Optimal execution cost for liquidation through a limit order market

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Abstract

We study the problem of optimally liquidating a large portfolio position in a limit order book market. We allow for both limit and market orders and the optimal solution is a combination of both types of orders. Market orders deplete the order book, making future trades more expensive, whereas limit orders can be entered at more favorable prices but are not guaranteed to be filled. We model the bid-ask spread with resilience by a jump-diffusion process, and the market order arrival process as a Poisson process. The objective is to minimize the execution cost of the strategy. We formulate the problem as a mixed stochastic continuous control and impulse problem for which the value function is shown to be the unique viscosity solution of the associated system of variational inequalities. We conclude with a calibration of the model on recent market data.

Keywords: Liquidity Risk, Limit Order Books, Impulse Control, Viscosity Solutions, System of Variational Inequalities.

MSC2000 subject classification: 60G40, 91B70, 93E20.

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1 Liquidity Risk in Limit Order Books

The study of market liquidity consists in quantifying the costs incurred by investors trading in markets in which supply or demand is finite, trading counterparties are not continuously available, or trading causes price impacts. Liquidity is a risk when the extent to which these properties are satisfied varies randomly through time. Liquidity and liquidity risk models vary considerably from one study to the next according to the problem at hand or the paradigm considered. For instance, Back [3] and Kyle [15] construct an equilibrium model for dealers markets with insider trading. Constantinides [9], Davis and Norman [11], and Shreve and Soner [22] study the portfolio selection problem with first order liquidity costs, namely proportional transaction costs arising from a bid-ask spread. There has also been a number of studies on large trader models ([4], [16], [20]), and dynamic supply curves ([7]), with a more recent emphasis on liquidation problems with market orders ([1], [17]). Some authors investigate liquidation problems with limit orders, in particular [5], and [12]. On the other hand, Avellaneda and Stoikov [2] and Guilbaud and Pham [14] consider a limit order market in which the optimal trading schedule of a market maker considers both limit and market orders. In this work, we also consider a limit order market in which both limit and market orders are allowed and study the problem of optimally liquidating a large portfolio position in this setting.

In most studies on portfolio optimization and, more specifically, on optimal portfolio liquidation, it is assumed that the agents are liquidity takers in the sense that they trade at the available prices, albeit with a liquidity premium that must be paid for immediacy of trading. However, in any financial market structure there must also necessarily exist market participants who are price setters (i.e. liquidity providers). For instance, in dealers markets, a market-maker (or specialist) quotes bids and offers and serves as the intermediary between public traders, see [8] and [13]. However, in limit order book markets, any public trader can also play the role of liquidity provider by posting prices and quantities at which he is willing to buy or sell while waiting for a counterparty to engage in that trade. On the other hand, a marketable limit order (i.e. a limit order that can be filled automatically against existing limit orders) is typically filled at a less favorable price as it depletes the order book, making additional trades more expensive. Limit orders can be entered at more favorable prices but are not guaranteed to be filled. The liquidity premium that must be

paid for immediacy of trading can therefore be defined as the best limit price minus the average price of a market order. In considering the liquidation problem of a large portfolio position, it is therefore desirable to work with an enlarged set of admissible trading strategies by including the possibility of making both limit orders and market orders. In this work, we model the bid-ask spread with resilience by a jump-diffusion process, and the market order arrival process as a Poisson process. See Section 2 for a description of the model. The objective is to liquidate a fixed number of shares of a risky asset by minimizing the expected liquidity premium paid. We formulate the problem in Section 3 as a mixed stochastic continuous control and impulse problem for which the value function is shown to be the unique viscosity solution of the associated system of variational inequalities. In Section 4, we calibrate the model to market data corresponding to four different firms traded on the NYSE exchange through the *ArcaBook*.

2 The Limit Order Book Market Model

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space supporting a one-dimensional Brownian motion W on a finite horizon $T < \infty$ and M a random Poisson measure on $\mathbb{R}^+ \times \mathbb{R}$ with mean measure $\gamma_t \, dt \, m(dz)$ where $\gamma : [0, T] \to (0, \bar{\gamma}]$ and m is a probability measure on $\mathbb{R}$, with $m(\mathbb{R}) < \infty$. We consider a market with a risky asset that can be traded through a limit order book. We consider a large investor whose goal is to liquidate a number $N > 0$ of shares of this risky asset. The investor sets a date T before which the position must be liquidated and attempts to minimize the price impact of the liquidation strategy. In a limit order book market, there are two types of transactions: limit orders in which quantities and prices to trade are constantly entered (and potentially later cancelled) in a limit order book, and market orders which are executed against the most favorable existing limit orders. The lowest limit order price to sell is the best ask price, whereas the highest limit order price to buy is the best bid. The trades that the investor who wants to liquidate a position can make are therefore sell limit orders, which will be executed when an incoming buy market order enters the system, and sell market orders, which are automatically executed against the existing buy limit orders. By definition, when an investor makes a market order, he sells his shares at a less favorable price than the best ask price. In this sense, he pays a liquidity cost for immediacy defined as the difference between the average price paid for this

transaction and the best ask price. The larger the number of shares he sells at this instant in time, the higher is the liquidity cost since his market order will be executed against the most favorable limit order prices in the book in a decreasing order. Thus, we see that liquidity costs of market orders are a function of transaction size. Moreover, the investor's market orders deplete the limit order book, thereby widening the bid-ask spread, which is defined as the difference between the best ask price and the best bid price. However, even when the investor does not trade or does not modify his limit orders, there are other market participants who will do so. As a result, the observed bid-ask spread evolves stochastically over time.

Mathematically, we define the limit order book as follows. Between the investor's market orders, we assume the dynamic of the bid-ask spread is given by the following positive process :

$$dX_t = \mu(t, X_{t^-})dt + \sigma(t, X_{t^-})dW_t + \int_{\mathbb{R}} X_{t-} g(X_{t-}, z)\tilde{M}(dt, dz), \tag{2.1}$$

where $\tilde{M}$ is the compensated random measure of M, and μ and σ are deterministic and Lipschitz continuous functions, satisfying the following growth condition in the second argument. There exist $\bar{\mu}$ and $\bar{\sigma} > 0$ such that

$$\sup_{t\in[0,T]} |\mu(t,x)| \leq \bar{\mu}(1+|x|) \text{ and } \sup_{t\in[0,T]} |\sigma(t,x)| \leq \bar{\sigma}(1+|x|), \text{ for all } x \geq 0.$$

We equally assume that μ, σ and g are such that X remains positive. Moreover, we shall impose integrability property on g, detailed in (2.3).

Remark 2.1. The tick size as a proportion of the asset price is a good indicator for the kind of model one should choose for X. Indeed, when the tick size is small with respect to the asset price, many different values of the spread can be observed and a diffusion may be a good model approximation for the stochastic behavior of the spread. In Section 4, we see that this is indeed appropriate for a stock like Google. On the other hand, if the asset price is small and the asset is liquid, the number of potentially observed spreads may be too small to be described by a diffusion, in which case the jump part of X should be used. By including both kinds of stochastic behaviors in our model for X, we cover these two scenarios.

