Optimal market making strategies under inventory constraints *

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Abstract

We study the problem of a market-maker acting as a liquidity provider by continuously setting bid and ask prices for an illiquid asset. We assume that the market maker has a contractual obligation to permanently quote bid and ask prices for the security and therefore to satisfy any sell or buy order from the asset's investors. On the opposite side of the trades, there are investors who act as liquidity takers by submitting either sell or buy market orders. The arrival of buy (and sell) market orders submitted by the investors is assumed to follow a Cox process with regime-shifting Markov intensity.

The role of the market maker is very important in the trading of illiquid assets as it acts as a facilitator of trades between different investors. The market maker may therefore benefit from the bid-ask spread but faces a number of constraints, in particular the liquidity and inventory constraints. The objective is to maximize the expected utility of the market maker's terminal wealth. We characterize our objective functions as unique viscosity solutions to the associated HJB system. We further enrich our study with some numerical results.

Keywords: stochastic control, dealer market, market maker, inventory risk, liquidity regime, viscosity solutions.

MSC2000 subject classification: 60G40, 91B70, 93E20.

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1 Introduction

We consider a financial market with a single dealer or market maker acting as a liquidity provider by continuously setting bid and ask prices for an illiquid asset. In most studies on financial markets, including most of the recent studies on liquidity risk, it is always assumed that investors are price-takers, i.e. liquidity takers, in the sense that they trade any financial asset at the available prices with a liquidity premium that must be paid for immediacy. It is clear from the structure of financial markets that, in addition to the presence of price-takers, there must necessarily exist market participants who are price-setters or liquidity providers. For instance, in limit order book markets, traders can post prices and quantities at which they are willing to buy or sell while waiting for a counterparty to engage in that trade. In dealer markets, a market-maker or specialist quotes bid and offer and serves as the intermediary between public traders. More precisely, the market-maker assumes the role of a counterparty when an investor wishes to buy or sell the securities.

Studying market making problems corresponds to considering an enlarged set of trading strategies by including limit orders. We may refer to [2], [3], and [7] where the authors study liquidation problems with limit orders. Some other papers consider optimal market making problem under a limit order book market. One such study is the market-making model of Guilbaud and Pham [9], where the authors consider the problem of a market maker whose objective is to maximize her expected utility from revenue over a short term horizon by submitting both limit and market orders. As mentioned above, in these models, the studies were carried under the framework of a limit order book market whereas our market making problem is solved in a dealer market. The structural constraints imposed upon a market maker in dealer markets are proved to be a major challenge. One such major constraint is the obligation of the market-maker to quote bid and ask prices and to act as a counterparty whenever an investor wishes to buy (to sell) at the ask price (at the bid price).

In this paper, we consider a financial market with a single dealer or market maker trading a single financial assets. Within this market, we study the problem of the market-maker acting as a liquidity provider by setting bid and ask prices for the risky asset. In conformity with a dealer market as mentioned above, we assume that the market maker has a contractual obligation to permanently quote bid and ask prices for this security and therefore to satisfy any sell and buy order from investors. On the opposite side of the trades, there are investors who act as liquidity takers, i.e. price-takers, by submitting either sell or buy market orders.

The role of the market maker is very important in the trading of illiquid assets as she acts as a facilitator of trades between different investors. The market maker may therefore benefit from the bid-ask spread but faces a number of constraints, in particular the liquidity and inventory constraints. Indeed, the obligation imposed upon the market maker to meet investors orders may make the position of the market maker very risky. For instance, when the market maker has to buy stocks successively due to investors' sell market orders, her stock holding position may become very large and positive, which is very risky in the event of a downturn of the market.

In the study of this market making/dealing problem, we may refer to [11], [8] and [12].

In these papers, the authors consider a market making problem as described above but within a financial market in which the risky asset has a reference price or a fair price S_t, which is assumed to follow an arithmetic brownian process. The market maker quotes her ask and bid prices as respectively $S_t + \delta_t^a$ and $S_t - \delta_t^b$, where (δ_t^a, δ_t^b) are both positive and represent the strategy control of the market maker. The price processes, ie. bid, ask or mid prices, are therefore mainly driven by the reference price process.

In our study, the market maker has the obligation to quote bid and ask prices, but we do not assume the existence of a reference price. The prices are therefore uniquely driven by the equilibrium between buy and sell market orders. An imbalance between buy and sell market orders, for instance, in the case when the arrivals of buy orders largely exceed those of the sell orders, is expected to move down the bid and ask prices quoted by the market maker. In our study, the arrival of buy (and sell) market orders submitted by the investors is assumed to follow a Cox process with a regime-shifting Markov intensity. One may expect the regime-shifting in intensity to strongly impact the trend of the bid and ask price processes. The objective is to maximize the expected utility of the market maker's terminal wealth. We characterize our objective functions as the unique viscosity solutions to the associated system of HJB equations. In the proof of the comparison theorem, a major problem is to circumvent the difficulty arising from the discontinuity of our HJB operator on some parts of the solvency region. One way to tackle this difficulty is to build specific test functions allowing us to prove the uniqueness by contradiction.

The rest of the paper is organized as follows. We define the model and formulate our optimal market making problem in the following section. In Sections 3 and 4, we obtain some analytical properties and prove the dynamic programming principle related to our control problem. These results enable us to obtain the characterization of the solution of the problem in terms of the unique viscosity solution to the associated system of HJB equations. Finally, in Section 5, we further enrich our study with numerical illustrations.

2 Problem formulation

Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with a right continuous filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ where T is a finite horizon. We assume that $\mathcal{F}_0$ contains all the P-null sets of $\mathcal{F}$. We consider a financial market operated as a single dealer market, in which there is a risky asset. In this dealer market, there is a market maker who has the obligation to permanently quote bid and ask prices and to act as a counterparty to investors' market orders. We equally assume that investors, considered as price-takers, may only submit either buy or sell market orders.

Trading orders. We denote by $(\theta_i^a)_{i \geq 1}$ (resp. $(\theta_i^b)_{i \geq 1}$) the sequence of non-decreasing $\mathbb{F}$-stopping times corresponding to the arrivals of buy (resp. sell) market orders. From the market maker's point of view, both sequences of stopping times correspond to trading times, i.e. the times when she has to act as a counterparty to investor's market orders. We denote by $(\xi_i)_{i \geq 1}$ the sequence of these trading times, which is the combined of sequences $(\theta_i^a)_{i \geq 1}$ and $(\theta_i^b)_{i \geq 1}$. When a buy (resp. sell) market order arrives at time θ_i^a (resp. θ_j^b), the market

maker has to sell (resp. buy) a unit of shares of the risky asset at the ask (resp. bid) price denoted by P^a (resp. P^b). We assume here that transactions are of constant size, scaled to 1 unit.

Market making strategies. We define a strategy control as being a $\mathbb{F}$-predictable process $\alpha = (\alpha_t)_{(0\leq t\leq T)} = (\epsilon^a_t, \epsilon^b_t, \eta^a_t, \eta^b_t)_{0\leq t\leq T}$ where the processes $\epsilon^a, \epsilon^b, \eta^a, \eta^b$ take values in $\{0, 1\}$.

For simplicity reason, we assume that the market maker may move the bid and ask prices by only one tick. More precisely, we assume that when a sell market order arrives at time θ^b_j, the market maker may either keep the bid and ask prices constant or decrease one or both of them by one tick δ, where δ is a strictly positive constant. On the opposite side, when a buy market order arrives at time θ^a_k, the market maker may either keep the bid and ask prices constant or increase one or both of them by one tick δ.

Bid-Ask spread modelling.
We denote by $P^a = (P^a_t)_{0\leq t\leq T}$ (resp. $P^b = (P^b_t)_{0\leq t\leq T}$) the price quoted by the market maker to buyers (resp. sellers). Notice that $P^a \geq P^b$.
The dynamics of $P^{a,b}$ evolves according to the following equations

$$\begin{aligned} dP^{a,b}_t &= 0,\ \xi_i < t < \xi_{i+1} \\ P^{a,b}_{\theta^b_{j+1}} &= P^{a,b}_{\theta^{b-}_{j+1}} - \delta\epsilon^{a,b}_{\theta^b_{j+1}} \\ P^{a,b}_{\theta^a_{k+1}} &= P^{a,b}_{\theta^{a-}_{k+1}} + \delta\eta^{a,b}_{\theta^a_{k+1}}. \end{aligned}$$

where i is the number of transactions before time t, j the number of buy transactions before time t for the market maker, k the number of sell transactions before time t, and δ represents one tick.

Stock holdings. The number of (units of) shares held by the market maker at time $t \in [0, T]$ is denoted by Y_t, and Y satisfies the following equations

$$dY_t = 0,\ \xi_i < t < \xi_{i+1} \tag{2.1}$$
$$Y_{\theta^b_{j+1}} = Y_{\theta^{b-}_{j+1}} + 1 \tag{2.2}$$
$$Y_{\theta^a_{k+1}} = Y_{\theta^{a-}_{k+1}} - 1, \tag{2.3}$$

Let $y_{min} < 0 < y_{max}$. We impose the following inventory constraint

$$y_{min} \leq Y_t \leq y_{max} \text{ a.s. } 0 \leq t \leq T. \tag{2.4}$$

This constraint is necessary since the number of stock shares of any asset is realistically finite.

Cash holdings. We denote by $r > 0$ the instantaneous interest rate. The bank account follows the below equation between two trading times

$$dX_t = rX_t dt,\ \xi_i < t < \xi_{i+1}. \tag{2.5}$$

When a discrete trading occurs at time θ^b_{j+1} (resp. θ^a_{k+1}), the cash amount becomes

$$X_{\theta^b_{j+1}} = X_{\theta^{b-}_{j+1}} - P^b_{\theta^{b-}_{j+1}} \tag{2.6}$$
$$X_{\theta^a_{k+1}} = X_{\theta^{a-}_{k+1}} + P^a_{\theta^{a-}_{k+1}}, \tag{2.7}$$

State process. We define the state process as follows:

$$Z = (X,\ Y,\ P := \frac{P^a + P^b}{2},\ S := P^a - P^b), \tag{2.8}$$

where P represents the mid-price and S the bid-ask spread of the stocks.
The dynamics of the process (P, S) is given by

$$\begin{aligned} dP_t &= 0,\ \xi_i < t < \xi_{i+1} && (2.9)\\ P_{\theta^b_{j+1}} &= P_{\theta^{b-}_{j+1}} - \frac{\delta}{2}(\epsilon^a_{\theta^b_{j+1}} + \epsilon^b_{\theta^b_{j+1}}) && (2.10)\\ P_{\theta^a_{k+1}} &= P_{\theta^{a-}_{k+1}} + \frac{\delta}{2}(\eta^a_{\theta^a_{k+1}} + \eta^b_{\theta^a_{k+1}}), && (2.11) \end{aligned}$$

$$\begin{aligned} dS_t &= 0,\ \xi_i < t < \xi_{i+1} && (2.12)\\ S_{\theta^b_{j+1}} &= S_{\theta^{b-}_{j+1}} - \delta(\epsilon^a_{\theta^b_{j+1}} - \epsilon^b_{\theta^b_{j+1}}) && (2.13)\\ S_{\theta^a_{k+1}} &= S_{\theta^{a-}_{k+1}} + \delta(\eta^a_{\theta^a_{k+1}} - \eta^b_{\theta^a_{k+1}}). && (2.14) \end{aligned}$$

Regime switching. We first consider the tick time clock associated to a Poisson process (N_t) with deterministic intensity λ, and representing the random times where the intensity of the orders arrival jumps.
We define a discrete-time stationary Markov chain $(\hat{I}_k)_{k\in\mathbb{N}}$, valued in the finite state space $\{1, ..., m\}$, with probability transition matrix $(p_{ij})_{1\le i,j\le m}$, i.e. $\mathbb{P}[\hat{I}_{k+1} = j|\hat{I}_k = i] = p_{ij}$ s.t. $p_{ii} = 0$, independent of N. We define the process

$$I_t = \hat{I}_{N_t},\ t \ge 0 \tag{2.15}$$

$(I_t)_t$ is a continuous time Markov chain with intensity matrix $\Gamma = (\gamma_{ij})_{1\le i,j\le m}$, where $\gamma_{ij} = \lambda p_{ij}$ for $i \neq j$, and $\gamma_{ii} = -\sum_{j\neq i} \gamma_{ij}$.
We model the arrivals of buy and sell market orders by two Cox processes N^a and N^b. The intensity rate of N^a_t and N^b_t is given respectively by $\lambda^a(I_t, P_t, S_t)$ and $\lambda^b(I_t, P_t, S_t)$ where λ^a and λ^b are deterministic functions valued in $\mathbb{R}$ and defined on $\{1, ..., m\} \times \frac{\delta}{2}\mathbb{N} \times \delta\mathbb{N}$. We assume that:

$$\bar{\lambda} := \sup_{\{1,...,m\}\times\frac{\delta}{2}\mathbb{N}\times\delta\mathbb{N}} \Big(\max(\lambda^a, \lambda^b, \lambda)\Big) < +\infty.$$

We now define θ^a_k (resp. θ^b_k) as the k^{th} jump time of N^a (resp. N^b), which corresponds to the k^{th} buy (sell) market order.
We introduce the following stopping times $\rho_j(t) = \inf\{u \ge t, I_u = j\}$ and $\rho(t) = \inf\{u \ge t, N_u > N_t\}$ for $0 \le t \le T$ and the notation $Z^{t,i,z,\alpha}$ is the state process associated to the control α such that $(I_t,\ Z^{t,i,z,\alpha}_t) = (i, z)$.

Cost of liquidation of the portfolio. If the current market price at time $t < T$ is p and the market maker decides to liquidate her portfolio, then we assume that the price she actually gets is

$$Q(t, y, p, s) = (p - \text{sign}(y)\frac{s}{2})f(t, y),$$

where f is an impact function defined from $[0,T] \times \mathbb{R}$ into $\mathbb{R}_+$.

We make the following assumption

Assumption (H1) *The impact function f is non-negative, non-increasing in y, and satisfies the following conditions*

$$\begin{aligned} f(t,y) &\leq f(t,y^{'}) \text{ if } y^{'} \leq y \\ yf(t^{'},y) &\leq yf(t,y) \text{ if } t^{'} \leq t. \end{aligned}$$

Remark 2.1 (H1) suggests that the further from maturity when the market maker liquidates her block of shares, the more she is penalized. In addition, the bigger the block of shares to liquidate, the more she is penalized. This form of impact function is consistent with both the asymmetric information and inventory motives in the market microstructure literature (see for instance [13], [5]).

Liquidation value and Solvency constraints. A key issue for the market maker is to maximize the value of the net wealth at time T. In our framework, we impose a constraint on the spread i.e.

$$0 < S_t \leq K\delta, \ 0 \leq t \leq T,$$

where K is a positive constant and $K\delta$ represents the maximum spread. This constraint is consistent with the idea to insure a good level of liquidity in the markets. This constraint is generally part of the commitments that market maker has taken in her contract with the financial Exchange. We also impose that the bid price remains positive, therefore the market maker has to use controls such that

$$P_t - S_t/2 > 0.$$

When the market maker has to liquidate her portfolio at time t, her wealth will be $L(t, X_t, Y_t, P_t, S_t)$ where L is the liquidation function defined as follows

$$L(t,x,y,p,s) = x + yQ(t,y,p,s).$$

We equally assume the following solvency constraint: $x \geq x_{min}$. Let $x_{min} < 0$. We assume that in the case that the cash held by the market maker falls under x_{min}, she has to liquidate her position. This constraint on x_{min} is a solvency constraint generally imposed by the market maker's employer since they do not have unlimited financing facilities. From the market maker's point of view, x_{min} is the threshold below which she shall not go. Otherwise, she might be replaced in the market making of these securities or her firm might terminate the current market making contract with the Exchange.
We may now introduce the following state space

$$\mathbb{S} := (x_{min}, +\infty) \times \{y_{min}, ..., y_{max}\} \times \frac{\delta}{2}\mathbb{N} \times \delta\{1, ..., K\}.$$

and then the solvency region

$$\mathcal{S} := \{(t,x,y,p,s) \in [0,T] \times \mathbb{S} : \; p - \frac{s}{2} \geq \delta\}.$$

We denote its boundary and its closure by

$$\partial_x \mathcal{S} \;\; := \;\; \Big\{(t,x,y,p,s) \in [0,T] \times \bar{\mathbb{S}} : \; x = x_{min}\Big\} \quad \text{and} \quad \bar{\mathcal{S}} \;=\; \mathcal{S} \cup \partial_x \mathcal{S}.$$

Admissible trading strategies. Given $(t,z) := (t,x,y,p,s) \in \mathcal{S}$, we say that the strategy $\alpha = (\epsilon^a_u, \epsilon^b_u, \eta^a_u, \eta^b_u)_{t \leq u \leq T}$ is admissible, if the processes $\epsilon^a, \epsilon^b, \eta^a, \eta^b$ are valued in $\{0,1\}$ and for all $u \in [t,T]$, $(u, Z^{t,i,z,\alpha}_{u^-}) \in \mathcal{S}$. We denote by $\mathcal{A}(t,z)$ the set of all these admissible policies. Notice that this set depends only on (t,p,s).

