}{\widetilde{Z}} I_C I_{]\!]0,\tau]\!]} + \frac{1}{1-\delta}\left(I_{]\!]0,\tau]\!]} \frac{1}{Z_-^2} \bullet [m]\right)^{1/2} =: V_1 + V_2.
\end{aligned}
$$

The last inequality above is due to $\sqrt{\sum(\Delta X)^2} \le \sum|\Delta X|$ and $\widetilde{Z} \ge Z_-(1-\delta)$ on C^c. Using the fact that $(Z_-)^{-1} I_{]\!]0,\tau]\!]}$ is $\mathbb{G}$-locally bounded and that m is an $\mathbb{F}$-locally bounded martingale, it follows that V_2 is $\mathbb{G}$-locally bounded. Hence, we focus on proving the $\mathbb{G}$-local integrability of V_1.

Consider a sequence of $\mathbb{G}$-stopping times $(\vartheta_n)_n$ that increases to $+\infty$ and

$$
\left((Z_-)^{-1} I_{]\!]0,\tau]\!]}\right)^{\vartheta_n} \le n.
$$

Also consider an $\mathbb{F}$-localizing sequence of stopping times, $(\tau_n)_n$, for the process $V_3 := \sum \frac{(\Delta m)^2}{1+|\Delta m|}$. Then, it is easy to prove

$$
U_n := \sum |\Delta m| I_{\{\Delta m < -\delta/n\}} \le \frac{n+\delta}{\delta} V_3,
$$

and conclude that $(U_n)^{\tau_n} \in \mathcal{A}^+(\mathbb{F})$. Therefore, due to

$$
\begin{aligned}
C \cap]\!]0,\tau]\!] \cap [\![0,\vartheta_n]\!] &= \{\Delta m < -\delta Z_-\} \cap]\!]0,\vartheta_n]\!] \cap]\!]0,\tau]\!] \\
&\subset]\!]0,\tau]\!] \cap]\!]0,\vartheta_n]\!] \cap \{\Delta m < -\tfrac{\delta}{n}\},
\end{aligned}
$$

we derive

$$(V_1)^{\vartheta_n\wedge\tau_n} \leq \Big(\widetilde{Z}\Big)^{-1} I_{]\!]0,\tau]\!]} \cdot (U_n)^{\tau_n}.$$

Since $(U_n)^{\tau_n}$ is $\mathbb{F}$-adapted, nondecreasing and integrable, then due to Lemma 3.3, we deduce that the process $V_1^{\vartheta_n\wedge\tau_n}$ is nondecreasing, $\mathbb{G}$-adapted and integrable. Since $\vartheta_n \wedge \tau_n$ increases to $+\infty$, we conclude that the process V_1 is $\mathbb{G}$-locally integrable. This completes the proof of $K \in {}^oL^1_{loc}(\widehat{m}^{(b)}, \mathbb{G})$, and the process $\widetilde{N}$ (given by (3.12)) is a well defined $\mathbb{G}$-local martingale by Definition 2.7.

Step 2: Herein, we shall prove assertion (a), then assertion (b). Due to (3.14), we have, on $]\!]0,\tau]\!]$,

$$-\Delta\widetilde{N} = K\Delta\widehat{m}^{(b)} - {}^{p,\mathbb{G}}\Big(K\Delta\widehat{m}^{(b)}\Big) = 1 - Z_-\Big(\widetilde{Z}\Big)^{-1} - {}^{p,\mathbb{F}}\Big(I_{\{\widetilde{Z}=0\}}\Big).$$

Thus, we deduce that $1 + \Delta\widetilde{N} > 0$, and assertion (a) is proved. In the rest of this proof, we will prove (3.13). To this end, let $M \in \mathcal{M}_{0,loc}(\mathbb{F})$ and put

$$\widehat{M} := M^{\mathbb{G}} := M^{\tau} - Z_-^{-1} I_{]\!]0,\tau]\!]} \cdot \langle M, m\rangle^{\mathbb{F}}.$$

Thanks to Proposition 2.8, (3.13) is equivalent to

$$K \cdot [\widehat{m}^{(b)}, \widehat{M}] \in \mathcal{A}_{loc}(\mathbb{G}) \text{ (or equivalently } V_4 := \frac{Z_-}{\widetilde{Z}} I_{]\!]0,\tau]\!]} \cdot [\widehat{m}^{(b)}, \widehat{M}] \in \mathcal{A}_{loc}(\mathbb{G})),$$

for any $M \in \mathcal{M}_{0,loc}(\mathbb{F})$. Then, it is easy to check that

$$\begin{aligned} V_4 &= \frac{Z_-}{\widetilde{Z}} I_{]\!]0,\tau]\!]} \cdot [\widehat{m}^{(b)}, \widehat{M}] = \frac{1}{\widetilde{Z}} I_{]\!]0,\tau]\!]} \cdot [m, \widehat{M}] - \frac{1}{Z_-\,\widetilde{Z}} I_{]\!]0,\tau]\!]} \cdot [\langle m\rangle^{\mathbb{F}}, \widehat{M}] \\ &= \frac{1}{\widetilde{Z}} I_{]\!]0,\tau]\!]} \cdot [m, M] - \frac{1}{Z_-\,\widetilde{Z}} I_{]\!]0,\tau]\!]} \cdot [m, \langle M, m\rangle^{\mathbb{F}}] \\ &\quad - \frac{1}{Z_-\,\widetilde{Z}} I_{]\!]0,\tau]\!]} \cdot [\langle m\rangle^{\mathbb{F}}, M] + \frac{1}{Z_-^2\,\widetilde{Z}} I_{]\!]0,\tau]\!]} \cdot [\langle m\rangle^{\mathbb{F}}, \langle M, m\rangle^{\mathbb{F}}]. \end{aligned}$$

Since m is an $\mathbb{F}$-locally bounded local martingale, all the processes

$$[m, M],\ [m, \langle M, m\rangle^{\mathbb{F}}],\ [\langle m\rangle^{\mathbb{F}}, M],\ \text{and } [\langle m\rangle^{\mathbb{F}}, \langle M, m\rangle^{\mathbb{F}}]$$

belong to $\mathcal{A}_{loc}(\mathbb{F})$. Thus, by combining this fact with Lemma 3.3 and the $\mathbb{G}$-local boundedness of $Z_-^{-p} I_{]\!]0,\tau]\!]}$ for any $p > 0$, it follows that $V_4 \in \mathcal{A}_{loc}(\mathbb{G})$. This ends the proof of the proposition. □

3.2 The Case of Quasi-Left Continuous Processes

This subsection focuses on processes that do not jump on predictable stopping times (i.e., quasi-left continuous processes). Herein, we prove that the NUPBR is preserved under random horizon for these processes. To this end, we start elaborating a preparatory theorem that is interesting in itself.

Theorem 3.6 *Let $\widetilde{R}_0$ be defined in Lemma 3.1–(b), and S be a quasi-left-continuous $\mathbb{F}$-local martingale. Consider the following assertions*
(a) $\mathcal{E}\left(\widetilde{N}^{(b)}\right) S^\tau$ *is a $\mathbb{G}$-local martingale (i.e., $\mathcal{E}\left(\widetilde{N}^{(b)}\right) \in \mathcal{L}_\sigma(\mathbb{G}, S^\tau)$).*
(b) $\mathcal{E}\left(I_{\{\widetilde{Z}=0<Z_-\}} \odot m^{(qc)}\right) S$ *is an $\mathbb{F}$-local martingale.*
(c) *S and τ satisfy*

$$\{\Delta S \neq 0\} \cap [\![\widetilde{R}_0]\!] \cap \{Z_- > 0\} = \{\Delta S \neq 0\} \cap \{Z_- > 0\} \cap \{\widetilde{Z} = 0\} = \emptyset. \quad (3.15)$$

Then, (c)$\Longrightarrow$ (b) $\Longleftrightarrow$ (a).
As a result, the process $\mathcal{E}\left(\widetilde{N}^{(b)}\right) \overline{S}^\tau$ is a $\mathbb{G}$-local martingale, where

$$\overline{S} := S^{\widetilde{R}_0-} + \left(\Delta S_{\widetilde{R}_0} I_{[\![\widetilde{R}_0,+\infty[\![}\right)^{p,\mathbb{F}}. \quad (3.16)$$

Proof We recall that $m^{(qc)}$ and Γ are defined in (2.3). In the proof, $\widetilde{N} := \widetilde{N}^{(b)}$ and $K := K^{(b)}$ are defined in Proposition 3.5. We start by giving some useful observations. Since S is $\mathbb{F}$-quasi-left-continuous, on the one hand we deduce that

$$\langle S, m\rangle^{\mathbb{F}} = \langle S, m^{(qc)}\rangle^{\mathbb{F}} = \langle S, I_{\Gamma^c} \bullet m\rangle^{\mathbb{F}}. \quad (3.17)$$

On the other hand, we notice that assertion (a) is equivalent to $\mathcal{E}(\widetilde{N}^{(qc)})S^\tau$ is a $\mathbb{G}$-local martingale, where $\widetilde{N}^{(qc)}$ is the quasi-left-continuous local martingale part of $\widetilde{N}$ given by $\widetilde{N}^{(qc)} := I_{\Gamma^c} \bullet \widetilde{N} = -K \odot \widehat{m}^{(qc)}$, and

$$\widehat{m}^{(qc)} := I_{\Gamma^c} \bullet \widehat{m}^{(b)} = (Z_-)^{-1} I_{]\!]0,\tau]\!]} \bullet m^{(qc)} - (Z_-)^{-2} I_{]\!]0,\tau]\!]} \bullet \langle m^{(qc)}\rangle^{\mathbb{F}}.$$

It is easy to check that (3.15) is equivalent to

$$I_{\{Z_->0 \ \&\ \widetilde{Z}=0\}} \bullet [S, m] = 0. \quad (3.18)$$

We now compute $-\langle \widetilde{N}^{(qc)}, \widehat{S}^\tau\rangle^{\mathbb{G}}$, where $\widehat{S}^\tau$ is the $\mathbb{G}$ local martingale given by

$$\widehat{S}^\tau = S^\tau - (Z_-)^{-1} I_{]\!]0,\tau]\!]} \bullet \langle S, m\rangle^{\mathbb{F}}.$$

Due to the quasi-left continuity of S and that of $m^{(qc)}$, the two processes $\langle S, m\rangle^{\mathbb{F}}$ and $\langle m^{(qc)}\rangle^{\mathbb{F}}$ are continuous and $[S, m^{(qc)}] = [S, m]$. Hence, we obtain

$$\begin{aligned} K \bullet [\widehat{S}, \widehat{m}^{(qc)}] &= K \bullet [S, \widehat{m}^{(qc)}] - K\Delta\widehat{m}^{(qc)}(Z_-)^{-1} \bullet \langle S, m\rangle^{\mathbb{F}} \\ &= (\widetilde{Z})^{-1} I_{]\!]0,\tau]\!]} \bullet [S, m^{(qc)}] = (\widetilde{Z})^{-1} I_{]\!]0,\tau]\!]} \bullet [S, m]. \end{aligned}$$

It follows that

$$\begin{aligned}
-\langle \widetilde{N}^{(qc)}, \widehat{S}\rangle^{\mathbb{G}} &= \Big(K \centerdot [\widehat{S}, \widehat{m}^{(qc)}]\Big)^{p,\mathbb{G}} = \Big((\widetilde{Z})^{-1} I_{]\!]0,\tau]\!]} \centerdot [S,m]\Big)^{p,\mathbb{G}} \\
&= (Z_-)^{-1} I_{]\!]0,\tau]\!]} \centerdot \Big(I_{\{\widetilde{Z}>0\}} \centerdot [S,m]\Big)^{p,\mathbb{F}} \\
&= (Z_-)^{-1} I_{]\!]0,\tau]\!]} \centerdot \langle S,m\rangle^{\mathbb{F}} - (Z_-)^{-1} I_{]\!]0,\tau]\!]} \centerdot \Big(I_{\{\widetilde{Z}=0<Z_-\}} \centerdot [S,m]\Big)^{p,\mathbb{F}} \\
&= (Z_-)^{-1} I_{]\!]0,\tau]\!]} \centerdot \langle S,m\rangle^{\mathbb{F}} + (Z_-)^{-1} I_{]\!]0,\tau]\!]} \centerdot \langle S, -I_{\{\widetilde{Z}=0<Z_-\}} \odot m^{(qc)}\rangle^{\mathbb{F}} \quad (3.19)
\end{aligned}$$

The first and the last equality follow from Proposition 2.8 applied to $\widetilde{N}^{(qc)} = K \odot \widehat{m}^{(qc)}$ and $-I_{\{\widetilde{Z}=0<Z_-\}} \odot m^{(qc)}$ respectively. The second and the third equalities are due to (3.17) and (3.8) respectively.

We now prove the theorem. Thanks to (3.19), it is obvious that assertion (a) is equivalent to $\langle S, -I_{\{\widetilde{Z}=0<Z_-\}} \odot m^{(qc)}\rangle^{\mathbb{F}} \equiv 0$ which in turn is equivalent to assertion (b).
Furthermore, it is clear that assertion (c) or equivalently (3.18) implies that assertion (b) holds. This ends the proof of the first part of the theorem. The second statement of the theorem follows immediately from the first statement and the fact that $\overline{S}$ is an $\mathbb{F}$-local martingale that is quasi-left-continuous and satisfies (3.15). This ends the proof of the theorem. □

The following theorem is the main result of this subsection.

Theorem 3.7 *If S is an $\mathbb{F}$-quasi-left-continuous process satisfying NUPBR($\mathbb{F}$), then the stopped process S^τ satisfies NUPBR($\mathbb{G}$).*

Proof For the sake of a clear exposition of ideas, the proof of this theorem will be achieved in five steps. In the first four steps, we will prove that S^τ satisfies the NUPBR under $\mathbb{G}$ for any $\mathbb{F}$-local martingale S that is quasi-left-continuous. Then, in the fifth and last step, we address the general case and conclude the proof of the theorem. The first four steps, in which S is fixed and is a quasi-left-continuous $\mathbb{F}$-local martingale, are the most important steps of the proof and are organized as follows. The first step will outline some decompositions and other structures for both S and m. The second step describes the martingale density for a specified local martingale part of S obtained from step 1, and on which the third step builds up a martingale density for S^τ under $\mathbb{G}$. The fourth step provides the technicalities required for the statements of the third step.
Step 1: Here we consider an $\mathbb{F}$-local martingale S which is quasi-left-continuous process. A direct application of Theorem A.1 (in Appendix) m, with $S = S_0 + S^c + x \star (\mu - \nu)$ and all other notation taken from the Appendix, we deduce the existence of a local martingale $m^\perp$ as well as two $\widetilde{\mathcal{P}}(\mathbb{F})$-measurable functionals f_m and a $\widetilde{\mathcal{O}}(\mathbb{F})$-measurable functional g_m such that $f_m \in \mathcal{G}^1_{loc}(\mu,\mathbb{F})$, $g_m \in \mathcal{H}^1_{loc}(\mu,\mathbb{F})$ and $\beta_m \in L(S^c)$ such that

$$m = \beta_m \centerdot S^c + f_m \star (\mu - \nu) + g_m \star \mu + m^\perp. \quad (3.20)$$

Recall that $\mathcal{G}^1_{loc}(\mu,\mathbb{H})$ (respectively $\mathcal{H}^1_{loc}(\mu,\mathbb{F})$ the set of all $\widetilde{\mathcal{P}}(\mathbb{H})$-measurable functions (respectively all $\widetilde{\mathcal{O}}(\mathbb{H})$-measurable functions) W such that $(\sum(W(t,S_t)-\int W(t,x)\nu^{\mathbb{H}}(\{t\},dx))^2)^{1/2}\in\mathcal{A}^+_{loc}(\mathbb{H})$ $\left(\left(W^2\star\mu\right)^{1/2}\in respectively(\nu^{\mathbb{H}}\mathcal{A}^+_{loc}(\mathbb{H})\right)$(recall that $\nu^{\mathbb{H}}$ is the $\mathbb{H}$-compensator of the random measure μ). In the following, we consider the decomposition of S given by

$$S^{\widetilde{R}_0}=\overline{S}+\Delta S_{\widetilde{R}_0}I_{[\![\widetilde{R}_0,+\infty[\![}-\left(\Delta S_{\widetilde{R}_0}I_{[\![\widetilde{R}_0,+\infty[\![}\right)^{p,\mathbb{F}}, \tag{3.21}$$

where $\overline{S}$ is given by (3.16).

Step 2: In this step we derive a characterization, using the predictable characteristics, for the martingale density of $\overline{S}^\tau$ under $\mathbb{G}$. Thanks to Theorem 3.6, we deduce that the set of martingale densities for the process $\overline{S}^\tau$ is not empty since it contains $\mathcal{E}(\widetilde{N}^{(b)})$. First notice that $\left(\Delta S_{\widetilde{R}_0}I_{[\![\widetilde{R}_0,+\infty[\![}\right)^\tau\equiv 0$ (due to $\tau<\widetilde{R}_0,\quad P-a.s.$), which leads to

$$S^\tau=\overline{S}^\tau-\widetilde{V}^\tau,\qquad \widetilde{V}:=\left(\Delta S_{\widetilde{R}_0}I_{[\![\widetilde{R}_0,+\infty[\![}\right)^{p,\mathbb{F}}\ \text{ and }\ V:=\Delta S_{\widetilde{R}_0}I_{[\![\widetilde{R}_0,+\infty[\![}.$$

Secondly, let $\mathcal{E}(\overline{N})$ a positive $\mathbb{G}$-local martingale. Then, a direct application of Theorem A.1 shows that there exists a positive $\widetilde{\mathcal{P}}(\mathbb{G})$-measurable functional $\overline{f}$ such that $(\overline{f}-1)\in\mathcal{G}^1_{loc}(\mu,\mathbb{G})$, a process $\overline{\beta}\in L(\widehat{S^c})$ and a local martingale $N^\perp$, such that

$$\overline{N}=\overline{\beta}\bullet\widehat{S^c}+(\overline{f}-1)I_{[\![0,\tau]\!]}\star(\mu^{\mathbb{G}}-\nu^{\mathbb{G}})+\overline{g}\star\mu^{\mathbb{G}}+N^\perp,$$

where (see the Appendix for the definition of A)

$$\widehat{S^c}:=I_{[\![0,\tau]\!]}\bullet S^c-\frac{1}{Z_-}I_{[\![0,\tau]\!]}\bullet\langle m,S^c\rangle=I_{[\![0,\tau]\!]}\bullet S^c-\frac{1}{Z_-}c\beta_mI_{[\![0,\tau]\!]}\bullet A\in\mathcal{M}_{loc}(\mathbb{G}).$$

Due to the following decomposition of S^τ under $\mathbb{G}$

$$S^\tau=\widehat{S^c}+x\star(\mu^{\mathbb{G}}-\nu^{\mathbb{G}})+c\beta_mZ_-^{-1}I_{[\![0,\tau]\!]}\bullet A+xI_{[\![0,\tau]\!]}\star(\nu^{\mathbb{G}}-\nu),$$

and $V=xI_{[\![\widetilde{R}_0]\!]}\star\mu$, we put $\psi:=M^P_\mu\left(I_{[\![0,\widetilde{R}_0[\![}\mid\widetilde{\mathcal{P}}\right)$ and get

$$\begin{aligned}\overline{S}^\tau&=S^\tau+\left(\int x(1-\psi(x))F(dx)\right)I_{[\![0,\tau]\!]}\bullet A,\\&=X^{\mathbb{G}}+\left(c\beta_mZ_-^{-1}+\int x\frac{f_m}{Z_-}F(dx)+\int x(1-\psi(x))F(dx)\right)I_{[\![0,\tau]\!]}\bullet A.\end{aligned}$$

Here $X^{\mathbb{G}}$ is a $\mathbb{G}$-local martingale, f_m is defined in (3.20) and is also given by $f_m:=M^P_\mu(\Delta m|\widetilde{\mathcal{P}})$ from which one can easily prove that $1+f_mZ_-^{-1}\geq 0,\ M^P_\mu-a.e$ on $]\!]0,\tau]\!]$. The random measure $\nu^{\mathbb{G}}$ is given by

$$\nu^{\mathbb{G}}(dt,dx)=\left(1+\frac{f_m(t,x)}{Z_{t-}}\right)I_{[\![0,\tau]\!]}\nu(dt,dx). \tag{3.22}$$

Then, $\mathcal{E}(\overline{N})\overline{S}^{\tau}$ is a local martingale if and only if $\overline{S}^{\tau}+[\overline{N},\overline{S}^{\tau}]$ is a $\mathbb{G}$-local martingale. This is equivalent to the fact that on $]\!]0,\tau]\!]$, $dA-a.e.$ we have

$$0\equiv c\Big(\frac{1}{Z_-}\beta_m+\overline{\beta}\Big)+\int x\Big\{\overline{f}(1+\frac{f_m}{Z_-})+[1-\psi(x)]-1\Big\}F(dx). \tag{3.23}$$

Step 3: This step builds up a martingale density for S^{τ} from that of $\overline{S}^{\tau}$ (characterized by (3.23)). Let $\mathcal{E}(N^{\mathbb{G}})$ be a positive $\mathbb{G}$-local martingale, where $N^{\mathbb{G}}$ has the form of

$$N^{\mathbb{G}}:=\beta\bullet\widehat{S}^c+(f-1)I_{]\!]0,\tau]\!]}\star(\mu-\nu^{\mathbb{G}}). \tag{3.24}$$

Then, using similar arguments as in step 2, we deduce that $\mathcal{E}(N^{\mathbb{G}})$ is a σ-martingale density for S^{τ} if and only if on $]\!]0,\tau]\!]$

$$0\equiv c\Big(\frac{1}{Z_-}\beta_m+\beta\Big)+\int x\Big\{f(1+\frac{f_m}{Z_-})-1\Big\}F(dx),\qquad dA\otimes P-a.e. \tag{3.25}$$

Thus to construct a martingale density for S^{τ} we need to choose β and f such that the above equation is satisfied and fulfilled the integrability required to define their integrals. Since $\overline{\beta}\in L(\widehat{S}^c)$, we choose

$$\beta:=\overline{\beta},\ \text{ and }\ f:=\overline{f}+\frac{1-\psi}{1+f_m/Z_-}I_{]\!]0,\tau]\!]}. \tag{3.26}$$

Therefore (3.25) will follow immediately from (3.23), and $\mathcal{E}(N^{\mathbb{G}})$ will constitute a $\mathbb{G}$-martingale density for S^{τ} as long as we prove that $(f-1)\in\mathcal{G}^1_{loc}(\mu,\mathbb{G})$ (Note that due to $\frac{(1-\psi)Z_-}{Z_-+f_m}\geq 0$, we have $f\geq\overline{f}>0$ and hence $\mathcal{E}(N^{\mathbb{G}})$ is positive). This will be the focus of the next step.