Liquidity cost

The liquidity cost due to a market order to sell is defined in terms of the structure of the limit order book. In some cases, it may be more relevant to construct a discrete limit order book structure, as done in Section 4, whereas for more liquid stocks with diffusion-like spread dynamics, it may be more relevant to consider an order book density as done in [1]. We summarize the information contained in the order book by a function $S(t, x, n)$ which gives the proceeds obtained for a sale of n shares at time t done through market orders when the spread equals x. In the order book density case, this corresponds to Equation 12 in [1]. Let A_t be a stochastic process representing the best ask price. We may then define the liquidity cost due to a market sell order of size n, denoted by $L(t, x, n)$, in terms of the best ask price as follows

$$L(t, x, n) := nA_t - S(t, x, n). \tag{2.2}$$

Let $p \in \mathbb{N}^*$. We introduce the set of functions from $[0, T] \times \mathbb{R}_+$ to $\mathbb{R}$ with at most polynomial growth of degree p in the second argument, uniformly in the first, and denote it by $\mathcal{P}$. For technical reasons (see Proposition (3.4)), we assume that for all $n \in \{0, .., N\}$, the function $L(., ., n)$ belongs to $\mathcal{P}$, and

$$\sup_{x \in \mathbb{R}^+} \int_{\mathbb{R}} |g(x, z)|^{p+1} m(dz) < \infty, \tag{2.3}$$

where g is defined in (2.1).

Example. The simplest example is a quadratic model with proportional transaction costs:

$$S(t, x, n) = (A_t - x)n + M_t n^2,$$

with A_t and M_t two stochastic processes representing the best ask price and a measure of illiquidity. This model arises from a limit order book with constant density as shown in [21]. In the quadratic model, $L(t, x, n) = xn + M_t n^2$.

Impact on the best bid

During a transaction, the investor's market orders are matched with the existing limit orders in

the order book so that the result is a shift in the best bid price to the left by an amount denoted by $I(t,x,n)$. In [1], this quantity is called the extra spread and denoted by D_t^B. The bid-ask spread will necessarily increase by the same amount. In the quadratic model for S, the quantity I is given by $I(t,x,n) = 2M_t n$ (c.f. [21]). See (4.13) and (4.14) below for a discrete model for L and I.

Liquidation Strategy

Assume that an investor has to sell a certain number N of shares of the risky asset before a fixed time $T > 0$ (The situation in which a number of shares is purchased can be treated in a similar fashion). The investor will pay fixed transaction costs and associated liquidity costs for each market order. The execution cost, also called slippage, is defined as the difference between the average price at which the orders are executed and the best ask price. The investor seeks to minimize expected slippage. In order for the problem to make sense, the investor needs an incentive to liquidate. This could be reflected in a positive drift in the bid-ask spread process, a short maturity T or a diminishing incoming market order intensity.

The investor's goal is to minimize slippage by balancing his actions between market orders, which are more expensive due to immediacy, and limit orders, for which the execution time is unknown and random but are executed at more favorable prices.

Market orders. The first type of trades the investor can make are market orders. Market orders also incur liquidity costs as well as fixed transaction costs defined in terms of the bid-ask spread. In other words, the slippage of a market order of size n is the fixed transaction cost, $k > 0$, plus the liquidity cost, i.e.

$$K(t,x,n) = k + L(t,x,n). \tag{2.4}$$

Let assume $L(t,0,1) = 0$ so that the first share bought through a market order costs the best ask price. The investor controls the time and the size of market orders. This is modelled by an impulse control strategy $\beta = (\tau_i,\ \xi_i)_{i \leq n}$ where the τ_i's are stopping times representing the intervention times of the investor and the ξ_i's are $\mathcal{F}_{\tau_i}$-measurable random variables valued in $\mathbb{N}$ and giving the number of shares sold by a market order at time τ_i.

Dynamics of the controlled bid-ask spread. As noted in the previous section, the bid-ask

spread increases after a market order. Although the limit price strategy has an impact on the observed bid-ask spread, we define X as the bid-ask spread that ignoring the investor's current limit price, which may be inside the spread. As such, the controlled bid-ask spread is independent of the investor's current limit price and we have the following dynamic for X^β:

$$\begin{cases} dX_t^\beta &= \mu(t, X_{t^-}^\beta)dt + \sigma(t, X_{t^-}^\beta)dW_t + \int_{\mathbb{R}} X_{t-}^\beta g(X_{t-}^\beta, z)\tilde{M}(dt, dz) \quad \text{if } \tau_n < t < \tau_{n+1} \\ X_{\tau_n}^\beta &= \check{X}_{\tau_n^-}^\beta + I(\tau_n, \check{X}_{\tau_n^-}^\beta, \xi_n), \end{cases} \tag{2.5}$$

where $\check{X}_{t^-}^\beta = X_{t^-}^\beta + \Delta X_t^\beta$, ΔX_t^β is the jump of the measure M at time t.

Limit Orders. The second type of trades the investor can make are limit orders. We denote by $\mathcal{A}_0$ a compact subset of $[0, \infty)$ representing the set of possible spreads *below the best ask price* at which the investor can place a limit order to sell in the order book. We also add the admissibility condition that the limit price is above the current best bid price, otherwise the limit order would in fact be a market order. In other words, the investor can choose to place his limit price anywhere inside the bid-ask spread or at the current best ask price. In practice, many limit orders could be placed, however we only consider one limit order at a time in our setting to keep the problem mathematically tractable. This is not a very big restriction as a new limit order can be placed immediately after the previous one is executed. Clearly, the higher the investor sets the price in the limit order the more profitable it is, however the less likely it is that the order will be executed. The limit price is a stochastic control denoted by $\alpha = (\alpha_t)_{t \leq T}$.

Market orders arrival. The investor's limit orders are matched against other market participants' market orders. We model this as follows.

The probability of a limit order being matched by an incoming market order depends on the strategy α and is constructed as follows. We start with a time inhomogeneous Poisson process $\mathcal{N}$, and independent w.r.t W and M, with intensity $\lambda(t, 0) \geq \underline{\lambda} > 0$, $t \geq 0$. The jumps of this Poisson process, denoted θ_i, $i \geq 1$ represent the arrival of market orders when the investor's limit order strategy is set at the best ask price. For all t, we let $\lambda(\cdot, a) : [0, T] \to [0, \infty)$ be an equicontinuous family of functions, bounded below and above by constants $\underline{\lambda}, \overline{\lambda} > 0$. If the investor chooses to

place a limit order at a spread α_t below the best ask price at time t, the likelihood of the execution of this order depends on the limit price and arrives with an intensity $\lambda(t, \alpha_t)$. At the time θ_i, the investor's limit order will go through for a random quantity equal to Y_i, independent of $\mathcal{F}_{\theta_i -}$. The fact that the jump intensity is time-dependent is particularly relevant in markets where there is well-known u-shaped trading volume pattern during the day.

Let $\frac{d\mathbb{P}^\alpha}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = Z_t^\alpha$ with $Z_0^\alpha = 1$ and

$$dZ_t^\alpha = Z_{t-}^\alpha \left(\frac{\lambda(t, \alpha_t)}{\lambda(t, 0)} - 1 \right) (d\mathcal{N}_t - \lambda(t, 0)dt) .$$

Then a control α changes the distribution of $\mathcal{N}$ under $\mathbb{P}$ to the distribution of $\mathcal{N}$ under $\mathbb{P}^\alpha$, by changing the intensity of $\mathcal{N}$ from $\lambda(t, 0)$ to $\lambda(t, \alpha_t)$.

Definition 2.1. (Investor's control strategy) *We define an investor's control strategy as being the full control available to the investor, thus given by a pair of processes* (α, β).