Objective functions. We consider an exponential utility function U i.e. there exists $\gamma > 0$ such that $U(x) = 1 - e^{-\gamma x}$ for $x \in \mathbb{R}$. We set $U_L = U o L$ and g a non-negative penalty function defined on $\{y_{min}, ..., y_{\max}\}$. This penalty function could be either imposed on the market maker or self-imposed by the market maker herself in order to reduce the inventory risk.

The objective of the market maker is to maximize the expected utility of her wealth at maturity, i.e, when her market making contract expired. As such, we consider the following value functions $(v_i)_{i \in \{1,...,m\}}$ which are defined on $\mathcal{S}$ by

$$v_i(t,z) := \sup_{\alpha \in \mathcal{A}(t,z)} J^\alpha_i(t,z)$$

where we have set

$$\begin{aligned} J^\alpha_i(t,z) \;\; &:= \;\; \mathbb{E}^{t,i,z}\left[U_L(T \wedge \tau^{t,i,z,\alpha}, Z^{t,i,z,\alpha}_{(T \wedge \tau^{t,i,z,\alpha})^-}) - \int_t^{T \wedge \tau^{t,i,z,\alpha}} g(Y^{t,i,y,\alpha}_s) ds\right], \\ \tau^{t,i,z,\alpha} \;\; &:= \;\; \inf\{u \geq t : X^{t,i,x,\alpha}_u \leq x_{min} \text{ or } Y^{t,i,y,\alpha}_u \in \{y_{min} - 1, y_{max} + 1\}\}. \end{aligned}$$

Remark 2.2 If the market maker fixes her trading strategy α randomly according to Bernoulli law, i.e. each component of the control $\epsilon^a_{\xi_i}, \epsilon^b_{\xi_i}, \eta^a_{\xi_i}$ and $\eta^b_{\xi_i}$ follows Bernoulli law for each transaction, then under some assumptions on the parameters of the model, we may prove the convergence of the mid-price of our model to the Black-Scholes model when the tick δ goes to zero. This remark is inspired by [1].

3 Analytical properties and dynamic programming principle

We use a dynamic programming approach to derive the system of partial derivative equations satisfied by the objective functions. First, we state the following Proposition in which we obtain some bounds of our objective functions

Proposition 3.1 *There exist nonnegative constants, C_1, C_2 and C_3, depending on the parameters of our problem, such that*

$$1 - C_1 - C_2 e^{C_3 p} \leq v_i(t,x,y,p,s) \leq 1, \quad \forall (i,t,x,y,p,s) \in \{1,...,m\} \times \mathcal{S},$$

Proof: Let $i \in \{1, ..., m\}$, $(t, z) := (t, x, y, p, s) \in \mathcal{S}$ and $\alpha \in \mathcal{A}(t, z)$. As U is lower than 1 and g positive, we obviously have $v_i(t, z) \leq 1$. Moreover, if we set $G = \sup_{y \in \{y_{min}, ..., y_{max}\}} g(y)$, we get

$$v_i(t,z) \geq 1 - \inf_{\alpha \in \mathcal{A}(t,z)} \mathbb{E}\left[\exp\left(-\gamma L(T \wedge \tau^{t,i,z,\alpha}, Z^{t,i,z,\alpha}_{(T\wedge\tau^{t,i,z,\alpha})^-})\right)\right] - GT.$$

We conclude the proof by applying the following Lemma. □

Lemma 3.1 *Let $\beta > 0$. For all $(i, t, z) := (i, t, x, y, p, s) \in \{1, ..., m\} \times \mathcal{S}$, we have*

$$u_i(t,z) := \sup_{\alpha \in \mathcal{A}} \mathbb{E}\left[\exp\left(-\beta L(T \wedge \tau^{t,i,z,\alpha}, Z^{t,i,z,\alpha}_{(T\wedge\tau^{t,i,z,\alpha})^-})\right)\right] \leq \exp\left(-\beta a + \bar{\lambda} T(e^{\beta b} - 1)\right) e^{\beta b p},$$

where we have set $a = x_{min} + y_{min} f(0, y_{min}) \frac{K\delta}{2}$ and $b = -y_{min} f(0, y_{min})$.

Proof: Let $i \in \{1, ..., m\}$, $(t, z) := (t, x, y, p, s) \in \mathcal{S}$ and $\alpha \in \mathcal{A}(t, z)$. We have

$$0 \leq S^{t,i,s,\alpha}_{(T\wedge\tau^{t,i,z,\alpha})^-} \leq K\delta, \quad y_{min} \leq Y^{t,i,y,\alpha}_{(T\wedge\tau^{t,i,z,\alpha})^-} \leq y_{max} \quad \text{and} \quad x_{min} \leq X^{t,i,x,\alpha}_{(T\wedge\tau^{t,i,z,\alpha})^-} .$$

Hence, we get

$$\begin{aligned} L\left(T \wedge \tau^{t,i,z,\alpha}, Z^{t,i,x,\alpha}_{(T\wedge\tau^{t,i,z,\alpha})^-}\right) &\geq x_{min} + y_{min} f(0, y_{min})(P^{t,i,p,\alpha}_{(T\wedge\tau^{t,i,z,\alpha})^-} + \frac{K\delta}{2}) \\ &\geq a - bP^{t,i,z,\alpha}_{(T\wedge\tau^{t,i,z,\alpha})^-} , \end{aligned}$$

where $a = x_{min} + y_{min} f(0, y_{min}) \frac{K\delta}{2}$ and $b = -y_{min} f(0, y_{min}) > 0$. Moreover, it follows from the definition of P that $P^{t,i,z,\alpha}_{(T\wedge\tau^{t,i,z,\alpha})^-} \leq p + N^{a,t,i,z,\alpha}_{(T\wedge\tau^{t,i,z,\alpha})^-}$. Therefore, we obtain

$$\begin{aligned} \mathbb{E}\left[\exp\left(-\beta L(T \wedge \tau^{t,i,z,\alpha}, Z^{t,i,z,\alpha}_{(T\wedge\tau^{t,i,z,\alpha})^-})\right)\right] &\leq e^{-\beta(a-bp)} \mathbb{E}\left[\exp\left(\beta b N^{a,t,i,z,\alpha}_{(T\wedge\tau^{t,z,\alpha})^-})\right)\right] \\ &\leq e^{-\beta(a-bp)} \mathbb{E}\left[\exp\left(\beta b \bar{N}_T)\right)\right], \end{aligned}$$

where $\bar{N}$ is a Poisson process with intensity $\bar{\lambda}$. We conclude the proof by observing that $\mathbb{E}\left[\exp\left(\beta b \bar{N}_T)\right)\right] = \exp\left((\bar{\lambda} T(e^{\beta b} - 1)\right)$. □

This technical Lemma equally allows us to show the next results on the Hölder continuity of the functions J_i and v_i. We begin with the following Lemma establishing the Hölder continuity with respect to the variable x for the functions J_i^α.

Lemma 3.2 *For $\xi \in [0, -x_{min} e^{-rT})$, we set*

$$\phi(\xi) = -\frac{1}{r} \ln\left(1 - \frac{\xi e^{rT}}{\mid x_{min} \mid}\right) \quad \text{and} \quad \psi(\xi) = \sqrt{\phi(\xi)} + \phi(\xi) + \xi.$$

Let $i \in \{1, ..., m\}$, $(t, z) := (t, x, y, p, s) \in \bar{\mathcal{S}}$ and $x < x' < x - x_{min} e^{-rT}$. For all $\alpha \in \mathcal{A}(t, z)$, we have

$$\mid J_i^\alpha(t, z') - J_i^\alpha(t, z) \mid \leq K_1(p) \psi(x' - x),$$

where $K_1(p)$ is a positive constant depending only on p and $z' = (x', y, p, s)$.

Proof: Let $\alpha \in \mathcal{A}(t,z) = \mathcal{A}(t,z')$. To simplify notations, we set $\tau = \tau^{t,i,z,\alpha}$ and $\tau' = \tau^{t,i,z',\alpha}$. We first notice that $\tau \leq \tau'$, then $(Y_\tau^{t,i,z,\alpha}, P_\tau^{t,i,z,\alpha}, S_\tau^{t,i,z,\alpha}) = (Y_\tau^{t,i,z',\alpha}, P_\tau^{t,i,z',\alpha}, S_\tau^{t,i,z',\alpha})$. Therefore we get:

$$\begin{aligned}
\Delta J_i^\alpha &:= J_i^\alpha(t,z') - J_i^\alpha(t,z) \\
&= \mathbb{E}\left[U_L\left(T\wedge\tau',\ Z^{t,i,z',\alpha}_{(T\wedge\tau')^-}\right) - U_L\left(T\wedge\tau,\ Z^{t,i,z,\alpha}_{(T\wedge\tau)^-}\right) - \int_{T\wedge\tau}^{T\wedge\tau'} g(Y_u^{t,i,z,\alpha})\,du\right] \\
&\geq \mathbb{E}\left[\left(U_L\left(T\wedge\tau',\ Z^{t,i,z',\alpha}_{(T\wedge\tau')^-}\right) - U_L\left(\tau,\ Z^{t,i,z,\alpha}_{\tau^-}\right) - \int_{\tau}^{T\wedge\tau'} g(Y_u^{t,i,z,\alpha})\,du\right)\mathbb{1}_{\{\tau<T\wedge\tau'\}}\right].
\end{aligned}$$

Indeed, on $\{\tau\wedge T = \tau'\wedge T\}$, we have

$$U(L(T\wedge\tau', Z^{t,i,z',\alpha}_{(T\wedge\tau')^-})) = U((x'-x)e^{r(T\wedge\tau-t)} + L(T\wedge\tau, Z^{t,i,z,\alpha}_{(T\wedge\tau)^-})) \geq U(L(T\wedge\tau, Z^{t,i,z,\alpha}_{(T\wedge\tau)^-})).$$

Notice that on $\{\tau<\tau'\}$, we have $y_{min} \leq Y_\tau^{t,i,z,\alpha} \leq y_{max}$ and therefore $X_\tau^{t,i,z,\alpha} \leq x_{min}$.
We introduce θ the first order arrival time after τ:

$$\theta := \inf\{u>\tau:\ N_u^a > N_{u^-}^a \text{ or } N_u^b > N_{u^-}^b\}.$$

As $x_{min}e^{-r\tau\wedge T} + (x'-x)e^{-rt} \leq e^{-rt}(x_{min}e^{-r(T-t)} + x' - x) < 0$, we can also define the following stopping time, greater than τ:

$$\nu := \tau - \frac{1}{r}\ln\left(1 - \frac{(x'-x)e^{r(T-t)}}{\mid x_{min}\mid}\right).$$

As g is bounded by $G>0$ and $U\leq 1$, we get $\Delta J_i^\alpha \geq \delta_1 - \delta_2 + \delta_3 - \delta_4$ where

$$\begin{aligned}
\delta_1 &:= \mathbb{E}\left[\left(U_L\left(T\wedge\tau',\ Z^{t,i,z',\alpha}_{(T\wedge\tau')^-}\right) - U_L\left(\tau,\ Z^{t,i,z,\alpha}_{\tau^-}\right)\right)\mathbb{1}_{\{\tau<T\wedge\tau'\}}\mathbb{1}_{\{\nu<\theta\}}\right] \\
\delta_2 &:= \mathbb{E}\left[\int_\tau^{T\wedge\tau'} g(Y_u^{t,i,z,\alpha})\,du\mathbb{1}_{\{\tau<T\wedge\tau'\}}\mathbb{1}_{\{\nu<\theta\}}\right] \\
\delta_3 &:= \mathbb{E}\left[\left(U_L\left(T\wedge\tau',\ Z^{t,i,z',\alpha}_{(T\wedge\tau')^-}\right) - U_L\left(\tau,\ Z^{t,i,z,\alpha}_{\tau^-}\right)\right)\mathbb{1}_{\{\tau<T\wedge\tau'\}}\mathbb{1}_{\{\theta\leq\nu\}}\right] \\
\delta_4 &:= \mathbb{E}\left[\int_\tau^{T\wedge\tau'} g(Y_u^{t,i,z,\alpha})\,du\mathbb{1}_{\{\tau<T\wedge\tau'\}}\mathbb{1}_{\{\theta\leq\nu\}}\right].
\end{aligned}$$

We first find a lower bound for δ_1 and an upper bound for δ_2. On $\{\nu<\theta\}\cap\{\tau<\tau'\wedge T\}$, we have

$$\begin{aligned}
X_\nu^{t,i,z',\alpha} &= (x'-x)e^{r(\nu-t)} + X_\tau^{t,i,z,\alpha}e^{r(\nu-\tau)} \\
&\leq e^{r\nu}\left((x'-x)e^{-rt} + x_{min}e^{-r\tau}\right) \\
&\leq x_{min}\frac{(x'-x)e^{r(\tau-t)} + x_{min}}{(x'-x)e^{r(T-t)} + x_{min}} \\
&\leq x_{min},
\end{aligned}$$

where the second inequality is deduced from the definition of ν.

