A Lévy HJM multiple-curve model with application to CVA computation

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October 7, 2013

Abstract

We consider the problem of valuation of interest rate derivatives in the post-crisis setup. We develop a multiple-curve model, set in the HJM framework and driven by a Lévy process. We proceed with joint calibration to caps and swaptions of different tenors, the calibration to caps guaranteeing that the model correctly captures volatility smile effects (in strike) and the calibration to at-the-money swaptions ensuring an appropriate term structure of the volatility in the model. To account for counterparty risk and funding issues, we use the calibrated multiple-curve model as an underlying model for CVA computation. We follow a reduced-form methodology through which the problem of pricing the counterparty risk and funding costs can be reduced to a pre-default Markovian BSDE, or an equivalent semi-linear PDE. As an illustration we study the case of a basis swap, for which we compute the counterparty risk and funding adjustments.

Keywords: interest rate derivative, multiple-curve term structure model, Lévy process, credit valuation adjustment (CVA), funding.

1 Introduction

As a consequence of the financial crisis, various new phenomena appeared in the fixed income markets. A variety of spreads have developed, notably Libor-OIS swap spreads and basis swap spreads, which has become known in the literature as a multiple-curve phenomenon (see, among others, Kijima, Tanaka, and Wong (2009), Kenyon (2010), Henrard (2007, 2010), Bianchetti (2010), Mercurio (2010b, 2010a), Fujii, Shimada, and Takahashi (2011, 2010), Moreni and Pallavicini (2013a)).

In addition, counterparty risk and funding costs have become major issues in OTC derivative transactions. To account for these, credit/debt valuation adjustments CVA/DVA and different kinds of funding valuation adjustment have been introduced (see e.g. the

*The research of S. Crépey benefited from the support of the "Chair Markets in Transition" under the aegis of Louis Bachelier laboratory, a joint initiative of École polytechnique, Université d'Évry Val d'Essonne and Fédération Bancaire Française

†Z. Grbac acknowledges the financial support from the DFG Research Center MATHEON, Project E13.

papers by Prisco and Rosen (2005), Brigo and Pallavicini (2008), Brigo and Capponi (2010), Burgard and Kjaer (2011a,2011b), Crépey (2012), Crépey, Gerboud, Grbac, and Ngor (2013), Pallavicini, Perini, and Brigo (2011, 2012); see also the books by Gregory (2012), Cesari, Aquilina, Charpillon, Filipovic, Lee, and Manda (2010), Brigo, Morini, and Pallavicini (2013) and Crépey, Bielecki, and Brigo (2013)). All these issues are related since the above-mentioned spreads are a consequence of banks' increased credit riskiness and funding costs (see e.g. Crépey and Douady (2013), Filipović and Trolle (2013) and Crépey, Grbac, and Nguyen (2012) for an economico-financial, econometrical and pricing perspective, respectively). We refer to an aggregate adjustment as a TVA, which stands for total valuation adjustment. It should be emphasized that we use this terminology in the paper and reserve the term CVA for a strict credit valuation adjustment. Note that in the literature, the acronym CVA is often used also in a broader sense (which then corresponds to our TVA). This is the reason why the name CVA was also preferred in the title and the abstract of the paper, however in the rest of the paper we consistently use the terms TVA and CVA in the sense explained above.

From a risk-management perspective, this new paradigm has created a need for interest rate models which, on the one hand, are multiple-curve models consistent with the reality of the post-crisis interest rate markets and, on the other hand, allow for practical TVA computations (for which, in particular, a short rate process r_t and a parsimonious Markov structure are required). This double requirement puts a lot of constraints on the model. A few suitable models which come to mind are the HJM multiple-curve model of Fujii and Takahashi (2011) and the "parsimonious" (in reference to the above Markov concern) models of Moreni and Pallavicini (2013a, 2013b). Moreni and Pallavicini (2013a) are the first to apply the HJM reconstruction formula, normally used to compute zero coupon bond prices, to define the FRA rates. Our model construction in this paper is similar in spirit. However, none of the mentioned papers studies the application to TVA computation.

To optimally meet all the conditions discussed above, a powerful modeling ingredient is the use of a more general class of driving processes, namely Lévy processes, as opposed to Brownian drivers in the three mentioned papers. Specifically, in this paper, we devise a Lévy HJM multiple-curve model driven by a two-dimensional NIG process, with a built-in HJM fit to the initial Libor and OIS term structures and with a two-dimensional Markov structure. Lévy drivers were already used with the same motivation in Crépey et al. (2012). However, there the Libor rates were defined in terms of the so-called Libor bonds reflecting the credit and liquidity risk of the Libor contributing banks. The main motivation was to explain in an economically satisfying way the spreads between the OIS and the Libor rates. But, since Libor bonds are not traded assets, we choose in this paper to follow an approach similar to Mercurio (2010a) and model the Libor FRA rates (see definition (9) below). The Libor FRA rates are directly observable up to the maturity of one year and for longer maturities they can be bootstrapped from Libor swap rates. The model of Mercurio (2010a) is developed in the standard Libor market model setup. This setup is less suited for TVA computations, as it does not allow for low-dimensional Markovian representations of the term structure of interest rates. In contrast, the HJM framework developed in the present paper has this property. In particular, we can access the short rate process r_t, which is needed for discounting in TVA computations.

The paper is organized as follows. In Section 2, we present a Lévy HJM multiple-curve model and provide pricing formulas for the most common interest rate derivatives such as swaps and basis swaps, caplets and swaptions. Section 4 deals with the calibration

of the model. The HJM framework yields an automatic fit to the initial Libor and OIS term structures and the dynamic parameters of the model are calibrated to option prices. We ensure joint calibration to caps and swaptions of different tenors, the calibration to caps guaranteeing that the model correctly captures volatility smile effects (in strike) and the calibration to at-the-money swaptions ensuring an appropriate term structure of the volatility in the model. This is important in view of the targeted application to TVA computations on multiple-curve products, which is the topic of Section 5, where we use the calibrated multiple-curve model as an underlying model for TVA computation. We follow the reduced-form methodology of Crépey (2012) (see also Pallavicini et al. (2012)) through which counterparty risk and funding adjustments are obtained as the solution to a related backward stochastic differential equation (BSDE). As an illustration we study the case of a basis swap, for which we compute the counterparty risk and funding adjustments.

2 Multiple-curve model

The post-crisis interest rate models have to account for the various spreads observed in the market, as discussed in the introduction. In particular, one needs to model not only the FRA rates related to the Libor rates, which are the underlying rates for most interest rate derivatives, but also the OIS rates implied by the overnight indexed swaps (OIS), which are used for discounting. In this section we develop a Lévy multiple-curve model for the FRA and the OIS rates, which is designed in the Heath–Jarrow–Morton (HJM) framework for term structure modeling. Furthermore, we provide the corresponding pricing formulas for the most common interest rate derivatives.

2.1 Driving process

We begin by introducing the class of driving processes that will be considered. Let a filtered probability space $(\Omega, \mathcal{F}_{\overline{T}}, \mathbb{F}, \mathbb{P})$, where $\overline{T}$ is a finite time horizon and $\mathbb{P}$ is a risk-neutral pricing measure, be fixed. The filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,\overline{T}]}$ satisfies the usual conditions. The driving process $Y = (Y_t)_{0\le t\le\overline{T}}$ is assumed to be an $\mathbb{F}$-adapted, $\mathbb{R}^n$-valued Lévy process. For the definition and main properties of Lévy processes we refer the reader to Cont and Tankov (2003) and Sato (1999). The characteristic function of Y_t is given by the Lévy-Khintchine formula, in which u denotes a row-vector in $\mathbb{R}^n$:

$$\mathbb{E}[e^{\mathrm{i}uY_t}] \;=\; \exp\left(t\left(\mathrm{i}ub - \frac{1}{2}ucu^\top + \int_{\mathbb{R}^n}\left(e^{\mathrm{i}ux} - 1 - \mathrm{i}uh(x)\right)F(dx)\right)\right), \tag{1}$$

where $b \in \mathbb{R}^n$, c is a symmetric, nonnegative definite real-valued n-dimensional matrix and F is a Lévy measure on $\mathbb{R}^n$, i.e. $F(\{0\}) = 0$ and $\int_{\mathbb{R}^n}(|x|^2 \wedge 1)F(dx) < \infty$. The function $h : \mathbb{R}^n \to \mathbb{R}^n$ is a truncation function.

We assume that there exist constants $\mathcal{K}, \varepsilon > 0$ such that

$$\int_{|x|>1} \exp(ux)F(dx) < \infty\,, \;\; u \in [-(1+\varepsilon)\mathcal{K}, (1+\varepsilon)\mathcal{K}]^n. \tag{2}$$

It is well-known that condition (2) guarantees the existence of exponential moments of the process Y: condition (2) holds if and only if $\mathbb{E}[\exp(uY_t)] < \infty$, for all $0 \le t \le \overline{T}$ and $u \in [-(1+\varepsilon)\mathcal{K}, (1+\varepsilon)\mathcal{K}]^n$ (cf. Theorem 25.3 in Sato (1999)). Moreover, (2) ensures that

$h(x) = x$ can be chosen as truncation function. Hence, Y is a special semimartingale, with the canonical representation

$$Y_t = bt + \sqrt{c}W_t + \int_0^t \int_{\mathbb{R}^n} x(\mu - \nu)(ds, dx), \qquad t \in [0, \overline{T}], \tag{3}$$

where μ is the random measure of the jumps of Y, ν is the $\mathbb{P}$-compensator of μ given by $\nu(ds, dx) = F(dx)ds$, $\sqrt{c}$ is a measurable version of a square-root of the matrix c, and W is a $\mathbb{P}$-standard Brownian motion.

The cumulant generating function associated with the Lévy process Y is denoted by ψ. Then for any row-vector $z \in \mathbb{C}^n$ such that $\Re z \in [-(1+\varepsilon)\mathcal{K}, (1+\varepsilon)\mathcal{K}]^n$, we have

$$\psi(z) = zb + \frac{1}{2}zcz^\top + \int_{\mathbb{R}^n} \left(e^{zx} - 1 - zx\right) F(dx). \tag{4}$$

Consequently, (1) is equivalently written as:

$$\mathbb{E}[e^{\mathrm{i}uY_t}] = \exp\left(t\psi(\mathrm{i}u)\right). \tag{5}$$

2.2 Multiple-curve dynamics

In this section we present the multiple-curve model for the OIS bond prices and the FRA rates. The dynamics of the OIS bond prices $(B_t(T))_{0\le t\le \overline{T}}$ are modeled in a HJM fashion as follows (see e.g. Eberlein and Raible (1999)):

$$B_t(T) = \frac{B_0(T)}{B_0(t)} \exp\left(\int_0^t (A(s,t) - A(s,T))ds + \int_0^t (\Sigma(s,t) - \Sigma(s,T))dY_s\right), \tag{6}$$

where $A(s,T)$ and $\Sigma(s,T)$ are deterministic, real-valued functions defined on the set $\{(s,T) \in [0,\overline{T}] \times [0,\overline{T}] : 0 \le s \le T\}$, whose paths are continuously differentiable in the second variable. We make a standing assumption that the volatility structure $\Sigma(\cdot,\cdot)$ is bounded such that $0 \le \Sigma^i(s,T) \le \frac{\mathcal{K}}{2}$ for every $0 \le s \le T \le \overline{T}$ and $i \in \{1, 2, \ldots, n\}$, where $\mathcal{K}$ is the constant from (2). The initial term structure $(B_0(T))_{0\le t\le \overline{T}}$ is assumed to be bounded. Moreover, $A(s,T) = \psi(-\Sigma(s,T))$, for every $s \le T$, which is a classical Lévy HJM drift condition guaranteeing absence of arbitrage between OIS bonds. The OIS discount factor process $\beta = (\beta_t)_{0\le t\le \overline{T}}$ defined by $\beta_t = \exp\left(-\int_0^t r_s ds\right)$, where r represents the short rate process, can be written as

$$\beta_t = B_0(t) \exp\left(-\int_0^t A(s,t)ds - \int_0^t \Sigma(s,t)dY_s\right) \tag{7}$$

and the connection between $B(T)$ and r is given by

$$B_t(T) = B_0(T) \exp\left(\int_0^t (r_s - A(s,T))ds - \int_0^t \Sigma(s,T)dY_s\right). \tag{8}$$