Dynamics of the remaining number of shares to liquidate $N^{\beta,n,t}$. To keep track of the portfolio through time, we define a pure jump process $N^{\beta,n,t}$ representing the remaining number of shares in the portfolio (taking into consideration transactions through both limit orders and market orders) when the portfolio starts with n remaining shares at time t. The process N thus starts at $N_t = n$ at time t, is piecewise constant, and jumps by $-(Y_i \wedge N_{\theta_i -}^{\beta,n,t})$ at time θ_i and by $-(\xi_i \wedge N_{\tau_i -}^{\beta,n,t})$ at time τ_i. This is understood to mean that the process jumps by $-(Y_i + \xi_j) \wedge N_{\theta_i -}^{\beta,n,t}$ if $\theta_i = \tau_j$ for some $i, j \geq 1$.

Admissible control strategies

Now, we define the set of admissible strategies. The limit orders control strategy is assumed to be a stochastic Markov control denoted by $\alpha = (\alpha_s)_{0 \leq s \leq T}$ such that $\alpha_t \leq X_{t-}^\beta$ for all $t \leq T$. We denote the set of Markov control by $\mathcal{A}$. Let $\mathcal{T}_{t,T}$ be the set of stopping times with values in $[t, T]$. The set of admissible strategies started at time $t \in [0, T]$ when the investor has n shares remaining

in the portfolio and that the spread is equal to x is defined as

$$\begin{aligned}\mathcal{AB}(t,n,x) &= \{(\alpha,\beta) : \alpha \in \mathcal{A}, \beta = (\tau_i,\xi_i)_{i\leq n}, \tau_i \in \mathcal{T}_{t,T};\ \xi_i \leq n \text{ is an } \mathbb{N}\text{-valued random variable} \\ &\qquad \mathcal{F}_{\tau_i^-} - \text{measurable s.t. } \tau^{\beta,n,t} \leq T\},\end{aligned}$$

where $\mathcal{T}_{t,T}$ is the set of stopping times with values in $[t,T]$, and $\tau^{\beta,n,t} = \inf\{s \geq t :\ N_s^{\beta,n,t} = 0\}$. The superscripts in most of the above expressions will often be omitted to alleviate the notations.

The Control Problem

The investor seeks to minimize expected slippage. For a strategy $(\alpha,\beta) \in \mathcal{AB}(t,n,x)$ started at time t, slippage is defined as

$$S_T^{\alpha,\beta} = \sum_{i=1}^{n} K(\tau_i, \check{X}^{\beta}_{\tau_i^-}, \xi_i)\mathbb{1}_{\tau_i \leq \tau^\beta} + \sum_{i\geq 1} \alpha_{\theta_i} Y_i \mathbb{1}_{\theta_i \leq \tau^\beta}.$$

For $(t,x,n) \in [0,T]\times[0,+\infty)\times\mathbb{N}$, we define the optimal expected slippage function in the following way:

$$C_n(t,x) = \inf_{(\alpha,\beta)\in\mathcal{AB}(t,x,n)} \mathbb{E}_{t,x,n,\alpha} S_T^{\alpha,\beta}, \tag{2.6}$$

with $\mathbb{E}_{t,x,n,\alpha}$ the expectation under $\mathbb{P}^\alpha$, given that $N_t = n$ and $X_t = x$. We extend this function to $[0,T]\times[0,+\infty)\times\mathbb{Z}$ by letting $C_{-i}(t,x) = 0$ for $i \in \mathbb{N}^*$. We have the following boundary condition:

$$C_n(T,x) = K(T,x,n) \text{ for all } n \in \mathbb{N}^*,$$

which follows readily from the fact that $\tau^{\beta,n,T} = T$, so that the investor must necessarily liquidate the remaining part of his portfolio with a market order at time T.

3 Characterization of the slippage function

In this section, we prove that the function C_n is the viscosity solution of an associated quasi-variational inequality. We first introduce the infinitesimal generator of the process $(t, X_t)_{t\geq 0}$ be-

tween two market orders:

$$\mathcal{L}u(t,x) = \frac{\partial u}{\partial t} + \frac{\sigma^2(t,x)}{2}\frac{\partial^2 u}{\partial x^2} + \mu(t,x)\frac{\partial u}{\partial x} + \gamma_t \int_{\mathbb{R}} (u(t, xg(x,z)) - u(t,x))m(dz),$$

and the limit orders operator:

$$\Delta_n^a u(t,x) = -\lambda(t,a)\left[f(a) + u(t,x) - \sum_{i=1}^{\infty} p_i C_{n-i}(t,x)\right],$$

in which $p_i = \mathbb{P}(Y_1 = i)$ $(i \geq 1)$ and $f(a) = a\sum_{i=1}^{\infty} ip_i$, $a \in \mathcal{A}_0$. Finally, define the impulse function for market orders:

$$\mathcal{M}_n(t,x) = \min_{i\in\{1,\dots,n\}} \left[C_{n-i}(t, x + I(t,x,i)) + K(t,i,x)\right].$$

Notice that, for all $(t,x,n) \in [0,T] \times \mathbb{R}_+ \times \mathbb{N}^*$, we deduce from (2.6) that

$$0 \leq C_n(t,x) \leq K(t,x,n) = \kappa + L(t,x,n).$$

Therefore, recalling that $\mathcal{P}$ is the set of functions from $[0,T] \times \mathbb{R}_+$ to $\mathbb{R}$ with at most polynomial growth of degree p in the second argument, we have $C_n \in \mathcal{P}$ for all $n \in \mathbb{N}$.

Our main result is the following theorem.

Theorem 3.1. *For all $n \geq 1$, C_n is the unique continuous viscosity solution in $\mathcal{P}$ of the following variational inequality:*

$$\begin{cases} \min\left(\mathcal{L}u + \min_{a\in\mathcal{A}_0} \Delta_n^a u;\ \mathcal{M}_n - u\right) = 0 & \text{on } [0,T) \times [0,\infty), \\ u(T,x) = K(T,n,x) & \text{for } x \geq 0. \end{cases} \tag{3.7}$$

The proof of Theorem 3.1 is based on the following dynamic programming principles (DPP):

Proposition 3.1. *Let $n \in \mathbb{N}$. For all $(t,x) \in [0,T) \times \mathbb{R}^+$,*

$$C_n(t,x) = \inf_{\alpha\in\mathcal{A},\nu\in\mathcal{T}_t} \mathbb{E}_{t,x,n,\alpha}[\mathcal{M}_{N_\nu}(\nu, X_\nu) - \int_t^\nu \lambda(s,\alpha_s) f(\alpha_s) ds].$$

(Here, N and X denote respectively the processes N^β and X^β obtained for $\beta \equiv 0$.) In particular, by the dynamic programming principle for optimal stopping problems we find that for all $\theta \in \mathcal{T}_t$, we have

$$C_n(t,x) = \inf_{\alpha\in\mathcal{A},\nu\in\mathcal{T}_t} \mathbb{E}_{t,x,n,\alpha}[C_{N_\theta}(\theta, X_\theta)\mathbb{1}_{\theta<\nu} + \mathcal{M}_{N_\nu}(\nu, X_\nu)\mathbb{1}_{\nu\leq\theta} - \int_t^{\nu\wedge\theta} \lambda(s,\alpha_s)f(\alpha_s)ds],$$

and for all $\epsilon > 0$,

$$C_n(t,x) = \inf_{\alpha\in\mathcal{A}} \mathbb{E}_{t,x,n,\alpha}[C_{N_\varsigma}(\varsigma, X_\varsigma) - \int_t^{\varsigma} \lambda(s,\alpha_s)f(\alpha_s)ds],$$

for all $\varsigma \leq \tau^\epsilon := \inf\{u \geq t : C_{N_u}(u, X_u) > \mathcal{M}_{N_u}(u, X_u) - \epsilon\}$.

