ROLE OF INFORMATION IN PRICING DEFAULT-SENSITIVE CONTINGENT CLAIMS

MONIQUE JEANBLANC, MARTA LENIEC

Abstract. We consider a financial market with a savings account and a stock S that follows a general diffusion. The default of the company, which issues the stock S, is modelled as a stopping time with respect to the filtration generated by the value of the firm that is not observable by regular investors. We assume that the stock price and the value of the firm are correlated. We study three investors with different information levels trading in the market who aim to price a general default-sensitive contingent claim. We use the density approach and Yor's method to solve the pricing problem. Specifically, we find the sets of equivalent martingale measures in three cases and, when needed, we choose one of them using f-divergence approach.

1. Model

Let $(\Omega, \mathcal{A}, \mathbb{A} = (\mathcal{A}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space, where $\mathbb{A}$ is the $\mathbb{P}$-augmentation of the filtration generated by a two-dimensional Brownian motion $(W^2, W^{2,\perp})$, where W^2 is independent of $W^{2,\perp}$. For a given stochastic process $X = (X_t)_{t\geq 0}$, we denote by $\mathbb{F}^X = (\mathcal{F}_t^X)_{t\geq 0}$ the natural filtration (satisfying the usual hypotheses of completion and right continuity) generated by X. Let $\rho \in (-1, 1)$ and W^1 denote the Brownian motion correlated with W^2 according to

$$W_t^1 = \rho W_t^2 + \sqrt{1-\rho^2} W_t^{2,\perp}, \tag{1}$$

so that $d\langle W^1, W^2\rangle_t = \rho dt$ and denote, for $i = 1, 2$, $\mathbb{F}^i = (\mathcal{F}_t^i)_{t\geq 0}$ the completed filtration generated by W^i. Similarly, the (completed) filtration $\mathbb{F}^{2,\perp}$ is generated by $W^{2,\perp}$. Moreover, let $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ denote the completed, right-continuous filtration generated by W^1 and W^2. Since the process $W^{2,\perp}$ satisfies

$$W_t^{2,\perp} = \frac{W_t^1 - \rho W_t^2}{\sqrt{1-\rho^2}},$$

the filtration $\mathbb{F}$ coincides with $\mathbb{A}$. In what follows, for a filtration $\mathbb{K}$, the set $\mathcal{P}(\mathbb{K})$ is the set of $\mathbb{K}$-predictable processes.

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Let S, V and τ denote the stock price, the value of the firm and the default time of the company, respectively.

We model the stock price, S, as a strictly positive $\mathbb{F}^1$-adapted diffusion driven by W^1. Namely, we assume that it satisfies the following stochastic differential equation:

$$dS_t = S_t(\mu(S_t)dt + \sigma(S_t)dW_t^1), \quad S_0 = s_0,$$

where s_0 is a positive deterministic constant, the two functions μ and $\sigma > 0$ satisfy the local Lipschitz condition and the condition of linear growth so that the solution is unique (see [10] for details).

We assume that the value of the firm, V, is an $\mathbb{F}^2$-adapted diffusion driven by W^2, starting at $V_0 = v_0$ which satisfies $\mathcal{F}_t^V = \mathcal{F}_t^2$, for all $t \geq 0$. The stock price process is correlated with the value of the firm (if $\rho \neq 0$).

We model the default of the firm as a strictly positive, finite $\mathbb{F}^2$-stopping time τ (which is $\mathcal{F}_\infty^2$-measurable). We assume that the law of τ is equivalent to the Lebesgue measure on $\mathbb{R}^+$, i.e., $\mathbb{P}(\tau \in du) \sim du$. Consequently, there exists a strictly positive function g (the density of the law of τ) such that, for every $u \geq 0$,

$$\mathbb{P}(\tau > u) = \int_u^\infty g(s)ds.$$

From the fact that τ is an $\mathbb{F}^2$-stopping time it is possible to know whether the default has already occurred or not by the observation of V. We note that, if $\rho \neq 0$ the default time τ fails to be independent of $\mathcal{F}_t^1$, for every $t \geq 0$.

We consider a financial market with the stock $S = (S_t)_{t\geq 0}$ and a bank account $B = (B_t)_{t\geq 0}$ assumed, for simplicity, to be constant equal to 1. In order that the financial market is arbitrage free, we assume the following:

Hypothesis 1. *The process $\theta = (\theta_t)_{t\geq 0}$, where $\theta_t = \frac{\mu(S_t)}{\sigma(S_t)}$, is such that the process $m = (m_t)_{t\geq 0}$, where $m_t = \mathcal{E}(-\int_0^\cdot \theta_s dW_s^1)_t$, $m_0 = 1$ is an $(\mathbb{F}^1, \mathbb{P})$-martingale.*

This is the case if θ satisfies the Novikov's condition or Kamazaki's condition (see [8], page 75), (e.g., if the coefficients μ and σ are such that θ is bounded). Then, m defines the unique change of probability measure on $\mathbb{F}^1$ by

$$d\mathbb{Q}|_{\mathcal{F}_t^1} = m_t d\mathbb{P}|_{\mathcal{F}_t^1} \quad \forall t \geq 0,$$

such that the process S is an $(\mathbb{F}^1, \mathbb{Q})$-(local) martingale and the set of equivalent martingale measures is non-empty and reduced to a singleton.

Definition 1.1. Let the maturity time $T > 0$ be fixed. A default-sensitive contingent claim is a random variable of the form

$$\Psi^\tau = Y_T^1 \mathbb{I}_{\tau > T} + Y_T^2(\tau)\mathbb{I}_{\tau \leq T}, \tag{2}$$

where, for $u \geq 0$, Y_T^1 and $Y_T^2(u)$ are integrable, $\mathcal{F}_T^1$-measurable random variables.

Let $\mathbb{K}$ denote a complete and right-continuous filtration such that $\mathbb{F}^1 \subset \mathbb{K}$ and $\mathcal{K}_\infty \subset \mathcal{A}_\infty$. We give a definition of a price of the default-sensitive contingent claim defined by (2). We denote by $\mathcal{M}(\mathbb{K})$ the set of equivalent martingale measures, or pricing measures, i.e., the set of probability measures equivalent to $\mathbb{P}$ in $\mathbb{K}$ such that if $\mathbb{Q} \in \mathcal{M}(\mathbb{K})$, then S is a $(\mathbb{K}, \mathbb{Q})$-(local) martingale.

Definition 1.2. If the set $\mathcal{M}(\mathbb{K})$ is non-empty and $\mathbb{Q} \in \mathcal{M}(\mathbb{K})$, a $(\mathbb{K}, \mathbb{Q})$-fair price at time $0 \leq t \leq T$ of the contingent claim Ψ^τ is the conditional expectation of Ψ^τ with respect to $\mathcal{K}_t$ under $\mathbb{Q}$, i.e.,

$$\mathbb{C}^{\mathbb{K},\Psi^\tau}_{t,\mathbb{Q}} = \mathbb{E}_{\mathbb{Q}}(\Psi^\tau | \mathcal{K}_t). \tag{3}$$

Note that if $\mathcal{M}(\mathbb{K})$ is empty, there will exist arbitrage opportunities in the enlarged filtration. This is studied in the literature under the name insider trading. If $\mathcal{M}(\mathbb{K})$ is non-empty, it is known that a fair price does not induce arbitrage opportunity.

2. Various flows of information

We model the flows of information available for various types of investors as filtrations larger that the reference filtration $\mathbb{F}^1$ generated by S.

In this section we introduce three different types of investors who, except for the observation of S, have some additional information about the time of default: investors with strong information about the time of default τ, regular investors who have information about the default when it occurs and agents with full information who have the strong information and observe the value of the firm V. In general, the stock price process, S, does not remain a semi-martingale in one of the enlarged filtration $\mathbb{K}$ defined above. However, under some specific hypotheses, any $(\mathbb{F}^1, \mathbb{P})$-semi-martingale remains a $(\mathbb{K}, \mathbb{P})$-semi-martingale. One of the hypotheses is Jacod's Hypothesis. We will assume that this hypothesis holds for the first two cases of investors. In the case of the third investor, the investor with full information, Jacod's Hypothesis is never satisfied. Hence, we recall Yor's method introduced in [15], which requires some other conditions. Then, under the appropriate hypotheses, we give the semi-martingale decomposition of S in the enlarged filtration $\mathbb{K}$. Furthermore, we characterize all strictly positive $(\mathbb{K}, \mathbb{P})$-(local) martingales and consider only those ones, which are $(\mathbb{K}, \mathbb{P})$-martingales and define the set $\mathcal{M}(\mathbb{K})$ of equivalent martingale measures, proving that this set is not empty.

We denote the set of all $\mathbb{P}$-integrable, $\mathcal{A}$-measurable random variables by $\mathcal{I}$.

2.1. Investors with strong information.

The investor with strong information observes the stock price process S as well as she possesses the knowledge of default time τ from time $t = 0$. Specifically, her information flow is modelled by the initially enlarged filtration $\mathbb{G}^\tau = (\mathcal{G}^\tau_t)_{t \geq 0}$, where

$$\mathcal{G}^\tau_t = \bigcap_{\epsilon > 0} \mathcal{F}^1_{t+\epsilon} \vee \sigma(\tau) \quad \forall t \geq 0. \tag{4}$$

See [9] for details on the initial enlargement. Note that $\mathcal{G}_0^\tau = \sigma(\tau)$ up to $\mathbb{P}$-negligible sets.
If $\rho = 0$, then τ is independent from $\mathbb{F}^1$ and the stock price process S, remains a semi-martingale in the filtration $\mathbb{G}^\tau$ with the same semi-martingale decomposition as in the reference filtration $\mathbb{F}^1$. However, it is not the case for a general $\rho \in (-1, 1)$. It was shown by Jacod in [7], that under the following Hypothesis 2, every $(\mathbb{F}^1, \mathbb{P})$-semi-martingale remains a $(\mathbb{G}^\tau, \mathbb{P})$-semi-martingale.