Hence, on $\{\nu < \theta\} \cap \{\tau < \tau' \wedge T\}$, we have $\tau' \leq \nu < \theta$ and it follows from the monotonicity of the function: $t \to yf(t,y)$, see Assumption (H1),

$$\begin{aligned} L(T \wedge \tau', Z^{t,i,z',\alpha}_{(T\wedge\tau')^-}) &\geq x_{min} + y(p - \text{sign}(y)\frac{s}{2})f(T \wedge \tau', y) \\ &\geq x_{min} + y(p - \text{sign}(y)\frac{s}{2})f(\tau, y) \\ &\geq L(\tau, Z^{t,i,z,\alpha}_{\tau}). \end{aligned}$$

Since U is non-decreasing, we have $\delta_1 \geq 0$. Moreover g is non-negative, as such, we get

$$\delta_2 \leq G\mathbb{E}[(\nu - \tau)\mathbb{1}_{\{\tau<T\wedge\tau'\}}\mathbb{1}_{\{\nu<\theta\}}] \leq -\frac{G}{r}\ln\left(1 - \frac{(x' - x)e^{r(T-t)}}{\mid x_{min} \mid}\right).$$

Now we find a lower bound for $\delta_3 - \delta_4$. As g is bounded by $G > 0$ and $U \leq 1$, we get

$$\delta_3 - \delta_4 \geq \mathbb{E}\left[U_L\left(T \wedge \tau',\ Z^{t,i,z',\alpha}_{(T\wedge\tau')^-}\right)\mathbb{1}_{\{\tau<T\wedge\tau'\}}\mathbb{1}_{\{\theta\leq\nu\}}\right] - (1 + TG)\mathbb{P}(\theta \leq \nu).$$

Therefore, we deduce from Cauchy-Schwarz inequality that

$$\delta_3 - \delta_4 \geq -\mathbb{E}\left[\left(U_L\left(T \wedge \tau',\ Z^{t,i,z',\alpha}_{(T\wedge\tau')^-}\right)\right)^2\right]^{\frac{1}{2}}(\mathbb{P}(\theta \leq \nu))^{\frac{1}{2}} - (1 + TG)\mathbb{P}(\theta \leq \nu).$$

Applying Lemma 3.1 with $\beta = 2\gamma$, we obtain that there exists $C(p) > 0$ such that.

$$\begin{aligned} \mathbb{E}\left[\left(U_L\left(T \wedge \tau',\ Z^{t,i,z',\alpha}_{(T\wedge\tau')^-}\right)\right)^2\right] &\leq 1 + \mathbb{E}[\exp\left(-2\gamma L\left(T \wedge \tau',\ Z^{t,i,z',\alpha}_{(T\wedge\tau')^-}\right)\right)] \\ &\leq 1 + C(p). \end{aligned}$$

Hence, we get $\delta_3 - \delta_4 \geq -(1 + C(p))^{\frac{1}{2}}(\mathbb{P}(\theta \leq \nu))^{\frac{1}{2}} - (1 + TG)\mathbb{P}(\theta \leq \nu)$. Moreover, we have

$$\begin{aligned} \mathbb{P}(\theta \leq \nu) &\leq \mathbb{P}(N^a_\nu - N^a_\tau > 0) + \mathbb{P}(N^b_\nu - N^b_\tau > 0) \\ &\leq 2\mathbb{P}(\bar{N}_\nu - \bar{N}_\tau > 0) \\ &\leq -\frac{2\bar{\lambda}}{r}\ln\left(1 - \frac{(x' - x)e^{r(T-t)}}{\mid x_{min} \mid}\right), \end{aligned}$$

where $\bar{N}$ is a Poisson process with intensity $\bar{\lambda}$. To conclude, as $\phi(x'-x) = -\frac{1}{r}\ln\left(1 - \frac{(x'-x)e^{r(T-t)}}{|x_{min}|}\right)$, we have shown that

$$\Delta J^\alpha_i \geq -G\phi(x' - x) - (2\bar{\lambda}(1 + C(p)))^{\frac{1}{2}}\left(\phi(x' - x)\right)^{\frac{1}{2}} - 2\bar{\lambda}(1 + TG)\phi(x' - x).$$

It remains to find an upper bound for ΔJ^α_i. As g is positive, we have

$$\begin{aligned} \Delta J^\alpha_i &\leq \mathbb{E}\left[U_L\left(T \wedge \tau',\ Z^{t,i,z',\alpha}_{(T\wedge\tau')^-}\right) - U_L\left(T \wedge \tau,\ Z^{t,i,z,\alpha}_{(T\wedge\tau)^-}\right)\right] \\ &\leq \hat{\delta}_1(\alpha) + \hat{\delta}_2(\alpha), \end{aligned}$$

where we have set

$$\begin{aligned} \hat{\delta}_1(\alpha) &:= \mathbb{E}\left[\left(U_L\left(T \wedge \tau',\ Z^{t,i,z',\alpha}_{(T\wedge\tau')^-}\right) - U_L\left(T \wedge \tau,\ Z^{t,i,z,\alpha}_{(T\wedge\tau)^-}\right)\right)\mathbb{1}_{\{\nu<\theta\}}\right] \\ \hat{\delta}_2(\alpha) &:= \mathbb{E}\left[\left(U_L\left(T \wedge \tau',\ Z^{t,i,z',\alpha}_{(T\wedge\tau')^-}\right) - U_L\left(T \wedge \tau,\ Z^{t,i,z,\alpha}_{(T\wedge\tau)^-}\right)\right)\mathbb{1}_{\{\theta\leq\nu\}}\right]. \end{aligned}$$

On $\{\nu < \theta\}$, we have seen that

$$L\left(T \wedge \tau',\ Z^{t,i,z',\alpha}_{(T\wedge\tau')^-}\right) = (x'-x)e^{r\tau'} + L\left(T \wedge \tau,\ Z^{t,i,z,\alpha}_{(T\wedge\tau)^-}\right).$$

Hence, it follows from the concavity of U and its monotony that

$$\begin{aligned}
\hat{\delta}_1(\alpha) &\leq (x'-x)e^{rT}\mathbb{E}\left[U'\left(L\left(T\wedge\tau,\ Z^{t,i,z,\alpha}_{T\wedge\tau}\right)\right)\mathbb{1}_{\{\nu<\theta\}}\right] \\
&\leq (x'-x)e^{rT}\mathbb{E}\left[U'\left(L\left(T\wedge\tau,\ Z^{t,i,z,\alpha}_{T\wedge\tau}\right)\right)\right] \\
&= \gamma(x'-x)e^{rT}\mathbb{E}\left[\exp\left(-\gamma L\left(T\wedge\tau,\ Z^{t,i,z,\alpha}_{T\wedge\tau}\right)\right)\right].
\end{aligned}$$

From Lemma 3.1, it follows that there exists $C(p) > 0$ such that $\hat{\delta}_1(\alpha) \leq C(p)(x'-x)$. Finally, we deduce from Cauchy-Schwarz inequality and Lemma 3.1 that there exists $C(p) > 0$ such that

$$\begin{aligned}
\hat{\delta}_2(\alpha) &\leq \mathbb{P}(\theta \leq \nu) + \left(\mathbb{E}\left[(U_L\left(T\wedge\tau,\ Z^{t,i,z,\alpha}_{(T\wedge\tau)^-}\right))^2\right]\right)^{\frac{1}{2}}\left(\mathbb{P}(\theta\leq\nu)\right)^{\frac{1}{2}} \\
&\leq 2\bar{\lambda}\phi(x'-x) + C(p)(2\bar{\lambda}\phi(x'-x))^{\frac{1}{2}}.
\end{aligned}$$

□

Now we turn to the Hölder continuity of the criterium function with respect to both time and cash variables.

Proposition 3.2
Let $i \in \{1,...,m\}$, $(t,z) := (t,x,y,p,s) \in \bar{\mathcal{S}}$ and (t',x') in $[0,T]\times(x_{min},+\infty)$ s.t.

$$x < x' < x - x_{min}e^{-rT},\ \textit{and} \tag{3.16}$$

$$\mid t-t' \mid < \min\left(\frac{\mid x_{min}\mid e^{-2rT}}{r\mid x'\mid}, \frac{1}{r}\mid \ln\left(\mid\frac{x'}{x_{min}}\mid\right)\mid\right),\ \textit{if } x' \neq 0. \tag{3.17}$$

For all $\alpha \in \mathcal{A}(t\wedge t',z)$ such that $\alpha_s = 0$ for all $s \in [t\wedge t', t\vee t']$, we have $\alpha \in \mathcal{A}(t,z)\cap\mathcal{A}(t',z')$ with $z' = (x',y,p,s)$ and

$$\mid J_i^\alpha(t,z) - J_i^\alpha(t',z')\mid \leq K_2(p)\left(\psi(re^{rT}\mid x'(t-t')\mid) + \psi(x'-x) + \mid t'-t\mid\right).$$

where $K_2(p)$ is a positive constant depending only on p.

Proof: Let $\alpha \in \mathcal{A}(t\wedge t',z)$ s. t. $\alpha_{|[t\wedge t',t\vee t']} = 0$. As y, p and s are fixed, we may write $J_i^\alpha(t,\zeta)$ instead of $J_i^\alpha(t,\zeta,y,p,s)$ for $\zeta \in [x_{min},+\infty)$.
We set $\hat{x}' := x'e^{r(t-t')}$. We have $\mid x'-\hat{x}'\mid \leq \mid x'\mid re^{rT}\mid t-t'\mid$ since $\mid e^\zeta - 1\mid \leq \mid\zeta\mid e^{|\zeta|}$. As such, from the condition on $|t-t'|$ in (3.17), we may apply Lemma 3.2 and obtain

$$\begin{aligned}
\mid J_i^\alpha(t',x') - J_i^\alpha(t,x)\mid &\leq \mid J_i^\alpha(t',x') - J_i^\alpha(t,\hat{x}')\mid \\
&\quad + \mid J_i^\alpha(t,\hat{x}') - J_i^\alpha(t,x')\mid + \mid J_i^\alpha(t,x') - J_i^\alpha(t,x)\mid \\
&\leq \mid J_i^\alpha(t',x') - J_i^\alpha(t,\hat{x}')\mid \\
&\quad + K_1(p)\left(\psi(\mid x'-\hat{x}'\mid) + \psi(x'-x)\right).
\end{aligned}$$

As ψ is increasing, we have

$$\psi(\mid x' - \hat{x}' \mid) \leq \psi(re^{rT} \mid x'(t - t^{'}) \mid).$$

Therefore, we just have to prove that there exists $C(p) > 0$ such that:

$$\mid J_i^\alpha(t^{'}, x') - J_i^\alpha(t, \hat{x}') \mid \leq C(p) \mid t' - t \mid . \tag{3.18}$$

We first set $t_0 = \min(t,\ t^{'}), \quad t_1 = max(t,\ t^{'})$ and

$$x_k = x' \exp\left(\frac{r}{2}(t - t^{'} - (-1)^k(t_1 - t_0))\right) \text{ for } k \in \{0, 1\}.$$

With these notations, if $t_0 = t^{'}$ then $x_0 = x'$ and $x_1 = \hat{x}'$ else $t_1 = t^{'}$, $x_1 = x'$ and $x_0 = \hat{x}'$. Hence we aim at proving that

$$\mid J_i^\alpha(t_1, z_1) - J_i^\alpha(t_0, z_0) \mid \leq C(p)(t_1 - t_0) \text{ with } x_1 = x_0 e^{r(t_1 - t_0)},\ z_k = (x_k, y, p, s)\ \forall k \in \{0, 1\}.$$

To simplify notations, we set $\tau^1 = \tau^{t_1,i,z_1,\alpha}$ and $\tau^0 = \tau^{t_0,i,z_0,\alpha}$. As $\alpha \in \mathcal{A}(t_0, z_0)$, we have

$$J_i^\alpha(t_0, z_0) \quad = \quad \mathbb{E}\left[U_L\left(T \wedge \tau^0,\ Z^{t_0,i,z_0,\alpha}_{(T\wedge\tau^0)^-}\right) - \int_{t_0}^{T\wedge\tau^0} g(Y_u^{t_0,i,z_0,\alpha})\, du\right].$$

Now, if we set $\hat{\theta} := \inf\{u \geq t_0 :\ N_u^a > N_{u^-}^a \text{ or } N_u^b > N_{u^-}^b \text{ or } N_u > N_{u^-}\}$. We have

$$\begin{aligned} J_i^\alpha(t_0, z_0) \quad = \quad & \mathbb{E}\left[\left(U_L\left(T \wedge \tau^0,\ Z^{t_0,i,z_0,\alpha}_{(T\wedge\tau^0)^-}\right) - \int_{t_0}^{T\wedge\tau^0} g(Y_u^{t_0,i,z_0,\alpha})\, du\right) \mathbb{1}_{\{\hat{\theta} \leq t_1\}}\right] \\ & + \mathbb{E}\left[\mathbb{E}\left[\left(U_L\left(T \wedge \tau^0,\ Z^{t_0,i,z_0,\alpha}_{(T\wedge\tau^0)^-}\right) - \int_{t_1}^{T\wedge\tau^0} g(Y_u^{t_0,i,z_0,\alpha})\, du\right) \mid \mathcal{F}_{t_1}\right] \mathbb{1}_{\{t_1 < \hat{\theta}\}}\right] \\ & - \mathbb{E}\left[\left(\int_{t_0}^{t_1} g(Y_u^{t_0,i,z_0,\alpha})\, du\right) \mathbb{1}_{\{t_1 < \hat{\theta}\}}\right]. \end{aligned}$$

We can notice that on $\{t_1 < \hat{\theta}\}$, we have

$$I_{t_1} = i, \quad \tau^1 = \tau^0 \quad \text{and} \quad \forall u \in [t_1, T \wedge \tau^1],\ Z_u^{t_0,i,z_0,\alpha} = Z_u^{t_1,i,z_1,\alpha}.$$

Therefore, we deduce from the Markov property that

$$\begin{aligned} J_i^\alpha(t_0, z_0) \quad = \quad & \mathbb{E}\left[\left(U_L\left(T \wedge \tau^0,\ Z^{t_0,i,z_0,\alpha}_{(T\wedge\tau^0)^-}\right) - \int_{t_0}^{T\wedge\tau^0} g(Y_u^{t_0,i,z_0,\alpha})\, du\right) \mathbb{1}_{\{\hat{\theta} \leq t_1\}}\right] \\ & + J_i^\alpha(t_1, z_1)\mathbb{P}(t_1 < \hat{\theta}) - \mathbb{E}\left[\left(\int_{t_0}^{t_1} g(Y_u^{t_0,i,z_0,\alpha})\, du\right) \mathbb{1}_{\{t_1 < \hat{\theta}\}}\right]. \end{aligned}$$

Then, it follows from Cauchy-Schwarz inequality and Lemma 3.1 that there exists $C(p) > 0$ such that

$$\begin{aligned} J_i^\alpha(t_0, z_0) \quad \geq \quad & -\mathbb{E}\left[\left(U_L\left(T \wedge \tau^0,\ Z^{t_0,i,z_0,\alpha}_{(T\wedge\tau^0)^-}\right)\right)^2\right]^{\frac{1}{2}} \left(\mathbb{P}(\hat{\theta} \leq t_1)\right)^{\frac{1}{2}} \\ & - TG\mathbb{P}(\hat{\theta} \leq t_1) + J_i^\alpha(t_1, z_1)\mathbb{P}(t_1 < \hat{\theta}) - G(t_1 - t_0) \\ \geq \quad & -C(p)\left(\mathbb{P}(\hat{\theta} \leq t_1)\right)^{\frac{1}{2}} - TG\mathbb{P}(\hat{\theta} \leq t_1) + J_i^\alpha(t_1, z_1)\mathbb{P}(t_1 < \hat{\theta}) - G(t_1 - t_0). \end{aligned}$$

Recalling that $J_i^\alpha(t_1, z_1) \leq 1$, we get

$$\begin{aligned} J_i^\alpha(t_1,z_1) - J_i^\alpha(t_0,z_0) &\leq C(p)\left(\mathbb{P}(\hat{\theta} \leq t_1)\right)^{\frac{1}{2}} + (J_i^\alpha(t_1,z_1) + TG)\mathbb{P}(\hat{\theta} \leq t_1) + G(t_1 - t_0) \\ &\leq C(p)(3\bar{\lambda})^{\frac{1}{2}} \mid t_1 - t_0 \mid^{\frac{1}{2}} + (3(1+TG)\bar{\lambda} + G) \mid t_1 - t_0 \mid . \end{aligned}$$

The last inequality follows from the following one

$$\begin{aligned} \mathbb{P}(\hat{\theta} \leq t_1) &\leq \mathbb{P}(N^a_{t_1} > N^a_{t_0}) + \mathbb{P}(N^b_{t_1} > N^b_{t_0}) + \mathbb{P}(N_{t_1} > N_{t_0}) \\ &\leq 3\mathbb{P}(\bar{N}_{t_1} > \bar{N}_{t_0}) \\ &\leq 3\bar{\lambda}(t_1 - t_0). \end{aligned}$$

Now we follow the same idea to find a lower bound for $J_i^\alpha(t_1, z_1) - J_i^\alpha(t_0, z_0)$. We have

$$J_i^\alpha(t_0,z_0) = E\left[U_L(T \wedge \tau^0,\ Z^{t_0,i,z_0,\alpha}_{(T\wedge\tau^0)^-}) - \int_{t_0}^{T\wedge\tau^0} g(Y_s^{t_0,i,z_0,\alpha})ds\right].$$

Once again, we notice that on $\{t_1 < \hat{\theta}\}$, we have

$$I_{t_1} = i, \quad \tau_1 = \tau_0 \quad \text{and} \quad \forall u \in [t_1, T \wedge \tau_1],\ Z_u^{t_0,i,z_0,\alpha} = Z_u^{t_1,i,z_1,\alpha}.$$