Proof: Set $H_t = \sum_{i\geq 1} Y_i \mathbb{1}_{\{\theta_i\leq t\}}$. From the fact that the Y_i's are independent of $\mathcal{F}_t$ for all t and identically distributed, we know that $H_t - \int_0^t \lambda(s,\alpha_s)\mathbf{E}Y_i ds$ is a $\mathbb{P}^\alpha$-martingale. From the fact that $\mathcal{M}_t(\alpha) := \int_0^t \alpha_s(dH_s - \lambda(s,\alpha_s)\mathbf{E}Y_i ds)$ is also a $\mathbb{P}^\alpha$-martingale for all adapted predictable processes α, we can write C_n as

$$C_n(t,x) = \inf_{(\alpha,\beta)\in\mathcal{AB}(t,x,n)} \mathbb{E}_{t,x,n,\alpha}\widetilde{S}_T^{\alpha,\beta}$$

with

$$\widetilde{S}_T^{\alpha,\beta} = \sum_{i=1}^n K(\tau_i, \check{X}^\beta_{\tau_i^-}, \xi_i)\mathbb{1}_{\tau_i\leq\tau^\beta} - \int_t^{\tau^\beta} \lambda(s,\alpha_s)f(\alpha_s)ds,$$

in which $f(a) = a\mathbf{E}(Y_1)$.

Recall that

$$dZ_t^\alpha = Z_{t-}^\alpha \left(\frac{\lambda(t,\alpha_t)}{\lambda(t,0)} - 1\right)(d\mathcal{N}_t - \lambda(t,0)dt).$$

Using this change of measure, we can write

$$\begin{aligned} C_n(t,x) &= \frac{1}{z}\inf_{(\alpha,\beta)\in\mathcal{AB}(t,x,n)} \mathbb{E}_{t,x,n,z} Z_T^\alpha \widetilde{S}_T^{\alpha,\beta} \\ &= \frac{1}{z}\inf_{(\alpha,\beta)\in\mathcal{AB}(t,x,n)} \mathbb{E}_{t,x,n,z} \sum_{i=1}^n Z_{\tau_i}^\alpha K(\tau_i, \check{X}^\beta_{\tau_i^-}, \xi_i)\mathbb{1}_{\tau_i\leq\tau^\beta} - \int_t^{\tau^\beta} Z_s^\alpha \lambda(s,\alpha_s)f(\alpha_s)ds \\ &:= \frac{1}{z}\widetilde{C}_n(t,x,z) \end{aligned}$$

with $\mathbb{E}_{t,x,n,z}$ the expectation under $\mathbb{P}$ given that $X_t = x, N_t = n$ and $Z_t = z$.

By Theorem 8.5 of Oksendal and Sulem [18], $\widetilde{C}_n(t,x,z)$ satisfies the following Dynamic Programming Principle:

$$\widetilde{C}_n(t,x,z) = \inf_{\alpha\in\mathcal{A},\nu\in\mathcal{T}_t} \mathbb{E}_{t,x,n}[\widetilde{\mathcal{M}}_{N_\nu}(\nu, X_\nu, Z^\alpha_\nu) - \int_t^\nu Z^\alpha_s \lambda(s,\alpha_s) f(\alpha_s) ds],$$

in which

$$\widetilde{\mathcal{M}}_x(t,x,z) = \min_{i\in\{1,\dots,n\}} \left[\widetilde{C}_{n-i}(t, x+I(t,x,i), z) + zK(t,i,x)\right].$$

Consequently,

$$\begin{aligned} C_n(t,x) &= \frac{1}{z}\widetilde{C}_n(t,x,z) = \frac{1}{z}\inf_{\alpha\in\mathcal{A},\nu\in\mathcal{T}_t} \mathbb{E}_{t,x,n}[Z^\alpha_\nu \mathcal{M}_{N_\nu}(\nu, X_\nu) - \int_t^\nu Z^\alpha_s \lambda(s,\alpha_s) f(\alpha_s) ds] \\ &= \inf_{\alpha\in\mathcal{A},\nu\in\mathcal{T}_t} \mathbb{E}_{t,x,n,\alpha}[\mathcal{M}_{N_\nu}(\nu, X_\nu) - \int_t^\nu \lambda(s,\alpha_s) f(\alpha_s) ds]. \end{aligned}$$

□

First we notice that $C_0 = 0$ is obviously a continuous solution of (3.7) for $n = 0$. We prove Theorem 3.1 using an induction argument on n and the following propositions.

Proposition 3.2 (Subsolution Property)**.** *Suppose C_k is a continuous function for all $k \in \{0,...,n-1\}$. The upper semi-continuous envelope of C_n, denoted by C^u_n is then a subsolution of (3.7).*

Proof: Let $(t_0, x_0) \in [0,\ T) \times (0, +\infty)$ and $\phi \in \mathcal{C}^{1,2}([0,\ T) \times (0, +\infty))$ such that

$$\phi(t_0,x_0) = C^u_n(t_0,x_0) \text{ and } \phi \geq C^u_n \text{ on } [0,\ T) \times (0,+\infty).$$

We have to prove that

$$\min\left(\mathcal{L}\phi(t_0,x_0) + \min_a \Delta^a_n \phi(t_0,x_0);\ [\mathcal{M}_n - \phi](t_0,x_0)\right) \geq 0.$$

However, we obviously have $\mathcal{M}_n \geq C_n$ and $\mathcal{M}_n$ is continuous so $\mathcal{M}_n(t_0,x_0) \geq C^u_n(t_0,x_0) = \phi(t_0,x_0)$. Hence, we just have to show that $\mathcal{L}\phi(t_0,x_0) + \Delta^a_n\phi(t_0,x_0) \geq 0$ for all $a \in \mathcal{A}_0$. We introduce a sequence $(t_m,x_m)_{m\geq 0}$ such that

$$\lim_{m\to+\infty}(t_m,x_m) = (t_0,x_0) \text{ and } \lim_{m\to+\infty} C_n(t_m,x_m) = C^u_n(t_0,x_0).$$

Let $\varepsilon > 0$. From the continuity of ϕ, $\mathcal{L}\phi$ and C_i for $0 \leq i < n$, we deduce that there exists $\eta > 0$ such that for all (t, x) such that $\mid t - t_0 \mid < \eta$ and $\mid x - x_0 \mid < \eta$, we have

$$\mid \phi(t,x) - \phi(t_0,x_0) \mid + \mid \mathcal{L}\phi(t,x) - \mathcal{L}\phi(t_0,x_0) \mid + \sum_{i=1}^{n-1} \mid C_i(t,x) - C_i(t_0,x_0) \mid \leq \varepsilon.$$

As ϕ is continuous, we have that $\gamma_m := C_n(t_m, x_m) - \phi(t_m, x_m)$ converges to 0 when m goes to infinity. Set $h_m = \sqrt{\gamma_m}$. Take m large enough so that $t_m + h_m < T \wedge (t_0 + \eta)$ and $\mathcal{B}(x_m, \frac{\eta}{2}) \subset \mathcal{B}(x_0, \eta)$.