Now we define the FRA rates $F_t(T,S)$ and specify their dynamics. More precisely, we introduce the following quantities:

$$F_t(T,S) = (S-T)\mathbb{E}_t^S\left[L_T(T,S)\right], \tag{9}$$

where $0 \leq t \leq T \leq S$. Here $L_T(T,S)$ denotes a T-spot Libor rate fixed at time T for the time interval $[T,S]$ and $\mathbb{E}_t^S$ denotes the $\mathcal{F}_t$-conditional expectation with respect to the S-forward martingale measure $\mathbb{P}^S$ (see (10)), where $\mathbb{F} = (\mathcal{F}_t)$ is the reference filtration. These quantities are exactly $(S-T)\times$ Libor FRA rates, as defined in Mercurio (2010a). Defined as $\mathbb{E}_t^S[L_T(T,S)]$ in (9), FRA rates are by their very definition local martingales under S-forward martingale measures, consistent with the model-free arbitrage formula (20) below for the price of an interest rate swap. Note that modeling $(S-T)\times$ FRA rates instead of the FRA rates themselves ("interest charge instead of interest rate") gives rise to slightly simpler formulas in our setup. From a practical point of view, modeling in terms of the FRA rates yields very similar numerical results. By a slight abuse of terminology, we will refer to $F_t(T,S)$ as the FRA rate.

Note that the definition of $F_t(T,S)$ implies the following two modeling requirements:

(i) $F_t(T,S) \geq 0$, for every t

(ii) $F_\cdot(T,S)$ is a $\mathbb{P}^S$-martingale, where the $\mathbb{P}^S$-forward martingale measure is characterized in terms of $\mathbb{P}$ by the following density process:

$$\frac{d\mathbb{P}^S}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \frac{\beta_t B_t(S)}{B_0(S)}, \quad 0 \leq t \leq S. \tag{10}$$

Both requirements are direct consequences of (9) if the Libor rate $L_T(T,S)$ is nonnegative, which is implied by market observations.

We model $F_t(T,S)$, for all $t \leq T \leq S$, as

$$F_t(T,S) = F_0(T,S) \exp\left(\int_0^t \alpha(s,T,S) ds + \int_0^t \varsigma(s,T,S) dY_s \right), \tag{11}$$

where $\alpha(s,T,S)$ is a drift term and $\varsigma(s,T,S)$ a volatility structure. We assume that $\alpha(s,T,S)$ and $\varsigma(s,T,S)$ are deterministic, real-valued functions defined on the set $\{(s,T,S) \in [0,\overline{T}] \times [0,\overline{T}] \times [0,\overline{T}] : 0 \leq s \leq T \leq S\}$ such that the above integrals are well-defined. The volatility structure is assumed to be bounded, i.e. $0 \leq \varsigma^i(s,T,S) \leq \frac{K}{2}$, for every $0 \leq s \leq T \leq S$ and $i \in \{1,\ldots,n\}$. The initial term structure $(F_0(T,S))_{0 \leq T \leq S \leq \overline{T}}$ is also bounded by assumption.

With this specification, requirement (i) is satisfied automatically and (ii) holds if the following drift condition is satisfied

$$\begin{aligned}
\alpha(s,T,S) &= -\varsigma(s,T,S) b^S - \frac{1}{2} \varsigma(s,T,S)\, c\, \varsigma(s,T,S)^\top \\
&\quad - \int_{\mathbb{R}^d} \left(e^{\varsigma(s,T,S)x} - 1 - \varsigma(s,T,S)x \right) F^S(dx) \\
&= -\psi^S(\varsigma(s,T,S)) \\
&= \psi(-\Sigma(s,S)) - \psi(\varsigma(s,T,S) - \Sigma(s,S))
\end{aligned} \tag{12}$$

where (b^S, c, F^S) is the Lévy triplet and ψ^S is the cumulant generating function of Y under the forward measure $\mathbb{P}^S$. This result is well-known for Lévy driven term structure models, compare for example Eberlein and Özkan (2005). The last equality follows from the connection between the cumulant generating functions ψ^S and ψ of Y under the measures $\mathbb{P}^S$ and $\mathbb{P}$ which is given by

$$\psi^S(z) = \psi(z - \Sigma(s,S)) - \psi(-\Sigma(s,S)) \tag{13}$$

for $z \in \mathbb{R}^n$ such that the above expressions are well-defined. To prove this result, it suffices to apply the definition of the cumulant generating function and equations (8) and (10).

Remark 2.1 Note that the following generalization of (11) can also be considered, which produces an equally tractable model:

$$F_t(T,S) + \Delta(T,S) = (F_0(T,S) + \Delta(T,S)) \exp\left(\int_0^t \alpha(s,T,S)ds + \int_0^t \varsigma(s,T,S)dY_s\right),$$

where $\Delta(T,S) \in \mathbb{R}$, for $T \leq S$, are constant shifts. This is known as a shifted model. The use of shifts allows one to recover a single-curve model as a special case of the multiple-curve model by setting $\Delta(T,S) = 1$, for all $T \leq S$, and $\varsigma(s,T,S) = \Sigma(s,S) - \Sigma(s,T)$ in the above specification. Then $F_t(T,S) = \frac{B_t(T)}{B_t(S)} - 1$, which is the classical relation from the single-curve model. Note, however, that in the shifted model the FRA rates can become negative with positive probability. More generally, the shifts can also be used to increase the flexibility of the model, which we found unnecessary in our case where sufficient flexibility is already ensured by maturity-dependent volatility specification for the FRA rate (cf. Sections 2.3 and 4). For different kinds of shifts used in the multiple-curve term structure literature see Mercurio (2010a) or Moreni and Pallavicini (2013a).

2.3 Lévy Hull-White specification

In the sequel we focus on the case where $Y = (Y^1, Y^2)$ is a two-dimensional Lévy process. The dependence between the two components can be modeled in an explicit way. For example one can use a common factor specification starting from a three-dimensional Lévy processes (Z^1, Z^2, Z^3) with independent components and setting $Y^1 := Z^1 + Z^3, Y^2 := Z^2 + Z^3$, or Lévy copulas (see Kallsen and Tankov (2006)). In the numerical part of the paper we find it sufficient to use a two-dimensional Lévy process, whose components Y^i are two independent NIG processes with cumulant generating function ψ^i of Y^i given, for $i = 1, 2$, by:

$$\psi^i(z) = -\nu_i\left(\sqrt{\nu_i^2 - 2z_i\theta_i - z_i^2\sigma_i^2} - \nu_i\right), \quad i = 1, 2, \tag{14}$$

where $\nu_i, \sigma_i > 0$ and $\theta_i \in \mathbb{R}$. We refer to the seminal paper by Barndorff-Nielsen (1997) for details and properties of NIG processes.[1]

Next we specify the volatility structures. We choose the Vasicek volatility structures, which is motivated by the desired Markovian representation mentioned in the introduction and used in Subsection 3.4. More precisely, the volatility of $B_t(T)$ is given by

$$\Sigma(s,T) = \left(\frac{\sigma}{a}\left(1 - e^{-a(T-s)}\right), 0\right) \tag{15}$$

and the volatility of $F_t(T,S)$ by

$$\varsigma(s,T,S) = \left(\frac{\sigma}{a}e^{as}\left(e^{-aT} - e^{-aS}\right), \frac{\sigma^*(T,S)}{a^*}e^{a^*s}\left(e^{-a^*T} - e^{-a^*S}\right)\right), \tag{16}$$

[1] The parametrization in equation (3.1) in Barndorff-Nielsen (1997) is recovered by setting $\mu = 0$, $\alpha = \frac{1}{\sigma}\sqrt{\frac{\theta_i^2}{\sigma_i^2} + \nu_i^2}$, $\beta = \frac{\theta_i}{\sigma_i^2}$ and $\delta = \sigma$.

where $\sigma, \sigma^*(T,S) > 0$ and $a, a^* \neq 0$ are real constants. Note that $\sigma^*(T,S)$ may depend on T and S, see Remark 2.1.

Denote by $f_t(T) = -\partial_T \log(B_t(T))$ the instantaneous continuously compounded forward rate, so that $r_t = f_t(t)$. Then the OIS bond price can be written as an exponentially-affine function of the short rate r:

$$B_t(T) = \exp(m(t,T) + n(t,T) r_t), \tag{17}$$

where

$$\begin{aligned} m(t,T) &= \log\left(\frac{B_0(T)}{B_0(t)}\right) - n(t,T)\left[f_0(t) + \psi^1\left(\frac{\sigma}{a}\left(e^{-at}-1\right)\right)\right] \\ &\quad - \int_0^t \left[\psi^1\left(\frac{\sigma}{a}\left(e^{-a(T-s)}-1\right)\right) - \psi^1\left(\frac{\sigma}{a}\left(e^{-a(t-s)}-1\right)\right)\right] ds \end{aligned}$$

and

$$n(t,T) = -e^{at}\int_t^T e^{-au}du = \frac{1}{a}\left(e^{-a(T-t)}-1\right).$$

The dynamics of the short rate r is described by the following SDE

$$dr_t = a(\rho(t) - r_t)dt + \sigma dY_t^1,$$

i.e.

$$r_t = e^{-at}\left(r_0 + a\int_0^t e^{as}\rho(s)ds + \sigma\int_0^t e^{as}dY_s^1\right)$$

with

$$\rho(t) = f_0(t) + \frac{1}{a}\frac{\partial}{\partial t}f_0(t) + \psi^1\left(\frac{\sigma}{a}\left(e^{-at}-1\right)\right) - (\psi^1)'\left(\frac{\sigma}{a}\left(e^{-at}-1\right)\right)\frac{\sigma}{a}e^{-at}. \tag{18}$$

Note that this is a Lévy Hull–White extended Vasicek model for the short rate r (cf. Eberlein and Raible (1999), equation (4.11), and Example 3.5 of Crépey et al. (2012)).

Furthermore, in a fashion similar to (17), the FRA rate given in (11) can be written as the following exponentially-affine function of the short rate r and another factor process which we denote by q:

$$\begin{aligned} F_t(T,S) &= F_0(T,S)\exp\left(\int_0^t \alpha(s,T,S)ds + \frac{\sigma}{a}\left(e^{-aT}-e^{-aS}\right)\int_0^t e^{as}dY_s^1\right. \\ &\quad \left. + \frac{\sigma^*(T,S)}{a^*}\left(e^{-a^*T}-e^{-a^*S}\right)\int_0^t e^{a^*s}dY_s^2\right) \\ &= \exp\left(m(t,T,S) + n(t,T,S)r_t + n^*(t,T,S)q_t\right), \end{aligned} \tag{19}$$

where

$$\begin{aligned} n(t,T,S) &= n(t,T) - n(t,S) \\ n^*(t,T,S) &= \frac{\sigma^*(T,S)}{a^*}\left(e^{-a^*(T-t)} - e^{-a^*(S-t)}\right) \end{aligned}$$

and

$$m(t,T,S) = \log(F_0(T,S)) + \int_0^t \alpha(s,T,S)ds - n(t,T,S)\left(f_0(t) + \psi^1\left(\frac{\sigma}{a}\left(e^{-at}-1\right)\right)\right).$$

The factor process q in (19) is defined by

$$q_t = e^{-a^* t} \int_0^t e^{a^* s} dY_s^2,$$

i.e. it satisfies the following SDE

$$dq_t = -a^* q_t dt + dY_t^2, \qquad q_0 = 0.$$

3 Pricing of interest rate derivatives

In this section we give an overview of the most common interest rate derivatives and provide pricing formulas in the Lévy multiple-curve model. Note that these are "clean prices", ignoring counterparty risk and assuming that funding is ensured at the OIS rate. Counterparty risk and funding valuation adjustments are computed separately, as will be explained in Section 5. The results of Subsections 3.1 and 3.2 below are valid for a general Lévy multiple-curve model and in Subsection 3.3 the assumptions of Section 2.3 are imposed.