Step 4: Since $(\overline{f}-1)\in\mathcal{G}^1_{loc}(\mu,\mathbb{G})$, $(f-1)\in\mathcal{G}^1_{loc}(\mu,\mathbb{G})$ is equivalent to

$$g:=\frac{1-\psi}{1+f_m/Z_-}I_{]\!]0,\tau]\!]}\in\mathcal{G}^1_{loc}(\mu,\mathbb{G}). \tag{3.27}$$

Due to the quasi-left-continuity, it is clear that (3.27) is equivalent to $\big(g^2\star\mu\big)^{1/2}\in\mathcal{A}^+_{loc}(\mathbb{G})$, which is implied by $g\star\mu\in\mathcal{A}^+_{loc}(\mathbb{G})$. In the remaining part of this proof, we prove this latter fact. To this end, consider the nondecreasing $\mathbb{F}$-adapted process

$$W:=\frac{1-\psi}{1+f_m/Z_-}I_{\{Z_->0\ \&\ Z_-+f_m>0\}}\star\mu.$$

It is clear that

$$\overline{W}:=g\star\mu=\frac{1-\psi}{1+f_m/Z_-}I_{]\!]0,\tau]\!]}\star\mu=W^{\tau}.$$

Then, we calculate

$$E\left(\overline{W}_\infty\right) = E\left(\widetilde{Z}\,.\,W_\infty\right) \leq E\left\{\left((1-\psi)I_{\{Z_->0\ \&\ Z_-+f_m>0\}} \star \nu\right)_\infty\right\}$$

$$\leq P\left(\Delta S_{\widetilde{R}_0} \neq 0\ \&\ \widetilde{R}_0 < +\infty\right) \leq 1.$$

Thus, $g \in \mathcal{G}^1_{loc}(\mu, \mathbb{G})$ and the proof of the first four steps is completed.
Step 5: Here we explain how the proof of the theorem will be achieved using the four previous steps proved above. Let S be an $\mathbb{F}$-quasi-left-continuous process satisfying the NUPBR($\mathbb{F}$). Thanks to Theorem 2.3, we deduce the existence of a real-valued $\mathbb{F}$-predictable process ϕ and a positive $\mathbb{F}$-local martingale Z such that

$$0 < \phi \leq 1 \qquad \text{and} \qquad Z(\phi\,.\,S) \quad \text{is a } \mathbb{F}-\text{local martingale.}$$

Then there exists a sequence of $\mathbb{F}$-stopping times that increases to infinity $(T_n)_n$ such that Z^{T_n} is a martingale. Put $Q_n := Z_{T_n} \cdot P \sim P$. Then, by working under Q_n, using $Z^{Q_n}_t := Q_n(\tau > t|\mathcal{F}_t)$ and its martingale part m^{Q_n}, and applying the previous four steps to $(\phi.S)^{T_n}$ under Q_n, we conclude that $(\phi.S)^{T_n\wedge\tau}$ satisfies the NUPBR($\mathbb{G}$). Thanks to Proposition 2.9, the NUPBR($\mathbb{G}$) of S^τ follows immediately. This ends the proof of theorem. □

Corollary 3.8 *The stopping time $\widetilde{R}_0$, given in Lemma 3.1b, is an accessible stopping time.*
As a consequence, all the three assertions (a), (b) *and* (c) *of Theorem 3.6 are equivalent and hold.*

Proof The stopping time $\widetilde{R}_0$, can be decomposed into a totally inaccessible part $\widetilde{R}^i_0$ and an accessible part $\widetilde{R}^a_0$ such that $\widetilde{R}_0 = \widetilde{R}^i_0 \wedge \widetilde{R}^a_0$. We consider the quasi-left-continuous $\mathbb{F}$-martingale

$$M = V - \widetilde{V} \in \mathcal{M}_{0,loc}(\mathbb{F})$$

where $V := I_{[\![\widetilde{R}^i_0,+\infty[\![}$ and $\widetilde{V} := (V)^{p,\mathbb{F}}$. Since $\tau < \widetilde{R}_0 \leq \widetilde{R}^i_0 \ \ P-a.s.$, we get

$$M^\tau = -\widetilde{V}^\tau \quad \text{is } \mathbb{G}\text{-predictable.} \tag{3.28}$$

Thanks to Theorem 3.7, we conclude that M^τ satisfies NUPBR($\mathbb{G}$). By combining this with (3.28), we conclude that this process is null (i.e. $\widetilde{V}^\tau = 0$) due to Lemma 2.10. Thus, we get

$$0 = E\left(\widetilde{V}_\tau\right) = E\left(\int_0^{+\infty} Z_{s-}d\widetilde{V}_s\right) = E\left(Z_{\widetilde{R}^i_0-}I_{\{\widetilde{R}^i_0<+\infty\}}\right),$$

or equivalently $Z_{\widetilde{R}^i_0-}I_{\{\widetilde{R}^i_0<+\infty\}} = 0$ that implies that $\{\widetilde{R}^i_0 < +\infty\} \subset \{\widetilde{R}^i_0 = \widehat{R}_0\}$. Hence $\widetilde{R}^i_0 = +\infty\ P-a.s.$ since $\widehat{R}_0$ is a predictable stopping time. This proves that $\widetilde{R}_0$ is an accessible stopping time. This ends the proof of the corollary. □

Remark 3.9 It is worthwhile to mention that beside being an important intermediatory result, Theorem 3.6 provides an explicit local martingale density for the class of

$$\mathcal{C}(\mathbb{G}) := \left\{ S^\tau \;\middle|\; S \text{ is a quasi-left-continuous } \mathbb{F}\text{-local martingale} \right\}.$$

3.3 The Case of Single Jump Process with Predictable Jump Time

This subsection investigates the class of processes that are single jump with predictable jump time. Herein, we elaborate two main results, Theorems 3.11 and 3.13. These results are based on the following proposition that is interesting in itself.

Proposition 3.10 *Let τ be a random time and M be an $\mathbb{F}$-martingale given by $M := XI_{[\![T,+\infty[\![}$, where T is an $\mathbb{F}$-predictable stopping time, and X is an $\mathcal{F}_T$-measurable random variable X. Then the following assertions are equivalent.*
(a) *M is a martingale under Q_T that is given by*

$$Q_T := \left(\frac{I_{\{\widetilde{Z}_T>0\}\cap\Gamma(T)}}{P\left(\widetilde{Z}_T > 0 \middle|\, \mathcal{F}_{T-}\right)} + I_{\Gamma^c(T)} \right) \cdot P, \tag{3.29}$$

and $\Gamma(T) := \{P(\widetilde{Z}_T > 0 \mid \mathcal{F}_{T-}) > 0\}$.
(b) *On the set $\{T < +\infty\}$, we have*

$$E\left(M_T I_{\{\widetilde{Z}_T=0<Z_{T-}\}} \middle|\, \mathcal{F}_{T-} \right) = 0, \quad P-a.s. \tag{3.30}$$

(c) *M^τ is a $\mathbb{G}$-martingale under $Q_T^{\mathbb{G}} := \left(L_b^{\mathbb{G}}(T)/E(L_b^{\mathbb{G}}(T)\middle|\, \mathcal{G}_{T-})\right) \cdot P$ where*

$$L_b^{\mathbb{G}}(T) := I_{\{T>\tau\}} + I_{\{T\leq\tau\}} \frac{Z_{T-}}{\widetilde{Z}_T} > 0. \tag{3.31}$$

Proof The proof will be achieved in two parts, part 1) and part 2).
Part 1) Here, we will prove the equivalence between assertions (a) and (b). First, for simplicity we denote by $Q := Q_T$, where Q_T is defined in (3.29), and remark that on $\{Z_{T-} = 0\}$, Q coincides with P and (3.30) holds. Thus, it is enough to prove the equivalence between (a) and (b) on the set $\{T < +\infty \;\&\; Z_{T-} > 0\}$. On this set, due to $E(X|\mathcal{F}_{T-}) = 0$, we derive

$$\begin{aligned} E^Q(X\big|\mathcal{F}_{T-}) &= E(XI_{\{\widetilde{Z}_T>0\}}\big|\mathcal{F}_{T-})\Big(P(\widetilde{Z}_T > 0\big|\mathcal{F}_{T-})\Big)^{-1} \\ &= -E(XI_{\{\widetilde{Z}_T=0\}}\big|\mathcal{F}_{T-})\left(P(\widetilde{Z}_T > 0\big|\mathcal{F}_{T-})\right)\Big)^{-1}. \end{aligned}$$

Therefore, we conclude that assertion (a) (or equivalently $E^Q(X|\mathcal{F}_{T-}) = 0$) is equivalent to (3.30). This ends the proof of the part 1).

Part 2) To prove (a)$\Longleftrightarrow$(c), we first notice that due to $(T \leq \tau) \subset (\widetilde{Z}_T > 0) \subset (Z_{T-} > 0)$, on $\{T \leq \tau\}$ we have

$$P\left(\widetilde{Z}_T > 0 \middle| \mathcal{F}_{T-}\right) E^{Q_T^{\mathbb{G}}}(X|\mathcal{G}_{T-}) = E\left(\frac{Z_{T-}}{\widetilde{Z}_T} X I_{\{T\leq\tau\}} | \mathcal{G}_{T-}\right) = E\left(X I_{\{\widetilde{Z}_T>0\}} | \mathcal{F}_{T-}\right)$$
$$= E^Q(X|\mathcal{F}_{T-})\, P\left(\widetilde{Z}_T > 0 \middle| \mathcal{F}_{T-}\right).$$

This equality proves that $M^\tau \in \mathcal{M}(Q^{\mathbb{G}}, \mathbb{G})$ –where $Q^{\mathbb{G}}$ is defined in (3.31)– if and only if $M \in \mathcal{M}(Q, \mathbb{F})$, and the proof of part 2) is completed. This ends the proof of the theorem. □

Below, we state our first main theorem of this subsection.

Theorem 3.11 *Consider an $\mathbb{F}$-predictable stopping time T and an $\mathcal{F}_T$-measurable random variable X such that $E(|X|\big|\ \mathcal{F}_{T-}) < +\infty\ P-a.s.$.*
If $S := X I_{\{Z_{T-}>0\}} I_{[\![T,+\infty[\![}$, then the following assertions are equivalent:
(a) *S^τ satisfies NUPBR($\mathbb{G}$),*
(b) *S satisfies NUPBR($\mathbb{F}, \widetilde{Q}_T$), where $\widetilde{Q}_T$ is defined by*

$$\widetilde{Q}_T := \left(\frac{\widetilde{Z}_T}{Z_{T-}} I_{\{Z_{T-}>0\}} + I_{\{Z_{T-}=0\}}\right) \cdot P, \tag{3.32}$$

(c) *S satisfies NUPBR($\mathbb{F}, Q_T$), where Q_T is defined in (3.29).*
(d) *The process $\widetilde{S} := X I_{\{\widetilde{Z}_T>0\}} I_{[\![T,+\infty[\![} = I_{\{\widetilde{Z}>0\}} \cdot S$ satisfies NUPBR($\mathbb{F}$).*

Proof The proof of this theorem will be achieved by proving (b) $\Longleftrightarrow$ (c) $\Longleftrightarrow$ (d) and (d) $\Longrightarrow$ (a) $\Longrightarrow$ (b), which will be carried out in four steps.

Step 1: In this step, we prove (b) $\Longleftrightarrow$ (c). Since S is a single jump process with predictable jump time T, then it is easy to see that S satisfies NUPBR under a probability R is equivalent to the fact that $I_A S$ and $I_{A^c} S$ satisfies NUPBR(R) for any $\mathcal{F}_{T-}$-measurable event A. Hence, it is enough to prove the equivalence between the assertions (b) and (c) separately on the events $\{Z_{T-} = 0\}$ and $\{Z_{T-} > 0\}$. Since $\{Z_{T-} = 0\} \subset \{\widetilde{Z}_T = 0\}$ and $E(\widetilde{Z}_T|\mathcal{F}_{T-}) = Z_{T-}$ on $\{T < +\infty\}$, by putting $\Gamma_0 := \left\{P(\widetilde{Z}_T > 0|\mathcal{F}_{T-}) = 0\right\}$, we derive

$$E\left(Z_{T-} I_{\Gamma_0 \cap \{T<+\infty\}}\right) = E\left(\widetilde{Z}_T I_{\Gamma_0 \cap \{T<+\infty\}}\right) = 0,$$

and

$$0 = P\left(\{Z_{T-} = 0\} \cap \{\widetilde{Z}_T > 0\} \cap \{T < +\infty\}\right)$$
$$= E\left(I_{\{Z_{T-}=0\}\cap\{T<+\infty\}} P\left(\widetilde{Z}_T > 0 | \mathcal{F}_{T-}\right)\right).$$

These equalities imply that on $\{T < +\infty\}$, $P-a.s.$, we have

$$\{Z_{T-} = 0\} = \Gamma_0 \subset \{\widetilde{Z}_T = 0\}. \tag{3.33}$$

Thus, on the set $\{T < +\infty\} \cap \Gamma_0$, the three probabilities P, Q_T and $\widetilde{Q}_T$ coincide, and the equivalence between assertions (b) and (c) is obvious. On the set $\{T < +\infty \ \& \ P[\widetilde{Z}_T > 0|\mathcal{F}_{T-}] > 0\}$, one has $\widetilde{Q}_T \sim Q_T$, and the equivalence between (b) and (c) is also obvious. This achieves this first step.

Step 2: This step will prove (c)$\Longleftrightarrow$ (d). Again thanks to (3.33), we deduce that on $\{Z_{T-} = 0\}$, $\widetilde{S} \equiv S \equiv 0$ and Q_T coincides with P as well. Hence, the equivalence between assertions (c) and (d) is obvious for this case. Thus, it is enough to prove the equivalence between these assertions on $\{T < +\infty \ \& \ P(\widetilde{Z}_T > 0|\mathcal{F}_{T-}) > 0\}$. Assume that (c) holds. Then, there exists a positive and $\mathcal{F}_T$–measurable random variable, Y, such that $P - a.s.$ on $\{T < +\infty\}$, we have

$$E^{Q_T}(Y|\mathcal{F}_{T-}) = 1, \ \ E^{Q_T}(Y|X||\mathcal{F}_{T-}) < +\infty, \ \ \& \ \ E^{Q_T}(YXI_{\{\widetilde{Z}_T>0\}}|\mathcal{F}_{T-}) = 0.$$

Since $Y > 0$ on $\{\widetilde{Z}_T > 0\}$, by putting

$$Y_1 := YI_{\{\widetilde{Z}_T>0\}} + I_{\{\widetilde{Z}_T=0\}} \ \text{ and } \ \widetilde{Y}_1 := \frac{Y_1}{E[Y_1|\mathcal{F}_{T-}]},$$

it is easy to check that $Y_1 > 0$, $\widetilde{Y}_1 > 0$,

$$E\left[\widetilde{Y}_1|\mathcal{F}_{T-}\right] = 1 \text{ and } E\left[\widetilde{Y}_1 XI_{\{\widetilde{Z}_T>0\}}|\mathcal{F}_{T-}\right] = \frac{E\left[YXI_{\{\widetilde{Z}_T>0\}}|\mathcal{F}_{T-}\right]}{E[Y_1|\mathcal{F}_{T-}]} = 0.$$

Therefore, $\widetilde{S}$ is a martingale under $R := \widetilde{Y}_1 \cdot P \sim P$, and hence $\widetilde{S}$ satisfies NUPBR($\mathbb{F}$). This proves assertion (d).
To prove the reverse sense, we suppose that assertion (d) holds. Then, there exists $0 < Y \in L^0(\mathcal{F}_T)$, such that $E[Y|X|I_{\{\widetilde{Z}_T>0\}}|\mathcal{F}_{T-}] < +\infty$, $E[Y|\mathcal{F}_{T-}] = 1$ and $E[YXI_{\{\widetilde{Z}_T>0\}}|\mathcal{F}_{T-}] = 0$. Then, consider

$$Y_2 := \frac{YI_{\{\widetilde{Z}_T>0\}}P(\widetilde{Z}_T > 0|\mathcal{F}_{T-})}{E[YI_{\{\widetilde{Z}_T>0\}}|\mathcal{F}_{T-}]} > 0, \qquad Q_T - a.s..$$

Then it is easy to verify that $Y_2 > 0 \ \ Q_T - a.s.$,

$$E^{Q_T}(Y_2|\mathcal{F}_{T-}) = 1, \qquad \text{and} \qquad E^{Q_T}(Y_2 X|\mathcal{F}_{T-}) = \frac{E\left[YXI_{\{\widetilde{Z}_T>0\}}|\mathcal{F}_{T-}\right]}{E[YI_{\{\widetilde{Z}_T>0\}}|\mathcal{F}_{T-}]} = 0.$$

This proves assertion (c), and the proof of (c)$\Longleftrightarrow$(d) is achieved.

Step 3: We will prove (a) $\Longrightarrow$ (b). Suppose S^τ satisfies NUPBR($\mathbb{G}$). Then there exists a positive $\mathcal{G}_T$-measurable random variable $Y^{\mathbb{G}}$ such that $E[XY^{\mathbb{G}}I_{\{T\leq\tau\}}|\mathcal{G}_{T-}] = 0$ on $\{T < +\infty\}$. Due to Lemma 2.13–(a), we deduce the existence of a positive $\mathcal{F}_T$-measurable variable $Y^{\mathbb{F}}$ such that $Y^{\mathbb{G}}I_{\{T\leq\tau\}} = Y^{\mathbb{F}}I_{\{T\leq\tau\}}$. Then, on $\{T < +\infty\}$ we obtain

$$0 = E[XY^{\mathbb{F}}I_{\{T\leq\tau\}}|\mathcal{G}_{T-}] = E[XY^{\mathbb{F}}\widetilde{Z}_T|\mathcal{F}_{T-}]\frac{I_{\{T\leq\tau\}}}{Z_{T-}}.$$

Therefore, by taking conditional expectation in the above equality, we get

$$0 = E[XY^{\mathbb{F}}\frac{\widetilde{Z}_T}{Z_{T-}}I_{\{Z_{T-}>0\}}|\mathcal{F}_{T-}] = E^{\widetilde{Q}_T}[XY^{\mathbb{F}}|\mathcal{F}_{T-}]I_{\{Z_{T-}>0\}} = E^{\widetilde{Q}_T}[S_TY^{\mathbb{F}}|\mathcal{F}_{T-}].$$

This proves that assertion (b) holds and the proof of (a)$\Longrightarrow$(b) is achieved.

Step 4: Herein, we will prove (d)$\Longrightarrow$(a). Suppose that $\widetilde{S}$ satisfies the NUPBR($\mathbb{F}$). Then, there exists $Y \in L^1(\mathcal{F}_T)$ such that on $\{T < +\infty\}$ we have

$$E[Y|\mathcal{F}_{T-}] = 1, \quad Y > 0, \qquad E[Y|X|I_{\{\widetilde{Z}_T>0\}}|\mathcal{F}_{T-}] < +\infty, \quad P-a.s.$$

and

$$E[YXI_{\{\widetilde{Z}_T>0\}}|\mathcal{F}_{T-}] = 0.$$

Then by considering $R := Y \cdot P \sim P$, we get

$$E^R\big[\widetilde{S}_T\big|\mathcal{F}_{T-}\big] = E^R\big[XI_{\{\widetilde{Z}_T>0\}}\big|\mathcal{F}_{T-}\big] = 0.$$

Therefore, assertion (a) follows directly from Proposition 3.10 applied to $M = \widetilde{S}$ under $R \sim P$ (it is easy to see that (3.30) holds for $(\widetilde{S}, R)$, i.e. $E^R(\widetilde{S}_T I_{\{\widetilde{Z}_T=0\}}|\mathcal{F}_{T-}) = 0$). This ends the fourth step and the proof of the theorem is completed. □

Remark 3.12 1) It is important to remark that on $\{T < +\infty\}$,

$$\frac{L_b^{\mathbb{G}}(T)}{E(L_b^{\mathbb{G}}(T)\big|\ \mathcal{G}_{T-})} \neq \frac{\mathcal{E}(\widetilde{N}^{(b)})_T}{\mathcal{E}(\widetilde{N}^{(b)})_{T-}}.$$

This highlights one of the main difficulties that we will face when we will formulate the results for possible many predictable jumps that might not be ordered. Simply, it might not be possible to piece up

$$L_b^{\mathbb{G}}(T_n) = 1 - \frac{\Delta m_{T_n}}{\widetilde{Z}_{T_n}}I_{\{T_n\leq\tau\}}, \quad n \geq 1$$

to form a positive $\mathbb{G}$-local martingale density for the process $(I_{\cup[\![T_n]\!]} \cdot S)^\tau$.
2) In Theorem 3.11, the choice of S having the form of $S := XI_{\{Z_{T-}>0\}}I_{[\![T,+\infty[\![}$ is not restrictive. This can be understood from the fact that any single jump process S can be decomposed as follows

$$S := XI_{[\![T,+\infty[\![} = XI_{\{Z_{T-}>0\}}I_{[\![T,+\infty[\![} + XI_{\{Z_{T-}=0\}}I_{[\![T,+\infty[\![} =: S_0 + S_1.$$

Thanks to $\{T \leq \tau\} \subset \{Z_{T-} > 0\}$, we have $S_0^\tau = XI_{\{Z_{T-}=0\}}I_{\{T\leq\tau\}}I_{[\![T,+\infty[\![} \equiv 0$ is a $\mathbb{G}$-martingale. Thus, the only part of S that requires careful attention is $S_1 := XI_{\{Z_{T-}>0\}}I_{[\![T,+\infty[\![}$.