We consider a strategy with no market orders before ν_m, the infimum between $t_m + h_m$ and the first exit time of the associated process X from $\mathcal{B}(x_m, \frac{\eta}{2}) \subset \mathcal{B}(x_0, \eta)$, i.e.

$$\nu_m = \inf\{t \geq t_m : \mid X_t - x_m \mid \geq \frac{\eta}{2}\} \wedge (t_m + h_m),$$

with $X_{t_m} = x_m$. We denote by $\theta(\alpha)$ the first jump time after t_m of $\mathcal{N}^\alpha$ and set $\hat{\nu}_m = \nu_m \wedge \theta(\alpha)$. From the DPP, we know that we have

$$\begin{aligned}
& \gamma_m + \phi(t_m, x_m) = C_n(t_m, x_m) \\
\leq\ & \inf_{\alpha \in \mathcal{A}} \mathbb{E}_{t_m, x_m, n}[C_{N^\alpha_{\hat{\nu}_m}}(\hat{\nu}_m, X_{\hat{\nu}_m}) - \int_{t_m}^{\hat{\nu}_m} \lambda(s, \alpha_s) f(\alpha_s) ds] \\
\leq\ & \mathbb{E}_{t_m, x_m, n}[C_n(\hat{\nu}_m, X_{\hat{\nu}_m}) \mathbb{1}_{\{\theta(a) > \hat{\nu}_m\}} + C_{N^\alpha_{\theta(a)}}(\theta(a), X_{\theta(a)}) \mathbb{1}_{\{\theta(a) = \hat{\nu}_m\}} - \int_{t_m}^{\hat{\nu}_m} \lambda(s,a) f(a) ds] \\
\leq\ & \mathbb{E}_{t_m, x_m, n}[\phi(\hat{\nu}_m, X_{\hat{\nu}_m}) \mathbb{1}_{\{\theta(a) > \hat{\nu}_m\}} + C_{N^\alpha_{\theta(a)}}(\theta(a), X_{\theta(a)}) \mathbb{1}_{\{\theta(a) = \hat{\nu}_m\}} - \int_{t_m}^{\hat{\nu}_m} \lambda(s,a) f(a) ds] \\
=\ & \mathbb{E}_{t_m, x_m, n}[\phi(\hat{\nu}_m, X_{\hat{\nu}_m}) + \left(C_{N^\alpha_{\theta(a)}}(\theta(a), X_{\theta(a)}) - \phi(\theta(a), X_{\theta(a)}) \right) \mathbb{1}_{\{\theta(a) = \hat{\nu}_m\}} - \int_{t_m}^{\hat{\nu}_m} \lambda(s,a) f(a) ds]
\end{aligned}$$

for all $a \in \mathcal{A}_0$. On the other hand, we can apply Itô's formula to the process $(\phi(t, X_t))_{t \geq 0}$ between t_m and $\hat{\nu}_m$. We get

$$\phi(t_m, x_m) = \mathbb{E}_{t_m, x_m, n}[\phi(\hat{\nu}_m, X_{\hat{\nu}_m}) - \int_{t_m}^{\hat{\nu}_m} \mathcal{L}\phi(t, X_t)\, dt].$$

Combining the last equations and inequalities, we obtain:

$$\begin{aligned}
& \mathbb{E}_{t_m,x_m,n}[\int_{t_m}^{\hat{\nu}_m} \mathcal{L}\phi(t,X_t)\,dt - \int_{t_m}^{\hat{\nu}_m} \lambda(s,a)f(a)ds] \\
\geq\ & \mathbb{E}_{t_m,x_m,n}[\Big(\phi(\theta(a),X_{\theta(a)}) - C_{N^a_{\theta(a)}}(\theta(a),X_{\theta(a)})\Big)\,\mathbb{1}_{\{\theta(a)=\hat{\nu}_m\}}] + \gamma_m \\
\geq\ & \gamma_m + (\phi(t_0,x_0)-\varepsilon)\,\mathbb{P}\left(\theta(a)=\hat{\nu}_m\right) \\
& - \sum_{k=1}^{\infty} \mathbb{E}_{t_m,x_m,n}\left[C_{n-k}(\theta(a),X_{\theta(a)})\mathbb{1}_{\{\theta(a)=\hat{\nu}_m\}}\right] p_k \\
\geq\ & \gamma_m + \left(\phi(t_0,x_0) - n\varepsilon - \sum_{k=1}^{\infty} C_{n-k}(t_0,x_0)p_k\right)\mathbb{P}\left(\theta(a)\leq\nu_m\right)
\end{aligned}$$

for any $a \in \mathcal{A}_0$. Dividing the last inequality by $\mathbb{E}_{t_m,x_m,n}(\hat{\nu}_m - t_m)$, letting m going to infinity and then ε to 0, we obtain the following inequality:

$$\mathcal{L}\phi(t_0,x_0) - \lambda(t_0,a)f(a) \geq \lambda(t_0,a)\left(\phi(t_0,x_0) - \sum_{k=1}^{\infty} C_{n-k}(t_0,x_0)p_k\right).$$

□

Proposition 3.3 (Supersolution Property)**.** *Suppose that for all $k \in \{0,...,n-1\}$ C_k is a continuous function. The lower semi-continuous envelope of C_n, denoted by C_n^l, is then a supersolution of (3.7).*

Proof: Let $(t_0,x_0) \in [0,\ T) \times (0,+\infty)$ and $\phi \in \mathcal{C}^{1,2}\left([0,\ T)\times(0,+\infty)\right)$ such that $\phi(t_0,x_0) = C_n^l(t_0,x_0)$ and $\phi \leq C_n^l$ on $[0,\ T)\times(0,+\infty)$. We have to prove that

$$\min\Big(\mathcal{L}\phi(t_0,x_0) + \min_a \Delta_n^a\phi(t_0,x_0);\ [\mathcal{M}_n - \phi](t_0,x_0)\Big) \leq 0.$$

If $[\mathcal{M}_n - \phi](t_0,x_0) \leq 0$, then it is automatically satisfied. Therefore, let us assume that $[\mathcal{M}_n - \phi](t_0,x_0) > 0$. Let $\epsilon = \frac{1}{2}[\mathcal{M}_n - \phi](t_0,x_0)$.

We introduce, as before, a sequence $(t_m,x_m)_{m\geq 0}$ such that

$$\lim_{m\to+\infty}(t_m,x_m) = (t_0,x_0) \text{ and } \lim_{m\to+\infty} C_n(t_m,x_m) = C_n^l(t_0,x_0) = \phi(t_0,x_0)$$

and take m large enough to satisfy

$$\mathcal{B}^m := (t_m, (t_m + \eta/2) \wedge T) \times (x_m - \eta/2, x_m + \eta/2) \subset (t_0, (t_0 + \eta) \wedge T) \times (x_0 - \eta, x_0 + \eta).$$

Define the stopping time $\nu_{\mathcal{B}^m}$ as the first exit time of $\mathcal{B}^m$.

Let $\varepsilon > 0$. From the continuity of ϕ, $\mathcal{L}\phi$ and C_i for $0 \leq i < n$, we deduce that for all $a \in \mathcal{A}_0$, there exists $\eta > 0$ such that for all (t, x) such that $\mid t - t_0 \mid < \eta$ and $\mid x - x_0 \mid < \eta$, we have

$$\mid \phi(t,x) - \phi(t_0,x_0) \mid + \mid \mathcal{L}\phi(t,x) - \mathcal{L}\phi(t_0,x_0) \mid + \sum_{i=1}^{n-1} \mid C_i(t,x) - C_i(t_0,x_0) \mid \leq \varepsilon.$$

As ϕ is continuous, we have that $\gamma_m := C_n(t_m, x_m) - \phi(t_m, x_m)$ converges to 0 when m goes to infinity. Set $h_m = \sqrt{\gamma_m}$. Take m large enough so that $t_m + h_m < T \wedge (t_0 + \eta)$.

