Arbitrages in a Progressive Enlargement Setting

Anna Aksamit*, Tahir Choulli†, Jun Deng †and Monique Jeanblanc *

November 26, 2013

Abstract

This paper completes the analysis of Choulli et al. [5] and contains two principal contributions. The first contribution consists in providing and analysing many practical examples of market models that admit classical arbitrages while they preserve the No Unbounded Profit with Bounded Risk (NUPBR hereafter) under random horizon and when an honest time is incorporated for particular cases of models. For these markets, we calculate explicitly the arbitrage opportunities. The second contribution lies in providing simple proofs for the stability of the No Unbounded Profit with Bounded Risk under random horizon and after honest time satisfying additional important condition for particular cases of models.

1 Introduction

This paper studies a financial market in which some assets, with prices adapted with respect to a reference filtration $\mathbb{F}$, are traded. One then assumes that an agent has some extra information, and may use strategies adapted to a larger filtration $\mathbb{G}$. This extra information is modeled by the knowledge of some random time τ, when this time occurs. We restrict our study to progressive enlargement of filtration setting, and we pay a particular attention to honest times. Our goal is to detect if the knowledge of τ allows for some arbitrage, i.e., if using $\mathbb{G}$-adapted strategies, the agent can make profit.

In this paper we consider two main notions of no-arbitrage, namely no classical arbitrage and No Unbounded Profit with Bounded Risk. To the best of our knowledge, there are no references for the case of classical arbitrages in a general setting. The goal of the present paper is firstly to introduce the problem, to solve it in some specific cases and to give some explicit examples of classical arbitrages (with a proof different from the one in [7]), and secondly to give, in some specific models, an easy proof of No Unbounded Profit with Bounded Risk condition.

In the case of honest times avoiding stopping times in a continuous filtration, the same problem was studied in Fontana et al. [7] where the authors have investigated several kinds of arbitrages. We refer the reader to that paper for an extensive list of related results in the literature.

The paper is organized as follows: Section 2 presents the problem and recalls some definitions and results on arbitrages and progressive enlargement of filtration. In Section 3 we study two classical situations in enlargment of filtration theory, namely immersion and positive density hypothesis cases. Section 4 concerns honest times, and we show that, in case of a complete market, there exist classical arbitrages before and after the honest time, and we give a way to construct these arbitrages. This fact is illustrated by many examples, where we exhibit these arbitrages in a closed form. In Section 5, we study some examples of non-honest times. In Section 6, we study NUPBR condition before a random time and after an honest time, in some specific examples.

*Laboratoire Analyse et Probabilités, Université d'Evry Val d'Essonne, Evry, France
†Mathematical and Statistical Sciences Depart., University of Alberta, Edmonton, Canada

2 General framework

We consider a filtered probability space $(\Omega, \mathcal{A}, \mathbb{F}, \mathbb{P})$ where the filtration $\mathbb{F}$ satisfies the usual hypotheses and $\mathcal{F}_\infty \subset \mathcal{A}$, and a random time τ (i.e., a positive $\mathcal{A}$-measurable random variable). We assume that the financial market where a risky asset with price S ($\mathbb{F}$-adapted positive process) and a riskless asset S^0 (assumed, for simplicity, to have a constant price so that the risk-free interest rate is null) are traded is arbitrage free. More precisely, without loss of generality we assume that S is a $(\mathbb{P}, \mathbb{F})$-(local) martingale. In this paper, the horizon is equal to ∞.

We denote by $\mathbb{G}$ the progressively enlarged filtration of $\mathbb{F}$ by τ, i.e., the smallest right-continuous filtration that contains $\mathbb{F}$ and makes τ a stopping time defined as

$$\mathcal{G}_t = \cap_{\epsilon>0} \mathcal{F}_{t+\epsilon} \vee \sigma(\tau \wedge (t+\epsilon)).$$

We recall that $(\mathcal{H}')$ hypothesis is said to hold between two filtrations $\mathbb{F}$ and $\mathbb{G}$ where $\mathbb{F} \subset \mathbb{G}$ if any $\mathbb{F}$-martingale is a $\mathbb{G}$-semimartingale. For a semimartingale X and a predictable process H, we use the notation $H \bullet X$ for the stochastic integral $\int_0^\cdot H_s dX_s$ when it exists.

We start by an elementary remark: assume that there are no arbitrages using $\mathbb{G}$-predictable strategies and that $\mathbb{P}$ is the unique probability measure making S an $\mathbb{F}$-martingale. So, in particular, the $(S, \mathbb{F})$ market is complete (i.e., the market where (S, S^0) are traded). Then, roughly speaking, S would be a $(\mathbb{Q}, \mathbb{G})$-martingale for some equivalent martingale measure $\mathbb{Q}$, hence would be also a $(\mathbb{Q}, \mathbb{F})$-martingale[1] and $\mathbb{Q}$ will coincide with $\mathbb{P}$ on $\mathbb{F}$. This implies that any $(\mathbb{F}, \mathbb{Q})$-martingale is a $(\mathbb{G}, \mathbb{Q})$-martingale.

Another trivial remark is that, in the particular case where τ is an $\mathbb{F}$-stopping time, the enlarged filtration and the reference filtration are the same. Therefore, no-arbitrage conditions hold before and after τ.

2.1 Illustrative examples

We study here two basic examples, in order to show in a first step how arbitrages can occur in a Brownian filtration, and in a second step that discontinuous models present some difficulties.

2.1.1 Brownian case

Let $dS_t = S_t \sigma dW_t$, where W is a Brownian motion and σ a constant, be the price of the risky asset. This martingale S goes to 0 a.s. when t goes to infinity, hence the random time $\tau = \sup\{t : S_t = S^*\}$ where $S^* = \sup_{s \geq 0} S_s$ is a finite honest time, and obviously leads to an arbitrage before τ: at time 0, buy one share of S (at price S_0), borrow S_0, then, at time τ, reimburse the loan S_0 and sell the share of the asset at price S_τ. The gain is $S_\tau - S_0 > 0$ with an initial wealth null. There are also arbitrages after τ: at time τ, take a short position on S, i.e., hold a self financing portfolio with value V such that $dV_t = -dS_t, V_\tau = 0$. Usually shortselling positions are not admissible, since $V_t = -S_t + S_\tau$ is not bounded below. Here $-S_t + S_\tau$ is positive, hence shortselling is an arbitrage opportunity.

2.1.2 Poisson case

Let N be a Poisson process with intensity λ and M be its compensated martingale. We define the price process S as $dS_t = S_{t-} \psi dM_t, S_0 = 1$ with ψ is a constant satisfying $\psi > -1$ and $\psi \neq 0$, so that

$$S_t = \exp(-\lambda\psi t + \ln(1+\psi) N_t)\,.$$

[1] Note that if S is a $(\mathbb{Q}, \mathbb{G})$-strict local martingale for some equivalent martingale measure $\mathbb{Q}$, one can not deduce that it is also a $(\mathbb{Q}, \mathbb{F})$-local martingale

Since $\frac{N_t}{t}$ goes to λ a.s. when t goes to infinity and $\ln(1+\psi)-\psi<0$, S_t goes to 0 a.s. when t goes to infinity. The random time

$$\tau = \sup\{t\,:\,S_t = S^*\}$$

with $S^* = \sup_{s\geq 0} S_s$ is a finite honest time.

If $\psi > 0$, then $S_\tau \geq S_0$ and an arbitrage opportunity is realized at time τ, with a long position in the stock. If $\psi < 0$, then the arbitrage is not so obvious. We shall discuss that with more details in Section 4.2.

There are arbitrages after τ, selling at time τ a contingent claim with payoff 1, paid at the first time ϑ after τ when $S_t > \sup_{s\leq\tau} S_s$. For $\psi > 0$, it reduces to $S_\tau = \sup_{s\leq\tau} S_s$, and, for $\psi < 0$, one has $S_{\tau-} = \sup_{s\leq\tau} S_s$. At time $t_0 = \tau$, the non informed buyer will agree to pay a positive price, the informed seller knows that the exercise will be never done.

2.2 Admissible portfolio and arbitrages opportunities

In this section, we recall the basic definitions on arbitrages, and we give sufficient conditions for no arbitrages in a market with zero interest rate. We refer to [7] for details.

Let $\mathbb{K}$ be one of the filtrations $\{\mathbb{F},\mathbb{G}\}$. Note that, in order that the integral $\theta \bullet S$ has a meaning for a $\mathbb{G}$ predictable process θ, one needs that S is a $\mathbb{G}$-semimartingale. This requires (on $\{t>\tau\}$) some hypotheses on τ.

For $a \in \mathbb{R}_+$, an element $\theta \in L^{\mathbb{K}}(S)$ is said to be an a-admissible $\mathbb{K}$-strategy if $(\theta \bullet S)_\infty := \lim_{t\to\infty}(\theta \bullet S)_t$ exists and $V_t(0,\theta) := (\theta \bullet S)_t \geq -a$ $\mathbb{P}$-a.s. for all $t \geq 0$. We denote by $\mathcal{A}_a^{\mathbb{K}}$ the set of all a-admissible $\mathbb{K}$-strategies. A process $\theta \in L^{\mathbb{K}}(S)$ is called an *admissible* $\mathbb{K}$*-strategy* if $\theta \in \mathcal{A}^{\mathbb{K}} := \bigcup_{a\in\mathbb{R}_+}\mathcal{A}_a^{\mathbb{K}}$.

An admissible strategy yields an Arbitrage Opportunity if $V(0,\theta)_\infty \geq 0$ $\mathbb{P}$-a.s. and $\mathbb{P}\big(V(0,\theta)_\infty > 0\big) > 0$. In order to avoid confusions, we shall call these arbitrages *classical arbitrages*. If there exists no such $\theta \in \mathcal{A}^{\mathbb{K}}$ we say that the financial market $\mathcal{M}(\mathbb{K}) := (\Omega,\mathbb{K},\mathbb{P};S)$ satisfies the No Arbitrage (NA) condition.

No Free Lunch with Vanishing Risk (NFLVR) holds in the financial market $\mathcal{M}(\mathbb{K})$ if and only if there exists an equivalent martingale measure in $\mathbb{K}$, i.e., a probability measure $\mathbb{Q}$, such that $\mathbb{Q}\sim\mathbb{P}$ and the process S is a $(\mathbb{Q},\mathbb{K})$-local martingale. If NFLVR holds, there are no classical arbitrages.

A non-negative $\mathcal{K}_\infty$-measurable random variable ξ with $\mathbb{P}(\xi>0)>0$ yields an Unbounded Profit with Bounded Risk if for all $x>0$ there exists an element $\theta^x \in \mathcal{A}_x^{\mathbb{K}}$ such that $V(x,\theta^x)_\infty := x + (\theta^x \bullet S)_\infty \geq \xi$ $\mathbb{P}$-a.s. If there exists no such random variable, we say that the financial market $\mathcal{M}(\mathbb{K})$ satisfies the No Unbounded Profit with Bounded Risk (NUPBR) condition.

We recall that NFLVR holds if and only if both NA and NUPBR hold (see [6] Corollary 3.4 and Proposition 3.6, [13]).

A strictly positive $\mathbb{K}$-local martingale $L = (L_t)_{t\geq 0}$ with $L_0 = 1$ and $L_\infty > 0$ $\mathbb{P}$-a.s. is said to be a local martingale deflator in $(S,\mathbb{K})$ on the time horizon $[0,\varrho]$ if the process LS^ϱ is a $\mathbb{K}$-local martingale; here ϱ is a $\mathbb{K}$-stopping time. The important result giving the characterisation of NUPBR condition for strictly positive price process is stated in Theorem 4.12 in [13] and then generalized in Theorem 5 in [16]. We recall it here.

Theorem 2.1 *Let S be a strictly positive $\mathbb{K}$-semimartingale. Then, the NUPBR condition holds in $\mathbb{K}$ if and only if there exists a local martingale deflator in $\mathbb{K}$.*

2.3 Enlargement of filtration results

We now recall some basic results on progressive enlargement of filtrations. The reader can refer to Jeulin [10] and Jeulin and Yor [11] for more information.

Let τ be a random time, i.e., a positive random variable. We define the right-continuous with left limits $\mathbb{F}$-supermartingale

$$Z_t := \mathbb{P}\left(\tau > t \mid \mathcal{F}_t\right).$$

Note that $Z_0 = 1$ if $\mathbb{P}(\tau > 0) = 1$. The optional decomposition of Z leads to an important $\mathbb{F}$-martingale that we denote by m, given by

$$m := Z + A^o, \tag{2.1}$$

where A^o is the $\mathbb{F}$-dual optional projection[2] of $A := \mathbb{1}_{[\![\tau,\infty[\![}$ (so A^o is a non-decreasing process). Note that m is non-negative: indeed $m_t = \mathbb{E}(A^o_\infty + Z_\infty|\mathcal{F}_t)$.