3.1 Interest rate derivatives with linear payoffs

For interest rate derivatives with linear payoffs the prices can be easily expressed in terms of $F_t(T,S)$, as we show below.

A **fixed-for-floating interest rate swap** is a financial contract between two parties to exchange a stream of fixed interest payments for a stream of floating payments linked to the Libor rates, based on a specified notional amount N. We assume, as standard, that the Libor rate is set in advance and the payments are made in arrears. The swap is initiated at time $T_0 \geq 0$. Denote by $T_1 < \cdots < T_n$, where $T_1 > T_0$, a collection of the payment dates and by K the fixed rate. Then the time-t value of the swap, where $t \leq T_0$, for the receiver of the floating rate is given by:

$$\begin{aligned} P_t^{sw} &= N \sum_{k=1}^{n} \delta_{k-1} B_t(T_k) \mathbb{E}_t^{T_k} [L_{T_{k-1}}(T_{k-1}, T_k) - K] \\ &= N \sum_{k=1}^{n} B_t(T_k) \left(F_t(T_{k-1}, T_k) - \delta_{k-1} K \right), \end{aligned} \tag{20}$$

where $\mathbb{E}^{T_k}$ is the expectation with respect to the forward measure $\mathbb{P}^{T_k}$ and $\delta_{k-1} = T_k - T_{k-1}$. The swap rate K_t^{sw} is given by

$$K_t^{sw} = \frac{\sum_{k=1}^{n} B_t(T_k) F_t(T_{k-1}, T_k)}{\sum_{k=1}^{n} \delta_{k-1} B_t(T_k)}. \tag{21}$$

Remark 3.1 The Libor-OIS swap spread mentioned in the introduction is by definition the difference between the swap rate (21) of the Libor-indexed interest rate swap and the OIS rate, where the latter is given by the classical formula:

$$K_t^{ois} = \frac{B_t(T_0) - B_t(T_n)}{\sum_{k=1}^{n} \delta_{k-1} B_t(T_k)}. \tag{22}$$

See Filipović and Trolle (2013) for details. Thus, the Libor-OIS swap spread is given, for $0 \leq t \leq T_0$, by

$$K_t^{sw} - K_t^{ois} = \frac{\sum_{k=1}^{n} B_t(T_k) F_t(T_{k-1}, T_k) - B_t(T_0) + B_t(T_n)}{\sum_{k=1}^{n} \delta_{k-1} B_t(T_k)}. \tag{23}$$

A **basis swap** is an interest rate swap, where two streams of floating payments linked to the Libor rates of different tenors are exchanged. Both rates are set in advance and paid in arrears. We consider a basis swap with two tenor structures denoted by $\mathcal{T}^1 = \{T_0^1 < \ldots < T_{n_1}^1\}$ and $\mathcal{T}^2 = \{T_0^2 < \ldots < T_{n_2}^2\}$, where $T_0^1 = T_0^2 \geq 0$, $T_{n_1}^1 = T_{n_2}^2$, and $\mathcal{T}^1 \subset \mathcal{T}^2$. The notional amount is denoted by N and the swap is initiated at time T_0^1, where the first payments are due at T_1^1 and T_1^2. The time-t value of the basis swap, for $t \leq T_0^1$, is given by:

$$\begin{aligned} P_t^{bsw} &= N\left(\sum_{i=1}^{n_1} \delta_{i-1}^1 B_t(T_i^1)\mathbb{E}_t^{T_i^1}[L_{T_{i-1}^1}(T_{i-1}^1, T_i^1)]\right. \\ &\quad \left. - \sum_{j=1}^{n_2} \delta_{j-1}^2 B_t(T_j^2)\mathbb{E}_t^{T_j^2}[L_{T_{j-1}^2}(T_{j-1}^2, T_j^2)]\right). \end{aligned} \tag{24}$$

Thus,

$$P_t^{bsw} = N\left(\sum_{i=1}^{n_1} B_t(T_i^1)F_t(T_{i-1}^1, T_i^1) - \sum_{j=1}^{n_2} B_t(T_j^2)F_t(T_{j-1}^2, T_j^2)\right). \tag{25}$$

The value of the basis swap after the initiation, i.e. the value at time t, for $T_0^1 \leq t < T_{n_1}^1$, is given by

$$\begin{aligned} P_t^{bsw} &= N\left(B_t(T_{i_t}^1)F_{T_{i_t-1}^1}(T_{i_t-1}^1, T_{i_t}^1) + \sum_{i=i_t+1}^{n_1} B_t(T_i^1)F_t(T_{i-1}^1, T_i^1)\right. \\ &\quad \left. - B_t(T_{j_t}^2)F_{T_{j_t-1}^2}(T_{j_t-1}^2, T_{j_t}^2) - \sum_{j=j_t+1}^{n_2} B_t(T_j^2)F_t(T_{j-1}^2, T_j^2)\right), \end{aligned} \tag{26}$$

where $T_{i_t}^1$, respectively $T_{j_t}^2$, denotes the smallest T_i^1, respectively T_j^2, which is strictly greater than t.

Note that a basis swap would have zero value (and zero TVA) at all times in the classical single-curve setup (see Section 4.4 in Crépey et al. (2012)). However, in a multiple-curve setup, the value of the basis swap at any time t is not zero and markets actually quote positive basis swap spreads, which, when added to the smaller tenor leg, make the value of the basis swap equal zero. More precisely, on the smaller tenor leg the floating interest rate $L_{T_{j-1}^2}(T_{j-1}^2, T_j^2)$ at T_j^2 is replaced by $L_{T_{j-1}^2}(T_{j-1}^2, T_j^2) + K_t^{bsw}$, for every $j = 1, \ldots, n_2$, where K_t^{bsw} is the fair basis swap spread given at time t by

$$K_t^{bsw} = \frac{\sum_{i=1}^{n_1} B_t(T_i^1)F_t(T_{i-1}^1, T_i^1) - \sum_{j=1}^{n_2} B_t(T_j^2)F_t(T_{j-1}^2, T_j^2)}{\sum_{j=1}^{n_2} \delta_{j-1}^2 B_t(T_j^2)}. \tag{27}$$

3.2 Caplets

Let us now consider a caplet with strike K and maturity T on the spot Libor rate for the period $[T, T+\delta]$, settled in arrears at time $T+\delta$. Its time-t price, for $t \leq T$, is given by

$$\begin{aligned} P_t^{cpl} &= \delta\, B_t(T+\delta)\mathbb{E}_t^{T+\delta}\left[(L_T(T, T+\delta) - K)^+\right] \\ &= B_t(T+\delta)\mathbb{E}_t^{T+\delta}\left[(F_T(T, T+\delta) - \delta K)^+\right], \end{aligned}$$

since

$$L_T(T,T+\delta) = \mathbb{E}_T^{T+\delta}\left[L_T(T,T+\delta)\right] = \delta^{-1}F_T(T,T+\delta).$$

The value of the caplet at time $t=0$ can be calculated using the Fourier transform method. One has

$$P_0^{cpl} \quad = \quad B_0(T+\delta)\mathbb{E}^{T+\delta}\left[\left(e^X-\overline{K}\right)^+\right],$$

where $\overline{K} := \delta K$ and

$$X := \log F_T(T,T+\delta) = \log F_0(T,T+\delta) + \int_0^T \alpha(s,T,T+\delta)ds + \int_0^T \varsigma(s,T,T+\delta)dY_s. \quad (28)$$

The moment generating function of the random variable X under the measure $\mathbb{P}^{T+\delta}$ is given by

$$\begin{aligned} M_X^{T+\delta}(z) &= \mathbb{E}^{T+\delta}\left[e^{zX}\right] \\ &= \exp\left(z\left(\log F_0(T,T+\delta) + \int_0^T \alpha(s,T,T+\delta)ds\right)\right) \\ &\quad \times \exp\left(\int_0^T \psi^{T+\delta}(z\varsigma(s,T,T+\delta))ds\right) \\ &= \exp\left(z\left(\log F_0(T,T+\delta) + \int_0^T \alpha(s,T,T+\delta)ds\right)\right) \quad (29) \\ &\quad \times \exp\left(-\int_0^T \psi(-\Sigma(s,T+\delta))ds\right) \\ &\quad \times \exp\left(\int_0^T \psi\left(z\varsigma(s,T,T+\delta) - \Sigma(s,T+\delta)\right)ds\right), \end{aligned}$$

for $z \in \mathbb{R}$ such that the above expectation exists and where ψ, resp. $\psi^{T+\delta}$, is the cumulant generating function of the process Y under the measure $\mathbb{P}$, resp. $\mathbb{P}^{T+\delta}$. The first equality follows by Lemma 3.1 in Eberlein and Raible (1999) and the second one by equation (13). The payoff function of the caplet

$$g(x) = (\mathrm{e}^x - \overline{K})^+,$$

has the (generalized) Fourier transform

$$\widehat{g}(z) = \int_{\mathbb{R}} e^{izx}g(x)dx = \frac{\overline{K}^{1+iz}}{iz(1+iz)},$$

for $z \in \mathbb{C}$ such that $\Im z > 1$. Therefore, applying Theorem 2.2 in Eberlein, Glau, and Papapantoleon (2010), we obtain the following result (cf. also Carr and Madan (1999)).

Proposition 3.2 *Assume that there exists a positive constant $\widetilde{\mathcal{K}} < \frac{\mathcal{K}}{2}$ such that $\Sigma(s,T) \leq \widetilde{\mathcal{K}}$ and $\varsigma(s,T,S) \leq \widetilde{\mathcal{K}}$ componentwise and for all $0 \leq s \leq T \leq S \leq \overline{T}$. Let $R \in (1, \frac{\mathcal{K}-\widetilde{\mathcal{K}}}{\widetilde{\mathcal{K}}})$ be an arbitrary real number. The price at time $t=0$ of the caplet with strike K and maturity T is given by*

$$\begin{aligned} P_0^{cpl} &= \frac{B_0(T+\delta)}{2\pi}\int_{\mathbb{R}} \widehat{g}(iR-v)M_X^{T+\delta}(R+iv)dv \\ &= \frac{B_0(T+\delta)}{2\pi}\int_{\mathbb{R}} \frac{\overline{K}^{1-\mathrm{i}v-R}M_X^{T+\delta}(R+\mathrm{i}v)}{(R+\mathrm{i}v)(R+\mathrm{i}v-1)}dv, \quad (30) \end{aligned}$$

Proof. The result follows by Theorem 2.2 in Eberlein, Glau, and Papapantoleon (2010). We only have to check that for $R \in (1, \frac{\mathcal{K}-\widetilde{\mathcal{K}}}{\widetilde{\mathcal{K}}})$ the moment generating function $M_X^{T+\delta}(R+\mathrm{i}v)$ is finite. A simple algebraic manipulation shows that $|R\varsigma^i(s,T,T+\delta) - \Sigma^i(s,T+\delta)| < \mathcal{K}$, which ensures that the cumulant generating function appearing in (29) has finite value for $z = R + \mathrm{i}v$. □

Remark 3.3 Alternatively, under the assumptions of Section 2.3, we can express the time-0 value of the caplet in terms of the factor process (r, q) as

$$P_0^{cpl} = B_0(T+\delta)\mathbb{E}^{T+\delta}\left[\left(e^{m(T,T,T+\delta)+n(T,T,T+\delta)r_T+n^*(T,T,T+\delta)q_T} - \overline{K}\right)^+\right],$$

where we have used (19).