Therefore, it follows from Markov property again that

$$\begin{aligned} J_i^\alpha(t_0,z_0) &\leq \mathbb{E}\left[\left(U_L\left(T\wedge\tau_0,\ Z^{t_0,i,z_0,\alpha}_{(T\wedge\tau_0)^-}\right) - \int_{t_0}^{T\wedge\tau_0} g(Y_u^{t_0,i,z_0,\alpha})\,du\right)\mathbb{1}_{\{\hat{\theta}\leq t_1\}}\right] \\ &\quad + \mathbb{E}\left[U_L(T\wedge\tau_1,\ Z^{t_1,i,z_1,\alpha}_{(T\wedge\tau_1)^-}) - \int_{t_1}^{T\wedge\tau_1} g(Y_s^{t_1,i,z_1,\alpha})ds\right]\mathbb{P}(t_1 < \hat{\theta}) \\ &\leq \mathbb{E}\left[\left(U_L\left(T\wedge\tau_0,\ Z^{t_0,i,z_0,\alpha}_{(T\wedge\tau_0)^-}\right) - \int_{t_0}^{T\wedge\tau_0} g(Y_u^{t_0,i,z_0,\alpha})\,du\right)\mathbb{1}_{\{\hat{\theta}\leq t_1\}}\right] \\ &\quad + J_i^\alpha(t_1,z_1)\mathbb{P}(t_1 < \hat{\theta}) \end{aligned}$$

As $U \leq 1$ and $g \geq 0$, we obtain

$$J_i^\alpha(t_0,z_0) \leq \mathbb{P}(\hat{\theta} \leq t_1) + J_i^\alpha(t_1,z_1)\mathbb{P}(t_1 < \hat{\theta}).$$

Hence, from the proof of Proposition 3.1, we know that there exists $C(p) > 0$ such that

$$\begin{aligned} J_i^\alpha(t_1,z_1) - J_i^\alpha(t_0,z_0) &\geq (J_i^\alpha(t_1,z_1) - 1)\mathbb{P}(\hat{\theta} \leq t_1) \\ &\geq -(C(p)+1)\mathbb{P}(\hat{\theta} \leq t_1) \\ &\geq -3(C(p)+1)\bar{\lambda}(t_1 - t_0). \end{aligned}$$

□

Proposition 3.3 ***Uniform Continuity of the objective functions***
Let $(i, y, p, s) \in \{1, ..., m\} \times \{y_{min}, .., y_{max}\} \times \frac{\delta}{2}\mathbb{N}^* \times \delta\{1, ..K\}$ *such that* $p - \frac{s}{2} > 0$.
The function $(t, x) \to v_i(t, x, y, p, s)$ *is uniformly continuous on* $[0, T] \times [x_{min}, +\infty)$.

Proof: Throughout the proof, we set $V(u,\xi) = v_i(u,\xi,y,p,s)$ on $[0,T]\times[x_{min},+\infty)$. Let $(t,z) := (t,x,y,p,s) \in \bar{\mathcal{S}}$ and (t',x') in $[0,T]\times(x_{min},+\infty)$ s.t. (t,x) and (t',x') satisfy conditions (3.16) and (3.17). We shall prove that

$$| V(t,x) - V(t',x') | \;\leq\; K_2(p)\Big(\psi(re^{rT} \mid x'(t-t^{'}) \mid) + \psi(x'-x)+ \mid t'-t \mid\Big).$$

where $z' = (x',y,p,s)$ and $K_2(p)$ is a positive constant depending only on p.

Let $\varepsilon > 0$ and $\alpha^{*\prime} \in \mathcal{A}(t',z')$ such that $V(t',x') \leq J_i^{\alpha^{*'}}(t',x') + \varepsilon$. For $u \in [t\wedge t', T]$, we set $\alpha_u = \alpha_u^{*\prime}\mathbb{1}_{\{u\geq t'\}}$. We have $\alpha \in \mathcal{A}(t',z') \cap \mathcal{A}(t,z)$, then it follows from Proposition 3.2 that

$$\begin{aligned} V(t',x') - V(t,x) &\leq J_i^{\alpha}(t',x') - J_i^{\alpha}(t,x) + \varepsilon \\ &\leq K_2(p)\left(\psi(re^{rT} \mid x'(t-t') \mid) + \psi(x'-x)+ \mid t'-t \mid\right) + \varepsilon. \end{aligned}$$

Now, we know that there exists $\alpha \in \mathcal{A}(t,z)$ such that $V(t,x) \leq J_i^{\alpha}(t,x)+\varepsilon$. For $u \in [t\wedge t', T]$, we set $\alpha_u = \alpha_u^{*}\mathbb{1}_{\{u\geq t\}}$. We have $\alpha \in \mathcal{A}(t',z') \cap \mathcal{A}(t,z)$, then it follows from Proposition 3.2 that

$$\begin{aligned} V(t',x') - V(t,x) &\geq J_i^{\alpha}(t',x') - J_i^{\alpha}(t,x) - \varepsilon \\ &\geq -K_2(p)\left(\psi(re^{rT} \mid x'(t-t') \mid) + \psi(x'-x)+ \mid t'-t \mid\right) - \varepsilon. \end{aligned}$$

Letting ε going to 0, we obtain the result. □

We shall adopt a dynamic programming approach to study this utility maximization problem. To this end, we now establish the dynamic programming principle for our stochastic control problem.

Theorem 3.1 ***Dynamic programming principle (DPP)***
Let $(i,t,z) := (i,t,x,y,p,s) \in \{1,...,m\}\times\mathcal{S}$. Let ν be a stopping time in $\mathcal{T}_{t,T}$, we have

$$v_i(t,z) = \sup_{\alpha\in\mathcal{A}(t,z)} \hat{J}_i^{\alpha,\nu}(t,z), \tag{3.19}$$

where, for $\alpha \in \mathcal{A}(t,z)$, we have set

$$\begin{aligned} \hat{J}_i^{\alpha,\nu}(t,z) \;=\; \mathbb{E}\Big[&- g(y)\left(\nu\wedge\hat{\theta}\wedge\hat{\tau}^{\alpha} - t\right) + v_{I_{\nu\wedge\hat{\theta}}}\left(\nu\wedge\hat{\theta},\; Z^{t,i,z,\alpha}_{\nu\wedge\hat{\theta}}\right)\mathbb{1}_{\{\nu\wedge\hat{\theta}<\hat{\tau}^{\alpha}\}} \\ &+ U_L\left(\hat{\tau}^{\alpha}, xe^{r(\hat{\tau}^{\alpha}-t)}, y, p, s\right)\mathbb{1}_{\{\hat{\tau}^{\alpha}\leq\nu\wedge\hat{\theta}\}}\Big], \end{aligned} \tag{3.20}$$

with $\hat{\tau}^{\alpha} = \tau^{t,i,z,\alpha}\wedge T$, $\rho = \inf\{u\geq t :\; N_u > N_{u^-}\}$, $\theta^w = \inf\{u\geq t :\; N_u^{w,i,t,z} > N_{u^-}^{w,i,t,z}\}$, for $w\in\{a,b\}$ and $\hat{\theta} = \rho\wedge\theta^a\wedge\theta^b$.

Proof: To establish the Dynamic Programming Principle, we may adapt the proof of **Theorem 5.2** in [4]. The proof relies on the continuity of the objective functions J_i and v_i established in Propositions 3.2 and 3.3.

First step: We prove that $v_i(t,z) \leq \sup_{\alpha \in \mathcal{A}(t,z)} \hat{J}_i^{\alpha,\nu}(t,z)$.

Let $\alpha \in \mathcal{A}(t,z)$. We have

$$\begin{aligned} J_i^\alpha(t,z) &= \mathbb{E}\left[U_L(\hat{\tau}^\alpha, Z^{t,i,z,\alpha}_{\hat{\tau}^{\alpha-}}) - \int_t^{\hat{\tau}^\alpha} g(Y_s^{t,i,y,\alpha})ds\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[U_L(\hat{\tau}^\alpha, Z^{t,i,z,\alpha}_{\hat{\tau}^{\alpha-}}) - \int_t^{\hat{\tau}^\alpha} g(Y_s^{t,i,y,\alpha})ds|\mathcal{F}_{\nu\wedge\hat{\theta}}\right]\mathbb{1}_{\{\nu\wedge\hat{\theta}<\hat{\tau}^\alpha\}}\right] \\ &\quad +\mathbb{E}\left[\left(U_L(\hat{\tau}^\alpha, xe^{r(\hat{\tau}^\alpha-t)}, y, p, s) - g(y)(\hat{\tau}^\alpha - t)\right)\mathbb{1}_{\{\hat{\tau}^\alpha \leq \nu\wedge\hat{\theta}\}}\right] \end{aligned}$$

Now, we shall work on $\{\nu\wedge\hat{\theta} < \hat{\tau}^\alpha\}$ and we have:

$$\mathbb{E}\left[U_L(\hat{\tau}^\alpha, Z^{t,i,z,\alpha}_{\hat{\tau}^{\alpha-}}) - \int_t^{\hat{\tau}^\alpha} g(Y_s^{t,i,y,\alpha})ds|\mathcal{F}_{\nu\wedge\hat{\theta}}\right] \leq v_{I_{\nu\wedge\hat{\theta}}}\left(\nu\wedge\hat{\theta},\ Z^{t,i,z,\alpha}_{\nu\wedge\hat{\theta}}\right) - (\nu\wedge\hat{\theta} - t)g(y).$$

Second step: We prove that $\hat{v}_\nu(i,t,z) := \sup_{\alpha\in\mathcal{A}(t,z)} \hat{J}_i^{\alpha,\nu}(t,z) \leq v_i(t,z)$.

Let $\varepsilon > 0$. We first notice that

$$X^{t,i,z,\alpha}_{\nu\wedge\hat{\theta}} \leq x^+e^{rT} + p + \frac{K}{2}\delta \text{ and } P^{t,i,z,\alpha}_{\nu\wedge\hat{\theta}} \leq p + \delta$$

Therefore $(\nu\wedge\hat{\theta}, Z^{t,i,z,\alpha}_{\nu\wedge\hat{\theta}})$ takes values in the bounded set $B(t,x,p)$ where

$$B(t,x,p) = \left\{(u,\xi,\zeta,\pi,\sigma) \in [t,T]\times\mathcal{S}:\ \xi \leq x^+e^{rT} + p + \frac{K}{2} \text{ and } \pi \leq p+\delta\right\}.$$

We now define a countable partition of $B(t,x,p)$ with Borel subsets $\mathcal{B}_k$ such that for all $k\in\mathbb{N}$, $\mathcal{B}_k = I\times J\times\{a\}\times\{b\}\times\{c\}$ where $I\times J \subset [t,T]\times(x_{min},+\infty)$, $a\in\{y_{min},...,y_{max}\}$, $b\in\frac{\delta}{2}\mathbb{N}$ and $c\in\delta,...,K\delta\}$. For $k\in\mathbb{N}$, we choose $((t_k,z_k) := (t_k,x_k,y_k,p_k,s_k) \in \bar{\mathcal{B}}_k$ such that t_k is the largest time in the trace of $\bar{\mathcal{B}}_k$ in $[t,T]$.
From Propositions 3.2 and 3.3, we can choose $\mathcal{B} = (\mathcal{B}_k)_k$ such that for all $k\in\mathbb{N}$, all $i\in\{1,...,m\}$, all (u,ξ,ζ,π,σ) in $\mathcal{B}_k$ and all $\alpha\in\mathcal{A}(u,z_k)$ s.t. $\alpha_{|[u,t_k]} = 0$, we have

$$|v_i(u,\xi,\zeta,\pi,\sigma) - v_i(t_k,z_k)| + |J_i^\alpha(u,\xi,\zeta,\pi,\sigma) - J_i^\alpha(t_k,z_k)| \leq \varepsilon. \tag{3.21}$$

Let $\alpha\in\mathcal{A}(t,z)$ such that

$$\hat{v}_\nu(i,t,z) \leq \varepsilon + \hat{J}_i^{\alpha,\nu}(t,z).$$

As $(\mathcal{B}_k)_{k\in\mathbb{N}}$ is a partition of $[t,T]\times\mathcal{S}$, we get

$$\begin{aligned} \hat{v}_\nu(i,t,z) &\leq \varepsilon + \sum_{k=0}^{+\infty}\mathbb{E}\Big[v_{I_{\nu\wedge\hat{\theta}}}\left(\nu\wedge\hat{\theta},\ Z^{t,i,z,\alpha}_{\nu\wedge\hat{\theta}}\right)\mathbb{1}_{\{\nu\wedge\hat{\theta}<\hat{\tau}^\alpha\}}\mathbb{1}_{\{(\nu\wedge\hat{\theta},\ Z^{t,i,z,\alpha}_{\nu\wedge\hat{\theta}})\in\mathcal{B}_k\}}\Big] \\ &\quad +\mathbb{E}\Big[-g(y)\left(\nu\wedge\hat{\theta}\wedge\hat{\tau}^\alpha - t\right) + U_L\left(\hat{\tau}^\alpha, xe^{r(\hat{\tau}^\alpha-t)}, y, p, s\right)\mathbb{1}_{\{\hat{\tau}^\alpha\leq\nu\wedge\hat{\theta}\}}\Big] \\ &\leq 2\varepsilon + \sum_{k=0}^{+\infty}\mathbb{E}\Big[v_{I_{\nu\wedge\hat{\theta}}}(t_k,z_k)\,\mathbb{1}_{\{\nu\wedge\hat{\theta}<\hat{\tau}^\alpha\}}\mathbb{1}_{\{(\nu\wedge\hat{\theta},\ Z^{t,i,z,\alpha}_{\nu\wedge\hat{\theta}})\in\mathcal{B}_k\}}\Big] \\ &\quad +\mathbb{E}\Big[-g(y)\left(\nu\wedge\hat{\theta}\wedge\hat{\tau}^\alpha - t\right) + U_L\left(\hat{\tau}^\alpha, xe^{r(\hat{\tau}^\alpha-t)}, y, p, s\right)\mathbb{1}_{\{\hat{\tau}^\alpha\leq\nu\wedge\hat{\theta}\}}\Big], \end{aligned}$$

where the latter inequality derives from (3.21).
Now, for $j \in \{1, ..., m\}$ and $k \in \mathbb{N}$, we introduce $\alpha^{j,k} \in \mathcal{A}(t_k, j, z_k)$ such that

$$v_j(t_k, z_k) \leq \varepsilon + J_j^{\alpha^{j,k}}(t_k, z_k).$$

Let $k \in \mathbb{N}$, we get

$$\begin{aligned} V_k &:= \mathbb{E}\Big[v_{I_{\nu\wedge\hat{\theta}}}(t_k, z_k)\, \mathbb{1}_{\{\nu\wedge\hat{\theta} < \hat{\tau}^\alpha\}} \mathbb{1}_{\{(\nu\wedge\hat{\theta},\ Z^{t,i,z,\alpha}_{\nu\wedge\hat{\theta}}) \in \mathcal{B}_k\}} \Big] \\ &\leq \sum_{j=1}^{m} \mathbb{E}\Big[\Big(\varepsilon + J_j^{\alpha^{j,k}}(t_k, z_k) \Big) \mathbb{1}_{\{I_{\nu\wedge\hat{\theta}} = j\}} \mathbb{1}_{\{\nu\wedge\hat{\theta} < \hat{\tau}^\alpha\}} \mathbb{1}_{\{(\nu\wedge\hat{\theta},\ Z^{t,i,z,\alpha}_{\nu\wedge\hat{\theta}}) \in \mathcal{B}_k\}} \Big]. \end{aligned}$$