A second important $\mathbb{F}$-supermartingale, defined through

$$\widetilde{Z}_t := \mathbb{P}\left(\tau \geq t \mid \mathcal{F}_t\right),$$

will play a particular rôle in the following. One has $\tilde{Z} = Z + \Delta A^o$, hence the supermartingale $\widetilde{Z}$ admits a decomposition as

$$\widetilde{Z} = m - A^o_- \,. \tag{2.2}$$

We start with the following obvious (but useful) result

Lemma 2.2 *Assume that the financial market $(S, \mathbb{F})$ is complete and let φ be the $\mathbb{F}$-predictable process satisfying $m = 1 + \varphi \bullet S$. If $m_\tau \geq 1$ and $\mathbb{P}(m_\tau > 1) > 0$, then, the $\mathbb{G}$-predictable process $\varphi \mathbb{1}_{[\![0,\tau]\!]}$ is a classical arbitrage strategy in the market "before τ", i.e., in $(S^\tau, \mathbb{G})$.*

Proof: The $\mathbb{F}$-predictable process φ exists due to the market completeness. Hence $\mathbb{1}_{[\![0,\tau]\!]}\varphi$ is a $\mathbb{G}$-predictable admissible self-financing strategy with initial value 1 and final value $m_\tau - 1$ satisfying $m_\tau - 1 \geq 0$ a.s. and $\mathbb{P}(m_\tau - 1 > 0) > 0$, so it is a classical arbitrage strategy in $(S^\tau, \mathbb{G})$. □

2.3.1 Decomposition formula before τ

In a first step, we restrict our attention to what happens before τ. Therefore, we do not require any extra hypothesis on τ, since, for any random time τ, any $\mathbb{F}$-martingale stopped at τ is a $\mathbb{G}$-semimartingale, as established by Jeulin [10, Prop. (4,16)]: to any $\mathbb{F}$-local martingale X, we associate the $\mathbb{G}$-local martingale $\widehat{X}$ (stopped at time τ)

$$\widehat{X}_t := X^\tau_t - \int_0^{t\wedge\tau} \frac{d\langle X, m\rangle^{\mathbb{F}}_s}{Z_{s-}}, \tag{2.3}$$

where, as usual, X^τ is the stopped process defined as $X^\tau_t = X_{t\wedge\tau}$.

An interesting case is the one of pseudo-stopping times. We recall that a random time τ is a pseudo-stopping time is any $\mathbb{F}$-martingale stopped at τ is a $\mathbb{G}$-martingale (see [15]). This is equivalent to the fact that the $\mathbb{F}$-martingale m is constantly equal to 1.

2.3.2 Honest times and decomposition formula after τ

We need to impose conditions on τ such that the ($\mathbb{F}$-martingale) price process S is a $\mathbb{G}$-semimartingale, so that one can define stochastic integrals of $\mathbb{G}$ predictable processes with respect to S. In this paper, we are not interested by necessary and sufficient conditions, these ones being far from tractable (see [10, III, 2,c]). Instead we focus here on honest times.

[2] See Appendix for the definition if needed

Theorem 2.3 *Let τ be a random time. Then, the following conditions are equivalent:*
(i) The random time τ is honest, i.e., for each $t \geq 0$, there exists an $\mathcal{F}_t$-measurable random variable τ_t such that $\tau = \tau_t$ on $\{\tau < t\}$.
(ii) $\widetilde{Z}_\tau = 1$ on $\{\tau < \infty\}$.
(iii) There exists an optional set Λ such that $\tau(\omega) = \sup\{t : (\omega, t) \in \Lambda\}$ on $\{\tau < \infty\}$.
(iv) $A^o_t = A^o_{t\wedge\tau}$.

PROOF: Equivalence between conditions $(i), (ii)$ and (iii) is stated in Theorem (5,1) from [10]. Implication $(i) \Rightarrow (iv)$ comes from analogous arguments as in [4]. To finish the proof, we show implication $(iv) \Rightarrow (iii)$. Let Λ be the support of the measure dA^o, i.e.,

$$\Lambda = \{(\omega, t)|\ \forall \varepsilon > 0\ A^o_t(\omega) > A^o_{t-\varepsilon}(\omega)\}.$$

The set Λ is optional since A^o is an optional process. Then, $[\![\tau]\!] \subset \Lambda$ and $A^o_t = A^o_{t\wedge\tau}$ imply that indeed τ is the end of Λ on $\{\tau < \infty\}$. □

In the case of honest time, any $\mathbb{F}$-martingale X is a $\mathbb{G}$-semimartingale with (predictable) decomposition [10, Prop. (5,10)]

$$X_t = \widehat{X}_t + \int_0^{t\wedge\tau} \frac{d\langle X, m\rangle^{\mathbb{F}}_s}{Z_{s-}} - \int_{t\wedge\tau}^t \frac{d\langle X, m\rangle^{\mathbb{F}}_s}{1 - Z_{s-}}, \tag{2.4}$$

where $\widehat{X}$ is a $\mathbb{G}$-local martingale.

We would like to emphasize the role of $\tilde{Z}$. As we shall see, this process will be important to prove the existence of arbitrage opportunities. We give also a simple characterisation of honest times avoiding $\mathbb{F}$- stopping times.

Lemma 2.4 *A random time τ is an honest time and avoids $\mathbb{F}$-stopping times if and only if $Z_\tau = 1$ a.s. on $(\tau < \infty)$.*

PROOF: Assume that τ is an honest time avoiding $\mathbb{F}$-stopping times. The honesty, by Theorem 2.3, implies that $\tilde{Z}_\tau = 1$ and the avoiding property implies the continuity of A^o since for each $\mathbb{F}$-stopping time T, $\mathbb{E}(\Delta A^o_T) = \mathbb{P}(\tau = T < \infty) = 0$. Then, the relation $\tilde{Z} = Z + \Delta A^o$ leads to the result.
Assume now that $Z_\tau = 1$ on the set $\{\tau < \infty\}$. Then, on $\{\tau < \infty\}$ we have $1 = Z_\tau \leq \widetilde{Z}_\tau \leq 1$, so $\widetilde{Z}_\tau = 1$ and τ is an honest time. Furthermore, as $\Delta A^o_\tau = \widetilde{Z}_\tau - Z_\tau = 0$, for each $\mathbb{F}$ stopping time T we have

$$\mathbb{P}(\tau = T < \infty) = \mathbb{E}(\mathbb{1}_{\{\tau=T\}}\mathbb{1}_{\{\Delta A^o_\tau = 0\}}\mathbb{1}_{(T<\infty)}) = \mathbb{E}(\int_0^\infty \mathbb{1}_{\{u=T\}}\mathbb{1}_{\{\Delta A^o_u = 0\}} dA^o_u) = 0.$$

So τ avoids $\mathbb{F}$ stopping times. □

3 Some particular cases

3.1 Immersion assumption, density hypothesis

We recall that the filtration $\mathbb{F}$ is immersed in $\mathbb{G}$ under $\mathbb{Q}$ if any $(\mathbb{F}, \mathbb{Q})$-local martingale is a $(\mathbb{G}, \mathbb{Q})$-local martingale.

Lemma 3.1 *If the immersion property is satisfied under a probability $\mathbb{Q}$ on $\mathbb{G}$, such that S is a $(\mathbb{F}, \mathbb{Q})$-martingale, all the three concepts of NFLVR, NA and NUPBR hold.*

PROOF: Let S be a $(\mathbb{F}, \mathbb{Q})$-local martingale, then it is a $(\mathbb{G}, \mathbb{Q})$-local martingale as well. □

One says that the random time τ satisfies the positive density hypothesis if there exists a positive $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^+)$-measurable function $(\omega, u) \to \alpha_t(\omega, u)$ which satisfies: for any Borel bounded function φ,

$$\mathbb{E}(\varphi(\tau)|\mathcal{F}_t) = \int_{\mathbb{R}_+} \varphi(u)\alpha_t(u)f(u)du, \quad \mathbb{P}-a.s.$$

where f is the density function of τ. In other terms, the conditional distribution of τ is characterized by the survival probability defined by

$$G_t(\theta) := \mathbb{P}(\tau > \theta|\mathcal{F}_t) = \int_\theta^\infty \alpha_t(u)f(u)du\,.$$

In that case Hypothesis $(\mathcal{H}')$ is satisfied (see [2] or [8]).

Lemma 3.2 *If S is a $(\mathbb{P}, \mathbb{F})$-martingale and if the conditional law of τ with respect to $\mathbb{F}$ satisfies the positive density hypothesis then NFLVR holds for $\mathbb{G}$. Thus both NA and NUPBR hold for $\mathbb{G}$ as well.*

Proof: Indeed, under the positive density hypothesis, it can be proved (see Amendinger's thesis [2] and Grorud and Pontier [8]), that the probability $\mathbb{P}^*$, defined on $\mathbb{F} \vee \sigma(\tau)$ as

$$d\mathbb{P}^*|_{\mathcal{F}_t \vee \sigma(\tau)} = \frac{1}{\alpha_t(\tau)} d\mathbb{P}|_{\mathcal{F}_t \vee \sigma(\tau)}$$

satisfies the following assertions

(i) Under $\mathbb{P}^*$, τ is independent from $\mathcal{F}_t$ for any t

(ii) $\mathbb{P}^*|_{\mathcal{F}_t} = \mathbb{P}|_{\mathcal{F}_t}$

(iii) $\mathbb{P}^*|_{\sigma(\tau)} = \mathbb{P}|_{\sigma(\tau)}$

Note that immersion is satisfied under $\mathbb{P}^*$. It is now obvious that, if S is a $(\mathbb{F}, \mathbb{P})$-martingale, NFLVR holds in the enlarged filtration $\mathbb{F} \vee \sigma(\tau)$, hence in $\mathbb{G}$. Indeed, the $(\mathbb{F}, \mathbb{P})$-martingale S is - using the independence property - an $(\mathbb{F}^\tau, \mathbb{P}^*)$-martingale, so that S, being $\mathbb{G}$ adapted, is a $(\mathbb{G}, \mathbb{P}^*)$-martingale and $\mathbb{P}^*$is an e.m.m. If S is only a $(\mathbb{F}, \mathbb{P})$-local martingale, it will be a $(\mathbb{F}^\tau, \mathbb{P}^*)$ but we can not conclude that is is a $(\mathbb{G}, \mathbb{P}^*)$-martingale. □

4 Classical arbitrages for a class of honest times

Herein, we generalize the results obtained in [7] – which are established for honest times avoiding $\mathbb{F}$-stopping times in a complete market with continuous filtration – to any complete market and to a much more broader class of honest times that will be defined below. Throughout this section, we denote by $\mathcal{T}_s$ the set of all $\mathbb{F}$-stopping times, $\mathcal{T}_h$ the set of all $\mathbb{F}$-honest times, and $\mathcal{R}$ the set of random times given by

$$\mathcal{R} := \Big\{\tau \ \text{ random time } \Big| \ \exists\, \Gamma \in \mathcal{A} \text{ and } T \in \mathcal{T}_s \text{ such that } \tau = T\mathbb{1}_\Gamma + \infty \mathbb{1}_{\Gamma^c}\Big\}, \tag{4.1}$$

Proposition 4.1 *The following inclusions hold*

$$\mathcal{T}_s \subset \mathcal{R} \subset \mathcal{T}_h. \tag{4.2}$$

Proof: The first inclusion is clear. For the inclusion $\mathcal{R} \subset \mathcal{T}_h$, we give, for ease of the reader two different proofs. Let us take $\tau \in \mathcal{R}$.
1) On $(\tau < t) = (T < t) \cap \Gamma$, we have $\tau = T \wedge t$ and $T \wedge t$ is $\mathcal{F}_t$-measurable. Thus, τ is an honest time.
2) We want to show that on $(\tau < \infty)$, $\widetilde{Z}_\tau = 1$. Indeed, $\widetilde{Z}_t = \mathbb{1}_{(T \geq t)}\mathbb{P}(\Gamma|\mathcal{F}_t) + \mathbb{P}(\Gamma^c|\mathcal{F}_t)$, so that

$$\mathbb{1}_{(\tau<\infty)}\widetilde{Z}_\tau = \mathbb{1}_\Gamma \mathbb{1}_{(T<\infty)}\widetilde{Z}_T = \mathbb{1}_\Gamma \mathbb{1}_{(T<\infty)}(\mathbb{1}_{(T\geq T)}\mathbb{P}(\Gamma|\mathcal{F}_T) + \mathbb{P}(\Gamma^c|\mathcal{F}_T)) = \mathbb{1}_\Gamma \mathbb{1}_{(T<\infty)} = \mathbb{1}_{(\tau<\infty)}.$$

This proves that τ is an honest time. □

The following theorem represents our principal result in the general framework.

Theorem 4.2 *Assume that $(S, \mathbb{F})$ is a complete market and let φ be an $\mathbb{F}$-predictable process satisfying $m = 1 + \varphi \bullet S$. Then the following assertions hold.*

(a) If τ is an honest time, and $\tau \notin \mathcal{R}$, then $\mathbb{G}$-predictable process $\varphi^b = \varphi \mathbb{1}_{[\![0,\tau]\!]}$ is a classical arbitrage strategy in the market "before τ", i.e., in $(S^\tau, \mathbb{G})$.

(b) If τ is an honest time, which is not an $\mathbb{F}$-stopping time, and if $\{\tau = \infty\} \in \mathcal{F}_\infty$, then the $\mathbb{G}$-predictable process $\varphi^a = -\varphi \mathbb{1}_{]\!]\tau,\nu]\!]}$, with $\mathbb{G}$-stopping time defined as

$$\nu := \inf\{t > \tau : \widetilde{Z}_t \leq \frac{1 - \Delta A^o_\tau}{2}\}, \tag{4.3}$$

is a classical arbitrage strategy in the market "after τ", i.e., in $(S - S^\tau, \mathbb{G})$.