Hypothesis 2. *The $\mathbb{F}^1$-(regular) conditional law of τ is equivalent to the law of τ, i.e.,*

$$\mathbb{P}(\tau \in du | \mathcal{F}_t^1) \sim \mathbb{P}(\tau \in du) \quad \forall t \geq 0, \forall u \geq 0, \quad \mathbb{P} - a.s.$$

We assume that Hypothesis 2 holds in Sections 2.1 and 2.2.
We recall results which establish the semi-martingale decomposition of S in the initially enlarged filtration $\mathbb{G}^\tau$. By Amendinger's reformulation of Jacod's result (see [1] and [5]): there exists a regular version of the conditional density, i.e., for any fixed $t \geq 0$ there exists an $\mathcal{F}_t^1 \otimes \mathcal{B}(\mathbb{R}^+)$-measurable, strictly positive function $(\omega, u) \mapsto p_t(\omega, u)$ such that for every $u \geq 0$, $p(u) = (p_t(u))_{t \geq 0}$ is an $(\mathbb{F}^1, \mathbb{P})$-martingale and

$$\mathbb{P}(\tau > u | \mathcal{F}_t^1) = \int_u^\infty p_t(s) g(s) ds, \quad \forall t \geq 0 \quad \mathbb{P} - a.s, \tag{5}$$

where $\mathcal{F}_t^1 \otimes \mathcal{B}(\mathbb{R}^+)$ is the product σ-algebra, generated by the sets $\{A \times B : A \in \mathcal{F}_t^1, B \in \mathcal{B}(\mathbb{R}^+)\}$.

By hypothesis on the law of τ, $\mathbb{P}(\tau > u | \mathcal{F}_0^1) = \mathbb{P}(\tau > u) = \int_u^\infty g(s) ds$. Thus, $p_0(u) = 1$, for all $u \geq 0$. The family of martingales $(p(u), u \geq 0)$ is called the conditional density of τ given $\mathbb{F}^1$. From the martingale representation theorem in a Brownian filtration and the strict positivity of p we have, for every $u \geq 0$, the existence of an ($\mathbb{F}^1$-predictable process $\beta(u)$) such that

$$dp_t(u) = p_t(u) \beta_t(u) dW_t^1, \quad p_0(u) = 1. \tag{6}$$

Hence,

$$p_t(u) = \exp\{ \int_0^t \beta_s(u) dW_s^1 - \frac{1}{2} \int_0^t \beta_s^2(u) ds \}.$$

By [1] and [5], Hypothesis 2 is satisfied if and only if there exists a probability measure $\mathbb{P}^*$ equivalent to $\mathbb{P}$ such that $\mathbb{F}^1$ and $\sigma(\tau)$ are independent under $\mathbb{P}^*$. They give the form of a specific choice of $\mathbb{P}^*$ in terms of the density p as follows:

Lemma 2.1. *Under Hypothesis 2, the probability measure $\mathbb{P}^*$ in $\mathbb{G}^\tau$ such that*

$$d\mathbb{P}^*|_{\mathcal{G}_t^\tau} = \frac{1}{p_t(\tau)} d\mathbb{P}|_{\mathcal{G}_t^\tau},$$

makes $\mathcal{F}^1$ and $\sigma(\tau)$ independent. Furthermore, the measure $\mathbb{P}^*$ is identical to $\mathbb{P}$ on $\mathbb{F}^1$ and on $\sigma(\tau)$.

In particular, the $(\mathbb{F}^1, \mathbb{P})$-Brownian motion W^1 is an $(\mathbb{F}^1, \mathbb{P}^*)$-Brownian motion and a $(\mathbb{G}, \mathbb{P}^*)$-Brownian motion.

Let $G = (G_t)_{t\geq 0}$ be the Azéma $\mathbb{F}^1$-supermartingale, i.e.,

$$G_t := \mathbb{P}(\tau > t|\mathcal{F}_t^1) = \int_t^\infty p_t(u)g(u)du \quad \forall t \geq 0, \tag{7}$$

where the integral representation holds by (5). For the proof of the following proposition see [11].

Proposition 2.2. *The process G defined by (7) starts at $G_0 = 1$ and has the following dynamics:*

$$dG_t = -p_t(t)g(t)dt + \Xi_t dW_t^1, \tag{8}$$

where

$$\Xi_t = -\int_0^t p_t(u)\beta_t(u)g(u)du. \tag{9}$$

The decreasing and continuous process $-\int_0^\cdot p_u(u)g(u)du = (-\int_0^t p_u(u)g(u)du)_{t\geq 0}$ is the $\mathbb{F}^1$-predictable, bounded variation part of the $\mathbb{F}^1$-Doob-Meyer decomposition of G.

By [7], we have the following lemma.

Lemma 2.3. *Let us assume that the Hypothesis 1 holds. The $(\mathbb{F}^1, \mathbb{P})$-Brownian motion W^1 has the following semi-martingale decomposition in the filtration $\mathbb{G}^\tau$ under the probability measure $\mathbb{P}$:*

$$W_t^1 = W_t^\tau + \int_0^t \frac{d\langle W^1, p(u)\rangle_s}{p_s(u)}|_{u=\tau} \quad \forall t \geq 0, \tag{10}$$

where $W^\tau = (W_t^\tau)_{t\geq 0}$ denotes a $(\mathbb{G}^\tau, \mathbb{P})$-Brownian motion.

As an immediate application, we obtain the semi-martingale decomposition of S in $\mathbb{G}^\tau$.

Theorem 2.4. *The process S is a $(\mathbb{G}^\tau, \mathbb{P})$-semi-martingale with the following dynamics:*

$$dS_t = S_t\Big((\mu(S_t) + \sigma(S_t)\beta_t(\tau))dt + \sigma(S_t)dW_t^\tau\Big),$$

where $W^\tau = (W_t^\tau)_{t\geq 0}$ is a $(\mathbb{G}^\tau, \mathbb{P})$-Brownian motion and $\beta(\tau)$ is defined in (6).

Proof: From (6), $d\langle W^1, p(u)\rangle_s = p_s(u)\beta_s(u)ds$ and from Lemma 2.3,

$$dW_t^1 = dW_t^\tau + \beta_t(\tau)dt. \tag{11}$$

Then,

$$dS_t = S_t\Big(\mu(S_t)dt + \sigma(S_t)(dW_t^\tau + \beta_t(\tau)dt)\Big).$$

□

Note that, if $\rho = 0$, τ is independent of $\mathbb{F}^1$, and the Hypothesis 1 is satisfied. In that trivial case, $p_t(u) = 1$ and $\beta_t(u) = 0$ for all $t \geq 0$ and $u \geq 0$. Hence, the $(\mathbb{F}^1, \mathbb{P})$-Brownian motion W^1 remains a $(\mathbb{G}^\tau, \mathbb{P})$-Brownian motion and S is a $(\mathbb{G}^\tau, \mathbb{P})$-semi-martingale with the same decomposition as in the filtration $\mathbb{F}^1$.

2.1.1. *Change of measure in* $\mathbb{G}^\tau$. Our aim is to characterize the set $\mathcal{M}(\mathbb{G}^\tau)$. Using a predictable representation theorem, we characterize all strictly positive $(\mathbb{G}^\tau, \mathbb{P})$-martingales.

As in [9], a process $\zeta(\tau)$ is $\mathbb{G}^\tau$-predictable if and only if it is of the form $\zeta(\tau) = (\zeta_t(\tau))_{t\geq 0}$, where $\zeta_t(\tau)(\omega) = \varsigma_t(\omega, \tau(\omega))$ and $(t, \omega, u) \mapsto \varsigma_t(\omega, u)$ is a $\mathcal{P}(\mathbb{F}^1) \otimes \mathcal{B}(\mathbb{R}^+)$-measurable function.

Let

$$\Delta = \{X, \sigma(\tau)\text{-measurable, strictly positive random variable}\},$$

By Proposition 4.3 in [3], every strictly positive $(\mathbb{G}^\tau, \mathbb{P})$-(local) martingale $Z(\tau)$ can be uniquely represented as

$$dZ_t(\tau) = Z_t(\tau)\alpha_t(\tau)dW_t^\tau,$$

where $Z_0(\tau) \in \Delta$ and $\alpha(\tau)$ is $\mathbb{G}^\tau$-predictable.

Let

$$\Gamma_{\mathbb{G}^\tau} = \{X \in \Delta : \mathbb{E}_{\mathbb{P}}\left(X\mathcal{E}\left(-\int_0^{\cdot}(\theta_s + \beta_s(\tau))dW_s^\tau\right)_t\right) = 1 \quad \forall t \geq 0\},$$

where $\mathcal{E}$ denotes the Doléans-Dade exponential.

Lemma 2.5. *The set* $\Gamma_{\mathbb{G}^\tau}$ *is not empty.*

Proof: Since m is an $(\mathbb{F}, \mathbb{P})$-martingale, hence a $(\mathbb{G}^\tau, \mathbb{P}^*)$-martingale, it makes sense to define the probability measure $\mathbb{Q}^*$ on $\mathcal{G}_t^\tau$ as $\frac{d\mathbb{Q}^*}{d\mathbb{P}^*} = m_t$, or $\frac{d\mathbb{Q}^*}{d\mathbb{P}} = m_t \frac{1}{p_t(\tau)} =: L_t$. An integration by parts and the equality (6) lead to

$$dL_t = -L_t(\theta_t + \beta_t(\tau))(dW_t - \beta_t(\tau)dt) = -L_t(\theta_t + \beta_t(\tau))dW_t^\tau$$

The measure $\mathbb{Q}^*$ being a probability measure, it follows that $\mathbb{E}_{\mathbb{P}}(L_T) = 1$, so $X = 1$ belongs to $\Gamma_{\mathbb{G}^\tau}$. □

Proposition 2.6. *Let* $Z_0(\tau) \in \Gamma_{\mathbb{G}^\tau}$. *The process* $Z(\tau) = (Z_t(\tau))_{t\geq 0}$, *where*

$$Z_t(\tau) = Z_0(\tau)\mathcal{E}\left(-\int_0^{\cdot}(\theta_s + \beta_s(\tau))dW_s^\tau\right)_t, \tag{12}$$

is a $(\mathbb{G}^\tau, \mathbb{P})$*-martingale and the probability measure* $\mathbb{Q}$ *defined by*

$$d\mathbb{Q}|_{\mathcal{G}_t^\tau} = Z_t(\tau)d\mathbb{P}|_{\mathcal{G}_t^\tau} \quad \forall t \geq 0$$

belongs to $\mathcal{M}(\mathbb{G}^\tau)$. *Conversely, any* $\mathbb{Q} \in \mathcal{M}(\mathbb{G}^\tau)$ *is given by (12).*

