Optimal selling time under stochastic liquidity discount and regime shifting

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Abstract

We study the problem of selling an illiquid asset in a market in which unfavorable bid prices are readily available and favorable offers enter the market less frequently. We setup a liquidity model in which the liquidity premium implicit in the bid prices evolves stochastically and the rate of arrival of favorable offers is governed by a regime-shifting Markov process. The objective is to maximize the expected utility of the proceeds received from the sale of the illiquid asset. We formulate this problem as a multidimensional optimal stopping time problem with random maturity. We characterize the objective function as the unique viscosity solution of the associated system of variational Hamilton–Jacobi–Bellman inequalities. We derive explicit solutions in the case of power and logarithmic utility functions when the liquidity premium factor follows a mean-reverting CIR process.

Keywords: stochastic control, liquidity discount, regime shifting, limit and market orders, viscosity solutions, system of variational inequalities.

MSC2000 subject classification: 60G40, 91B70, 93E20.

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1 Introduction

Many classical models in mathematical finance assume infinite liquidity or perfect elasticity of traded assets. In particular, it is often assumed that there is a unique market price for a given asset at which traders may buy or sell any quantity instantaneously. Relaxing this assumption is crucial in the study of many financial problems, including optimal allocation and liquidation problems for large portfolios. The market liquidity crisis of 2007-2008 and its contagion effect have highlighted the pressing need for better liquidity management. Since then, there has been an increasing number of studies on liquidity risk and its management in the mathematical finance literature.

In this paper, we consider the problem of filling a sale order of an illiquid asset traded in a market in which unfavorable bid prices are readily available and favorable offers enter the market less frequently. On one hand, the investor may choose to accept the current highest bid price, which is offered however at a (liquidity) discount with respect to a fair price. On the other hand, the investor may also choose to wait until an investor agrees to buy the asset at the fair price. Our analysis applies particularly well to over-the-counter markets, block trades markets, private equity and real estate markets in which liquidity risk is an important factor.

The main goal in the study of liquidity risk is to find the best way to quantify the costs incurred by investors trading in markets in which supply is finite, trade counterparts are not always available, or trading causes price impacts. Furthermore the extent to which these above properties are present varies randomly through time. Liquidity models vary considerably among academic studies, and each model is adapted to the problem at hand. We may, for instance, point out the literature on informed trading ([3], [14]) and bid-ask spreads ([10], [13], [20]). More recently studies on large trader models ([4], [15], [19]), and dynamic supply curves ([7]), with an emphasis on liquidation problems ([1], [17]), have been given a considerable amount of attention.

In most studies mentioned so far, with the exception of the insider trading models of Back [3] and Kyle [14], it is always assumed that investors are price-takers, i.e. liquidity takers, in the sense that they trade at the available prices with a liquidity premium that must be paid for immediacy. It is clear from the structure of financial markets that in addition to the presence of price-takers there must necessarily exist market participants who are price-setters or liquidity providers. In less liquid markets such as real estate markets and many other over-the-counter markets, individual traders must meet and negotiate for a transaction to occur, so that the risk that a counterparty does not arrive in the short term is a non-negligible factor. It is important to highlight that in many markets, participants may be simultaneously price-takers, by trading with readily available counterparties, and price-setters, by submitting their own bids and offers. It is therefore desirable to consider an enlarged set of admissible trading strategies by including the possibility of trading at the current best bid (market price) or submitting a more favorable ask price (limit price). Several research papers investigate liquidation problems with limit orders, see for instance [2], [5], [6], and [11]. However, in our view, there are still few papers in the fields of mathematical finance which consider the problem of an investor acting as both price-setters

and price-takers. One such study is the market-making model of Guilbaud and Pham [12], where the authors consider the problem of a market maker whose objective is to maximize her expected utility from revenue over a short term horizon by submitting both limit and market orders in a limit order book market. In their model, the arrival intensity is assumed to be deterministic and the bid-ask spread is modeled by a time-changed Markov chain.

In our study, we model both the liquidity discount and the limit price as diffusion processes. Furthermore, we assume that the market order arrival time, i.e. the time at which the limit price is matched, by means of an intensity function depending on the current state of a Markov chain. This intensity process can be thought as a measure of liquidity given current market conditions. The objective is to maximize the expected utility of the wealth received from the sale of the asset. We formulate this two-dimensional stochastic control problem as an optimal stopping time problem with random maturity and regime shifting. Studying a mixed optimal stopping problem with random maturity and regime switching problem in a multidimensional setting is a major mathematical challenge. It is not an easy task to rigorously prove the PDE characterization of the value functions, and in particular the continuity property, in such a general framework, given the absence of the smooth fit property in general. The main challenge in this setting is to overcome the non-linearity of the lower and upper bounds of the value functions when building a strict supersolution to our HJB system in order to prove the comparison principle. We show that the supermeanvalued property of the utility function (see Assumption 2.2 below) is one of the main hypotheses that one needs to make to derive these results.

The rest of the paper is organized as follows. We define the model and formulate our optimal stopping problem in the following section. In Section 3, we characterize the solution of the problem in terms of the unique viscosity solution to the associated HJB system and obtain some qualitative description of these functions. In Sections 4 and 5, we derive explicit solutions in the case of power and logarithmic utility functions when the liquidity discount factor follows a mean-reverting CIR process.

2 The Illiquid Market Model

Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a probability space equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, satisfying the usual conditions. It is assumed that all random variables and stochastic processes are defined on the stochastic basis $(\Omega, \mathbb{F}, \mathbb{P})$. We denote by $\mathcal{T}$ the collection of all $\mathbb{F}-$stopping times. Let W and B be two correlated $\mathbb{F}$–Brownian motions, with correlation ρ, i.e. $d[W, B]_t = \rho dt$ for all t. We consider a financial market, in which there is an illiquid asset with a theoretical or fair value evolving according to a positive process S, which may be written as $S_t := \exp(X_t)$. The process X is assumed to follow the following SDE

$$\begin{aligned} dX_t &= \mu(X_t)dt + \sigma(X_t)dB_t \\ X_0 &= x, \end{aligned} \tag{2.1}$$

with μ and σ two Lipschitz functions on $\mathbb{R}$ and satisfying the following growth condition

$$\lim_{|x|\to\infty} \frac{|\mu(x)| + |\sigma(x)|}{|x|} = 0. \tag{2.2}$$

The investor holding the asset may not currently be able to sell it at its fair price as this asset is illiquid. To be more specific, the investor has two ways of disposing of this asset, he can either wait until an "impatient" investor decides to buy it at the fair price or sell it at any desired time with a liquidity premium for immediacy. In general, our model applies to any illiquid market structure. Examples of such assets include real estate, private equity, credit and exotic options and other over-the-counter markets for other illiquid products.

Liquidity discount factor. We model the liquidity discount factor as a given process $(f(Y_t))_{t\geq 0}$, where f is strictly decreasing C^2 function defined on $\mathbb{R}^+ \to [0,1]$, and satisfies the following conditions:

$$\begin{aligned} f(0) = 1 \text{ and } \lim_{y\to\infty} f(y) &= 0 \\ \exists\, c_0 > 0, \text{ such that } \lim_{y\to\infty} f(y)\exp(y^{c_0}) &= 0. \end{aligned} \tag{2.3}$$

The liquidity discount factor, given by $f(Y_t)$ at time t, is defined in terms of the mean-reverting non-negative process Y which is governed by the following SDE:

$$\begin{aligned} dY_t &= \alpha(Y_t)dt + \gamma(Y_t)dW_t, \\ Y_0 &= y, \end{aligned} \tag{2.4}$$

where α is a Lipschitz function on $\mathbb{R}^+$ and, for any $\varepsilon > 0$, γ is a Lipschitz function on $[\varepsilon, \infty)$.We assume that α and γ satisfy the following growth condition

$$\limsup_{|y|\to\infty} \frac{|\alpha(y)| + |\gamma(y)|}{|y|} < +\infty. \tag{2.5}$$

Furthermore, to insure the mean-reverting property, we assume that there exists $\beta > 0$ such that $(\beta - y)\alpha(y)$ is positive for all $y \geq 0$. Should the investor decide to sell the asset at the highest available bid price, i.e. with a liquidity discount, he would obtain a cash flow of $S_t f(Y_t)$.

The process Y is a measure of illiquidity. Indeed, when Y goes to infinity, the discount factor $f(Y)$ goes to zero, whereas the liquidation cost vanishes when $Y \to 0$, i.e. the highest bid price converges to the fair price of the asset. The main example is $f(y) = \exp(-y)$ with the process Y given as a CIR process:

$$\begin{aligned} dY_t &= \kappa\,(\beta - Y_t)\,dt + \gamma\sqrt{Y_t}dW_t, \\ Y_0 &= y, \end{aligned} \tag{2.6}$$

with κ, β and γ positive constants.

Market order arrival. We define the market order arrival time as the moment when a trader is willing to buy the asset at its fair price S_t from the investor. We model the market order arrival time, denoted by τ, by means of an intensity function λ_i depending on the current state i of a continuous-time, time-homogenous, irreducible Markov chain L, independent of W and B, with $m+1$ states. The states of the chain represent liquidity states of the financial market. The generator of the chain L under $\mathbb{P}$ is denoted by $A =$

$(\vartheta_{i,j})_{i,j=0,\dots m}$. Here $\vartheta_{i,j}$ is the constant intensity of transition of the chain L from state i to state j $(0 \leq i, j \leq m)$. Without loss of generality we assume

$$\lambda_0 > \lambda_1 > \ldots > \lambda_m > 0. \tag{2.7}$$

Utility function. We let U denote the utility function of the investor. We assume that U satisfies the following assumptions.

Assumption 2.1 *$U : \mathbb{R}^+ \to \mathbb{R}$ is non-decreasing, concave and twice continuously differentiable, and satisfies*

$$\lim_{x \to 0} x\, U'(x) \quad < \quad +\infty. \tag{2.8}$$

Assumption 2.2 *U is supermeanvalued w.r.t. S, i.e.*

$$U(S_t) \geq \mathbb{E}[U(S_\theta)|\mathcal{F}_t] \tag{2.9}$$

for any stopping time $\theta \in \mathcal{T}$.

The financial interpretation of the supermeanvalued property of U w.r.t. S is as follows: it is always better to accept an incoming market order then to wait for a later one. Indeed, if a buy order for the fair value S_t arrives at time t, then the utility of the obtained price S_t, is greater that the expected utility obtained at any fair price in the future. For more details on the supermeanvalued property, which is closely related to the concept of superharmonicity, we may refer to Dynkin [9] and Okendal [18].

Objective function. The objective of the investor is to maximize the expected utility of the wealth obtained from the sales of the illiquid asset. As such, we consider the following value function:

$$v(i,x,y) := \sup_{\theta \in \mathcal{T}} \mathbb{E}^{i,x,y}\left[h(X_\theta, Y_\theta)\, \mathbb{1}_{\theta \leq \tau} + U(e^{X_\tau})\, \mathbb{1}_{\theta > \tau}\right], \quad x \in \mathbb{R},\, y \in \mathbb{R}^+,\, i \in \{0, \dots, m\} \tag{2.10}$$

where $\mathbb{E}^{i,x,y}$ stands for the expectation with initial conditions $X_0 = x$, $Y_0 = y$ and $L_0 = i$, and $h(x,y) = U(e^x f(y))$. Recall that τ, the market order arrival time, is defined through the Markov chain L.

For the rest of the paper, we sometimes write $v_i(x,y)$ instead of $v(i,x,y)$ depending on the context.

3 Characterization of the value function

In this section, we characterize the solution of the problem, i.e. the value function v, in terms of the unique viscosity solution to the associated HJB system and obtain some descriptive properties of these functions including the monotonicity and continuity of the functions v_i.

3.1 First analytical properties of the value function

We denote by $\mathcal{L}$ the second order differential operator associated to the state processes (X, Y):

$$\mathcal{L}\phi(x,y) = \mu(x)\frac{\partial\phi}{\partial x} + \alpha(y)\frac{\partial\phi}{\partial y} + \frac{1}{2}\sigma^2(x)\frac{\partial^2\phi}{\partial x^2} + \rho\gamma(y)\sigma(x)\frac{\partial^2\phi}{\partial x\partial y} + \frac{1}{2}\gamma^2(y)\frac{\partial^2\phi}{\partial y^2}\,. \tag{3.11}$$

The main result of this section is that the value function v is the unique viscosity solution of the following variational inequality:

$$\min\Big[-\mathcal{L}v(i,x,y) - \mathcal{G}_i v(.,x,y) - \mathcal{J}_i v(i,x,y)\,,\; v(i,x,y) - h(x,y)\Big] = 0, \tag{3.12}$$

in which the operators $\mathcal{G}_i$ and $\mathcal{J}_i$ are defined as

$$\begin{aligned}\mathcal{G}_i\varphi(.,x,y) &= \sum_{j\neq i}\vartheta_{i,j}\left(\varphi(j,x,\,y) - \varphi(i,x,\,y)\right)\\ \mathcal{J}_i\varphi(i,x,y) &= \lambda_i\left(U(e^x) - \varphi(i,x,\,y)\right).\end{aligned}$$

We can also say that the family of value functions $v_i (i = 0,\ldots,m)$ is the unique solution of the above system of variational inequalities, meaning that each v_i satisfies the variational inequality (3.12) for $i = 0,\ldots,m$. In the same reasoning, we sometimes write $\mathcal{G}_i v_.(x,y)$ instead of $\mathcal{G}_i v(.,x,y)$ when referring to the v_i's as a family of functions. Before stating the main result, we derive a number of analytical properties of the value functions.

Proposition 3.1 *The value functions v_i are non-decreasing in x and non-increasing in y and verify the following inequalities*

$$Max\big(h(x,y), \mathbb{E}^x[U(e^{X_\tau})]\big) \leq v_i(x,y) \leq U(e^x)\ on\ \mathbb{R}\times\mathbb{R}^+\,. \tag{3.13}$$

Proof. From the definition of the value function, by considering $\theta = 0$, it is obvious that $v_i(x,y) \geq h(x,y)$.
For any $t > 0$, we also have

$$v_i(x,y) \geq \mathbb{E}^{i,x,y}\left[h(X_t,Y_t)\,\mathbb{1}_{t\leq\tau}\mathbb{1}_{Y_t>0} + U(e^{X_\tau})\,\mathbb{1}_{t>\tau}\right].$$

As τ is almost surely finite, letting t going to $+\infty$, we find that $v_i(x,y) \geq \mathbb{E}^x[U(e^{X_\tau})]$. Since U is non-decreasing and $0 \leq f(Y) \leq 1$, we also have the following inequalities:

$$\begin{aligned}v_i(x,y) &\leq \sup_{\theta\in\mathcal{T}}\mathbb{E}\left[U(e^{X_\theta})\mathbb{1}_{\theta\leq\tau} + U(e^{X_\tau})\mathbb{1}_{\theta>\tau}\right]\\ &\leq \sup_{\theta\in\mathcal{T}}\mathbb{E}\left[U(e^{X_{\theta\wedge\tau}})\right]\\ &\leq \sup_{\theta\in\mathcal{T}}\mathbb{E}\left[U(e^{X_\theta})\right].\end{aligned}$$

Using the supermeanvalued property of U w.r.t S, we obtain $v_i(x,y) \leq U(e^x)$.