3.3 Swaptions

In this section we consider a swaption, which is an option to enter an interest rate swap with swap rate K and maturity T_n at a pre-specified date $T = T_0 \geq 0$. The underlying swap is defined in Section 3 and we consider the notional amount $N = 1$. The value at time $t \leq T$ of the swaption is given by

$$P_t^{swn} = B_t(T)\sum_{j=1}^{n}\delta_{j-1}\mathbb{E}_t^T\left[B_T(T_j)\left(K_T^{sw} - K\right)^+\right],$$

since the swaption can be seen as a sequence of fixed payments $\delta_{j-1}\left(K_T^{sw} - K\right)^+$, $j = 1, \ldots, n$, received at dates $T_1, \ldots, T_n$, where K_T^{sw} denotes the swap rate of the underlying swap at time T. We have at time $t = 0$

$$\begin{aligned} P_0^{swn} &= B_0(T)\mathbb{E}^T\left[\sum_{j=1}^{n}\delta_{j-1}B_T(T_j)\left(K_T^{sw} - K\right)^+\right] \\ &= B_0(T)\mathbb{E}^T\left[\left(\sum_{j=1}^{n}B_T(T_j)F_T(T_{j-1},T_j) - \sum_{j=1}^{n}\delta_{j-1}B_T(T_j)K\right)^+\right], \end{aligned} \tag{31}$$

which follows by (21).

We will work under the assumptions of Section 2.3. The driving process Y is a two-dimensional Lévy process and we assume the Vasicek volatility structures:

$$\begin{aligned} \Sigma(t,T) &= \left(\frac{\sigma}{a}\left(1 - e^{-a(T-t)}\right), 0\right) \\ \varsigma(t,T,S) &= \left(\frac{\sigma}{a}e^{at}\left(e^{-aT} - e^{-aS}\right), \frac{\sigma^*(T,S)}{a^*}e^{a^*t}\left(e^{-a^*T} - e^{-a^*S}\right)\right). \end{aligned}$$

Recall from equations (6) and (11) that for each j, $B_T(T_j)$ is given by

$$B_T(T_j) = \frac{B_0(T_j)}{B_0(T)}\exp\left(\int_0^T (A(s,T) - A(s,T_j))ds + \int_0^T (\Sigma(s,T) - \Sigma(s,T_j))dY_s\right)$$

and $F_T(T_{j-1},T_j)$ is given by

$$F_T(T_{j-1},T_j) \;=\; F_0(T_{j-1},T_j)\exp\left(\int_0^T \alpha(T,T_{j-1},T_j)ds+\int_0^T \varsigma(T,T_{j-1},T_j)dY_s\right).$$

Introducing an $\mathcal{F}_T$-measurable random vector

$$Z=(Z^1,Z^2)=\left(\int_0^T e^{as}dY_s^1,\int_0^T e^{a^*s}dY_s^2\right),$$

we obtain for each j

$$\begin{aligned}B_T(T_j)&=c^{j,0}e^{c^{j,1}Z^1}\\ F_T(T_{j-1},T_j)&=\bar{c}^{j,0}e^{\bar{c}^jZ},\end{aligned}$$

where

$$\begin{aligned}c^{j,0}&=\frac{B_0(T_j)}{B_0(T)}\exp\left(\int_0^T(A(s,T)-A(s,T_j))ds\right),\\ c^{j,1}&=\frac{\sigma}{a}\left(e^{-aT_j}-e^{-aT}\right),\\ \bar{c}^{j,0}&=F_0(T_{j-1},T_j)\exp\left(\int_0^T\alpha(T,T_{j-1},T_j)ds\right),\\ \bar{c}^j&=(\bar{c}^{j,1},\bar{c}^{j,2})=\left(\frac{\sigma}{a}\left(e^{-aT_{j-1}}-e^{-aT_j}\right),\frac{\sigma^*(T_{j-1},T_j)}{a^*}\left(e^{-a^*T_{j-1}}-e^{-a^*T_j}\right)\right)\end{aligned}$$

are deterministic constants. Therefore, (31) becomes

$$\begin{aligned}P_0^{swn} &= B_0(T)\mathbb{E}^T\left[\left(\sum_{j=1}^n c^{j,0}e^{c^{j,1}Z^1}\bar{c}^{j,0}e^{\bar{c}^jZ}-\sum_{j=1}^n c^{j,0}\delta_{j-1}Ke^{c^{j,1}Z^1}\right)^+\right]\\ &= B_0(T)\mathbb{E}^T\left[\left(\sum_{j=1}^n a^{j,0}e^{a^{j,1}Z^1+a^{j,2}Z^2}-\sum_{j=1}^n b^{j,0}e^{b^{j,1}Z^1}\right)^+\right],\end{aligned}\tag{32}$$

with $a^{j,0}=c^{j,0}\bar{c}^{j,0}$, $a^{j,1}=c^{j,1}+\bar{c}^{j,1}$, $a^{j,2}=\bar{c}^{j,2}$, $b^{j,0}=c^{j,0}\delta_{j-1}K$ and $b^{j,1}=c^{j,1}$. One can compute the above expectation applying the Fourier transform method, cf. Theorem 3.2 in Eberlein, Glau, and Papapantoleon (2010). Note that the value of the expectation depends only on the distribution of the random vector Z under the measure $\mathbb{P}^T$. The moment generating function M_Z^T of Z under $\mathbb{P}^T$ is given by

$$\begin{aligned}M_Z^T(z) &= \mathbb{E}^T\left[e^{z_1Z^1+z_2Z^2}\right]\\ &= \mathbb{E}^T\left[e^{\int_0^T z_1e^{as}dY_s^1+\int_0^T z_2e^{a^*s}dY_s^2}\right]\\ &= \exp\left(\int_0^T\psi^T(z_1e^{as},z_2e^{a^*s})ds\right)\\ &= \exp\left(-\int_0^T\psi\left(\left(\frac{\sigma}{a}(e^{-a(T-s)}-1),0\right)\right)ds\right)\\ &\qquad\times\exp\left(\int_0^T\psi\left(z_1e^{as}-\frac{\sigma}{a}(1-e^{-a(T-s)}),z_2e^{a^*s}\right)ds\right),\end{aligned}\tag{33}$$

for any $z \in \mathbb{R}^2$ such that the expectation above is finite. The first equality follows again from Lemma 3.1 in Eberlein and Raible (1999) and the second one by (13).

Letting

$$f(x) = f(x_1, x_2) := \left(\sum_{j=1}^{n} a^{j,0} e^{a^{j,1}x_1 + a^{j,2}x_2} - \sum_{j=1}^{n} b^{j,0} e^{b^{j,1}x_1} \right)^+,$$

its (generalized) Fourier transform is given by

$$\widehat{f}(z) = \int_{\mathbb{R}^2} e^{izx} f(x) dx, \tag{34}$$

for a row vector $z \in \mathbb{C}^2$ such that the above integral is finite; see Remark 3.5 for more details on the computation of $\widehat{f}$. We obtain the following result:

Proposition 3.4 *The time-0 price of the swaption is given by the following formula:*

$$P_0^{swn} = \frac{B_0(T)}{(2\pi)^2} \int_{\mathbb{R}^2} M_Z^T(R + \mathrm{i}u) \widehat{f}(\mathrm{i}R - u) du, \tag{35}$$

where the row vector $R \in \mathbb{R}^2$ is such that $M_Z^T(R + \mathrm{i}u)$ exists and the function $g(x) := e^{-Rx} f(x)$ satisfies the prerequisites of Theorem 3.2 in Eberlein, Glau, and Papapantoleon (2010).

Proof. Follows directly from Eberlein, Glau, and Papapantoleon (2010, Theorem 3.2) applied to (32). □

Remark 3.5 On the theoretical side, Proposition 3.4 is the swaption counterpart of Proposition 3.2 for a caplet. However, as opposed to the caplet formula (30), which can be readily implemented using one-dimensional FFT, the corresponding swaption formula (35) is not so practical. First, the expression in (35) involves a numerical two-dimensional integration of an integrand which is the product of two functions, M_Z^T and the Fourier transform $\widehat{f}$. Second, computing M_Z^T involves a one-dimensional integral of a smooth function requiring very few function evaluations, while computing $\widehat{f}$ in (34) is more complicated since it involves integrating an oscillating function in dimension two. Since $\widehat{f}$ itself is also highly oscillatory we end up with a significant computational burden in solving the two-dimensional integral in (35). However, swaptions can instead be efficiently valued numerically based on (32), rewritten in the notation introduced above as:

$$\begin{aligned} P_0^{swn} &= B_0(T) \left(\sum_{j=1}^{n} a^{j,0} \mathbb{E}^T \left[e^{a^{j,1}Z^1 + a^{j,2}Z^2} \mathbb{1}_{\{\widetilde{f}(Z) \geq 0\}} \right] \right. \\ &\quad \left. - \sum_{j=1}^{n} b^{j,0} \mathbb{E}^T \left[e^{b^{j,1}Z^1} \mathbb{1}_{\{\widetilde{f}(Z) \geq 0\}} \right] \right). \end{aligned} \tag{36}$$

Here the function $\widetilde{f}$ is given by

$$\widetilde{f}(x_1, x_2) = \sum_{j=1}^{n} a^{j,0} e^{a^{j,1}x_1 + a^{j,2}x_2} - \sum_{j=1}^{n} b^{j,0} e^{b^{j,1}x_1}.$$

Using (33) the expectations in (36) can be calculated using the standard transform methodology of Duffie, Pan, and Singleton (2000) after having replaced $\widetilde{f}$ with a linear approximation in the domain of integration $\{\widetilde{f} \geq 0\}$. The construction of the linear approximation is described in Singleton and Umantsev (2002) and it is known as the linear boundary approximation method. The approximation has a high degree of accuracy in our setup, as we will see in Section 4.3.

Remark 3.6 Similarly to the caplet in Remark 3.3, the time-0 value of the swaption can be expressed in terms of the factor process (r, q) as follows:

$$\begin{aligned} P_0^{swn} &= B_0(T)\mathbb{E}^T\left[\left(\sum_{j=1}^{n} e^{m(T,T_j)+n(T,T_j)r_T} e^{m(T,T_{j-1},T_j)+n(T,T_{j-1},T_j)r_T+n^*(T,T_{j-1},T_j)q_T}\right.\right. \\ &\quad \left.\left. - \sum_{j=1}^{n} \delta_{j-1} K e^{m(T,T_j)+n(T,T_j)r_T}\right)^+\right] \\ &= B_0(T)\mathbb{E}^T\left[\left(\sum_{j=1}^{n} \alpha^{j,0} e^{\alpha^{j,1}r_T+\alpha^{j,2}q_T} - \sum_{j=1}^{n} \beta^{j,0} e^{\beta^{j,1}r_T}\right)^+\right], \end{aligned}$$

where we have used (17) and (19). Here $\alpha^{j,0} = e^{m(T,T_j)+m(T,T_{j-1},T_j)}$, $\alpha^{j,1} = n(T,T_j) + n(T,T_{j-1},T_j)$, $\alpha^{j,2} = n^*(T,T_{j-1},T_j)$, $\beta^{j,0} = e^{m(T,T_j)}\delta_{j-1}K$ and $\beta^{j,1} = n(T,T_j)$. One can proceed as in Proposition 3.4 and obtain the Fourier pricing formula alternative to (35). Again for the same reason as explained in Remark 3.5, the linear boundary approximation method is preferred also in this case.