Proof: (a) From $m = \widetilde{Z} + A^o_-$ and $\widetilde{Z}_\tau = 1$, we deduce that $m_\tau \geq 1$. Since $\tau \notin \mathcal{R}$, one has $\mathbb{P}(m_\tau > 1) = \mathbb{P}(A^o_{\tau-} > 0) > 0$. Then, by Lemma 2.2, process $\varphi^b = \varphi \mathbb{1}_{[\![0,\tau]\!]}$ is an arbitrage strategy in $(S^\tau, \mathbb{G})$.
(b) From $m = Z + A^o$ and Theorem 2.3 (iv), one obtains that, for $t > \tau$, $m_t - m_\tau = Z_t - Z_\tau \geq -1$. On the other hand, using $m = \widetilde{Z} + A^o_-$, one obtains that, for $t > \tau$, $m_t - m_\tau = \widetilde{Z}_t - 1 + \Delta A^o_\tau$. Assumption $\{\tau = \infty\} \in \mathcal{F}_\infty$ ensures that $\widetilde{Z}_\infty = \mathbb{1}_{\{\tau=\infty\}}$ and in particular $\{\tau < \infty\} \subset \{\widetilde{Z}_\infty = 0\}$. So, $\mathbb{G}$-stopping time ν defined in (4.3) satisfies $\{\nu < \infty\} = \{\tau < \infty\}$. Then,

$$m_\nu - m_\tau = \widetilde{Z}_\nu - 1 + \Delta A^o_\tau \leq \frac{\Delta A^o_\tau - 1}{2} \leq 0,$$

and, as τ is not an $\mathbb{F}$-stopping time,

$$\mathbb{P}(m_\nu - m_\tau < 0) = \mathbb{P}(\Delta A^o_\tau < 1) > 0.$$

Hence $-\int_\tau^{t\wedge\nu} \varphi_s dS_s = m_{\tau\wedge t} - m_{t\wedge\nu}$ is the value of an admissible self-financing strategy $\varphi^a = -\varphi \mathbb{1}_{]\!]\tau,\nu]\!]}$ with initial value 0 and terminal value $m_\tau - m_\nu \geq 0$ satisfying $\mathbb{P}(m_\tau - m_\nu > 0) > 0$. This ends the proof of the theorem. □

Remark 4.3 We recall that if τ is a finite honest time (hence $\mathcal{F}_\infty$-measurable) and is not an $\mathbb{F}$-stopping time, then the density hypothesis is not satisfied and immersion does not hold. Indeed:

(i) Density hypothesis would hold if, under some equivalent probability measure, τ would be independent from $\mathcal{F}_\infty$.

(ii) The immersion property is equivalent to $\mathbb{P}(\tau > t|\mathcal{F}_t) = \mathbb{P}(\tau > t|\mathcal{F}_\infty)$ which, for a finite honest time is $\mathbb{1}_{\tau>t}$. Then, one should have $\mathbb{P}(\tau > t|\mathcal{F}_t) = \mathbb{1}_{\tau>t}$ and τ would be a stopping time.

Remark 4.4 The completeness of the market is an obvious condition to conclude. See [7] for a counter example.

In the following two subsections we explore several examples of honest times. Each of them is defined as an end of optional set, so by Theorem 2.3 (iii), is indeed an honest time.

4.1 Classical arbitrage opportunities in a Brownian filtration

In this subsection, we develop practical market models S and honest times τ within the Brownian filtration for which one can compute explicitly the arbitrage opportunities for both before and after τ. For other examples of honest times, and associated classical arbitrages we refer the reader to [7] (note that the arbitrages constructed in that paper are different from our arbitrages). Throughout this subsection, we assume given a one-dimensional Brownian motion W and $\mathbb{F}$ is its augmented natural filtration. The market model is represented by the bank account whose process is the constant one and one stock whose price process is given by

$$S_t = \exp(\sigma W_t - \frac{1}{2}\sigma^2 t), \qquad \sigma > 0 \text{ given.}$$

It is worth mentioning that in this context of Brownian filtration, for any process V with locally integrable variation, its $\mathbb{F}$-dual optional projection is equal to its $\mathbb{F}$-dual predictable projection, i.e., $V^{o,\mathbb{F}} = V^{p,\mathbb{F}}$.

4.1.1 Last passage time at a given level

Proposition 4.5 *Consider the following random times*

$$\tau := \sup\{t : S_t = a\} \qquad \textit{and} \qquad \nu := \inf\{t > \tau \;\big|\; S_t \leq \frac{a}{2}\},$$

where $0 < a < 1$. Then, the following assertions hold.
(a) The model "before τ" $(S^\tau, \mathbb{G})$ admits a classical arbitrage opportunity given by the $\mathbb{G}$-predictable process

$$\varphi^b = \frac{1}{a}\mathbb{1}_{\{S<a\}} I_{]\!]0,\tau]\!]}.$$

(b) The model "after τ" $(S - S^\tau, \mathbb{G})$ admits a classical arbitrage opportunity given by $\mathbb{G}$-predictable process

$$\varphi^a = -\frac{1}{a}\mathbb{1}_{\{S<a\}} I_{]\!]\tau,\nu]\!]}.$$

Proof: Since $\tau \in \mathcal{T}_h \backslash \mathcal{R}$ we make a use of Theorem 4.2. We compute the predictable process φ such that $m = 1 + \varphi \bullet S$. To this end, we calculate Z as follows. Using [9, exercise 1.2.3.10], we derive

$$1 - Z_t := \mathbb{P}\left(\tau \leq t | \mathcal{F}_t\right) = \mathbb{P}\left(\sup_{t<u} S_u \leq a | \mathcal{F}_t\right) = \mathbb{P}\left(\sup_u \widetilde{S}_u \leq \frac{a}{S_t} | \mathcal{F}_t\right) = \Phi\left(\frac{a}{S_t}\right)$$

where $\widetilde{S}_u = \exp(\sigma\widetilde{W}_u - \frac{1}{2}\sigma^2 u)$, $\widetilde{W}$ independent of $\mathcal{F}_t$ and $\Phi(x) = \mathbb{P}\left(\sup_u \widetilde{S}_u \leq x\right) = \mathbb{P}(\frac{1}{U} \leq x) = \mathbb{P}(\frac{1}{x} \leq U) = (1 - \frac{1}{x})^+$, where U is a random variable with uniform law. Thus we get $Z_t = 1 - (1 - \frac{S_t}{a})^+$ (in particular $Z_\tau = \widetilde{Z}_\tau = 1$), and

$$dZ_t = \mathbb{1}_{\{S_t<a\}} \frac{1}{a} dS_t - \frac{1}{2a} d\ell^a_t$$

where ℓ^a is the local time of the S at the level a. Therefore, we deduce that

$$m = 1 + \varphi \bullet S.$$

Note that $\nu := \inf\{t > \tau \;\big|\; S_t \leq \frac{a}{2}\} = \inf\{t > \tau \mid 1 - (1 - \frac{S_t}{a})^+ \leq \frac{1}{2}\}$, so ν coincides with (4.3). Theorem 4.2 ends the proof of the proposition. □

4.1.2 Last passage time at a level before maturity

Our second example of random time, in this subsection, takes into account finite horizon. In this example, we introduce the following notation

$$H(z,y,s) := e^{-zy}\mathcal{N}\left(\frac{zs-y}{\sqrt{s}}\right) + e^{zy}\mathcal{N}\left(\frac{-zs-y}{\sqrt{s}}\right), \tag{4.4}$$

where $\mathcal{N}(x)$ is the cumulative distribution function of the standard normal distribution.

Proposition 4.6 *Consider the following random time (an honest time)*

$$\tau_1 := \sup\{t \leq 1 \,:\, S_t = b\}$$

where b is a positive real number, $0 < b < 1$. Let V and β be given by

$$V_t := \alpha - \gamma t - W_t \quad \textit{with } \alpha = \frac{\ln b}{\sigma} \textit{ and } \gamma = -\frac{\sigma}{2}$$

$$\beta_t := e^{\gamma V_t}\left(\gamma H(\gamma, |V_t|, 1-t) - sgn(V_t)\, H'_x(\gamma, |V_t|, 1-t)\right),$$

with H defined in (4.4), and let ν be as in (4.3). Then, the following assertions hold.
(a) The model "before τ_1" $(S^{\tau_1}, \mathbb{G})$ admits a classical arbitrage opportunity given by the $\mathbb{G}$-predictable process

$$\varphi^b := \frac{1}{\sigma S_t}\beta_t I_{[\![0,\tau_1]\!]}.$$

(b) The model "after τ_1" $(S - S^{\tau_1}, \mathbb{G})$ admits a classical arbitrage opportunity given by $\mathbb{G}$-predictable process

$$\varphi^a := -\frac{1}{\sigma S_t}\beta_t I_{]\!]\tau_1,\nu]\!]}.$$

Proof: The proof of this proposition follows from Theorem 4.2 as long as we can write the martingale m as an integral stochastic with respect to S. This is the main focus of the remaining part of this proof. By Theorem 2.3 (iii), the time τ_1 is honest and finite. Honest time τ_1 can be seen as

$$\tau_1 \quad = \quad \sup\{t \leq 1 \,:\, \gamma t + W_t = \alpha\} = \sup\{t \leq 1 \,:\, V_t = 0\}.$$

Setting $T_0(V) = \inf\{t \,:\, V_t = 0\}$, we obtain, using standard computations (see [9] p. 145-148)

$$1 - Z_t = \mathbb{P}(\tau_1 \leq t | \mathcal{F}_t) = (1 - e^{\gamma V_t} H(\gamma, |V_t|, 1-t))\mathbb{1}_{\{T_0(V) \leq t \leq 1\}} + \mathbb{1}_{\{t>1\}},$$

where H is given in (4.4). In particular $Z_\tau = \widetilde{Z}_\tau = 1$. Using Itô's lemma, we obtain the decomposition of $1 - e^{\gamma V_t} H(\gamma, |V_t|, 1-t)$ as a semimartingale. The martingale part of Z is given by $dm_t = \beta_t dW_t = \frac{1}{\sigma S_t}\beta_t dS_t$, which ends the proof. □

4.2 Arbitrage opportunities in a Poisson filtration

Throughout this subsection, we suppose given a Poisson process N, with intensity rate $\lambda > 0$, and natural filtration $\mathbb{F}$. The stock price process is given by

$$dS_t = S_{t-}\psi dM_t, \quad S_0 = 1, \qquad M_t := N_t - \lambda t, \tag{4.5}$$

or equivalently $S_t = \exp(-\lambda\psi t + \ln(1+\psi)N_t)$, where $\psi > -1$. In what follows, we introduce the notation

$$\alpha := \ln(1+\psi), \qquad \mu := \frac{\lambda\psi}{\ln(1+\psi)} \text{ and } Y_t := \mu t - N_t, \tag{4.6}$$

so that $S_t = \exp(-\ln(1+\psi)Y_t)$. We associate to the process Y its ruin probability, denoted by $\Psi(x)$ given by

$$\Psi(x) = \mathbb{P}(T^x < \infty), \quad \text{with} \quad T^x = \inf\{t : x + Y_t < 0\} \quad \text{and} \quad x \geq 0. \tag{4.7}$$

Below, we describe our first example of honest time and the associated arbitrage opportunity.

4.2.1 Last passage time at a given level

Proposition 4.7 *Suppose that $\psi > 0$ and let φ be*

$$\varphi := \frac{\Psi(Y_- - a - 1)\mathbb{1}_{\{Y_- \geq a+1\}} - \Psi(Y_- - a)\mathbb{1}_{\{Y_- \geq a\}} + \mathbb{1}_{\{Y_- < a+1\}} - \mathbb{1}_{\{Y_- < a\}}}{\psi S_-}.$$

For $0 < b < 1$, consider the following random time

$$\tau := \sup\{t : \, S_t \geq b\} = \sup\{t : Y_t \leq a\}, \tag{4.8}$$

with $a := -\frac{1}{\alpha} \ln b$. Then the following assertions hold.
(a) The model "before τ" $(S^\tau, \mathbb{G})$ admits a classical arbitrage opportunity given by the $\mathbb{G}$-predictable process $\varphi^b := \varphi I_{[\![0,\tau_1]\!]}$.
(b) The model "after τ" $(S - S^\tau, \mathbb{G})$ admits a classical arbitrage opportunity given by $\mathbb{G}$-predictable process $\varphi^a := -\varphi I_{]\!]\tau_1,\nu]\!]}$, with ν as in (4.3).