It follows from the DPP that

$$C_n(t_m, x_m) = \inf_{\alpha \in \mathcal{A}} \mathbb{E}_{t_m, x_m, n}[C_{N^\alpha_\varsigma}(\varsigma, X_\varsigma) - \int_{t_m}^{\varsigma} \lambda(s, \alpha_s) f(\alpha_s) ds],$$

for all $\varsigma \leq \tau^{\epsilon,m,\alpha} := \inf\{u \geq t_m : C_{N^\alpha_u}(u, X_u) > \mathcal{M}_{N^\alpha_u}(u, X_u) - \epsilon\}$. Note that $\tau^{\epsilon,m,\alpha} > t_m$ a.s. for m large enough. We define the following stopping time $\tilde{\nu}^m = \tau^{\epsilon,m,\alpha} \wedge \nu_{\mathcal{B}^m} \wedge (t_m + h_m)$. From the above DPP, it follows that

$$\begin{aligned} C_n(t_m, x_m) &= \inf_{\alpha \in \mathcal{A}} \mathbb{E}_{t_m, x_m, n}[C_{N^\alpha_{\theta(\alpha) \wedge \tilde{\nu}^m}}(\theta(\alpha) \wedge \tilde{\nu}^m, X_{\theta(\alpha) \wedge \tilde{\nu}^m}) - \int_{t_m}^{\theta(\alpha) \wedge \tilde{\nu}^m} \lambda(s, \alpha_s) f(\alpha_s) ds] \\ &= \inf_{\alpha \in \mathcal{A}} \mathbb{E}\Big[C_n(\tilde{\nu}^m, X_{\tilde{\nu}^m}) \mathbb{1}_{\{\tilde{\nu}^m < \theta(\alpha)\}} + C_{N^\alpha_{\theta(\alpha)}}(\theta(\alpha), X_{\theta(\alpha)}) \mathbb{1}_{\{\theta(\alpha) \leq \tilde{\nu}^m\}} \\ &\qquad - \int_{t_m}^{\theta(\alpha) \wedge \tilde{\nu}^m} \lambda(s, \alpha_s) f(\alpha_s) ds\Big] \end{aligned}$$

in which $\theta(\alpha)$ is the first jump time after t_m of the process H. However, we have $C_n \geq C_n^l \geq \phi$, so

$$\begin{aligned} & C_n(t_m, x_m) \\ \geq & \inf_{\alpha \in \mathcal{A}} \mathbb{E}[\phi(\tilde{\nu}^m, X_{\tilde{\nu}^m}) \mathbb{1}_{\{\tilde{\nu}^m < \theta(\alpha)\}} + C_{N^\alpha_{\theta(\alpha)}}(\theta(\alpha), X_{\theta(\alpha)}) \mathbb{1}_{\{\theta(\alpha) \leq \tilde{\nu}^m\}} - \int_{t_m}^{\theta(\alpha) \wedge \tilde{\nu}^m} \lambda(s, \alpha_s) f(\alpha_s) ds] \\ = & \inf_{\alpha \in \mathcal{A}} \mathbb{E}[\phi(\tilde{\nu}^m \wedge \theta(\alpha), X_{\tilde{\nu}^m \wedge \theta(\alpha)}) + [C_{N^\alpha_{\theta(\alpha)}} - \phi](\theta(\alpha), X_{\theta(\alpha)}) \mathbb{1}_{\{\theta(\alpha) \leq \tilde{\nu}^m\}} - \int_{t_m}^{\theta(\alpha) \wedge \tilde{\nu}^m} \lambda(s, \alpha_s) f(\alpha_s) ds]. \end{aligned}$$

Hence, we can apply Itô's formula to the process $(\phi(t,X_t))_{t_m\le t\le T}$ between t_m and $\tilde\nu^m\wedge\theta(\alpha)$ to obtain:

$$\gamma_m \ge \inf_{\alpha\in\mathcal{A}} \mathbb{E}[\int_{t_m}^{\tilde\nu^m\wedge\theta(\alpha)} \mathcal{L}\phi(s,X_s)\,ds + [C_{N^\alpha_{\theta(\alpha)}} - \phi](\theta(\alpha),X_{\theta(\alpha)})\mathbb{1}_{\{\theta(\alpha)\le\tilde\nu^m\}} - \int_{t_m}^{\theta(\alpha)\wedge\tilde\nu^m} \lambda(s,\alpha_s)f(\alpha_s)ds].$$

We also have

$$\begin{aligned}\mathbb{E}[[C_{N^\alpha_{\theta(\alpha)}} - \phi](\theta(\alpha),X_{\theta(\alpha)})\mathbb{1}_{\{\theta(\alpha)\le\tilde\nu^m\}}] &= \sum_{k=1}^{+\infty}\mathbb{E}\left[[C_{n-k}-\phi](\theta(\alpha),X_{\theta(\alpha)})\mathbb{1}_{\{\theta(\alpha)\le\tilde\nu^m\}}\right]p_k \\ &\ge \mathbb{E}\left[\int_{t_m}^{\tilde\nu^m\wedge\theta(\alpha)}\lambda(s,\alpha_s)\,ds\right]\left(\sum_{k=1}^{+\infty}[C_{n-k}-\phi](t_m,X_{t_m})p_k - 2\varepsilon\right).\end{aligned}$$

For any $\alpha\in\mathcal{A}$, the result is that the quantity

$$\frac{\mathbb{E}[\int_{t_m}^{\tilde\nu^m\wedge\theta(\alpha)}\mathcal{L}\phi(s,X_s)\,ds + [C_{N^\alpha_{\theta(\alpha)}} - \phi](\theta(\alpha),X_{\theta(\alpha)})\mathbb{1}_{\{\theta(\alpha)\le\tilde\nu^m\}} - \int_{t_m}^{\theta(\alpha)\wedge\tilde\nu^m}\lambda(s,\alpha_s)f(\alpha_s)ds]}{\mathbb{E}_{t_m,x_m,n}(\tilde\nu^m\wedge\theta(\alpha)-t_m)}$$

converges to

$$\mathcal{L}\phi(t_0,x_0) - \lambda(t_0,\alpha_0)\left(f(\alpha_0)+\phi(t_0,x_0) - \sum_{k=1}^{\infty}C_{n-k}(t_0,x_0)p_k\right)$$

when m goes to infinity and then ε to 0. We finally obtain

$$\mathcal{L}\phi(t_0,x_0) + \min_a \Delta_n^a\phi(t_0,x_0)\le 0$$

by taking the minimum over $\mathcal{A}_0$. □

For the following comparison to hold, we need an additional assumption on g, namely there exists $C>0$ and $\tilde g$ such that for all $x,y\in\mathbb{R}_+$

$$\mid g(x,z)-g(y,z)\mid \le C\mid x-y\mid \tilde g(z) \tag{3.8}$$

such that $\int_{\mathbb{R}}\tilde g(z)m(dz)<\infty$.

Proposition 3.4 (Comparison Principle)**.** *Assume that for all $k\in\{0,...,n-1\}$, C_k is a continuous function and that (3.8) holds. Then if v is viscosity subsolution of (3.7) and w is a viscosity supersolution of (3.7), such that $\lim_{x\to 0}v^u\le\lim_{x\to 0}w^l$, and $v,\ w\in\mathcal{P}$ then $v^u\le w^l$.*

Proof: For ease of exposition, we omit the superscripts denoting the semi-continuous envelopes of v and w.

We assumed that v and w have at most polynomial growth of order $p > 0$. Let $\varepsilon = \min(1, 1/\underline{\lambda})$, $b > 0$ and define $\varphi(t,x) = -e^{-bt}(1 + x^{p+1}) - 1/\underline{\lambda}$, for $x \geq 0$, $t \leq T$.