Proof: Let $Z(\tau)$ be any strictly positive $(\mathbb{G}^\tau, \mathbb{P})$-(local) martingale of the form $dZ_t(\tau) = Z_t(\tau)\alpha_t(\tau)dW_t^\tau$. The process $Z(\tau)$ defines a measure from $\mathcal{M}(\mathbb{G}^\tau)$ if and only if $\mathbb{E}_\mathbb{P}(Z_t(\tau)) = 1$ for any $t \geq 0$ (i.e., the martingale property is satisfied) and $Z(\tau)S = (Z_t(\tau)S_t)_{t\geq 0}$ is a $(\mathbb{G}^\tau, \mathbb{P})$-(local) martingale. Using integration by parts formula,

$$\begin{aligned} d(Z_t(\tau)S_t) &= Z_t(\tau)dS_t + S_t dZ_t(\tau) + d\langle Z(\tau), S\rangle_t \\ &= Z_t(\tau)S_t\,(\,(\mu(S_t) + \sigma(S_t)(\beta_t(\tau) + \alpha_t(\tau)))dt + (\alpha_t(\tau) + \sigma(S_t))dW_t^\tau) \end{aligned}$$

Hence, the latter condition is satisfied if and only if $\alpha(\tau) = -\theta - \beta(\tau)$. Moreover, since $Z_0(\tau) \in \Gamma_{\mathbb{G}^\tau}$, we have that $\mathbb{E}_\mathbb{P}(Z_t(\tau)) = 1$ for any $t \geq 0$. □

Since the processes θ and $\beta(\tau)$ are known, we have obtained an infinite set of martingale measures which elements differ only in the starting value of the Radon-Nikodym density, i.e., $Z_0(\tau) \in \Gamma_{\mathbb{G}^\tau}$. However, we shall see later that this fact does not imply any incompleteness of the market.
Since the measures $\mathbb{P}^*$ and $\mathbb{Q}$ are equivalent to $\mathbb{P}$, $\mathbb{Q}$ is equivalent to $\mathbb{P}^*$. We give the Radon-Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}^*$.

Proposition 2.7. *The Radon-Nikodym derivative for the change of measure from* $\mathbb{Q} \in \mathcal{M}(\mathbb{G}^\tau)$ *to* $\mathbb{P}^*$ *on* $\mathbb{G}^\tau$ *is given by*

$$Y_t(\tau) = Z_0(\tau)m_t. \tag{13}$$

Proof: Let $\mathbb{Q} \in \mathcal{M}(\mathbb{G}^\tau)$. By Proposition 2.6,

$$\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{G}_t^\tau} = Z_t(\tau),$$

where $Z(\tau)$ is defined as in (12). Moreover, by Lemma 2.5,

$$\frac{d\mathbb{Q}}{d\mathbb{P}^*}|_{\mathcal{G}_t^\tau} = Z_0(\tau)m_t.$$

□

2.1.2. *Pricing in* $\mathbb{G}^\tau$. The change of measure defined by the process $Y(\tau)$ in (13) allows us to give a price of the default-sensitive contingent claim Ψ^τ.

Proposition 2.8. *Let* $0 \leq t < T$. *The price,* $\mathbb{C}_{t,\mathbb{Q}}^{\mathbb{G}^\tau,\Psi^\tau}$, *of the default-sensitive contingent claim* $\Psi^\tau = Y_T^1\mathbb{I}_{\tau>T} + Y_T^2(\tau)\mathbb{I}_{\tau\leq T}$, *where* Y_T^1 *and* $Y_T^2(u)$ *are integrable,* $\mathcal{F}_T^1$*-measurable random variables, is for any* $\mathbb{Q} \in \mathcal{M}(\mathbb{G}^\tau)$ *uniquely given by:*

$$\mathbb{C}_{t,\mathbb{Q}}^{\mathbb{G}^\tau,\Psi^\tau} = \mathbb{I}_{\tau>T}\frac{\mathbb{E}_\mathbb{P}(Y_T^1 m_T|\mathcal{F}_t^1)}{m_t} + \mathbb{I}_{\tau\leq T}\frac{\mathbb{E}_\mathbb{P}(Y_T^2(u)m_T|\mathcal{F}_t^1)_{|u=\tau}}{m_t},$$

where $m = (m_t)_{t\geq 0}$ *is defined in Hypothesis 1.*

Proof: By the change of measure defined by (13) and by (4) we get, for any $\mathbb{Q} \in \mathcal{M}(\mathbb{G}^\tau)$,

$$\mathbb{E}_{\mathbb{Q}}(\Psi^\tau|\mathcal{G}_t^\tau) = (Y_0(\tau))^{-1} m_t^{-1} \mathbb{E}_{\mathbb{P}^*}(\Psi^\tau Y_0(\tau) m_T|\mathcal{G}_t^\tau).$$

The random variable $Y_0(\tau)$ is $\mathcal{G}_t^\tau$-measurable for every $t \geq 0$. Thus,

$$\mathbb{E}_{\mathbb{Q}}(\Psi^\tau|\mathcal{G}_t^\tau) = m_t^{-1} \mathbb{E}_{\mathbb{P}^*}(\Psi^\tau m_T|\mathcal{G}_t^\tau).$$

Since $\mathcal{F}_t^1$ and $\sigma(\tau)$ are independent under $\mathbb{P}^*$ for every $t \geq 0$, and $m_t \in \mathcal{F}_t^1$ we have

$$\mathbb{E}_{\mathbb{P}^*}(\Psi^\tau m_T|\mathcal{G}_t^\tau) = \mathbb{E}_{\mathbb{P}^*}(\Psi^u m_T|\mathcal{F}_t^1)_{|u=\tau}.$$

From the fact that $\mathbb{P}$ and $\mathbb{P}^*$ coincide on $\mathcal{F}_T^1$ and the process m is $\mathbb{F}^1$-adapted,

$$\mathbb{E}_{\mathbb{P}^*}(\Psi^u m_T|\mathcal{F}_t^1) = \mathbb{E}_{\mathbb{P}}(\Psi^u m_T|\mathcal{F}_t^1),$$

which equals to

$$\mathbb{E}_{\mathbb{P}}((Y_T^1 \mathbb{I}_{u>T} + Y_T^2(u)\mathbb{I}_{u\leq T}) m_T|\mathcal{F}_t^1).$$

□

Example 2.9. The price at time $0 \leq t \leq T$ of the default-sensitive zero-recovery zero-coupon bond which pays $\mathbb{I}_{\tau>T}$ at maturity is $\mathbb{I}_{\tau>T}$.

As already discussed, the set $\mathcal{M}(\mathbb{G}^\tau)$ is infinite. Each measure $\mathbb{Q} \in \mathcal{M}(\mathbb{G}^\tau)$ is defined by (12). However, the randomness due to τ does not imply any incompleteness. Indeed, as shown in Example 2.9, the price of a default-sensitive zero-coupon bond is the (random) quantity $\mathbb{I}_{\tau>T}$ and the market is complete. The uniqueness of equivalent martingale measure implies completeness if and only if the initial σ-algebra is trivial.

2.2. **Regular investors.** The regular investor observes the stock price process S and the default when it happens. We model her information by the progressive enlargement $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$, where

$$\mathcal{G}_t = \bigcap_{\epsilon>0} \mathcal{F}_{t+\epsilon}^1 \vee \sigma(\tau \wedge (t+\epsilon)) \quad \forall t \geq 0.$$

See [3] for results on the progressive enlargement. Under Hypothesis 1, by Stricker's Theorem, the process S is a $(\mathbb{G}, \mathbb{P})$-semi-martingale. Moreover, we have the following representation of the $(\mathbb{F}^1, \mathbb{P})$-Brownian motion W^1 in terms of a $(\mathbb{G}, \mathbb{P})$-Brownian motion:

Lemma 2.10. *Let us assume that the Hypothesis* 1 *holds. The* $(\mathbb{F}^1, \mathbb{P})$*-Brownian motion* W^1 *has the following semi-martingale decomposition in filtration* $\mathbb{G}$*:*

$$W_t^1 = B_t + \int_0^{\tau\wedge t} \frac{d\langle W^1, G\rangle_s}{G_s} + \int_{\tau\wedge t}^t \frac{d\langle W^1, p(u)\rangle_s}{p_s(u)}|_{u=\tau},$$

where $B = (B_t)_{t\geq 0}$ *is a* $(\mathbb{G}, \mathbb{P})$*-Brownian motion.*

Theorem 2.11. *The process S is a $(\mathbb{G},\mathbb{P})$-semi-martingale with the following decomposition:*

$$dS_t = S_t\Big((\mu(S_t)+\sigma(S_t)\frac{\Xi_t}{G_t}\mathbb{I}_{t\leq\tau}+\sigma(S_t)\beta_t(\tau)\mathbb{I}_{t>\tau}))dt+\sigma(S_t)dB_t\Big),$$

where $B=(B_t)_{t\geq 0}$ is a $(\mathbb{G},\mathbb{P})$-Brownian motion and Ξ is defined in (9).

Proof: From Lemma 2.10, (8) and (6),

$$dW_t^1 = dB_t + \Big(\frac{\Xi_t}{G_t}\mathbb{I}_{t\leq\tau}+\beta_t(\tau)\mathbb{I}_{t>\tau}\Big)dt, \tag{14}$$

which leads to the result. □

We note that, if $\rho=0$, then $\beta_t(\tau)=0$ and $\Xi_t=0$ for $t\geq 0$. Hence, S is a $(\mathbb{G},\mathbb{P})$-semi-martingale with the same decomposition as in the filtration $\mathbb{F}^1$.
Recall that a process ζ is $\mathbb{G}$-predictable if and only if it is of the form $\zeta=(\zeta_t)_{t\geq 0}$, where $\zeta_t=\tilde{\zeta}_t\mathbb{I}_{t\leq\tau}+\hat{\zeta}_t(\tau)\mathbb{I}_{t>\tau}$ and $\tilde{\zeta}$ is $\mathbb{F}^1$-predictable and $(t,\omega,u)\mapsto\hat{\zeta}_t(\omega,u)$ is a $\mathcal{P}(\mathbb{F}^1)\otimes\mathcal{B}(\mathbb{R}^+)$-measurable function, where $\mathcal{P}(\mathbb{F}^1)$ denotes the predictable σ-algebra in $\mathbb{F}^1$ (see Proposition 1.3 in [3]).
We have the following decomposition of G (see (8)):

$$dG_t = \Xi_t dW_t^1 - p_t(t)g(t)dt =: \Xi_t dW_t^1 - dA_t$$

In [8] the following was shown, using the fact that G is continuous.