The following states our second principal result of this subsection.

Theorem 3.13 *Let T be an $\mathbb{F}$-predictable stopping time. Then, the following are equivalent:*
(a) *On $\{T<+\infty\}$, we have*

$$\Big\{\widetilde{Z}_T=0\Big\}\subset\Big\{Z_{T-}=0\Big\}. \tag{3.34}$$

(b) *The probability $\widetilde{Q}(T)$, defined in (3.32), is equivalent to P on $\{T<+\infty\}$.*
(c) *For any $M:=XI_{[\![T,+\infty[\![}$ where $X\in L^\infty(\mathcal{F}_T)$ such that $E(X|\mathcal{F}_{T-})=0$, M^τ satisfies NUPBR($\mathbb{G}$).*

Proof Suppose that (3.34) holds. Then it is obvious that this property implies (3.30) for any random variable X. Therefore, a direct application of Theorem 3.10 implies assertion (c). This proves (a)$\Longrightarrow$ (c). To prove the reverse, it is enough to consider

$$M:=XI_{[\![T,+\infty[\![},\qquad \text{where}\ \ X:=I_{\{\widetilde{Z}_T=0\}}-P(\widetilde{Z}_T=0|\mathcal{F}_{T-}).$$

Then, due to $\{T\leq\tau\}\subset\{\widetilde{Z}_T>0\}\subset\{Z_{T-}>0\}$, we calculate

$$M^\tau=-P(\widetilde{Z}_T=0|\mathcal{F}_{T-})I_{\{T\leq\tau\}}I_{[\![T,+\infty[\![}\ \text{ is }\mathbb{G}\text{-predictable.}$$

Therefore, M^τ satisfies NUPBR($\mathbb{G}$) if and only if it is constant in time equal to $M_0=0$ (due to Lemma 2.10). This is equivalent to

$$0=E\Big[P(\widetilde{Z}_T=0|\mathcal{F}_{T-})I_{\{T\leq\tau\}}I_{[\![T,+\infty[\![}\Big]=E\left(Z_{T-}I_{\{\widetilde{Z}_T=0\ \&\ T<+\infty\}}\right).$$

It is obvious that this equality is equivalent to (3.34), and assertion (a) follows. The equivalence between assertions (a) and (b) is obvious from (3.33). This ends the proof of the theorem. □

3.4 The General Framework

This subsection extends the previous subsection to the case where there are countable many predictable jumps, and summarizes the main results into unified general statements. This achievement relies deeply on the following proposition which is important in itself. The highly innovative assertions of this proposition are assertions (b) and (c), where we translate the localization (respectively the local integrability) under $\mathbb{G}$ into a localization (respectively a local integrability) under $\mathbb{F}$.

Proposition 3.14 *The following assertions hold:*
(a) *Let h be a $\widetilde{\mathcal{P}}(\mathbb{H})$-measurable functional. Then, $\sqrt{(h-1)^2\star\mu}\in\mathcal{A}^+_{loc}(\mathbb{H})$ iff*

$$(h-1)^2I_{\{|h-1|\leq\alpha\}}\star\mu\ \text{ and }\ |h-1|I_{\{|h-1|>\alpha\}}\star\mu\ \ \text{belong to}\ \ \mathcal{A}^+_{loc}(\mathbb{H}).$$

(b) *Let* $(\sigma_n^{\mathbb{G}})_n$ *be a sequence of* $\mathbb{G}$*-stopping times that increases to infinity. Then, there exists a nondecreasing sequence of* $\mathbb{F}$*-stopping times,* $(\sigma_n^{\mathbb{F}})_{n\geq 1}$*, satisfying the following properties*

$$\sigma_n^{\mathbb{G}} \wedge \tau = \sigma_n^{\mathbb{F}} \wedge \tau, \quad \sigma_\infty := \sup_n \sigma_n^{\mathbb{F}} \geq \widehat{R} \ \ P-a.s., \tag{3.35}$$

$$\text{and} \qquad Z_{\sigma_\infty -} = 0 \quad P-a.s. \quad \text{on} \quad \Sigma \cap (\sigma_\infty < +\infty), \tag{3.36}$$

where $\Sigma := \bigcap_{n\geq 1} (\sigma_n^{\mathbb{F}} < \sigma_\infty)$.

(c) *Let* h *be a nonnegative and* $\widetilde{\mathcal{P}}(\mathbb{F})$*-measurable functional. Then,* $hI_{\rrbracket 0,\tau\rrbracket} \star \mu \in \mathcal{A}^+_{loc}(\mathbb{G})$ *if and only if for all* $\delta > 0$*,* $hI_{\{Z_- \geq \delta\}} \star \mu^1 \in \mathcal{A}^+_{loc}(\mathbb{F})$*, where* $\mu^1 := \widetilde{Z} \cdot \mu$.

(d) *Let* f *be positive and* $\widetilde{\mathcal{P}}(\mathbb{F})$*-measurable, and* $\mu^1 := \widetilde{Z} \cdot \mu$*. Then* $\sqrt{(f-1)^2 I_{\rrbracket 0,\tau\rrbracket} \star \mu} \in \mathcal{A}^+_{loc}(\mathbb{G})$ *iff* $\sqrt{(f-1)^2 I_{\{Z_- \geq \delta\}} \star \mu^1} \in \mathcal{A}^+_{loc}(\mathbb{F})$*, for all* $\delta > 0$.

Proof (a) Put $W := (h-1)^2 \star \mu = W_1 + W_2$, where $W_1 := (h-1)^2 I_{\{|h-1| \leq \alpha\}} \star \mu$, $W_2 := (h-1)^2 I_{\{|h-1| > \alpha\}} \star \mu$ and $W_2' := |h-1| I_{\{|h-1| > \alpha\}} \star \mu$. Note that

$$\sqrt{W} \leq \sqrt{W_1} + \sqrt{W_2} \leq \sqrt{W_1} + W_2'.$$

Therefore $\sqrt{W_1}, W_2' \in \mathcal{A}^+_{loc}$ imply $\sqrt{W}$ is locally integrable.
Conversely, if $\sqrt{W} \in \mathcal{A}^+_{loc}$, $\sqrt{W_1}$ and $\sqrt{W_2}$ are both locally integrable. Since W_1 is locally bounded and has finite variation, W_1 is locally integrable. In the following, we focus on the proof of the local integrability of W_2'. Denote

$$\tau_n := \inf\{t \geq 0 : \ V_t > n\}, \quad V := W_2.$$

It is easy to see that τ_n increases to infinity and $V_- \leq n$ on the set $\rrbracket 0, \tau_n \rrbracket$. On the set $\{\Delta V > 0\}$, we have $\Delta V \geq \alpha^2$. By using the elementary inequality $\sqrt{1 + \frac{n}{\alpha^2}} - \sqrt{\frac{n}{\alpha^2}} \leq \sqrt{1+x} - \sqrt{x} \leq 1$, when $0 \leq x \leq \frac{n}{\alpha^2}$, we have

$$\sqrt{V_- + \Delta V} - \sqrt{V_-} \geq \beta_n \sqrt{\Delta V} \ \text{ on } \rrbracket 0, \tau_n \rrbracket, \ \text{where } \ \beta_n := \sqrt{1 + \frac{n}{\alpha^2}} - \sqrt{\frac{n}{\alpha^2}},$$

and

$$(W_2')^{\tau_n} = \Big(\sum \sqrt{\Delta V}\Big)^{\tau_n} \leq \frac{1}{\beta_n} \Big(\sum \Delta \sqrt{V}\Big)^{\tau_n} = \frac{1}{\beta_n} \Big(\sqrt{W_2}\Big)^{\tau_n} \in \mathcal{A}^+_{loc}(\mathbb{H})$$

Therefore $W_2' \in (\mathcal{A}^+_{loc}(\mathbb{H}))_{loc} = \mathcal{A}^+_{loc}(\mathbb{H})$.

(b) Due to Jeulin [28], there exists a sequence of $\mathbb{F}$-stopping times $(\sigma_n^{\mathbb{F}})_n$ such that

$$\sigma_n^{\mathbb{G}} \wedge \tau = \sigma_n^{\mathbb{F}} \wedge \tau. \tag{3.37}$$

By putting $\sigma_n := \sup_{k \leq n} \sigma_k^{\mathbb{F}}$, we shall prove that

$$\sigma_n^{\mathbb{G}} \wedge \tau = \sigma_n \wedge \tau, \tag{3.38}$$

or equivalently $\{\sigma_n^{\mathbb{F}} \wedge \tau < \sigma_n \wedge \tau\}$ is negligible. Due to (3.37) and $\sigma_n^{\mathbb{G}}$ is nondecreasing, we derive

$$\{\sigma_n^{\mathbb{F}} < \tau\} = \{\sigma_n^{\mathbb{G}} < \tau\} \subset \bigcap_{i=1}^{n} \{\sigma_i^{\mathbb{G}} = \sigma_i^{\mathbb{F}}\} \subset \{\sigma_n^{\mathbb{F}} = \sigma_n\}.$$

This implies that,

$$\{\sigma_n^{\mathbb{F}} \wedge \tau < \sigma_n \wedge \tau\} = \{\sigma_n^{\mathbb{F}} < \tau, \ \& \ \sigma_n^{\mathbb{F}} < \sigma_n\} = \emptyset,$$

and the proof of (3.38) is completed. Without loss of generality we assume that the sequence $\sigma_n^{\mathbb{F}}$ is nondecreasing. By taking limit in (3.37), we obtain $\tau = \sigma_\infty \wedge \tau, P-$a.s. which is equivalent to $\sigma_\infty \geq \tau, P-$a.s. Since $\widehat{R}$ is the smallest $\mathbb{F}$-stopping time greater or equal than τ almost surely, we obtain, $\sigma_\infty \geq \widehat{R} \geq \tau \ P-a.s.$. This achieves the proof of (3.35).
On the set Σ, it is easy to show that

$$I_{[\![0,\sigma_n^{\mathbb{F}}]\!]} \longrightarrow I_{[\![0,\sigma_\infty^{\mathbb{F}}[\![}, \quad \text{when } n \text{ goes to } +\infty.$$

Then, thanks again to (3.37) (by taking $\mathbb{F}$-predictable projection and let n go to infinity afterwards), we obtain

$$Z_- = Z_- I_{[\![0,\sigma_\infty^{\mathbb{F}}[\![}, \qquad \text{on} \quad \Sigma. \tag{3.39}$$

Hence, (3.36) follows immediately, and the proof of assertion (b) is completed.

(c) Suppose that $hI_{[\![0,\tau]\!]} \star \mu \in \mathcal{A}^+_{loc}(\mathbb{G})$. Then, there exists a sequence of $\mathbb{G}$-stopping times $(\sigma_n^{\mathbb{G}})$ increasing to infinity such that $hI_{[\![0,\tau]\!]} \star \mu^{\sigma_n^{\mathbb{G}}}$ is integrable. Consider (σ_n) a sequence of $\mathbb{F}$-stopping times satisfying (3.35)–(3.36) (its existence is guaranteed by assertion (b)). Therefore, for any fixed $\delta > 0$

$$W^n := M_\mu^P \left(\widetilde{Z}|\widetilde{\mathcal{P}}\right) I_{\{Z_- \geq \delta\}} h \star \nu^{\sigma_n} \in \mathcal{A}^+(\mathbb{F}), \tag{3.40}$$

or equivalently, this process is càdlàg predictable with finite values. Thus, it is obvious that the proof of assertion (iii) will follow immediately if we prove that the $\mathbb{F}$-predictable and nondecreasing process

$$W := M_\mu^P \left(\widetilde{Z}|\widetilde{\mathcal{P}}\right) I_{\{Z_- \geq \delta\}} h \star \nu \quad \text{is càdlàg with finite values.} \tag{3.41}$$

To prove this last fact, we consider the random time τ^δ defined by

$$\tau^\delta := \sup\{t \geq 0 : Z_{t-} \geq \delta\}.$$

Then, it is clear that $I_{]\!]\tau^\delta,+\infty[\![} \cdot W \equiv 0$ and

$$\tau^\delta \leq \widehat{R} \leq \sigma_\infty \quad \text{and} \quad Z_{\tau^\delta -} \geq \delta \quad P\text{–}a.s. \qquad \text{on } \{\tau^\delta < +\infty\}.$$

The proof of (3.41) will be achieved by considering three sets, namely $\{\sigma_\infty = \infty\}$, $\Sigma \cap \{\sigma_\infty < +\infty\}$, and $\Sigma^c \cap \{\sigma_\infty < +\infty\}$. It is obvious that (3.41) holds

on $\{\sigma_\infty = \infty\}$. Due to (3.36), we deduce that $\tau^\delta < \sigma_\infty, P-$a.s. on $\Sigma \cap \{\sigma_\infty < +\infty\}$. Since W is supported on $[\![0, \tau^\delta]\!]$, then (3.41) follows immediately on the set $\Sigma \cap \{\sigma_\infty < +\infty\}$. Finally, on the set

$$\Sigma^c \cap \{\sigma_\infty < +\infty\} = \left(\bigcup_{n \geq 1} \{\sigma_n = \sigma_\infty\} \right) \cap \{\sigma_\infty < +\infty\},$$

the sequence σ_n increases stationarily to σ_∞, and thus (3.41) holds on this set. This completes the proof of (3.41), and hence $hI_{\{Z_- \geq \delta\}} \star (\widetilde{Z} \cdot \mu)$ is locally integrable, for any $\delta > 0$.

Conversely, if $hI_{\{Z_- \geq \delta\}} \widetilde{Z} \star \mu \in \mathcal{A}^+_{loc}(\mathbb{F})$, there exists a sequence of $\mathbb{F}$-stopping times $(\tau_n)_{n \geq 1}$ that increases to infinity and $\left(hI_{\{Z_- \geq \delta\}} \widetilde{Z} \star \mu \right)^{\tau_n} \in \mathcal{A}^+(\mathbb{F})$. Then, we have

$$E\left[hI_{\{Z_- \geq \delta\}} I_{[\![0,\tau]\!]} \star \mu(\tau_n) \right] = E\left[hI_{\{Z_- \geq \delta\}} \widetilde{Z} \star \mu(\tau_n) \right] < +\infty. \tag{3.42}$$

This proves that $hI_{\{Z_- \geq \delta\}} I_{[\![0,\tau]\!]} \star \mu$ is $\mathbb{G}$-locally integrable, for any $\delta > 0$. Since $(Z_-)^{-1} I_{[\![0,\tau]\!]}$ is $\mathbb{G}$-locally bounded, then there exists a family of $\mathbb{G}$-stopping times $(\tau_\delta)_{\delta > 0}$ that increases to infinity when δ decreases to zero, and

$$[\![0, \tau \wedge \tau_\delta]\!] \subset \{Z_- \geq \delta\}.$$

This implies that the process $\left(hI_{[\![0,\tau]\!]} \star \mu \right)^{\tau_\delta}$ is $\mathbb{G}$-locally integrable, and hence the assertion (c) follows immediately.

(d) The proof of assertion (d) follows from combining assertions (a) and (b). This ends the proof of the proposition. □

In the following we elaborate our first main result in the general setting.

Theorem 3.15 *Let S be a process that satisfies NUPBR($\mathbb{F}$). Then, the following are equivalent.*
(a) *The process S^τ satisfies NUPBR($\mathbb{G}$).*
(b) *For any $\delta > 0$, there exists a positive $\mathbb{F}$-local martingale, Y, such that $[Y, S] \in \mathcal{A}_{loc}(\mathbb{F})$ and*

$$^{p,\mathbb{F}}\left(Y \Delta S I_{\{\widetilde{Z} > 0 \ \&\ Z_- \geq \delta\}} \right) = 0. \tag{3.43}$$

Proof We start by outlining three remarks that simplify the overall proof of the theorem. The first remark consists of

$$\{\widetilde{Z}^Q_T = 0\} = \{\widetilde{Z}_T = 0\}, \quad \text{for any } Q \sim P \text{ and any } \mathbb{F}\text{-stopping time } T, \tag{3.44}$$

where $\widetilde{Z}^Q_t := Q[\tau \geq t | \mathcal{F}_t]$. This equality follows from

$$E\left[\widetilde{Z}_T I_{\{\widetilde{Z}^Q_T = 0\}} \right] = E\left[I_{\{\tau \geq T\}} I_{\{\widetilde{Z}^Q_T = 0\}} \right] = 0$$

—which implies $\{\widetilde{Z}^Q = 0\} \subset \{\widetilde{Z} = 0\}$— and the symmetric role of Q and P. The second remark guarantees the existence of a sequence of $\mathbb{F}$-predictable stopping times $(T_n)_{n\geq 1}$ that exhausts the predictable jumps of S, and if we put

$$S^i = S - I_{\cup_n [\![T_n]\!]} \centerdot S \quad \text{and} \quad S^a := I_{\cup_n [\![T_n]\!]} \centerdot S,$$

then the process $(S^i)^\tau$ satisfies the NUPBR($\mathbb{G}$) (thanks to Theorem 3.7). The last remark lies in the fact that assertion (b) is equivalent to the existence of a positive $\mathbb{F}$-local martingale such that $[Y, S^a] \in \mathcal{A}(\mathbb{F})$

$$^{p,\mathbb{F}}\left(Y \Delta S^a I_{\{\widetilde{Z}>0 \ \& \ Z_- \geq \delta\}}\right) = 0.$$

Therefore, due to the last two remarks, we conclude that it is enough to prove the theorem for S^a. Thus in the remaining part of this proof we will assume that $S = S^a$. The rest of the proof contains two parts 1) and 2) that prove (b)⇒ (a) and (a)⇒ (b) respectively.

1) We prove (b)⇒ (a). Suppose that assertion (b) holds, and consider a sequence of $\mathbb{F}$-stopping times $(\tau_n)_n$ that increases to infinity such that Y^{τ_n} is an $\mathbb{F}$-martingale. Then, by putting $Q_n := Y_{\tau_n}/Y_0 \cdot P$, and using (3.44) and Proposition 2.9, we deduce that there is no loss of generality in assuming $Y \equiv 1$. Put

$$X^n = \Delta S_{T_n} I_{\{\widetilde{Z}_{T_n}>0\}} \quad \text{and} \quad M^n = X^n I_{[\![T_n,+\infty[\![}.$$

Hence, the condition (3.30) in Theorem 3.10 holds for X^n and M^n. Therefore, for each n, $(1+\Delta N^{(b)}_{T_n} - \Delta V^{(b)}_{T_n})\Delta S_{T_n} I_{\{T_n\leq\tau\}} I_{[\![T_n,+\infty[\![}$ is a $\mathbb{G}$-martingale. Then, a direct application of Yor's exponential formula, we get

$$\mathcal{E}\left(I_\Gamma \centerdot N^{(b)} - I_\Gamma \centerdot V^{(b)}\right) \mathcal{E}\left(\theta I_\Gamma \centerdot S^\tau\right) = \mathcal{E}\left(W^{(b)}\right)$$

where $W^{(b)} := I_\Gamma \centerdot N^{(b)} - I_\Gamma \centerdot V^{(b)} + \sum_{n\geq 1} \theta_{T_n}\left(1 + \Delta N^{(b)}_{T_n} - \Delta V^{(b)}_{T_n}\right) \Delta S_{T_n} I_{\{T_n\leq\tau\}} I_{[\![T_n,+\infty[\![}$.