From the uniqueness of the solution of the stochastic differential equation (2.1) combined with the non-decreasing property of U, we obtain that v_i is non-decreasing in x. We may apply the same argument to obtain that v_i is non-increasing in y, but one should be careful

when using the uniqueness of the trajectory of Y which only holds up to $\xi_y := \inf\{t > 0, Y_t^y = 0\}$. See Remark 3.1 below in this regard. Since $f(0) = \sup_{y\in\mathbb{R}^+} f(y) = 1$, the non-increasing property of v_i in y is verified. □

From Proposition 3.1, we obviously obtain that $v_i(x, 0) = U(e^x)$, which states that when the liquidity is infinite (i.e., $y = 0$) or when the best bid price matches the "fair price" of the asset, it is optimal to immediately sell the asset. Furthermore, if $\mathbb{E}^x[U(e^{X_\tau})] = U(e^x)$, we find that $v_i(x, y) = U(e^x)$ so that the optimal policy is to wait until τ, i.e. the arrival of a market buy order.

In the following remark, we recall some properties on the continuity of the stochastic flow of processes X and Y.

Remark 3.1 Noticing that the coefficients of the SDE governing X are Lipschitz continuous, we have the continuity of $X(t, x) := X_t^x$ in variables (t, x), for almost all ω. In particular, for any given $t > 0$, the mapping which associates x to the trajectory of X:

$$\begin{aligned} \mathbb{R} &\to C([0, t], \mathbb{R}) \\ x &\mapsto X(., \omega, x) \end{aligned}$$

is continuous. In here, $C([0, t], \mathbb{R})$ denotes the space of continuous real functions defined on $[0, t]$. On the other hand, since the coefficients of the SDE of Y are only locally Lipschitz, the mapping which associates y to the trajectory of Y:

$$\begin{aligned} \mathbb{R} &\to C([0, t], \mathbb{R}) \\ y &\mapsto Y(., \omega, y) \end{aligned}$$

is continuous only on the open set $A_y := \{y : \xi_y > t\}$, where as above $\xi_y = \inf\{t > 0, Y_t^y = 0\}$.

Before turning to the continuity of the value functions, we show the existence of an optimal stopping time.

Lemma 3.1 *There exists a stopping time $\theta^*_{i,x,y}$ such that*

$$v(i, x, y) = \mathbb{E}^{i,x,y}\left[h(X_{\theta^*_{i,x,y}}, Y_{\theta^*_{i,x,y}})\,\mathbb{1}_{\theta^*_{i,x,y}\le\tau\wedge\xi_y} + U(e^{X_\tau})\,\mathbb{1}_{\theta^*_{i,x,y}>\tau\wedge\xi_y}\right]. \tag{3.14}$$

Moreover, on $\{\xi_y \le \tau\}$, we have $\theta^*_{i,x,y} \le \xi_y$.

Proof. We have

$$v(i, x, y) = \sup_{\theta\in\mathcal{T}} \mathbb{E}^{i,x,y}\left[h(X_\theta, Y_\theta)\,\mathbb{1}_{\theta\le\tau} + U(e^{X_\tau})\,\mathbb{1}_{\theta>\tau}\right].$$

We consider the process Z defined as

$$Z_t = h(X_t, Y_t)\,\mathbb{1}_{t\le\tau} + U(e^{X_\tau})\,\mathbb{1}_{t>\tau}.$$

The process $(v(L_t, X_t, S_t))_{t\ge0}$ is the Snell envelop of Z. As such,

$$v(i, x, y) = \mathbb{E}^{i,x,y}\left[Z_{\theta^*_{i,x,y}}\right]$$

where

$$\theta^*_{i,x,y} = \inf\{t \geq 0; v(L^i_t, X^x_t, Y^y_t) \leq Z_t\}.$$

From the definition of the stopping time ξ_y and since $v(i,x,0) = U(e^x)$, we have

$$\begin{aligned} v(L^i_{\xi_y}, X^x_{\xi_y}, Y^y_{\xi_y}) &= v(L^i_{\xi_y}, X^x_{\xi_y}, 0) \\ &= U(e^{X^x_{\xi_y}})\mathbb{1}_{\xi_y \leq \tau} + U(e^{X^x_{\xi_y}})\mathbb{1}_{\tau < \xi_y} \\ &= Z_{\xi_y} + \left(U(e^{X^x_{\xi_y}}) - U(e^{X^x_\tau})\right)\mathbb{1}_{\tau<\xi_y}. \end{aligned}$$

Therefore, on the set $\{\xi_y \leq \tau\}$, we have $v(L^i_{\xi_y}, X^x_{\xi_y}, Y^y_{\xi_y}) = Z_{\xi_y}$ so that $\theta^*_{i,x,y} \leq \xi_y$. Moreover, we find

$$\{\theta^*_{i,x,y} \leq \tau\} = \{\theta^*_{i,x,y} \leq \tau \wedge \xi_y\} \qquad \text{and} \qquad \{\theta^*_{i,x,y} > \tau\} = \{\theta^*_{i,x,y} > \tau \wedge \xi_y\}.$$

It allows us to conclude the proof, by observing that

$$\begin{aligned} v(i,x,y) &= \mathbb{E}^{i,x,y}\left[Z_{\theta^*_{i,x,y}}\right] \\ &= \mathbb{E}^{i,x,y}\left[h(X_{\theta^*_{i,x,y}}, Y_{\theta^*_{i,x,y}})\,\mathbb{1}_{\theta^*_{i,x,y} \leq \tau\wedge\xi_y} + U(e^{X_\tau})\,\mathbb{1}_{\theta^*_{i,x,y}>\tau\wedge\xi_y}\right]. \end{aligned}$$

□

We now prove the continuity of the value functions.

Proposition 3.2 *The value functions v_i are continuous on $\mathbb{R}\times\mathbb{R}^+$ and satisfy:*

$$\lim_{(u,y)\to(x,0^+)} v_i(u,y) = v_i(x,0) = U(e^x).$$

Proof. Since both $h(x,y)$ and $U(e^x)$ are continuous, using relation (3.13), we obtain

$$\lim_{(u,y)\to(x,0^+)} v_i(u,y) = v_i(x,0) = U(e^x),$$

leading to the continuity of v_i on $\mathbb{R}\times\{0\}$.

We now examine the continuity of v_i at a given $(x,y) \in \mathbb{R}\times(0,\infty)$ and $i \in \{0,...,m\}$. We consider a sequence $(x_n,y_n)_{n\geq 0}$ which converges to (x,y). Without loss of generality, we may consider $(x_n,y_n) \in (x-1,x+1)\times((y-1)_+, y+1)$. We need to show that $\lim_{n\to\infty} v_i(x_n,y_n) = v_i(x,y)$, which we will show in two steps.

Step 1. We first show that for a given $\varepsilon > 0$, there exists an $N > 0$, such that $\forall n \geq N$, we have

$$v_i(x_n,y_n) - v_i(x,y) \leq \varepsilon.$$

We separate the sequence (x_n,y_n) into two subsequences,

- $(\tilde{x}_n,\tilde{y}_n)$, the subsequence containing only $y_n \geq y$ and

- $(\bar{x}_n, \bar{y}_n)$ the subsequence containing only $y_n < y$.

(a) Sequence $(\tilde{x}_n, \tilde{y}_n)$. Since v_i is non-increasing in y, we have

$$v_i(\tilde{x}_n, \tilde{y}_n) - v_i(x, y) \leq v_i(\tilde{x}_n, y) - v_i(x, y).$$

For n, such that $\tilde{x}_n \leq x$, we have

$$v_i(\tilde{x}_n, y) - v_i(x, y) \leq 0. \tag{3.15}$$

For all n such that $\tilde{x}_n > x$, from Lemma 3.1, there exists θ_n, such that

$$v_i(\tilde{x}_n, y) = \mathbb{E}^{i,\tilde{x}_n,y}\left[h(X_{\theta_n}, Y_{\theta_n})\mathbb{1}_{\theta_n \leq \xi_y \wedge \tau} + U(e^{X_\tau})\mathbb{1}_{\tau \wedge \xi_y < \theta_n}\right].$$

As such, we have

$$\begin{aligned} 0 \leq v_i(\tilde{x}_n, y) - v_i(x, y) &\leq \mathbb{E}^i\Big[\left(h(X^{\tilde{x}_n}_{\theta_n}, Y^y_{\theta_n}) - h(X^x_{\theta_n}, Y^y_{\theta_n})\right)\mathbb{1}_{\theta_n \leq \tau \wedge \xi_y} \\ &\quad + \left(U(e^{X^{\tilde{x}_n}_\tau}) - U(e^{X^x_\tau})\right)\mathbb{1}_{\tau \wedge \xi_y < \theta_n}\Big]. \end{aligned}$$

We let

$$\begin{aligned} A_n &:= h(X^{\tilde{x}_n}_{\theta_n}, Y^y_{\theta_n}) - h(X^x_{\theta_n}, Y^y_{\theta_n}) \geq 0, \\ B_n &:= \left(U(e^{X^{\tilde{x}_n}_\tau}) - U(e^{X^x_\tau})\right) \geq 0. \end{aligned}$$

We first notice h is continuous in both variables and, in particular, continuous in the first variable, uniformly on any compact set of the second variable. Using Remark 3.1 and noticing that the function f is valued in the compact $[0, 1]$, we obtain

$$\lim_{n\to\infty} A_n = 0, \ a.s.$$

Furthermore, by well-known comparison theorems for SDEs, for n big enough, we have

$$\begin{aligned} |A_n| &= h(X^{\tilde{x}_n}_{\theta_n}, Y^y_{\theta_n}) - h(X^x_{\theta_n}, Y^y_{\theta_n}) \\ &\leq h(X^{x+1}_{\theta_n}, Y^y_{\theta_n}) - h(X^x_{\theta_n}, Y^y_{\theta_n}) \\ &\leq \sup_{t \leq \tau}\left[h(X^{x+1}_t, Y^y_t) - h(X^x_t, Y^y_t)\right]. \end{aligned}$$

Using the properties of the utility function U, which is non-decreasing and concave, there exists $s_0 > 0$ and $M > 0$ such that $\forall\, 0 < s < s_0$, we have $sU'(s) < M$. As such, we have

$$\begin{aligned} h(X^{x+1}_t, Y^y_t) - h(X^x_t, Y^y_t) &< f(Y^y_t)\left(e^{X^{x+1}_t} - e^{X^x_t}\right)U'(f(Y^y_t)e^{X^x_t})\mathbb{1}_{f(Y^y_t)e^{X^x_t} < s_0} \\ &+ f(Y^y_t)\left(e^{X^{x+1}_t} - e^{X^x_t}\right)U'(s_0)\mathbb{1}_{f(Y^y_t)e^{X^x_t} \geq s_0} \\ &\leq M\left(e^{X^{x+1}_t - X^x_t} - 1\right) + \left(e^{X^{x+1}_t} - e^{X^x_t}\right)U'(s_0). \end{aligned}$$

Then, we obtain

$$|A_n| \leq \sup_{t \leq \tau}\left[M\left(e^{X^{x+1}_t - X^x_t} - 1\right) + \left(e^{X^{x+1}_t} - e^{X^x_t}\right)U'(s_0)\right],$$

which is integrable.
It is equally clear that $\lim_{n\to\infty} B_n = 0$, $a.s.$ and $|B_n| \leq \left|U(e^{X_\tau^{x+1}}) - U(e^{X_\tau^x})\right|$, which is integrable.

Applying the dominated convergence theorem, we obtain that for all $\varepsilon > 0$, there exists $N > 0$, such that $\forall n \geq N$,

$$v_i(\tilde{x}_n, \tilde{y}_n) - v_i(x, y) \leq \varepsilon. \tag{3.16}$$

(b) Sequence $(\bar{x}_n, \bar{y}_n)$, i.e. the sequence for which $\bar{y}_n \leq y$. We set $\xi_n := \inf\{t \geq 0; Y_t^{\bar{y}_n} = 0\}$. We have

$$v_i(\bar{x}_n, \bar{y}_n) - v_i(x, y) \leq [v_i(\bar{x}_n, \bar{y}_n) - v_i(x, \bar{y}_n)] + [v_i(x, \bar{y}_n) - v_i(x, y)].$$

We first consider the term $v_i(\bar{x}_n, \bar{y}_n) - v_i(x, \bar{y}_n)$. For n such that $\bar{x}_n \leq x$, we have

$$v_i(\bar{x}_n, \bar{y}_n) - v_i(x, \bar{y}_n) \leq 0.$$

For all n such that $\bar{x}_n > x$, from Lemma 3.1, there exists θ_n such that

$$\begin{aligned} v_i(\bar{x}_n, \bar{y}_n) - v_i(x, \bar{y}_n) \quad \leq \quad & \mathbb{E}^i\Big[\left(h(X_{\theta_n}^{\bar{x}_n}, Y_{\theta_n}^{\bar{y}_n}) - h(X_{\theta_n}^x, Y_{\theta_n}^{\bar{y}_n})\right) \mathbb{1}_{\theta_n \leq \tau \wedge \xi_n} \\ & + \left(U(e^{X_\tau^{\bar{x}_n}}) - U(e^{X_\tau^x})\right) \mathbb{1}_{\tau < \theta_n \wedge \xi_n}\Big]. \end{aligned}$$

We let

$$\begin{aligned} C_n \quad &:= \quad h(X_{\theta_n}^{\bar{x}_n}, Y_{\theta_n}^{\bar{y}_n}) - h(X_{\theta_n}^x, Y_{\theta_n}^{\bar{y}_n}) > 0, \\ D_n \quad &:= \quad U(e^{X_\tau^{\bar{x}_n}}) - U(e^{X_\tau^x}) > 0. \end{aligned}$$

Using the same argument as in (a), we have $\lim_{n\to\infty} C_n(\omega) = 0$, $\lim_{n\to\infty} D_n(\omega) = 0$, and C_n and D_n are dominated by integrable random variables. Applying the dominated convergence theorem, we obtain for a given ε, there exists an $N > 0$, such that $\forall n \geq N$, we have

$$v_i(\bar{x}_n, \bar{y}_n) - v_i(x, \bar{y}_n) \leq \frac{\varepsilon}{2}. \tag{3.17}$$