3.4 Markovian perspective

Recall that the previously described Lévy Hull-White model produces the following two-dimensional factor process $X_t = (r_t, q_t)$ driven by two $(\mathbb{F}, \mathbb{P})$-Lévy processes Y^1 and Y^2:

$$\begin{aligned} dr_t &= a(\rho(t) - r_t)dt + \sigma dY_t^1\,, \;\; r_0 = \text{const.} \\ dq_t &= -a^* q_t dt + dY_t^2\,, \;\; q_0 = 0. \end{aligned}$$

Under the assumptions of Section 2.3, the price $B_t(T)$ of the OIS bond (resp. the FRA rate $F_t(T,S)$) can be written as an exponential-affine function of r_t (resp. of r_t and q_t); see equations (17) and (19). To obtain the clean prices P_t for interest rate derivatives with linear payoffs, one simply has to insert the above representations for $B_t(T)$ and $F_t(T,S)$ into corresponding equations for each product. Consequently, all these prices can be represented by explicit formulas of the form

$$P_t = P(t, X_t), \qquad t \in [0, \overline{T}], \tag{37}$$

where P is a deterministic function and X_t is a relevant Markovian factor process. For instance, in the case of the basis swap on which TVA computations will be performed in Section 5, the equation (26) yields $X_t = (r_t, q_t, r_t^1, q_t^1, r_t^2, q_t^2)$, where

$$r_t^1 = r_{T_{i_t-1}}\,, \;\; q_t^1 = q_{T_{i_t-1}}\,, \;\; r_t^2 = r_{T_{j_t-1}}\,, \;\; q_t^2 = q_{T_{j_t-1}}, \tag{38}$$

which is a six-dimensional Markovian factor process. The path-dependence which is reflected by the last four factors in X_t is due to the fact that both legs of the basis swap deliver payments in arrears.

Remark 3.7 Since OIS rates are stochastic in reality, it is natural to have them stochastic in the model. However, until today, there are no liquid OIS option data. Therefore, deterministic OIS rates could also be considered without major inconsistency in the model since the OIS volatility parameters cannot be identified from the market anyway; see Subsection 4.2 regarding the way we fix them in the numerical implementation. If further simplicity is desired, deterministic OIS rates can be obtained by letting σ be zero in (15)-(16), which results in a one-factor Markov multiple-curve model q_t driven by Y^2 (note that Y^1 plays no role when σ is zero). The computations of the next sections were done also for this one-factor specification and we found very little difference in the results, except of course for the short rate process in the upper graph of Figure 6, which in this case collapses to the process mean function (black curve in this graph). In particular, this means that there is little model risk of this kind in the TVA computations of Section 5.

4 Model calibration

Recalling that a TVA can be viewed as a long-term option on the underlying contract(s), both smile and term structure volatility effects matter for TVA computations. Furthermore, calibration to instruments with different underlying tenors is necessary in view of the multiple-curve feature. Among these, the 3m- and 6m-tenors underlie the most liquid instruments. Therefore, in view of the targeted application to TVA computations, it is important to achieve a joint calibration to caps and swaptions of 3m- and 6m-tenors, the calibration to caps guaranteeing that the model correctly captures volatility smile effects (in strike) and the calibration to at-the-money swaptions ensuring a correct term structure of volatility in the model. In this section, we calibrate a two-dimensional Lévy model with independent NIG components to EUR market Bloomberg data of January 4, 2011: Eonia, 3m-Euribor and 6m-Euribor initial term structures on the one hand, 3m- and 6m-tenor caps and at-the-money swaptions on the other hand. In a first step, we calibrate the non-maturity/tenor dependent parameters, as well as the values of $\sigma^*(2.75, 3)$ and $\sigma^*(2.5, 3)$, to the corresponding 3m- and 6m-tenor caplets with settlement date 3 years. In a second step, we use at-the-money swaptions on 3m- and 6m-tenor swaps either ending or starting at the 3 year maturity in order to identify remaining values $\sigma^*(T, S)$ not identified by the caplet calibration.

Note that market quotes typically reflect prices of fully collateralized transactions, which can be considered as clean prices (see Sect. 3.3 in Crépey et al. (2013)). The clean price of the previous sections is thus the relevant notion of valuation at the stage of model calibration.

4.1 Initial term structures

The initial term structures of the Eonia, 3m-Euribor and 6m-Euribor markets (and therefore also the initial values of 3m-6m basis swaps in view of (25)) are fitted automatically in our HJM setup. More precisely, Bloomberg provides zero coupon discount bond price $B_0^i(T)$ and yield $R_0^i(T) = -\ln(B_0^i(T))/T$ for any tenor $i = \mathrm{d}$, 3m and 6m ("d" stands for "one day", in reference to the OIS market) and for any maturity T desired, constructed in a manner described in Akkara (2012).

For the Eonia curve, we explicitly need the initial instantaneous forward rate curve $f_0(T) = \frac{\partial}{\partial T}(TR^{\mathrm{d}}(T))$ and its derivative $\frac{\partial}{\partial T} f_0(T)$ for insertion into $\rho(\cdot)$ in (18). These can be recovered by fitting the data, using the least squares method, to the following Nelson-

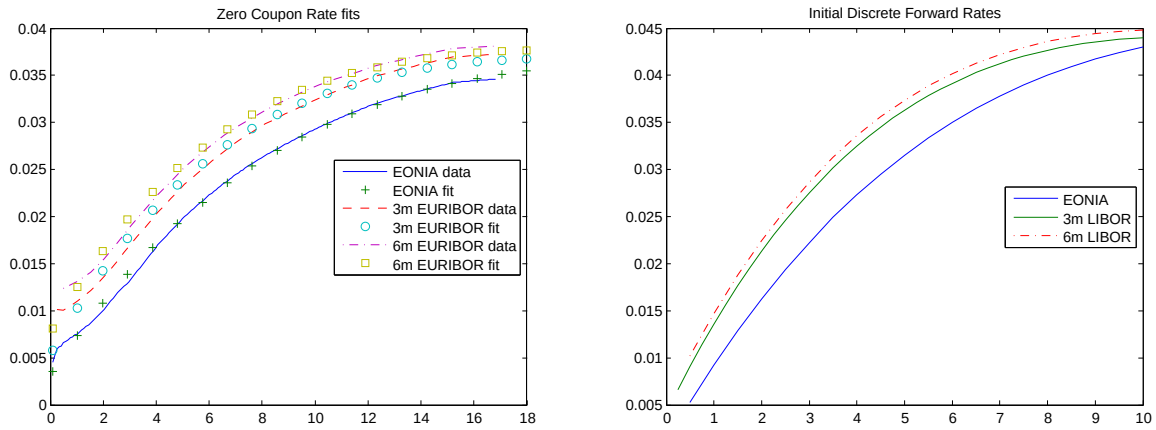


Figure 1: Initial term structures. *Left*: Zero coupon rates. *Right*: Discrete forward rates.

Siegel-Svensson parametrization (for $i = \mathrm{d}$):

$$R_0^i(T) = \beta_0^i + \beta_1^i \left(\frac{1 - e^{-T\lambda_1^i}}{T\lambda_1^i} \right) + \beta_2^i \left(\frac{1 - e^{-T\lambda_1^i}}{T\lambda_1^i} - e^{-T\lambda_1^i} \right) + \beta_3^i \left(\frac{1 - e^{-T\lambda_2^i}}{T\lambda_2^i} - e^{-T\lambda_2^i} \right). \quad (39)$$

For $i = 3\mathrm{m}$ and $6\mathrm{m}$, in principle we only need a term structure of the discrete FRA rates $\delta_i^{-1} F_0^i(T)$, where by definition of Bloomberg's "3m- and 6m-discount bonds":

$$F_0^i(T_k^i) = \frac{B_0^i(T_{k-1}^i)}{B_0^i(T_k^i)} - 1. \quad (40)$$

A common problem with a direct computation based on (40) is that a smooth $R^i(T)$ curve does not necessarily result in a smooth FRA curve (see for example Hagan and West (2006)). Thus, a procedure yielding regular FRA curves is preferred since it typically gives rise to a more stable calibration. To bypass this problem, we also fit the 3m- and 6m-tenor curves to a Nelson-Siegel-Svensson parametrizations of the form (39) before applying (40). The results are plotted in Figure 1, which corresponds to the following parameters:

	β_0	β_1	β_2	β_3	λ_1^i	λ_2^i
Eonia	1.042×10^{-5}	0.00319198	0.095675	0.02100	0.07264	0.08429
3m-Euribor	$7.8542 \times 10^{-}4$	0.004575	0.2384	0.096819	0.0012073	0.10283
6m-Euribor	0.007656	6.4046×10^{-5}	0.04434	0.092094	0.006675	0.10531

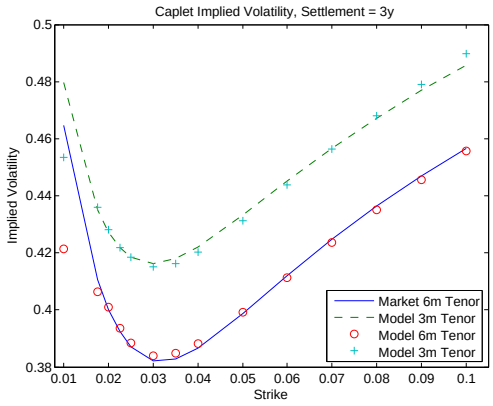
4.2 Calibration to caps

Given the smoothed initial interest rate term structures, we then fix the following parameters in (14)–(16):

$$\sigma = 0.005, \quad a = 0.5, \quad \nu_1 = 1.0, \quad \theta_1 = 0, \quad \sigma_1 = 1, \quad a^* = 0.05, \quad \sigma_2 = 1.0. \quad (41)$$

Indeed, since there are no liquid option data on the OIS market, the first five parameters cannot be determined from the market (see Remark 3.7), while a^* and σ_2 are not identifiable

Figure 2: Market versus calibrated caplet implied volatilities.



in a model with varying $\sigma^*(T,S)$. Note that the above parameters yield an implied volatility of 23% for the at-the-money-caplet written on the OIS discrete forward rate $(\frac{1}{B_{2.5}(3)}-1)/0.5$. This implied volatility corresponds roughly to the implied volatility of the 3-year Euribor caplet, pre-2007, when the OIS discrete forward rate levels were within a few basis points of the Euribor rate.

It remains to calibrate the residual parameters of the model to Euribor option data. Bloomberg provides quotes for caps, but it is more convenient to calibrate to caplets, which contain information concerning a single maturity. We therefore convert cap data to caplet data using the bootstrapping procedure described in Levin (2012). The following parameters are then obtained by least squares calibration to caplet market implied volatilities, using Proposition 3.2 with the cumulant generating functions (14) for the embedded caplet pricing task in our model:

$$\nu_2 = 0.6502, \quad \theta_2 = -0.2643, \quad \sigma^*(2.75,3) = 1.9813, \quad \sigma^*(2.5,3) = 0.9158.$$

As can be seen in Figure 2, except for the most extreme strikes, the two-dimensional Lévy model shows a very good fit to the 3m- and 6m-tenor caplet implied volatilities.

4.3 Calibration to swaptions

We proceed using the parameters found in the previous subsection and calibrate the remaining volatility parameters $\sigma^*(T,S)$ from a subset of 3m- and 6m-tenor at-the-money swaptions, using the linear boundary approximation method described in Remark 3.5 for pricing the swaptions.