Proof: Since $\psi > 0$, one has $\mu > \lambda$ so that Y goes to $+\infty$ as t goes to infinity, and τ is finite. The supermartingale Z associated with the time τ is

$$Z_t = \mathbb{P}(\tau > t|\mathcal{F}_t) = \Psi(Y_t - a)\mathbb{1}_{\{Y_t \geq a\}} + \mathbb{1}_{\{Y_t < a\}} = 1 + \mathbb{1}_{\{Y_t \geq a\}} \left(\Psi(Y_t - a) - 1\right),$$

where Ψ is defined in (4.7) (see [1] for more details on this example). We set $\theta = \dfrac{\mu}{\lambda} - 1$, and deduce that $\Psi(0) = (1 + \theta)^{-1}$ (see [3]). Define $\vartheta_1 = \inf\{t > 0 : Y_t = a\}$ and then, for each $n > 1$, $\vartheta_n = \inf\{t > \vartheta_{n-1} : Y_t = a\}$. It can be proved that the times ϑ_n are predictable $\mathbb{F}$-stopping times ([1]). The $\mathbb{F}$-dual optional projection A^o of the process $\mathbb{1}_{[\![\tau,\infty]\!]}$ equals

$$A^o = \frac{\theta}{1 + \theta} \sum_n \mathbb{1}_{[\![\vartheta_n,\infty]\!]}.$$

Indeed, for any $\mathbb{F}$-optional process U we have

$$\mathbb{E}(U_\tau) = \mathbb{E}(\sum \mathbb{1}_{\{\tau=\vartheta_n\}} U_{\vartheta_n}) = \mathbb{E}(\sum \mathbb{E}(\mathbb{1}_{\{\tau=\vartheta_n\}}|\mathcal{F}_{\vartheta_n}) U_{\vartheta_n})$$

and $\mathbb{E}(\mathbb{1}_{\{\tau=\vartheta_n\}}|\mathcal{F}_{\vartheta_n}) = \mathbb{P}(T^0 = \infty) = 1 - \Psi(0) = 1 - \frac{1}{1+\theta}$.
As a result the process A^o is predictable, and hence $Z = m - A^o$ is the Doob-Meyer decomposition of Z. Thus we can get

$$\Delta m = Z - {}^pZ$$

where ${}^pZ$ is the $\mathbb{F}$-predictable projection of Z [3]. To calculate ${}^pZ$, we write the process Z in a more adequate form. To this end, we first remark that

$$\mathbb{1}_{\{Y \geq a\}} = \mathbb{1}_{\{Y_- \geq a+1\}} \Delta N + (1 - \Delta N)\mathbb{1}_{\{Y_- \geq a\}} \quad \text{and} \quad \mathbb{1}_{\{Y < a\}} = \mathbb{1}_{\{Y_- < a+1\}} \Delta N + (1 - \Delta N)\mathbb{1}_{\{Y_- < a\}}.$$

Then, we obtain

$$\begin{aligned} \Delta m &= \left(\Psi(Y_- - a - 1)\mathbb{1}_{\{Y_- \geq a+1\}} - \Psi(Y_- - a)\mathbb{1}_{\{Y_- \geq a\}} + \mathbb{1}_{\{Y_- < a+1\}} - \mathbb{1}_{\{Y_- < a\}}\right) \Delta N \\ &= \psi S_- \varphi \Delta M = \varphi \Delta S. \end{aligned}$$

Since the two martingales m and S are discontinuous, we deduce that $m = 1 + \varphi \bullet S$. Therefore, the proposition follows from Theorem 4.2. □

[3] Note that here we talk about predictable projection and not about dual predictable projection.

4.2.2 Time of supremum on fixed time horizon

The second example requires the following notations

$$S_t^* := \sup_{s\le t} S_s, \quad \Psi(x,t) := \mathbb{P}(S_t^* > x), \quad \widehat{\Phi}(t) := \mathbb{P}(\sup_{s<t} S_s \le 1), \quad \widetilde{\Phi}(x,t) := \mathbb{P}(\sup_{s<t} S_s < x) \tag{4.9}$$

Proposition 4.8 *Consider the random time τ defined by*

$$\tau = \sup\{t \le 1 : S_t = S_t^*\}, \tag{4.10}$$

where $S_t^* = \sup_{s\le t} S_s$. Then, the following assertions hold.
a) For $\psi > 0$, define the $\mathbb{G}$-predictable process φ as

$$\varphi_t := \mathbb{1}_{\{t<1\}} \left[\Psi\left(\max(\tfrac{S^*_{t-}}{S_{t-}(1+\psi)}, 1), 1-t\right) - \Psi\left(\tfrac{S^*_{t-}}{S_{t-}}, 1-t\right)\right] + \mathbb{1}_{\{S^*_{t-}<S_{t-}(1+\psi)\}}\, \widehat{\Phi}(1-t)$$
$$+ \left[\mathbb{1}_{\{\max(S^*_{1-},S_{1-}(1+\psi))=S_0\}} - \mathbb{1}_{\{\max(S^*_{1-},S_{1-})=S_0\}}\right] \mathbb{1}_{\{t=1\}}$$

Then, $\varphi^b := \varphi \mathbb{1}_{[\![0,\tau]\!]}$ is an arbitrage opportunity for the model $(S^\tau, \mathbb{G})$, and $\varphi^a := -\varphi I_{]\!]\tau,\nu]\!]}$ is an arbitrage opportunity for the model $(S - S^\tau, \mathbb{G})$. Here Ψ and $\widehat{\Phi}$ are defined in (4.9), and ν is defined similarly as in (4.3).
b) For $-1 < \psi < 0$, define the $\mathbb{G}$-predictable process

$$\varphi_t := \frac{\psi I_{\{S_t^*=S_{t-}\}} \widehat{\Phi}(\frac{1}{1+\psi}, 1-t) + \Psi(\frac{S_t^*}{S_{t-}(1+\psi)}, 1-t) - \Psi(\frac{S_t^*}{S_{t-}}, 1-t)}{\psi S_{t-}}.$$

Then, $\varphi^b := \varphi \mathbb{1}_{[\![0,\tau]\!]}$ is an arbitrage opportunity for the model $(S^\tau, \mathbb{G})$, and $\varphi^a := -\varphi I_{]\!]\tau,\nu]\!]}$ is an arbitrage opportunity for the model $(S - S^\tau, \mathbb{G})$.

Proof: Note that, if $-1 < \psi < 0$ the process S^* is continuous, $S_\tau < S_\tau^* = \sup_{t\in[0,1]} S_t$ on the set $(\tau < 1)$ and $S_{\tau-} = S^*_{\tau-} = \sup_{t\in[0,1]} S_t$. If $\psi > 0$, $S_{\tau-} < S^*_{\tau-} < \sup_{t\in[0,1]} S_t$ on the set $(\tau < 1)$.

Define the sets $(E_n)_{n=0}^\infty$ such that $E_0 = \{\tau = 1\}$ and $E_n = \{\tau = T_n\}$ with $n \ge 1$. The sequence $(E_n)_{n=0}^\infty$ forms a partition of Ω. Then, $\tau = \mathbb{1}_{E_0} + \sum_{n=1}^\infty T_n \mathbb{1}_{E_n}$. Note that τ is not an $\mathbb{F}$ stopping time since $E_n \notin \mathcal{F}_{T_n}$ for any $n \ge 1$.

The supermartingale Z associated with the honest time τ is

$$Z_t = \mathbb{P}(\tau > t|\mathcal{F}_t) = \mathbb{P}(\sup_{s\in(t,1]} S_s > \sup_{s\in[0,t]} S_s|\mathcal{F}_t) = \mathbb{P}(\sup_{s\in[0,1-t]} \widehat{S}_s > \frac{S_t^*}{S_t}|\mathcal{F}_t) = \mathbb{1}_{(t<1)}\Psi(\frac{S_t^*}{S_t}, 1-t),$$

with $\widehat{S}$ an independent copy of S and $\Psi(x,t)$ is given by (4.9).
As $\{\tau = T_n\} \subset \{\tau \le T_n\} \subset \{Z_{T_n} < 1\}$, we have

$$Z_\tau = \mathbb{1}_{\{\tau=1\}} Z_1 + \sum_{n=1}^\infty \mathbb{1}_{\{\tau=T_n\}} Z_{T_n} < 1, \quad \text{and} \quad \{\widetilde{Z} = 0 < Z_-\} = \emptyset.$$

In the following we will prove assertion a). Thus, we suppose that $\psi > 0$, and we calculate

$$\begin{aligned} A_t^o &= \mathbb{P}(\tau = 1|\mathcal{F}_1)\mathbb{1}_{\{t\ge1\}} + \sum_n \mathbb{P}(\tau = T_n|\mathcal{F}_{T_n})\mathbb{1}_{\{t\ge T_n\}} \\ &= \mathbb{1}_{\{S_1^*=S_0\}}\mathbb{1}_{\{t\ge1\}} + \sum_n \mathbb{1}_{\{T_n<1\}}\mathbb{1}_{\{S^*_{T_n-}<S_{T_n}\}}\mathbb{P}(\sup_{s\in[T_n,1[} S_s \le S_{T_n}|\mathcal{F}_{T_n})\mathbb{1}_{\{t\ge T_n\}} \\ &= \mathbb{1}_{\{S_1^*=S_0\}}\mathbb{1}_{\{t\ge1\}} + \sum_n \mathbb{1}_{\{T_n<1\}}\mathbb{1}_{\{S^*_{T_n-}<S_{T_n-}(1+\psi)\}}\, \widehat{\Phi}(1-T_n)\mathbb{1}_{\{t\ge T_n\}}, \end{aligned}$$

with $\widehat{\Phi}$ is given by (4.9). As before, we write

$$A_t^o = \mathbb{1}_{\{S_1^*=S_0\}}\mathbb{1}_{\{t\geq 1\}} + \sum_{s\leq t}\mathbb{1}_{\{s<1\}}\mathbb{1}_{\{S_{s-}^*<S_{s-}(1+\psi)\}}\,\widehat{\Phi}(1-s)\Delta N_s$$
$$= \mathbb{1}_{\{S_1^*=S_0\}}\mathbb{1}_{\{t\geq 1\}} + \int_0^{t\wedge 1}\mathbb{1}_{\{S_{s-}^*<S_{s-}(1+\psi)\}}\,\widehat{\Phi}(1-s)\,dM_s + \lambda\int_0^{t\wedge 1}\mathbb{1}_{\{S_{s-}^*<S_{s-}(1+\psi)\}}\,\hat{\Phi}(1-s)ds.$$

Remark that we have

$$\mathbb{1}_{\{S_1^*=S_0\}} = \left[\mathbb{1}_{\{\max(S_{1-}^*,S_{1-}(1+\psi))=S_0\}} - \mathbb{1}_{\{\max(S_{1-}^*,S_{1-})=S_0\}}\right]\Delta M_1 + \mathbb{1}_{\{\max(S_{1-}^*,S_{1-})=S_0\}}.$$

and

$$\Delta m = \Delta Z + \Delta A^o = Z - {}^p(Z) + \Delta A^o - {}^p(\Delta A^o).$$

Then we re-write the process Z as follows

$$Z = \mathbb{1}_{[\![0,1[\![}\Psi\left(\max(\frac{S_-^*}{S_-(1+\psi)},1),1-t\right)\Delta M + (1-\Delta M)I_{[\![0,1[\![}\Psi\left(\frac{S_-^*}{S_-},1-t\right).$$

This implies that

$$Z - {}^p(Z) = \mathbb{1}_{[\![0,1[\![}\left[\Psi\left(\max(\frac{S_-^*}{S_-(1+\psi)},1),1-t\right) - \Psi\left(\frac{S_-^*}{S_-},1-t\right)\right]\Delta M.$$

Thus by combining all these remarks, we deduce that

$$\Delta m = Z - {}^p(Z) + \Delta A^o - {}^p(\Delta A^o) = \varphi\Delta S.$$

Then, the assertion a) follows immediately from Theorem 4.2.
Next, we will prove assertion b). Suppose that $-1<\psi<0$, and we calculate

$$A_t^o = \mathbb{P}(\tau=1|\mathcal{F}_1)\mathbb{1}_{\{t\geq 1\}} + \sum_n \mathbb{P}(\tau=T_n|\mathcal{F}_{T_n})\mathbb{1}_{\{t\geq T_n\}}$$
$$= \mathbb{1}_{\{S_1^*=S_1\}}\mathbb{1}_{\{t\geq 1\}} + \sum_n \mathbb{1}_{\{T_n<1\}}\mathbb{1}_{\{S_{T_n}^*=S_{T_n-}\}}\mathbb{P}(\sup_{s\in[T_n,1[}S_s < S_{T_n-}|\mathcal{F}_{T_n})\mathbb{1}_{\{t\geq T_n\}}$$
$$= \mathbb{1}_{\{S_1^*=S_1\}}\mathbb{1}_{t\geq 1\}} + \sum_n \mathbb{1}_{\{T_n<1\}}\mathbb{1}_{\{S_{T_n}^*=S_{T_n-}\}}\widetilde{\Phi}(\frac{S_{T_n-}}{S_{T_n}},1-T_n)\mathbb{1}_{\{t\geq T_n\}},$$

with $\widetilde{\Phi}(x,t)$ is given by (4.9). In order to find the compensator of A^o, we write

$$A_t^o = \mathbb{1}_{\{S_1^*=S_1\}}\mathbb{1}_{\{t\geq 1\}} + \sum_{s\leq t}\mathbb{1}_{\{s<1\}}\mathbb{1}_{\{S_s^*=S_{s-}\}}\widetilde{\Phi}(\frac{1}{1+\psi},1-s)\,\Delta N_s$$
$$= \mathbb{1}_{\{S_1^*=S_1\}}\mathbb{1}_{\{t\geq 1\}} + \int_0^{t\wedge 1}\mathbb{1}_{\{S_s^*=S_{s-}\}}\widetilde{\Phi}(\frac{1}{1+\psi},1-s)\,dM_s + \lambda\int_0^{t\wedge 1}\mathbb{1}_{\{S_s^*=S_{s-}\}}\widetilde{\Phi}(\frac{1}{1+\psi},1-s)\,ds.$$

As a result, due to the continuity of the process S^*, we get

$$\begin{aligned} A_t^o - {}^p(A^o)_t &= I_{\{S_t^*=S_{t-}\}}\widetilde{\Phi}(\frac{1}{1+\psi},1-t)\Delta M_t, \\ Z_t - {}^pZ_t &= \left[\Psi(\frac{S_t^*}{S_{t-}(1+\psi)},1-t) - \Psi(\frac{S_t^*}{S_{t-}},1-t)\right]\Delta N_t. \end{aligned}$$