We now consider the term $v_i(x, \bar{y}_n) - v_i(x, y)$. From Lemma 3.1, for all $n \geq 1$ there exists θ_n such that

$$v_i(x, \bar{y}_n) - v_i(x, y) \leq \mathbb{E}^i\left[\left(h(X_{\theta_n}^x, Y_{\theta_n}^{\bar{y}_n}) - h(X_{\theta_n}^x, Y_{\theta_n}^y)\right) \mathbb{1}_{\theta_n \leq \tau \wedge \xi_n}\right].$$

Since $\bar{y}_n < y$, we have $\xi_n \leq \xi_y$, and thanks to the monotonicity of h, we may write

$$E_n := \left(h(X_{\theta_n}^x, Y_{\theta_n}^{\bar{y}_n}) - h(X_{\theta_n}^x, Y_{\theta_n}^y)\right) \mathbb{1}_{\theta_n \leq \tau \wedge \xi_n} \leq \left(h(X_{\theta_n}^x, Y_{\theta_n}^{\bar{y}_n}) - h(X_{\theta_n}^x, Y_{\theta_n}^y)\right) \mathbb{1}_{\theta_n \leq \tau \wedge \xi_y}.$$

Moreover, it follows from Lemma 3.1 that $\{\theta_n = \xi_y \leq \tau\} = \{\xi_n = \theta_n = \xi_y \leq \tau\}$. Therefore, we find

$$\begin{aligned} E_n \quad \leq \quad & \left[h(X_{\theta_n}^x, Y_{\theta_n}^{\bar{y}_n}) - h(X_{\theta_n}^x, Y_{\theta_n}^y)\right] \mathbb{1}_{\theta_n < \tau \wedge \xi_y} \\ & + \left[h(X_{\theta_n}^x, 0) - h(X_{\theta_n}^x, 0)\right] \mathbb{1}_{\theta_n = \xi_n = \xi_y \leq \tau} + \left[h(X_\tau^x, Y_\tau^{\bar{y}_n}) - h(X_\tau^x, Y_\tau^y)\right] \mathbb{1}_{\tau < \xi_y} \\ = \quad & \left[h(X_{\theta_n}^x, Y_{\theta_n}^{\bar{y}_n}) - h(X_{\theta_n}^x, Y_{\theta_n}^y)\right] \mathbb{1}_{\theta_n < \tau \wedge \xi_y} + \left[h(X_\tau^x, Y_\tau^{\bar{y}_n}) - h(X_\tau^x, Y_\tau^y)\right] \mathbb{1}_{\tau < \xi_y}. \end{aligned}$$

Using Remark 3.1 and the same convergence argument as above, we obtain the pointwise convergence of E_n. Noticing that, for n big enough,

$$\begin{aligned}\left[h(X^x_{\theta_n}, Y^{\bar{y}_n}_{\theta_n}) - h(X^x_{\theta_n}, Y^y_{\theta_n})\right] \mathbb{1}_{\theta_n<\tau\wedge\xi_y} &\leq \sup_{t<\tau\wedge\xi_y}\left[h(X^x_t, Y^{\frac{y}{2}}_t) - h(X^x_t, Y^y_t)\right]\mathbb{1}_{t<\tau\wedge\xi_y},\\ \text{and } \left[h(X^x_\tau, Y^{\bar{y}_n}_\tau) - h(X^x_\tau, Y^y_\tau)\right] \mathbb{1}_{\tau<\xi_y} &\leq \left[h(X^x_\tau, Y^{\frac{y}{2}}_\tau) - h(X^x_\tau, Y^y_\tau)\right]\mathbb{1}_{\tau<\xi_y},\end{aligned}$$

we obtain an integrable upper bound for $|E_n|$, leading therefore to the desired results, i.e. there exists an $N > 0$, such that $\forall n \geq N$, we have

$$v_i(x, \bar{y}_n) - v_i(x, y) \leq \frac{\varepsilon}{2}. \tag{3.18}$$

Combining inequalities (3.16), (3.17) and (3.18), we obtain that there exists $N > 0$, such that $\forall n \geq N$, we have

$$v_i(x_n, y_n) - v_i(x, y) \leq \varepsilon. \tag{3.19}$$

Step 2. We use the same arguments as in Step 1 to show that for a given $\varepsilon > 0$, there exists an $N > 0$, such that $\forall n \geq N$, we have $v_i(x, y) - v_i(x_n, y_n) \leq \varepsilon$. This part of the proof is easier as the optimal stopping time from Lemma 3.1 does not depend on n in some cases.

Combining the two steps, we obtain the continuity of v_i on $\mathbb{R} \times \mathbb{R}^+$. □

3.2 Viscosity Characterization of the value function

In order to prove that the value function v is the unique solution of the HJB system, we shall assume that the following dynamic programming principle holds: for any $(i, x, y) \in \{0, \ldots, m\} \times \mathbb{R} \times (0, \infty)$, for all $\nu \in \mathcal{T}$, we have

$$(\mathbf{DP}) \quad v(i, x, y) = \sup_{\theta\in\mathcal{T}} \mathbb{E}^{i,x,y}\left[h(X_\theta, Y_\theta)\mathbb{1}_{\theta\leq\tau\wedge\nu} + U(e^{X_\tau})\mathbb{1}_{\tau<\theta\wedge\nu} + v(L_\nu, X_\nu, Y_\nu)\mathbb{1}_{\nu<\theta}\mathbb{1}_{\nu\leq\tau}\right].$$

We then have the PDE characterization of the value functions.

Theorem 3.1 *The value functions v_i, $i \in \{0, \ldots, m\}$, are continuous on $\mathbb{R} \times \mathbb{R}^+$, and constitute the unique viscosity solution on $\mathbb{R} \times \mathbb{R}^+$ with growth condition*

$$|v_i(x, y)| \leq |U(e^x)| + |U(e^x)f(y)|,$$

and boundary condition

$$\lim_{y\downarrow 0} v_i(x, y) = U(e^x),$$

to the system of variational inequalities :

$$\begin{aligned}\min\Big[- \mathcal{L}v_i(x, y) - \mathcal{G}_i v_\cdot(x, y) - \mathcal{J}_i v_i(x, y)\,,\, v_i(x, y) - U(e^x f(y))\Big] = 0,\\ \forall\,(x, y) \in \times\mathbb{R} \times \mathbb{R}^+_*,\ \textit{and } i \in \{0, \ldots, n\}.\end{aligned} \tag{3.20}$$

The proof of Theorem 3.1 is based on the following lemmas.

Lemma 3.2 *The value functions $(v_i)_{0\leq i\leq m}$ constitute a subsolution to the system of variational inequalities (3.20).*

Proof of lemma 3.2: We prove the subsolution property by contradiction. Suppose that the claim is not true. Then there exists $(\bar{i},\bar{x},\bar{y})\in\{0,1,\dots,m\}\times\mathbb{R}\times\mathbb{R}^+$, a neighborhood of $B_{(\bar{x},\bar{y})}(\delta):=\{(x,y)\in\mathbb{R}\times\mathbb{R}^+;|x-\bar{x}|\leq\delta;|y-\bar{y}|\leq\delta\}$ where $\delta>0$, C^2 functions φ_i $(0\leq i\leq m)$ with $(\varphi_{\bar{i}}-v_{\bar{i}})(\bar{x},\bar{y})=0$, and $\varphi_i\geq v_i$ on $B_{(\bar{x},\bar{y})}(\delta)$ $(i\in\{0,1,\dots,m\})$, and $\eta>0$, such that for all $(x,y)\in\bar{B}_{(\bar{x},\bar{y})}(\delta)$, we have

$$-\mathcal{L}\varphi_{\bar{i}}(x,y)-\mathcal{G}_{\bar{i}}\varphi_{\cdot}(x,y)-\mathcal{J}_{\bar{i}}\varphi_{\bar{i}}(x,y) > \eta, \tag{3.21}$$

$$\varphi_{\bar{i}}(x,y)-U(e^x f(y)) > \eta. \tag{3.22}$$

We consider the exit time

$$\tau_\delta=\inf\{t\geq 0\ ;\ (L_t^{\bar{i}},X_t^{\bar{x}},Y_t^{\bar{y}})\notin\{\bar{i}\}\times B_{(\bar{x},\bar{y})}(\delta)\}.$$

Let $\theta\in\mathcal{T}$, and apply Itô's Formula to φ between 0 and $\gamma_\delta:=\tau_\delta\wedge\theta\wedge\tau$. Taking an expectation, we obtain

$$\mathbb{E}^{\bar{i},\bar{x},\bar{y}}\left[\varphi_{L_{\gamma_\delta}}(X_{\gamma_\delta},Y_{\gamma_\delta})\right] = \varphi_{\bar{i}}(\bar{x},\bar{y})+\mathbb{E}^{\bar{i},\bar{x},\bar{y}}\left[\int_0^{\gamma_\delta}\left(\mathcal{L}\varphi_{\bar{i}}(X_t,Y_t)+\mathcal{G}_{\bar{i}}\varphi_{\cdot}(X_t,Y_t)\right)dt\right].$$

From relation (3.21), the above inequality becomes

$$\begin{aligned}\varphi_{\bar{i}}(\bar{x},\bar{y}) &\geq \mathbb{E}^{\bar{i},\bar{x},\bar{y}}\left[\varphi_{L_{\gamma_\delta}}(X_{\gamma_\delta},Y_{\gamma_\delta})+\int_0^{\gamma_\delta}\left(\eta+\mathcal{J}_{\bar{i}}\varphi_{\bar{i}}(X_t,Y_t)\right)dt\right]\\ &\geq \mathbb{E}^{\bar{i},\bar{x},\bar{y}}\left[\varphi_{L_{\gamma_\delta}}(X_{\gamma_\delta},Y_{\gamma_\delta})+\left(U(e^{X_\tau})-\varphi_{\bar{i}}(X_\tau,Y_\tau)\right)\mathbb{1}_{\tau<\theta\wedge\tau_\delta}\right]+\eta\mathbb{E}^{\bar{i},\bar{x},\bar{y}}[\gamma_\delta]\end{aligned}$$

since $\left(U(e^{X_\tau})-\varphi_{\bar{i}}(X_\tau,Y_\tau)\right)\mathbb{1}_{\tau\leq t}-\int_0^t\mathcal{J}_{\bar{i}}\varphi_{\bar{i}}(X_s,Y_s)ds$ is a martingale on $[0,\gamma_\delta]$. Hence,

$$\begin{aligned}\varphi_{\bar{i}}(\bar{x},\bar{y}) &\geq \mathbb{E}^{\bar{i},\bar{x},\bar{y}}\left[U(e^{X_\tau})\mathbb{1}_{\tau<\theta\wedge\tau_\delta}+\varphi_{L_{\gamma_\delta}}(X_{\theta\wedge\tau_\delta},Y_{\theta\wedge\tau_\delta})\mathbb{1}_{\tau\geq\theta\wedge\tau_\delta}\right]+\eta\mathbb{E}^{\bar{i},\bar{x},\bar{y}}[\gamma_\delta]\\ &\geq \mathbb{E}^{\bar{i},\bar{x},\bar{y}}\left[U(e^{X_\tau})\mathbb{1}_{\tau<\theta\wedge\tau_\delta}+\varphi_{\bar{i}}(X_\theta,Y_\theta)\mathbb{1}_{\theta\leq\tau\wedge\tau_\delta}+\varphi_{L_{\gamma_\delta}}(X_{\tau_\delta},Y_{\tau_\delta})\mathbb{1}_{\tau_\delta<\theta}\mathbb{1}_{\tau_\delta\leq\tau}\right]\\ &\quad+\eta\mathbb{E}^{\bar{i},\bar{x},\bar{y}}[\gamma_\delta].\end{aligned}$$

Using (3.22) and the fact that $\varphi_i\geq v_i$ on $B_{(\bar{x},\bar{y})}(\delta)$ for all $i\leq m$, we obtain for any $\theta\in\mathcal{T}$

$$\begin{aligned}\varphi_{\bar{i}}(\bar{x},\bar{y}) &\geq \mathbb{E}^{\bar{i},\bar{x},\bar{y}}\left[U(e^{X_\tau})\mathbb{1}_{\tau<\theta\wedge\tau_\delta}+\left(U(e^{X_\theta}f(Y_\theta))+\eta\right)\mathbb{1}_{\theta\leq\tau\wedge\tau_\delta}\right]\\ &\quad+\mathbb{E}^{\bar{i},\bar{x},\bar{y}}\left[v(L_{\tau_\delta},X_{\tau_\delta},Y_{\tau_\delta})\mathbb{1}_{\tau_\delta<\theta}\mathbb{1}_{\tau_\delta\leq\tau}\right]+\eta\mathbb{E}^{\bar{i},\bar{x},\bar{y}}[\gamma_\delta]\\ &\geq \mathbb{E}^{\bar{i},\bar{x},\bar{y}}\left[U(e^{X_\tau})\mathbb{1}_{\tau<\theta\wedge\tau_\delta}+U(e^{X_\theta}f(Y_\theta))\mathbb{1}_{\theta\leq\tau\wedge\tau_\delta}\right]\\ &\quad+\mathbb{E}^{\bar{i},\bar{x},\bar{y}}\left[v(L_{\tau_\delta},X_{\tau_\delta},Y_{\tau_\delta})\mathbb{1}_{\tau_\delta<\theta}\mathbb{1}_{\tau_\delta\leq\tau}\right]+\eta\mathbb{E}^{\bar{i},\bar{x},\bar{y}}[1\wedge\tau_\delta\wedge\tau].\end{aligned}$$

Using the Dynamic Programming Principle, we obtain

$$\varphi_{\bar{i}}(\bar{x},\bar{y}) \geq v_{\bar{i}}(\bar{x},\bar{y})+\eta\mathbb{E}[1\wedge\tau_\delta\wedge\tau].$$

Noticing that $\eta\mathbb{E}[1\wedge\tau_\delta\wedge\tau]>0$, we obtain the contradiction and therefore leading us to the subsolution property.