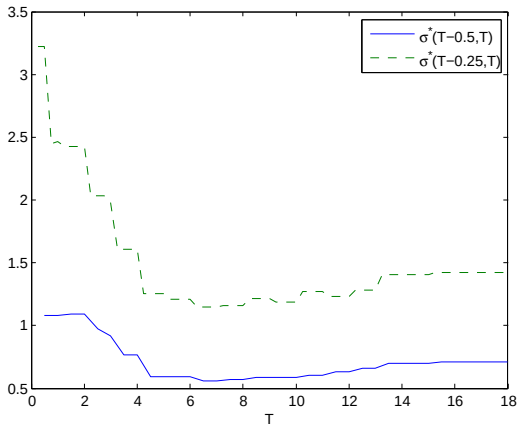
We first consider 3y$\times\Delta$ at-the-money swaptions with Δ = 1y, 2y,..., 10y, 12y, 15y (these are available written on both 3m- and 6m-Euribor). We find the corresponding $\sigma^*(T,S)$-parameters sequentially, starting with the 3y$\times$1y swaption written on 6m-rates which contains two floating payments $L_3(3,3.5)$ and $L_{3.5}(3.5,4)$. We assume $\sigma^*(3,3.5) = \sigma^*(3.5,4)$ and this common value is then calibrated to the at-the-money volatility of the 3y$\times$1y swaption. Next, to calibrate to the 3y$\times$2y swaption, we use the value of $\sigma^*(3,3.5)$ already determined and calibrate $\sigma^*(4,4.5)$ and $\sigma^*(4.5,5)$ by again assuming them equal. We continue in this manner up to the 3y$\times$15y swaption. This gives us the values of $\sigma^*(T,T+0.5)$ for T =3y, 3.5y,..., 17.5y. For swaptions written on 3m-rates, we have four payments

per year, hence four free parameters to determine for each swaption. We reduce this to one parameter by assuming $\sigma^*(T, T+0.25) = \sigma^*(T+0.25, T+0.5) = \sigma^*(T+0.5, T+0.75) = \sigma^*(T+0.75, T+1)$ for each T =3y, 4y,..., 17y. The parameters are then determined sequentially, analogously to the 6m-case.

Finally, we determine the parameters $\sigma^*(T, T+\delta)$, where $T <$ 3y and $\delta =$ 3m or 6m. For the 3m-market, we determine these from $T\times$1y swaptions, where $T =$ 3m, 6m, 9m, 1y, 2y; the 6m-parameters are determined from $T\times$1y swaptions where T =6m, 1y, 2y. In both cases we use a sequential approach analogous to the one already used and described above. The calibrated values of $\sigma^*(T, S)$ are plotted in Figure 3 and the market and model implied volatilities of the 3y$\times\Delta$ swaptions can be seen in the left panel of Figure 4. For completeness, we also show the resulting swaption implied smile of 3y$\times$15y swaptions in the right panel (note these are "repriced" implied volatilities, there are no market quotes to compare with them). Figure 5 further shows that the linear boundary approximation method we use for swaption valuation is sufficiently accurate, comparing with reference implied volatilities obtained by Monte Carlo simulation using $n = 10^3$ time steps and $m = 5 \times 10^6$ paths (denoted MC in Figure 5). The graphs of Figure 5 demonstrate the accuracy of the approximation, even for a long swap maturity (up to 15 years), where similar approximations sometimes have poorer performance (see e.g. Schrager and Pelsser (2006)). Other numerical experiments (not shown) verify that the approximation works equally well for other model parameters and swaption strikes, maturities and swap lengths.

Note that the computation time for estimating each expectation in the formula (36) after

Figure 3: Calibrated values of $\sigma^*(T, S)$



applying the linear boundary approximation (see Remark 3.5) is roughly the one needed for valuing a caplet, which could be time-consuming. However, since we basically calibrate one $\sigma^*(T, S)$ parameter per swaption, our swaption calibration is quite fast. Moreover, several optimizations in the swaption pricing scheme could also be considered.

5 Counterparty risk and funding adjustments

In this section we consider the computation of counterparty risk and funding adjustments to the above multi-curve "clean" prices.

Figure 4: Model swaption implied volatility. *Left:* Implied volatility of at-the-money swaptions with maturity of 3 years and varying swap lengths compared to market data. 3m and 6m refer to the tenor of the floating Libor rate of the underlying swap. *Right:* Implied volatility of a 3y×15y swaption with varying strikes.

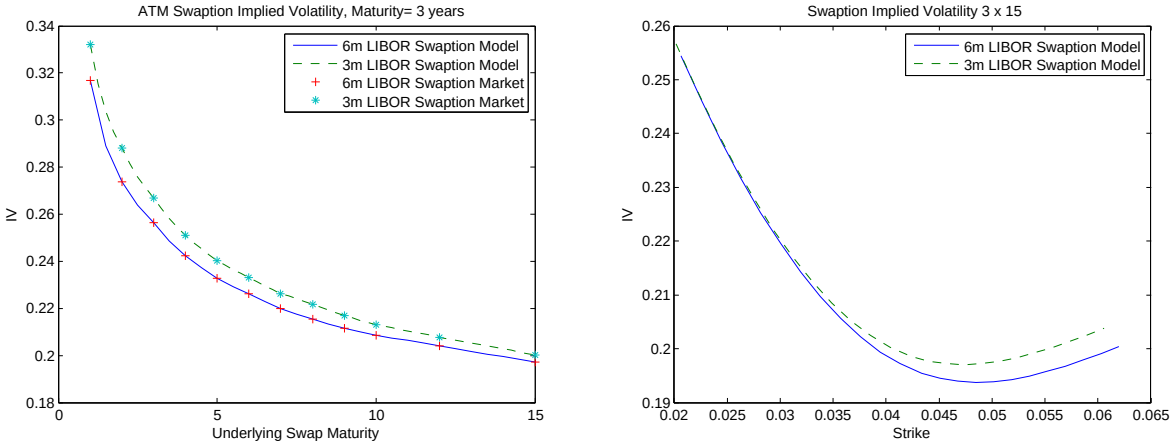
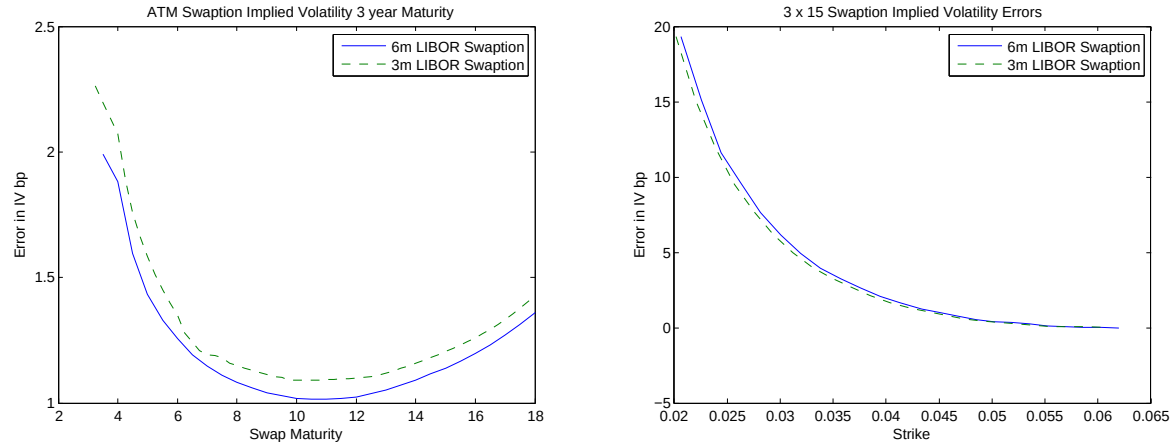


Figure 5: Errors due to the approximation in (36) calculated with calibrated parameters. *Left*: Implied volatility error in basis points of at-the-money swaptions with varying swap lengths. The error is calculated as $10^4\times$ (MC impld vol - linear boundary approx impld vol). 3m and 6m refer to the tenor. *Right*: Implied volatility error for a 3y×15y swaption with varying strikes.



5.1 TVA equations

Different interdependent valuation adjustments can be computed on top of the clean price P in order to account for (possibly bilateral) counterparty risk and funding constraints: credit/debt valuation adjustment CVA/DVA and liquidity funding valuation adjustment LVA, as well as replacement cost RC, with TVA for total valuation adjustment in aggregate, i.e.

$$\text{TVA} = \text{CVA} + \text{DVA} + \text{LVA} + \text{RC}. \tag{42}$$

Each of these dependent terms has received a lot of attention in the recent literature. We refer to sections 1-3 of Crépey et al. (2013) and the references therein for details. Here we only recall that the TVA can be viewed as the price of an option on the "clean" value of the contract P (for a basis swap considered here see (25) and (26)), at the first-to-default time τ of a party. This option also pays dividends, which correspond to the funding benefit (in excess over the OIS rate r_t). More specifically, the TVA equation reads as the following backward stochastic differential equation (BSDE) under the risk-neutral measure $\mathbb{P}$:

$$\Theta_t = \mathbb{E}_t\left(\int_t^{\overline{T}} g_s(\Theta_s)ds\right), \quad t \in [0, \overline{T}], \tag{43}$$

where Θ_t, P_t and g_t represent the TVA process, the clean price process and the counterparty risk and funding coefficient, respectively. The overall (selling) price of the contract for the bank (cost of the corresponding hedge, including the counterparty risk and funding features) is

$$\Pi = P - \Theta. \tag{44}$$

The coefficient g_t in (43) is given, for every real ϑ (representing the TVA Θ_t that one is looking for in the probabilistic interpretation), by:

$$\begin{aligned} g_t(\vartheta) + r_t\vartheta = & \underbrace{-\gamma_t^c(1-\rho_t^c)(Q_t-\Gamma_t)^-}_{CVA\ coeff.} + \underbrace{\gamma_t^b(1-\rho_t^b)(Q_t-\Gamma_t)^+}_{DVA\ coeff.} \\ & + \underbrace{b_t\Gamma_t^+ - \bar{b}_t\Gamma_t^- + \lambda_t(P_t-\vartheta-\Gamma_t)^+ - \widetilde{\lambda}_t(P_t-\vartheta-\Gamma_t)^-}_{LVA\ coeff.} \\ & + \underbrace{\gamma_t\,(P_t-\vartheta-Q_t)}_{RC\ coeff.}, \end{aligned} \tag{45}$$

where:

- γ_t^b, γ_t^c and γ_t are the default intensities of the bank, of its counterparty and their first-to-default intensity (in models where the bank and the counterparty can default together, γ_t can be less than $\gamma_t^b + \gamma_t^c$),
- ρ_t^b and ρ_t^c are the recovery rates of the bank and the counterparty to each other,
- Q_t is the value of the contract according to the scheme used by the liquidator in case $t = \tau < \overline{T}$, e.g. $Q_t = P_t$ (used henceforth unless otherwise stated) or $Q_t = P_t - \Theta_t$,

- $\Gamma_t = \Gamma_t^+ - \Gamma_t^-$, where Γ_t^+ (resp. Γ_t^-) is the value of the collateral posted by the bank to the counterparty (resp. by the counterparty to the bank), e.g. $\Gamma_t = 0$ (used henceforth unless otherwise stated) or $\Gamma_t = Q_t$,

- b_t and $\bar{b}_t$ are the spreads over the short rate r_t for the remuneration of the collateral Γ_t^+ and Γ_t^- posted by the bank and the counterparty to each other,

- λ_t (resp. $\widetilde{\lambda}_t$) is the liquidity funding spread over the short rate r_t corresponding to the remuneration of the external funding loan (resp. debt) of the bank. By liquidity funding spreads we mean that these are free from credit risk, i.e.

$$\widetilde{\lambda}_t = \bar{\lambda}_t - \gamma_t^b(1 - \mathfrak{r}^b), \tag{46}$$

  where $\bar{\lambda}_t$ is the funding borrowing spread (all inclusive) of the bank and $\mathfrak{r}^b$ stands for a recovery rate of the bank to its unsecured funder. In the case of λ_t there is no credit risk involved anyway since the funder of the bank is assumed risk-free.