Now, we define the random variable κ with values in $\mathbb{N}$, null on $\{\nu \wedge \hat{\theta} \geq \hat{\tau}^\alpha\}$ and such that for $\omega \in \{\nu \wedge \hat{\theta} < \hat{\tau}^\alpha\}$, $(\nu \wedge \hat{\theta},\ Z^{t,i,z,\alpha}_{\nu\wedge\hat{\theta}})(\omega) \in \mathcal{B}_{\kappa(\omega)}$. Notice that κ is $\mathcal{F}_{\nu\wedge\hat{\theta}}$-measurable.
For $\omega \in \{\nu \wedge \hat{\theta} < \hat{\tau}^\alpha\}$, we set:

$$\hat{\alpha}_s(\omega) := \begin{cases} \alpha_s(\omega) & \text{if} \quad t \leq s \leq \nu \wedge \hat{\theta}(\omega) \\ 0 & \text{if} \quad \nu \wedge \hat{\theta}(\omega) < s < t_{\kappa(\omega)}. \\ \alpha_s^{j,\kappa(\omega)}(\omega) & \text{if} \quad t_{\kappa(\omega)} \leq s \text{ and } I_{\nu\wedge\hat{\theta}}(\omega) = j \end{cases} \tag{3.22}$$

On $\{\nu \wedge \hat{\theta} \geq \hat{\tau}^\alpha\}$, we set $\hat{\alpha} = \alpha$. To simplify notations, we shall write $\hat{\alpha}$ for $\hat{\alpha}_{|[t_\kappa, T]}$ and $\hat{\alpha}_{|[\nu\wedge\hat{\theta}, T]}$. We get

$$V_k \leq \mathbb{E}\Big[\Big(\varepsilon + J^{\hat{\alpha}}_{I_{\nu\wedge\hat{\theta}}}(t_\kappa, z_\kappa) \Big) \mathbb{1}_{\{\nu\wedge\hat{\theta} < \hat{\tau}^\alpha\}} \mathbb{1}_{\{(\nu\wedge\hat{\theta},\ Z^{t,i,z,\alpha}_{\nu\wedge\hat{\theta}}) \in \mathcal{B}_\kappa\}} \Big].$$

From the definition of $\hat{\alpha}$, we have $\hat{\alpha} \in \mathcal{A}\left(\nu \wedge \hat{\theta}, I_{\nu\wedge\hat{\theta}}, Z^{t,i,z,\alpha}_{\nu\wedge\hat{\theta}}\right)$, therefore we deduce from the last inequality that

$$\begin{aligned} \sum_{k=0}^{+\infty} V_k &\leq \varepsilon + \sum_{k=0}^{+\infty} \mathbb{E}\Big[J^{\hat{\alpha}}_{I_{\nu\wedge\hat{\theta}}}(t_\kappa, z_\kappa)\, \mathbb{1}_{\{\nu\wedge\hat{\theta} < \hat{\tau}^\alpha\}} \mathbb{1}_{\{(\nu\wedge\hat{\theta},\ Z^{t,i,z,\alpha}_{\nu\wedge\hat{\theta}}) \in \mathcal{B}_\kappa\}} \Big] \\ &\leq 2\varepsilon + \sum_{k=0}^{+\infty} \mathbb{E}\Big[J^{\hat{\alpha}}_{I_{\nu\wedge\hat{\theta}}}\left(\nu \wedge \hat{\theta},\ Z^{t,i,z,\alpha}_{\nu\wedge\hat{\theta}}\right) \mathbb{1}_{\{\nu\wedge\hat{\theta} < \hat{\tau}^\alpha\}} \mathbb{1}_{\{(\nu\wedge\hat{\theta},\ Z^{t,i,z,\alpha}_{\nu\wedge\hat{\theta}}) \in \mathcal{B}_\kappa\}} \Big] \\ &= 2\varepsilon + \mathbb{E}\Big[J^{\hat{\alpha}}_{I_{\nu\wedge\hat{\theta}}}\left(\nu \wedge \hat{\theta},\ Z^{t,i,z,\alpha}_{\nu\wedge\hat{\theta}}\right) \mathbb{1}_{\{\nu\wedge\hat{\theta} < \hat{\tau}^\alpha\}} \Big] \\ &= 2\varepsilon + \mathbb{E}\Big[\left(U_L(\hat{\tau}^{\hat{\alpha}}, Z^{\nu\wedge\hat{\theta}, I_{\nu\wedge\hat{\theta}}, Z^{t,i,z,\alpha}_{\nu\wedge\hat{\theta}}, \hat{\alpha}}_{\hat{\tau}^{\hat{\alpha}}-}) - \int_{\nu\wedge\hat{\theta}}^{\hat{\tau}^{\hat{\alpha}}} g(Y_s^{\nu\wedge\hat{\theta}, I_{\nu\wedge\hat{\theta}}, Z^{t,i,z,\alpha}_{\nu\wedge\hat{\theta}}, \hat{\alpha}}) ds \right) \mathbb{1}_{\{\nu\wedge\hat{\theta} < \hat{\tau}^\alpha\}} \Big] \\ &= 2\varepsilon + \mathbb{E}\Big[\left(U_L(\hat{\tau}^{\hat{\alpha}}, Z^{t,i,z,\hat{\alpha}}_{\hat{\tau}^{\hat{\alpha}}-}) - \int_{\nu\wedge\hat{\theta}}^{\hat{\tau}^{\hat{\alpha}}} g(Y_s^{t,i,z,\hat{\alpha}}) ds \right) \mathbb{1}_{\{\nu\wedge\hat{\theta} < \hat{\tau}^\alpha\}} \Big], \end{aligned}$$

where the last equality is obtained from the definition in (3.22). To conclude we notice that on $\{\hat{\tau}^\alpha \leq \nu \wedge \hat{\theta}\}$, we have $\hat{\tau}^\alpha = \hat{\tau}^{\hat{\alpha}}$ and then

$$\begin{aligned} \hat{v}_\nu(i,t,z) &\leq 4\varepsilon + \mathbb{E}\Big[-g(y)\left(\nu \wedge \hat{\theta} \wedge \hat{\tau}^\alpha - t\right) + U_L\left(\hat{\tau}^\alpha, xe^{r(\hat{\tau}^\alpha - t)}, y, p, s\right) \mathbb{1}_{\{\hat{\tau}^\alpha \leq \nu\wedge\hat{\theta}\}} \Big] \\ &\quad + \mathbb{E}\Big[\left(U_L(\hat{\tau}^{\hat{\alpha}}, Z^{t,i,z,\hat{\alpha}}_{\hat{\tau}^{\hat{\alpha}}-}) - \int_{\nu\wedge\hat{\theta}}^{\hat{\tau}^{\hat{\alpha}}} g(Y_s^{t,i,z,\hat{\alpha}}) ds \right) \mathbb{1}_{\{\nu\wedge\hat{\theta} < \hat{\tau}^\alpha\}} \Big] \end{aligned}$$

$$
\begin{aligned}
&= 4\varepsilon + \mathbb{E}\Big[U_L(\hat{\tau}^{\hat{\alpha}}, Z^{t,i,z,\hat{\alpha}}_{\hat{\tau}^{\hat{\alpha}-}}) - \int_t^{\hat{\tau}^{\hat{\alpha}}} g(Y^{t,i,z,\hat{\alpha}}_s)ds\Big] \\
&= 4\varepsilon + J_i^{\hat{\alpha}}(t,z) \\
&\leq 4\varepsilon + v_i(t,z).
\end{aligned}
$$

Sending ε to zero, we may conclude the proof. □

4 Viscosity characterization of the objective functions

The objective of this section is to provide a rigorous characterization for the value function by means of viscosity solutions to the HJB equation (4.23) together with the appropriate boundary terminal conditions. The uniqueness property is particularly crucial to numerically solve the associated HJB. Since the value functions $(v_i)_{i\in\{1,...,m\}}$ are continuous, we shall work with the notion of continuous viscosity solutions.

We first define the following set

$$
\begin{aligned}
A(t,z) := \{\alpha = (\varepsilon^a, \varepsilon^b, \eta^a, \eta^b) \in \{0,1\}^4 \quad \text{s.t.} \quad & p - \frac{s}{2} - \delta\varepsilon^b \geq \delta, \\
& \delta \leq s - \delta(\varepsilon^a - \varepsilon^b) \leq K\delta,\ \delta \leq s + \delta(\eta^a - \eta^b) \leq K\delta\}.
\end{aligned}
$$

For all $(i,t,x,y,p,s) := (i,t,z) \in \{1,...,m\} \times \mathcal{S}$ and $\alpha := \{\varepsilon^a, \varepsilon^b, \eta^a, \eta^b) \in A(t,z)$, we introduce the two following operators

$$
\mathcal{A}v_i(t,z,\alpha) = \begin{cases} U_L(t,x,y_{min},p,s) & \text{if} \quad y = y_{min} \\ v_i(t,x+p+\frac{s}{2}, y-1, p+\frac{\delta}{2}(\eta^a+\eta^b), s+\delta(\eta^a-\eta^b)) & \text{else} \quad . \end{cases}
$$

$$
\mathcal{B}v_i(t,z,\alpha) = \begin{cases} U_L(t,x,y_{max},p,s) & \text{if} \quad y = y_{max} \\ U_L(t,z) & \text{if} \quad x < x_{min} + p - \frac{s}{2} \\ U_L(t,z) & \text{if} \quad x = x_{min} + p - \frac{s}{2} < 0 \\ v_i(t,x-p+\frac{s}{2}, y+1, p-\frac{\delta}{2}(\varepsilon^a+\varepsilon^b), s-\delta(\varepsilon^a-\varepsilon^b)) & \text{else} \quad . \end{cases}
$$

We now denote by $\mathcal{H}_i$ the Hamiltonian associated with state i

$$
\begin{aligned}
\mathcal{H}_i(t,z,v,\frac{\partial v}{\partial x}) &= rx\frac{\partial v_i}{\partial x}(t,x,y,p,s) + \sum_{j\neq i}\gamma_{ij}\left(v_j(t,x,y,p,s) - v_i(t,x,y,p,s)\right) - g(y) \\
&\quad + \sup_{\alpha\in A(t,z)}\Big[\lambda_i^a(p,s)\left(\mathcal{A}v_i(t,x,y,p,s,\alpha) - v_i(t,x,y,p,s)\right) \\
&\quad + \lambda_i^b(p,s)\left(\mathcal{B}v_i(t,x,y,p,s,\alpha) - v_i(t,x,y,p,s)\right)\Big].
\end{aligned}
$$

Notice that we may often refer to the value function v as the family of value functions $(v_i)_{i\in\{1,...,m\}}$. We may also refer to $v(i,t,z)$ instead of $v_i(t,z)$.
The main result of this section is that the value function $v = (v_i)_{i\in\{1,...,m\}}$ is the unique viscosity solution of the following HJB system

$$
-\frac{\partial v_i}{\partial t} - \mathcal{H}_i(t,z,v,\frac{\partial v}{\partial x}) = 0,\ (i,t,z) \in \{1,...,m\} \times \mathcal{S}, \tag{4.23}
$$

We now state our Theorem related to the characterization of our value functions.

Theorem 4.2 *The family of objective functions $v = (v_i)_{1\leq i\leq m}$ is the unique family of functions such that*

i) *Continuity condition: For all $(i,y,p,s) \in \{1,..,m\}\times\{y_{min},..,y_{max}\}\times\frac{\delta}{2}\mathbb{N}\times\delta\{1,..,K\}$, $(t,x)\to v_i(t,x,y,p,s)$ is continuous on $\{(t,x)\in[0,T)\times[x_{min},+\infty) : (t,x,y,p,s)\in\mathcal{S}\}$.*

ii) *Growth condition: There exist C_1, C_2 and C_3 positive constants such that*

$$1 - C_1 - C_2 e^{C_3 p} \leq v_i(t,x,y,p,s) \leq 1, \qquad \text{on } \{1,..,m\}\times\mathcal{S}. \tag{4.24}$$

iii) *Boundary and terminal conditions:*

$$v_i(t,x_{min},y,p,s) = U_L(t,x_{min},y,p,s) \text{ and } v_i(T,x,y,p,s) = U_L(T,x,y,p,s). \tag{4.25}$$

iv) *Viscosity solution: $(v_i)_{1\leq i\leq m}$ is a viscosity solution of the system of HJB equations(4.23) on $\{1,...,m\}\times\mathcal{S}$.*

Proof: Assertions *i*), *ii*) and *iii*) respectively follow from Proposition 3.3, Proposition 3.1 and the definition of objective functions. Therefore, only assertion *iv*) and the uniqueness result still need to be established. We shall divide our proof in three lemmas: first we prove that $(v_i)_{1\leq i\leq m}$ is a viscosity subsolution (see Lemma 4.3) then a supersolution (see Lemma 4.4) and finally we prove a comparison theorem (see Lemma 4.6) which will lead to the uniqueness result.

Lemma 4.3 *The family of objective functions $(v_i)_{1\leq i\leq m}$ is a subsolution of the system of variational inequalities (4.23) on $\{1,...,m\}\times\mathcal{S}$*

Proof: Let $(i,t,z)\in\{1,..,m\}\times\mathcal{S}$ and $(\psi_j)_{1\leq j\leq m}$ a family of functions such that for all $(j,\zeta,\pi,\sigma)\in\{1,..,m\}\times\{y_{min},..,y_{max}\}\times\frac{\delta}{2}\mathbb{N}\times\delta\{1,..,K\}$, $(u,\xi)\to\psi_j(u,\xi,\zeta,\pi,\sigma)$ is a $\mathcal{C}^1$ function on $\{(u,\xi)\in[0,T)\times[x_{min},+\infty) : (u,\xi,\zeta,\pi,\sigma)\in\mathcal{S}\}$ and $v-\psi$ has a global maximum at $(i,t,z)\in\{1,...,m\}\times\mathcal{S}$. Without loss of generality, we assume that $0 = (v-\psi)(i,t,z)$.

Let $0 < h < T-t$ such that

$$\begin{aligned} &\text{If} \quad x < x_{min} + p - \tfrac{s}{2} \quad \text{then } xe^{ru} < x_{min} + p - \frac{s}{2} \text{ for all } u\in[0,h] \\ &\text{If} \quad x > x_{min} + p - \tfrac{s}{2} \quad \text{then } xe^{ru} > x_{min} + p - \frac{s}{2} \text{ for all } u\in[0,h]. \end{aligned} \tag{4.26}$$

Notice that if $x = x_{min} + p - \frac{s}{2} < 0$ then $xe^{ru} < x_{min} + p - \frac{s}{2}$ for all $u\in(0,h]$ and if $x = x_{min} + p - \frac{s}{2} \geq 0$ then $xe^{ru} \geq x_{min} + p - \frac{s}{2}$ for all $u\in[0,h]$.