Lemma 2.12. *Let $H_t=\mathbb{I}_{\tau\leq t}$ and $M_t=H_t-\int_0^{\tau\wedge t}\frac{dA_s}{G_s}$. Then, the process $M=(M_t)_{t\geq 0}$ is a $(\mathbb{G},\mathbb{P})$-martingale, orthogonal to the $(\mathbb{G},\mathbb{P})$-Brownian motion B.*

2.2.1. *Change of measure in $\mathbb{G}$.* In this section, we characterize the set $\mathcal{M}(\mathbb{G})$. We we give the result of the predictable representation theorem which characterizes all strictly positive $(\mathbb{G},\mathbb{P})$-(local) martingales. Then, we give a hypothesis which has to be satisfied if we want to avoid arbitrages.
Let

$$\Gamma=\{\upsilon=(\upsilon_t)_{t\geq 0}\in\mathcal{P}(\mathbb{F}^1):\upsilon_t>-1 \text{ and } \int_0^t|\upsilon_s|\lambda_s ds<\infty \quad \forall t\geq 0\}. \tag{15}$$

By predictable representation theorem (see Proposition 4.3 in [3]), every strictly positive $(\mathbb{G},\mathbb{P})$-(local) martingale K starting at $K_0=1$ can be uniquely represented as

$$dK_t = K_{t^-}\Big(\vartheta_t dB_t+\upsilon_t dM_t\Big), \quad K_0=1, \tag{16}$$

where $\vartheta\in\mathcal{P}(\mathbb{G})$, $\upsilon\in\Gamma$, M is defined as in Lemma 2.12 and B is the $(\mathbb{G},\mathbb{P})$-Brownian motion defined in Lemma 2.10.

Remark 2.13. The process υ in (16) is in general stated to be $\mathbb{G}$-predictable in a predictable representation theorem. However, from [8], there exists an $\mathbb{F}^1$-predictable process $\tilde{\upsilon}$ such that $\upsilon_t \mathbb{I}_{t\leq\tau} = \tilde{\upsilon}_t \mathbb{I}_{t\leq\tau}$. Then, $\upsilon_t dM_t = \tilde{\upsilon}_t dM_t$.

Let the process $\lambda = (\lambda_t)_{t\geq 0}$ be defined by $\lambda_t = \frac{p_t(t)g(t)}{G_t}$ and

$$\gamma_t = \frac{\Xi_t}{G_t}\mathbb{I}_{t\leq\tau} + \beta_t(\tau)\mathbb{I}_{t>\tau}. \tag{17}$$

Define

$$\Gamma_{\mathbb{G}} = \{\upsilon \in \Gamma : \mathbb{E}_{\mathbb{P}}\left(\mathcal{E}\left(-\int_0^{\cdot}(\theta_s+\gamma_s)dB_s\right)_t \exp\{-\int_0^{t\wedge\tau}\upsilon_s\lambda_s ds + \ln(1+\upsilon_\tau)\mathbb{I}_{t>\tau}\}\right) = 1\}.$$

Obviously, $\upsilon = 0$ belongs to $\Gamma_{\mathbb{G}}$.

Proposition 2.14. *Let Hypothesis 2 hold and $\upsilon \in \Gamma_{\mathbb{G}}$. The process $K = (K_t)_{t\geq 0}$, defined as*

$$K_t = \mathcal{E}\left(-\int_0^{\cdot}(\theta_s+\gamma_s)dB_s\right)_t \exp\{-\int_0^{t\wedge\tau}\upsilon_s\lambda_s ds + \ln(1+\upsilon_\tau)\mathbb{I}_{t>\tau}\}, \tag{18}$$

is a $(\mathbb{G},\mathbb{P})$-martingale and the probability measure $\mathbb{Q}$ defined by

$$d\mathbb{Q}|_{\mathcal{G}_t} = K_t d\mathbb{P}|_{\mathcal{G}_t} \quad \forall t \geq 0$$

belongs to $\mathcal{M}(\mathbb{G})$. All the elements of $\mathcal{M}(\mathbb{G})$ are given by processes (18).

Proof: Let K be a process of the form (18). Then $\mathbb{E}_{\mathbb{P}}(K_t) = 1$ and the process K is a $(\mathbb{G},\mathbb{P})$-martingale. Moreover, by taking into account that S is continuous, we have

$$\frac{d(S_tK_t)}{S_tK_{t-}} = \Big(\sigma(S_t) - \theta_t - \frac{\Xi_t}{G_t}\mathbb{I}_{t\leq\tau} - \beta_t(\tau)\mathbb{I}_{t>\tau}\Big)dB_t + \upsilon_t dM_t.$$

Hence, the process $SK = (S_tK_t)_{t\geq 0}$ is a $(\mathbb{G},\mathbb{P})$-(local) martingale and the process K defines a measure from $\mathcal{M}(\mathbb{G})$. Using the Predictable Representation property (16), any Radon-Nykodym density has the form (18). □

However, there exists more than one element $\upsilon \in \Gamma_{\mathbb{G}}$ such that K defines a probability measure from the set $\mathcal{M}(\mathbb{G})$. Indeed, we show in Section 3, that for the log utility function, one has $\upsilon = 0$. Moreover, for the power utility function, we prove that $\upsilon \neq 0$ (see Remark 3.5). Hence, there exist at least two elements of $\mathcal{M}(\mathbb{G})$. In particular, from the fact that the initial σ-algebra $\mathcal{G}_0$ is trivial, the market is incomplete.

2.3. **Investors with full information.** We say that an investor possesses the full information, when she observes the stock price process S, the firm value process V and knows τ from time $t = 0$. We denote the information flow by $\mathbb{G}^F = (\mathcal{G}_t^F)_{t\geq 0}$, where

$$\mathcal{G}_t^F = \bigcap_{\epsilon>0} \mathcal{F}_{t+\epsilon}^1 \vee \mathcal{F}_{t+\epsilon}^2 \vee \sigma(\tau) \quad \forall t \geq 0.$$

We note that $\mathcal{G}_0^F = \sigma(\tau)$, up to $\mathbb{P}$-negligible sets. The following proposition will establish that, since τ is an $\mathbb{F}^2$-stopping time, Jacod's hypothesis does not hold for the $\mathbb{F}$-conditional law of τ, where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ and

$$\mathcal{F}_t = \bigcap_{\epsilon>0} \mathcal{F}_{t+\epsilon}^1 \vee \mathcal{F}_{t+\epsilon}^2 \quad \forall t \geq 0.$$

Proposition 2.15. *The $\mathbb{F}$-conditional law of τ is given by:*

$$\mathbb{P}(\tau > u|\mathcal{F}_t) = \begin{cases} \mathbb{I}_{\tau>t}\mathbb{P}(\tau > u|\mathcal{F}_t^2), & u > t, \\ \mathbb{I}_{\tau>u}, & u \leq t. \end{cases} \quad \mathbb{P} - a.s.$$

Proof: From the fact that the filtration $\mathbb{F}$ coincides with $\mathbb{A}$ and τ is an $\mathbb{F}^2$-stopping time, we have:

$$\mathbb{P}(\tau > u|\mathcal{F}_t) = \mathbb{P}(\tau > u|\mathcal{A}_t) = \mathbb{P}(\tau > u|\mathcal{F}_t^2),$$

where the last equality holds by the fact that the filtration $\mathbb{A} = (\mathcal{A}_t)_{t\geq 0}$ is generated by the two-dimensional Brownian motion $(W^2, W^{2,\perp})$ and τ is independent of $\mathcal{F}_t^{2,\perp}$, for any $t \geq 0$.

We write the conditional probability on two sets:

$$\mathbb{P}(\tau > u|\mathcal{F}_t^2) = \mathbb{I}_{u>t}\mathbb{P}(\tau > u|\mathcal{F}_t^2) + \mathbb{I}_{u\leq t}\mathbb{P}(\tau > u|\mathcal{F}_t^2).$$

On the one hand, from the fact that for $u > t$, $\{\tau > u\} \subset \{\tau > t\}$ and $\{\tau > t\} \in \mathcal{F}_t^2$, we get:

$$\mathbb{I}_{u>t}\mathbb{P}(\tau > u|\mathcal{F}_t^2) = \mathbb{I}_{\tau>t}\mathbb{I}_{u>t}\mathbb{P}(\tau > u|\mathcal{F}_t^2).$$

On the other hand, from the fact that τ is an $\mathbb{F}^2$-stopping time, $\{\tau > u\} \in \mathcal{F}_t^2$ for $u \leq t$. We obtain:

$$\mathbb{I}_{u\leq t}\mathbb{P}(\tau > u|\mathcal{F}_t^2) = \mathbb{I}_{u\leq t}\mathbb{I}_{\tau>u},$$

which gives the result.

□

From Proposition 2.15, Hypothesis 1 is not satisfied, since obviously $\mathbb{I}_{\tau>u}$ is not absolutely continuous with respect to any probability law with density on $\mathbb{R}^+$.