Consider the following $\mathbb{G}$-predictable process

$$\phi = \sum_{n\geq 1} \xi_n I_{[\![T_n]\!]\cap[\![0,\tau]\!]} + I_{\Gamma^c \cup]\!]\tau,+\infty[\![}, \qquad \text{where}$$

$$\xi_n := \frac{2^{-n}\left(1 + \mathcal{E}(W^{(b)})_{T_n-}\right)^{-1}}{\left(1 + E\left[|\Delta N^{(b)}_{T_n}| \,\middle|\, \mathcal{G}_{T_n-}\right] + \Delta V^{(b)}_{T_n-} + E\left[|\theta_{T_n} \frac{Z_{T_n-}}{\widetilde{Z}_{T_n}} I_{\{T_n\leq\tau\}} \Delta S_{T_n}| \middle| \mathcal{G}_{T_n-}\right]\right)}.$$

Then, it is easy to verify that ϕ is $\mathbb{G}$-predictable, $0 < \phi \leq 1$ and $E\left(|\phi \cdot \mathcal{E}(W^{(b)})|_{var}(+\infty)\right) \leq \sum_{n\geq 1} 2^{-n} = 1$. Hence, $\phi \centerdot \mathcal{E}(W^{(b)}) \in \mathcal{A}(\mathbb{G})$. Since, $\Delta N^{(b)}_{T_n} I_{[\![T_n,+\infty[\![}$ and $(1 + \Delta N^{(b)}_{T_n} - \Delta V^{(b)}_{T_n})\Delta S_{T_n} I_{\{T_n\leq\tau\}} I_{[\![T_n,+\infty[\![}$ are $\mathbb{G}$-martingales, we derive

$$\left(\phi \centerdot \mathcal{E}(W^{(b)})\right)^{p,\mathbb{G}} = \sum_n \phi_{T_n} \mathcal{E}_{T_n-}(W) E(\Delta W^{(b)}_{T_n} | \mathcal{G}_{T_n-}) I_{[\![T_n,+\infty[\![} = -\phi \mathcal{E}_-(W^{(b)}) \centerdot V^{(b)} \leq 0.$$

This proves that $\mathcal{E}(W^{(b)})$ is a positive σ-supermartingale. Thus, thanks to Kallsen [30], we conclude that it is a supermartingale and $\left(I_{\{Z_-\geq\delta\}}\bullet S\right)^\tau$ admits a $\mathbb{G}$-deflator. Then, thanks to Proposition 2.12, we deduce that $\left(I_{\{Z_-\geq\delta\}}\bullet S\right)^\tau$ satisfies NUPBR($\mathbb{G}$). Remark that, due to the $\mathbb{G}$-local boundedness of $(Z_-)^{-1}I_{[\![0,\tau]\!]}$, there exists a family of $\mathbb{G}$-stopping times $\tau_\delta,\ \delta>0$ such that τ_δ converges almost surely to infinity when δ goes zero and

$$[\![0,\tau\wedge\tau_\delta]\!]\subset\{Z_-\geq\delta\}.$$

This implies that $S^{\tau\wedge\tau_\delta}$ satisfies NUPBR($\mathbb{G}$), and the assertion (a) follows from Proposition 2.9 (by taking $Q_n=P$ for all $n\geq 1$). This ends the proof of (b)⇒(a).

2) We will prove (a)⇒(b). Suppose that S^τ satisfies NUPBR($\mathbb{G}$). Then, there exists a σ-martingale density under $\mathbb{G}$, for $I_{\{Z_-\geq\delta\}}\bullet S^\tau$, $(\delta>0)$, that we denote by $D^{\mathbb{G}}$. Then, from a direct application of Theorem A.1 and Theorem A.4, we deduce the existence of a positive $\widetilde{\mathcal{P}}(\mathbb{G})$-measurable functional, $f^{\mathbb{G}}$, such that $D^{\mathbb{G}}:=\mathcal{E}(N^{\mathbb{G}})>0$, with

$$N^{\mathbb{G}}:=W^{\mathbb{G}}\star(\mu^{\mathbb{G}}-\nu^{\mathbb{G}}),\ W^{\mathbb{G}}:=f^{\mathbb{G}}-1+\frac{\widehat{f^{\mathbb{G}}}-a^{\mathbb{G}}}{1-a^{\mathbb{G}}}I_{\{a^{\mathbb{G}}<1\}},$$

and

$$xf^{\mathbb{G}}I_{\{Z_-\geq\delta\}}\star\nu^{\mathbb{G}}=xf^{\mathbb{G}}\left(1+\frac{f_m}{Z_-}\right)I_{[\![0,\tau]\!]}I_{\{Z_-\geq\delta\}}\star\nu\equiv 0. \tag{3.45}$$

Here,

$$\nu^{\mathbb{G}}=\left(1+\frac{f_m}{Z_-}\right)I_{[\![0,\tau]\!]}\cdot\nu\ \text{ and }\ f_m:=M_\mu^P\left(\Delta m|\widetilde{\mathcal{P}}(\mathbb{F})\right).$$

Thanks to Lemma 2.13, we conclude the existence of a positive $\widetilde{\mathcal{P}}(\mathbb{F})$-measurable functional, f, such that $f^{\mathbb{G}}I_{[\![0,\tau]\!]}=fI_{[\![0,\tau]\!]}$. Thus (3.45) becomes

$$U:=xf\left(1+\frac{f_m}{Z_-}\right)I_{[\![0,\tau]\!]}I_{\{Z_->0\}}\star\nu\equiv 0.$$

Introduce the following notations

$$\begin{aligned}&\mu_0:=I_{\{\tilde{Z}>0\ \&\ Z_-\geq\delta\}}\cdot\mu,\ \ \nu_0:=h_0I_{\{Z_-\geq\delta\}}\cdot\nu,\ h_0:=M_\mu^P\left(I_{\{\tilde{Z}>0\}}|\widetilde{\mathcal{P}}\right),\\ &g:=\frac{f(1+\frac{f_m}{Z_-})}{h_0}I_{\{h_0>0\}}+I_{\{h_0=0\}},\ \ a_0(t):=\nu_0(\{t\},\mathbb{R}^d),\end{aligned} \tag{3.46}$$

and assume that

$$\sqrt{(g-1)^2\star\mu_0}\in\mathcal{A}^+_{loc}(\mathbb{F}). \tag{3.47}$$

Then, thanks to Lemma A.5, we deduce that $W:=(g-1)/(1-a^0+\widehat{g})\in\mathcal{G}^1_{loc}(\mu_0,\mathbb{F})$, and the local martingales

$$N^0:=\frac{g-1}{1-a^0+\widehat{g}}\star(\mu_0-\nu_0),\ \ Y^0:=\mathcal{E}(N^0), \tag{3.48}$$

are well defined satisfying $1+\Delta N^0>0$, $[N^0,S]\in\mathcal{A}(\mathbb{F})$, and on $\{Z_->0\}$ we have

$$\frac{{}^{p,\mathbb{F}}\left(Y^0\Delta SI_{\{\widetilde{Z}>0\}}\right)}{Y^0_-}={}^{p,\mathbb{F}}\left((1+\Delta N^0)\Delta SI_{\{\widetilde{Z}>0\}}\right)={}^{p,\mathbb{F}}\left(\frac{g}{1-a^0+\widehat{g}}\Delta SI_{\{\widetilde{Z}>0\}}\right)$$
$$=\Delta\frac{gxh_0}{1-a^0+\widehat{g}}\star\nu=\Delta\frac{xf(1+f_m/Z_-)}{1-a^0+\widehat{g}}\star\nu=Z_-^{-1}\frac{{}^{p,\mathbb{F}}(\Delta U)}{1-a^0+\widehat{g}}\equiv 0.$$

This proves that assertion (b) holds under the assumption (3.47). The remaining part of the proof will show that this assumption always holds. To this end, we start by noticing that on the set $\{h_0>0\}$,

$$g-1=\frac{f(1+\frac{f_m}{Z_-})}{h_0}-1=\frac{(f-1)(1+\frac{f_m}{Z_-})}{h_0}+\frac{f_m}{Z_-h_0}+\frac{M^P_\mu\left(I_{\{\widetilde{Z}=0\}}|\widetilde{\mathcal{P}}\right)}{h_0}$$
$$:=g_1+g_2+g_3.$$

Since $\left((f-1)^2I_{]\!]0,\tau]\!]}\star\mu\right)^{1/2}\in\mathcal{A}^+_{loc}(\mathbb{G})$, then due to Proposition 3.14–(4)

$$\sqrt{(f-1)^2I_{\{Z_-\geq\delta\}}\star(\widetilde{Z}\cdot\mu)}\in\mathcal{A}^+_{loc}(\mathbb{F}),\quad\text{for any }\ \delta>0.$$

Then, a direct application of Proposition 3.14–(1), for any $\delta>0$, we have

$$(f-1)^2I_{\{|f-1|\leq\alpha\ \&\ Z_-\geq\delta\}}\star(\widetilde{Z}\cdot\mu),\ \ |f-1|I_{\{|f-1|>\alpha\ \&\ Z_-\geq\delta\}}\star(\widetilde{Z}\cdot\mu)\in\mathcal{A}^+_{loc}(\mathbb{F}).$$

By stopping, without loss of generality, we assume the above two processes and $[m,m]$ belong to $\mathcal{A}^+(\mathbb{F})$ and remark that $Z_-+f_m=M^P_\mu\left(\widetilde{Z}|\widetilde{\mathcal{P}}\right)\leq M^P_\mu\left(I_{\{\widetilde{Z}>0\}}|\widetilde{\mathcal{P}}\right)=h_0$ that follows from $\widetilde{Z}\leq I_{\{\widetilde{Z}>0\}}$. Therefore, we derive

$$E\left[g_1^2I_{\{|f-1|\leq\alpha\}}\star\mu_0(\infty)\right]=E\left[\frac{(f-1)^2(1+\frac{f_m}{Z_-})^2}{h_0^2}I_{\{|f-1|\leq\alpha\}}\star\mu_0(\infty)\right]$$
$$=E\left[\frac{(f-1)^2(1+\frac{f_m}{Z_-})^2}{h_0^2}I_{\{|f-1|\leq\alpha\}}\star\nu_0(\infty)\right]$$
$$\leq\delta^{-2}E\left[(f-1)^2(Z_-+f_m)I_{\{|f-1|\leq\alpha\ \&\ Z_-\geq\delta\}}\star\nu(\infty)\right]$$
$$=\delta^{-2}E\left[(f-1)^2I_{\{|f-1|\leq\alpha\}}\star(\widetilde{Z}I_{\{Z_-\geq\delta\}}\cdot\mu)(\infty)\right]<+\infty,$$

and

$$E\left[g_1I_{\{|f-1|>\alpha\}}\star\mu_0(\infty)\right]=E\left[\frac{|f-1|(1+\frac{f_m}{Z_-})}{h_0}I_{\{|f-1|>\alpha\}}\star\mu_0(\infty)\right]$$
$$=E\left[|f-1|(1+\frac{f_m}{Z_-})I_{\{|f-1|>\alpha\}}I_{\{Z_-\geq\delta\}}\star\nu_0(\infty)\right]$$
$$\leq\delta^{-1}E\left[|f-1|I_{\{|f-1|>\alpha\}}\star(\widetilde{Z}I_{\{Z_-\geq\delta\}}\cdot\mu)(\infty)\right]<+\infty.$$

Here μ_0 and ν_0 are defined in (3.46). Therefore, again by Proposition 3.14–(a), we conclude that $\sqrt{g_1^2 \star \mu_0} \in \mathcal{A}^+_{loc}(\mathbb{F})$.

Notice that $g_2 + g_3 = \frac{M_\mu^P\left(\Delta m I_{\{\widetilde{Z}>0\}}|\widetilde{\mathcal{P}}\right)}{Z_- h_0}$, and due to Lemma A.2, we derive

$$\begin{aligned} E\left[(g_2+g_3)^2 \star \mu_0(\infty)\right] &= E\left[\frac{M_\mu^P\left(\Delta m I_{\{\widetilde{Z}>0\}}|\widetilde{\mathcal{P}}\right)^2}{Z_-^2 h_0^2} \star \mu_0(\infty)\right] \\ &\leq E\left[\frac{M_\mu^P\left((\Delta m)^2|\widetilde{\mathcal{P}}\right) M_\mu^P\left(I_{\{\widetilde{Z}>0\}}|\widetilde{\mathcal{P}}\right)}{Z_-^2 h_0^2} \star \mu_0(\infty)\right] \\ &= E\left[\frac{M_\mu^P\left((\Delta m)^2|\widetilde{\mathcal{P}}\right)}{Z_-^2} I_{\{Z_-\geq\delta\}} \star \mu(\infty)\right] \\ &\leq \delta^{-2} E\left[[m,m]_\infty\right] < +\infty. \end{aligned}$$

Hence, we conclude that $\sqrt{(g-1)^2 \star \mu_0} \in \mathcal{A}^+_{loc}(\mathbb{F})$. This ends the proof of (3.47), and the proof of the theorem is completed. □

The following generalizes Theorem 3.13 to the case of countable many predictable jumps, and constitutes our second main result of this subsection.

Theorem 3.16 *The following are equivalent:*
(a) *The thin set* $\left\{\widetilde{Z} = 0 \ \&\ Z_- > 0\right\}$ *is evanescent.*
(b) *For any (bounded) process S satisfying NUPBR($\mathbb{F}$), the process S^τ satisfies NUPBR($\mathbb{G}$).*

Proof We start by proving (a)$\Longrightarrow$(b). Suppose that assertion (a) holds. Then for any bounded S satisfying NUPBR($\mathbb{F}$), the condition (3.43) holds with $Y \equiv 1$. Thus, S^τ satisfies NUPBR($\mathbb{G}$). This proves assertion (b).
To prove (b)$\Longrightarrow$(a), we consider an $\mathbb{F}$-stopping times T such that $[\![T]\!] \subset \{\widetilde{Z} = 0 < Z_-\}$. Put

$$V := I_{[\![T,+\infty[\![}, \quad \widetilde{V} := (V)^{p,\mathbb{F}}, \ \text{ and } S := V - \widetilde{V}.$$

Then, due to $\{T < +\infty\} \subset \{\widetilde{Z}_T = 0\} \subset \{T \geq \widetilde{R}_0 > \tau\}$, it is easy to see that

$$V^\tau = I_{\{T\leq\tau\}} I_{[\![T,+\infty[\![} \equiv 0.$$

Therefore, we deduce that $X^\tau = -\widetilde{V}^\tau$ which is $\mathbb{G}$-predictable. Thus, X^τ satisfies NUPBR($\mathbb{G}$) if and only if it is a null process, due to Lemma 2.10. This is equivalent to

$$0 = E\left(\widetilde{V}_\tau\right) = E\left(\int_0^\infty Z_{s-} d\widetilde{V}_s\right) = E\left(Z_{T-} I_{\{T<+\infty\}}\right),$$

which –in turn– is equivalent to $T = +\infty,\ P-a.s.$ since $[\![T]\!] \subset \{\widetilde{Z} = 0 < Z_-\}$. Thus $\{\widetilde{Z} = 0\ \&\ Z_- > 0\}$ is evanescent, and the proof of the theorem is completed. □

Corollary 3.17 *The following assertions hold.*
(a)*If either m is quasi-left-continuous or Z is positive, S^τ satisfies NUPBR($\mathbb{G}$) for any process S that satisfies NUPBR($\mathbb{F}$).*
(b) *If S is continuous and satisfies NUPBR($\mathbb{F}$), then for any random time τ, S^τ satisfies NUPBR($\mathbb{G}$).*

Proof The proof of the assertion (a) of the corollary follows easily from Theorem 3.16, and either $\Delta m = \widetilde{Z} - Z_-$, which implies that $\{\widetilde{Z} = 0 < Z_-\} = \emptyset$, or $\{\widetilde{Z} = 0\} \subset \{Z = 0\} = \emptyset$.
The proof of assertion (b) follows from Theorem 3.16 and the fact that (3.43) is valid with $Y \equiv 1$. This ends the proof of the corollary. □

Remark 3.18 Any one of the two assertions of the above Corollary generalizes the main result of Fontana et al.[21], where the authors assume the predictable representation property on the market model, and other heavy assumptions on the random time τ and the market model as well. The original proof of the first assertion of this corollary was obtained in a very simple way by the third author, Jun Deng, and contained the main idea behind the obtained general development of this section.

4 Non-Arbitrage after Specific Honest Times

In this section, we focus on a specific class of honest times (see Definition 2.5) τ, for which the associated supermartingale Z satisfies

$$Z_\tau < 1, \quad P-a.s. \tag{4.1}$$

Remark 4.1 1) It is proved in Kardaras [33] that an honest time satisfies $Z_\tau = 1 \quad P-a.s.$ if and only if it avoids $\mathbb{F}$-stopping times. It is obvious that any stopping time is a honest time satisfying (4.1), while there are many honest times satisfying (4.1) which are not stopping times. For details and examples about these random times, we refer the reader to Aksamit et al. [2]. More importantly, for this latter paper, the condition (4.1) has been the main leitmotif and an important hint.
2) Our condition (4.1) represents our first original contribution in this section. Indeed, Imkeller [25] (see also Fontana et al. [21]) proved that for honest times avoiding stopping times in a Brownian filtration, the NUPBR is violated after τ. Up to our knowledge, there is no single result —in the literature— that suggests an alternative for the part after τ. Herein, via our condition (4.1), we are suggesting a class of honest times that might preserve the NUPBR for any quasi-left-continuous process.

This section contains four subsections, where we study the NUPBR condition after an honest time τ satisfying (4.1). Like in the previous section, the first subsection presents new stochastic developments for the part after τ which will be of paramount importance for the proofs of the results . These innovative results are summarized in two lemmas (see Lemmas 4.2, 4.3) and two propositions (see Proposition 4.4-4.5). The second subsection focuses on the quasi-left-continuous processes, while the third subsection deals with the single jump processes. Finally, the last subsection states the results for the general case.

4.1 Comparing Dual Predictable Projections under $\mathbb{G}$ and $\mathbb{F}$ after τ

Lemma 4.2 *Suppose that (4.1) holds, and denote*

$$J := 1 - Z_- - \Delta m = 1 - \widetilde{Z}\,. \tag{4.2}$$

Then the following assertions hold.
(a) *The $\mathbb{G}$-predictable process $\frac{1}{1-Z_-} I_{\rrbracket \tau, +\infty \llbracket}$ is $\mathbb{G}$-locally bounded.*
(b) *For any $\mathbb{F}$-adapted process, V, with locally integrable variation we have*

$$I_{\rrbracket \tau, +\infty \llbracket} \cdot V^{p,\mathbb{G}} = I_{\rrbracket \tau, +\infty \llbracket} \frac{1}{1 - Z_-} \cdot (J \cdot V)^{p,\mathbb{F}}\ . \tag{4.3}$$

(c) *For any process V as in* (b)*, one has, on $\rrbracket \tau, +\infty \llbracket$*

$$^{p,\mathbb{G}}(\Delta V) = \frac{1}{1 - Z_-}\ ^{p,\mathbb{F}}(J \Delta V)\ . \tag{4.4}$$

(d) *For any $\mathbb{F}$-local martingale M, one has, on $\rrbracket \tau, +\infty \llbracket$*

$$^{p,\mathbb{G}}\left(\frac{\Delta M}{1 - \widetilde{Z}}\right) = \frac{^{p,\mathbb{F}}\left(\Delta M I_{\{\widetilde{Z} < 1\}}\right)}{1 - Z_-}, \quad \textit{and} \quad ^{p,\mathbb{G}}\left(\frac{1}{J}\right) = \frac{^{p,\mathbb{F}}\left(I_{\{\widetilde{Z} < 1\}}\right)}{1 - Z_-}. \tag{4.5}$$

(e) *For any quasi-left-continuous $\mathbb{F}$-local martingale M, one has, on $\rrbracket \tau, +\infty \llbracket$*

$$^{p,\mathbb{G}}\left(\frac{\Delta M}{J}\right) = 0, \quad \textit{and} \quad ^{p,\mathbb{G}}\left(\frac{1}{1 - Z_- - \Delta m^{(qc)}}\right) = \frac{1}{1 - Z_-}, \tag{4.6}$$

where $m^{(qc)}$ is the quasi-left-continuous part of m, defined in 2.3.

Proof The proof of the lemma will be achieved in four steps.
1) Herein we prove assertion (a). It is known [14, Chapter XX] that $Z = \widetilde{Z}$ on $\rrbracket \tau, +\infty \llbracket$, and

$$\rrbracket \tau, +\infty \llbracket \subset \{Z_- < 1\} \cap \{\widetilde{Z} < 1\} = \{Z_- < 1\} \cap \{Z < 1\}\ .$$

Then due to (4.1), we deduce that $[\![\tau, +\infty \llbracket \subset \{Z < 1\}$, and hence the process

$$X := (1 - Z)^{-1} I_{[\![\tau, +\infty \llbracket},$$

is càdlàg $\mathbb{G}$-adapted. Combining these with $]\!]\tau,+\infty[\![\subset \{Z_- < 1\}$, we easily prove that $\sup_{0\le u\le t} X_u < +\infty,\ P-a.s.$ This is equivalent to say that X is pre-locally bounded and hence its left limit process $X_- = (1-Z_-)^{-1} I_{]\!]\tau,+\infty[\![}$ is locally bounded. This proves assertion (a).

2) This part proves assertions (b) and (c). Since

$$I_{]\!]\tau,+\infty[\![} \centerdot V - I_{]\!]\tau,+\infty[\![} \centerdot V^{p,\mathbb{F}} + I_{]\!]\tau,+\infty[\![}(1-Z_-)^{-1} \centerdot \langle V, m\rangle^{\mathbb{F}}$$

is a $\mathbb{G}$-local martingale, we obtain

$$\begin{aligned}
\left(I_{]\!]\tau,+\infty[\![} \centerdot V\right)^{p,\mathbb{G}} &= I_{]\!]\tau,+\infty[\![} \centerdot V^{p,\mathbb{F}} - I_{]\!]\tau,+\infty[\![}(1-Z_-)^{-1} \centerdot \langle V, m\rangle^{\mathbb{F}},\\
&= I_{]\!]\tau,+\infty[\![} \centerdot V^{p,\mathbb{F}} - I_{]\!]\tau,+\infty[\![}(1-Z_-)^{-1} \centerdot (\Delta m \centerdot V)^{p,\mathbb{F}},\\
&= I_{]\!]\tau,+\infty[\![}(1-Z_-)^{-1} \centerdot \Big((1-Z_- - \Delta m) \centerdot V\Big)^{p,\mathbb{F}}.
\end{aligned}$$

This ends the proof of (4.3). The equality (4.4) follows immediately from (4.3) by taking the jumps in both sides.