This implies that

$$\begin{aligned} \Delta m_t &= Z_t - {}^pZ_t + A_t^o - {}^p(A^o)_t \\ &= \left\{\psi I_{\{S_t^*=S_{t-}\}}\widetilde{\Phi}(\frac{1}{1+\psi},1-t) + \Psi\left(\frac{S_t^*}{S_{t-}(1+\psi)},1-t\right) - \Psi\left(\frac{S_t^*}{S_{t-}}\right),1-t\right\}\Delta N_t. \end{aligned}$$

Since m and S are pure discontinuous $\mathbb{F}$-local martingales, we conclude that m can be written in the form of

$$m = m_0 + \varphi \cdot S,$$

and the proof of the assertion b) follows immediately from Theorem 4.2. This ends the proof of the proposition. □

4.2.3 Time of overall supremum

Below, we will present our last example of this subsection. The analysis of this example is based on the following three functions.

$$\Psi(x) = \mathbb{P}(S^* > x) = \mathbb{P}(\sup_s S_s > x), \quad \widehat{\Phi} = \mathbb{P}(\sup_s S_s \leq 1), \text{ and } \quad \widetilde{\Phi}(x) = \mathbb{P}(\sup_s S_s < x). \tag{4.11}$$

Proposition 4.9 *Consider the random time τ given by*

$$\tau = \sup\{t : S_t = S_t^*\}. \tag{4.12}$$

Then, the following assertions hold.
a) For $\psi > 0$, define the $\mathbb{G}$-predictable process φ as

$$\varphi_t := \frac{\mathbb{1}_{\{S^*_{t-} < S_{t-}(1+\psi)\}}\widehat{\Phi} + \Psi\left(\max(\frac{S^*_{t-}}{S_{t-}(1+\psi)}, 1\right) - \Psi(\frac{S^*_{t-}}{S_{t-}})}{S_{t-}\psi}.$$

Then, $\varphi^b := \varphi \mathbb{1}_{[\![0,\tau]\!]}$ is an arbitrage opportunity for the model $(S^\tau, \mathbb{G})$, and $\varphi^a := -\varphi I_{]\!]\tau,\nu]\!]}$ is an arbitrage opportunity for the model $(S - S^\tau, \mathbb{G})$. Here Ψ and $\widehat{\Phi}$ are defined in (4.11), and ν is defined in similar way as in (4.3).
b) For $-1 < \psi < 0$, define the $\mathbb{G}$-predictable process φ as

$$\varphi := \frac{\Psi(\frac{S^*}{S_-(1+\psi)}) - \Psi(\frac{S^*}{S_-}) + \mathbb{1}_{\{S^* = S_-\}}\widetilde{\Phi}(\frac{1}{1+\psi})\psi}{\psi S_-}.$$

Then, $\varphi^b := \varphi \mathbb{1}_{[\![0,\tau]\!]}$ is an arbitrage opportunity for the model $(S^\tau, \mathbb{G})$, and $\varphi^a := -\varphi I_{]\!]\tau,\nu]\!]}$ is an arbitrage opportunity for the model $(S - S^\tau, \mathbb{G})$. Here again ν is defined as in (4.3).

Proof: Let us note that τ is finite and, as before, if $-1 < \psi < 0$, $S_\tau < S^*_\tau = \sup_t S_t$ and S^* is continuous and if $\psi > 0$, $S_\tau = S^*_\tau = \sup_t S_t$.
The supermartingale Z associated with the honest time τ is

$$Z_t = \mathbb{P}(\tau > t|\mathcal{F}_t) = \mathbb{P}(\sup_{s\in(t,\infty]} S_s > \sup_{s\in[0,t]} S_s|\mathcal{F}_t) = \mathbb{P}(\sup_{s\in[0,\infty]} \widehat{S}_s > \frac{S^*_t}{S_t}|\mathcal{F}_t) = \Psi(\frac{S^*_t}{S_t}),$$

with $\widehat{S}$ an independent copy of S and Ψ is given by (4.11). As a result, we deduce that $Z_\tau < 1$. In the following, we will prove assertion a). We suppose that $\psi > 0$, denoting by $(T_n)_n$ the sequence of jumps of the Poisson process N, we derive

$$\begin{aligned} A^o_t &= \sum_n \mathbb{P}(\tau = T_n|\mathcal{F}_{T_n})\mathbb{1}_{\{t\geq T_n\}} = \sum_n \mathbb{1}_{\{S^*_{T_n-} < S_{T_n}\}}\mathbb{P}(\sup_{s\geq T_n} S_s \leq S_{T_n}|\mathcal{F}_{T_n})\mathbb{1}_{\{t\geq T_n\}} \\ &= \sum_n \mathbb{1}_{\{S^*_{T_n-} < S_{T_n-}(1+\psi)\}}\widehat{\Phi}\mathbb{1}_{\{t\geq T_n\}}, \end{aligned}$$

with $\widehat{\Phi} = \mathbb{P}(\sup_s S_s \leq 1)$ given by (4.11).

We continue to find compensator of A^o

$$\begin{aligned} A_t^o &= \sum_{s\leq t} \mathbb{1}_{\{S^*_{s-}<S_{s-}(1+\psi)\}}\widehat{\Phi}\Delta N_s \\ &= \int_0^t \mathbb{1}_{\{S^*_{s-}<S_{s-}(1+\psi)\}}\widehat{\Phi}dM_s + \lambda\int_0^t \mathbb{1}_{\{S^*_{s-}<S_{s-}(1+\psi)\}}\widehat{\Phi}ds. \end{aligned}$$

Now as we did for the previous propositions, we calculate the jumps of m. To this end, we re-write Z as follows

$$Z = \left[\Psi\left(\max(\frac{S^*_-}{S_-(1+\psi)},1)\right) - \Psi(\frac{S^*_-}{S_-})\right]\Delta M + \Psi(\frac{S^*_-}{S_-}).$$

This implies that

$$Z - {}^pZ = \left[\Psi\left(\max(\frac{S^*_-}{S_-(1+\psi)},1)\right) - \Psi(\frac{S^*_-}{S_-})\right]\Delta M.$$

Hence, we derive

$$\Delta m = \left[\mathbb{1}_{\{S^*_{s-}<S_{s-}(1+\psi)\}}\widehat{\Phi} + \Psi\left(\max(\frac{S^*_-}{S_-(1+\psi)},1)\right) - \Psi(\frac{S^*_-}{S_-})\right]\Delta M.$$

Since both martingales m and M are purely discontinuous, we deduce that $m = m_0 + \varphi \bullet S$. Then, the proposition follows immediately from Theorem 4.2.
In the following, we will prove assertion b). To this end, we suppose that $\psi < 0$, and we calculate

$$\begin{aligned} A_t^o &= \sum_n \mathbb{P}(\tau = T_n|\mathcal{F}_{T_n})\mathbb{1}_{\{t\geq T_n\}} = \sum_n \mathbb{1}_{\{S^*_{T_n}=S_{T_n-}\}}\mathbb{P}(\sup_{s\geq T_n} S_s < S_{T_n-}|\mathcal{F}_{T_n})\mathbb{1}_{\{t\geq T_n\}} \\ &= \sum_n \mathbb{1}_{\{S^*_{T_n}=S_{T_n-}\}}\widetilde{\Phi}(\frac{S_{T_n-}}{S_{T_n}})\mathbb{1}_{\{t\geq T_n\}}, \end{aligned}$$

with $\widetilde{\Phi}(x) = \mathbb{P}(\sup_s S_s < x)$. Therefore,

$$\begin{aligned} A_t^o &= \sum_{s\leq t} \mathbb{1}_{\{S^*_s=S_{s-}\}}\widetilde{\Phi}(\frac{1}{1+\psi})\Delta N_s \\ &= \int_0^t \mathbb{1}_{\{S^*_s=S_{s-}\}}\widetilde{\Phi}(\frac{1}{1+\psi})dM_s + \lambda\int_0^t \mathbb{1}_{\{S^*_s=S_{s-}\}}\widetilde{\Phi}(\frac{1}{1+\psi})ds. \end{aligned}$$

Since in the case of $\psi < 0$, the process S^* is continuous, we obtain

$$Z - {}^pZ = \left[\Psi(\frac{S^*}{S_-(1+\psi)}) - \Psi(\frac{S^*}{S_-})\right]\Delta N, \quad A^o - {}^p(A^o) = \mathbb{1}_{\{S^*=S_-\}}\widetilde{\Phi}(\frac{1}{1+\psi})\Delta M.$$

Therefore, we conclude that

$$\Delta m = Z - {}^pZ + A^o - {}^p(A^o) = \left\{\Psi(\frac{S^*}{S_-(1+\psi)}) - \Psi(\frac{S^*}{S_-}) + \mathbb{1}_{\{S^*=S_-\}}\widetilde{\Phi}(\frac{1}{1+\psi})\psi\right\}\Delta N.$$

This implies that the martingale m has the form of $m = 1 + \varphi \cdot S$, and assertion b) follows immediately from Theorem 4.2, and the proof of the proposition is completed. □

5 Arbitrage opportunities for non-honest random times

This section is our second main part of the corp of the paper. Herein, we develop a number of practical examples of market models and examples of random times that are not honest times and we study the existence of classical arbitrages. This section contains two subsections that treat two different situations.

5.1 In a Brownian filtration: Emery's example

We present here an example where τ is a pseudo stopping-time.

Let S be defined through $dS_t = \sigma S_t dW_t$, where W is a Brownian motion and σ a constant.

Let $\tau = \sup\{t \leq 1 : S_1 - 2S_t = 0\}$, that is the last time before 1 at which the price is equal to half of its terminal value at time 1.

Proposition 5.1 *In the above model NA holds before τ. There are classical arbitrages after τ.*

Proof: Note that

$$\{\tau \leq t\} = \{\inf_{t\leq s\leq 1} 2S_s \geq S_1\} = \{\inf_{t\leq s\leq 1} 2\frac{S_s}{S_t} \geq \frac{S_1}{S_t}\}$$

Since $\frac{S_s}{S_t}, s \geq t$ and $\frac{S_1}{S_t}$ are independent from $\mathcal{F}_t$,

$$\mathbb{P}(\inf_{t\leq s\leq 1} 2\frac{S_s}{S_t} \geq \frac{S_1}{S_t}|\mathcal{F}_t) = \mathbb{P}(\inf_{t\leq s\leq 1} 2S_{s-t} \geq S_{1-t}) = \Phi(1-t)$$

where $\Phi(u) = \mathbb{P}(\inf_{s\leq u} 2S_s \geq S_u)$. It follows that the supermartingale Z is a deterministic decreasing function, hence, τ is a pseudo-stopping time and S is a $\mathbb{G}$-martingale up to time τ and there are no arbitrages up to τ.

There are obviously arbitrages after τ, since, at time τ, one knows the value of S_1 and $S_1 > S_\tau$. In fact, for $t > \tau$, one has $S_t > S_\tau$, and the arbitrage occurs at any time before 1. □

5.2 In a Poisson filtration

This subsection develops similar examples of random times – as in the Brownian filtration of the previous subsection – and shows that the effects of these random times on the market's economic structure differ tremendously from the one of the previous subsection.

In this section, we will work on a Poisson process N with intensity λ and the compensated martingale $M_t = N_t - \lambda t$. Denote

$$T_n = \inf\{t \geq 0 : N_t \geq n\}, \text{ and } H_t^n = \mathbb{1}_{\{T_n\leq t\}}, \quad n = 1, 2.$$

The stock price S is described by

$$dS_t = S_{t-}\psi dM_t, \quad \text{where,} \quad \psi > -1, \text{ and } \psi \neq 0. \tag{5.1}$$

or equivalenty, $S_t = S_0 \exp(-\lambda\psi t + \ln(1+\psi)N_t)$. Then,

$$M_t^1 := H_t^1 - \lambda(t \wedge T_1) := H_t^1 - A_t^1, \text{ and } M_t^2 := H_t^2 - (\lambda(t \wedge T_2) - \lambda(t \wedge T_1))) := H_t^2 - A_t^2$$

are two $\mathbb{F}$-martingales. Remark that if $\psi \in (-1, 0)$, between T_1 and T_2, the stock price increases; if $\psi > 0$, between T_1 and T_2, the stock process decreases. This would be the starting point of the existence of arbitrages.

5.2.1 Average of two jump times

Below, we present an example of random time that avoids stopping times and the non-arbitrage property fails.