2.3.1. *Yor's method.* From the fact that the equivalence between the $\mathbb{F}$-conditional and unconditional law of τ does not hold, we cannot apply Jacod's result for the decomposition of W^1 and W^2 in $\mathbb{G}^F$. However, since we work with a Brownian filtration, we can use Yor's method (see [15]). For our purpose we fix $u \geq 0$ and consider the process $P(u) = (P_t(u))_{t\geq 0}$, where $P_t(u) = \mathbb{P}(\tau > u|\mathcal{F}_t)$, which, as in the

proof of Proposition 2.15, coincides with $\mathbb{P}(\tau > u|\mathcal{F}_t^2)$. The process $P(u)$ is an $(\mathbb{F}^2, \mathbb{P})$-square integrable martingale (as discussed in Section , also an $(\mathbb{A}, \mathbb{P})$-martingale) and by the martingale representation theorem, it can be written as:

$$P_t(u) = P_0(u) + \int_0^t Q_s(u) dW_s^2, \tag{19}$$

where the process $Q(u) = (Q_t(u))_{t\geq 0} \in \mathcal{P}(\mathbb{A})$. There exists a family of measures $P(dx) = (P_t(dx))_{t\geq 0}$ such that

$$P_t(u) = \int_u^\infty P_t(dx). \tag{20}$$

Similarly as in [15], we work under following hypothesis:

Hypothesis 3. *There exists an $\mathbb{A}$-predictable family of measures $Q(dx) = (Q_t(dx))_{t\geq 0}$ such that*

i) *the process $Q(u)$ defined in (19) can be written as $Q_t(u) = \int_u^\infty Q_t(dx)$,*
ii) *the measure $Q_t(dx)$ is absolutely continuous with respect to $P_t(dx)$ for every $t \geq 0$,*
iii) *the process $\zeta(\tau)$ defined by $Q_t(dx) = \zeta_t(x)P_t(dx)$ satisfies $\int_0^t |\zeta_s(\tau)|ds < \infty$ for every $t \geq 0$.*

We assume that Hypothesis 3 holds in Section 2.3.
Now, we state a part of Theorem 1.6 in [15]:

Lemma 2.16. *The $(\mathbb{A}, \mathbb{P})$-Brownian motion W^2 decomposes as:*

$$W_t^2 = W_t^{2,\tau} + \int_0^t \zeta_s(\tau) ds,$$

where $W^{2,\tau}$ is a $(\mathbb{G}^F, \mathbb{P})$-Brownian motion.

Theorem 2.17. *The process $W^{2,\perp}$ is a $(\mathbb{G}^F, \mathbb{P})$-Brownian motion. The $(\mathbb{A}, \mathbb{P})$-Brownian motion W^1 decomposes as*

$$W_t^1 = W_t^{1,\tau} + \rho \int_0^t \zeta_s(\tau) ds,$$

where $W^{1,\tau}$ denotes a $(\mathbb{G}^F, \mathbb{P})$-Brownian motion.

Proof: From the fact that τ is an $\mathbb{F}^2$-stopping time, we have

$$\mathbb{P}(\tau > u|\mathcal{F}_t^{2,\perp}) = \mathbb{P}(\tau > u).$$

Hence, the $(\mathbb{A},\mathbb{P})$-Brownian motion $W^{2,\perp}$ remains a $(\mathbb{G}^F,\mathbb{P})$-Brownian motion independent from $W^{2,\tau}$ and

$$W_t^1=\rho W_t^2+\sqrt{1-\rho^2}W_t^{2,\perp}=\rho W_t^{2,\tau}+\sqrt{1-\rho^2}W_t^{2,\perp}+\rho\int_0^t\zeta_s(\tau)ds.$$

The process $W^{1,\tau}=\rho W^{2,\tau}+\sqrt{1-\rho^2}W^{2,\perp}$ is a $(\mathbb{G}^F,\mathbb{P})$-Brownian motion and the result holds. □

We have that $d\langle W^{1,\tau},W^{2,\tau}\rangle_t=d\langle W^1,W^2\rangle_t=\rho dt$. Hence, there exists a $(\mathbb{G}^F,\mathbb{P})$-Brownian motion $W^{1,\tau,\perp}$, independent of $W^{1,\tau}$, such that

$$W_t^{2,\tau}=\rho W_t^{1,\tau}+\sqrt{1-\rho^2}W_t^{1,\tau,\perp}$$

and the filtration $\mathbb{G}^F$ coincides with the initial enlargement of the two-dimensional Brownian motion $(W^{1,\tau},W^{1,\tau,\perp})$.

Theorem 2.18. *The process S is a $(\mathbb{G}^F,\mathbb{P})$-semi-martingale with the following decomposition:*

$$dS_t=S_t\Big(\Big(\mu(S_t)+\sigma(S_t)\rho\zeta_t(\tau)\Big)dt+\sigma(S_t)dW_t^{1,\tau}\Big). \tag{21}$$

2.3.2. *Change of measure in $\mathbb{G}^F$.* Hereafter, we characterize the set $\mathcal{M}(\mathbb{G}^{\mathbb{F}})$. We give a predictable representation theorem which characterizes all strictly positive $(\mathbb{G}^{\mathbb{F}},\mathbb{P})$-(local) martingales. Then, we give an hypothesis which has to be satisfied if we want to avoid arbitrages opportunities.

Analogously to Proposition 4.3 in [3], we note that every strictly positive $(\mathbb{G}^F,\mathbb{P})$-(local) martingale $L(\tau)$ can be uniquely represented as

$$dL_t(\tau)=L_t(\tau)\Big(a_t(\tau)dW_t^{1,\tau}+b_t(\tau)dW_t^{1,\tau,\perp}\Big), \tag{22}$$

where $a(\tau)\in\mathcal{P}(\mathbb{G}^F)$, $b(\tau)\in\mathcal{P}(\mathbb{G}^F)$, $W^{1,\tau}$ and $W^{1,\tau,\perp}$ denote the orthogonal $(\mathbb{G}^F,\mathbb{P})$-Brownian motions and $L_0(\tau)\in\Delta$. Let us define the following sets:

$$\hat{\Gamma}=\{(b,X):(b,X)\in\mathcal{P}(\mathbb{G}^F)\times\Delta\}$$

and

$$\Gamma_{\mathbb{G}^F}=\{(b,X)\in\hat{\Gamma}:\mathbb{E}_{\mathbb{P}}\left(X\mathcal{E}\left(-\int_0^{\cdot}\Big(\theta_s+\rho\zeta_s(\tau)\Big)dW_s^{1,\tau}\right)_t\cdot\mathcal{E}\left(\int_0^{\cdot}b_s(\tau)dW_s^{1,\tau,\perp}\right)_t\right)=1\}.$$

Hypothesis 4. *The set $\Gamma_{\mathbb{G}^F}$ is not empty*

Proposition 2.19. *Let Hypotheses 3 and 4 hold. For any $(b,L_0(\tau))$ from the set $\Gamma_{\mathbb{G}^F}$, the process $L(\tau)$ given by*

$$L_t(\tau)=L_0(\tau)\mathcal{E}\left(-\int_0^{\cdot}\Big(\theta_s+\rho\zeta_s(\tau)\Big)dW_s^{1,\tau}\right)_t\cdot\mathcal{E}\left(\int_0^{\cdot}b_s(\tau)dW_s^{1,\tau,\perp}\right)_t \tag{23}$$

is a $(\mathbb{G}^F,\mathbb{P})$*-martingale and the probability measure* $\mathbb{Q}$ *defined by*

$$d\mathbb{Q}|_{\mathcal{G}_t^F} = L_t(\tau)d\mathbb{P}|_{\mathcal{G}_t^F} \quad \forall t \geq 0$$

belongs to $\mathcal{M}(\mathbb{G}^F)$*. Conversely, any* $\mathbb{Q} \in \mathcal{M}(\mathbb{G}^F)$ *is given by (23).*

Proof: Let $L(\tau)$ be any strictly positive $(\mathbb{G}^F,\mathbb{P})$-(local) martingale of the form $dL_t(\tau) = L_t(\tau)\Big(a_t(\tau)dW_t^{1,\tau} + b_t(\tau)dW_t^{1,\tau,\perp}\Big)$. The process $L(\tau)$ defines a measure from $\mathcal{M}(\mathbb{G}^F)$ if and only if $\mathbb{E}_{\mathbb{P}}(L_t(\tau)) = 1$ for any $t \geq 0$ and $L(\tau)S = (L_t(\tau)S_t)_{t\geq 0}$ is a $(\mathbb{G}^F,\mathbb{P})$-(local) martingale. The latter is satisfied if and only if $a(\tau) = -\theta - \rho\zeta(\tau)$. Moreover, since $(b, L_0(\tau)) \in \Gamma_{\mathbb{G}^F}$, we have that $\mathbb{E}_{\mathbb{P}}(L_t(\tau)) = 1$ for any $t \geq 0$. □

The processes θ and $\zeta(\tau)$ in (23) are known. However, there exist infinitely many processes $(b(\tau), L_0(\tau)) \in \Gamma_{\mathbb{G}^F}$ such that $L(\tau)$ defines a probability measure from the set $\mathcal{M}(\mathbb{G}^F)$. Hence, the set $\mathcal{M}(\mathbb{G}^F)$ has infinitely many elements.

2.3.3. *Pricing in* $\mathbb{G}^F$*.* We recall that we work under Hypothesis 3 and Hypothesis 4.