3) Now, we prove assertion (d). By applying (4.3) for $V_{\epsilon,\delta} \in \mathcal{A}_{loc}(\mathbb{F})$ given by

$$V_{\epsilon,\delta} := \sum \frac{\Delta M}{J} I_{\{|\Delta M|\ge\epsilon,\ J\ge\delta\}},$$

we get, on $]\!]\tau,+\infty[\![$,

$$^{p,\mathbb{G}}\left(\frac{\Delta M}{J} I_{\{|\Delta M|\ge\epsilon,\ J\ge\delta\}}\right) = (1-Z_-)^{-1}\ {}^{p,\mathbb{F}}\left(\Delta M I_{\{|\Delta M|\ge\epsilon,\ J\ge\delta\}}\right).$$

Then, by letting ϵ and δ go to zero, on $]\!]\tau,+\infty[\![$ we obtain

$$^{p,\mathbb{G}}\left(\frac{\Delta M}{J}\right) = (1-Z_-)^{-1}\ {}^{p,\mathbb{F}}\left(\Delta M I_{\{J>0\}}\right).$$

This proves the first equality in (4.5). The remaining equality in (4.5) follows from the first equality and the fact that ${}^{p,\mathbb{F}}(\Delta m) = 0$. Indeed, we have, on $]\!]\tau,+\infty[\![$,

$$\begin{aligned}
^{p,\mathbb{G}}\left(\frac{1}{J}\right) &= (1-Z_-)^{-1} + (1-Z_-)^{-1}\ {}^{p,\mathbb{G}}\left(\frac{\Delta m}{J}\right)\\
&= (1-Z_-)^{-1} + (1-Z_-)^{-2}\ {}^{p,\mathbb{F}}\left(\Delta m I_{\{J>0\}}\right)\\
&= (1-Z_-)^{-1} - (1-Z_-)^{-1}\ {}^{p,\mathbb{F}}\left(I_{\{J=0\}}\right) = (1-Z_-)^{-1}\ {}^{p,\mathbb{F}}\left(I_{\{\widetilde{Z}<1\}}\right).
\end{aligned}$$

This ends the proof of assertion (d).

4) The proof of assertion (e) follows immediately from assertion (d) and the fact that the thin process ${}^{p,\mathbb{F}}\left(\Delta M I_{\{J>0\}}\right)$ may take nonzero values on countably many predictable stopping times only, on which ΔM already vanishes. Hence the first equality in (4.6) follows. A direct application of this equality implies that on $\{Z_- < 1\}$, we have

$${}^{p,\mathbb{F}}\left(\frac{1}{1-Z_- - \Delta m^{(qc)}}\right) = \frac{1}{1-Z_-} + \frac{1}{1-Z_-}\,{}^{p,\mathbb{F}}\left(\frac{\Delta m^{(qc)}}{1-Z_- - \Delta m^{(qc)}}\right) = \frac{1}{1-Z_-}.$$

This completes the proof of (4.6) as well as the proof of the lemma. □

The next lemma focuses on the integrability of the process $J^{-1}I_{]\!]\tau,+\infty[\![}$ with respect to any process with $\mathbb{F}$-locally integrable variation.

Lemma 4.3 *Let V be a càdlàg $\mathbb{F}$-adapted process. Then, the following assertions hold.*
(a) *If V is nondecreasing and locally integrable (respectively integrable), then*

$$U := (1-Z)^{-1} I_{]\!]\tau,+\infty[\![} \cdot V = J^{-1} I_{]\!]\tau,+\infty[\![} \cdot V, \tag{4.7}$$

is $\mathbb{G}$-locally integrable (respectively $\mathbb{G}$-integrable).
(b) *If $V \in \mathcal{A}_{loc}(\mathbb{F})$, then U is a well defined process, $U \in \mathcal{A}_{loc}(\mathbb{G})$ and*

$$(U)^{p,\mathbb{G}} = I_{]\!]\tau,+\infty[\![}(1-Z_-)^{-1} \cdot (I_{\{\widetilde{Z}<1\}} \cdot V)^{p,\mathbb{F}}. \tag{4.8}$$

Proof (a) Let $(\vartheta_n)_{n\geq 1}$ be a sequence of $\mathbb{F}$-stopping times that increases to $+\infty$ such that

$$E\left(V_{\vartheta_n}\right) < +\infty.$$

Then the lemma follows immediately if we prove $E\left(U_{\vartheta_n}\right) \leq E\left(V_{\vartheta_n}\right)$. To this end, we use the fact that $Z = \widetilde{Z}$ after τ to obtain

$$\begin{aligned} E\left(U_{\vartheta_n}\right) &= E\left(\int_0^{\vartheta_n} I_{\{t>\tau\}} \frac{1}{1-\widetilde{Z}_t} dV_t\right) = E\left(\int_0^{\vartheta_n} \frac{P(\tau < t|\mathcal{F}_t)}{1-\widetilde{Z}_t} I_{\{Z_t<1\}} dV_t\right) \\ &= E\left(\int_0^{\vartheta_n} I_{\{Z_s<1\}} dV_s\right) \leq E\left(V_{\vartheta_n}\right). \end{aligned}$$

Since ϑ_n increases to $+\infty$, we get $U \in \mathcal{A}^+_{loc}(\mathbb{G})$. This proves assertion (a).
(b) Suppose that $V \in \mathcal{A}_{loc}(\mathbb{F})$. Then, $\mathrm{Var}(V) = V^+ + V^- \in \mathcal{A}^+_{loc}(\mathbb{F})$, and hence due to assertion (a), we deduce that U has a $\mathbb{G}$-local integrable variation that coincides with

$$\mathrm{Var}(U) = (1-Z)^{-1} I_{]\!]\tau,+\infty[\![} \cdot \mathrm{Var}(V).$$

For any $n \geq 1$, put

$$U_n := (1-Z)^{-1} I_{]\!]\tau,+\infty[\![} I_{\{\widetilde{Z}\leq 1-\frac{1}{n}\}} \cdot V = \left(1-\widetilde{Z}\right)^{-1} I_{]\!]\tau,+\infty[\![} I_{\{\widetilde{Z}\leq 1-\frac{1}{n}\}} \cdot V.$$

Then, thanks to (4.3), we derive

$$(U)^{p,\mathbb{G}} = \lim_{n\longrightarrow+\infty} (U_n)^{p,\mathbb{G}}$$

$$= \lim_{n\longrightarrow+\infty} (1-Z_-)^{-1} I_{]\!]\tau,+\infty[\![} \cdot \left(I_{\{\widetilde{Z}\leq 1-\frac{1}{n}\}} \cdot V\right)^{p,\mathbb{F}}.$$

This clearly implies (4.8), and the proof of the lemma is completed. □

Similarly as in Propositions 3.4–3.5, for the case before τ, we will elaborate two important propositions that are crucial for the analysis in the forthcoming subsections.

Proposition 4.4 *Let τ be an honest time satisfying (4.1). Then, the following nondecreasing process*

$$V_t^{(a)} := \sum_{0<u\leq t} {}^{p,\mathbb{F}}\left(I_{\{\widetilde{Z}=1\}}\right)_u I_{\{u>\tau\}}, \tag{4.9}$$

is $\mathbb{G}$-predictable, càdlàg, and locally bounded.

Proof Since under assumption (4.1) the process $(1-Z_-)^{-1}I_{]\!]\tau,+\infty[\![}$ is $\mathbb{G}$-locally bounded, then there exists a sequence of $\mathbb{G}$-stopping times $(\tau_n)_n$ that increases to infinity such that

$$\left((1-Z_-)^{-1}I_{]\!]\tau,+\infty[\![}\right)^{\tau_n} \leq n+1.$$

Consider a sequence of $\mathbb{F}$-stopping times, $(\sigma_n)_n$, that increases to infinity and $\langle m,m\rangle_{\sigma_n} \leq n$. Then, for any nonnegative $\mathbb{F}$-predictable process H which is bounded by $C>0$, we derive

$$\begin{aligned}(H\cdot V^{(a)})_{\sigma_n\wedge\tau_n} &\leq \sum_{0\leq u\leq\sigma_n} H_u\ {}^{p,\mathbb{F}}\left(I_{\{\widetilde{Z}=1>Z_-\}}\right)_u I_{\{1-Z_{u-}\geq\frac{1}{n+1}\}}\\ &= \sum_{0\leq u\leq\sigma_n} H_u\ {}^{p,\mathbb{F}}\left(I_{\{\Delta m=1-Z_-\geq\frac{1}{n+1}\}}I_{\{Z_-\leq 1-\frac{1}{n+1}\}}\right)_u\\ &\leq \sum_{0\leq u\leq\sigma_n} H_u\ {}^{p,\mathbb{F}}\left(I_{\{\Delta m\geq\frac{1}{n+1}\}}\right)_u \leq (n+1)^2 H\cdot\langle m,m\rangle_{\sigma_n} \leq C(n+1)^3.\end{aligned}$$

This ends the proof of the proposition. □

Proposition 4.5 *Consider*

$$\widehat{m}^{(a)} := \frac{1}{1-Z_-} I_{]\!]\tau,+\infty[\![}\cdot m + \frac{1}{(1-Z_-)^2} I_{]\!]\tau,+\infty[\![}\cdot\langle m\rangle^{\mathbb{F}}, \tag{4.10}$$

and the space ${}^oL^1_{loc}(\widehat{m}^{(a)},\mathbb{G})$ defined in Definition 2.7. Then, we have

$$K^{(a)} := \frac{(1-Z_-)^3}{(1-Z_-)^2+\Delta\langle m\rangle^{\mathbb{F}}}\,\frac{1}{1-\widetilde{Z}}\,I_{]\!]\tau,+\infty[\![} \in {}^oL^1_{loc}(\widehat{m}^{(a)},\mathbb{G}). \tag{4.11}$$

Furthermore, the associated $\mathbb{G}$-local martingale,

$$\widetilde{N}^{(a)} := K^{(a)} \odot \widehat{m}^{(a)}, \tag{4.12}$$

satisfies the following
(a) $\mathcal{E}(\widetilde{N}^{(a)}) > 0$ *(or equivalently* $1 + \Delta\widetilde{N}^{(a)} > 0$*) and* $I_{]\!]0,\tau]\!]} \cdot \widetilde{N}^{(a)} = 0$.
(b) *For any* $M^{\mathbb{G}} := I_{]\!]\tau,+\infty[\![} \cdot M + (1 - Z_-)^{-1} I_{]\!]\tau,+\infty[\![} \cdot \langle M, m\rangle^{\mathbb{F}}$, *where* $M \in \mathcal{M}_{0,loc}(\mathbb{F})$, *we have*

$$[\widetilde{N}^{(a)}, M^{\mathbb{G}}] \in \mathcal{A}_{loc}(\mathbb{G}) \quad \left(i.e.\ \langle \widetilde{N}^{(a)}, M^{\mathbb{G}}\rangle^{\mathbb{G}} \ \ exists\right). \tag{4.13}$$

Proof In the proof, for the sake of simplicity, we note $K := K^{(a)}, \widehat{m} := \widehat{m}^{(a)}$. Thanks to (4.5), we derive

$$0 \leq^{p,\mathbb{G}} (K) \leq \frac{(1 - Z_-)^2}{(1 - Z_-)^2 + \Delta\langle m\rangle^{\mathbb{F}}} I_{]\!]\tau,+\infty[\![} \leq 1.$$

Thus, we conclude that $K \in {}^{o}L^1_{loc}(\widehat{m}, \mathbb{G})$ if and only if

$$\left(\sum \left(\frac{1}{J}\Delta\widehat{m} - {}^{p,\mathbb{G}}\left(\frac{1}{J}\Delta\widehat{m}\right)\right)^2\right)^{1/2} \in \mathcal{A}^+_{loc}(\mathbb{G}). \tag{4.14}$$

To prove this fact, on $]\!]\tau, +\infty[\![$, we calculate, making use of Lemma 4.2

$$\begin{aligned} J^{-1}\Delta\widehat{m} - {}^{p,\mathbb{G}}\left(J^{-1}\Delta\widehat{m}\right) &= \frac{\Delta m}{(1 - Z_-)J} + (1 - Z_-)^{-2}\frac{1}{J}\Delta\langle m\rangle^{\mathbb{F}} \\ &\quad -(1 - Z_-)^{-3}\,\Delta\langle m\rangle^{\mathbb{F}}\ {}^{p,\mathbb{F}}(I_{\{J>0\}}) - (1 - Z_-)^{-2}\ {}^{p,\mathbb{F}}(\Delta m I_{\{J>0\}}) \\ &= \frac{\Delta m}{1 - \widetilde{Z}} + {}^{p,\mathbb{F}}\left(I_{\{\widetilde{Z}=1\}}\right). \end{aligned} \tag{4.15}$$

Therefore, the proof of (4.14) follows the same steps as in the proof of Proposition 3.5 by using $C := \{\Delta m > (1 - \delta)Z_-\}$, the $\mathbb{G}$-localizing sequence of stopping time $(\tau_n)_{n\geq 1}$ for the process $(1 - Z_-)^{-1} I_{]\!]\tau,+\infty[\![}$, and Proposition 4.4 instead. This proves (4.11).
Assertion (a) follows from $I_{]\!]0,\tau]\!]} \cdot \widehat{m} = 0$ and (4.15) that implies that $1 + \Delta\widetilde{N} = 1 + \frac{\Delta m}{1-\widetilde{Z}} I_{]\!]\tau,+\infty[\![} + {}^{p,\mathbb{F}}\left(I_{\{\widetilde{Z}=1\}}\right) I_{]\!]\tau,+\infty[\![} > 0$.
We now prove (4.13). Due to Proposition 2.8 and the $\mathbb{G}$-local boundedness of $(1 - Z_-)^{-2} I_{]\!]\tau,+\infty[\![}$ (see Lemma 4.2(a)), we deduce that $[\widetilde{N}^{(a)}, M^{\mathbb{G}}] \in \mathcal{A}_{loc}(\mathbb{G})$ if and only if $J^{-1} \cdot [m^{\mathbb{G}}, M^{\mathbb{G}}] \in \mathcal{A}_{loc}(\mathbb{G})$, where

$$m^{\mathbb{G}} := I_{]\!]\tau,+\infty[\![} \cdot m + (1 - Z_-)^{-1} I_{]\!]\tau,+\infty[\![} \cdot \langle m\rangle^{\mathbb{F}}. \tag{4.16}$$

We calculate this quantity

$$\begin{aligned} J^{-1} \cdot [m^{\mathbb{G}}, M^{\mathbb{G}}] &= J^{-1} I_{]\!]\tau,+\infty[\![} \cdot [m, M] + J^{-1}(1 - Z_-)^{-1}\Delta m I_{]\!]\tau,+\infty[\![} \cdot \langle m, M\rangle^{\mathbb{F}} \\ &\quad + \frac{\Delta M}{J(1 - Z_-)} I_{]\!]\tau,+\infty[\![} \cdot \langle m\rangle^{\mathbb{F}} + \frac{\Delta\langle m\rangle^{\mathbb{F}}}{J(1 - Z_-)^2} I_{]\!]\tau,+\infty[\![} \cdot \langle m, M\rangle^{\mathbb{F}}. \end{aligned}$$

On the one hand,

$$(1-Z_-)^{-1}\Delta m I_{]\!]\tau,+\infty[\![},\ \Delta\langle m\rangle^{\mathbb{F}}(1-Z_-)^{-2}I_{]\!]\tau,+\infty[\![} \text{ and } (1-Z_-)^{-1}I_{]\!]\tau,+\infty[\![}$$

are $\mathbb{G}$-locally bounded. On the other hand, since $[m,M]$, $\Delta M \centerdot \langle m\rangle^{\mathbb{F}}$ and $\langle m,M\rangle^{\mathbb{F}}$ belong to $\mathcal{A}_{loc}(\mathbb{F})$, thanks to Lemma 4.3(a) we conclude that

$$J^{-1}I_{]\!]\tau,+\infty[\![}\centerdot[m,M],\ J^{-1}I_{]\!]\tau,+\infty[\![}\centerdot\left(\Delta M\centerdot\langle m\rangle^{\mathbb{F}}\right) \text{ and } J^{-1}I_{]\!]\tau,+\infty[\![}\centerdot\langle m,M\rangle^{\mathbb{F}}$$

belong to $\mathcal{A}_{loc}(\mathbb{G})$. Therefore, this achieves the proof of (4.13), and the proof of the proposition is completed. □

4.2 The Case of Quasi-Left-Continuous Processes

Theorem 4.6 *Let τ be an honest time satisfying (4.1), and $\widetilde{N}^{(a)}$ be the $\mathbb{G}$-local martingale defined by (4.12). Let M be a quasi-left-continuous $\mathbb{F}$-local martingale. If $\{\widetilde{Z}=1>Z_-\ \&\ \Delta M\neq 0\}$ is evanescent, the following two equivalent assertions hold.*
(a) $\mathcal{E}(\widetilde{N}^{(a)})\,(M-M^\tau)$ *is a $\mathbb{G}$-local martingale.*
(b) $\mathcal{E}\left(I_{\{\widetilde{Z}=1>Z_-\}}\odot m\right)M$ *is an $\mathbb{F}$-local martingale.*

Proof Suppose that $\{\widetilde{Z}=1>Z_-\ \&\ \Delta M\neq 0\}$ is evanescent. Then $\sum I_{\{\widetilde{Z}=1>Z_-\}}\Delta M(1-Z_-)=I_{\{\widetilde{Z}=1>Z_-\}}\centerdot[M,m]\equiv 0$, and due to the quasi-left-continuity of M and Proposition 2.8-(b), we easily derive

$$\langle I_{\{\widetilde{Z}=1>Z_-\}}\odot m,M\rangle^{\mathbb{F}}={}^{p,\mathbb{F}}\left(I_{\{\widetilde{Z}=1>Z_-\}}\centerdot[M,m]\right)=0,$$

which is equivalent to assertion (b). To achieve the proof of the theorem, we will prove that both assertions (a) and (b) are equivalent. In the remaining part of the proof, we shall denote $\widehat{N}=\widehat{m}^{(a)}$, $\widetilde{N}=\widetilde{N}^{(a)}$ and $K=K^{(a)}$.
Notice that the quasi-left-continuity of M implies $[\langle m\rangle,M]=[m,\langle m,M\rangle]=[\langle m\rangle,\langle m,M\rangle]=0$. Hence

$$K\centerdot[\widehat{N},M^{\mathbb{G}}]=\frac{K}{1-Z_-}\centerdot[m,M]=\frac{1}{1-\widetilde{Z}}I_{]\!]\tau,+\infty[\![}\centerdot[m,M],\tag{4.17}$$

where $M^{\mathbb{G}}$ is the $\mathbb{G}$-local martingale associated to M and is given by

$$M^{\mathbb{G}}=I_{]\!]\tau,+\infty[\![}\centerdot M+(1-Z_-)^{-1}I_{]\!]\tau,+\infty[\![}\centerdot\langle m,M\rangle^{\mathbb{F}}.$$

Then, by combining (4.17), Proposition 2.8-(b) and (4.3), we calculate

$$\begin{aligned}\langle\widetilde{N},M^{\mathbb{G}}\rangle^{\mathbb{G}}&=\left(K\centerdot[\widehat{N},M^{\mathbb{G}}]\right)^{p,\mathbb{G}}=\left((1-\widetilde{Z})^{-1}I_{]\!]\tau,+\infty[\![}\centerdot[m,M]\right)^{p,\mathbb{G}}\\&=\frac{I_{]\!]\tau,+\infty[\![}}{1-Z_-}\centerdot\langle m,M\rangle^{\mathbb{F}}-\frac{I_{]\!]\tau,+\infty[\![}}{1-Z_-}\centerdot\left(I_{\{\widetilde{Z}=1>Z_-\}}\centerdot[M,m]\right)^{p,\mathbb{F}}\\&=\frac{I_{]\!]\tau,+\infty[\![}}{1-Z_-}\centerdot\langle m,M\rangle^{\mathbb{F}}-\frac{I_{]\!]\tau,+\infty[\![}}{1-Z_-}\centerdot\langle I_{\{\widetilde{Z}=1>Z_-\}}\odot m,M\rangle^{\mathbb{F}}.\end{aligned}$$

Then, from the above equality, the equivalence between the two assertions (a) and (b) follows immediately. □

Below, we elaborate our principal result of this subsection.