Lemma 3.3 *The value functions v_i, $i \in \{0,1,\dots,m\}$ constitute a supersolution to the system of variational inequalities (3.20).*

Proof of lemma 3.3: We consider the C^2 test functions φ_i ($i \in \{0,1,\dots,m\}$), such that $v_{\bar{i}}(\bar{x},\bar{y}) = \varphi_{\bar{i}}(\bar{x},\bar{y})$ and $\varphi_i \leq v_i$ ($i \in \{0,1,\dots,m\}$). We can also assume w.l.o.g. that $\varphi_i(x,y) < v_i(x,y)$ on $\{0,1,\dots,m\} \times \mathbb{R} \times \mathbb{R}^+ \setminus (\bar{i},\bar{x},\bar{y})$. We have to prove that

$$\min\Big[-\mathcal{L}\varphi_{\bar{i}}(\bar{x},\bar{y}) - \mathcal{G}_{\bar{i}}\varphi_{\cdot}(\bar{x},\bar{y}) - \mathcal{J}_{\bar{i}}\varphi_{\bar{i}}(\bar{x},\bar{y})\,,\, \varphi_{\bar{i}}(\bar{x},\bar{y}) - U(e^{\bar{x}}f(\bar{y}))\Big] \geq 0.$$

We first note that $\varphi_{\bar{i}}(\bar{x},\bar{y}) = v_{\bar{i}}(\bar{x},\bar{y}) \geq U(e^{\bar{x}}f(\bar{y}))$, so we just have to show that

$$-\mathcal{L}\varphi_{\bar{i}}(\bar{x},\bar{y}) - \mathcal{G}_{\bar{i}}\varphi_{\cdot}(\bar{x},\bar{y}) - \mathcal{J}_{\bar{i}}\varphi_{\bar{i}}(\bar{x},\bar{y}) \geq 0.$$

For the state variables starting initially from $(\bar{i},\bar{x},\bar{y})$ and a stopping time $\theta \in \mathcal{T}$, we consider the exit time

$$\tau_\delta = \inf\{t \geq 0\ ;\ (L_t^{\bar{i}}, X_t^{\bar{x}}, Y_t^{\bar{y}}) \notin \{\bar{i}\} \times B_{(\bar{x},\bar{y})}(\delta)\},$$

where, as before, $B_{(\bar{x},\bar{y})}(\delta) := \{(x,y) \in \mathbb{R} \times \mathbb{R}^+; |x-\bar{x}| \leq \delta; |y-\bar{y}| \leq \delta\}$.
Using the dynamic programming principle for v applied to the stopping time $\tau_\delta \wedge t$, with $t > 0$, we find

$$\begin{aligned}
\varphi_{\bar{i}}(\bar{x},\bar{y}) &= v_{\bar{i}}(\bar{x},\bar{y}) \\
&\geq \mathbb{E}^{\bar{i},\bar{x},\bar{y}}\Big[U(e^{X_\tau})\mathbb{1}_{\tau<\theta\wedge\tau_\delta\wedge t} + U(e^{X_\theta}f(Y_\theta))\mathbb{1}_{\theta\leq\tau\wedge\tau_\delta\wedge t} \\
&\quad + v(L_{\tau_\delta\wedge t}, X_{\tau_\delta\wedge t}, Y_{\tau_\delta\wedge t})\mathbb{1}_{\tau_\delta\wedge t<\theta}\mathbb{1}_{\tau_\delta\wedge t\leq\tau}\Big] \\
&\geq \mathbb{E}^{\bar{i},\bar{x},\bar{y}}\Big[U(e^{X_\tau})\mathbb{1}_{\tau<\theta\wedge\tau_\delta\wedge t} + U(e^{X_\theta}f(Y_\theta))\mathbb{1}_{\theta\leq\tau\wedge\tau_\delta\wedge t} \\
&\quad + \varphi_{L_{\tau_\delta\wedge t}}(X_{\tau_\delta\wedge t}, Y_{\tau_\delta\wedge t})\mathbb{1}_{\tau_\delta\wedge t<\theta}\mathbb{1}_{\tau_\delta\wedge t\leq\tau}\Big], \qquad (3.23)
\end{aligned}$$

for any $\theta \in \mathcal{T}$.

Now applying Itô's formula to φ between 0 and $\gamma_\delta := \tau_\delta \wedge \tau \wedge t$, we obtain by taking an expectation

$$\mathbb{E}\left[\varphi_{L_{\gamma_\delta}}(X_{\gamma_\delta}, Y_{\gamma_\delta})\right] = \varphi_{\bar{i}}(\bar{x},\bar{y}) + \mathbb{E}\left[\int_0^{\gamma_\delta} (\mathcal{L}\varphi_{\bar{i}} + \mathcal{G}_{\bar{i}}\varphi_{\cdot})(X_t,Y_t)dt\right],$$

and, with inequality (3.23) with $\theta > \tau_\delta \wedge t$, we obtain

$$\begin{aligned}
0 &\geq \mathbb{E}\left[\int_0^{\gamma_\delta} (\mathcal{L}\varphi_{\bar{i}} + \mathcal{G}_{\bar{i}}\varphi_{\cdot})(X_t,Y_t)dt\right] + \mathbb{E}\left[\big(U(e^{X_\tau}) - \varphi_{\bar{i}}(X_\tau,Y_\tau)\big)\mathbb{1}_{\tau<\tau_\delta\wedge t}\right] \\
&\geq \mathbb{E}\left[\int_0^{\gamma_\delta} (\mathcal{L}\varphi_{\bar{i}} + \mathcal{G}_{\bar{i}}\varphi_{\cdot} + \mathcal{J}_{\bar{i}}\varphi_{\bar{i}})(X_t,Y_t)dt\right].
\end{aligned}$$

From the definition of τ_δ, we readily see that the integrand part of (3.24) is bounded. Dividing the previous inequality by t and taking t to 0, we may apply the dominated convergence theorem and obtain

$$-(\mathcal{L}\varphi_{\bar{i}} + \mathcal{G}_{\bar{i}}\varphi_{\cdot} + \mathcal{J}_{\bar{i}}\varphi_{\bar{i}})(\bar{x},\bar{y}) \geq 0,$$

leading us to the supersolution property.

□

The following lemma is the key to show the uniqueness of the solution.

Lemma 3.4 *Let $(w_i)_{0\le i\le m}$ be a continuous viscosity supersolution to the system of variational inequalities (3.20) on $\mathbb{R}\times\mathbb{R}^+$, and consider the following C^2 function:*

$$g(x,y)=\begin{cases} ax^4+by^n+k+U(1)\theta(0)+A_1x+\frac{1}{2}A_2x^2 & x\le 0\\ ax^4+by^n+k+U(e^x)\theta(x) & x>0. \end{cases} \tag{3.24}$$

with $\theta(x)=ln(4+x)$, $A_1=U'(1)\theta(0)+U(1)\theta'(0)$, $A_2=U''(1)\theta(0)+2U'(1)\theta'(0)+U(1)\theta''(0)$, a, b and k strictly positive constants and $n\ge c_0$, with c_0 as defined in (2.3).

Let $w_i^\gamma:=(1-\gamma)w_i+\gamma g$, $0\le i\le m$. Then, $(w_i^\gamma)_{0\le i\le m}$ is strict supersolution to the HJB system, i.e., there exists some $\delta>0$ such that $(w_i^\gamma)_{0\le i\le m}$ is a supersolution of

$$\min\Big[-\mathcal{L}w_i^\gamma(x,y)-\mathcal{G}_iw_\cdot^\gamma(x,y)-\mathcal{J}_iw_i^\gamma(x,y)\,,\,w_i^\gamma(x,y)-U(e^xf(y))\Big]\ge\delta, \tag{3.25}$$

$(i,x,y)\in\{0,\dots,m\}\times\mathbb{R}\times\mathbb{R}^+$.

Proof The proof of this lemma is quite straightforward and is therefore omitted. □

Remark 3.2 We notice that the function g dominates the upper bound $U(e^x)$ and the lower bound $U(f(y)e^x)$ of the value functions when $|x|$ and y go to ∞, i.e.,

$$\lim_{|x|,y\to\infty}\frac{|U(f(y)e^x)|+|U(e^x)|}{g(x,y)}=0.$$

Indeed, g has been precisely constructed to satisfy the above property as well as the strict supersolution property defined in Lemma 3.4.

We can now show a comparison theorem from which the uniqueness of (3.20) follows.

Lemma 3.5 *Let $(u_i)_{0\le i\le m}$ a continuous viscosity subsolution to the system of variational inequalities (3.20) on $\mathbb{R}\times\mathbb{R}^+$, and $(w_i)_{0\le i\le m}$ a continuous viscosity supersolution to the system of variational inequalities (3.20) on $\mathbb{R}\times\mathbb{R}^+$, satisfying the boundary conditions $\lim_{y\downarrow 0}u_i(x,y)\le\lim_{y\downarrow 0}w_i(x,y)$, $i\in\{0,\dots,m\},x\in\mathbb{R}$, and the following growth condition*

$$|u_i(x,y)|+|w_i(x,y)|\le|U(e^x)|+|U(e^xf(y)|.$$

Then,

$$u_i(x,y)\le w_i(x,y),\ on\ \ \mathbb{R}\times\mathbb{R}^+,\ i\in\{0,\dots,m\}.$$

The proof of Lemma 3.5 is postponed in Appendix A.

3.3 Execution and continuation regions

We now prove useful qualitative properties of the execution regions of the optimal stopping problem. We introduce the following execution and continuation regions:

$$\begin{aligned}\mathcal{E} &= \left\{(i,x,y) \in \{0,...,m\} \times \mathbb{R} \times \mathbb{R}^+ \,|\, v(i,x,y) = h(x,y)\right\} \\ \mathcal{C} &= \{0,...,m\} \times \mathbb{R} \times \mathbb{R}^+ \setminus \mathcal{E}.\end{aligned}$$

Clearly, outside the execution region $\mathcal{E}$, it is never optimal to withdraw the limit order and use a market order. Moreover, the smallest optimal stopping time θ^*_{ixy} verifies

$$\theta^*_{ixy} = \inf\left\{u \geq 0 \,|\, \left(L^i_u, X^x_u, Y^y_u\right) \in \mathcal{E}\right\}.$$

We define the $(i,x)-$sections for every $(i,x) \in \{0,...,m\} \times \mathbb{R}$ by

$$\mathcal{E}_{(i,x)} = \{y \geq 0 \,|\, v(i,x,y) = h(x,y)\} \text{ and } \mathcal{C}_{(i,x)} = \mathbb{R}^+ \setminus \mathcal{E}_{(i,x)}.$$

Proposition 3.3 (Properties of execution region)

i) *$\mathcal{E}$ is closed in $\{0,...,m\} \times \mathbb{R} \times (0,+\infty)$,*

ii) *Let $(i,x) \in \{0,...,m\} \times \mathbb{R}$.*

- *If $\mathbb{E}^{i,x}[U(e^{X_\tau})] = U(e^x)$, then, for all $y \in \mathbb{R}^+$, $v(i,x,y) = U(e^x)$ and $\mathcal{E}_{(i,x)} = \{0\}$.*

- *If $\mathbb{E}^{i,x}[U(e^{X_\tau})] < U(e^x)$, then there exists $x_0 \in \mathbb{R}$ such that $\mathcal{E}_{(i,x_0)} \setminus \{0\} \neq \emptyset$ and $\bar{y}^*(i,x) := \sup \mathcal{E}_{(i,x)} < +\infty$.*

Proof: The proof follows some ideas presented in [22].

i) For all $i \in \{0,...,m\}$, v_i, U and f are continuous, then $\mathcal{E} = \bigcup_{i=0}^m \left[v_i - h\right]^{-1}(0)$ is a closed set.

ii) Let $(i,x) \in \{0,...,m\} \times \mathbb{R}$. If $\mathbb{E}^{i,x}[U(e^{X_\tau})] = U(e^x)$, it follows from Proposition 3.1 that $v_i(x,y) = U(e^x)$, for all $y \geq 0$. As $f < 1$ on $(0,+\infty)$ and U is increasing, we obviously have $\mathcal{E}_{(i,x)} = \{0\}$.

Now, we assume that $\mathbb{E}^{i,x}[U(e^{X^x_\tau})] < U(e^x)$ and that for all $x_0 \in \mathbb{R}$, $\mathcal{E}_{(i,x_0)} \setminus \{0\} = \emptyset$. Let $y \in (0,+\infty)$. We find

$$\begin{aligned}T_t(i,x,y) &:= \mathbb{E}^{i,x,y}\left[h\left(X_t, Y_t\right) \mathbb{1}_{t \leq \tau \wedge \xi_y} + U(e^{X_\tau}) \mathbb{1}_{t > \tau \wedge \xi_y}\right] \\ &\geq \mathbb{E}^{i,x,y}\left[h(X_t, Y_t) \mathbb{1}_{t \leq \tau \wedge \xi_y} + v(L_\tau, X_\tau, Y_\tau) \mathbb{1}_{t > \tau \wedge \xi_y}\right].\end{aligned}$$

Therefore, letting t going to $+\infty$, we have

$$\liminf_{t \to +\infty} T_t(i,x,y) \geq \mathbb{E}^{i,x,y}\left[v(L_{\tau \wedge \xi_y}, X_{\tau \wedge \xi_y}, Y_{\tau \wedge \xi_y})\right] = v(i,x,y).$$

The last equality comes from the fact that τ is almost surely finite and that the process $(v(L_t, X_t, Y_t))_{0 \leq t}$ is a martingale up to time $\xi_y = \inf\{t \geq 0 : Y^y_t = 0\}$. Indeed, we

have $\xi_y = \theta^*_{ixy}$ since we have assumed that for all $x_0 \in \mathbb{R}$, $\mathcal{E}_{(i,x_0)} \setminus \{0\} = \emptyset$. On the other hand, from the above assumption, we derive the following relation:

$$\limsup_{t\to+\infty} T_t(i,x,y) = \mathbb{E}^{i,x}\left[U(e^{X_\tau})\right] < U(e^x).$$

Therefore we have proved that, for all $y > 0$,

$$h(x,y) < v(i,x,y) \le \liminf_{t\to+\infty} T_t(i,x,y) \le \limsup_{t\to+\infty} T_t(i,x,y) \le \mathbb{E}^{i,x}\left[U(e^{X_\tau})\right].$$

Since $f(0) = 1$, by taking y going to 0, we obtain the following contradiction $U(e^x) \le \mathbb{E}^{i,x}\left[U(e^{X_\tau})\right] < U(e^x)$.

Finally, we recall that U is increasing and $\lim_{y\to+\infty} f(y) = 0$. Therefore, we have

$$\lim_{y\to+\infty} v(i,x,y) \ge \mathbb{E}^{i,x}\left[U(e^{X_\tau})\right] > U(0) = \lim_{y\to+\infty} h(x,y).$$

It obviously follows that $\bar{y}^*(i,x) := \sup \mathcal{E}_{(i,x)} < +\infty$.

□

4 Logarithmic utility

Throughout this section, we assume that the diffusion processes X and Y are governed by the following SDE, which are particular cases of (2.1) and (2.4)

$$\begin{array}{rcl} dX_t &=& \mu dt + \sigma(X_t) dB_t;\ X_0 = x \\ dY_t &=& \kappa\left(\beta - Y_t\right) dt + \gamma\sqrt{Y_t} dW_t;\ Y_0 = y \end{array} \tag{D-1}$$

where μ, κ, β and γ are constant. We first notice that the supermeanvalued assumption combined with the logarithmic utility function implies that $\mu \le 0$. Moreover, if $\mu = 0$, we have seen that $v(i,x,y) = U(e^x)$ and $\mathcal{E}_{(i,x)} = \{0\}$ (see Proposition 3.3), so we shall assume throughout this section that $\mu < 0$.