Proposition 5.2 *Consider the random time $\tau = k_1T_1 + k_2T_2$ that avoids $\mathbb{F}$ stopping times, where $k_1 + k_2 = 1$ and $k_1, k_2 > 0$. Then the following properties hold:*
(a) *The random time τ is not an honest time.*

(b) $\widetilde{Z}_\tau = Z_\tau = e^{-\lambda k_1 (T_2 - T_1)} < 1$, *and* $\{\widetilde{Z} = 0 < Z_-\} = \emptyset$.
(c) *There is a classical arbitrage before* τ*, given by*

$$\varphi_t := -e^{-\lambda \frac{k_2}{k_1}(t-T_1)} \left(\mathbb{1}_{\{N_{t-} \geq 1\}} - \mathbb{1}_{\{N_{t-} \geq 2\}} \right) \frac{1}{\psi S_{t-}} \mathbb{1}_{\{t \leq \tau\}}. \tag{5.2}$$

(d) *There exist arbitrages after* τ*: if* $\psi \in (-1, 0)$*, buy at* τ *and sell before* T_2*; if* $\psi > 0$*, short sell at* τ *and buy back before* T_2*.*

Proof: First, we compute the supermartingale Z:

$$\mathbb{P}(\tau > t | \mathcal{F}_t) \quad = \quad \mathbb{1}_{T_1 > t} + \mathbb{1}_{\{T_1 \leq t\}} \mathbb{1}_{\{T_2 > t\}} \mathbb{P}(k_1 T_1 + k_2 T_2 > t | \mathcal{F}_t)$$

On the set $E = (T_1 \leq t) \cap (T_2 > t)$, the quantity $\mathbb{P}(k_1 T_1 + k_2 T_2 > t | \mathcal{F}_t)$ is $\mathcal{F}_{T_1}$-measurable. It follows that, on E,

$$\mathbb{P}(k_1 T_1 + k_2 T_2 > t | \mathcal{F}_t) = \frac{\mathbb{P}(k_1 T_1 + k_2 T_2 > t, T_2 > t | \mathcal{F}_{T_1})}{\mathbb{P}(T_2 > t | \mathcal{F}_{T_1})} = \frac{e^{-\lambda \frac{1}{k_2}(t - T_1)}}{e^{-\lambda (t - T_1)}} = e^{-\lambda \frac{k_1}{k_2}(t - T_1)},$$

where we used the independence property of T_1 and $T_2 - T_1$. Therefore, we deduce that,

$$\mathbb{P}(\tau > t | \mathcal{F}_t) \quad = \quad \mathbb{1}_{\{T_1 > t\}} + \mathbb{1}_{\{T_1 \leq t\}} \mathbb{1}_{\{T_2 > t\}} e^{-\lambda \frac{k_1}{k_2}(t - T_1)}.$$

Since $Z_t = (1 - H_t^1) + H_t^1 (1 - H_t^2) e^{-\lambda \frac{k_1}{k_2}(t - T_1)}$, we deduce, using the fact that $e^{-\lambda (t - T_1)} dH_t^1 = dH_t^1$,

$$\begin{aligned}
dZ_t &= -dH_t^1 + e^{-\lambda \frac{k_1}{k_2}(t - T_1)} ((1 - H_t^2) dH_t^1 - H_t^1 dH_t^2)) - \lambda \frac{k_1}{k_2} H_t^1 (1 - H_t^2) e^{-\lambda \frac{k_1}{k_2}(t - T_1)} dt \\
&= e^{-\lambda \frac{k_1}{k_2}(t - T_1)} (-H^2 dH_t^1 - H_t^1 dH_t^2) - \lambda \frac{k_1}{k_2} H_t^1 (1 - H_t^2) e^{-\lambda \frac{k_1}{k_2}(t - T_1)} dt \\
&= -e^{-\lambda \frac{k_1}{k_2}(t - T_1)} dH_t^2 - \lambda \frac{k_1}{k_2} H_t^1 (1 - H_t^2) e^{-\lambda \frac{k_1}{k_2}(t - T_1)} dt \\
&= dm_t - e^{-\lambda \frac{k_1}{k_2}(t - T_1)} dA_t^2 - \lambda \frac{k_1}{k_2} H_t^1 (1 - H_t^2) e^{-\lambda \frac{k_1}{k_2}(t - T_1)} dt,
\end{aligned}$$

where

$$dm_t = -e^{-\lambda \frac{k_1}{k_2}(t - T_1)} dM_t^2.$$

Hence

$$m_\tau = 1 - \int_0^\tau e^{-\lambda \frac{k_1}{k_2}(t - T_1)} dM_t^2 = 1 + \int_{T_1}^\tau e^{-\lambda \frac{k_1}{k_2}(t - T_1)} \frac{\lambda^2 t}{1 + \lambda t} dt > 1.$$

Now we will start proving the proposition.
i) Since τ avoids stopping times, $Z = \widetilde{Z}$. Note that $\widetilde{Z}_\tau = Z_\tau = e^{-\lambda k_1 (T_2 - T_1)} < 1$. Hence, τ is not an honest time. Since $Z > 0$, we deduce that both assertions (a) and (b) hold.

ii) Now, we will prove assertion (c). We will describe explicitly the arbitrage strategy. Note that $\{T_2 \leq t\} = \{N_t \geq 2\}$. We deduce that

$$M_t^2 = \mathbb{1}_{\{T_2 \leq t\}} - A_t^2 = \mathbb{1}_{\{N_t \geq 2\}} - A_t^2 = \mathbb{1}_{\{N_{t-} \geq 1\}} \Delta N_t + \mathbb{1}_{\{N_{t-} \geq 2\}} (1 - \Delta N_t) - A_t^2. \tag{5.3}$$

Hence,

$$\Delta M_t^2 = M_t^2 - {}^p(M^2)_t = \left(\mathbb{1}_{\{N_{t-} \geq 1\}} - \mathbb{1}_{\{N_{t-} \geq 2\}} \right) \Delta N_t = \left(\mathbb{1}_{\{N_{t-} \geq 1\}} - \mathbb{1}_{\{N_{t-} \geq 2\}} \right) \Delta M_t. \tag{5.4}$$

Since M^2 and M are both purely discontinuous, we have $m_t = 1 + (\phi \bullet M)_t = 1 + (\varphi \bullet S)_t$, where

$$\phi_t = -e^{-\lambda \frac{k_1}{k_2}(t - T_1)} \left(I_{\{N_{t-} \geq 1\}} - I_{\{N_{t-} \geq 2\}} \right), \text{ and } \varphi_t = \phi_t \frac{1}{\psi S_{t-}}. \tag{5.5}$$

iii) Arbitrages after τ: At time τ, the value of T_2 is known for the one who has $\mathbb{G}$ information. The price process decreases before time T_2, however, waiting up time T_2 does not lead to an arbitrage Setting $\Delta = T_2 - \tau$ (which is known at time τ), there is an arbitrage selling short S at time τ for a delivery at time $\tau + \frac{1}{2}\Delta$. The strategy is admissible, since between T_1 and T_2, the quantity S_t is bounded by $S_0(1+\varphi)$. This ends the proof of the proposition. □

5.2.2 Minimum of two scaled jump times

We give now an example of a non honest random time, which does not avoid $\mathbb{F}$ stopping time and induces classical arbitrage opportunities.

Proposition 5.3 *Consider the same market as before, and define $\tau = T_1 \wedge aT_2$, where $0 < a < 1$. Then, the following properties hold:*
(a) *τ is not an honest time and does not avoid $\mathbb{F}$-stopping times,*
(b) *$Z_\tau = \mathbb{1}_{\{T_1 > aT_2\}} e^{-\beta a T_2}(\beta a T_2 + 1) < 1$ and $\widetilde{Z}_\tau = e^{-\beta a T_2}(\beta a T_2 + 1) < 1$, and $\{\widetilde{Z} = 0 < Z_-\} = \emptyset$.*
(c) *There exists a classical arbitrage before τ given by*

$$\varphi_t = -e^{-\beta t}(\beta t + 1)\left(\mathbb{1}_{\{N_{t-} \geq 0\}} - \mathbb{1}_{\{N_{t-} \geq 1\}}\right) \frac{1}{\psi S_{t-}} \mathbb{1}_{\{t \leq \tau\}}, \quad \text{where } \beta = \lambda(1/a - 1). \tag{5.6}$$

(d) *There exist arbitrages after τ: if $\psi \in (-1, 0)$, buy at τ and sell before τ/a; if $\psi > 0$, short sell at τ and buy back before τ/a.*

Proof: First, let us compute the supermartingale Z,

$$\begin{aligned} Z_t &= \mathbb{1}_{\{T_1 > t\}} \mathbb{P}(aT_2 > t | \mathcal{F}_t) = \mathbb{1}_{T_1 > t} \frac{\mathbb{P}(aT_2 > t, T_1 > t)}{\mathbb{P}(T_1 > t)} \\ &= \mathbb{1}_{\{T_1 > t\}} e^{\lambda t}\, \mathbb{E}(\mathbb{1}_{T_1 > t} e^{-\lambda(\frac{t}{a} - T_1)^+}) \\ &= \mathbb{1}_{\{T_1 > t\}} e^{\lambda t} \int_t^{t/a} e^{-\lambda(\frac{t}{a} - x)} \lambda e^{-\lambda x} dx + \mathbb{1}_{\{T_1 > t\}} e^{\lambda t} \int_{t/a}^{\infty} \lambda e^{-\lambda y} dy \\ &= \mathbb{1}_{\{T_1 > t\}} e^{-\beta t}(\beta t + 1), \end{aligned}$$

where $\beta = \lambda(1/a - 1)$. In particular $Z_\tau = \mathbb{1}_{T_1 > aT_2} e^{-\beta a T_2}(\beta a T_2 + 1) < 1$. Similar computation as above leads to $\widetilde{Z}_t = Z_{t-} = \mathbb{1}_{\{T_1 \geq t\}} e^{-\beta t}(\beta t + 1)$. This proves assertions (a) and (b).
i) Here, we will prove assertion (c). Thanks to Itô's formula, we have

$$dZ_t = -e^{-\beta t}(\beta t + 1) dH_t^1 - \mathbb{1}_{t \leq T_1} \beta^2 e^{-\beta t} t dt = -e^{-\beta t}(\beta t + 1) dM_t^1 - e^{-\beta t}(\beta t + 1) dA_t^1 - \mathbb{1}_{t \leq T_1} \beta^2 e^{-\beta t} t dt.$$

Therefore,

$$dm_t = -e^{-\beta t}(\beta t + 1) dM_t^1.$$

Hence

$$\begin{aligned} m_\tau &= 1 + \mathbb{1}_{\{aT_2 < T_1\}} \lambda \left(\frac{2(1 - e^{-\beta a T_2})}{\beta} - a T_2 e^{-\beta a T_2} \right) \\ &\quad + \mathbb{1}_{\{T_1 < aT_2\}} \left(2\lambda \frac{1 - e^{-\beta T_1}}{\beta} - \lambda T_1 e^{-\beta T_1} - T_1 \beta e^{-\beta T_1} - e^{-\beta T_1} + \mathbb{1}_{\{T_1 = 0\}} \right) \end{aligned} \tag{5.7}$$

and, using the fact that when $x > 0$, $1 - e^{-x} - x e^{-x} > 0$ and $2\lambda\beta e^{-\beta x} - \lambda\beta x e^{-\beta x} - x\beta^2 e^{-\beta x} - \beta e^{-\beta x} + \beta I_{x=0} > 0$, one obtains $m_\tau > 1$; hence the existence of classical arbitrages.

Now, we describe explicitly the arbitrage strategy. Notice that $\{T_1 \leq t\} = \{N_t \geq 1\}$. We deduce that

$$M_t^1 = \mathbb{1}_{\{T_1 \leq t\}} - A_t^1 = \mathbb{1}_{\{N_t \geq 1\}} - A_t^1 = \mathbb{1}_{\{N_{t-} \geq 0\}} \Delta N_t + \mathbb{1}_{\{N_{t-} \geq 1\}}(1 - \Delta N_t) - A_t^1. \tag{5.8}$$

Hence,

$$\Delta M_t^1 = M_t^1 - {}^p(M^1)_t = \left(\mathbb{1}_{\{N_{t-}\geq 0\}} - \mathbb{1}_{\{N_{t-}\geq 1\}}\right)\Delta N_t = \left(I_{\{N_{t-}\geq 0\}} - I_{\{N_{t-}\geq 1\}}\right)\Delta M_t. \tag{5.9}$$

Since M^1 and M are both purely discontinuous, we have $m = 1 + \varphi \bullet S$, where

$$\varphi_t = -e^{-\beta t}(\beta t + 1)\left(\mathbb{1}_{\{N_{t-}\geq 0\}} - \mathbb{1}_{\{N_{t-}\geq 1\}}\right)\frac{1}{\psi S_{t-}}. \tag{5.10}$$

ii) The proof of assertion (d) follows the same proof of assertion (d) of Proposition 5.6. This ends the proof of the proposition. □

5.2.3 Maximum of two scaled jump times

I think one can delete this example; same kind of example than the previous one Yes, I agree.