Theorem 4.7 *Let τ be an honest time satisfying (4.1). Then, for any quasi-left-continuous process, S, satisfying NUPBR($\mathbb{F}$), $S-S^\tau$ satisfies NUPBR($\mathbb{G}$).*

Proof The proof of this theorem follows the same pattern of the five steps of the proof of Theorem 3.7 (the part before τ), while the difference between the two proofs lies in the arguments used to achieve each step. Thus, by arguing as in the fifth step of the proof of Theorem 3.7, there is no loss of generality in assuming –for the rest of this proof– that S is a quasi-left-continuous $\mathbb{F}$-local martingale. Introduce

$$h := I_{\{\widetilde{Z}=1>Z_-\}} \quad \text{and} \quad \overline{S} := S - h \odot S = (1-h) \odot S.$$

For any $\mathbb{F}$-local martingale, M, we denote by $M^{\mathbb{G}}$ the $\mathbb{G}$-local martingale

$$M^{\mathbb{G}} := I_{]\!]\tau,+\infty[\![} \cdot M + (1-Z_-)^{-1} I_{]\!]\tau,+\infty[\![} \cdot \langle M, m\rangle^{\mathbb{F}}.$$

By applying this to the two $\mathbb{F}$-local martingales S and $h \odot S$, we obtain

$$\begin{aligned} I_{]\!]\tau,+\infty[\![} \cdot S &= S^{\mathbb{G}} - \frac{1}{1-Z_-} I_{]\!]\tau,+\infty[\![} \cdot \langle S, m\rangle^{\mathbb{F}}, \\ &= \mathbb{G} - \text{local martingale} - \frac{c\beta_m}{1-Z_-} I_{]\!]\tau,+\infty[\![} \cdot A - \frac{x f_m}{1-Z_-} I_{]\!]\tau,+\infty[\![} \star \nu \\ I_{]\!]\tau,+\infty[\![} \cdot (h \odot S) &= \widehat{h \odot S}^{\mathbb{G}} - \frac{1}{1-Z_-} I_{]\!]\tau,+\infty[\![} \cdot \langle h \odot S, m\rangle^{\mathbb{F}} \\ &= \mathbb{G} - \text{local martingale} - I_{]\!]\tau,+\infty[\![} x M_\mu^P \left(h \middle| \widetilde{\mathcal{P}}\right) \star \nu. \end{aligned}$$

The last equality follows for the fact that the set $\{\widetilde{Z} = 1 > Z_-\}$ is thin and coincides with $\{\Delta m = 1 - Z_- > 0\}$. Thus, since S is quasi-left-continuous, we deduce that $h \odot S$ is a purely discontinuous local martingale and its jumps given by $\Delta(h \odot S) = h\Delta S$ vanish at any $\mathbb{F}$-predictable stopping time. Hence, it is obvious that both $(h \odot S)$ and $\overline{S}$ are quasi-left-continuous $\mathbb{F}$-local martingales, and

$$\{\Delta \overline{S} \neq 0\} \cap \{\widetilde{Z} = 1 > Z_-\} = \emptyset.$$

Therefore, a direct application of Theorem 4.6 for the process $\overline{S}$, implies the existence of a positive $\mathbb{G}$-local martingale $\mathcal{E}(N_1)$ such that $\mathcal{E}(N_1) I_{]\!]\tau,+\infty[\![} \cdot \overline{S}$ is a $\mathbb{G}$-local martingale. This is equivalent to the existence of a positive $\widetilde{\mathcal{P}}(\mathbb{G})$-measurable functional f_1 and $\beta_1 \in L\left((S^c)^{\mathbb{G}}\right)$ such that $N_1 = \beta_1 \cdot (S^c)^{\mathbb{G}} + (f_1 - 1) \star (\mu^{\mathbb{G}} - \nu^{\mathbb{G}})$, and $dA - a.e.$ on $]\!]\tau, +\infty[\![$, we have

$$\frac{c\beta_m}{1-Z_-} = c\beta_1 + \int x \left\{ f_1 \left(1 - \frac{f_m}{1-Z_-}\right) + M_\mu^P \left(h \middle| \widetilde{\mathcal{P}}\right) - 1 \right\} F(dx). \quad (4.18)$$

Here $\mu^{\mathbb{G}} := I_{]\!]\tau,+\infty[\![}\mu$ and $\nu^{\mathbb{G}}$ its $\mathbb{G}$-compensator.
Now, we construct a martingale density for $S - S^{\tau}$. Let $\mathcal{E}(N)$ be a positive $\mathbb{G}$-local martingale (with Jacod's parameters —guaranteed by Theorem A.1– $(\beta, f, 0, 0)$) such that

$$N := \beta \bullet (S^c)^{\mathbb{G}} + (f-1) \star (\mu^{\mathbb{G}} - \nu^{\mathbb{G}}). \tag{4.19}$$

Then $\mathcal{E}(N)(S - S^{\tau})$ is a $\mathbb{G}$-local martingale if and only if

$$\frac{c\beta_m}{1-Z_-} = c\beta + \int x \left\{ f\left(1 - \frac{f_m}{1-Z_-}\right) - 1 \right\} F(dx). \tag{4.20}$$

Thus, by choosing β and f as

$$\beta := \beta_1, \text{ and } \ f := f_1 + \frac{M_{\mu}^{P}\left(h \middle| \widetilde{\mathcal{P}}\right)}{1 - \frac{f_m}{1-Z_-}} I_{\{Z_- < 1 \ \& \ f_m < 1 - Z_-\}}. \tag{4.21}$$

we deduce that $f \geq f_1 > 0$ and (4.20) holds due to (4.18) and $]\!]\tau, +\infty[\![\subset \{Z_- < 1 \ \& \ f_m < 1 - Z_-\}$. Therefore, the proof will be completed once we show that

$$f - 1 \in \mathcal{G}^1_{loc}(\mu^{\mathbb{G}}, \mathbb{G}). \tag{4.22}$$

To this end, consider two sequences of $\mathbb{F}$-stopping times and $\mathbb{G}$-stopping times $(\sigma_n^{\mathbb{F}})_n$ and $(\tau_n^{\mathbb{G}})_n$ that increase to infinity and satisfy

$$\left((1-Z_-)^{-1} I_{]\!]\tau,+\infty[\![}\right)^{\tau_n^{\mathbb{G}}} \leq n \quad \text{and} \quad [m,m]_{\sigma_n^{\mathbb{F}}} \leq n.$$

Then, since $f_1 - 1 \in \mathcal{G}^1_{loc}(\mu^{\mathbb{G}}, \mathbb{G})$, we focus on $g := \frac{M_{\mu}^{P}\left(h \middle| \widetilde{\mathcal{P}}\right)}{1 - \frac{f_m}{1-Z_-}} I_{\{Z_- < 1 \ \& \ f_m < 1 - Z_-\}}$ and we derive

$$\begin{aligned} E\Big(g \star \mu^{\mathbb{G}}(\sigma_n^{\mathbb{F}} \wedge \tau_n^{\mathbb{G}})\Big) &\leq E\Big(g I_{\{1-Z_- \geq n^{-1}\}} \star \mu(\sigma_n^{\mathbb{F}})\Big) \\ &\leq E\Big(h I_{\{1-Z_- \geq n^{-1}\}} I_{]\!]\tau,+\infty[\![} \star \mu(\sigma_n^{\mathbb{F}})\Big) \leq E\Big(\sum_{0<u\leq\sigma_n^{\mathbb{F}}} I_{\{\Delta m \geq n^{-1}\}}\Big) \leq n^3. \end{aligned}$$

This proves that $g \star \mu^{\mathbb{G}} \in \mathcal{A}^{+}_{loc}(\mathbb{G})$. Hence, a combination of this fact and $\sqrt{\sum (\Delta X)^2} \leq \sum |\Delta X|$, the condition (4.22) follows immediately, and the proof of the theorem is completed. □

Corollary 4.8 *Let τ be an honest time satisfying (4.1). Then, the thin set $\{\widetilde{Z} = 1 > Z_-\}$ is accessible.*

Proof Consider a sequence of stopping times $(T_n)_{n\geq 1}$ that exhausts the thin set $\{\widetilde{Z} = 1 \ \& \ Z_- < 1\}$ (i.e $\left\{\widetilde{Z} = 1 \ \& \ Z_- < 1\right\} = \bigcup_{n=1}^{+\infty} [\![T_n]\!]$). Then, each T_n —that we denote by T for the sake of simplicity— can be decomposed into a totally

inaccessible part T^i and and an accessible part T^a such that $T = T^i \wedge T^a$. Consider the following quasi-left-continuous processes

$$V := I_{[\![T^i,+\infty[\![}, \quad \text{and} \quad M := V - (V)^{p,\mathbb{F}} =: V - \widetilde{V}.$$

Then, since on $\{T^i < +\infty\} \subset \{\widetilde{Z}_{T^i} = 1\}$, we deduce that $\{T^i < +\infty\} \subset \{\tau \geq T^i\}$ and hence

$$I_{]\!]\tau,+\infty[\![} \cdot M = -I_{]\!]\tau,+\infty[\![} \cdot \widetilde{V} \quad \text{ is } \mathbb{G}\text{-predictable.}$$

Then, the process $I_{]\!]\tau,+\infty[\![} \cdot M$ satisfies NUPBR($\mathbb{G}$) if and only if it is null (see Lemma 2.10), or equivalently

$$0 = E\left(I_{]\!]\tau,+\infty[\![} \cdot \widetilde{V}_\infty\right) = E\left(\int_0^\infty (1 - Z_{s-}) d\widetilde{V}_s\right) = E\left((1 - Z_{T^i-}) I_{\{T^i<+\infty\}}\right).$$

Therefore, we conclude that $T^i = +\infty,\ P-a.s.$, and the stopping time T is an accessible stopping time. This proves the corollary. □

4.3 The Single Jump Process with Predictable Jump Time

This subsection investigates the NUPBR for processes having the form of $X_t = \xi I_{[\![T,+\infty[\![}(t)$, where ξ is an $\mathcal{F}_T$-measurable random variable that is bounded or integrable, and T is an $\mathbb{F}$-predictable stopping time. It is important to highlight the fact that in this subsection, we do **not** assume the condition (4.1) on τ. Below, we start with a preparatory result that is interesting in itself.

Theorem 4.9 *Let τ be an honest time. Consider an $\mathbb{F}$-predictable stopping time T and an $\mathcal{F}_T$-measurable random variable X such that $E[|X||\mathcal{F}_{T-}] < +\infty, P$-a.s. Define $M := X I_{\{Z_{T-}<1\}} I_{[\![T,+\infty[\![}$,*

$$\begin{aligned} \frac{dQ^{\mathbb{F}}(T)}{dP} := D^{\mathbb{F}} &:= \frac{I_{\{\widetilde{Z}_T<1 \ \&\ P(\widetilde{Z}_T<1|\mathcal{F}_{T-})>0\}}}{P(\widetilde{Z}_T < 1|\mathcal{F}_{T-})} + I_{\{P(\widetilde{Z}_T<1|\mathcal{F}_{T-})=0\}}, \quad \textit{and} \\ \frac{dQ^{\mathbb{G}}(T)}{dP} := D^{\mathbb{G}} &:= \frac{1 - Z_{T-}}{(1 - \widetilde{Z}_T)P(\widetilde{Z}_T < 1|\mathcal{F}_{T-})} I_{\{T>\tau\}} + I_{\{T\leq\tau\}}. \end{aligned} \tag{4.23}$$

Then the following assertions are equivalent.
(a) *M is a $(Q^{\mathbb{F}}(T), \mathbb{F})$-martingale.*
(b) *On $\{Z_{T-} < 1\}$, we have*

$$E\left(X I_{\{\widetilde{Z}_T<1\}} \mid \mathcal{F}_{T-}\right) = 0, \qquad P - a.s. \tag{4.24}$$

(c) *$(M - M^\tau)$ is a $(Q^{\mathbb{G}}(T), \mathbb{G})$-martingale.*

Proof For the sake of simplicity, throughout the proof we will use the notations $Q_1 := Q^{\mathbb{F}}(T)$ and $Q_2 := Q^{\mathbb{G}}(T)$. We start by remarking that

$$\{Z_{T-} = 1\} = \{P(\widetilde{Z}_T < 1|\mathcal{F}_{T-}) = 0\} \subset \{\widetilde{Z}_T = 1\}. \tag{4.25}$$

Indeed, $\{Z_{T-} = 1\} \subset \{P(\widetilde{Z}_T < 1|\mathcal{F}_{T-}) = 0\}$ follows from $\{Z_{T-} = 1\} \subset \{\widetilde{Z}_T = 1\}$ and $P\left(\{Z_{T-} = 1\} \cap \{\widetilde{Z}_T < 1\}\right) = E[I_{\{Z_{T-}=1\}} P(\widetilde{Z}_T < 1|\mathcal{F}_{T-})]$, while the equalities

$$E[I_{\{\widetilde{Z}_T<1\}} I_{\{P(\widetilde{Z}_T<1|\mathcal{F}_{T-})=0\}}] = E[P(\widetilde{Z}_T < 1|\mathcal{F}_{T-}) I_{\{P(\widetilde{Z}_T<1|\mathcal{F}_{T-})=0\}}] = 0,$$
$$\text{and} \quad E[(1 - \widetilde{Z}_T) I_{\{P(\widetilde{Z}_T<1|\mathcal{F}_{T-})=0\}}] = E[(1 - Z_{T-}) I_{\{P(\widetilde{Z}_T<1|\mathcal{F}_{T-})=0\}}]$$

imply $\{P(\widetilde{Z}_T < 1|\mathcal{F}_{T-}) = 0\} \subset \{\widetilde{Z}_T = 1\}$ and $\{P(\widetilde{Z}_T < 1|\mathcal{F}_{T-}) = 0\} \subset \{Z_{T-} = 1\}$ respectively. This completes the proof of (4.25).
Thanks to $E[D^{\mathbb{F}}|\mathcal{F}_{T-}] = 1$ and (4.25), we obtain

$$E^{Q_1}[X I_{\{Z_{T-}<1\}}|\mathcal{F}_{T-}] = E\left[D^{\mathbb{F}} X I_{\{Z_{T-}<1\}}|\mathcal{F}_{T-}\right] = \frac{E\left[X I_{\{\widetilde{Z}_T<1\}}|\mathcal{F}_{T-}\right]}{P(\widetilde{Z}_T < 1|\mathcal{F}_{T-})} I_{\{Z_{T-}<1\}}.$$

Therefore, the equivalence between assertions (a) and (b) follows from a combination of this equality and the fact that M is a $(Q_1, \mathbb{F})$-martingale if and only if $E^{Q_1}(M_T \mid \mathcal{F}_{T-}) = 0$ on $\{T < +\infty\}$.

In the remaining part of this proof we will prove (b)$\Longleftrightarrow$ (c). To this end, we first notice that $M - M^{\tau}$ is $(Q_2, \mathbb{G})$-martingale if and only if $E^{Q_2}[X I_{\{Z_{T-}<1 \ \& \ T>\tau\}}|\mathcal{G}_{T-}] = 0$ on $\{T < +\infty\}$. Furthermore, due to (4.25) and $E[D^{\mathbb{G}}|\mathcal{G}_{T-}] = 1$ we have

$$\begin{aligned} E^{Q_2}[X I_{\{Z_{T-}<1 \ \& \ T>\tau\}}|\mathcal{G}_{T-}] &= E\left[D^{\mathbb{G}} X I_{\{Z_{T-}<1\}} I_{\{T>\tau\}}|\mathcal{G}_{T-}\right] \\ &= E\left[\frac{X I_{\{T>\tau\}}}{1 - \widetilde{Z}_T}\Big|\mathcal{G}_{T-}\right] \frac{1 - Z_{T-}}{P(\widetilde{Z}_T < 1|\mathcal{F}_{T-})} I_{\{Z_{T-}<1 \ \& \ T>\tau\}} \\ &= \frac{E\left[X I_{\{\widetilde{Z}_T<1\}}|\mathcal{F}_{T-}\right]}{P(\widetilde{Z}_T < 1|\mathcal{F}_{T-})} I_{\{Z_{T-}<1\}} I_{\{T>\tau\}}, \end{aligned} \tag{4.26}$$

where the last equality in (4.26) follows from the fact that

$$E\left(K \mid \mathcal{G}_{T-}\right) I_{\{T>\tau\}} = E\left(K \frac{1 - \widetilde{Z}_T}{1 - Z_{T-}} I_{\{T>\tau\}} \mid \mathcal{F}_{T-}\right) I_{\{T>\tau\}}.$$

for any random variable K for which the above conditional expectations exist (see Proposition 5.3 of [28]). Therefore, if assertion (b) holds, then assertion (c) follows immediately from (4.26). Conversely, if assertion (c) holds, then $E^{Q_2}[X I_{\{Z_{T-}<1\}} I_{\{T>\tau\}}|\mathcal{G}_{T-}] = 0$. Thus, a combination of this with (4.26) leads to $E\left[X I_{\{\widetilde{Z}_T<1\}}|\mathcal{F}_{T-}\right](1 - Z_{T-}) = 0$. This proves assertion (ii), and the proof of the theorem is completed. □

Remark 4.10 It is important to notice that

$$D^{\mathbb{G}}(T) := \left(1 + \frac{\Delta m_T}{1 - \widetilde{Z}_T}\right) I_{\{T>\tau\}} + I_{\{T\leq\tau\}} \neq \frac{\mathcal{E}_T\left(\widetilde{N}^{(a)}\right)}{\mathcal{E}_{T-}\left(\widetilde{N}^{(a)}\right)}.$$

This explains one of the main difficulties that we will face when dealing with countable many predictable jumps. In fact, it might not be possible to piece-up the sequence $\left(D^{\mathbb{G}}(T_n),\ n \geq 1\right)$ into a local martingale density when the stopping times are not ordered.

The following theorem is one of the principal result of this subsection.

Theorem 4.11 *Consider an $\mathbb{F}$-predictable stopping time T, an honest time τ, and an $\mathcal{F}_T$-measurable random variable X such that $E(|X|\big|\ \mathcal{F}_{T-}) < +\infty$ $P-$ a.s.. If $S := XI_{\{Z_{T-}<1\}}I_{[\![T,+\infty[\![}$, then the following assertions are equivalent.*
(a) *$S - S^{\tau}$ satisfies NUPBR($\mathbb{G}$).*
(b) *S satisfies NUPBR($\mathbb{F}, \widetilde{Q}'_T$), where*

$$\widetilde{Q}'_T := \Big(\frac{1 - \widetilde{Z}_T}{1 - Z_{T-}} I_{\{Z_{T-}<1\}} + I_{\{Z_{T-}=1\}}\Big) \cdot P. \tag{4.27}$$

(c) *S satisfies NUPBR($\mathbb{F}, Q'_T$), where*

$$Q'_T := \Big(\frac{I_{\{\widetilde{Z}_T<1\}\cap\Gamma_1(T)}}{P(\widetilde{Z}_T < 1\big|\mathcal{F}_{T-})} + I_{\Gamma_1(T)^c}\Big)\cdot P \quad \textit{and}\ \ \Gamma_1(T) := \{P(\widetilde{Z}_T < 1\big|\mathcal{F}_{T-}) > 0\}. \tag{4.28}$$

(d) $\widetilde{S} := XI_{\{\widetilde{Z}_T<1\}}I_{[\![T,+\infty[\![}$ satisfies NUPBR($\mathbb{F}$).

Proof The proof consists of four steps.

Step 1: Herein, we will prove (b)$\Leftrightarrow$(c). Due to $\{Z_{T-} = 1\} = \{P(\widetilde{Z}_T < 1|\mathcal{F}_{T-}) = 0\} \subset \{\widetilde{Z}_T = 1\}$ (see (4.25) for its proof), it is easy to see that $\widetilde{Q}'_T \sim Q'_T \ll P$. Therefore, the equivalence between (b) and (c) follows immediately.

Step 2: This step focuses on proving (c)$\Rightarrow$(d). Suppose (c) holds. Then, there exists an $\mathcal{F}_T$-measurable random variable $Y_T > 0$, Q'_T-a.s. such that $E^{Q'_T}[S_T Y_T|\mathcal{F}_{T-}] = 0$, or equivalently

$$E[XY_T I_{\{\widetilde{Z}_T<1\}}|\mathcal{F}_{T-}]I_{\{Z_{T-}<1\}} = 0 \ \text{ and } \ E[XY_T|\mathcal{F}_{T-}]I_{\{Z_{T-}=1\}} = 0. \tag{4.29}$$

Since, on the set $\{Z_{T-} = 1\}$, $\widetilde{S} \equiv 0$, it is enough to focus on the part corresponding to $\{Z_{T-} < 1\}$. Put

$$\widetilde{Y}_T := Y_T I_{\{\widetilde{Z}_T<1\}} + I_{\{\widetilde{Z}_T=1\}} \quad \text{ and } \ Q_1 := \widetilde{Y}_T / E(\widetilde{Y}_T|\mathcal{F}_{T-}) \cdot P \sim P.$$

Then, from (4.29), we derive that $E^{Q_1}[\widetilde{S}_T|\mathcal{F}_{T-}] = E^{Q_1}[XI_{\{\widetilde{Z}_T<1\}}|\mathcal{F}_{T-}] = 0$. Therefore, $\widetilde{S}$ is a $(Q_1, \mathbb{F})$-martingale and assertion (d) follows.

Step 3: This step proves (d)⇒ (a). Suppose that $\widetilde{S}$ satisfies NUPBR($\mathbb{F}$). Then, there exists an $\mathcal{F}_T$-measurable $Y_3 > 0$ such that $E[Y_3 X I_{\{\widetilde{Z}_T<1\}}|\mathcal{F}_{T-}] = 0$. Put $Q_3 := Y_3/E(Y_3|\mathcal{F}_{T-}) \cdot P \sim P$ and remark that $\{\widetilde{Z}_T < 1\} = \{\widetilde{Z}_T^{Q_3} < 1\}$, where $\widetilde{Z}_t^{Q_3} := Q_3(\tau \geq t|\mathcal{F}_t)$. Therefore, a direct application of Theorem 4.9 under Q_3, we conclude that $S-S^\tau = \widetilde{S}-\widetilde{S}^\tau$ satisfies NUPBR($\mathbb{G}$), and assertion (a) holds.

Step 4: This last step will prove (a)⇒ (b). Suppose $S-S^\tau$ satisfies NUPBR($\mathbb{G}$). There exists a positive $\mathcal{G}_T$-measurable $Y^{\mathbb{G}}$ such that $E[XY^{\mathbb{G}}I_{\{T>\tau\}}|\mathcal{G}_{T-}] = 0$. Then, thanks to Jeulin [28] (Proposition 5.3), we deduce the existence of a positive and $\mathcal{F}_T$-measurable $\overline{Y}^{\mathbb{F}}$ such that $Y^{\mathbb{G}}I_{\{T>\tau\}} = Y^{\mathbb{F}}I_{\{T>\tau\}}$. Then, we calculate

$$\begin{aligned} 0 &= E[XY^{\mathbb{G}}I_{\{T>\tau\}}|\mathcal{G}_{T-}] = E[XY^{\mathbb{F}}(1-\widetilde{Z}_T)|\mathcal{F}_{T-}]\frac{I_{\{T>\tau\}}}{1-Z_{T-}} \\ &= E^{\widetilde{Q}'(T)}\left(XY^{\mathbb{F}}\middle|\ \mathcal{F}_{T-}\right) I_{\{T>\tau\}}. \end{aligned}$$

Therefore, by taking conditional expectation and using the fact that $S_T I_{\{Z_{T-}=0\}} = 0$, we obtain

$$(1-Z_{T-})E^{\widetilde{Q}'(T)}[XY^{\mathbb{F}}|\mathcal{F}_{T-}] = 0, \text{ or equivalently } \ E^{\widetilde{Q}'(T)}[S_T Y^{\mathbb{F}}|\mathcal{F}_{T-}] = 0 \ P\text{−}a.s.$$

This proves that S satisfies NUPBR($\mathbb{F}, \widetilde{Q}'_T$) and the proof of the theorem is achieved. □

Below, we state our second main result of this subsection.