The following theorem shows that in the logarithmic case, we can reduce the dimension of the problem by factoring out the x-variable. For this purpose, we define $\mathcal{T}_{L,W}$ the set of stopping times with respect to the filtration generated by (L,W), and the differential operator $\overline{\mathcal{L}}\phi(y) := \frac{1}{2}\gamma^2 y \frac{\partial^2\phi}{\partial y^2} + \kappa(\beta - y)\frac{\partial\phi}{\partial y} + \mu$, for $\phi \in \mathcal{C}^2(\mathbb{R}^+)$.

Theorem 4.2 *For $(i,y) \in \{1,...,m\} \times \mathbb{R}^+$ we define the function:*

$$w(i,y) = \sup_{\theta\in\mathcal{T}_{L,W}} \mathbb{E}^{i,y}[\mu(\theta\wedge\tau) + \ln\left(f(Y_\theta)\right) \mathbb{1}_{\{\theta\le\tau\}}].$$

Then,

$$v(i,x,y) = x + w(i,y) \text{ on } \{0,...,m\} \times \mathbb{R} \times \mathbb{R}^+,$$

with w the unique viscosity solution to the system of equations:

$$\min\Big[-\overline{\mathcal{L}}w(i,y)+\lambda_i w(i,y)-\sum_{j\neq i}\vartheta_{i,j}\left(w(j,\,y)-w(i,\,y)\right)\,,\,w(i,y)-g(y)\Big]=0, \qquad (4.26)$$

where $g(y):=\ln(f(y))$ Moreover, the functions $w(i,.)$ are of class C^1 on $\mathbb{R}^+$ and C^2 on the open set $\mathcal{C}_{(i,x)}\cup \mathrm{Int}(\mathcal{E}_{(i,x)})$.

Proof: We first notice that

$$v(i,x,y)=\sup_{\theta\in\mathcal{T}}\mathbb{E}^{i,x,y}[X_{\theta\wedge\tau}+\ln\left(f(Y_\theta)\right)\mathbb{1}_{\{\theta\leq\tau\}}] \text{ on } \{0,...,m\}\times\mathbb{R}\times\mathbb{R}^+.$$

Moreover, for $(i,x,x',y)\in\{0,...,m\}\times\mathbb{R}^2\times\mathbb{R}^+$, we have

$$v(i,x',y)-v(i,x,y)\leq\sup_{\theta\in\mathcal{T}}\mathbb{E}^{i,x',y}[X_{\theta\wedge\tau}|L_0=i]-\mathbb{E}^{i,x,y}[X_{\theta\wedge\tau}|L_0=i]=x'-x.$$

On the other hand, we have

$$v(i,x',y)-v(i,x,y)\geq\mathbb{E}^{i,x',y}[X_{\theta^*_{ixy}\wedge\tau}|L_0=i]-\mathbb{E}^{i,x,y}[X_{\theta^*_{ixy}\wedge\tau}|L_0=i]=x'-x.$$

It follows that there exists a function w defined on $\{0,...,m\}\times\mathbb{R}^+$ such that $v(i,x,y)=x+w(i,y)$. Then, $\theta^*_{ixy}:=\inf\left\{u\geq 0\,|\,\left(L^i_u,X^x_u,Y^y_u\right)\in\mathcal{E}\right\}=\inf\{t\geq 0:\ w(L^i_t,Y^y_t)=\ln(f(Y^y_t))\}:=\theta^*_{iy}$, belongs to $\mathcal{T}_{L,W}$. Hence, we have

$$w(i,y)=\sup_{\theta\in\mathcal{T}_{L,W}}\mathbb{E}^{i,y}[\mu(\theta\wedge\tau)+\ln\left(f(Y_\theta)\right)\mathbb{1}_{\{\theta\leq\tau\}}].$$

We deduce from Theorem 3.1 that $(w_i)_{0\leq i\leq m}$ is the unique continuous viscosity solution of the system of equation (4.26). We conclude the proof by asserting that the fact that $w(i,\cdot)$ is of class C^1 on $\mathbb{R}^+$ and C^2 on the open set $\mathcal{C}_{(i,x)}\cup\mathrm{Int}(\mathcal{E}_{(i,x)})$ for all $i\in\{1,...,m\}$ can be established by following the proof of Proposition 3.3 in [16]. □

Remark 4.3 From Theorem 4.2, we notice that the (i,x)-sections of the execution region $\mathcal{E}_{(i,x)}$ do not depend on x. For convenience we denote them by $\mathcal{E}_{(i,.)}$ in this section. In the same way, we write $\mathcal{C}_{(i,.)}:=\mathcal{C}_{(i,x)}$.

4.1 Execution region

In the logarithmic case, the execution region can be characterized in more details.

Proposition 4.4 *Let $i\in\{0,...,m\}$ and set*

$$\hat{y}_i=\inf\{y\geq 0:\ \mathcal{H}_i g(y)\geq 0\} \text{ with } \mathcal{H}_i g(y)=\overline{\mathcal{L}}g(y)-\lambda_i g(y)+\sum_{j\neq i}\vartheta_{i,j}\left(w(j,\,y)-g(y)\right).$$

There exists $y^*_i\geq 0$ such that $[0,y^*_i]=\mathcal{E}_{(i,.)}\cap[0,\hat{y}_i]$. Moreover, $w(i,\cdot)-g(\cdot)$ is non-decreasing on $[y^*_i,\hat{y}_i]$.

Proof: Let $y_i^* = \inf\{y \geq 0 :\ w(i,y) > g(y)\}$. Notice that

$$\mathcal{H}_i g(0) = \mu + \kappa\beta g'(0) < 0,$$

since $\mu < 0$ and $g'(0) < 0$. Hence, $\hat{y}_i > 0$. As we have $\mathcal{H}_i w(i,y) \leq 0$ on $\mathbb{R}^+$, we have $y_i^* \leq \hat{y}_i$. If $y_i^* = \hat{y}_i$, the result is obvious so we shall assume that $y_i^* < \hat{y}_i$. For all $z \in (y_i^*, \hat{y}_i) \cap \mathcal{C}_{(i,\cdot)}$, we have

$$\frac{\gamma^2 z}{2}\frac{\partial^2 w}{\partial y^2}(i,z) = -\kappa(\beta - z)\frac{\partial w}{\partial x}(i,z) + \lambda_i w(i,z) - \sum_{j\neq i}\vartheta_{i,j}\left(w(j,\,z) - w(i,\,z)\right) - \mu.$$

Therefore, if we set $d_i = w(i,\cdot) - g(\cdot)$, we find

$$\frac{\gamma^2 z}{2} d_i''(z) \geq -\kappa(\beta - z) d_i'(z) + \left(\lambda_i + \sum_{j\neq i}\vartheta_{i,j}\right) d_i(x) - \sup_{z' \in [y_i^*, z]} \mathcal{H}_i g(z'). \tag{4.27}$$

From the definition of y_i^*, there exists a sequence $(z_n)_{n\geq 0}$ taking values in $(y_i^*, \hat{y}_i) \cap \mathcal{C}_{(i,\cdot)}$ and such that $\lim_{n\to+\infty} z_n = y_i^*$. It follows from the smooth fit property and from (4.27) that

$$\lim_{z \searrow y_i^*} \frac{\gamma^2 z}{2} d_i''(z) \geq -\mathcal{H}_i g(y_i^*) > 0.$$

It implies that $y_i^* < \xi_i$ where $\xi_i := \inf\{z > y_i^* :\ d_i(z) = 0 \text{ or } d_i'(z) < 0\}$. Assume that $\xi_i < \hat{y}_i$. As d_i is increasing on (y_i^*, ξ_i), we have $d_i(\xi_i) > 0$ and $d_i'(\xi_i) = 0$. However, it leads to a contradiction because (4.27) implies that $d_i''(\xi_i) > 0$. □

When $\overline{\mathcal{L}}g(y)$ is non-decreasing in y, the previous result can be specified further.

Proposition 4.5 *Assume that the function $y \mapsto \overline{\mathcal{L}}g(y)$ is non-decreasing on $\mathbb{R}^+$, then for all $i \in \{0, ..., m\}$, $w(i,\cdot) - g(\cdot)$ is non-decreasing on $\mathbb{R}^+$ and we have $\mathcal{E}_{(i,\cdot)} = [0, y_i^*]$, with $y_i^* > 0$.*

Proof: Let $i \in \{0, ..., m\}$ and $0 \leq y < z$. We introduce the following stopping time

$$\theta_{yz} = \inf\{t \geq 0 :\ Y_t^y = Y_t^z\}.$$

As g is non increasing, we have

$$\begin{aligned} w(i,z) - w(i,y) &\geq \mathbb{E}\Big[\left(g(Y^z_{\theta^*_{iy}}) - g(Y^y_{\theta^*_{iy}})\right)\mathbb{1}_{\{\theta^*_{iy} < \tau\}}\Big] \\ &= \mathbb{E}\Big[\left(g(Y^z_{\theta^*_{iy}\wedge\theta_{yz}}) - g(Y^y_{\theta^*_{iy}\wedge\theta_{yz}})\right)\mathbb{1}_{\{\theta^*_{iy} < \tau\}}\Big] \\ &\geq \mathbb{E}\Big[\left(g(Y^z_{\theta^*_{iy}\wedge\theta_{yz}}) - g(Y^y_{\theta^*_{iy}\wedge\theta_{yz}})\right)\Big]. \end{aligned}$$

Applying Itô formula, it follows from the fact that $\overline{\mathcal{L}}g$ is non decreasing that

$$w(i,z) - w(i,y) \geq g(z) - g(y) + \mathbb{E}\Big[\int_0^{\theta^*_{iy}\wedge\theta_{yz}} \overline{\mathcal{L}}g(Y_u^z) - \overline{\mathcal{L}}g(Y_u^y)\,du\Big] \geq g(z) - g(y).$$

□

Remark 4.4 If we set $f(y) = e^{-y}$ on $\mathbb{R}^+$, we have $\overline{\mathcal{L}}g(y) = \kappa(y - \beta)$ so the assumption of Proposition 4.5 is satisfied.

4.2 Logarithmic utility with no-switch (case $\vartheta_{i,j} = 0\ \forall i \neq j$)

Fix $i \leq m$ and assume that $\vartheta_{i,j} = 0\ \forall i \neq j$. In this case, an explicit solution of the HJB system can be obtained.

Proposition 4.6 *If $\mathcal{E}_{(i,\cdot)} = [0, y_i^*]$ then y_i^* is the solution of the following equation*

$$\frac{g(y_i^*) - \frac{\mu}{\lambda_i}}{g'(y_i^*)} = -\frac{\gamma^2}{2\lambda_i} \frac{\Psi\left(\frac{\lambda_i}{\kappa}, \frac{2\kappa\beta}{\gamma^2}, \frac{2\kappa}{\gamma^2} y_i^*\right)}{\Psi\left(\frac{\lambda_i}{\kappa} + 1, \frac{2\kappa\beta}{\gamma^2} + 1, \frac{2\kappa}{\gamma^2} y_i^*\right)}, \tag{4.28}$$

and $w(i,\cdot)$ is given by

$$w(i,y) = \begin{cases} g(y) & y \leq y_i^* \\ \dfrac{g(y_i^*) - \frac{\mu}{\lambda_i}}{\Psi\left(\frac{\lambda_i}{\kappa}, \frac{2\kappa\beta}{\gamma^2}, \frac{2\kappa}{\gamma^2} y_i^*\right)} \Psi\left(\frac{\lambda_i}{\kappa}, \frac{2\kappa\beta}{\gamma^2}, \frac{2\kappa}{\gamma^2} y\right) + \dfrac{\mu}{\lambda_i} & y > y_i^*, \end{cases} \tag{4.29}$$

where Ψ denotes the confluent hypergeometric function of second kind (see Appendix B).

Proof: We start analyzing the following differential equation:

$$-\frac{1}{2}\gamma^2 y \phi''(y) - \kappa(\beta - y)\phi'(y) + \lambda_i \phi(y) - \mu = 0\,. \tag{4.30}$$

The solution is $\eta_i(y) = a_i \Phi\left(\frac{\lambda_i}{\kappa}, \frac{2\kappa\beta}{\gamma^2}, \frac{2\kappa}{\gamma^2} y\right) + b_i \Psi\left(\frac{\lambda_i}{\kappa}, \frac{2\kappa\beta}{\gamma^2}, \frac{2\kappa}{\gamma^2} y\right) + \frac{\mu}{\lambda_i}$, where $(a_i, b_i) \in \mathbb{R}^2$ and Φ and Ψ are respectively the confluent hypergeometric functions of the first and second kind, see Appendix B. As $\mathcal{E}_{(i,\cdot)} = [0, y_i^*]$, there exists $a_i, b_i \in \mathbb{R}$ such that

$$w(i,y) = \begin{cases} g(y) & \text{for} \quad 0 \leq y \leq y_i^* \\ \eta_i(y) & \text{for} \quad y > y_i^*. \end{cases}$$

From Proposition 3.1, $w(i,\cdot)$ is non-increasing on $[0, +\infty)$, as such $\lim_{y\to\infty} w(i,y)$ exists. The coefficient a_i of the confluent hypergeometric function of first kind is then equal to zero, since Φ does not admit a limit, see Appendix B.

From Theorem 4.2, $w(i,\cdot)$ is $C^1([0, +\infty))$. Using the continuity of $w(i,\cdot)$ and $w'(i,\cdot)$ at y_i^*, we obtain two conditions which depend linearly on the parameter b_i, as such, we obtain relation (4.28). For explicit details on the derivatives of the confluent hypergeometric functions, we refer to Appendix B.

□

4.3 Logarithmic utility with switch between two regimes

Now, we assume that there are two liquidity regimes (i.e., $m = 1$) and $\vartheta_{0,1}\vartheta_{1,0} \neq 0$. We also assume that, for both $i = 0, 1$, there exists $y_i^* > 0$ such that $\mathcal{E}_{(i,\cdot)} = [0, y_i^*]$.
Let Λ be the matrix

$$\Lambda = \begin{pmatrix} \lambda_0 + \vartheta_{0,1} & -\vartheta_{0,1} \\ -\vartheta_{1,0} & \lambda_1 + \vartheta_{1,0} \end{pmatrix}. \tag{4.31}$$

As $\vartheta_{0,1}\vartheta_{1,0} > 0$ it is easy to check that Λ has two eigenvalues $\widetilde{\lambda}_0$ and $\widetilde{\lambda}_1 < \widetilde{\lambda}_0$. Let $\widetilde{\Lambda} = P^{-1}\Lambda P$ be the diagonal matrix with diagonal $(\widetilde{\lambda}_0, \widetilde{\lambda}_1)$. The transition matrix P is denoted by

$$P = \begin{pmatrix} p_0^0 & p_1^0 \\ p_0^1 & p_1^1 \end{pmatrix}. \tag{4.32}$$

Without loss of generality, we shall assume that $p_0^0 + p_1^0 = 1 = p_0^1 + p_1^1$, indeed $(1, -1)$ is not an eigenvector of Λ as $\lambda_0 > \lambda_1$.