The data Q_t, Γ_t, b_t and $\bar{b}_t$ are specified in a so-called Credit Support Annex (CSA) contracted between the two parties, where the motivation of the CSA is to mitigate counterparty risk.

Remark 5.1 The above presentation corresponds to a pre-default reduced-form modeling approach, under the immersion hypothesis between the reference filtration $\mathbb{F}$ and a full model filtration $\mathbb{G}$, which is given as $\mathbb{F}$ progressively enlarged by the default times of the parties. From the financial point of view, the immersion hypothesis is a "moderate dependence" assumption between the counterparty risk and the underlying contract exposure, financially acceptable in the case of counterparty risk on interest rate derivatives. For more details the reader is referred to Remark 2.1 in Crépey (2012). Note that we write $\widetilde{\mathbb{F}}$ and $\mathbb{F}$ there instead of, respectively, $\mathbb{F}$ and $\mathbb{G}$ here. Since, in the present paper, we only work with pre-default values, we likewise denote here by $\mathbb{E}_t$, Θ_t and g_t what is denoted by $\widetilde{\mathbb{E}}_t$, $\widetilde{\Theta}_t$ and $\widetilde{g}_t$ there.

In the numerical implementation which follows we set the above parameters equal to the following constants:

$$\begin{aligned}
&\gamma^b = 5\%,\ \gamma^c = 7\%,\ \gamma = 10\% \\
&\rho^b = \rho^c = 40\% \\
&b = \bar{b} = \lambda = \widetilde{\lambda} = 1.5\%.
\end{aligned}$$

In view of the Markovian perspective of Subsection 3.4, observe that $g_t(\vartheta) = \widehat{g}(t, X_t, \vartheta)$, for a suitable function $\widehat{g}$. Thus, (43) is rewritten as the following TVA Markovian BSDE under $\mathbb{P}$:

$$\Theta(t, X_t) = \mathbb{E}_t\left(\int_t^{\overline{T}} \widehat{g}(s, X_s, \Theta(s, X_s))ds\right), \quad t \in [0, \overline{T}]. \tag{47}$$

Note that due to the specific choice of numerical parameters $\lambda = \widetilde{\lambda}$ above, the coefficients of the $(P_t - \vartheta - \Gamma_t)^{\pm}$-terms coincide in (45), so that this is the case of a "linear TVA" where the coefficient g depends linearly on ϑ. This will allow us to validate the results of the numerical BSDE scheme (48) for (47) by a standard Monte Carlo procedure.

5.2 Numerical results

We illustrate numerically the above methodology on a basis swap, a new product that appeared in the markets due to the multi-curve discrepancy (as recalled after (26) a basis swap would have zero value and TVA at all times in a classical single-curve setup). We consider a basis swap with notional $N = 100$ and maturity $\overline{T} = 10$y, in the calibrated model of Section 4. The pricing formula (25) yields the time-0 value of the basis swap equal to 1€23, which is the difference between a 6m-leg worth 28€89 and a 3m-leg worth 27€65. The corresponding time-0 basis swap spread (cf. (27)) is $K_0^{bsw} = 14$bps.

We study the computation of the TVA for the bank having sold this basis swap to its counterparty, i.e. one short basis swap position for the bank. The first step ("forwardation") is to simulate, forward in time by an Euler scheme, a stochastic grid with n (fixed to 100 throughout) uniform time steps and m (set equal to 10^4 or 10^5 below) scenarios for the processes r_t and q_t and for the corresponding values P_t of the basis swap, based on $P_t = P(t, X_t)$, with $X_t = (r_t, q_t, r_t^1, q_t^1, r_t^2, q_t^2)$ (cf. (38)). The results are plotted in Figure 6. The second step is to compute the TVA process, backward in time, by nonlinear regression on the time-space grid generated in the first step. We thus approximate $\Theta_t(\omega)$ in (47) by $\widehat{\Theta}_i^j$ on the corresponding time-space grid, where the time-index i runs from 1 to n and the space-index j runs from 1 to m. Denoting by $\widehat{\Theta}_i = (\widehat{\Theta}_i^j)_{1\le j\le m}$ the vector of TVA values on the space grid at time i, we have $\widehat{\Theta}_n = 0$ and then, for every $i = n-1, \cdots, 0$ and $j = 1, \cdots, m$

$$\widehat{\Theta}_i^j = \widehat{\mathbb{E}}_i^j \left(\widehat{\Theta}_{i+1} + \widehat{g}_{i+1}\big(\widehat{X}_{i+1}, \widehat{\Theta}_{i+1}\big)h \right), \tag{48}$$

for the time-step $h = \frac{\overline{T}}{n} = 0.1$y. We use a d-nearest neighbor average non-parametric regression estimate (see e.g. Hastie, Tibshirani, and Friedman (2009)), with $d = 1$ which was found the most efficient. This means that a conditional expectation of the form $\mathbb{E}(Y|X = x)$ is estimated by an empirical average of the three realizations of Y associated with the three realizations of X closest to x in the Euclidean norm.

Note that in view of (37) and (38), in the present case of a basis swap, this regression must in principle be performed against the "full" Markovian vector factor process $X_t = (r_t, q_t, r_t^1, q_t^1, r_t^2, q_t^2)$, where the last four components reflect the path-dependence of payments in arrears. However, as already noted in the single-curve setup of Crépey et al. (2013), these extra factors have a limited impact in practice. To illustrate this, Figure 7 shows the TVA processes obtained by regression for $m = 10^4$ (top) and 10^5 (bottom) against r_t and q_t only (left) and against the whole vector X_t (right). Note that one should really target $m = 10^4$ because in the industry practice one typically cannot afford much more on such applications, where not a single basis swap, but the whole OTC derivative book of the bank has to be dealt with. Table 1 displays the time-0 value Θ_0 of the TVA, CVA, DVA, LVA and RC, where the last four are obtained by plugging for ϑ in the respective term of (45) the TVA process Θ_t computed in the first place (see Subsect. 5.2 in Crépey et al. (2013) for the details of this procedure). The sum of the CVA, DVA, LVA and RC, which in theory sum up to the TVA, is shown in column 8. Therefore, columns 3, 8 and 9 yield three different estimates for the time-0 TVA. Table 2 displays the relative differences between these estimates, as well as the Monte Carlo confidence interval in a comparable scale in the last column. The nonlinear regression TVA estimates (time-0 TVA computed by regression or repriced as the sum of its four components) are all outside the corresponding Monte Carlo confidence intervals, but this is not surprising since these estimates entail a space-regression

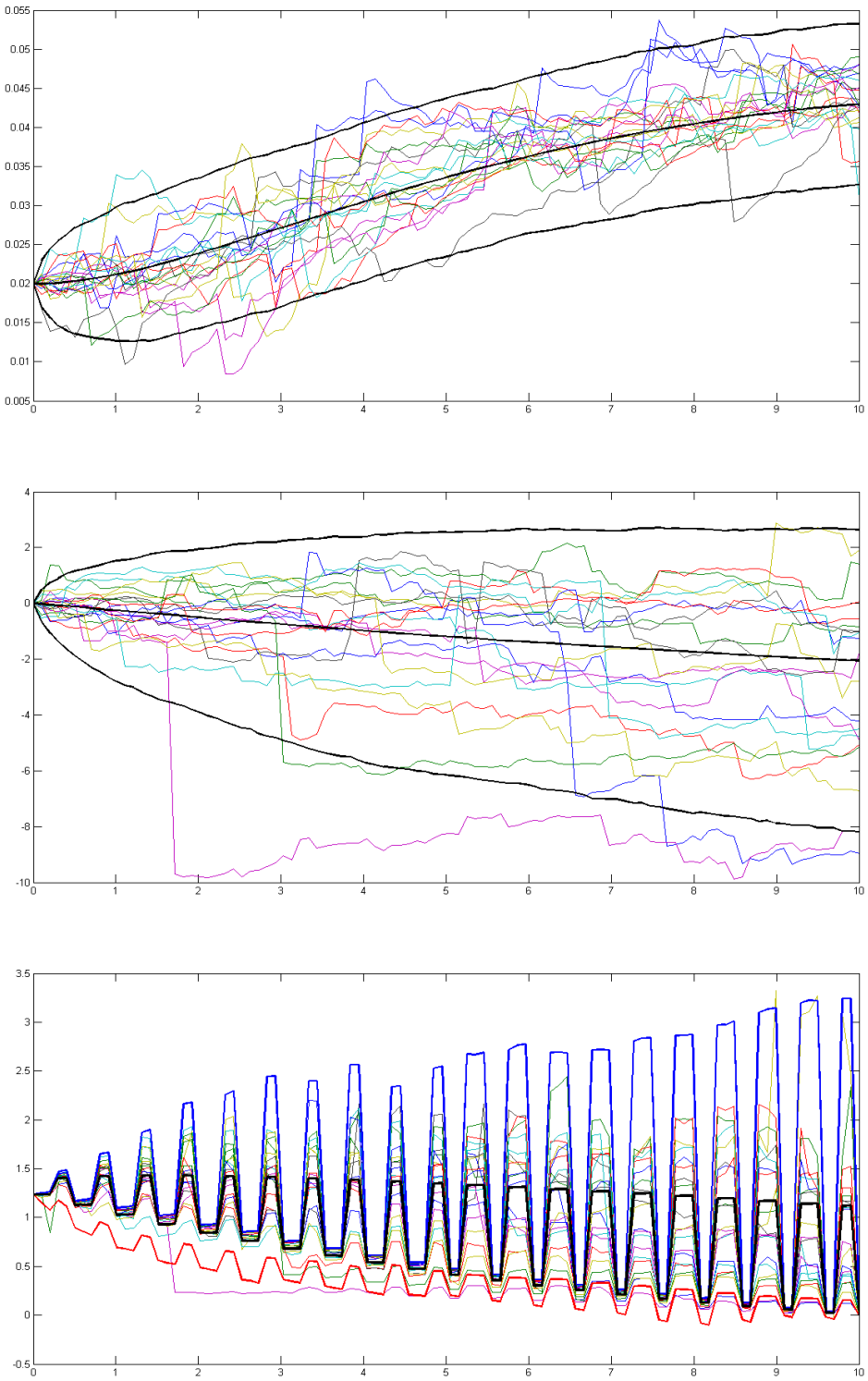


Figure 6: Clean valuation of the basis swap. *Top panels:* Processes r_t and q_t. *Bottom panel:* Clean price process $P_t = P(t, X_t)$ of the basis swap where $X_t = (r_t, q_t, r_t^1, q_t^1, r_t^2, q_t^2)$. Each panel shows twenty paths simulated with $n = 100$ time points, along with the process mean and 2.5 / 97.5-percentiles computed as function of time over $m = 10^4$ simulated paths.