Theorem 4.12 *Consider an $\mathbb{F}$-predictable stopping time T, and an honest time τ. Then, the following are equivalent*
(a) *On $\{T < +\infty\}$, we have*

$$\left\{\widetilde{Z}_T = 1\right\} \subset \{Z_{T-} = 1\}. \tag{4.30}$$

(b) *For any $X \in L^\infty(\mathcal{F}_T)$ such that $E(X\big|\ \mathcal{F}_{T-}) = 0$, the process $M - M^\tau$ satisfies NUPBR($\mathbb{G}$), where $M := XI_{[\![T,+\infty[\![}$.*

Proof Suppose that assertion (a) holds. Then it is obvious that (4.30) implies that $\{\widetilde{Z}_T < 1\} = \{Z_{T-} < 1\}$ and hence $Q'(T)$ defined in (4.27) coincides with P. Thus any $(P, \mathbb{F})$-martingale is a $(Q'(T), \mathbb{F})$-martingale, and assertion (b) follows from Theorem 4.11. This proves (a)⟹ (b). To prove the converse, we assume that assertion (b) holds, and we consider the $\mathcal{F}_T$-measurable and bounded random variable $X := (I_{\{\widetilde{Z}_T=1\}} - P(\widetilde{Z}_T = 1|\mathcal{F}_{T-}))I_{\{T<+\infty\}}$ and the

bounded $\mathbb{F}$-martingale $M := XI_{[\![T,+\infty[\![}$. Then, $M - M^\tau$ satisfies NUPBR($\mathbb{G}$) on the one hand. On the other hand, due to $\{T > \tau\} \subset \{\widetilde{Z}_T < 1\}$,

$$M - M^\tau = -P(\widetilde{Z}_T = 1|\mathcal{F}_{T-}))I_{\{T>\tau\}}I_{[\![T,+\infty[\![} \text{ is } \mathbb{G} - \text{predictable.}$$

Thus, this process is null (see Lemma 2.10), or equivalently $\{Z_{T-} < 1\} \subset \{\widetilde{Z}_T < 1\}$ $P - a.s.$ on $\{T < +\infty\}$. This proves assertion (a), and the proof of the theorem is completed. □

4.4 The General Framework

In this subsection, we will address the NUPBR for the general case. For this framework, we derive two main results which are presented in Theorems 4.14–4.15. The proof of these theorems relies on results (Proposition 4.13) that are interesting in themselves. Below, we start developing these preparatory results.

Proposition 4.13 *Suppose that τ is a finite honest time satisfying (4.1). Then, the following properties hold.*
(a) *Let $\left(\sigma_n^{\mathbb{G}}\right)_{n\geq 1}$ be a sequence of $\mathbb{G}$-stopping times that increase to infinity. Then, there exists a sequence of $\mathbb{F}$-stopping times $\left(\sigma_n^{\mathbb{F}}\right)_{n\geq 1}$ that increase to infinity such that $\sigma_n^{\mathbb{G}} \vee \tau = \sigma_n^{\mathbb{F}} \vee \tau$.*
(b) *Let V be an $\mathbb{F}$-adapted process with finite variation. Then $I_{]\!]\tau,+\infty[\![} \cdot V$ belongs to $\mathcal{A}_{loc}(\mathbb{G})$ if and only if $(1 - \widetilde{Z}) \cdot V$ belongs to $\mathcal{A}_{loc}(\mathbb{F})$.*
(c) *Let f be a $\widetilde{\mathcal{P}}(\mathbb{F})$-measurable and positive functional, and $\overline{\mu} := (1 - \widetilde{Z}) \cdot \mu$. Then $\sqrt{(f-1)^2 I_{]\!]\tau,+\infty[\![}} \star \mu \in \mathcal{A}^+_{loc}(\mathbb{G})$ if and only if $\sqrt{(f-1)^2 \star \overline{\mu}} \in \mathcal{A}^+_{loc}(\mathbb{F})$.*

Proof (a) Due to Jeulin [28] (see Proposition 5.3), for all n, there exists an $\mathbb{F}$-stopping time $\sigma_n^{\mathbb{F}}$ such that $[\![0, \sigma_n^{\mathbb{G}}]\!] \cap]\!]\tau, +\infty[\![= [\![0, \sigma_n^{\mathbb{F}}]\!] \cap]\!]\tau, +\infty[\![$. As a result, we get

$$\sigma_n^{\mathbb{G}} \vee \tau = \sigma_n^{\mathbb{F}} \vee \tau, \quad P - a.s. \tag{4.31}$$

This implies that $\{\sigma_n^{\mathbb{G}} > \tau\} = \{\sigma_n^{\mathbb{F}} > \tau\}$, and in the following we will prove that

$$\sigma_n^{\mathbb{G}} \vee \tau = \sigma_n \vee \tau, \quad \text{where} \quad \sigma_n := \sup_{k\leq n} \sigma_k^{\mathbb{F}}. \tag{4.32}$$

To this end, we will prove that $\{\sigma_n^{\mathbb{F}} \vee \tau < \sigma_n \vee \tau\}$ is negligible. Due to (4.31), on the event $\{\sigma_i^{\mathbb{F}} > \tau\}$, we have

$$\sigma_i^{\mathbb{F}} = \sigma_i^{\mathbb{G}} \leq \sigma_n^{\mathbb{G}} = \sigma_n^{\mathbb{F}}, \quad P - a.s.$$

This implies that $\{\sigma_i^{\mathbb{F}} > \sigma_n^{\mathbb{F}}\} \cap \{\sigma_i^{\mathbb{F}} > \tau\}$ is negligible. Therefore,

$$\{\sigma_n^{\mathbb{F}} \vee \tau < \sigma_n \vee \tau\} = \{\sigma_n > \sigma_n^{\mathbb{F}}\} \cap \{\sigma_n > \tau\} = \bigcup_{i=1}^{n-1} \{\sigma_i^{\mathbb{F}} > \sigma_n^{\mathbb{F}}\} \cap \{\sigma_i^{\mathbb{F}} > \tau\} = \emptyset,$$

and (4.32) holds. By letting n go to infinity in (4.32), we deduce that $\sigma_\infty = \sup_{n\geq 1} \sigma_n = +\infty, \quad P - a.s.$ since $\tau < +\infty, \ P - a.s.$ This ends the proof of

the assertion (a).

(b) By working with $Var(V) := V^+ + V^-$ instead of V, there is no loss of generality in assuming that V is nondecreasing. Suppose $I_{]\!]\tau,+\infty[\![}\cdot V \in \mathcal{A}^+_{loc}(\mathbb{G})$. Then, there exists a sequence of $\mathbb{G}$-stopping time $\left(\sigma^{\mathbb{G}}_n\right)_{n\geq 1}$ that increases to infinity such that $\left(I_{]\!]\tau,+\infty[\![}\cdot V\right)^{\sigma^{\mathbb{G}}_n} \in \mathcal{A}^+(\mathbb{G})$. Thanks to assertion (a), we deduce the existence of the $\mathbb{F}$-stopping times $(\sigma^{\mathbb{F}}_n)_{n\geq 1}$ increasing to infinity and

$$\left(I_{]\!]\tau,+\infty[\![}\cdot V\right)^{\sigma^{\mathbb{G}}_n} = \left(I_{]\!]\tau,+\infty[\![}\cdot V\right)^{\sigma^{\mathbb{F}}_n}.$$

Then, we derive that

$$E\left[I_{]\!]\tau,+\infty[\![}\cdot V_{\sigma^{\mathbb{G}}_n}\right] = E\left[I_{]\!]\tau,+\infty[\![}\cdot V_{\sigma^{\mathbb{F}}_n}\right] = E\left[(1-\widetilde{Z})\cdot V_{\sigma^{\mathbb{F}}_n}\right] < +\infty.$$

This proves that $(1-\widetilde{Z})\cdot V \in \mathcal{A}^+_{loc}(\mathbb{F})$ as long as $I_{]\!]\tau,+\infty[\![}\cdot V \in \mathcal{A}^+_{loc}(\mathbb{G})$. The proof of the reverse sense is obvious and will be omitted. This completes the proof of the assertion (b).
(c) The proof of assertion (c) follows from assertion (b) and Proposition 3.14–(a). This ends the proof of the proposition. □

In the following, we will state the first main result of this subsection.

Theorem 4.14 *Let S be a process satisfying NUPBR($\mathbb{F}$), and τ be a finite honest time satisfying (4.1). Then, the following are equivalent.*
(a) *The process $S - S^\tau$ satisfies NUPBR($\mathbb{G}$).*
(b) *There exists a positive $\mathbb{F}$-local martingale Y such that $[Y,S] \in \mathcal{A}_{loc}(\mathbb{F})$ and*

$$^{p,\mathbb{F}}\left(Y\Delta S I_{\{\widetilde{Z}<1\}}\right) = 0.$$

Proof Before proving the equivalence between the two assertions of the theorem, we will start outlining a number of remarks that simplify tremendously the proof. It is easy to prove that on $\{T<+\infty\}$ we have

$$\{\widetilde{Z}^Q_T = 1\} = \{\widetilde{Z}_T = 1\} \quad \text{for any } Q\sim P \text{ and any } \mathbb{F}-\text{stopping time } T, \tag{4.33}$$

where $\widetilde{Z}^Q_t := E^Q[I_{\{\tau\geq t\}}|\mathcal{F}_t]$. Indeed, due to

$$E\left[(1-\widetilde{Z}_T)I_{\{\widetilde{Z}^Q_T=1\}}\right] = E\left[I_{\{\tau<T\}}I_{\{\widetilde{Z}^Q_T=1\}}\right] = 0,$$

the inclusion $\{\widetilde{Z}^Q_T = 1\} \subset \{\widetilde{Z}_T = 1\}$ follows, while the reverse inclusion follows by symmetry. This proves (4.33). Let $(T_n)_{n\geq 1}$ be a sequence of $\mathbb{F}$-predictable stopping times that exhausts the predictable jumps of S. Put

$$S^a := I_\Gamma \bullet S, \quad \Gamma := \bigcup_{n=1}^{+\infty} [\![T_n]\!], \qquad \text{and} \quad S^i := S - S^a.$$

Then, thanks to Theorem 4.7, the process $S^i - (S^i)^\tau$ satisfies NUPBR($\mathbb{G}$) due to the assumption (4.1) on τ. Thus assertion (a) is equivalent to $S^a - (S^a)^\tau$ satisfies NUPBR($\mathbb{G}$). On the other hand, assertion (b) is equivalent to the existence of a positive $\mathbb{F}$-local martingale Y such that

$$^{p,\mathbb{F}}\left(Y\Delta S^a I_{\{\widetilde{Z}<1\}}\right) = 0.$$

These two last remarks prove that it is enough to prove the theorem for S^a, and without loss of generality we will assume that $S = S^a$ in the rest of the proof. The remaining part of this proof contains two parts 1) and 2) where we prove (b)⇒ (a) and (a)⇒ (b) respectively.

1) Here we prove (b)⇒ (a). Suppose that assertion (b) holds, and consider a sequence of $\mathbb{F}$-stopping times $(\sigma_n)_n$ such that Y^{σ_n} is a martingale, and put $Q_n := (Y_{\sigma_n}/Y_0)\cdot P$. Therefore, thanks to Proposition 2.9, it is enough to prove that assertion (a) holds true under Q_n for S^{σ_n}. Therefore, without loss of generality, we assume $Y \equiv 1$. Put

$$X^n = \Delta S_{T_n} I_{\{\widetilde{Z}_{T_n}<1\}} \quad \text{and} \quad M^n = X^n I_{[\![T_n,+\infty[\![}.$$

Hence, the condition (4.24) in Theorem 4.9 is trivially satisfied for X^n and M^n. Thus, we deduce that for each n,

$$(1 + \Delta\widetilde{N}^{(a)}_{T_n} - \Delta V^{(a)}_{T_n})\Delta S_{T_n} I_{\{T_n>\tau\}} I_{[\![T_n,+\infty[\![},$$

is a $\mathbb{G}$-martingale. Then, for each $\theta \in L(S - S^\tau)$ such that $\theta^T \Delta S > -1$, Yor's formula implies that

$$\mathcal{E}\left(I_\Gamma \bullet \widetilde{N}^{(a)} - I_\Gamma \cdot V^{(a)}\right)\mathcal{E}\left(\theta I_\Gamma \bullet (S - S^\tau)\right) = \mathcal{E}\left(W\right),$$

where

$$W := I_\Gamma \bullet (\widetilde{N}^{(a)} - V^{(a)}) + \sum_{n\geq 1} \theta_{T_n}\left(1 + \Delta\widetilde{N}^{(a)}_{T_n} - \Delta V^{(a)}_{T_n}\right)\Delta S_{T_n} I_{\{T_n>\tau\}} I_{[\![T_n,+\infty[\![},$$

and $\widetilde{N}^{(a)}$ and $V^{(a)}$ are defined in (4.12) and (4.9) respectively. Put

$$\phi = \sum_{n\geq 1} \frac{2^{-n}\left(1 + \mathcal{E}(W)_{T_n-}\right)^{-1}}{(1 + \Xi_n)} I_{]\!]\tau,+\infty[\![\cap[\![T_n]\!]} + I_{\Gamma^c \cup [\![0,\tau]\!]} \quad \text{and}$$

$$\Xi_n := E\left[|\Delta\widetilde{N}^{(a)}_{T_n}|\,\Big|\mathcal{G}_{T_n-}\right] + \Delta V^{(a)}_{T_n} + E\left[\frac{|\theta_{T_n}\Delta S_{T_n}|(1 - Z_{T_n-})}{1 - \widetilde{Z}_{T_n}}\Big|\mathcal{G}_{T_n-}\right] I_{\{T_n<+\infty\}}.$$

Then, it is easy to check that ϕ is $\mathbb{G}$-predictable, $0 < \phi \leq 1$, and $E\left(|\phi \bullet \mathcal{E}(W)|_{var}(+\infty)\right) \leq \sum_{n\geq 1} 2^{-n} = 1$. Hence $\phi \bullet \mathcal{E}(W) \in \mathcal{A}(\mathbb{G})$, and using the fact that $\Delta\widetilde{N}^{(a)}_{T_n} I_{[\![T_n,+\infty[\![}$ and $(1 + \Delta\widetilde{N}^{(a)}_{T_n} - \Delta V^{(a)}_{T_n})\Delta S_{T_n} I_{\{T_n>\tau\}} I_{[\![T_n,+\infty[\![}$ are $\mathbb{G}$-martingales, we get

$$\left(\phi \bullet \mathcal{E}(W)\right)^{p,\mathbb{G}} = \sum_{n\geq 1} \phi_{T_n} E\left(\Delta W_{T_n}\big|\mathcal{G}_{T_n-}\right) I_{[\![T_n,+\infty[\![} = -\phi\mathcal{E}_-(W) \bullet V^{(a)}$$

is a nonincreasing process. This proves that $\mathcal{E}(W)$ is a positive σ-supermartingale, and hence it is a supermartingale due to Kallsen [30]. Thus, $S - S^\tau$ admits a deflator, and the proof of (b)⇒(a) follows immediately from Proposition 2.12.

2) Here, we will prove (a)⇒(b). Suppose that $S-S^\tau$ satisfies NUPBR($\mathbb{G}$), and thanks to Theorem 2.3 there exists a σ-martingale density $D^{\mathbb{G}} := \mathcal{E}(N^{\mathbb{G}})$. A direct application of Theorem A.1 and Theorem A.4, we deduce the existence of a $\widetilde{\mathcal{P}}(\mathbb{G})$-measurable functional $f^{\mathbb{G}}$ such that $f^{\mathbb{G}} > 0$,

$$N^{\mathbb{G}} := W^{\mathbb{G}} \star (\mu^{\mathbb{G}} - \nu^{\mathbb{G}}), \quad W^{\mathbb{G}} := f^{\mathbb{G}} - 1 + \frac{\widehat{f^{\mathbb{G}}} - a^{\mathbb{G}}}{1 - a^{\mathbb{G}}} I_{\{a^{\mathbb{G}}<1\}},$$

and

$$x f^{\mathbb{G}} \star \nu^{\mathbb{G}} = x f^{\mathbb{G}} \left(1 - \frac{f_m}{1 - Z_-}\right) I_{]\!]\tau,+\infty[\![} \star \nu \equiv 0. \tag{4.34}$$

Here

$$\mu^{\mathbb{G}} := I_{]\!]\tau,+\infty[\![} \cdot \mu, \ \nu^{\mathbb{G}} = \left(1 - \frac{f_m}{1 - Z_-}\right) I_{]\!]\tau,+\infty[\![} \cdot \nu, \quad f_m := M_\mu^P(\Delta m | \widetilde{\mathcal{P}}(\mathbb{F})).$$

Thanks to Lemma 2.13, there exists a positive and $\mathcal{P}(\mathbb{F})$-measurable functional f such that $f^{\mathbb{G}} = f$ on the stochastic interval $]\!]\tau, +\infty[\![$, and (4.34) becomes

$$x f \left(1 - \frac{f_m}{1 - Z_-}\right) I_{]\!]\tau,+\infty[\![} \star \nu \equiv 0. \tag{4.35}$$

Due to Proposition 4.13 and $\mathbb{G}$-locally boundedness of $1/(1 - Z_-)$ on $I_{]\!]\tau,+\infty[\![}$, we could find a sequence of $\mathbb{F}$-stopping time $(\sigma_n^{\mathbb{F}})_{n\geq 1}$ that increases to infinity and $\frac{1}{1-Z_-} I_{[\![0,\sigma_n^{\mathbb{F}}]\!]} I_{]\!]\tau,+\infty[\![}$ is bounded by $(n+1)$. Also, since $\left((f-1)^2 I_{]\!]\tau,+\infty[\![} \star \mu\right)^{1/2} \in \mathcal{A}^+_{loc}(\mathbb{G})$, thanks to Proposition 4.13 we deduce the existence of a sequence of $\mathbb{F}$-stopping times $(\tau_n)_n$ that increases to infinity such that

$$[m,m]^{\tau_n}, \quad (f-1)^2 I_{\{|f-1|\leq\alpha\}} \star ((1-\widetilde{Z}) \cdot \mu)^{\tau_n} \quad \text{and} \quad |f-1| I_{\{|f-1|>\alpha\}} \star ((1-\widetilde{Z}) \cdot \mu)^{\tau_n}$$

are integrable. Consider the following notations

$$\mu_1 := I_{\{\widetilde{Z}<1\}} \cdot \mu, \quad \nu_1 := M_\mu^P \left(I_{\{\widetilde{Z}<1\}} | \widetilde{\mathcal{P}}\right) \cdot \nu, \ h_1 := M_\mu^P \left(I_{\{\widetilde{Z}<1\}} | \widetilde{\mathcal{P}}\right),$$
$$g := \frac{f(1 - \frac{f_m}{1-Z_-})}{h_1} I_{\{h_1>0 \ \& \ Z_-<1\}} + I_{\{h_1=0 \text{ or } Z_-=1\}},$$

and suppose that

$$W^{(1)}(t,x) := \frac{g_t(x) - 1}{1 - a_t^{(1)} + \widehat{g}_t} \in \mathcal{G}^1_{loc}(\mu_1, \mathbb{F}), \tag{4.36}$$

where $a_t^{(0)} := \nu_0(\{t\}, \mathbb{R}^d)$ and $\widehat{g}_t := \int g_t(x) \nu_1(\{t\}, dx)$.