Proposition 4.7 *With the above assumptions, we obtain $y_0^* \leq y_1^*$.*

Proof: Assume that $y_1^* < y_0^*$ and set $d(y) := w(0,y) - w(1,y)$ on $\mathbb{R}^+$. We obviously have $d'(y_1^*) \leq 0$ and we set $\hat{y} := \inf\{y > y_1^* : \ d'(y) = 0\}$.
As we have $\lim_{y\to+\infty} w(1,y) = \dfrac{\mu}{\lambda_1} < \dfrac{\mu}{\lambda_0} = \lim_{y\to+\infty} w(0,y) < 0$, we know that $\hat{y} < +\infty$. From Proposition 4.4, we know that the function $y \mapsto w(1,y) - g(y)$ is increasing on $(y_1^*, \hat{y}_1)$. Moreover, for $y \leq y_0^*$, we have

$$\begin{aligned} 0 &\geq \mathcal{H}_0 g(y) \\ &= \mathcal{H}_1 g(y) + \vartheta_{0,1}(w(1,y) - g(y)) - (\lambda_0 - \lambda_1) g(y) \\ &\geq \mathcal{H}_1 g(y). \end{aligned}$$

Therefore, we find $y_0^* < \hat{y}$ and $\hat{y} \in \mathcal{C}_{(0,\cdot)} \cap \mathcal{C}_{(1,\cdot)}$. We then obtain

$$\begin{aligned} 0 &= \mathcal{H}_1 w(1,\hat{y}) - \mathcal{H}_0 w(0,\hat{y}) \\ &= \frac{\gamma^2 \hat{y}}{2} d''(\hat{y}) - \lambda_1 w(1,\hat{y}) + \lambda_0 w(0,\hat{y})) - (\vartheta_{0,1} + \vartheta_{1,0}) d(\hat{y}). \end{aligned}$$

Hence, we have

$$\frac{\gamma^2 \hat{y}}{2} d''(\hat{y}) = (\lambda_0 + \vartheta_{0,1} + \vartheta_{1,0}) d(\hat{y}) - (\lambda_0 - \lambda_1) w(1,\hat{y})) < 0,$$

which leads to a contradiction. □

As before, the value function can be written in terms of the confluent hypergeometric functions.

Proposition 4.8 *The function w is given by*

$$w(0,y) = \begin{cases} g(y) & y \in [0, y_0^*] \\ \widehat{c}\Phi\left(\dfrac{\lambda_0 + \vartheta_{0,1}}{\kappa}, \dfrac{2\kappa\beta}{\gamma^2}, \dfrac{2\kappa}{\gamma^2} y\right) + \widehat{d}\Psi\left(\dfrac{\lambda_0 + \vartheta_{0,1}}{\kappa}, \dfrac{2\kappa\beta}{\gamma^2}, \dfrac{2\kappa}{\gamma^2} y\right) & y \in (y_0^*, y_1^*] \\ \quad + \mathcal{I}\left(\dfrac{2\kappa}{\gamma^2}, \beta, -2\dfrac{\lambda_0 + \vartheta_{0,1}}{\gamma^2}, 2\dfrac{\vartheta_{0,1} g(\cdot) + \mu}{\gamma^2}\right)(y) & \\ p_0^0 \left[\widehat{e}\Psi\left(\dfrac{\widetilde{\lambda}_0}{\kappa}, \dfrac{2\kappa\beta}{\gamma^2}, \dfrac{2\kappa}{\gamma^2} x\right) + \dfrac{\mu}{\widetilde{\lambda}_0}\right] & y \in (y_1^*, \infty) \\ \quad + p_1^0 \left[\widehat{f}\Psi\left(\dfrac{\widetilde{\lambda}_1}{\kappa}, \dfrac{2\kappa\beta}{\gamma^2}, \dfrac{2\kappa}{\gamma^2} x\right) + \dfrac{\mu}{\widetilde{\lambda}_1}\right] & \end{cases} \tag{4.33}$$

$$w(1,y) = \begin{cases} g(y) & y \in [0, y_1^*] \\ p_0^1 \left[\widehat{e}\Psi\left(\dfrac{\widetilde{\lambda}_0}{\kappa}, \dfrac{2\kappa\beta}{\gamma^2}, \dfrac{2\kappa}{\gamma^2} y \right) + \dfrac{\mu}{\widetilde{\lambda}_0} \right] & y \in (y_1^*, \infty) \\ \qquad + p_1^1 \left[\widehat{f}\Psi\left(\dfrac{\widetilde{\lambda}_1}{\kappa}, \dfrac{2\kappa\beta}{\gamma^2}, \dfrac{2\kappa}{\gamma^2} y \right) + \dfrac{\mu}{\widetilde{\lambda}_1} \right], \end{cases}$$

where Φ and Ψ denote respectively the confluent hypergeometric function of first and second kind, and $\mathcal{I}$ is a particular solution to the non-homogeneous confluent differential equation. Moreover, $(y_0^*, y_1^*, \widehat{c}, \widehat{d}, \widehat{e}, \widehat{f})$ are such that $w(0,y)$ and $w(1,y)$ belong to $C^1(\mathbb{R}^+)$.

Proof: We have

$$-\overline{\mathcal{L}} w(i,y) - \sum_{j \neq i} \vartheta_{i,j}\big(w(j,y) - w(i,y)\big) + \lambda_i w(i,y) - \mu = 0 \qquad \forall i = 0,1 \quad \forall y > y_i^* \,.$$

From Proposition 4.7, we have $y_0^* \leq y_1^*$. We may therefore distinguish two regions:

- $\mathcal{C}_0 \cap \mathcal{E}_1$, the region where is optimal to execute when the state of liquidity is in regime 1 and not to execute when it is in regime 0. This region is the interval $(y_0^*, y_1^*]$, which may be empty when $y_1^* = y_0^*$.
- $\mathcal{C}_0 \cap \mathcal{C}_1$ which corresponds to $(y_1^*, +\infty)$, where it is never optimal to execute regardless of the liquidity state.

We start with an analysis of the region $\mathcal{C}_0 \cap \mathcal{E}_1$. For all $y \in \mathcal{C}_0 \cap \mathcal{E}_1$ we have

$$\begin{aligned} w(1,y) &= g(y) \\ \overline{\mathcal{L}} w(0,y) &= \vartheta_{0,1}\big(w(0,y) - g(y)\big) + \lambda_0 w(0,y) - \mu \,. \end{aligned}$$

The function $w(0,\cdot)$ is solution of a non-homogeneous ordinary differential equation. The general solution is a linear combination of the two confluent hypergeometric functions and a particular solution in order to verify the non-homogeneous part. This particular solution may be obtained with the usual method of variation of parameters. See Appendix B in this regard. A straightforward computation shows that the expression for $w(1,\cdot)$ as given in (4.33) satisfies the ODE.
We now analyze the region $\mathcal{C}_0 \cap \mathcal{C}_1$. For all $y \in \mathcal{C}_0 \cap \mathcal{C}_1$ we have

$$\begin{aligned} \overline{\mathcal{L}} w(1,y) &= -\vartheta_{1,0}\big(w(0,y) - w(1,y)\big) + \lambda_1 w(1,y) - \mu \\ \overline{\mathcal{L}} w(0,y) &= -\vartheta_{0,1}\big(w(1,y) - w(0,y)\big) + \lambda_0 w(0,y) - \mu \,. \end{aligned}$$

We recall that the operator $\overline{\mathcal{L}}$ does not depend on the liquidity state i. Then, we consider the two linear combinations $\widehat{w}(0,y) = p_0^0 w(0,y) + p_1^0 w(1,y)$ and $\widehat{w}(1,y) = p_0^1 w(0,y) + p_1^1 w(1,y)$. As such, the pair $(\widehat{w}(0,y), \widehat{w}(1,y))$ satisfies

$$\begin{cases} \overline{\mathcal{L}} \widehat{w}(1,y) &= \widetilde{\lambda}_1 \widehat{w}(1,y) - \mu \\ \overline{\mathcal{L}} \widehat{w}(0,y) &= \widetilde{\lambda}_0 \widehat{w}(0,y) - \mu. \end{cases}$$

The above two ODEs are independent and are of the confluent hypergeometric kind. The general solution is a linear combination of the two confluent hypergeometric functions plus a particular solution, which could be chosen as a constant. Moreover, since the value function is decreasing in y and therefore admits a limit when y goes to infinity, the coefficient of the confluent hypergeometric function of the first kind must be zero since this function does not have a limit when y goes to infinity. We therefore obtain the expressions for $w(0,\cdot)$ and $w(1,\cdot)$ on the interval (y_1^*,∞) as written in (4.33).

Finally, the two functions $w(0,\cdot)$ and $w(1,\cdot)$ belong to $\mathcal{C}^1$, so that the free parameters $(y_0^*, y_1^*, \widehat{c}, \widehat{d}, \widehat{e}, \widehat{f})$ may be chosen in order to preserve the continuity and the differentiability of the two functions at points y_0^* and y_1^*. □

Corollary 4.1 *Assume $f(y) = e^{-y}$, we have*

$$\mathcal{I}\left(\frac{2\kappa}{\gamma^2}, \beta, -2\frac{\lambda_0+\vartheta_{0,1}}{\gamma^2}, 2\frac{\vartheta_{0,1}g(\cdot)+\mu}{\gamma^2}\right)(y) = \frac{\mu - \kappa\beta\frac{\vartheta_{0,1}}{\kappa+\lambda_0+\vartheta_{0,1}}}{\lambda_0+\vartheta_{0,1}} - \frac{\vartheta_{0,1}}{\kappa+\lambda_0+\vartheta_{0,1}}y. \quad (4.34)$$

The explicit system of equations satisfied by $(y_0^*, y_1^*, \widehat{c}, \widehat{d}, \widehat{e}, \widehat{f})$ is linear with respect to $(\widehat{c}, \widehat{d}, \widehat{e}, \widehat{f})$ and is detailed in Appendix B (see (B.49)*).*

5 Power utility

Throughout this section, we assume that $U(s) = s^a$ with $0 < a \leq 1$ and that μ and σ are constant. The diffusion processes X and Y are then governed by the following SDE, which are particular cases of (2.1) and (2.4),

$$\begin{array}{rcl} dX_t &=& \mu dt + \sigma dB_t \\ dY_t &=& \kappa\left(\beta - Y_t\right)dt + \gamma\sqrt{Y_t}dW_t. \end{array} \quad \text{(D-2)}$$

We first notice that the supermeanvalued assumption implies that $\mu a + \frac{\sigma^2 a^2}{2} \leq 0$. If $\mu a + \frac{\sigma^2 a^2}{2} = 0$, we have seen that $v(i,x,y) = U(e^x)$ and $\mathcal{E}_{(i,x)} = \{0\}$ (see Proposition 3.3). We shall then assume throughout this section that $\mu a + \frac{\sigma^2 a^2}{2} < 0$.

Recall that $\mathcal{T}_{L,W}$ is the set of stopping times with respect to the filtration generated by (L, W). In the power utility case, the differential operator $\tilde{\mathcal{L}}$ is given by

$$\tilde{\mathcal{L}}\phi(y) = \frac{1}{2}\gamma^2 y\frac{\partial^2\phi}{\partial y^2} + \Big[\kappa(\beta - y) + \rho\sigma\gamma a\sqrt{y}\Big]\frac{\partial\phi}{\partial y} + \Big[\frac{\sigma^2 a^2}{2} + \mu a\Big]\phi(y).$$

Theorem 5.3 *For $(i,y) \in \{1,...,m\} \times \mathbb{R}^+$, define*

$$u(i,y) = \sup_{\theta\in\mathcal{T}_{L,W}} \mathbb{E}^{i,y}[e^{(\mu a + (1-\rho^2)\frac{\sigma^2 a^2}{2})(\theta\wedge\tau) + \rho\sigma a W_{\theta\wedge\tau}}\left(\mathbb{1}_{\{\theta>\tau\}} + g(Y_\theta)\mathbb{1}_{\{\theta\leq\tau\}}\right)].$$

Then,

$$v(i,x,y) = e^{ax}u(i,y) \text{ on } \{1,...,m\} \times \mathbb{R} \times \mathbb{R}^+,$$

with u the unique viscosity solution of the system of equations:

$$\min\Big[-\tilde{\mathcal{L}}u(i,y) - \lambda_i(1-u(i,y)) - \sum_{j\neq i}\vartheta_{i,j}\left(u(j,\,y) - u(i,\,y)\right),\, u(i,y) - g(y)\Big] = 0 \quad (5.35)$$

where $g(y) := (f(y))^a$. *Moreover, the functions* $u(i,.)$ *are of class* C^1 *on* $\mathbb{R}^+$ *and* C^2 *on the open set* $\mathcal{C}_{(i,x)} \cup \mathrm{Int}(\mathcal{E}_{(i,x)})$.