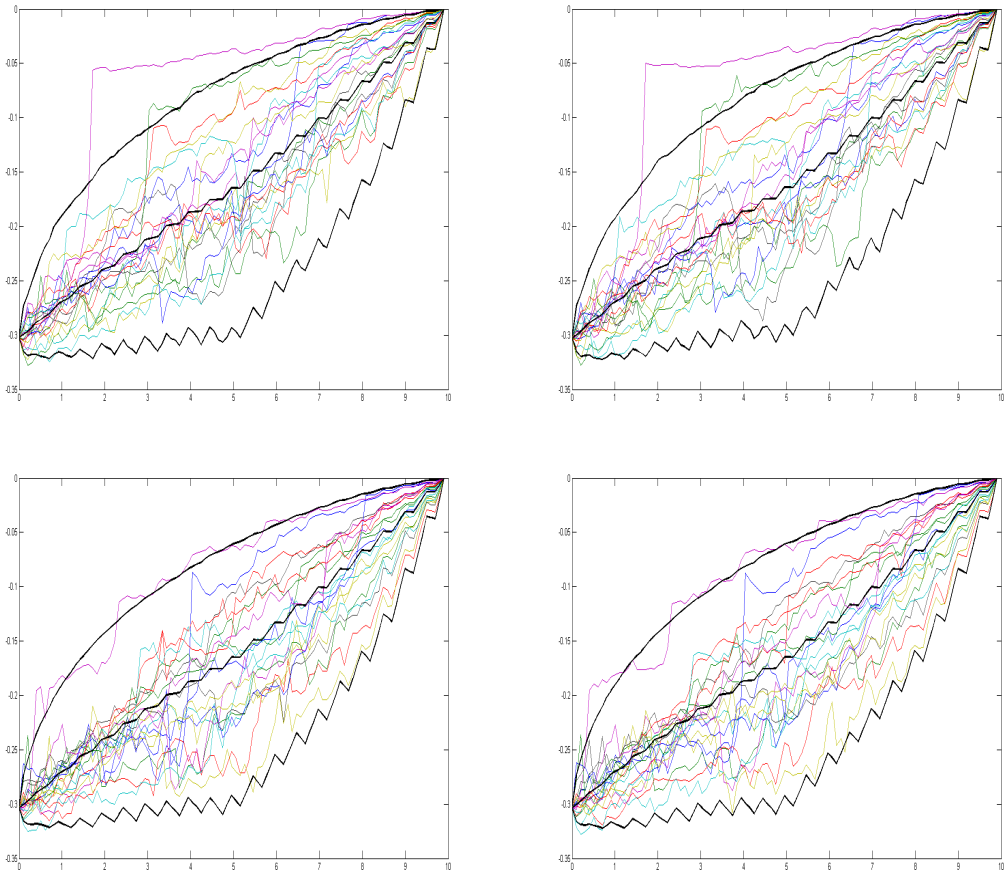


Figure 7: TVA process Θ_t of the basis swap. All graphs show twenty paths of the TVA process at $n = 100$ time points, along with the process mean and 2.5 / 97.5-percentiles as a function of time. *Top graphs:* $m = 10^4$ simulated paths. *Bottom graphs:* $m = 10^5$ simulated paths. *Left graphs*: "Reduced" regression against (r_t, q_t). *Right graphs*: "Full" regression against $X_t = (r_t, q_t, r_t^1, q_t^1, r_t^2, q_t^2)$.

bias when compared with a standard Monte Carlo estimate. Note that of course the latter is no longer available in nonlinear cases.

The expected exposure profiles (see Crépey et al. (2013) for the details about such representations) corresponding to the TVA decompositions of columns 4 to 7 in Table 1 are shown in Figure 8 for the regression against (r, q) with $m = 10^4$ paths. We only show the results in this case since the profiles in all the other three cases of Table 1 are visually indistinguishable from the former.

m	Regr	regr TVA	CVA	DVA	LVA	RC	Sum	MC TVA
10^4	r,q	-0.3026	-0.3442	9.75E-04	-0.1011	0.1423	-0.302	-0,2996
	PD	-0.3027	-0.3442	9.75E-04	-0.1011	0.1421	-0.3022	
10^5	r,q	-0.3035	-0.3447	8.66E-04	-0.1013	0.1426	-0.3026	-0,3001
	PD	-0.3034	-0.3447	8.66E-04	-0.1013	0.1425	-0.3026	

Table 1: Time-0 TVA and its decomposition (all in €) computed by regression for $m = 10^4$ or 10^5 against r_t and q_t only (rows "r,q") or against the whole vector X_t accounting for path-dependence due to the payments in arrears (rows "PD"). *Column 3*: Time-0 regressed TVA. *Columns 4 to 7*: TVA decomposition into time-0 CVA, DVA, LVA and RC repriced individually by plugging Θ_t for ϑ in the respective term of (45). *Column 8*: Sum of the four components. *Column 9*: Monte Carlo TVA.

m	Regr	Sum/TVA	TVA/MC	Sum/MC	CI//|MC|
10^4	r,q	-0.20%	1,00%	0.80%	0.367%
	PD	-0.17%	1.03%	0.87%	
10^5	r,q	-0.30%	1.13%	0.83%	0.117%
	PD	-0.26%	1.10%	0.83%	

Table 2: Time-0 TVA relative errors corresponding to the results of Table 1. "A/B" represents the relative difference $\frac{(A-B)}{B}$. "CI//|MC|" in the last column refers to the half-size of the 95%-Monte Carlo confidence interval divided by the absolute value of the standard Monte Carlo estimate of the time-0 TVA.

Finally, to provide an insight about the TVA under alternative CSA specifications, we repeat the above numerical implementation in each of the following five cases, with $\bar{\lambda}$ set equal to 4.5% everywhere (and the other parameters as before):

$$\begin{array}{lllll} 1. & (\mathfrak{r}^b, \rho^b, \rho^c) = (100, 40, 40)\%, & Q = P, & \Gamma = 0 \\ 2 & (\mathfrak{r}^b, \rho^b, \rho^c) = (100, 40, 40)\%, & Q = P, & \Gamma = Q = P \\ 3. & (\mathfrak{r}^b, \rho^b, \rho^c) = (40, 40, 40)\%, & Q = P, & \Gamma = 0 \\ 4. & (\mathfrak{r}^b, \rho^b, \rho^c) = (100, 100, 40)\%, & Q = P, & \Gamma = 0 \\ 5. & (\mathfrak{r}^b, \rho^b, \rho^c) = (100, 100, 40)\%, & Q = \Pi, & \Gamma = 0. \end{array} \tag{49}$$

Note that in case 3, we have by (46):

$$\widetilde{\lambda} = 4.5\% - 0.6 \times 0.5 \times 10\% = 1.5\% = \lambda,$$

so this is the linear TVA case considered above. Table 3 shows an analog of the first row of Table 1 (time-0 TVA and its decomposition for $m = 10^4$ and regression against (r, q))

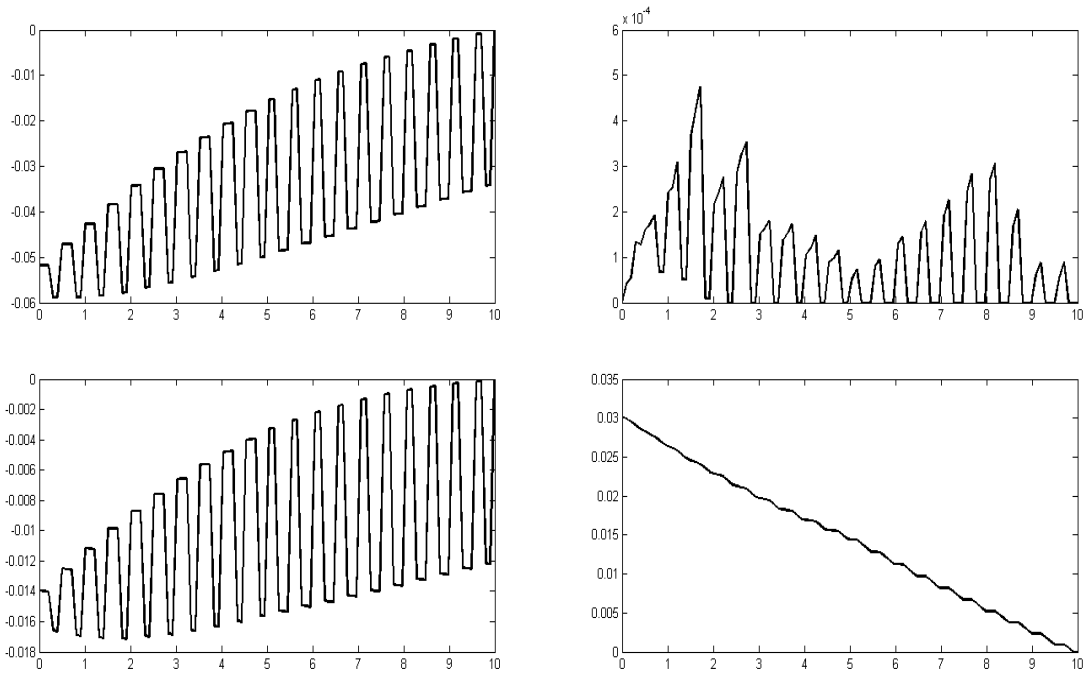


Figure 8: Expected exposures of the TVA components of the basis swap. *Top panels*: CVA (*left*) and DVA (*right*) exposures. *Bottom panels*: LVA (*left*) and RC (*right*) exposures.

in each of the five cases. Moving from case 1 to 2, there are no CVA and DVA anymore and the dominant effect is the cancelation of the previously highly negative, costly CVA, resulting in a higher TVA, hence a lower (selling) price for the bank. Moving from 1 to 3, a funding benefit at own default is acknowledged by the bank, resulting in a higher TVA, hence a lower selling price for the bank. Moving from 1 to 4, the beneficial DVA at own default is ignored by the bank, being considered as fake benefit, which results in a lower TVA, but negligibly so, since the DVA was very small anyway. The related LVA numbers are very close (both equal to -0.2784 at the four-digit accuracy of the table) because the parameter ρ^b, which changes between these two cases, has no direct impact on the LVA and the indirect impact through the change of the TVA in the second row of (45) is limited (as the parameters λ and $\widetilde{\lambda}$ are not so large). Finally, 5 represents a (slightly artificial) case of a bank in a "dominant" position, able to enforce the value of the contract Π from its own perspective (see (44)) for the CSA close-out valuation process Q. Hence, a zero RC for the bank follows in this case.

Conclusion

In this paper the valuation of interest rate derivatives in the post-crisis setup is studied. To this end, a multiple-curve model, which takes into account the spreads that have emerged since the crisis (notably Libor-OIS swap spreads and basis swap spreads), is developed. Counterparty risk and funding adjustments are modeled on the top of a clean multiple-curve price. The main contributions of this work are, on the one hand, a parsimonious Markovian multiple-curve model jointly calibrated to caplets and swaptions and, on the other hand,

Case	regr TVA	CVA	DVA	LVA	RC	Sum	Sum/TVA
1	-0.4198	-0.3442	9.75E-04	-0.2784	0.2026	-0.4191	-0.17%
2	-0.0796	0	0	-0.1168	0.0374	-0.0794	-0.25%
3	-0.3026	-0.3442	9.75E-04	-0.1011	0.1423	-0.302	-0.20%
4	-0.4202	-0.3442	0	-0.2784	0.2028	-0.4198	-0.10%
5	-0.5122	-0.2477	0	-0.264	0	-0.5117	-0.10%

Table 3: Time-0 TVA and its decomposition (all in €) computed by regression for $m = 10^4$ against r_t and q_t. *Column 2*: Time-0 regressed TVA Θ_0. *Columns 3 to 6*: TVA decomposition into time-0 CVA, DVA, LVA and RC repriced individually by plugging Θ_t for ϑ in the respective term of (45). *Column 7*: Sum of the four components. *Column 8*: Relative difference between columns 7 and 2.

BSDE-based numerical computations of counterparty risk and funding adjustments, as well as the interpretation and comments of the numerical results.

Acknowledgements

We warmly thank Giovanni Cesari and Andrea Pallavicini for their comments on preliminary versions of this paper, as well as Rama Cont and other participants of the Conference on Current Topics in Mathematical Finance (WU Vienna, April 2013), where this work was first presented, for useful questions and remarks.

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