Then, we can easily prove that assertion (b) holds. In fact, we take

$$N^{(1)} := \frac{g-1}{1-a^{(1)}+\widehat{g}} \star (\mu_1 - \nu_1) \quad \text{and} \quad Y := \mathcal{E}(N^{(1)}).$$

Then, it is clear that

$$1 + \Delta N^{(1)} = \frac{1}{1-a^{(1)}+\widehat{g}} I_{\{\Delta S=0 \text{ or } \widetilde{Z}=1\}} + \frac{g(\Delta S)}{1-a^{(1)}+\widehat{g}} I_{\{\Delta S\neq 0 \ \& \ \widetilde{Z}<1\}} > 0,$$

and on $\{Z_- < 1\}$ we get

$$\begin{aligned}
{}^{p,\mathbb{F}}\left(Y\Delta S I_{\{\widetilde{Z}<1\}}\right)_t &= Y_{t-} \ {}^{p,\mathbb{F}}\left((1+\Delta N^{(1)})\Delta S I_{\{\widetilde{Z}<1\}}\right)_t = \frac{Y_{t-} \ {}^{p,\mathbb{F}}\left(g(\Delta S)\Delta S I_{\{\widetilde{Z}<1\}}\right)_t}{1-a_t^{(1)}+\widehat{g}_t} \\
&= \frac{Y_{t-}}{1-a_t^{(1)}+\widehat{g}_t} \int g_t(x) x h_1(t,x) \nu(\{t\}, dx) \\
&= \frac{Y_{t-}}{1-a_t^{(1)}+\widehat{g}_t} \int x f_t(x) \left(1 - \frac{f_m(t,x)}{1-Z_-}\right) \nu(\{t\}, dx) \equiv 0.
\end{aligned}$$

The last equality in the above string of equalities follows directly from (4.35). Therefore, assertion (b) will follow immediately as long as we prove (4.36). To this end, on $\{h_1 > 0 \ \& \ Z_- < 1\}$ we calculate

$$\begin{aligned}
g - 1 &= \frac{f(1-Z_- - f_m)}{h_1(1-Z_-)} - 1 = \frac{(f-1)(1-Z_- - f_m)}{h_1(1-Z_-)} - \frac{M_\mu^P\left(\Delta m I_{\{\widetilde{Z}<1\}} | \widetilde{\mathcal{P}}\right)}{(1-Z_-)h_1} \\
&=: g_1 + g_2,
\end{aligned}$$

and remark that $\{1 - Z_- - f_m > 0\} \subset \{h_1 > 0\}$ which is due to

$$1 - Z_- - f_m = 1 - M_\mu^P\left(\widetilde{Z}|\widetilde{\mathcal{P}}\right) \leq M_\mu^P\left(I_{\{\widetilde{Z}<1\}}|\widetilde{\mathcal{P}}\right) = h_1,$$

that is implied by $I_{\{\widetilde{Z}=1\}} \leq \widetilde{Z}$. Therefore, we derive that

$$\begin{aligned}
E\left[g_1^2 I_{\{|f-1|\leq\alpha\}} \star \mu_1(\sigma_n \wedge \tau_n)\right] &= E\left[\frac{(f-1)^2(1-Z_- - f_m)^2}{h_1^2(1-Z_-)^2} I_{\{|f-1|\leq\alpha\}} I_{\{\widetilde{Z}<1\}} \star \mu(\sigma_n \wedge \tau_n)\right] \\
&= E\left[\frac{(f-1)^2(1-Z_- - f_m)^2}{h_1(1-Z_-)^2} I_{\{|f-1|\leq\alpha\}} I_{\{Z_-<1\}} \star \nu(\sigma_n \wedge \tau_n)\right] \\
&\leq E\left[(f-1)^2 \frac{1-Z_- - f_m}{(1-Z_-)^2} I_{\{|f-1|\leq\alpha\}} I_{\{Z_-<1\}} \star \nu(\sigma_n \wedge \tau_n)\right] \\
&\leq E\left[(f-1)^2 \frac{1}{(1-Z_-)^2} I_{\{|f-1|\leq\alpha\}} \star ((1-\widetilde{Z}) \cdot \mu(\sigma_n \wedge \tau_n))\right] \\
&= E\left[(f-1)^2 \frac{1}{(1-Z_-)^2} I_{\{|f-1|\leq\alpha\}} I_{]\!]\tau,+\infty[\![} \star \mu(\sigma_n \wedge \tau_n)\right] \\
&\leq (n+1)^2 E\left[(f-1)^2 I_{\{|f-1|\leq\alpha\}} \star \mu(\tau_n)\right] < +\infty.
\end{aligned}$$

and

$$\begin{aligned}
E\left[g_1 I_{\{|f-1|>\alpha\}} \star \mu_1(\sigma_n \wedge \tau_n)\right] &= E\left[\frac{|f-1|(1-Z_- - f_m)}{h_1(1-Z_-)} I_{\{|f-1|>\alpha\}} I_{\{\widetilde{Z}<1\}} \star \mu(\sigma_n \wedge \tau_n)\right] \\
&= E\left[|f-1|\frac{1-Z_- - f_m}{1-Z_-} I_{\{|f-1|>\alpha\}} I_{\{Z_-<1\}} \star \nu(\sigma_n \wedge \tau_n)\right] \\
&\leq E\left[|f-1|\frac{1}{1-Z_-} I_{\{|f-1|>\alpha\}} \star ((1-\widetilde{Z})\cdot\mu)(\sigma_n \wedge \tau_n)\right] \\
&= E\left[|f-1|\frac{1}{1-Z_-} I_{\{|f-1|>\alpha\}} I_{]\!]\tau,+\infty[\![} \star \mu(\sigma_n \wedge \tau_n)\right] \\
&\leq (n+1)E\left[|f-1| I_{\{|f-1|>\alpha\}} \star ((1-\widetilde{Z})\cdot\mu)(\tau_n)\right] < +\infty.
\end{aligned}$$

A combination of these inequalities and $\sqrt{g_1^2 \star \mu_1} \leq \sqrt{2}\sqrt{g_1^2 I_{\{|f-1|>\alpha\}} \star \mu_1} + \sqrt{2}\left(g_1 I_{\{|f-1|>\alpha\}} \star \mu_1\right)$, we deduce that $\sqrt{g_1^2 \star \mu_1} \leq \sqrt{2} \in \mathcal{A}^+_{loc}(\mathbb{F})$. To prove $\sqrt{g_2^2 \star \mu_1} \in \mathcal{A}^+_{loc}(\mathbb{F})$, we derive

$$\begin{aligned}
E\left[(g_2)^2 \star \mu_1(\sigma_n \wedge \tau_n)\right] &= E\left[\frac{M_\mu^P\left(\Delta m I_{\{\widetilde{Z}<1\}}|\widetilde{\mathcal{P}}\right)^2}{(1-Z_-)^2 h_1^2} I_{\{\widetilde{Z}<1 \ \&\ Z_-<1\}} \star \mu(\sigma_n \wedge \tau_n)\right] \\
&\leq E\left[\frac{M_\mu^P\left((\Delta m)^2|\widetilde{\mathcal{P}}\right) M_\mu^P\left(I_{\{\widetilde{Z}<1 \ \&\ Z_-<1\}}|\widetilde{\mathcal{P}}\right)^2}{(1-Z_-)^2 h_1^2} \star \nu(\sigma_n \wedge \tau_n)\right] \\
&= E\left[\frac{M_\mu^P\left((\Delta m)^2|\widetilde{\mathcal{P}}\right)}{(1-Z_-)^2} I_{\{Z_-<1\}} \star \nu(\sigma_n \wedge \tau_n)\right] \\
&= E\left[\frac{1}{(1-Z_-)^3} I_{]\!]\tau,+\infty[\![} \cdot \left(M_\mu^P\left((\Delta m)^2|\widetilde{\mathcal{P}}\right) \star \nu\right)_{\sigma_n \wedge \tau_n}\right] \\
&\leq (n+1)^3 E([m,m]_{\tau_n}) < +\infty.
\end{aligned}$$

Hence, $\sqrt{(g-1)^2 \star \mu_1} \in \mathcal{A}^+_{loc}(\mathbb{F})$ follows. Thanks to Lemma A.5 (see Choulli and Schweizer(2012) [8]), (4.36) follows immediately. This ends the proof of the theorem. □

Finally, we conclude the paper with our second main result of this subsection.

Theorem 4.15 *Let τ be an honest time satisfying (4.1). Then, the following are equivalent.*
(a) *$S - S^\tau$ satisfies NUPBR($\mathbb{G}$), for any S satisfying NUPBR($\mathbb{F}$).*
(b) *The thin set $\{\widetilde{Z} = 1 \ \&\ Z_- < 1\}$ is evanescent.*

Proof The implication (b) ⇒ (a) is a trivial consequence of Theorem 4.14 by taking Y any σ-martingale density for S (which is guaranteed by Theorem 2.3). To prove the reverse, we assume that assertion (a) holds, and consider

$V := I_{[\![T,+\infty[\![}$ and $M = V - (V)^{p,\mathbb{F}}$, where T is any $\mathbb{F}$-stopping time such that $[\![T]\!] \subset \{\widetilde{Z} = 1 > Z_-\}$. Since $\{T > \tau\} \subset \{\widetilde{Z}_T < 1\}$, we derive

$$M - M^\tau = XI_{\{T>\tau\}}I_{[\![T,+\infty[\![} = -P\left(\widetilde{Z}_T = 1 \middle| \mathcal{F}_{T-}\right) I_{\{T>\tau\}}I_{[\![T,+\infty[\![},$$

and conclude that $M - M^\tau$ is $\mathbb{G}$-predictable. Thus, $M - M^\tau$ satisfies NUPBR($\mathbb{G}$) if and only if it is a null process (see Lemma 2.10), or equivalently

$$0 = E\left(P\left(\widetilde{Z}_T = 1 \middle| \mathcal{F}_{T-}\right) I_{\{+\infty>T>\tau\}}\right) = E\left((1 - Z_{T-})I_{\{\widetilde{Z}_T=1\}}I_{\{T<+\infty\}}\right).$$

This implies that $T = +\infty,\ P - a.s.$ Therefore, assertion (b) follows immediately from a combination of this and the fact that $\{\widetilde{Z} = 1 > Z_-\} \subset \{\Delta m \neq 0\}$ is a thin set. This ends the proof of the theorem. □

A Appendix

This section recall an important result on representation of local martingales. This result relies on the continuous local martingale part and the jump random measure of a given semimartingale. Thus, throughout this section, we suppose given a d-dimensional semimartingale, $S = (S_t)_{0\leq t\leq T}$. To this semimartingale, we associate its **predictable characteristics** that we will present below (for more details about these and other related issues, we refer the reader to Section II.2 of [27]). The random measure μ associated to the jumps of S is defined by

$$\mu(dt,\ dx) = \sum I_{\{\Delta S_s\neq 0\}}\delta_{(s,\ \Delta S_s)}(dt,\ dx),$$

with δ_a the Dirac measure at point a. The continuous local martingale part of S is denoted by S^c. This leads to the following decomposition, called "*the canonical representation*" (see Theorem 2.34, Section II.2 of [27]), namely,

$$S = S_0 + S^c + h(x) \star (\mu - \nu) + (x - h(x)) \star \mu + B, \tag{A.1}$$

where the random measure ν is the compensator of the random measure μ, the function $h(x)$ is the truncation function given by $h(x) = xI_{\{|x|\leq 1\}}$, and B is a predictable process with bounded variation. For the matrix C with entries $C^{ij} := \langle S^{c,i}, S^{c,j}\rangle$, the triple $(B,\ C,\ \nu)$ is called *predictable characteristics* of S. Furthermore, we can find a version of the characteristics triple satisfying

$$B = b \bullet A,\ \ C = c \bullet A \ \text{ and } \ \nu(\omega,\ dt,\ dx) = dA_t(\omega)F_t(\omega,\ dx). \tag{A.2}$$

Here A is an increasing and predictable process which is continuous if and only if S is quasi-left continuous, b and c are predictable processes, $F_t(\omega,\ dx)$ is a predictable kernel, $b_t(\omega)$ is a vector in $\mathbb{R}^d$ and $c_t(\omega)$ is a symmetric $d \times d$-matrix , for all $(\omega,\ t) \in \Omega \times [0,\ T]$. In the sequel we will often drop ω and t and write, for instance, $F(dx)$ as a shorthand for $F_t(\omega, dx)$. The characteristics, $B,\ C$, and ν, satisfy

$$F_t(\omega,\ \{0\}) = 0, \qquad \int (|x|^2 \wedge 1)F_t(\omega,\ dx) \leq 1, \qquad \Delta B_t = b\Delta A = \int h(x)\nu(\{t\}, dx),$$

$$c = 0 \quad \text{on} \ \ \{\Delta A \neq 0\}.$$

We set

$$\nu_t(dx) := \nu(\{t\}, dx), \quad a_t := \nu_t(\mathbb{R}^d) = \Delta A_t F_t(\mathbb{R}^d) \leq 1.$$

For the following representation theorem, we refer to [26, Theorem 3.75, page 103] and to [27, Lemma 4.24, Chap III]. Below $\widetilde{\mathcal{O}}$ and $\widetilde{\mathcal{P}}$ denote the σ-fields $\mathcal{O} \otimes \mathcal{B}(\mathbb{R}^d)$ and $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$ respectively.

Theorem A.1 *Let $N \in \mathcal{M}_{0,loc}$. Then, there exist a predictable S^c-integrable process β, $N^\perp \in \mathcal{M}_{0,loc}$ with $N^\perp$ and S orthogonal and functionals $f \in \widetilde{\mathcal{P}}$ and $g \in \widetilde{\mathcal{O}}$ such that*

(a) $\left(\sum_{s\leq 0} f_s(\Delta S_s)^2 I_{\{\Delta S_s\neq 0\}}\right)^{1/2}$ *and* $\left(\sum_{s\leq t} g_s(\Delta S_s)^2 I_{\{\Delta S_s\neq 0\}}\right)^{1/2}$ *belong to* $\mathcal{A}^+_{loc}$.

(b) $M^P_\mu(g \mid \widetilde{\mathcal{P}}) = 0, \quad M^P_\mu - a.e.$, *where* $M^P_\mu := P \otimes \mu$.

(c) *The process N satisfies*

$$N = \beta \bullet S^c + W \star (\mu - \nu) + g \star \mu + N^\perp, \quad \text{where } W = f + \frac{\widehat{f}}{1-a} I_{\{a<1\}}. \tag{A.3}$$

Here $\widehat{f}_t = \int f_t(x)\nu(\{t\}, dx)$ and f has a version such that $\{a = 1\} \subset \{\widehat{f} = 0\}$. Moreover

$$\Delta N_t = \Big(f_t(\Delta S_t) + g_t(\Delta S_t)\Big) I_{\{\Delta S_t\neq 0\}} - \frac{\widehat{f}_t}{1-a_t} I_{\{\Delta S_t=0\}} + \Delta N^\perp_t. \tag{A.4}$$

The quadruplet $\left(\beta, f, g, N^\perp\right)$ are called the Jacod's parameters of the local martingale N with respect to S.

The following is a simple but useful result on the conditional expectation with respect to M^P_μ.

Lemma A.2 *Consider a filtration $\mathbb{H}$ satisfying the usual conditions. Let f and g two non-negative $\widetilde{\mathcal{O}}(\mathbb{H})$-measurable functionals. Then we have*

$$M^P_\mu\left(fg\Big|\ \widetilde{\mathcal{P}}\right)^2 \leq M^P_\mu\left(f^2\Big|\ \widetilde{\mathcal{P}}\right) M^P_\mu\left(g^2\Big|\ \widetilde{\mathcal{P}}\right), \qquad M^P_\mu\text{–}a.e. \tag{A.5}$$

Proof The proof is the same as the one of the regular Cauchy-Schwarz formula, by putting $\bar{f} := f/M^P_\mu\left(f^2\Big|\ \widetilde{\mathcal{P}}\right)$ and $\bar{g} := g/M^P_\mu\left(g^2\Big|\ \widetilde{\mathcal{P}}\right)$ and using the simple inequality $xy \leq (x^2 + y^2)/2$. This ends the proof of the lemma. □

The following lemma is borrowed from Jacod's Theorem 3.75 in [26] (see also Proposition 2.2 in [8]).

Lemma A.3 *Let $\mathcal{E}(N)$ be a positive local martingale and (β, f, g, N') be the Jacod's parameters of N. Then $\mathcal{E}(N) > 0$ (or equivalently $1 + \Delta N > 0$) implies that*

$$f > 0, \quad M^P_\mu - a.e.$$

Theorem A.4 *Let S be a semi-martingale with predictable characteristic triplet $(b, c, \nu = A \otimes F)$, N be a local martingale such that $\mathcal{E}(N) > 0$, and (β, f, g, N') be its Jacod's parameters. Then the following assertions hold.*
1) $\mathcal{E}(N)$ is a σ-martingale density of S if and only if the following two properties hold:

$$\int |x - h(x) + xf(x)|F(dx) < +\infty, \quad P \otimes A - a.e. \tag{A.6}$$

and

$$b + c\beta + \int \Big(x - h(x) + xf(x)\Big) F(dx) = 0, \qquad P \otimes A - a.e. \tag{A.7}$$

2) In particular, we have

$$\int x(1 + f_t(x))\nu(\{t\}, dx) = \int x(1 + f_t(x))F_t(dx)\Delta A_t = 0, \qquad P - a.e. \tag{A.8}$$

Proof The proof can be found in Choulli et al. [7, Lemma 2.4], and also Choulli and Schweizer [8]. □

Lemma A.5 (see Choulli and Schweizer [8]): *Consider a filtration $\mathbb{H}$ satisfying the usual conditions. Let f be a $\widetilde{\mathcal{P}}(\mathbb{H})$-measurable functional such that $f > 0$ and*

$$\left[(f-1)^2 \star \mu\right]^{1/2} \in \mathcal{A}^+_{loc}(\mathbb{H}). \tag{A.9}$$

Then, the $\mathbb{H}$-predictable process $\left(1 - a^{\mathbb{H}} + \widehat{f}^{\mathbb{H}}\right)^{-1}$ is locally bounded, and hence

$$W_t(x) := \frac{f_t(x) - 1}{1 - a^{\mathbb{H}}_t + \widehat{f}^{\mathbb{H}}_t} \in \mathcal{G}^1_{loc}(\mu, \mathbb{H}). \tag{A.10}$$

Here, $a^{\mathbb{H}}_t := \nu^{\mathbb{H}}(\{t\}, \mathbb{R}^d)$, $\widehat{f}^{\mathbb{H}}_t := \int f_t(x)\nu^{\mathbb{H}}(\{t\}, dx)$ and $\nu^{\mathbb{H}}$ is the $\mathbb{H}$-predictable random measure compensator of μ under $\mathbb{H}$.

Proof The proof of this lemma can be found in [8]. Below we provide this proof for the sake of completeness. In this proof we put

$$U_t(x) = 1 - f_t(x), \quad \text{and} \quad \widehat{U}_t := a^{\mathbb{H}}_t - \widehat{f}^{\mathbb{H}}_t.$$

We start by remarking that (A.10) follows from the combination of (A.9) and the local boundedness of $1/(1-\widehat{U})$. Therefore, in what follows, we will focus on proving this latter fact. Consider $\delta \in (0,1)$, $\eta \in (0,1)$, and the stopping times and processes defined by

$$T_0 = 0, \quad T_{n+1} := \inf\left\{ t > T_n \;\Big|\; \sum_{T_n < v \le t} \left(U_v(\Delta S_v) I_{\{\Delta S_v \neq 0\}}\right)^2 > \delta^2 \right\},$$

$$V_n(t) := \left[\sum_{T_n < v \le t} \left(U_v(\Delta S_v) I_{\{\Delta S_v \neq 0\}}\right)^2 \right]^{1/2}$$

Remark that —since for each $n \ge 0$, the process $(V_n(t))^2$ is RCLL and nondecreasing real-valued process—- we have

$$(V_n(T_{n+1}))^2 := \sum_{T_n < v \le T_{n+1}} \left(U_v(\Delta S_v) I_{\{\Delta S_v \neq 0\}}\right)^2 \ge \delta^2 \;\text{ on }\; \{T_{n+1} < +\infty\}.$$

This implies that T_n increases to $+\infty$ almost surely, and

$$V_n(t-) \le \delta, \quad P-a.s. \quad \text{for all } t \le T_{n+1}.$$

Due to $0 \le (1-\widehat{U})^{-1} I_{\{\widehat{U} < 1-\eta\}} \le \eta^{-1}$ and

$$(1-\widehat{U})^{-1} = (1-\widehat{U})^{-1} I_{\{\widehat{U} \ge 1-\eta\}} + (1-\widehat{U})^{-1} I_{\{\widehat{U} < 1-\eta\}},$$

we deduce that the proof of the lemma will achieved once we prove that

$$Y := \frac{1}{1-\widehat{U}} I_{\{\widehat{U} \ge 1-\eta\}}$$

is locally bounded. Thanks to Lenglart's result (see [13], page 337) which claims that a predictable process is locally bounded if and only if it is pre-locally bounded , or equivalently

$$\sup_{0 \le u \le t} Y_u < +\infty \;\; P-a.s. \quad \text{for any } t \in (0, +\infty).$$

Since T_n increases to ∞ almost surely, then this fact is implied by

$$\sup_{T_n \le u \le t \wedge T_{n+1}} Y_u < +\infty \;\; P-a.s. \quad \text{on} \;\; \{t > T_n\}.$$

Simple calculation leads to

$$\widehat{U}_s \leq V_n(s-) + {}^{p,\mathbb{H}}(\Delta V_n)_s, \quad \text{for all } T_n < s \leq T_{n+1}.$$

Thus, it is easy to see that for $\delta + \eta < 1$,

$$\{s \in]T_n, T_{n+1}] \;\Big|\; \widehat{U}_s \geq 1-\eta\} \subset \{s \in]T_n, T_{n+1}] \;\Big|\; {}^{p,\mathbb{H}}(\Delta V_n)_s \geq 1-\eta-V_n(s-)\}$$

$$\subset \{s \in]T_n, T_{n+1}] \;\Big|\; \Delta\left((V_n)^{p,\mathbb{H}}\right) = {}^{p,\mathbb{H}}(\Delta V_n)_s \geq 1-\eta-\delta\} =: \Gamma_n.$$

It is obvious that $\#\left(\Gamma_n \cap [0,t]\right) < +\infty \; P-a.s.$ since $(V_n)^{p,\mathbb{H}}$ is a càdlàg process. Thus, we deduce that

$$\sup_{T_n \leq u \leq t \wedge T_{n+1}} Y_u = \max_{T_n \leq u \leq t \wedge T_{n+1}} Y_u < +\infty.$$

This ends the proof of the lemma. □

Acknowledgements The research of Tahir Choulli and Jun Deng is supported financially by the Natural Sciences and Engineering Research Council of Canada, through Grant G121210818. The research of Anna Aksamit and Monique Jeanblanc is supported by Chaire Markets in transition, French Banking Federation.

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