Proposition 2.20. *Let* $0 \leq t \leq T$*. The price,* $\mathbb{C}_{t,\mathbb{Q}}^{\mathbb{G}^F,\Psi^\tau}$*, of the default-sensitive contingent claim* $\Psi^\tau = Y_T^1\mathbb{I}_{\tau>T} + Y_T^2(\tau)\mathbb{I}_{\tau\leq T}$*, where* Y_T^1 *and* $Y_T^2(u)$ *are integrable,* $\mathcal{F}_T^1$*-measurable random variables is uniquely given by:*

$$\mathbb{C}_{t,\mathbb{Q}}^{\mathbb{G}^F,\Psi^\tau} = \mathbb{I}_{T<\tau}\mathbb{C}_{1,t}^{\mathbb{G}^F} + \mathbb{I}_{T\geq\tau}\mathbb{C}_{2,t}^{\mathbb{G}^F}, \tag{24}$$

where

$$\begin{aligned}
\mathbb{C}_{1,t}^{\mathbb{G}^F} &= \frac{\mathbb{E}_{\mathbb{P}}(Y_T^1 L_T(u)\Pi_T(u)|\mathcal{F}_t^1 \vee \mathcal{F}_t^2)|_{u=\tau}}{L_t(\tau)\Pi_t(\tau)},\\
\mathbb{C}_{2,t}^{\mathbb{G}^F} &= \frac{\mathbb{E}_{\mathbb{P}}(Y_T^2(u) L_T(u)|\mathcal{F}_t^1 \vee \mathcal{F}_t^2)|_{u=\tau}}{L_t(\tau)\Pi_t(\tau)},\\
L_t(\tau) &= \mathcal{E}\left(-\int_0^{\cdot}\Big(\theta_s + \rho\zeta_s(\tau)\Big)dW_s^{1,\tau}\right)_t \quad \forall t \geq 0
\end{aligned}$$

and $\Pi_t(u)du = \mathbb{P}(\tau \in du|\mathcal{F}_t^2)$*.*

Proof: We have that

$$\mathbb{C}_{t,\mathbb{Q}}^{\mathbb{G}^F,\Psi^\tau} = \frac{\mathbb{E}_{\mathbb{P}}((Y_T^1\mathbb{I}_{\tau>T} + Y_T^2(\tau)\mathbb{I}_{\tau\leq T})L_T(\tau)|\mathcal{G}_t^F)}{L_t(\tau)}.$$

Let us denote $\Pi_t(u)du = \mathbb{P}(\tau \in du|\mathcal{F}_t^2)$. On $\tau \leq T$, $\Pi_t(u)du = \delta_\tau(du)$. Hence, on $\tau > T$

$$\mathbb{C}_{1,t}^{\mathbb{G}^F} = \frac{\mathbb{E}_{\mathbb{P}}(Y_T^1 L_T(\tau)|\mathcal{G}_t^F)}{L_t(\tau)} = \frac{\mathbb{E}_{\mathbb{P}}(Y_T^1 L_T(u)\Pi_T(u)|\mathcal{F}_t^1 \vee \mathcal{F}_t^2)|_{u=\tau}}{L_t(\tau)\Pi_t(\tau)},$$

and on $\tau \leq T$

$$\mathbb{C}_{2,t}^{\mathbb{G}^F} = \frac{\mathbb{E}_{\mathbb{P}}(Y_T^2(\tau)L_T(\tau)|\mathcal{G}_t^F)}{L_t(\tau)} = \frac{\mathbb{E}_{\mathbb{P}}(Y_T^2(u)L_T(u)|\mathcal{F}_t^1 \vee \mathcal{F}_t^2)|_{u=\tau}}{L_t(\tau)\Pi_t(\tau)}.$$

□

3. f-DIVERGENCE METHODOLOGY

We saw in Sections 2.1, 2.2 and 2.3 that the sets $\mathcal{M}(\mathbb{G}^\tau)$, $\mathcal{M}(\mathbb{G})$ and $\mathcal{M}(\mathbb{G}^F)$ have infinitely many elements. However, in the case of the filtrations $\mathbb{G}^\tau$ and $\mathbb{G}^F$, the price of the default-sensitive contingent claim Ψ^τ is unique (and $\sigma(\tau)$-measurable). To give a price of Ψ^τ in the case of the filtration $\mathbb{G}$, one needs to choose one of the measures from $\mathcal{M}(\mathbb{G})$ in an appropriate manner. There are various approaches introduced in the literature, for example: the minimal entropy martingale measure or the f-divergence minimal martingale measure (see [2]). In [13], the problem of pricing defaultable contingent claims was studied beyond the standard risk-neutral models. Namely, the authors used as numeraire the growth optimal portfolio, i.e., the portfolio that maximizes expected logarithmic utility from terminal wealth. Then, default-sensitive claims are priced under the real-world probability measure. This methodology corresponds to the benchmark approach introduced by Platen (see [12]).

In our paper, we consider the pricing via the f-divergence approach.

Definition 3.1. Let f be a convex function on $[0, \infty)$ and let $\mathbb{Q}$ and $\mathbb{P}$ be probability measures such that $\mathbb{Q} << \mathbb{P}$ and let $\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{K}_T}$ be the Radon-Nikodym density of $\mathbb{Q}$ with respect to $\mathbb{P}$ on $\mathcal{K}_T$. We call

$$\mathbb{E}_{\mathbb{P}}(f(\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{K}_T}))$$

the f-divergence of $\mathbb{Q}$ with respect to $\mathbb{P}$ on $\mathcal{K}_T$.

As in [4], we use $\mathbb{Q}^* \in \mathcal{M}(\mathbb{K})$ such that

$$\mathbb{E}_{\mathbb{P}}(f(\frac{d\mathbb{Q}^*}{d\mathbb{P}}|_{\mathcal{K}_T})) = \inf_{\mathbb{Q}\in\mathcal{M}(\mathbb{K})} \mathbb{E}_{\mathbb{P}}(f(\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{K}_T})). \tag{25}$$

It was shown in [2] that the f-divergence minimization is equivalent to the utility maximization where f is the Legendre transform of a utility function $\hat{f}$. In our study we take under consideration the utility function $\hat{f}(x) = \ln(x)$. Hence, in the f-divergence minimization problem (25) we investigate $f(x) = -\ln(x) - 1$, or equivalently $f(x) = -\ln(x)$. We emphasise the analogy to the benchmark approach, since the growth optimal portfolio is defined as the portfolio that maximizes expected logarithmic utility from terminal wealth.

3.1. Pricing in $\mathbb{G}$.

The set $\mathcal{M}(\mathbb{G})$ has infinitely many elements. To choose a pricing measure $\mathbb{Q}^* \in \mathcal{M}(\mathbb{G})$ we deal with following problem:

$$\mathbb{E}_{\mathbb{P}}(-\ln(K_T^*)) = \inf_{\upsilon\in\Gamma_{\mathbb{G}}} \mathbb{E}_{\mathbb{P}}(-\ln(K_T)) = \sup_{\upsilon\in\Gamma_{\mathbb{G}}} \mathbb{E}_{\mathbb{P}}(\ln(K_T)), \tag{26}$$

where $K_T^* = \mathcal{E}\left(-\int_0^{\cdot}(\theta_s+\gamma_s)dB_s\right)_T \exp\{-\int_0^{T\wedge\tau} \upsilon_s^* \frac{p_s(s)g(s)}{G_s}ds + \ln(1+\upsilon_\tau^*)\mathbb{I}_{T>\tau}\}$ and γ is defined in (17).

Proposition 3.2. *The optimization problem (26) is solved by*

$$K_T^* = \mathcal{E}\left(-\int_0^{\cdot}(\theta_s+\gamma_s)dB_s\right)_T, \tag{27}$$

i.e., $\upsilon^* = 0$, where B is the $(\mathbb{G},\mathbb{P})$-Brownian motion. More precisely, we have

$$d\mathbb{Q}^*|_{\mathcal{G}_T} = \mathcal{E}\left(-\int_0^{\cdot}\big(\theta_s + \frac{\Xi_s}{G_s}\mathbb{I}_{s\leq\tau} + \beta_s(\tau)\mathbb{I}_{s>\tau}\big)dB_s\right)_T d\mathbb{P}|_{\mathcal{G}_T} \quad \forall t\geq 0,$$

where Ξ is defined in (9) and $\theta_t = \frac{\mu(S_t)}{\sigma(S_t)}$ and β is defined as in (6).

Proof: By (18), we have

$$\ln(K_T) = \ln\left(\mathcal{E}\left(-\int_0^{\cdot}(\theta_s+\gamma_s)dB_s\right)_T\right) + \ln(1+\upsilon_\tau)\mathbb{I}_{T>\tau} - \int_0^{T\wedge\tau}\upsilon_s\lambda_s ds.$$

From the fact that the process ϑ is uniquely defined and fixed, the optimization problem (26) reduces to

$$\mathbb{E}_{\mathbb{P}}\left(\ln(1+\upsilon_\tau^*)\mathbb{I}_{T>\tau} - \int_0^{T\wedge\tau}\upsilon_s^*\lambda_s ds\right) = \sup_{\upsilon\in\Gamma_{\mathbb{G}}}\mathbb{E}_{\mathbb{P}}\left(\ln(1+\upsilon_\tau)\mathbb{I}_{T>\tau} - \int_0^{T\wedge\tau}\upsilon_s\lambda_s ds\right),$$

where $\Gamma_{\mathbb{G}}$ is defined as in (15) and $\lambda_t = \frac{p_t(t)g(t)}{G_t}$. Denote by $I = \ln(1+\upsilon_\tau)\mathbb{I}_{T>\tau} - \int_0^{T\wedge\tau}\upsilon_s\lambda_s ds$. By the tower property of the expectation, we have that $\mathbb{E}_{\mathbb{P}}(I) = \mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}(I|\mathcal{F}_T^1))$. Then, using (5), we get:

$$\mathbb{E}_{\mathbb{P}}(I) = \mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}(I|\mathcal{F}_T^1)) = \mathbb{E}_{\mathbb{P}}\left(\int_0^{\infty}\Big(\ln(1+\upsilon_u)\mathbb{I}_{T>u} - \int_0^{T}\mathbb{I}_{s\leq u}\upsilon_s\lambda_s ds\Big)p_T(u)g(u)du\right).$$

By Fubini's Theorem we get that the right-hand side of the equation above equals

$$\mathbb{E}_{\mathbb{P}}\left(\int_0^{T}\Big(\ln(1+\upsilon_u)p_T(u)g(u) - \lambda_u\upsilon_u\int_u^{\infty}p_T(x)g(x)dx\Big)du\right).$$

Applying Fubini's Theorem and tower property, we get:

$$\mathbb{E}_{\mathbb{P}}(I) = \int_0^{T}\left[\mathbb{E}_{\mathbb{P}}\left(\ln(1+\upsilon_u)g(u)\mathbb{E}_{\mathbb{P}}(p_T(u)|\mathcal{F}_u^1) - \lambda_u\upsilon_u\int_u^{\infty}\mathbb{E}_{\mathbb{P}}(p_T(x)|\mathcal{F}_u^1)g(x)dx\right)\right]du.$$

Then, by the fact that $p(u)$ is an $(\mathbb{F}^1, \mathbb{P})$-martingale, $\lambda_t = \frac{p_t(t)g(t)}{G_t}$, and $G_t = \int_t^\infty p_t(x)g(x)dx$ and using Fubini's Theorem we obtain:

$$\mathbb{E}_{\mathbb{P}}(I) = \mathbb{E}_{\mathbb{P}}\left(\int_0^T \Big(p_u(u)g(u)(\ln(1+\upsilon_u) - \upsilon_u)\Big)du\right).$$