We choose an admissible strategy $\alpha\in\mathcal{A}(t,z)$ and set $\widehat{\tau}^\alpha := \tau^{i,t,z,\alpha}\wedge T$ such that the dynamic programming principle (3.19) implies

$$\begin{aligned} \psi_i(t,z) &= v_i(t,z) \\ &\leq \mathbb{E}\Big[-g(y)(\nu-t) + v_{I_{(t+h)\wedge\hat{\theta}}}\Big((t+h)\wedge\hat{\theta},\ Z^{t,i,z,\alpha}_{(t+h)\wedge\hat{\theta}}\Big)\mathbb{1}_{\{(t+h)\wedge\hat{\theta}<\hat{\tau}^\alpha\}} \end{aligned}$$

$$
\begin{aligned}
&+U_L\left(\hat{\tau}^\alpha, xe^{r(\hat{\tau}^\alpha - t)}, y, p, s\right)\mathbb{1}_{\{\hat{\tau}^\alpha \le (t+h)\wedge\hat{\theta}\}}\Big] + h^2,\\
\le\ & \mathbb{E}\Big[-g(y)\left((t+h)\wedge\hat{\theta}\wedge\hat{\tau}^\alpha - t\right) + \psi_{I_\nu}(\nu, Z_\nu) \qquad (4.27)\\
&+\left(U_L\left(\hat{\tau}^\alpha, xe^{r(\hat{\tau}^\alpha - t)}, y, p, s\right) - \psi_i(\hat{\tau}^\alpha, xe^{r(\hat{\tau}^\alpha - t)}, y, p, s)\right)\mathbb{1}_{\{\hat{\tau}^\alpha \le (t+h)\wedge\hat{\theta}\}}\Big] + h^2,
\end{aligned}
$$

where we have set $\nu := (t+h)\wedge\hat{\theta}\wedge\hat{\tau}^{\alpha-}$. Applying Itô's formula to $\psi_{I_u}(u, Z_u)$ between t and ν , we have

$$
\begin{aligned}
\psi_{I_\nu}(\nu, Z_\nu) &= \psi_i(t,z) + \int_t^\nu \frac{\partial\psi_i}{\partial t}(u, Z_u)du + \int_t^\nu \frac{\partial\psi_i}{\partial x}(u, Z_u)rX_u du\\
&+ \sum_{t\le u\le\nu}\Big(\psi_{I_u}(u, Z_u) - \psi_{I_{u-}}(u, Z_{u-})\Big)\\
&= \psi_i(t,z) + \int_t^\nu \frac{\partial\psi_i}{\partial t}(u, Z_u)du + \int_t^\nu \frac{\partial\psi_i}{\partial x}(u, Z_u)rX_u du\\
&+ \int_t^\nu \Big(\psi_{I_u}(u, Z_u) - \psi_i(u, Z_{u-})\Big)dN_u^a + \int_t^\nu \Big(\psi_{I_u}(u, Z_u) - \psi_i(u, Z_{u-})\Big)dN_u^b\\
&+ \int_t^\nu \Big(\psi_{I_u}(u, Z_u) - \psi_i(u, Z_{u^-})\Big)dN_u.
\end{aligned}
$$

By taking expectation, we obtain

$$
\begin{aligned}
\mathbb{E}[\psi_{I_\nu}(\nu, Z_\nu)] &= \psi_i(t,z) + \mathbb{E}\Big[\int_t^\nu \frac{\partial\psi_i}{\partial t}(u, Z_u)du + \int_t^\nu \frac{\partial\psi_i}{\partial x}(u, Z_u)rX_u du\Big] \qquad (4.28)\\
&+ \mathbb{E}\Big[\int_t^\nu \lambda_i^a(p,s)\Big(\psi_i(u, X_{u-} + p + \frac{s}{2}, y-1, p + \frac{\delta}{2}(\eta_u^a + \eta_u^b), s + \delta(\eta_u^a - \eta_u^b))\\
&\qquad -\psi(u, X_{u-}, y, p, s)\Big)\mathbb{1}_{\{y > y_{min}\}}du\Big]\\
&+ \mathbb{E}\Big[\int_t^\nu \lambda_i^b(p,s)\Big(\psi(u, X_{u-} - p + \frac{s}{2}, y+1, p - \frac{\delta}{2}(\eta_u^a + \eta_u^b), s - \delta(\eta_u^a - \eta_u^b))\\
&\qquad -\psi(u, X_{u-}, y, p, s)\Big)\mathbb{1}_{\{y < y_{max} \text{ and } X_{u-} - p + \frac{s}{2} \ge x_{min}\}}du\Big]\\
&+ \sum_{j\ne i}\mathbb{E}\Big[\int_t^\nu \gamma_{i,j}\Big(\psi_j(u, Z_u) - \psi_i(u, Z_u)\Big)\,du\Big].
\end{aligned}
$$

Plugging (4.28) into (4.27), we obtain

$$
\begin{aligned}
\psi_i(t,z) &\le \mathbb{E}[-g(y)\,(\nu - t)] + \psi_i(t,z) \qquad (4.29)\\
&+ \mathbb{E}\Big[\int_t^\nu \frac{\partial\psi_i}{\partial t}(u, Z_u)du + \int_t^\nu \frac{\partial\psi_i}{\partial x}(u, Z_u)rX_u du\Big]\\
&+ \mathbb{E}\Big[\int_t^\nu \lambda_i^a(p,s)\Big(\psi_i(u, X_{u-} + p + \frac{s}{2}, y-1, p + \frac{\delta}{2}(\eta_u^a + \eta_u^b), s + \delta(\eta_u^a - \eta_u^b))\\
&\qquad -\psi_i(u, X_{u-}, y, p, s)\Big)\mathbb{1}_{\{y > y_{min}\}}du\Big]\\
&+ \mathbb{E}\Big[\int_t^\nu \lambda_i^b(p,s)\Big(\psi_i(u, X_{u-} - p + \frac{s}{2}, y+1, p - \frac{\delta}{2}(\eta_u^a + \eta_u^b), s - \delta(\eta_u^a - \eta_u^b))\\
&\qquad -\psi_i(u, X_{u-}, y, p, s)\Big)\mathbb{1}_{\{y < y_{max} \text{ and } X_{u-} - p + \frac{s}{2} \ge x_{min}\}}du\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \sum_{j\neq i} \mathbb{E}\Big[\int_t^\nu \gamma_{i,j}\Big(\psi_j(u,Z_u)-\psi_i(u,Z_u)\Big)\,du\Big]\\
&+ \quad R_i(t,z)+h^2,
\end{aligned}
$$

where we have set

$$
R_i(t,z)=\mathbb{E}\Big[\Big(U_L\Big(\hat\tau^\alpha, xe^{r(\hat\tau^\alpha-t)},y,p,s\Big)-\psi_i(\hat\tau^\alpha,xe^{r(\hat\tau^\alpha-t)},y,p,s)\Big)\,\mathbb{1}_{\{\hat\tau^\alpha\le(t+h)\wedge\hat\theta\}}\Big].
$$

As $x>x_{min}$, we have $xe^{r(\hat\tau^\alpha-t)}>x_{min}$ on $\{\hat\tau^\alpha\le t+h\}$ for h small enough. Hence, we have:

$$
\begin{aligned}
R_i(t,z) &= \mathbb{E}\Big[[U_L-\psi_i]\Big(\theta^a,xe^{r(\theta^a-t)},y_{min},p,s\Big)\,\mathbb{1}_{\{\hat\tau^\alpha=\theta^a\le(t+h)\wedge\hat\theta;\ y=y_{\min}\}}\Big]\\
&\quad +\mathbb{E}\Big[[U_L-\psi_i]\Big(\theta^b,xe^{r(\theta^b-t)},y,p,s\Big)\,\mathbb{1}_{\{\hat\tau^\alpha=\theta^b\le(t+h)\wedge\hat\theta;\ y=y_{max}\ \text{or}\ xe^{r(\hat\tau^\alpha-t)}<x_{min}+p-\frac{s}{2}\}}\Big]\\
&= \mathbb{E}\Big[\int_t^\nu \lambda_i^a(p,s)\Big[U_L-\psi_i\Big](u,X_{u-},y,p,s)\mathbb{1}_{\{y=y_{min}\}}du\Big] \qquad (4.30)\\
&\quad +\mathbb{E}\Big[\int_t^\nu \lambda_i^b(p,s)\Big[U_L-\psi_i\Big](u,X_{u-},y,p,s)\Big)\mathbb{1}_{\{y=y_{max}\ \text{or}\ xe^{r(u-t)}<x_{min}+p-\frac{s}{2}\}}du\Big]
\end{aligned}
$$

At this point we remark that, for $u\in(t,\nu]$,

$$
\begin{aligned}
\mathbb{1}_{\{xe^{r(u-t)}\ge x_{min}+p-\frac{s}{2}\}} &= 1_{\{x>x_{min}+p-\frac{s}{2}\ \text{or}\ x=x_{min}+p-\frac{s}{2}\ge 0\}}\\
\mathbb{1}_{\{xe^{r(u-t)}<x_{min}+p-\frac{s}{2}\}} &= 1_{\{x<x_{min}+p-\frac{s}{2}\ \text{or}\ x=x_{min}+p-\frac{s}{2}<0\}}.
\end{aligned}
$$

Therefore, plugging (4.30) into (4.29), it follows from the definition of the Hamiltonian that

$$
0\le \mathbb{E}[\int_t^\nu -g(y)+\mathcal{H}_i(u,Z_{u-},\psi,\frac{\partial\psi}{\partial x})\,du].
$$

From the right continuity of the processes $(N_t)_t$, $(N_t^a)_t$ and $(N_t^b)_t$, we get

$$
\lim_{h\to0^+}\frac{1}{h}[\int_t^\nu -g(y)+\mathcal{H}_i(u,Z_{u-},\psi,\frac{\partial\psi}{\partial x})\,du]=-g(y)+\mathcal{H}_i(t,z,\psi,\frac{\partial\psi}{\partial x}),\quad \text{a.s.}
$$

Since ψ is smooth with respect to the variables t and x and the process $(u,Z_u)_{t\le u\le t+h}$ is bounded on $\{u\le\nu\}$, we deduce from the dominated convergence theorem that

$$
0\le\lim_{h\to0^+}\frac{1}{h}\mathbb{E}[\int_t^\nu -g(y)+\mathcal{H}_i(u,Z_{u-},\psi,\frac{\partial\psi}{\partial x})\,du]=-g(y)+\mathcal{H}_i(t,z,\psi,\frac{\partial\psi}{\partial x}).
$$

Therefore $(v_i)_{1\le i\le m}$ is a subsolution of the system of variational inequalities (4.23) on $\{1,...,m\}\times\mathcal{S}$.

□

Lemma 4.4 *The family of objective functions $(v_i)_{1\le i\le m}$ is a supersolution of the system of variational inequalities (4.23) on $\{1,...,m\}\times\mathcal{S}$.*

Proof: The proof is very similar to the one of the previous Lemma. Indeed, let $(i,t,z) \in \{1,..,m\} \times \mathcal{S}$ and $(\psi_j)_{1\leq j\leq m}$ a family of functions such that for all $(j,\zeta,\pi,\sigma) \in \{1,..,m\} \times \{y_{min},..,y_{max}\} \times \frac{\delta}{2}\mathbb{N} \times \delta\{1,..,K\}$, $(u,\xi) \to \psi_j(u,\xi,\zeta,\pi,\sigma)$ is a $\mathcal{C}^1$ function on $\{(u,\xi) \in [0,T) \times [x_{min},+\infty) : (u,\xi,\zeta,\pi,\sigma) \in \mathcal{S}\}$ and $v - \psi$ has a global minimum at $(i,t,z) \in \{1,...,m\} \times \mathcal{S}$. Without loss of generality we assume that $0 = (v-\psi)(i,t,z)$.
Let $0 < h < T - t$, $\alpha \in \mathcal{A}(t,z)$ an admissible strategy and set $\widehat{\tau}^\alpha := \tau^{i,t,z,\alpha} \wedge T$. The dynamic programming principle (3.19) implies the opposite inequality of (4.27) without the term h^2. Then, we may apply Itô's formula to $\psi_{I_u}(u, Z_u)$ between t and ν and by taking expectation, we obtain equation (4.28). Finally, we obtain the opposite inequality of (4.29) for any admissible strategy. Therefore, we get

$$0 \geq \mathbb{E}[\int_t^\nu -g(y) + \mathcal{H}_i(u, Z_{u-}, \psi, \frac{\partial \psi}{\partial x})\, du]$$

and we conclude by dividing by h and letting h going to 0.

□

We now turn to the uniqueness result. First, we give an equivalent formulation of the viscosity solutions which is useful to prove the comparison result, see [14].

Lemma 4.5
Let $(\phi_i)_{1\leq i\leq m}$ a family of functions defined on $\mathcal{S}$. ϕ is a viscosity supersolution (resp. subsolution) of the system of variational inequalities (4.23) on $\{1,...,m\} \times \mathcal{S}$ if,

$$-\frac{\partial \psi_{i_0}}{\partial t}(t_0,z_0) - \mathcal{H}_{i_0}\left(t_0, z_0, \phi, \frac{\partial \psi}{\partial x}\right) \geq 0, \quad \textit{(resp. } \leq 0\textit{)} \tag{4.31}$$

whenever, for all $(j,y,p,s) \in \{1,..,m\} \times \{y_{min},..,y_{max}\} \times \frac{\delta}{2}\mathbb{N} \times \delta\{1,..,K\}$, $(t,x) \to \psi_j(t,x,y,p,s)$ is a $\mathcal{C}^1$ function on $\{(t,x) \in [0,T) \times [x_{min},+\infty) : (t,x,y,p,s) \in \mathcal{S}\}$ and $\phi - \psi$ has a global minimum (resp. maximum) at $(i_0,t_0,z_0) \in \{1,...,m\} \times \mathcal{S}$.

With this equivalent definition of viscosity solutions, we are now able to establish the following comparison result.

Lemma 4.6 *Let $(u_i)_{1\leq i\leq m}$ (resp. $(w_i))_{1\leq i\leq m}$) be a viscosity subsolution (resp. supersolution) of (4.23) on $[0,T] \times \mathcal{S}$ satisfying the growth condition (4.24) and such that for all $(j,y,p,s) \in \{1,..,m\} \times \{y_{min},..,y_{max}\} \times \frac{\delta}{2}\mathbb{N} \times \delta\{1,..,K\}$, $(t,x) \to u_j(t,x,y,p,s)$ (resp. $w_j(t,x,y,p,s)$) is a continuous function on $\{(t,x) \in [0,T) \times [x_{min},+\infty) : (t,x,y,p,s) \in \mathcal{S}\}$. Assume that for all $(i,t,z) \in \{1,..,m\} \times \mathcal{S}$ we have*

$$u_i(T,z) \leq w_i(T,z), \tag{4.32}$$

$$u_i(t,x_{min},y,p,s) \leq w_i(t,x_{min},y,p,s) \tag{4.33}$$

then we have $u_i(t,z) \leq w_i(t,z)$, for all $(i,t,z) \in \{1,..,m\} \times \mathcal{S}$.

Proof: Let β_1, β_2 and β_3 be positive constants such that $\beta_3 > C_3$. We set $h(t,z) = e^{\beta_2(T-t)}\left(\beta_1 + x^2 + e^{\beta_3 p}\right)$, for $(t,z) \in \mathcal{S}$ and $w_i^\gamma = (1-\gamma)w_i + \gamma h$ for $i \in \{1,..,m\}$ and

$\gamma \in (0,1)$.
First we show that h is a supersolution of (4.23) on $\mathcal{S}$. For $(i,t,z) \in \{1,..,m\} \times \mathcal{S}$, we have

$$\begin{aligned} -\frac{\partial h}{\partial t}(t,z) - \mathcal{H}_i\left(t,z,h,\frac{\partial h}{\partial x}\right) &= \beta_2 e^{\beta_2(T-t)}\left(\beta_1 + x^2 + e^{\beta_3 p}\right) - e^{\beta_2(T-t)} 2rx^2 + g(y) \\ &\quad - \sup_{\alpha \in A(t,z)} \Big[\lambda_i^a(p,s)\left(\mathcal{A}h(t,z,\alpha) - h(t,z)\right) \\ &\quad + \lambda_i^b(p,s)\left(\mathcal{B}h(t,z,\alpha) - h(t,z)\right)\Big]. \end{aligned} \tag{4.34}$$

For $\alpha \in A(t,z)$ and $\beta_1 > 1$, we have

$$\begin{aligned} \mathcal{A}h(t,z,\alpha) - h(t,z) &= \Big[U_L(t,z) - h(t,z)\Big]\mathbb{1}_{y=y_{min}} \\ &\quad e^{\beta_2(T-t)}\Big[(x+p+\frac{s}{2})^2 + e^{\beta_3\left(p+\frac{\delta}{2}(\eta^a+\eta^b)\right)} - x^2 - e^{\beta_3 p}\Big]\mathbb{1}_{y\neq y_{min}} \\ &\leq e^{\beta_2(T-t)}\Big[(p+\frac{s}{2})(2x+p+\frac{s}{2}) + e^{\beta_3 p}\left(e^{\beta_3\delta}-1\right)\Big]\mathbb{1}_{y\neq y_{min}} \end{aligned} \tag{4.35}$$