Let $m \geq 1$. We need to show that $\varrho := \sup_{(t,x)} v_m - w \leq 0$, with $v_m := v + \frac{1}{m}\varphi$. Suppose on the contrary that $\varrho > 0$. Since,

$$\lim_{x\to\infty} v_m - w = -\infty \text{ and } \lim_{x\to 0} v_m - w \leq 0, \tag{3.9}$$

it is clear that this supremum is attained at some point $(t_0, x_0) \in [0,T) \times \mathcal{O}$ in which $\mathcal{O}$ is an open subset of $\mathbb{R}_+$, i.e. $\varrho = v_m(t_0,x_0) - w(t_0,x_0)$, with $t_0 < T$ and $x_0 > 0$.

For $i \geq 1$, define $\Phi_i(t,x,y) = v_m(t,x) - w(t,y) - \frac{i}{2} \mid x - y \mid^2$. Let $\varrho_i = \sup_{[0,T]\times\mathcal{O}^2} \Phi_i(t,x,y)$, which we can assume is attained at some point $(\hat{t}_i, \hat{x}_i, \hat{y}_i) \in [0,T] \times \overline{\mathcal{O}}^2$. By taking a subsequence, we can also assume there exists a point $(\hat{t}_0, \hat{x}_0, \hat{y}_0)$ to which $(\hat{t}_i, \hat{x}_i, \hat{y}_i)$ converges as $i \to \infty$. For i large enough, we can then assume that $\hat{t}_i < T$, and $\hat{x}_i > 0$. The goal is to apply Theorem 8.3 of [10] to the functions Φ_i for each $i \geq 1$ and take a limit as $i \to \infty$. In order to do so, we first want to show that v_m is a subsolution of (3.7).

We begin by proving that the function $v_m = v + \frac{1}{m}\varphi$ is a strict subsolution of (3.7) in the sense that

$$\min(\mathcal{L}v_m + \min_{a\in\mathcal{A}_0} \Delta_n^a v_m, \mathcal{M}_n - v_m) \geq \frac{1}{m}\varepsilon > 0.$$

Since v is a subsolution, $\mathcal{M}_n(t,x) - v(t,x) \geq 0$ which implies that $\mathcal{M}_n(t,x) - v(t,x) - \frac{1}{m}\varphi(t,x) \geq \frac{1}{m}\varepsilon$. On the other hand, on $[0,T] \times \mathbb{R}_+$, we calculate

$$\begin{aligned} e^{bt}\mathcal{L}\varphi(t,x) &\geq b(1+x^{p+1}) - \frac{\sigma(t,x)^2}{2}p(p+1)x^{p-1} - \mu(t,x)(p+1)x^p - \gamma_t x^{p+1}\int_{\mathbb{R}}(g(z)^{p+1} - 1)m(dz) \\ &\geq \Big[b - \frac{\overline{\sigma}^2}{2}p(p+1) - \overline{\mu}(p+1) - \overline{\gamma}\int_{\mathbb{R}} |g(z)|^{p+1} m(dz)\Big]x^{p+1} + c(p)x^p + d(p)x^{p-1} + b, \end{aligned}$$

where $c(p)$ and $d(p)$ are constants depending on model parameters. Therefore, for b large enough,

we get $\mathcal{L}\varphi(t,x) \geq 0$ and, consequently,

$$\mathcal{L}[v+\frac{1}{m}\varphi] + \min_{a\in\mathcal{A}_0} \Delta_n^a[v+\frac{1}{m}\varphi] \quad \geq \quad \min_{a\in\mathcal{A}_0} \Delta_n^a[v+\frac{1}{m}\varphi] - \min_{a\in\mathcal{A}_0} \Delta_n^a v$$

because v is a subsolution which implies that $\mathcal{L}v \geq -\min_{a\in\mathcal{A}_0} \Delta_n^a v$. Moreover, $v+\frac{1}{m}\varphi \leq v$ so that $\Delta_n^a[v+\frac{1}{m}\varphi] \geq \Delta_n^a v$ and

$$\begin{aligned}\mathcal{L}[v+\frac{1}{m}\varphi] + \min_{a\in\mathcal{A}_0} \Delta_n^a[v+\frac{1}{m}\varphi] &\geq \min_{a\in\mathcal{A}_0} \left(\Delta_n^a[v+\frac{1}{m}\varphi] - \Delta_n^a v\right) \\ &\geq \min_{a\in\mathcal{A}_0} \lambda(t,a)\frac{1}{\underline{\lambda} m} \geq 1/m \geq \varepsilon/m.\end{aligned}$$

In order to show that $\lim_i \hat{x}_i = \lim_i \hat{y}_i = \hat{x}_0$, consider the following inequality:

$$\Phi_i(\hat{t}_0,\hat{x}_0,\hat{x}_0) \leq \Phi_i(\hat{t}_i,\hat{x}_i,\hat{y}_i).$$

In particular,

$$\frac{i}{2} \mid \hat{x}_i - \hat{y}_i \mid^2 \quad \leq \quad -v_m(\hat{t}_0,\hat{x}_0) + w(\hat{t}_0,\hat{x}_0) + v_m(\hat{t}_i,\hat{x}_i) - w(\hat{t}_i,\hat{y}_i)$$

and there exists $C>0$ such that

$$\mid \hat{x}_i - \hat{y}_i \mid^2 \quad \leq \quad \frac{C}{i}$$

since v_m and w are bounded on $\mathcal{O}$. Taking i to ∞, we find $\hat{x}_0 := \lim_i \hat{x}_i = \hat{y}_0 := \lim_i \hat{y}_i$. Finally, we show that $\varrho_i \to \varrho$ when $i\to\infty$. To do so, note that $\varrho \leq \varrho_i$ since $\Phi_i(\hat{t}_i,\hat{x}_i,\hat{y}_i) \geq \Phi_i(t_0,x_0,x_0) = v_m(t_0,x_0) - w(t_0,x_0) = \varrho$. In particular, we can bound ϱ by

$$\varrho_i \leq v_m(\hat{t}_i,\hat{x}_i) - w(\hat{t}_i,\hat{y}_i) - \frac{i}{2} \mid \hat{x}_i - \hat{y}_i \mid^2 \leq v_m(\hat{t}_i,\hat{x}_i) - w(\hat{t}_i,\hat{y}_i)$$

which converges to a limit $A \leq v_m(\hat{t}_0,\hat{x}_0) - w(\hat{t}_0,\hat{x}_0) \leq \varrho$ since v_m is upper semi-continuous and w is lower semi-continuous. Since this limit is less or equal to ϱ, we conclude that $\varrho_i \to \varrho$ and $\frac{i}{2} \mid \hat{x}_i - \hat{y}_i \mid^2 \to 0$ when $i\to\infty$, and $v_m(\hat{t}_0,\hat{x}_0) - w(\hat{t}_0,\hat{x}_0) = \varrho$.