Proof: We first notice that

$$v(i,x,y) = \sup_{\theta \in \mathcal{T}} \mathbb{E}^{i,x,y}[e^{aX_{\theta\wedge\tau}} \left\{[(f(Y_\theta))^a - 1]\, \mathbb{1}_{\{\theta \le \tau\}} + 1\right\}] \text{ on } \{0,...,m\} \times \mathbb{R} \times \mathbb{R}^+.$$

Moreover, for $(i,x,x',y) \in \{0,...,m\} \times \mathbb{R}^2 \times \mathbb{R}^+$, we have

$$e^{-ax'} v(i,x',y) - e^{-ax} v(i,x,y) = 0.$$

Indeed, if we set $\hat{B} = \frac{1}{\sqrt{1-\rho^2}}(B - \rho W)$, we have

$$\begin{aligned}
e^{-ax} v(i,x,y) &= \sup_{\theta \in \mathcal{T}} \mathbb{E}^{i,y}[e^{\mu a(\theta\wedge\tau) + \sigma a B_{\theta\wedge\tau}} \left(\mathbb{1}_{\{\theta > \tau\}} + g(Y_\theta)\mathbb{1}_{\{\theta \le \tau\}}\right)] \\
&= \sup_{\theta \in \mathcal{T}} \mathbb{E}^{i,y}[e^{\mu a(\theta\wedge\tau) + \rho\sigma a W_{\theta\wedge\tau} + \sqrt{1-\rho^2}\sigma a \hat{B}_{\theta\wedge\tau}} \left(\mathbb{1}_{\{\theta > \tau\}} + g(Y_\theta)\mathbb{1}_{\{\theta \le \tau\}}\right)] \\
&= \sup_{\theta \in \mathcal{T}} \mathbb{E}^{i,y}[e^{(\mu a + (1-\rho^2)\frac{\sigma^2 a^2}{2})(\theta\wedge\tau) + \rho\sigma a W_{\theta\wedge\tau}} \left(\mathbb{1}_{\{\theta > \tau\}} + g(Y_\theta)\mathbb{1}_{\{\theta \le \tau\}}\right)].
\end{aligned}$$

It follows that there exists a function u defined on $\{0,...,m\} \times \mathbb{R}^+$ such that $v(i,x,y) = e^{ax}u(i,y)$ and $\theta^*_{ixy} = \inf\{t \ge 0 :\ u(L^i_t, Y^y_t) = (f(Y^y_t))^a\} := \theta^*_{iy}$, belongs to the set of stopping times with respect to the filtration generated by (L,X), denoted by $\mathcal{T}_{L,W}$. Hence, we have $v(i,x,y) = e^{ax}u(i,y)$ where

$$u(i,y) = \sup_{\theta \in \mathcal{T}_{L,W}} \mathbb{E}^{i,y}[e^{(\mu a + (1-\rho^2)\frac{\sigma^2 a^2}{2})(\theta\wedge\tau) + \rho\sigma a W_{\theta\wedge\tau}} \left(\mathbb{1}_{\{\theta > \tau\}} + g(Y_\theta)\mathbb{1}_{\{\theta \le \tau\}}\right)].$$

We deduce from Theorem 3.1 that $(u(i,\cdot))_{0\le i\le m}$ are the unique continuous viscosity solutions of the system of equations (5.35). We conclude the proof by asserting that $u(i,\cdot)$ is of class C^1 on $\mathbb{R}^+$ and C^2 on the open set $\mathcal{C}_{(i,x)} \cup \mathrm{Int}(\mathcal{E}_{(i,x)})$ for all $i \in \{1,...,m\}$. It can be established by following the proof of Proposition 3.3 in [16]. □

In Proposition 5.9, 5.10, and 5.11, we give results on the execution region and the explicit solution of the HJB equation that are similar to those presented in the previous section. We shall omit the proofs as they may be obtained using the same arguments. We begin by the next proposition which summarizes some criteria implying that the execution region is an interval.

Proposition 5.9 *(Execution region)*
Let $i \in \{0,...,m\}$ *and set* $\hat{y}_i = \inf\{y \ge 0 :\ \mathcal{H}_i g(y) \ge 0\}$ *with* $\mathcal{H}_i g(y) = \tilde{\mathcal{L}} g(y) + \lambda_i(1 - g(y)) + \sum_{j\ne i} \vartheta_{i,j}\,(u(j,\,y) - g(y))\,.$
There exists $y^*_i \ge 0$ *such that for all* $x \in \mathbb{R}$, $[0, y^*_i] = \mathcal{E}_{(i,x)} \cap [0, \hat{y}_i]$. *Moreover,* $u(i,\cdot) - g(\cdot)$ *is non-decreasing on* $[y^*_i, \hat{y}_i]$.
Assume that the function $y \to \tilde{\mathcal{L}} g(y)$ *is non decreasing on* $\mathbb{R}^+$, *then for all* $i \in \{0,...,m\}$, $u(i,\cdot) - g(\cdot)$ *is non decreasing on* $\mathbb{R}^+$. *Especially, for all* $x \in \mathbb{R}$, $[0, y^*_i] = \mathcal{E}_{(i,x)}$.

As before, the value function can be written in terms of the confluent hypergeometric functions.

Proposition 5.10 *(Power utility with no-switch)*
Let $(i,x) \in \{1,...,m\} \times \mathbb{R}$. We assume that $\rho = 0$, $\vartheta_{i,j} = 0$ for all $j \neq i$ and that there exists $y_i^* \geq 0$ such that $\mathcal{E}_{(i,x)} = [0, y_i^*]$. If we set $\lambda_i^{(a)} := \lambda_i - \frac{\sigma^2}{2}a^2 - \mu a$, then y_i^* is the solution of

$$\frac{g(y_i^*) - \frac{\lambda_i}{\lambda_i^{(a)}}}{g'(y_i^*)} = -\frac{\gamma^2}{2\lambda_i^{(a)}} \frac{\Psi\left(\frac{\lambda_i^{(a)}}{\kappa}, \frac{2\kappa\beta}{\gamma^2}, \frac{2\kappa}{\gamma^2}y_i^*\right)}{\Psi\left(\frac{\lambda_i^{(a)}}{\kappa} + 1, \frac{2\kappa\beta}{\gamma^2} + 1, \frac{2\kappa}{\gamma^2}y_i^*\right)}. \tag{5.36}$$

The function $u(i,\cdot)$ is given by

$$u(i,y) = \begin{cases} g(y) & y \leq y_i^* \\ \dfrac{g(y_i^*) - \frac{\lambda_i}{\lambda_i^{(a)}}}{\Psi\left(\frac{\lambda_i^{(a)}}{\kappa}, \frac{2\kappa\beta}{\gamma^2}, \frac{2\kappa}{\gamma^2}y_i^*\right)} \Psi\left(\frac{\lambda^{(a)}}{\kappa}, \frac{2\kappa\beta}{\gamma^2}, \frac{2\kappa}{\gamma^2}y\right) + \frac{\lambda}{\lambda^{(a)}} & y > y_i^*. \end{cases} \tag{5.37}$$

where Ψ denotes the confluent hypergeometric function of second kind (see Appendix B).

Proposition 5.11 *(The two-regime case)*
Assume that $m = 1$, $\rho = 0$ and, for all $i \in \{0,1\}$, $\mathcal{E}_{(i,s)} = [0, y_i^*]$. We then have $y_0^* \leq y_1^*$ and the function u is given by

$$u(0,y) = \begin{cases} g(y) & y \in [0, y_0^*] \\ \widehat{c}\Phi\left(\dfrac{\lambda_0^{(a)} + \vartheta_{0,1}}{\kappa}, \dfrac{2\kappa\beta}{\gamma^2}, \dfrac{2\kappa}{\gamma^2}y\right) + \widehat{d}\Psi\left(\dfrac{\lambda_0^{(a)} + \vartheta_{0,1}}{\kappa}, \dfrac{2\kappa\beta}{\gamma^2}, \dfrac{2\kappa}{\gamma^2}y\right) & y \in]y_0^*, y_1^*] \\ \quad + \mathcal{I}\left(\dfrac{2\kappa}{\gamma^2}, \beta, -2\dfrac{\lambda_0^{(a)} + \vartheta_{0,1}}{\gamma^2}, 2\dfrac{\vartheta_{0,1}g(\cdot) + \lambda_0}{\gamma^2}\right)(y) & \\ p_0^0\left[\widehat{e}\Psi\left(\dfrac{\widetilde{\lambda}_0^{(a)}}{\kappa}, \dfrac{2\kappa\beta}{\gamma^2}, \dfrac{2\kappa}{\gamma^2}y\right) + \dfrac{\widetilde{\lambda}_0}{\widetilde{\lambda}_0^{(a)}}\right] & y \in (y_1^*, \infty) \\ \quad + p_1^0\left[\widehat{f}\Psi\left(\dfrac{\widetilde{\lambda}_1^{(a)}}{\kappa}, \dfrac{2\kappa\beta}{\gamma^2}, \dfrac{2\kappa}{\gamma^2}y\right) + \dfrac{\widetilde{\lambda}_1}{\widetilde{\lambda}_1^{(a)}}\right] & \end{cases} \tag{5.38}$$

$$u(1,y) = \begin{cases} g(y) & x \in [0, y_1^*] \\ p_0^1\left[\widehat{e}\Psi\left(\dfrac{\widetilde{\lambda}_0^{(a)}}{\kappa}, \dfrac{2\kappa\beta}{\gamma^2}, \dfrac{2\kappa}{\gamma^2}y\right) + \dfrac{\widetilde{\lambda}_0}{\widetilde{\lambda}_0^{(a)}}\right] & y \in (y_1^*, \infty) \\ \quad + p_1^1\left[\widehat{f}\Psi\left(\dfrac{\widetilde{\lambda}_1^{(a)}}{\kappa}, \dfrac{2\kappa\beta}{\gamma^2}, \dfrac{2\kappa}{\gamma^2}y\right) + \dfrac{\widetilde{\lambda}_1}{\widetilde{\lambda}_1^{(a)}}\right], & \end{cases}$$

where Φ and Ψ denote respectively the confluent hypergeometric function of first and second kind and $(y_0^*, y_1^*, \widehat{c}, \widehat{d}, \widehat{e}, \widehat{f})$ are such that $u(0,y)$ and $u(1,y)$ belong to $C^1(\mathbb{R}^+)$.

A Proof of comparison principle

Proof of lemma 3.5: In order to prove the comparison principle, it suffices to show that for all $\gamma \in (0,1)$:

$$\max_{i\in\{0,\dots,m\}} \sup_{\mathbb{R}\times\mathbb{R}^+} (u_i - w_i^\gamma) \leq 0,$$

since the required result is obtained by letting γ to 0.

We argue by contradiction and suppose that there exist some $\gamma \in (0,1)$ and $i \in \{0,\dots,m\}$, s.t.

$$\theta := \max_{j\in\{0,\dots,m\}} \sup_{\mathbb{R}\times\mathbb{R}^+} (u_j - w_j^\gamma) = \sup_{\mathbb{R}\times\mathbb{R}^+} (u_i - w_i^\gamma) > 0. \tag{A.39}$$

Let $z = (x,y)$. Notice that $u_i(z) - w_i^\gamma(z)$ goes to $-\infty$ when $|z|$ goes to infinity, as pointed out in Remark 3.2. We also have $\lim_{y\downarrow 0} u_i(x,y) - \lim_{y\downarrow 0} w_i^\gamma(x,y) \leq 0$ by assumption. Hence, by continuity of the functions u_i and w_i^γ, there exists $z_0 \in \mathbb{R}\times(0,\infty)$ s.t.

$$\theta = u_i(z_0) - w_i^\gamma(z_0).$$

For any $\varepsilon > 0$, we consider the functions

$$\begin{aligned} \Phi_\varepsilon(z,z') &= u_i(z) - w_i^\gamma(z') - \phi_\varepsilon(z,z'), \\ \phi_\varepsilon(z,z') &= \frac{1}{4}|z-z_0|^4 + \frac{1}{2\varepsilon}|z-z'|^2, \end{aligned}$$

for all $z, z' \in \mathbb{R}\times(0,\infty)$. By standard arguments of comparison principles, the function Φ_ε attains a maximum in $(z_\varepsilon, z'_\varepsilon) \in (\mathbb{R}\times(0,\infty))^2$, which converges (up to a subsequence) to (z_0,z_0) when ε goes to zero. Moreover,

$$\lim_{\varepsilon\to 0} \frac{|z_\varepsilon - z'_\varepsilon|^2}{\varepsilon} = 0. \tag{A.40}$$

Applying Theorem 3.2 of [8], we obtain the existence of 2×2 matrices $M_\varepsilon = (M_{\varepsilon jl})_{1\leq j,l\leq 2}$, $M'_\varepsilon = (M'_{\varepsilon jl})_{1\leq j,l\leq 2}$ such that:

$$\begin{aligned} (p_\varepsilon, M_\varepsilon) &\in J^{2,+}u_i(z_\varepsilon), \\ (p'_\varepsilon, M'_\varepsilon) &\in J^{2,-}w_i^\gamma(z'_\varepsilon), \end{aligned}$$

and

$$\begin{pmatrix} M_\varepsilon & 0 \\ 0 & -M'_\varepsilon \end{pmatrix} \leq D^2_{z,z'}\phi_\varepsilon(z_\varepsilon, z'_\varepsilon) + \varepsilon\left(D^2_{z,z'}\phi_\varepsilon(z_\varepsilon, z'_\varepsilon)\right)^2, \tag{A.41}$$

where

$$\begin{aligned} p_\varepsilon = (p_{\varepsilon j})_{1\leq j\leq 2} &= D_z\phi_\varepsilon(z_\varepsilon, z'_\varepsilon), \\ p'_\varepsilon = (p'_{\varepsilon j})_{1\leq j\leq 2} &= -D_{z'}\phi_\varepsilon(z_\varepsilon, z'_\varepsilon). \end{aligned}$$

By writing the viscosity subsolution property of u_i and the strict viscosity supersolution property (3.25) of w_i^γ, we have the following inequalities:

$$\min\Big[-p_{\varepsilon 1}\mu(x_\varepsilon) - p_{\varepsilon 2}\alpha(y_\varepsilon) - \frac{1}{2}\sigma^2(x_\varepsilon)M_{\varepsilon 11} - \rho\gamma(y_\varepsilon)\sigma(x_\varepsilon)M_{\varepsilon 12} - \frac{1}{2}\gamma^2(y_\varepsilon)M_{\varepsilon 22} - \mathcal{G}_i u(.,x_\varepsilon,y_\varepsilon) - \mathcal{J}_i u(i,x_\varepsilon,y_\varepsilon)\,,\, u_i(x_\varepsilon,y_\varepsilon) - U(e^{x_\varepsilon}f(y_\varepsilon))\Big] \le 0, \quad \text{(A.42)}$$

$$\min\Big[-p'_{\varepsilon 1}\mu(x'_\varepsilon) - p'_{\varepsilon 2}\alpha(y'_\varepsilon) - \frac{1}{2}\sigma^2(x'_\varepsilon)M'_{\varepsilon 11} - \rho\gamma(y'_\varepsilon)\sigma(x'_\varepsilon)M'_{\varepsilon 12} - \frac{1}{2}\gamma^2(y'_\varepsilon)M'_{\varepsilon 22} - \mathcal{G}_i w^\gamma(.,x'_\varepsilon,y'_\varepsilon) - \mathcal{J}_i w^\gamma(i,x'_\varepsilon,y'_\varepsilon)\,,\, w_i^\gamma(x'_\varepsilon,y'_\varepsilon) - U(e^{x'_\varepsilon}f(y'_\varepsilon))\Big] \ge \delta. \quad \text{(A.43)}$$

We then distinguish the following two cases :