The function $g(z) = \ln(1+z) - z$ has a local maximum at $z = 0$. We conclude that (27) holds. □

From [11] we have:

Lemma 3.3. *a) Let U be a $\mathcal{G}_T$-measurable, $\mathbb{P}$-integrable random variable. Then, for every $0 \le t < T$,*

$$\mathbb{E}_{\mathbb{P}}(U|\mathcal{G}_t)\mathbb{I}_{t<\tau} = \mathbb{I}_{t<\tau}\frac{\mathbb{E}_{\mathbb{P}}(U\mathbb{I}_{t<\tau}|\mathcal{F}_t^1)}{G_t}. \tag{28}$$

b) Let $Y_T(u)$ be an $\mathcal{F}_T^1$-measurable random variable, where $u \ge 0$. Then, for every $0 \le t < T$,

$$\mathbb{E}_{\mathbb{P}}(Y_T(\tau)|\mathcal{G}_t)\mathbb{I}_{t<\tau} = \mathbb{I}_{t<\tau}\frac{1}{G_t}\mathbb{E}_{\mathbb{P}}\left(\int_t^\infty Y_T(u)p_T(u)g(u)du|\mathcal{F}_t^1\right)$$

and

$$\mathbb{E}_{\mathbb{P}}(Y_T(\tau)|\mathcal{G}_t)\mathbb{I}_{t\ge\tau} = \mathbb{I}_{t\ge\tau}\frac{\mathbb{E}_{\mathbb{P}}(Y_T(u)p_T(u)|\mathcal{F}_t^1)|_{u=\tau}}{p_t(\tau)}.$$

Proposition 3.4. *Let $0 \le t < T$ and $f(x) = -\ln(x)$ in the f-divergence minimization problem. Then, the price, $\mathbb{C}_{t,\mathbb{Q}^*}^{\mathbb{G},\Psi^\tau}$, of the default-sensitive contingent claim $\Psi^\tau = Y_T^1\mathbb{I}_{\tau>T} + Y_T^2(\tau)\mathbb{I}_{\tau\le T}$, where Y_T^1 and $Y_T^2(u)$ are integrable, $\mathcal{F}_T^1$-measurable random variables, is given by:*

$$\mathbb{C}_{t,\mathbb{Q}^*}^{\mathbb{G},\Psi^\tau} = \mathbb{I}_{t<\tau}\mathbb{C}_{1,t}^{\mathbb{G}} + \mathbb{I}_{t\ge\tau}\mathbb{C}_{2,t}^{\mathbb{G}},$$

where

$$\mathbb{C}_{1,t}^{\mathbb{G}} = \frac{\mathbb{E}_{\mathbb{P}}(\int_t^T Y_T^2(u)\hat{K}_T^*(u)p_T(u)g(u)du|\mathcal{F}_t^1) + \mathbb{E}_{\mathbb{P}}\left(Y_T^1\tilde{K}_T^*G_T|\mathcal{F}_t^1\right)}{G_t\tilde{K}_t^*},$$

$$\mathbb{C}_{2,t}^{\mathbb{G}} = \frac{\mathbb{E}_{\mathbb{P}}(Y_T^2(u)\hat{K}_T^*(u)p_T(u)|\mathcal{F}_t^1)|_{u=\tau}}{p_t(\tau)\hat{K}_t^*(\tau)},$$

the processes G and $p(\tau)$ are defined by (7) and (6), respectively, and $K_t^* = \mathcal{E}\left(\int_0^\cdot \vartheta_s dB_s\right)_t = \tilde{K}_t^*\mathbb{I}_{t\le\tau} + \hat{K}_t^*(\tau)\mathbb{I}_{t>\tau}$.

Remark 3.5. The case with the power utility function was discussed in [6]. The BSDEs methodology proves that, in general, $\upsilon^* \neq 0$. Furthermore, to the best of our knowledge, no closed-form formula can be obtained. Namely, for $f(x) = x^q$, where $q < 0$, it is proved in [6] that, on the set $t \leq \tau$

$$v_t^* = \left(\frac{v_t^0}{v_t^1(t)}\right)^{\frac{1}{q-1}} - 1.$$

where $v^1(u)$ and v^0 are parts of solutions of the BSDEs

$$\begin{aligned}
dv_t^1(u) &= \left[-\frac{1}{2}q(q-1)|\hat{\vartheta}_t(u)|^2 v_t^1(u) + q\hat{\vartheta}_t(u)\hat{K}_t^1(u)\right]dt + \hat{K}_t^1(u)dB_t,\\
v_T^1(u) &= 1\\
dv_t^0 &= \Big[\big(-\frac{1}{2}q(q-1)|\tilde{\vartheta}_t|^2 + (1-q)\lambda_t\big)v_t^0 + q\tilde{\vartheta}_t\hat{K}_t^0\\
&\qquad -(1-q)\lambda_t\left(v_t^1(t)\right)^{1-p}(v_t^0)^p\big)\Big]dt + \hat{K}_t^0 dB_t,\\
v_T^0 &= 1,
\end{aligned}$$

where $\tilde{\vartheta}_t = -\theta_t - \frac{\Xi_t}{G_t}$ and $\hat{\vartheta}_t(\tau) = -\theta_t - \beta_t(\tau)$. In particular $v_t^* = 0$ if and only if $v_t^0 = v_t^1(t)$ which leads to

$$\begin{aligned}
dv_t^1(u) &= \left[\left(-\frac{1}{2}q(q-1)|\hat{\vartheta}_t(u)|^2\right)v_t^1(u) + q\hat{\vartheta}_t(u)\hat{K}_t^1(u)\right]dt + \hat{K}_t^1(u)dB_t,\\
v_T^1(u) &= 1\\
dv_t^0 &= \Big[-\frac{1}{2}q(q-1)|\tilde{\vartheta}_t|^2 v_t^0 + q\tilde{\vartheta}_t\hat{K}_t^0\Big]dt + \hat{K}_t^0 dB_t,\\
v_T^0 &= 1.
\end{aligned}$$

The two equations can be solved in a closed form as

$$v_t^1(u) = E\left(\Gamma_t^T(u)|\mathcal{F}_t\right),$$

where

$$\Gamma_t^T(u) = \exp\left(\int_t^T \frac{1}{2}q(q-1)|\hat{\vartheta}_v(u)|^2 dv\right)\mathcal{E}\left(-\int_t^{\cdot} q\hat{\vartheta}_v(u)dB_v\right)_T.$$

so that

$$v_t^1(t) = E\left(\Gamma_t^T(t)|\mathcal{F}_t\right)$$

and

$$v_t^0 = E\left(\Upsilon_t^T|\mathcal{F}_t\right),$$

where

$$\Upsilon_t^T = \exp\left(\int_t^T \frac{1}{2}q(q-1)|\tilde{\vartheta}_v|^2)dv\right)\mathcal{E}\left(-\int_t^{\cdot} q\tilde{\vartheta}_v dB_v\right)_T.$$

These solutions are equal if and only if $\hat{\vartheta}_v(t) = \tilde{\vartheta}_v$ for any v and any t. Equivalently, $\beta_t(\tau) = \frac{\Xi_t}{G_t}$ for $t \leq \tau$.

4. Example

A particular case of the model considered in Section is the following: take S and V which satisfy the following stochastic differential equations:

$$\begin{aligned} dS_t &= S_t\sigma_1 dW_t^1, \quad S_0 = s_0 \\ dV_t &= V_t\sigma_2 dW_t^2, \quad V_0 = v_0, \end{aligned}$$

where $d\langle W^1, W^2\rangle_t = \rho dt$ such that $s_0, v_0, \sigma_1 > 0, \sigma_2 > 0$, and ρ are constants (hence, the SDEs have strong solutions) and define τ as the first hitting time of a constant barrier:

$$\tau = \inf\{t \geq 0 : V_t \leq a\},$$

where $a \in (0, v_0)$.
Let us consider the investor with full information, defined in Section 2.3. We use Yor's method described in Section 2.3.1. Let us denote

$$P_t(u) = \mathbb{P}(\tau > u|\mathcal{F}_t^2),$$

which, by Proposition 2.15, can be written as

$$P_t(u) = \begin{cases} \mathbb{I}_{\tau>t}P_t(u), & u > t, \\ \mathbb{I}_{\tau>u}, & u \leq t. \end{cases} \quad \mathbb{P} - a.s.$$

We consider $\mathbb{I}_{u>t}P_t(u)$, which equals $\mathbb{I}_{\tau>t}\mathbb{I}_{u>t}\mathbb{P}(\tau_t > u|\mathcal{F}_t^2)$, where

$$\tau_t = \inf\{s \geq t : V_s \leq a\}.$$

We have that

$$\mathbb{I}_{u>t}\mathbb{P}(\tau_t > u|\mathcal{F}_t^2) = \mathbb{I}_{u>t}\mathbb{P}(\inf_{t\leq s\leq u} V_s > a|\mathcal{F}_t^2) = \mathbb{I}_{u>t}\mathbb{P}(\inf_{0\leq s\leq u-t} U_s > \frac{a}{z})|_{z=V_t},$$

where $U_t = \exp\{-\frac{1}{2}\sigma_2^2 t + \sigma_2 W_t^2\}$. The last equality is by the fact that, for $s > t$: $V_s = V_t \exp\{-\frac{1}{2}\sigma_2^2(s-t) + \sigma_2(W_s^2 - W_t^2)\} = V_t U_{s-t}$, V_t is $\mathcal{F}_t^2$-measurable and independent from U_{s-t} and the $(\mathbb{A}, \mathbb{P})$-Brownian motion W^2 has stationary and independent increments. Then, we get (by [8]) that

$$\mathbb{I}_{u>t}P_t(u) = \mathbb{I}_{u>t}\mathbb{I}_{\tau>t}\Big(\Phi(X_t(u)) - \frac{V_t}{a}\Phi(\hat{X}_t(u))\Big) = \mathbb{I}_{u>t}\mathbb{I}_{\tau>t}\int_u^\infty q_t(s)ds,$$

where

$$\begin{aligned} X_t(u) &= \frac{\ln(\frac{V_t}{a}) - \frac{1}{2}\sigma_2^2(u-t)}{\sigma_2\sqrt{u-t}}, \\ \hat{X}_t(u) &= -X_t(u) - \sigma_2\sqrt{u-t}, \\ \Phi(x) &= \int_{-\infty}^x \frac{1}{\sqrt{2\pi}}\exp\{-\frac{y^2}{2}\}dy \end{aligned}$$

and, from technical computations, for $u > t$

$$q_t(u) = \frac{\ln(\frac{V_t}{a})}{\sigma_2(u-t)^{\frac{3}{2}}\sqrt{2\pi}} \exp\{-\frac{(\ln(\frac{a}{V_t}) + \frac{1}{2}\sigma_2^2(u-t))^2}{2\sigma_2^2(u-t)}\}. \tag{29}$$