Proposition 5.4 *Consider the same market as before. and define $\tau = T_1 \vee aT_2$, where $0 < a < 1$. Then, the following properties hold:*
(a) *τ is not an honest time and does not avoid $\mathbb{F}$-stopping times.*
(b) *$Z_\tau = \frac{\lambda\tau e^{-\lambda\tau/a}}{1-e^{-\lambda\tau}} < 1$, and $\widetilde{Z}_\tau = I_{\{T_1\geq aT_2\}} + I_{\{T_1<aT_2\}}Z_\tau \not\equiv 1$, and $\{\widetilde{Z} = 0 < Z_-\} = \emptyset$.*
(c) *There exists a classical arbitrage before τ given by*

$$\varphi_t = -\left(1 - \frac{\lambda t e^{-\lambda\frac{t}{a}}}{1-e^{-\lambda t}}\right)\left(I_{\{N_{t-}\geq 0\}} - I_{\{N_{t-}\geq 1\}}\right)\frac{1}{\psi S_{t-}}. \tag{5.11}$$

(d) *There exist classical arbitrages after τ: if $\psi \in (0,1)$ and $T_1 < aT_2$, buy at τ and sell before τ/a; if $\psi > 0$ and $T_1 < aT_2$, short sell at τ and buy back before τ/a.*

Proof: First, let us compute the supermartingale Z,

$$\begin{aligned}
1 - Z_t &= P\left(\tau \leq t|\mathcal{F}_t\right) = P\left(T_1 \vee aT_2 \leq t|\mathcal{F}_t\right) \\
&= \mathbb{1}_{\{T_1\leq t\}}P\left(T_2 \leq \frac{t}{a}\middle|\mathcal{F}_t\right) = \mathbb{1}_{\{T_1\leq t\}}\frac{P\left(T_2 \leq \frac{t}{a}, T_1 \leq t\right)}{P\left(T_1 \leq t\right)} \\
&= \mathbb{1}_{\{T_1\leq t\}}\frac{1}{1-e^{-\lambda t}}\int_0^t \left(1 - e^{-\lambda\left(\frac{t}{a}-y\right)}\right)\lambda e^{-\lambda y}dy \\
&= \mathbb{1}_{\{T_1\leq t\}}\left(1 - \frac{\lambda t e^{-\lambda\frac{t}{a}}}{1-e^{-\lambda t}}\right).
\end{aligned}$$

Therefore

$$Z_t = 1 - \mathbb{1}_{\{T_1\leq t\}}\left(1 - \frac{\lambda t e^{-\lambda\frac{t}{a}}}{1-e^{-\lambda t}}\right) = \mathbb{1}_{\{T_1>t\}} + I_{\{T_1\leq t\}}\frac{\lambda t e^{-\lambda\frac{t}{a}}}{1-e^{-\lambda t}}, \tag{5.12}$$

and

$$Z_\tau = \frac{\lambda(T_1 \vee aT_2)e^{-\lambda\frac{T_1\vee aT_2}{a}}}{1-e^{-\lambda(T_1\vee aT_2)}} < 1.$$

Using the same type of arguments give

$$\widetilde{Z}_t = \mathbb{1}_{\{T_1\geq t\}} + \mathbb{1}_{\{T_1<t\}}\frac{\lambda t e^{-\lambda\frac{t}{a}}}{1-e^{-\lambda t}}, \quad \text{and} \quad \widetilde{Z}_\tau = \mathbb{1}_{\{T_1\geq aT_2\}} + \mathbb{1}_{\{T_1<aT_2\}}\frac{\lambda(T_1 \vee aT_2)e^{-\lambda\frac{T_1\vee aT_2}{a}}}{1-e^{-\lambda(T_1\vee aT_2)}}. \tag{5.13}$$

These allows us to conclude that both assertions (a0 and (b) hold. The proof of the assertion (d) is similar to that asertion (d) of the previous proposition. The remaining part of the proof will address

assertion (c).
By putting $K_t = 1 - \frac{\lambda t e^{-\lambda \frac{t}{a}}}{1-e^{-\lambda t}}$, $Z_t = 1 - H_t^1 K_t$ and applying Itô formula, we derive that

$$dZ_t = -d(H^1 K)_t = -K_t dH_t^1 - H_t^1 dK_t = -K_t dM_t^1 - K_t dA_t^1 - H_t^1 dK_t. \tag{5.14}$$

Hence,

$$m_t = 1 - \int_0^t K_t dM_t^1. \tag{5.15}$$

Now, we describe explicitly the arbitrage strategy. Notice that $\{T_1 \leq t\} = \{N_t \geq 1\}$. We deduce that

$$M_t^1 = I_{\{T_1 \leq t\}} - A_t^1 = I_{\{N_t \geq 1\}} - A_t^1 = I_{\{N_{t-} \geq 0\}} \Delta N_t + I_{\{N_{t-} \geq 1\}} (1 - \Delta N_t) - A_t^1. \tag{5.16}$$

Hence,

$$\Delta M_t^1 = M_t^1 - {}^p(M^1)_t = \left(I_{\{N_{t-} \geq 0\}} - I_{\{N_{t-} \geq 1\}}\right) \Delta N_t = \left(I_{\{N_{t-} \geq 0\}} - I_{\{N_{t-} \geq 1\}}\right) \Delta M_t. \tag{5.17}$$

Since M_t^1 and M_t are both purely discontinuous, we have $m_t = 1 + \varphi \cdot S_t$, where

$$\varphi_t = -K_t \left(I_{\{N_{t-} \geq 0\}} - I_{\{N_{t-} \geq 1\}}\right) \frac{1}{\psi S_{t-}}. \tag{5.18}$$

□

6 NUPBR for particular models

In this section, we address some interesting practical models, for which we prove that the NUPBR remains valid up to τ. The originality of this part – as we mentioned in the introduction and the abstract – lies in the simplicity of the proof. A general and complete analysis about the NUPBR is addressed in full generality in Choulli et al. (2013).

6.1 Before τ

Let $\widehat{m}$ be the $\mathbb{G}$-martingale stopped at time τ associated with m by (2.3), on $\{t \leq \tau\}$

$$\widehat{m}_t := m_t^\tau - \int_0^t \frac{d\langle m, m\rangle_s^{\mathbb{F}}}{Z_s}.$$

6.1.1 Case of continuous filtration

We start with the particular case of continuous martingales and prove that, for any random time τ, NUPBR holds before τ.

We note that the continuity assumption implies that the martingale part of Z is continuous and that the optional and Doob-Meyer decompositions of Z are the same.

Proposition 6.1 *Assume that all $\mathbb{F}$-martingales are continuous. Then, for any random time τ, NUPBR holds before τ. A $\mathbb{G}$-local martingale deflator for S^τ is given by $dL_t = -\frac{L_t}{Z_t} d\widehat{m}_t$.*

Proof: We make a use of Theorem 2.1 and we provide a $\mathbb{G}$-local martingale deflator for S^τ. Define the positive $\mathbb{G}$-local martingale L as $dL_t = -\frac{L_t}{Z_t} d\widehat{m}_t$. Then, if SL is a $\mathbb{G}$-local martingale, NUPBR holds. Recall that, using (2.3) again,

$$\widehat{S}_t := S_t^\tau - \int_0^{t\wedge\tau} \frac{d\langle S, m\rangle_s^{\mathbb{F}}}{Z_s}$$

is a $\mathbb{G}$-local martingale. From integration by parts, we obtain (using that the bracket of continuous martingales does not depend on the filtration)

$$\begin{aligned} d(LS^\tau)_t &= L_t dS^\tau_t + S_t dL_t + d\langle L, S^\tau\rangle^{\mathbb{G}}_t \\ &\overset{\mathbb{G}-\text{mart}}{=} L_t \frac{1}{Z_t} d\langle S, m\rangle^{\mathbb{F}}_t + \frac{1}{Z_t} L_t d\langle S, \widehat{m}\rangle^{\mathbb{G}}_t \\ &\overset{\mathbb{G}-\text{mart}}{=} L_t \frac{1}{Z_t} \left(d\langle S, m\rangle_t - d\langle S, m\rangle_t\right) = 0 \end{aligned}$$

where $X \overset{\mathbb{G}-\text{mart}}{=} Y$ is a notation for $X - Y$ is a $\mathbb{G}$-local martingale. □

Remark 6.2 If τ is an honest time and Predictable Representation Property holds with respect to S then, as a consequence of Theorem 4.2, the NA condition does not hold, hence NFLVR condition does not hold neither. That in turn implies that all the $\mathbb{G}$-local martingale deflators for S^τ are strict $\mathbb{G}$-local martingales.

6.1.2 Case of a Poisson filtration

We assume that S is an $\mathbb{F}$-martingale of the form $dS_t = S_{t-}\psi_t dM_t$, where ψ is a predictable process, satisfying $\psi > -1$ and $\psi \neq 0$, where M is the compensated martingale of a standard Poisson process.

In a Poisson setting, from Predictable Representation Property, $dm_t = \nu_t dM_t$ for some $\mathbb{F}$-predictable process ν, so that, on $t \leq \tau$,

$$d\widehat{m}_t = dm_t - \frac{1}{Z_{t-}} d\langle m, m\rangle_t = dm_t - \frac{1}{Z_{t-}} \lambda \nu_t^2 dt$$

Proposition 6.3 *In a Poisson setting, for any random time τ, NUPBR holds before τ since*

$$L = \mathcal{E}\left(-\frac{1}{Z_- + \nu} \cdot \widehat{m}\right) = \mathcal{E}\left(-\frac{\nu}{Z_- + \nu} \cdot \widehat{M}\right),$$

is a $\mathbb{G}$-local martingale deflator for S^τ.

Proof: We make a use of Theorem 2.1 and we are looking for a $\mathbb{G}$-local martingale deflator of the form $dL_t = L_{t-}\kappa_t d\widehat{m}_t$ (and $\psi_t\kappa_t > -1$) so that L is positive and $S^\tau L$ is a $\mathbb{G}$-local martingale. Integration by parts formula leads to (on $t \leq \tau$)

$$\begin{aligned} d(LS)_t &= L_{t-} dS_t + S_{t-} dL_t + d[L, S]_t \\ &\overset{\mathbb{G}-\text{mart}}{=} L_{t-} S_{t-} \psi_t \frac{1}{Z_{t-}} d\langle M, m\rangle_t + L_{t-} S_{t-} \kappa_t \psi_t \nu_t dN_t \\ &\overset{\mathbb{G}-\text{mart}}{=} L_{t-} S_{t-} \psi_t \frac{1}{Z_{t-}} \nu_t \lambda dt + L_{t-} S_{t-} \kappa_t \psi_t \nu_t \lambda (1 + \frac{1}{Z_{t-}} \nu_t) dt \\ &= L_{t-} S_{t-} \psi_t \nu_t \lambda \left(\frac{1}{Z_{t-}} + \kappa_t (1 + \frac{1}{Z_{t-}} \nu_t)\right) dt. \end{aligned}$$

Therefore, for $\kappa_t = -\frac{1}{Z_{t-} + \nu_t}$, one obtains a deflator. Note that

$$dL_t = L_{t-}\kappa_t d\widehat{m}_t = -L_{t-} \frac{1}{Z_{t-} + \nu_t} \nu_t d\widehat{M}_t$$

is indeed a positive $\mathbb{G}$-local martingale, since $\frac{1}{Z_{t-} + \nu_t} \nu_t < 1$. □

Remark 6.4 If τ is an honest time and Predictable Representation Property holds with respect to S then all the $\mathbb{G}$-local martingale deflators for S^τ are strict $\mathbb{G}$-local martingales.

6.1.3 Lévy processes

Assume that $S = \psi \star (\mu - \nu)$ where μ is the jump measure of a Lévy process and ν its compensator. Here, $\psi \star (\mu - \nu)$ is the process $\int_0^{\cdot} \int \psi(x,s)(\mu(dx,ds) - \nu(dx,ds))$. The martingale m admits a representation as $m = \psi^m \star (\mu - \nu)$. Then, using (2.3), the $\mathbb{G}$-compensator of μ is $\nu^{\mathbb{G}}$ where

$$\nu^{\mathbb{G}}(dt,dx) = \frac{1}{Z_{t-}} \left(Z_{t-} + \psi^m(t,x)\right) \nu(dt,dx)$$

i.e., S admits a $\mathbb{G}$-semimartingale decomposition of the form

$$S = \psi \star (\mu - \nu^{\mathbb{G}}) - \psi \star (\nu - \nu^{\mathbb{G}})$$

Proposition 6.5 *Consider the positive $\mathbb{G}$-local martingale*

$$L := \mathcal{E}\left(-\frac{\psi^m}{Z_- + \psi^m} I_{[\![0,\tau]\!]} \star (\mu - \nu^{\mathbb{G}}) \right).$$

Then L is a $\mathbb{G}$-local martingale deflator for S^τ, and hence S^τ satisfies NUPBR.

Proof: We make a use of Theorem 2.1 and our goal is to find a positive $\mathbb{G}$-local martingale L of the form

$$dL_t = L_{t-} \kappa_t d\widehat{m}_t$$

so that LS^τ is a $\mathbb{G}$-local martingale.

From integration by parts formula

$$\begin{aligned} d(SL) &\overset{\mathbb{G}-\text{mart}}{=} -L_- \psi \star (\nu - \nu^{\mathbb{G}}) + d[S,L] = -L_- \psi \star (\nu - \nu^{\mathbb{G}}) + L_- \psi \psi^m \kappa \star \mu \\ &\overset{\mathbb{G}-\text{mart}}{=} -L_- \psi \star (\nu - \nu^{\mathbb{G}}) + L_- \psi \psi^m \kappa \star \nu^{\mathbb{G}} \\ &= -L_- \psi \left(1 - (1 + \psi^m \kappa) \frac{1}{Z_-} \left(Z_- + \psi^m\right) \right) \star \nu \end{aligned}$$

Hence the possible choice $\kappa = -\frac{1}{Z_- + \psi^m}$. It can be checked that indeed, L is a positive $\mathbb{G}$-local martingale See [5]. □

6.2 After τ

We now assume that τ is an honest time, which satisfies $Z_\tau < 1$ (for integrability reasons). This condition and Lemma 2.4 imply in particular that τ does not avoids $\mathbb{F}$-stopping times. For the further discussion on the condition $Z_\tau < 1$ we refer the reader to [1]. Note also that, in the case of continuous filtration, and $Z_\tau = 1$, NUPBR fails to hold after τ (see [7]).