In the same way, if we set $\mathcal{C} = \{z \in \mathbb{S} :\ y \neq y_{min} \text{ or } x > x_{min}+p-\frac{s}{2} \text{ or } x = x_{min}+p-\frac{s}{2} \geq 0\}$ then we have

$$\begin{aligned} \mathcal{B}h(t,z,\alpha) - h(t,z) &\leq e^{\beta_2(T-t)}\Big[(x-p+\frac{s}{2})^2 + e^{\beta_3\left(p-\frac{\delta}{2}(\varepsilon^a+\varepsilon^b)\right)} - x^2 - e^{\beta_3 p}\Big]\mathbb{1}_{z\in\mathcal{C}} \\ &\leq e^{\beta_2(T-t)}\Big[(-p+\frac{s}{2})(2x-p+\frac{s}{2})\Big]\mathbb{1}_{z\in\mathcal{C}} \end{aligned} \tag{4.36}$$

Now we plug inequalities (4.35) and (4.36) in inequality (4.34), and obtain that for β_1 and β_2 large enough, there exists $\eta > 0$ such that

$$-\frac{\partial h}{\partial t}(t,z) - \mathcal{H}_i\left(t,z,h,\frac{\partial h}{\partial x}\right) > \eta.$$

To prove Lemma 4.6, it suffices to show that for all $\gamma \in (0,1)$, we have

$$\max_{j\in\{1,..,m\}} \sup_{(t,z)\in\mathcal{S}} \left(u_j - w_j^\gamma\right) \leq 0,$$

since the required result is obtained by letting γ going to 0.
We shall argue by contradiction and assume that

$$\zeta := \max_{j\in\{1,..,m\}} \sup_{(t,z)\in\mathcal{S}} \left(u_j - w_j^\gamma\right) > 0.$$

From the growth condition satisfied by u and w, we deduce that for all $(j,t,z) \in \{1,..,m\} \times \mathcal{S}$,

$$\left(u_j - w_j^\gamma\right)(t,z) \leq C_1 + C_2 e^{C_3 p} - \gamma e^{\beta_2(T-t)}\left(\beta_1 + x^2 + e^{\beta_3 p}\right).$$

Hence, we have $\lim_{p+x\to+\infty}\left(u_j - w_j^\gamma\right)(t,z) = -\infty$ and this implies that there exists $(i^*,t^*,z^*) \in \{1,..,m\} \times \mathcal{S}$ such that

$$\left(u_{i^*} - w_{i^*}^\gamma\right)(t^*,z^*) = \zeta > 0.$$

If $t^* = T$ or $x^* = x_{min}$, we deduce from the boundary conditions satisfied by u and w that

$$0 < \zeta \leq \gamma \Big[u_{i^*}(t^*, z^*) - e^{\beta_2 (T - t^*)} \beta_1 \Big] \leq 0,$$

since $u_i \leq 1 \leq \beta_1$. Therefore, we have $t^* < T$ and $x^* > x_{min}$.
We now distinguish the following two cases.

First case: Assume that $x^* = x_{min} + p^* - \frac{s^*}{2}$.
We introduce a sequence $(x_n)_{n \geq 0}$ taking values in $(x_{min}, +\infty) \backslash \{x^*\}$ and such that $\lim_{n \to +\infty} x_n = x^*$. We set $z_n = (x_n, y^*, p^*, s^*)$ and, for $n \in \mathbb{N}$,

$$\zeta_n = \max_{j \in \{1, .., m\}} \sup_{(t,z) \in \mathcal{S}} \left[u_j(t,z) - w_j^\gamma(t,z) - \frac{\|z_n - z\|^2}{\|z_n - z^*\|^2} \right].$$

It follows from the growth conditions satisfied by u and w, that there exists a sequence $(i_n^*, t_n^*, z_n^*)_{n \geq 0}$ taking values in $\{1, .., m\} \times \mathcal{S}$ and such that, for all $n \in \mathbb{N}$,

$$\zeta_n = u_{i_n^*}(t_n^*, z_n^*) - w_{i_n^*}^\gamma(t_n^*, z_n^*) - \frac{\|z_n - z_n^*\|^2}{\|z_n - z^*\|^2}.$$

We notice that

$$\zeta \geq \zeta_n \geq (u_{i^*} - w_{i^*}^\gamma)(t^*, z_n).$$

From the continuity properties of u and w, it follows that $\lim_{n \to +\infty} \zeta_n = \zeta > 0$ and

$$\lim_{n \to +\infty} \frac{\|z_n - z_n^*\|^2}{\|z_n - z^*\|^2} \leq \lim_{n \to +\infty} \zeta - \zeta_n = 0.$$

Therefore, there exists $N \in \mathbb{N}$ such that, for all $n \geq N$, we have $\zeta_n > 0$, $(y_n^*, p_n^*, s_n^*) = (y^*, p^*, s^*)$ and $x_n^* \in (x_{min}, +\infty) \setminus \{x^*\}$. Let $n \geq N$, if $t_n^* = T$, we would have the following contradiction

$$0 < \zeta_n \leq \gamma \Big[u_{i_n^*}(t_n^*, z_n^*) - \beta_1 \Big] \leq 0.$$

therefore $t_n^* < T$.

For $(t, z, z') \in [0, T] \times \mathbb{S}^2$, we define the function:

$$\Phi_n(t, z, z') \quad = \quad u_{i_n^*}(t, z) - w_{i_n^*}^\gamma(t, z') - \frac{\|z - z'\|^2}{\varepsilon} - \frac{\|z_n - z\|^2}{\|z_n - z^*\|^2} - \frac{\|z_n - z'\|^2}{\|z_n - z^*\|^2},$$

where $\varepsilon > 0$.

By a classical argument in the theory of viscosity solutions, we can show that there exists $(t_n(\varepsilon), z_n^1(\varepsilon), z_n^2(\varepsilon))_{n \geq 0} \in [0, T] \times \mathbb{S}^2$ such that

$$\Phi_n(t_n(\varepsilon), z_n^1(\varepsilon), z_n^2(\varepsilon)) = \sup_{(t,z,z') \in [0,T] \times \mathbb{S}^2} \Phi_n(t, z, z').$$

Moreover we can prove that

$$\lim_{\varepsilon \to 0} (t_n(\varepsilon), z_n^1(\varepsilon), z_n^2(\varepsilon)) = (t_n^*, z_n^*, z_n^*) \quad \text{and} \quad \lim_{\varepsilon \to 0} \frac{\|z_n^1(\varepsilon) - z_n^2(\varepsilon)\|^2}{\varepsilon} = 0.$$

To simplify notations, we shall omit to precise the dependency on ε.

Notice that y, p and s are discrete variables thus, for ε small enough, we have

$$(y_n^1, p_n^1, s_n^1) = (y^*, p^*, s^*) = (y_n^2, p_n^2, s_n^2).$$

We introduce two functions defined on $\{1, .., m\} \times \mathcal{S}$ by

$$\begin{aligned}
\psi_i(t,z) &= w_{i_n^*}^\gamma(t_n, z_n^2) + \frac{\|z - z_n^2\|^2}{\epsilon} + \frac{\|z_n - z\|^2}{\|z_n - z^*\|^2} + \frac{\|z_n - z_n^2\|^2}{\|z_n - z^*\|^2} \\
\phi_i(t,z) &= u_{i_n^*}(t_n, z_n^1) - \frac{\|z_n^1 - z\|^2}{\epsilon} - \frac{\|z_n - z_n^1\|^2}{\|z_n - z^*\|^2} - \frac{\|z_n - z\|^2}{\|z_n - z^*\|^2}.
\end{aligned}$$

From the definition of ψ and ϕ, $u - \psi$ has a local maximum at (i_n^*, t_n, z_n^1) and $w^\gamma - \phi$ has a local minimum at (i_n^*, t_n, z_n^2), which implies from the equivalent formulation of the viscosity solutions that

$$- \frac{\partial \psi_{i_n^*}}{\partial t}(t_n, z_n^1) - \mathcal{H}_{i_n^*}\left(t_n, z_n^1, u, \frac{\partial \psi}{\partial x}\right) \leq 0, \tag{4.37}$$

$$- \frac{\partial \phi_{i_n^*}}{\partial t}(t_n, z_n^2) - \mathcal{H}_{i_n^*}\left(t_n, z_n^2, w^\gamma, \frac{\partial \phi}{\partial x}\right) \geq \gamma\eta > 0. \tag{4.38}$$

From inequalities (4.37) and (4.38), we have

$$\begin{aligned}
-\gamma\eta &\geq -\frac{\partial \psi_{i_n^*}}{\partial t}(t_n, z_n^1) + \frac{\partial \phi_{i_n^*}}{\partial t}(t_n^2, z_n^2) - \mathcal{H}_{i_n^*}\left(t_n, z_n^1, u, \frac{\partial \psi}{\partial x}\right) + \mathcal{H}_{i_n^*}\left(t_n, z_n^2, w^\gamma, \frac{\partial \phi}{\partial x}\right) \\
&\geq \Delta_1 + \Delta_2 + \Delta_3,
\end{aligned} \tag{4.39}$$

where we have set

$$\begin{aligned}
\Delta_1 &= -\frac{\partial \psi_{i_n^*}}{\partial t}(t_n, z_n^1) + \frac{\partial \phi_{i_n^*}}{\partial t}(t_n, z_n^2) - r x_n^1 \frac{\partial \psi_{i_n^*}}{\partial x}(t_n, z_n^1) + r x_n^2 \frac{\partial \phi_{i_n^*}}{\partial x}(t_n, z_n^2) \\
&= -\frac{2r}{\varepsilon}(x_n^1 - x_n^2)^2 + \frac{2r}{(x_n - x^*)^2}\left(x_n^1(x_n - x_n^1) - x_n^2(x_n - x_n^2)\right), \\
\Delta_2 &= \sum_{j \neq i_n^*} \gamma_{i_n^*, j}\left(\left[u_{i_n^*}(t_n, z_n^1) - w_{i_n^*}^\gamma(t_n, z_n^2)\right] - \left[u_j(t_n, z_n^1) - w_j^\gamma(t_n, z_n^2)\right]\right) \geq 0 \\
\Delta_3 &= \sup_{\alpha \in A(t_n, z_n^2)}\left[\lambda_{i_n^*}^a\left(\mathcal{A} w_{i_n^*}^\gamma(t_n, z_n^2, \alpha) - w_{i_n^*}^\gamma(t_n, z_n^2)\right) + \lambda_{i_n^*}^b\left(\mathcal{B} w_{i_n^*}^\gamma(t_n, z_n^2, \alpha) - w_{i_n^*}^\gamma(t_n, z_n^2)\right)\right] \\
&\quad - \sup_{\alpha \in A(t_n, z_n^1)}\left[\lambda_{i_n^*}^a\left(\mathcal{A} u_{i_n^*}(t_n, z_n^1, \alpha) - u_{i_n^*}(t_n, z_n^1)\right) + \lambda_{i_n^*}^b\left(\mathcal{B} u_{i_n^*}(t_n, z_n^1, \alpha) - u_{i_n^*}(t_n, z_n^1)\right)\right].
\end{aligned}$$

We have $\lim_{\varepsilon \to 0} \Delta_1 = 0$ and $\lim_{\varepsilon \to 0} \Delta_2 \geq 0$. Indeed, we have

$$\begin{aligned}
\lim_{\varepsilon \to 0} \Delta_2 &= \sum_{j \neq i_n^*} \gamma_{i_n^*, j}\left(\left[u_{i_n^*}(t_n^*, z_n^*) - w_{i_n^*}^\gamma(t_n^*, z_n^2)\right] - \left[u_j(t_n^*, z_n^*) - w_j^\gamma(t_n^*, z_n^*)\right]\right) \\
&= \sum_{j \neq i_n^*} \gamma_{i_n^*, j}\left(\zeta_n - \left[u_j(t_n^*, z_n^*) - w_j^\gamma(t_n^*, z_n^*) - \frac{\|z_n - z_n^*\|^2}{\|z_n - z^*\|^2}\right]\right) \\
&\geq 0.
\end{aligned}$$

We may conclude the proof by proving that $\liminf_{\varepsilon\to 0}\Delta_3 \geq 0$. As $(y_n^1,p_n^1,s_n^1)=(y^*,p^*,s^*)=(y_n^2,p_n^2,s_n^2)$, there exists $\alpha^*\in A(t_n,z_n^1)=A(t_n,z_n^2)$ such that $\Delta_3\geq \lambda_{i_n^*}^a\delta_1+\lambda_{i_n^*}^b\delta_2$, where

$$\begin{aligned}\delta_1 &= \Big[\mathcal{A}w_{i_n^*}^\gamma(t_n,z_n^2,\alpha^*)-w_{i_n^*}^\gamma(t_n,z_n^2)\Big]-\Big[\mathcal{A}u_{i_n^*}(t_n,z_n^1,\alpha^*)-u_{i_n^*}(t_n,z_n^1)\Big]\\ \delta_2 &= \Big[\mathcal{B}w_{i_n^*}^\gamma(t_n,z_n^2,\alpha^*)-w_{i_n^*}^\gamma(t_n,z_n^2)\Big]-\Big[\mathcal{B}u_{i_n^*}(t_n,z_n^1,\alpha^*)-u_{i_n^*}(t_n,z_n^1)\Big].\end{aligned}$$

We set $\alpha^*=(\varepsilon^a,\varepsilon^b,\eta^a,\eta^b)$ and for $z=(x,y,p,s)\in\mathbb{S}$, we shall use the following notations:

$$\begin{aligned}\bar{z} &= \Big(x+p+\frac{s}{2},y-1,p+\frac{\delta}{2}(\eta^a+\eta^b),s+\delta(\eta^a-\eta^b)\Big)\\ \hat{z} &= \Big(x-p+\frac{s}{2},y+1,p-\frac{\delta}{2}(\varepsilon^a+\varepsilon^b),s+\delta(\varepsilon^a-\varepsilon^b)\Big).\end{aligned}$$

If $y^*=y_{min}$, we have

$$\begin{aligned}\lim_{\varepsilon\to 0}\delta_1 &= \lim_{\varepsilon\to 0}\Big[U_L(t_n,z_n^2)-w_{i_n^*}^\gamma(t_n,z_n^2)\Big]-\Big[U_L(t_n,z_n^1)-u_{i_n^*}(t_n,z_n^1)\Big]\\ &= \lim_{\varepsilon\to 0}\Big[u_{i_n^*}(t_n,z_n^1)-w_{i_n^*}^\gamma(t_n,z_n^2)\Big]\\ &= u_{i_n^*}(t_n^*,z_n^*)-w_{i_n^*}^\gamma(t_n^*,z_n^*)\\ &= \zeta_n+\frac{\|z_n-z_n^*\|^2}{\|z_n-z^*\|^2}>0.\end{aligned}$$

If $y^*>y_{min}$, we then have

$$\begin{aligned}\lim_{\varepsilon\to 0}\delta_1 &= \lim_{\varepsilon\to 0}\Big[w_{i_n^*}^\gamma(t_n,\bar{z}_n^2)-w_{i_n^*}^\gamma(t_n,z_n^2)\Big]-\Big[u_{i_n^*}(t_n,\bar{z}_n^1)-u_{i_n^*}(t_n,z_n^1)\Big]\\ &= \zeta_n-\Big[u_{i_n^*}(t^*,\bar{z}_n^*)-w_{i_n^*}^\gamma(t_n^*,\bar{z}_n^*)\Big]+\frac{\|z_n-z_n^*\|^2}{\|z_n-z^*\|^2}\\ &\geq \zeta_n-\zeta.\end{aligned}$$

On the other hand, if $y^*=y_{max}$ or $x_n^*<x_{min}+p^*-\frac{s^*}{2}$, then, for n large enough, we get

$$\lim_{\varepsilon\to 0}\delta_2=\lim_{\varepsilon\to 0}\Big[U_L(t_n,z_n^2)-w_{i_n^*}^\gamma(t_n,z_n^2)\Big]-\Big[U_L(t_n,z_n^1)-u_{i_n^*}(t_n,z_n^1)\Big]\geq\zeta_n>0.$$

Moreover, if $y^*<y_{max}$ and $x_n^*>x_{min}+p^*-\frac{s^*}{2}$, we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0}\delta_2 &= \lim_{\varepsilon\to 0}\Big[w_{i_n^*}^\gamma(t_n,\hat{z}_n^2)-w_{i_n^*}^\gamma(t_n,z_n^2)\Big]-\Big[u_{i_n^*}(t_n,\hat{z}_n^1)-u_{i^*}(t_n,z_n^1)\Big]\\ &\geq \zeta_n-\Big[u_{i_n^*}(t^*,\hat{z}_n^*)-w_{i_n^*}^\gamma(t_n^*,\hat{z}_n^*)\Big]\\ &\geq \zeta_n-\zeta.\end{aligned}$$

Hence, if we let ε going to 0 in inequality (4.39), we get $-\gamma\eta\geq 2(\zeta_n-\zeta)$. We obtain a contradiction by letting n going to $+\infty$.