Finally, we get

$$P_t(u) = \begin{cases} \mathbb{I}_{\tau>t} \ln(\frac{V_t}{a}) \frac{1}{\sigma_2\sqrt{2\pi}} \int_u^\infty \frac{1}{(s-t)^{3/2}} \exp\{-\frac{X_t^2(s)}{2}\} ds, & u > t, \\ \mathbb{I}_{\tau>u}, & u \le t. \end{cases} \quad \mathbb{P}-a.s.$$

On the one hand

$$P_t(u) = \int_u^\infty P_t(ds),$$

where

$$P_t(ds) = \begin{cases} \mathbb{I}_{\tau>t} \frac{\ln(\frac{V_t}{a})}{\sigma_2\sqrt{2\pi(s-t)^3}} \exp\{-\frac{X_t^2(s)}{2}\} ds, & s > t, \\ \delta_\tau(ds), & s \le t, \end{cases}$$

and δ denotes the Dirac measure.
On the other hand, from the fact that $P(u)$ is an $(\mathbb{F}^2, \mathbb{P})$-martingale and applying Itô's formula, we get, for $t < \tau$

$$P_t(u) = P_0(u) + \int_0^t \Big(\frac{2}{\sqrt{2\pi(u-s)}} \exp\{-\frac{X_s^2(u)}{2}\} - \frac{V_s}{a}\sigma_2 \Phi(-X_s(u) - \sigma_2\sqrt{u-s}) \Big) dW_s^2.$$

Hypothesis 3 holds, for $t < \tau$

$$Q_t(du) = \mathbb{I}_{t<u} \frac{1}{\sqrt{2\pi(u-t)^3}} \exp\{-\frac{X_t^2(u)}{2}\} \Big(\frac{(\ln(\frac{V_t}{a}))^2}{\sigma_2^2(u-t)} - \frac{\ln(\frac{V_t}{a})}{2} - 1 \Big) du$$

and

$$\zeta_t(\tau) = \mathbb{I}_{\tau>t} \frac{Q_t(du)}{P_t(du)}|_{u=\tau} = \mathbb{I}_{\tau>t} \Big(\frac{1}{W_\tau^2 - W_t^2 - \frac{1}{2}\sigma_2(\tau-t)} - \frac{W_\tau^2 - W_t^2}{\tau - t} \Big). \tag{30}$$

To show that

$$\int_0^t |\zeta_s(\tau)| ds < \infty \quad \mathbb{P}-\text{a.s.} \quad \forall t \ge 0,$$

we denote

$$\begin{aligned} x_t(\tau) &= \mathbb{I}_{\tau>t} \Big| \frac{1}{W_\tau^2 - W_t^2 - \frac{1}{2}\sigma_2(\tau-t)} \Big|, \\ y_t(\tau) &= \mathbb{I}_{\tau>t} \Big| \frac{W_\tau^2 - W_t^2}{\tau - t} \Big| \end{aligned}$$

and consider them separately.
To prove that

$$\int_0^t x_s(\tau)ds < \infty \quad \mathbb{P}-\text{a.s.} \quad \forall t \geq 0, \tag{31}$$

we use the fact that there exists a probability measure $\hat{\mathbb{P}}$, equivalent to $\mathbb{P}$, under which the process $U = (U_t)_{t\geq 0}$, where $\tilde{W}_t = W_t^2 - \frac{1}{2}\sigma_2 t$, is a standard Brownian motion. Let us denote $\alpha = \frac{1}{\sigma_2}\ln\frac{a}{v_0}$ and define the hitting times

$$\tau_\alpha = \inf\{t \geq 0 : \tilde{W}_t = \alpha\}$$

and

$$\tau_\beta = \inf\{t \geq 0 : \tilde{W}_t = \beta\},$$

where $\beta > 0$. Hence, (31) holds if and only if

$$\int_0^{\tau_\alpha \wedge t} \frac{1}{\tilde{W}_s - \alpha} ds < \infty \quad \hat{\mathbb{P}}-\text{a.s.} \quad \forall t \geq 0 \tag{32}$$

By Exercise (2.8) on p. 244 [14],

$$\mathbb{E}_{\hat{\mathbb{P}}}\left(\int_0^{\tau_\alpha\wedge\tau_\beta} \frac{1}{\tilde{W}_s - \alpha} ds\right) = -\frac{2\alpha}{\beta-\alpha}\left(\beta + \int_0^\beta \frac{\beta - x}{x-\alpha}dx\right) = -2\alpha\ln\Big(\frac{\beta-\alpha}{-\alpha}\Big) < \infty,$$

for every $\beta > 0$. Hence,

$$\int_0^{\tau_\alpha\wedge\tau_\beta} \frac{1}{\tilde{W}_s - \alpha} ds < \infty \quad \hat{\mathbb{P}}-\text{a.s.} \quad \forall t \geq 0$$

and, from the fact that $\beta > 0$ is arbitrary, (32) holds for any $t \geq 0$. Moreover, from the fact that Brownian motion is Hölder continuous with exponent γ, where $\gamma \leq \frac{1}{2}$, we have that

$$\int_0^t y_s(\tau)ds < \infty \quad \mathbb{P}-\text{a.s.} \quad \forall t \geq 0.$$

Finally,

$$\int_0^t |\zeta_s(\tau)|ds \leq \int_0^t x_s(\tau)ds + \int_0^t y_s(\tau)ds < \infty \quad \mathbb{P}-\text{a.s.} \quad \forall t \geq 0.$$

By Lemma 2.16 and from the fact that $P(u) = (P_t(u))_{t\geq 0}$, is an $(\mathbb{F}^2, \mathbb{P})$-martingale, we get the enlargement decomposition formulae for the $(\mathbb{F}^1, \mathbb{P})$-Brownian motion W^1:

$$W_t^1 = W_t^{1,\tau} + \rho\int_0^{t\wedge\tau}\Big(\frac{1}{W_\tau^2 - W_s^2 - \frac{1}{2}\sigma_2(\tau - s)} - \frac{W_\tau^2 - W_s^2}{\tau - s}\Big)ds,$$

where $W^{1,\tau}$ denotes a $(\mathbb{G}^F,\mathbb{P})$-Brownian motion. Moreover, the process S is a $(\mathbb{G}^F,\mathbb{P})$-semi-martingale with the following decomposition:

$$dS_t = \sigma_1 S_t \left(dW_t^{1,\tau} + \mathbb{I}_{t<\tau}\rho \left(\frac{1}{W_\tau^2 - W_t^2 - \frac{1}{2}\sigma_2(\tau - t)} - \frac{W_\tau^2 - W_t^2}{\tau - t} \right) dt \right)$$

and it is a $(\mathbb{G}^F,\mathbb{Q})$-(local) martingale if and only if the processes $a(\tau)$ and $b(\tau)$, defined in Section 2.3.2, are such that

$$a_t(\tau) = -\rho\zeta_t(\tau).$$

Thus, the process $L(\tau)$, defined in Section 2.3.2, can be written as follows:

$$L_t(\tau) = L_0(\tau)\mathcal{E}\left(-\rho \int_0^{\cdot} \zeta_s(\tau) dW_s^{1,\tau} \right)_t \cdot \mathcal{E}\left(\int_0^{\cdot} b_s(\tau) dW_s^{1,\tau,\perp} \right)_t .$$

The price of the default-sensitive contingent claim $\Psi^\tau = Y_T^1 \mathbb{I}_{\tau>T} + Y_T^2(\tau)\mathbb{I}_{\tau\leq T}$, where Y_T^1 and $Y_T^2(u)$ are integrable, $\mathcal{F}_T^1$-measurable, is uniquely given by

$$\mathbb{C}_{t,\mathbb{Q}}^{\mathbb{G}^F,\Psi^\tau} = \mathbb{I}_{T<\tau}\mathbb{C}_{1,t}^{\mathbb{G}^F} + \mathbb{I}_{T\geq\tau}\mathbb{C}_{2,t}^{\mathbb{G}^F}, \tag{33}$$

where

$$\mathbb{C}_{1,t}^{\mathbb{G}^F} = \frac{\mathbb{E}_{\mathbb{P}}(Y_T^1 L_T(u)\Pi_T(u)|\mathcal{F}_t^1 \vee \mathcal{F}_t^2)|_{u=\tau}}{L_t(\tau)\Pi_t(\tau)},$$

$$\mathbb{C}_{2,t}^{\mathbb{G}^F} = \frac{\mathbb{E}_{\mathbb{P}}(Y_T^2(u) L_T(u)|\mathcal{F}_t^1 \vee \mathcal{F}_t^2)|_{u=\tau}}{L_t(\tau)\Pi_t(\tau)},$$

$$L_t(u) = \mathcal{E}\left(-\rho \int_0^{\cdot\wedge u} \Big(\frac{1}{W_u^2 - W_s^2 - \frac{1}{2}\sigma_2(u-s)} - \frac{W_u^2 - W_s^2}{u-s} \Big) dW_s^{1,\tau} \right)_t$$

and $\Pi(u) = (\Pi_t(u))_{t\geq 0}$ satisfies $\Pi_t(u)du = \mathbb{P}(\tau \in du|\mathcal{F}_t^2)$.

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M. Jeanblanc: Université d'Evry Val d'Essonne, monique.jeanblanc@univ-evry.fr
M. Leniec: Uppsala University, marta.leniec@math.uu.se