After (2.4), for any $\mathbb{F}$-martingale X (in particular for m and S)

$$\widehat{X}_t := X_t^\tau - \int_0^{t\wedge\tau} \frac{d\langle X,m\rangle_s^{\mathbb{F}}}{Z_s} + \int_{t\wedge\tau}^t \frac{d\langle X,m\rangle_s^{\mathbb{F}}}{1 - Z_s}$$

is a $\mathbb{G}$-local martingale.

6.2.1 Case of continuous filtration

We start with the particular case of continuous martingales and prove that, for any honest time τ such that $Z_\tau < 1$, NUPBR holds after τ.

Proposition 6.6 *Assume that τ is an honest time, which satisfies $Z_\tau < 1$ and that all $\mathbb{F}$-martingales are continuous. Then, for any honest time τ, NUPBR holds after τ. A $\mathbb{G}$-local martingale deflator for $S - S^\tau$ is given by $dL_t = -\frac{L_t}{1-Z_t} d\widehat{m}_t$.*

Proof: We use Theorem 2.1 as usual. The proof is based on Itô's calculus. Looking for a $\mathbb{G}$-local martingale deflator of the form $dL_t = L_t \kappa_t d\widehat{m}_t$, and using integration by parts formula, we obtain that, for $\kappa = -(1-Z)^{-1}$, the process $L(S - S^\tau)$ is a $\mathbb{G}$-local martingale. □

Remark 6.7 If Predictable Representation Property holds with respect to S then, as a consequence of Theorem 4.2, the NA condition does not hold, hence NFLVR condition does not hold neither. That in turn implies that all the $\mathbb{G}$-local martingale deflators for $S - S^\tau$ are strict $\mathbb{G}$-local martingales.

6.2.2 Case of a Poisson filtration

We assume that S is an $\mathbb{F}$-martingale of the form $dS_t = S_{t-}\psi_t dM_t$, with ψ is a predictable process, satisfying $\psi > -1$.

The decomposition formula (2.4) reads after τ as

$$\widehat{S}_t = (\mathbb{1}_{]\tau,\infty[} \cdot S)_t + \int_{t\vee\tau}^t \frac{1}{1-Z_{s-}} d\langle S, m\rangle_s = (\mathbb{1}_{]\tau,\infty[} \cdot S)_t + \lambda \int_{t\vee\tau}^t \frac{1}{1-Z_{s-}} \nu_s \psi_s S_{s-} ds.$$

Proposition 6.8 *Let $\mathbb{F}$ be a Poisson filtration and τ is an honest time satisfying $Z_\tau < 1$. Then, NUPBR holds after τ since*

$$L = \mathcal{E}\left(\frac{1}{1-Z_- - \nu} \cdot \widehat{m}\right) = \mathcal{E}\left(\frac{\nu}{1-Z_- - \nu} \mathbb{1}_{]\tau,\infty[} \cdot \widehat{M}\right),$$

is a $\mathbb{G}$-local martingale deflator for $S - S^\tau$.

Proof: We make a use of Theorem 2.1 and we are looking for a $\mathbb{G}$-local martingale deflator of the form $dL_t = L_{t-}\kappa_t d\widehat{m}_t$ (and $\psi_t\kappa_t > -1$) so that L is positive $\mathbb{G}$-local martingale and $(S - S^\tau)L$ is a $\mathbb{G}$-local martingale. Integration by parts formula leads to

$$\begin{aligned}
d(L(S-S^\tau))_t &= L_{t-} d(S-S^\tau)_t + (S_{t-} - S^\tau_{t-}) dL_t + d[L, S-S^\tau]_t \\
&\overset{\mathbb{G}-\text{mart}}{=} -\lambda L_{t-} S_{t-} \nu_t \psi_t \frac{1}{1-Z_{t-}} \mathbb{1}_{\{t>\tau\}} dt + L_{t-} S_{t-} \kappa_t \psi_t \nu_t \mathbb{1}_{\{t>\tau\}} dN_t \\
&\overset{\mathbb{G}-\text{mart}}{=} -\lambda L_{t-} S_{t-} \nu_t \psi_t \frac{1}{1-Z_{t-}} \mathbb{1}_{\{t>\tau\}} dt + \lambda L_{t-} S_{t-} \kappa_t \psi_t \nu_t \mathbb{1}_{\{t>\tau\}} (1 - \frac{1}{1-Z_{t-}} \nu_t) dt \\
&= \lambda L_{t-} S_{t-} \psi_t \nu_t \mathbb{1}_{\{t>\tau\}} \left(-\frac{1}{1-Z_{t-}} + \kappa_t (1 - \frac{1}{1-Z_{t-}} \nu_t) \right) dt.
\end{aligned}$$

Therefore, for $\kappa_t = \frac{1}{1-Z_{t-}-\nu_t}$, one obtains a $\mathbb{G}$-local martingale deflator. Note that

$$dL_t = L_{t-}\kappa_t d\widehat{m}_t = L_{t-} \frac{1}{1-Z_{t-}-\nu_t} \nu_t \mathbb{1}_{\{t>\tau\}} d\widehat{M}_t$$

is indeed a positive $\mathbb{G}$-local martingale, since $\frac{1}{1-Z_{t-}-\nu_t} \nu_t \Delta N_t > -1$. □

Remark 6.9 If Predictable Representation Property holds with respect to S then, all the $\mathbb{G}$-local martingale deflators for $S - S^\tau$ are strict $\mathbb{G}$-local martingales.

6.2.3 Lévy processes

Assume that $S = \psi \star (\mu - \nu)$ where μ is the jump measure of a Lévy process and ν its $\mathbb{F}$-compensator. Then, by (2.4), the $\mathbb{G}$-compensator of μ is $\nu^{\mathbb{G}}$ where

$$\nu^{\mathbb{G}}(dt,dx) = \left(1 + \mathbb{1}_{\{t\leq\tau\}}\frac{1}{Z_{t-}}\psi^m(t,x) - \mathbb{1}_{\{t>\tau\}}\frac{1}{1-Z_{t-}}\psi^m(t,x)\right)\nu(dt,dx)$$

i.e., S admits a $\mathbb{G}$-semimartingale decomposition of the form

$$S = \psi \star (\mu - \nu^{\mathbb{G}}) - \psi \star (\nu - \nu^{\mathbb{G}})$$

Proposition 6.10 *Assume that τ be an honest time satisfying $Z_\tau < 1$ in a Lévy framework. Then, the positive $\mathbb{G}$-local martingale*

$$L := \mathcal{E}\left(\frac{\psi^m}{1-Z_- - \psi^m} I_{\rrbracket\tau,\infty\rrbracket} \star (\mu - \nu^{\mathbb{G}})\right),$$

is a $\mathbb{G}$-local martingale deflator for $S - S^\tau$, and hence $S - S^\tau$ satisfies NUPBR.

Proof: We use of Theorem 2.1 again. Our goal is to find a positive $\mathbb{G}$-local martingale L of the form

$$dL_t = L_{t-}\kappa_t \mathbb{1}_{\{t>\tau\}} d\widehat{m}_t$$

so that $L(S - S^\tau)$ is a $\mathbb{G}$-local martingale.

From integration by parts formula

$$\begin{aligned}
d(L(S-S^\tau)) &\overset{\mathbb{G}-\text{mart}}{=} -L_- d(S-S^\tau) + d[S,L] \\
&= -L_-\psi\frac{\psi^m}{1-Z_-}\mathbb{1}_{]\tau,\infty[} \star \nu + L_-\kappa\psi\psi^m\mathbb{1}_{]\tau,\infty[} \star \mu \\
&\overset{\mathbb{G}-\text{mart}}{=} -L_-\psi\frac{\psi^m}{1-Z_-}\mathbb{1}_{]\tau,\infty[} \star \nu + L_-\kappa\psi\psi^m\mathbb{1}_{]\tau,\infty[} \star \nu^{\mathbb{G}} \\
&= -L_-\psi\psi^m\mathbb{1}_{]\tau,\infty[}\left(-\frac{1}{1-Z_-} + \kappa(1 - \frac{\psi^m}{1-Z_-})\right) \star \nu
\end{aligned}$$

Hence the possible choice $\kappa = \frac{1}{1-Z_- - \psi^m}$. □

Conclusions

In this paper we have treated the question whether the no-arbitrage conditions are stable with respect to progressive enlargement of filtration. We focused on two components of No Free Lunch with Vanishing Risk concept, namely on No Arbitrage Opportunity and No Unbounded Profit with Bounded Risk. The problem was divided into stability before and after random time containing extra information.

The question regarding No Arbitrage Opportunity condition was answered in the case of Brownian filtration and Poisson filtration for special case of honest time, moreover particular examples of non-honest times were described. Both, Brownian and Poisson filtrations possess an important, and crucial from our problem point of view, characteristic of Predictable Representation Property. One may further investigate similar problem without assuming market completness. One may as well consider other example/classes of non-honest random times.

Afterwards, we handeled with stability of NUPBR concept in very particular situations, namely in continuous martingale case, standard Poisson process case and Lévy process case. We provided reults with simple proofs in those particular situations. We emphasize again that in full generality

the problem is solved in [5] revealing as well results within progressive enlargement of filtration theory.

Combining results on NA and NUPBR conditions we concluded (in Remarks 6.2, 6.4, 6.7, 6.9) that some $\mathbb{G}$-local martingales are in fact $\mathbb{G}$-strict local martingales. That provides a way to construct strict local martingale in enlarged Brownian and Poisson filtrations.

A Appendix

Let $(A_t,\, t \geq 0)$ be an integrable increasing process (not necessarily $\mathbb{F}$-adapted). There exists a unique integrable $\mathbb{F}$-optional increasing process $(A^o_t, t \geq 0)$, called the dual optional projection of A such that

$$\mathbb{E}\left(\int_{[0,\infty[} U_s dA_s\right) = \mathbb{E}\left(\int_{[0,\infty[} U_s dA^o_s\right)$$

for any positive $\mathbb{F}$-optional process U.
There exists a unique integrable $\mathbb{F}$-predictable increasing process $(A^p_t, t \geq 0)$, called the dual predictable projection of A such that

$$\mathbb{E}\left(\int_{[0,\infty[} U_s dA_s\right) = \mathbb{E}\left(\int_{[0,\infty[} U_s dA^p_s\right)$$

for any positive $\mathbb{F}$-predictable process U.

Acknowledgement: This research benefited from the support of the "Chair Markets in Transition", under the aegis of Louis Bachelier laboratory, a joint initiative of Ecole Polytechnique, Université d'Evry Val d'Essonne and Fedération Bancaire Française.

References

[1] Aksamit, A., Choulli, T., and Jeanblanc, M. *Thin random times and their applications to finance*, Working paper, 2013.

[2] Amendinger, J., *Initial enlargement of filtrations and additional information in financial markets*, PhD Thesis, Technischen Universität Berlin, 1999.

[3] Asmussen, S., *Ruin Probability*, World Scientific, 2000.

[4] Azéma, J., *Quelques applications de la théorie générale des processus. I*, Inventiones mathematicae 18, 293–336, Springer, 1972.

[5] Choulli T., Aksamit A., Deng J., and Jeanblanc M. *Non-Arbitrage up to Random Horizons and after Honest Times for Semimartingale Models* Preprint, 2013.
http://arxiv.org/abs/1310.1142

[6] Delbaen, F., Schachermayer, W., *A general version of the fundamental theorem of asset pricing*, Mathematische Annalen, 300: 463-520, 1994.

[7] Fontana, C., Jeanblanc, M. and Song, S. *On arbitrages arising with honest times.* To appear in Finance and Stochastics.

[8] Grorud, A., and Pontier, M., *Insider trading in a continuous time market model*, International Journal of Theoretical and Applied Finance, 1, 331-347, 1998.

[9] Jeanblanc, M., Yor, M., and Chesney, M., *Mathematical Methods for financial Markets*, Springer, 2009.

[10] Jeulin, T., *Semi-Martingales et Grossissement d'une Filtration*, Lecture Notes in Mathematics, vol. 833, Springer, Berlin - Heidelberg - New York, 1980.

[11] Jeulin, T., and Yor, M., *Grossissement d'une filtration et semi-martingales : formules explicites.* In C. Dellacherie, P-A. Meyer, and M. Weil, editors, *Séminaire de Probabilités XII*, volume 649 of *Lecture Notes in Mathematics*, pages 78–97. Springer-Verlag, 1978.

[12] Kabanov, Y., *On the FTAP of Kreps-Delbaen-Schachermayer*, In: Kabanov, Y. et al. (eds.): Statistics and Control of Stochastic Processes: The Liptser Festschrift, pp. 191–203. World Scientific, Singapore, 1997.

[13] Karatzas, I., and Kardaras, C., *The numeraire portfolio in semimartingale financial models*, Finance and Stochastics 11, 447–493, 2007.

[14] Kardaras, C., *Finitely additive probabilities and the fundamental theorem of asset pricing* In: Chiarella, C., Novikov, A. (eds.): Contemporary Quantitative Finance: Essays in Honour of Eckhard Platen, pp. 19–34. Springer, Berlin Heidelberg, 2010.

[15] Nikeghbali, A., and Yor, M., *A definition and some characteristic properties of pseudo-stopping times*, Annals of Probability, 33(5), 1804-1824, 2005.

[16] Takaoka, K., *A note on the condition of no unbounded profit with bounded risk*, To appear in Finance and Stochastics, 2013.