$\star$ *Case 1* : $u_i(x_\varepsilon,y_\varepsilon) - U(e^{x_\varepsilon}f(y_\varepsilon)) \le 0$ in (A.42).
From the continuity of u_i and by sending $\varepsilon \to 0$, this implies

$$u_i(x_0,y_0) - U(e^{x_0}f(y_0)) \le 0. \quad \text{(A.44)}$$

On the other hand, from (A.43), we also have

$$w_i^\gamma(x'_\varepsilon,y'_\varepsilon) - U(e^{x'_\varepsilon}f(y'_\varepsilon)) \ge \delta,$$

which implies, by sending $\varepsilon \to 0$ and using the continuity of w_i :

$$w_i^\gamma(x_0,y_0) - U(e^{x_0}f(y_0)) \ge \delta. \quad \text{(A.45)}$$

Combining (A.44) and (A.45), we obtain

$$\theta = u_i(z_0) - w_i^\gamma(z_0) \quad \le \quad -\delta,$$

which is a contradiction.
$\star$ *Case 2* : $\big[-p_{\varepsilon 1}\mu(x_\varepsilon) - p_{\varepsilon 2}\alpha(y_\varepsilon) - \frac{1}{2}\sigma^2(x_\varepsilon)M_{\varepsilon 11} - \rho\gamma(y_\varepsilon)\sigma(x_\varepsilon)M_{\varepsilon 12} - \frac{1}{2}\gamma^2(y_\varepsilon)M_{\varepsilon 22} - \mathcal{G}_i u(.,x_\varepsilon,y_\varepsilon) - \mathcal{J}_i u(i,x_\varepsilon,y_\varepsilon)\big] \le 0$ in (A.42)

From (A.43), we have

$$\Big[-p'_{\varepsilon 1}\mu(x'_\varepsilon) - p'_{\varepsilon 2}\alpha(y'_\varepsilon) - \frac{1}{2}\sigma^2(x'_\varepsilon)M'_{\varepsilon 11} - \rho\gamma(y'_\varepsilon)\sigma(x'_\varepsilon)M'_{\varepsilon 12} - \frac{1}{2}\gamma^2(y'_\varepsilon)M'_{\varepsilon 22} - \mathcal{G}_i w^\gamma(.,x'_\varepsilon,y'_\varepsilon) - \mathcal{J}_i w^\gamma(i,x'_\varepsilon,y'_\varepsilon)\Big] \ge \delta.$$

Combining the two above inequalities and using relation (A.41) and the continuity of u_i and w_i^γ, we obtain the required contradiction : $\delta \le 0$. This ends the proof. □

B Confluent Hypergeometric Functions

In this appendix, we discuss the solution of the following class of ordinary differential equation:

$$yf''(y) + a(b-y)f'(y) + cf(y) + l(y) = 0\,. \quad \text{(B.46)}$$

We refer mainly to [21] for more complete details in the resolution of this type of equations. We start analyzing the associated homogeneous ED

$$y f_0''(y) + a(b-y) f_0'(y) + c f_0(y) = 0\,.$$

Let $J(A,C,y)$ be the solution of the confluent hypergeometric differential equation

$$y J''(A,C,y) + (C-y) J'(A,C,y) - A J(A,C,y) = 0\,,$$

then, it is easy to verify that

$$f_0(y) = J\Big(-\frac{c}{a}, ab, ay\Big)\,.$$

In the rest of this appendix, we will assume that $a,b,c \neq 0$ and $\frac{c}{a} \notin \mathbb{N}$. In the other cases, the solution is either polynomial and exponential functions or a linear combination of confluent hypergeometric functions of first kind. A direct application of the separation of variable method gives the following solution

$$f(y) = k f_0(y) + \mathcal{I}(a,b,c,l(\cdot))(y) \tag{B.47}$$

where

$$\begin{aligned} h_0(y) &= exp\left\{\int^y \frac{a(b-z)f_0(z) + 2z f_0'(z)}{z f_0(z)} dz\right\} \\ h_1(y) &= -\int^y \frac{l(z)}{z h_0(z)} dz \\ \mathcal{I}(a,b,c,l(\cdot))(y) &= \int^y (k_0 + h_1(z))\, h_0(z) dz \end{aligned}$$

with k and k_0 constant.

$$f(y) = J\Big(-\frac{c}{a}, ab, ay\Big) - \frac{d}{c}\,. \tag{B.48}$$

The general solution J of the confluent hypergeometric differential equation is generally written as a linear combination of the Kummer function Φ and the Tricomi function Ψ.

We summarize here some properties of functions Φ and Ψ:

$$\begin{aligned} \Phi'(A,C,y) &= \frac{A}{C}\Phi(A+1,C+1,y) \\ \Psi'(A,C,y) &= -A\Psi(A+1,C+1,y) \\ \lim_{y\to 0} \Phi(A,C,y) &= 1 \\ \lim_{y\to 0} \Psi(A,C,y) &\simeq \begin{cases} \frac{\Gamma(1-C)}{\Gamma(A-C+1)} & \text{if } C<1 \\ \\ \frac{\Gamma(C-1)}{\Gamma(A)} y^{1-C} & \text{if } C>1 \end{cases} \\ \lim_{y\to\infty} \Phi(A,C,y) &\quad \text{does not exist} \\ \lim_{y\to\infty} \Psi(A,C,y) &\simeq y^{-A} \end{aligned}$$

B.1 System of equations verified by the parameters of the value function

The following system of equations provides us with the values $(y_0^*, y_1^*, \widehat{c}, \widehat{d}, \widehat{e}, \widehat{f})$ needed to define completely the value function in Proposition 4.8.

The first equation is derived from the continuity of $w(0,y)$ at y_0^*:

$$\begin{aligned} 0 &= \frac{\kappa+\lambda_0}{\kappa+\lambda_0+\vartheta_{0,1}} y_0^* + \widehat{c}\Phi\left(\frac{\lambda_0+\vartheta_{0,1}}{\kappa}, \frac{2\kappa\beta}{\gamma^2}, \frac{2\kappa}{\gamma^2} y_0^*\right) \\ &\quad + \widehat{d}\Psi\left(\frac{\lambda_0+\vartheta_{0,1}}{\kappa}, \frac{2\kappa\beta}{\gamma^2}, \frac{2\kappa}{\gamma^2} y_0^*\right) + \frac{\mu - \kappa\beta\frac{\vartheta_{0,1}}{\kappa+\lambda_0+\vartheta_{0,1}}}{\lambda_0+\vartheta_{0,1}}. \end{aligned} \tag{B.49a}$$

The second equation is obtained from the continuity of the derivative of $w(0,y)$ at y_0^*:

$$\begin{aligned} 0 &= \frac{\kappa+\lambda_0}{\kappa+\lambda_0+\vartheta_{0,1}} + \widehat{c}\frac{\lambda_0+\vartheta_{0,1}}{\kappa\beta}\Phi\left(\frac{\lambda_0+\vartheta_{0,1}}{\kappa}+1, \frac{2\kappa\beta}{\gamma^2}+1, \frac{2\kappa}{\gamma^2} y_0^*\right) \\ &\quad - 2\widehat{d}\frac{\lambda_0+\vartheta_{0,1}}{\gamma^2}\Psi\left(\frac{\lambda_0+\vartheta_{0,1}}{\kappa}+1, \frac{2\kappa\beta}{\gamma^2}+1, \frac{2\kappa}{\gamma^2} y_0^*\right). \end{aligned} \tag{B.49b}$$

The third equation is obtained from the continuity of $w(0,y)$ at y_1^*:

$$\begin{aligned} 0 &= \frac{\vartheta_{0,1}}{\kappa+\lambda_0+\vartheta_{0,1}} y_1^* - \widehat{c}\Phi\left(\frac{\lambda_0+\vartheta_{0,1}}{\kappa}, \frac{2\kappa\beta}{\gamma^2}, \frac{2\kappa}{\gamma^2} y_1^*\right) \\ &\quad - \widehat{d}\Psi\left(\frac{\lambda_0+\vartheta_{0,1}}{\kappa}, \frac{2\kappa\beta}{\gamma^2}, \frac{2\kappa}{\gamma^2} y_1^*\right) - \frac{\mu - \kappa\beta\frac{\vartheta_{0,1}}{\kappa+\lambda_0+\vartheta_{0,1}}}{\lambda_0+\vartheta_{0,1}} \\ &\quad + p_0^0\left[\widehat{e}\Psi\left(\frac{\widetilde{\lambda}_0}{\kappa}, \frac{2\kappa\beta}{\gamma^2}, \frac{2\kappa}{\gamma^2} y_1^*\right) + \frac{\mu}{\widetilde{\lambda}_0}\right] + p_1^0\left[\widehat{f}\Psi\left(\frac{\widetilde{\lambda}_1}{\kappa}, \frac{2\kappa\beta}{\gamma^2}, \frac{2\kappa}{\gamma^2} y_1^*\right) + \frac{\mu}{\widetilde{\lambda}_1}\right]. \end{aligned} \tag{B.49c}$$

The fourth equation is obtained from the continuity of the derivative of $w(0,y)$ at y_1^*:

$$\begin{aligned} 0 &= \frac{\vartheta_{0,1}}{\kappa+\lambda_0+\vartheta_{0,1}} - \widehat{c}\frac{\lambda_0+\vartheta_{0,1}}{\kappa\beta}\Phi\left(\frac{\lambda_0+\vartheta_{0,1}}{\kappa}+1, \frac{2\kappa\beta}{\gamma^2}+1, \frac{2\kappa}{\gamma^2} y_1^*\right) \\ &\quad + 2\widehat{d}\frac{\lambda_0+\vartheta_{0,1}}{\gamma^2}\Psi\left(\frac{\lambda_0+\vartheta_{0,1}}{\kappa}+1, \frac{2\kappa\beta}{\gamma^2}+1, \frac{2\kappa}{\gamma^2} y_1^*\right) \\ &\quad - 2p_0^0\widehat{e}\frac{\widetilde{\lambda}_0}{\gamma^2}\Psi\left(\frac{\widetilde{\lambda}_0}{\kappa}+1, \frac{2\kappa\beta}{\gamma^2}+1, \frac{2\kappa}{\gamma^2} y_1^*\right) \\ &\quad - 2p_1^0\widehat{f}\frac{\widetilde{\lambda}_1}{\gamma^2}\Psi\left(\frac{\widetilde{\lambda}_1}{\kappa}+1, \frac{2\kappa\beta}{\gamma^2}+1, \frac{2\kappa}{\gamma^2} y_1^*\right). \end{aligned} \tag{B.49d}$$

The fifth equation gives the continuity of $w(1,y)$ at y_1^*:

$$\begin{aligned} 0 &= y_1^* + p_0^1\left[\widehat{e}\Psi\left(\frac{\widetilde{\lambda}_0}{\kappa}, \frac{2\kappa\beta}{\gamma^2}, \frac{2\kappa}{\gamma^2} y_1^*\right) + \frac{\mu}{\widetilde{\lambda}_0}\right] \\ &\quad + p_1^1\left[\widehat{f}\Psi\left(\frac{\widetilde{\lambda}_1}{\kappa}, \frac{2\kappa\beta}{\gamma^2}, \frac{2\kappa}{\gamma^2} y_1^*\right) + \frac{\mu}{\widetilde{\lambda}_1}\right]. \end{aligned} \tag{B.49e}$$

The sixth equation gives the continuity of the derivative of $w(1,y)$ at y_1^*:

$$\begin{aligned} 1 &= 2p_0^1\widehat{e}\frac{\widetilde{\lambda}_0}{\gamma^2}\Psi\left(\frac{\widetilde{\lambda}_0}{\kappa}+1, \frac{2\kappa\beta}{\gamma^2}+1, \frac{2\kappa}{\gamma^2} y_1^*\right) \\ &\quad + 2p_1^1\widehat{f}\frac{\widetilde{\lambda}_1}{\gamma^2}\Psi\left(\frac{\widetilde{\lambda}_1}{\kappa}+1, \frac{2\kappa\beta}{\gamma^2}+1, \frac{2\kappa}{\gamma^2} y_1^*\right). \end{aligned} \tag{B.49f}$$

References

[1] A. Alfonsi, A. Fruth and A. Schied (2010): "Optimal execution strategies in limit order books with general shape functions", *Quantitative Finance*, **10**, 143-157.

[2] M. Avellaneda and S. Stoikov (2008) : "High-frequency trading in a limit order book", *Quantitative Finance*, **8**, 217-224.

[3] K. Back (1992) : "Insider trading in continuous time", *Review of Financial Studies*, **5**, 387-409.

[4] P. Bank and D. Baum (2004): "Hedging and portfolio optimization in illiquid financial markets with a large trader", *Mathematical Finance*, **14**, 1-18.

[5] E. Bayraktar and M. Ludkovski (2012) : "Liquidation in Limit Order Books with Controlled Intensity", to appear in *Mathematical Finance*.

[6] B. Bouchard, N. M. Dang and C.A. Lehalle (2011) : "Optimal control of trading algorithms: a general impulse control approach", SIAM Journal on Financial Mathematics, 2011, 4, 404-438.

[7] U. Cetin, R. Jarrow and P. Protter (2004): "Liquidity risk and arbitrage pricing theory", *Finance and Stochastics*, **8**, 311-341.

[8] M. Crandall, Ishii H. and P.L. Lions (1992): "User's guide to viscosity solutions of second order partial differential equations", *Bull. Amer. Math. Soc.*, **27**, 1-67.

[9] E. B. Dynkin (1965), Markov Processes, Vol. II, Springer Verlag.

[10] N. C. Framstad, B. Oksendal, and A. Sulem (2001): "Optimal consumption and portfolio in a jump diffusion market with proportional transaction costs", *Journal of Mathematical Economics*, **35**, 233-257.

[11] O. Guéant, C. Lehalle, and J. Tapia (2011): "Optimal portfolio liquidation with limit orders", *SIAM J. Finan. Math.*, **3**, 740-764.

[12] F. Guilbaud and H. Pham (2013): "Optimal high frequency trading with limit and market orders", *Quantitative Finance*, **13**, 79-94.

[13] R. Korn (1998): "Portfolio optimization with strictly positive transaction costs and impulse control", *Finance and Stochastics*, **2**, 85-114.

[14] A. Kyle (1985): "Continuous actions and insider trading", *Econometrica*, **53**, 1315-1335.

[15] V. Ly Vath, M. Mnif and H. Pham (2007): "A Model of Optimal Portfolio Selection under Liquidity Risk and Price Impact", *Finance and Stochastics*, **11**, 51-90.

[16] V. Ly Vath, H. Pham and S. Villeneuve (2008): "A mixed singular/switching control problem for a dividend policy with reversible technology investment", *Annals of Applied Probability*, **18**, 1164-1200.

[17] A. Obizhaeva and J. Wang (2013): "Optimal Trading Strategy and Supply/Demand Dynamics", *Journal of Financial Markets*, **16**, 1-32.

[18] B. Oksendal (2003) Stochastic Differential Equations, 6th ed., Springer, New York.

[19] A. Roch (2011): "Liquidity risk, price impacts and the replication problem", Finance and Stochastics, **15**, 399-419.

[20] S. E. Shreve and H. M. Soner (1994): "Optimal investment and consumption with transaction costs" *The Annals of Applied Probability* **4**, 609-692.

[21] F. G. Tricomi (1960): Fonctions hypergéométriques confluentes, Gauthier-Villars, Paris, France.

[22] S. Villeneuve (2007): "On the Threshold Strategies ans Smooth-Fit Principle for Optimal Stopping Problems", *Journal of Applied Probability*, **44**, 181-198.