Second case: We assume that $x^*\neq x_{min}+p^*-\frac{s^*}{2}$. As we shall work far from the set of discontinuity of the operator $\mathcal{B}$, this case is more simple and we just give the sketch of the proof. For $(t,z,z')\in[0,T]\times\mathbb{S}^2$, we define the function:

$$\Phi(t,z,z') = u_{i^*}(t,z)-w_{i^*}^\gamma(t,z')-\frac{\|z-z'\|^2}{\varepsilon},$$

where $\varepsilon > 0$.

By a classical argument in the theory of viscosity solutions, we can show that there exists $(t(\varepsilon), z^1(\varepsilon), z^2(\varepsilon)) \in [0,T] \times \mathbb{S}^2$ such that

$$\Phi(t(\varepsilon), z^1(\varepsilon), z^2(\varepsilon)) = \sup_{(t,z,z') \in [0,T] \times \mathbb{S}^2} \Phi(t, z, z').$$

Moreover we can prove that

$$\lim_{\varepsilon \to 0} (t(\varepsilon), z^1(\varepsilon), z^2(\varepsilon)) = (t^*, z^*, z^*) \quad \text{and} \quad \lim_{\varepsilon \to 0} \frac{\|z^1(\varepsilon) - z^2(\varepsilon)\|^2}{\epsilon} = 0.$$

The function $\hat{\phi}$ and $\hat{\psi}$ defined by

$$\begin{aligned} \hat{\psi}_i(t,z) &= w^{\gamma}_{i^*}(t(\epsilon), z^2(\varepsilon)) + \frac{\|z - z^2(\varepsilon)\|^2}{\epsilon} \\ \hat{\phi}_i(t,z) &= u_{i^*}(t(\epsilon), z^1(\epsilon)) - \frac{\|z^1(\epsilon) - z\|^2}{\epsilon} \end{aligned}$$

are respectively super and sub solution of equation (4.23) then we get

$$-\gamma\eta \geq -\frac{\partial \hat{\psi}_{i^*}}{\partial t}(t(\varepsilon), z^1(\varepsilon)) + \frac{\partial \hat{\phi}_{i^*}}{\partial t}(t(\varepsilon), z^2(\varepsilon)) - \mathcal{H}_{i^*}\left(t(\varepsilon), z^1(\varepsilon), u, \frac{\partial \hat{\psi}}{\partial x}\right) + \mathcal{H}_{i^*}\left(t(\varepsilon), z^2(\varepsilon), w^{\gamma}, \frac{\partial \hat{\phi}}{\partial x}\right).$$

We conclude the proof by letting ε going to and get the following contradiction $0 > -\gamma\eta \geq 0$

□

5 Numerical Results

In this section, we present the results of a numerical method used to approximate the solution of the system of equations (4.23). This method is based on the following iterative scheme allowing us to obtain the HJB (4.23) as a limit of HJB equations (we refer to [6] for more details on this procedure):
$\forall i \in \{1, ..., m\}$:

$$\begin{aligned} v^0_i(t,z) &= U_L(t,z), \qquad (t,z) \in \bar{\mathcal{S}} \\ -\frac{\partial v^n_i}{\partial t} - \hat{\mathcal{H}}_i(t, z, v^n, v^{n-1}, \frac{\partial v^n}{\partial x}) &= 0, \qquad (t,z) \in \mathcal{S}, \end{aligned}$$

where $\hat{\mathcal{H}}_i$ is the Hamiltonian associated with state i and defined as follows:

$$\begin{aligned} \hat{\mathcal{H}}_i(t, z, v^n, v^{n-1}, \frac{\partial v^n}{\partial x}) &= rx\frac{\partial v^n_i}{\partial x} + \sum_{j \neq i} \gamma_{ij}\left(v^n_j(t,x,y,p,s) - v^n_i(t,x,y,p,s)\right) - g(y) \\ &\quad + \sup_{\alpha \in A(t,z)} \Big[\lambda^a_i(p,s)\left(\mathcal{A}v^{n-1}_i(t,x,y,p,s,\alpha) - v^n_i(t,x,y,p,s)\right) \\ &\quad + \lambda^b_i(p,s)\left(\mathcal{B}v^{n-1}_i(t,x,y,p,s,\alpha) - v^n_i(t,x,y,p,s)\right)\Big] = 0. \end{aligned}$$

The boundary and terminal conditions are given by :

$$\begin{aligned} v_i^n(t, x_{min}, y, p, s) &= U_L(t, x_{min}, y, p, s) \\ v_i^n(T, p, s, x, y) &= U_L(T, x, y, p, s). \end{aligned}$$

After localizing the problem on the discretized grid $(x_{min}, x_{max}] \times \{y_{min}, ..., y_{max}\} \times [p_{min}, p_{max}] \times \delta\{1, ..., K\}$, where $x_{max} \geq x_{min}$ and p_{min} and p_{max} are nonnegative constant, each HJB equation is approximated by a finite difference scheme assuming a Dirichlet boundary condition on the localized boundary

$$v_i^n(t, x_{max}, y, p, s) = U_L(t, x_{max}, y, p, s).$$

Numerical tests are performed for a two-regime case with the following numerical data:

- **Market values**:

  $\rightarrow$ Initial conditions: $x = 5, \quad y = 2, \quad p = 1, \quad s = 0.02.$

  $\rightarrow$ $r = 0.05, \quad \delta = 0.02, \quad \lambda = 20.$

  $\rightarrow$ Impact function: $f(t, y) = \exp(0.09y(T - t))$.

  $\rightarrow$ Intensity functions:

$$\begin{aligned} \lambda_i^a(p, s) &= \frac{\psi_i^a}{p} \exp(-s - 0.01(p-1)) \\ \lambda_i^b(p, s) &= \psi_i^b p \exp(-s + 0.01(p-1)), \end{aligned}$$

$$\text{where } \psi_1^a = 120, \quad \psi_2^a = 80, \quad \psi_1^b = 80, \quad \psi_2^b = 120.$$

- **Constraints**:

  $\rightarrow$ $x_{min} = -20, \quad y_{min} = -10, \quad y_{max} = 10, \quad K = 5, \quad T = 1.$

  $\rightarrow$ Penalty function: $g(y) = y^2 \times 10^{-3}$.

  $\rightarrow$ Utility function: $U(l) = 1 - e^{-0.01l}$ i.e. $\gamma = 0.01$.

- **Numerical values**:

  $\rightarrow$ Localisation: $x_{max} = 20, \quad p_{min} = 1 - 20 \times \frac{\delta}{2}, \quad p_{max} = 1 + 20 \times \frac{\delta}{2}$

  $\rightarrow$ Discretization: $n_x = 40$ and $n_t = 20$.

Remark 5.3 1.) The impact function f and the intensity functions λ_i^a and λ_i^b are respectively inspired from the models studied in [10] and [2].
2.) With these choices of λ^a and λ^b, we suggest that the intensity of the buy market orders is non-increasing with respect to the mid price while the intensity of the sell market orders is non-decreasing with respect to the mid price. Both intensities are non-increasing with respect to the spread.
3.) This choice matches with the following financial interpretation : when the price gets higher, we are likely to have many more investors willing to sell and fewer willing to buy. On the other hand, when the spread gets higher, fewer trading orders are expected.

Figures 1 and 2 : Simulated paths of the cash and the stock inventory position We represent in Figure 1 (resp. Figure 2) a simulated trajectory of market maker's cash position (resp. stock inventory). The cash increases when a buy market order arrives and decreases when a sell market order arrives and vice versa for the stock inventory position.

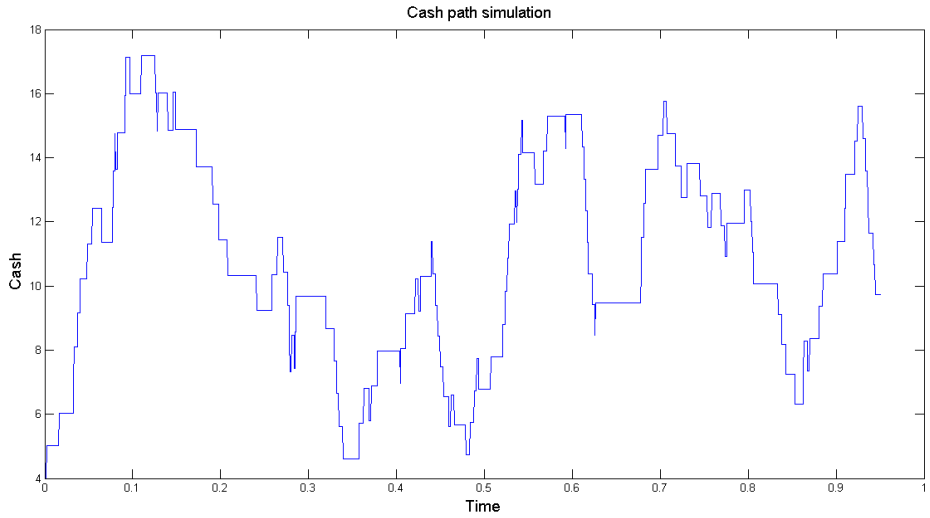


Figure 1: *A simulated trajectory of the market maker's cash position*

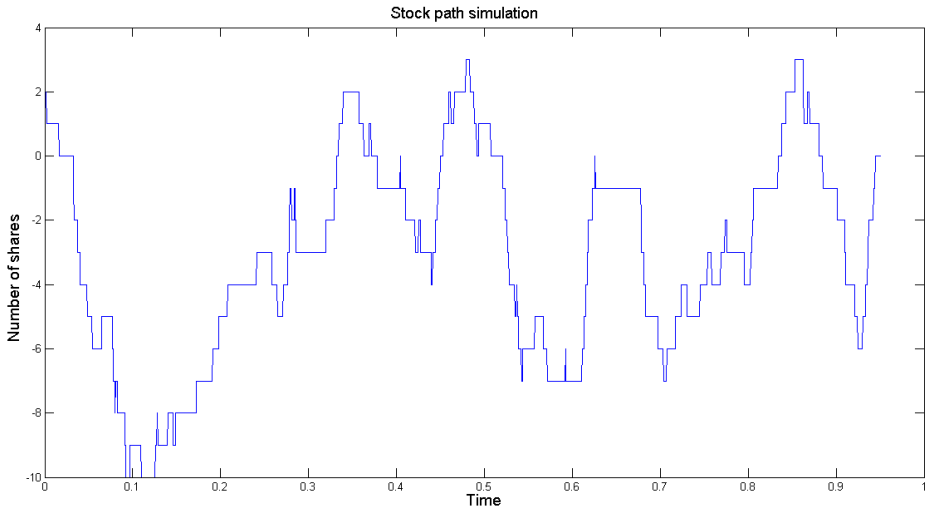


Figure 2: *A simulated trajectory of the market maker's stock inventory position*

Figure 3 and Figure 4: Simulated paths of the bid and ask Prices and the Net Wealth

We represent in Figure 3 and Figure 4 respectively the bid and ask prices, and the market maker's net wealth trajectories according to the optimal policy computed by the iterative procedure described above and used to solve the HJB (4.23).

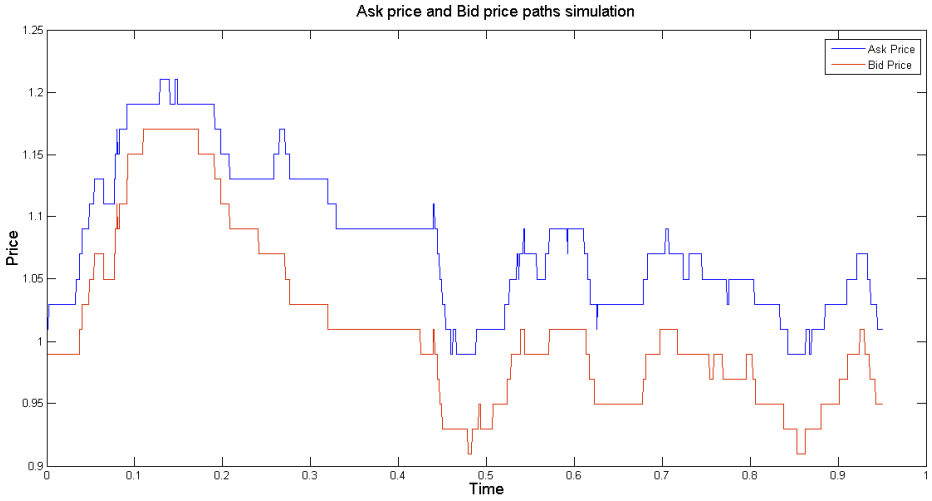


Figure 3: *Bid and ask price paths*

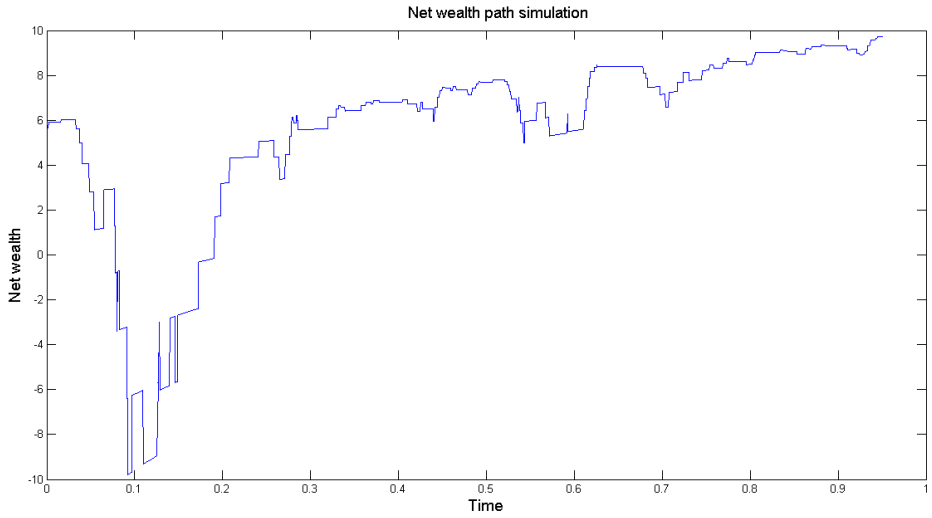


Figure 4: *A path of market maker net wealth,* $(L(t, Z_t)_{t\geq 0}$

Remark 5.4 Financial interpretations:

- Figure 2 shows that between $t = 0$ and $t = 0.1$, there is clearly an imbalance between buy and sell market orders (with buy market orders largely exceeding sell orders). Therefore, the market maker has to short sell the stock in order to satisfy those buy market orders, resulting in an increase of the bid and ask prices (see Figure 3). As the results of the price increase combined with her short position, the market maker's net wealth sharply decreases, see Figure 4. This clearly highlights the difficulty to control the inventory risk the market maker resulting in potential losses when the market is very volatile.

- Between $t = 0.1$ and $t = 0.4$, there is a reversal of the situation with an imbalance of orders in favor of the sell market orders. The market maker takes that opportunity to buy back the shares and controls her inventory risk by keeping her stockholder position near to zero.

- Between $t = 0.4$ and $t = 1$, there are alternatively bull and bear markets,but there is no sharp imbalance between buy and sell market orders as between $t = 0$ and $t = 0.1$. As such, the market maker may be able to constantly increase her net wealth while controlling her inventory position.

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