We can now apply Theorem 8.3 of [10] at the point $(\hat{t}_i,\hat{x}_i,\hat{y}_i)$: for all $\epsilon>0$, there exist p_0, M

and $M' \in \mathbb{R}$ such that

$$\begin{pmatrix} -\frac{1}{\epsilon} - \mid\mid A \mid\mid & 0 \\ 0 & -\frac{1}{\epsilon} - \mid\mid A \mid\mid \end{pmatrix} \leq \begin{pmatrix} M & 0 \\ 0 & M' \end{pmatrix} \leq A + \epsilon A^2 \tag{3.10}$$

with $A = D^2\phi_i(\hat{x}_i, \hat{y}_i) = \begin{pmatrix} i & -i \\ -i & i \end{pmatrix}$, $(\phi_i(x,y) := i/2 \mid x - y \mid^2)$, and, using Lemma 2.2 of [19], which gives the relation between the notion of superjets and our definition of viscosity supersolutions, we find

$$\begin{aligned} \min\left(\mathcal{K}[p_0, \frac{\partial\phi_i}{\partial\hat{x}_i}, M](\hat{t}_i, \hat{x}_i) + \gamma_{\hat{t}_i} I[v_m](\hat{t}_i, \hat{x}_i) + \min_{a\in\mathcal{A}_0} \Delta_n^a v_m(\hat{t}_i, \hat{x}_i);\ [\mathcal{M}_n - v_m](\hat{t}_i, \hat{x}_i)\right) \geq \varepsilon/m \\ \min\left(\mathcal{K}[p_0, \frac{\partial\phi_i}{\partial\hat{y}_i}, M'](\hat{t}_i, \hat{y}_i) + \gamma_{\hat{t}_i} I[w](\hat{t}_i, \hat{y}_i) + \min_{a\in\mathcal{A}_0} \Delta_n^a w(\hat{t}_i, \hat{y}_i);\ [\mathcal{M}_n - w](\hat{t}_i, \hat{y}_i)\right) \leq 0 \end{aligned}$$

in which $\mathcal{K}[p,q,M](t,x) = p + \frac{\sigma^2(t,x)}{2}M + \mu(t,x)q$ and $I[\psi](t,x) = \int_{\mathbb{R}}(\psi(t, xg(x,z)) - \psi(t,x))m(dz)$. Subtracting the last two inequalities, we conclude that either

1. $\mathcal{M}_n(\hat{t}_i, \hat{x}_i) - v_m(\hat{t}_i, \hat{x}_i) - (\mathcal{M}_n(\hat{t}_i, \hat{y}_i) - w(\hat{t}_i, \hat{y}_i)) \geq \varepsilon/m$, or
2. $p_0 + \frac{\partial\phi_i}{\partial\hat{t}_i} + \frac{\sigma^2(\hat{t}_i,\hat{x}_i)}{2}M + \mu(\hat{t}_i, \hat{x}_i)\frac{\partial\phi_i}{\partial\hat{x}_i} + \gamma_{\hat{t}_i} I[v_m](\hat{t}_i, \hat{x}_i) + \min_{a\in\mathcal{A}_0}\Delta_n^a v_m - (p_0 + \frac{\sigma^2(\hat{t}_i,\hat{y}_i)}{2}M' - \mu(\hat{t}_i, \hat{y}_i)\frac{\partial\phi_i}{\partial\hat{y}_i} + \gamma_{\hat{t}_i} I[w](\hat{t}_i, \hat{y}_i) + \min_{a\in\mathcal{A}_0}\Delta_n^a w) \geq \varepsilon/m$.

In the first case, we simply note that $\mathcal{M}_n$ is continuous by assumption, so that $\mathcal{M}_n(\hat{t}_i, \hat{x}_i) - v_m(\hat{t}_i, \hat{x}_i) - (\mathcal{M}_n(\hat{t}_i, \hat{y}_i) - w(\hat{t}_i, \hat{y}_i))$ converges to $w(\hat{t}_0, \hat{x}_0) - v_m(\hat{t}_0, \hat{x}_0) = -\varrho$. Consequently, the limit $\varrho \leq -\varepsilon/m$, a contradiction. In the second case, we calculate:

$$\begin{aligned} & \mathcal{K}[p_0, \frac{\partial\phi_i}{\partial\hat{x}_i}, M](\hat{t}_i, \hat{x}_i) + \min_{a\in\mathcal{A}_0}\Delta_n^a v_m(\hat{t}_i, \hat{x}_i) - \mathcal{K}[p_0, \frac{\partial\phi_i}{\partial\hat{y}_i}, M'](\hat{t}_i, \hat{y}_i) - \min_{a\in\mathcal{A}_0}\Delta_n^a w(\hat{t}_i, \hat{y}_i) \\ & \gamma_{\hat{t}_i} I[v_m](\hat{t}_i, \hat{x}_i) - \gamma_{\hat{t}_i} I[w](\hat{t}_i, \hat{y}_i) \\ = \ & \frac{\sigma^2(\hat{t}_i, \hat{x}_i)}{2}M - \frac{\sigma^2(\hat{t}_i, \hat{y}_i)}{2}M' + i(\mu(\hat{t}_i, \hat{x}_i) - \mu(\hat{t}_i, \hat{y}_i))(\hat{x}_i - \hat{y}_i) \\ & + \gamma_{\hat{t}_i}\int_{\mathbb{R}}(v_m(\hat{t}_i, \hat{x}_i g(\hat{x}_i, z)) - w(\hat{t}_i, \hat{y}_i g(\hat{y}_i, z)))m(dz) - \gamma_{\hat{t}_i}\int_{\mathbb{R}}(v_m(\hat{t}_i, \hat{x}_i) - w(\hat{t}_i, \hat{y}_i))m(dz) \\ & + \min_{a\in\mathcal{A}_0}\Delta_n^a v_m - \min_{a\in\mathcal{A}_0}\Delta_n^a w. \end{aligned}$$

Hence,

$$
\begin{aligned}
& \mathcal{K}[p_0, \frac{\partial \phi_i}{\partial \hat{x}_i}, M](\hat{t}_i, \hat{x}_i) + \min_{a \in \mathcal{A}_0} \Delta_n^a v_m(\hat{t}_i, \hat{x}_i) - \mathcal{K}[p_0, \frac{\partial \phi_i}{\partial \hat{y}_i}, M'](\hat{t}_i, \hat{y}_i) - \min_{a \in \mathcal{A}_0} \Delta_n^a w(\hat{t}_i, \hat{y}_i) \\
& \gamma_{\hat{t}_i} I[v_m](\hat{t}_i, \hat{x}_i) - \gamma_{\hat{t}_i} I[w](\hat{t}_i, \hat{y}_i) \\
\leq \; & \frac{\sigma^2(\hat{t}_i, \hat{x}_i)}{2} M - \frac{\sigma^2(\hat{t}_i, \hat{y}_i)}{2} M' + iC \mid \hat{x}_i - \hat{y}_i \mid^2 \\
& + \gamma_{\hat{t}_i} \int_{\mathbb{R}} \frac{i}{2} (\mid \hat{x}_i g(\hat{x}_i, z) - \hat{y}_i g(\hat{y}_i, z) \mid^2 - \mid \hat{x}_i - \hat{y}_i \mid^2) m(dz) \\
& \underline{\lambda}(w(t_i, \hat{y}_i) - v_m(t_i, \hat{x}_i)) + \overline{\lambda} \sum_{i=1}^{n-1} \left| C_{n-i}(\hat{t}_i, \hat{x}_i) - C_{n-i}(\hat{t}_i, \hat{y}_i) \right| \qquad (3.11) \\
\leq \; & \frac{\sigma^2(\hat{t}_i, \hat{x}_i)}{2} M - \frac{\sigma^2(\hat{t}_i, \hat{y}_i)}{2} M' + iC \mid \hat{x}_i - \hat{y}_i \mid^2 \\
& + \gamma_{\hat{t}_i} \int_{\mathbb{R}} \frac{i}{2} (\mid \hat{x}_i \mid^2 \mid g(\hat{x}_i, z) - g(\hat{y}_i, z) \mid^2 + g(\hat{y}_i, z)^2 \mid \hat{x}_i - \hat{y}_i \mid^2 - \mid \hat{x}_i - \hat{y}_i \mid^2) m(dz) \\
& \underline{\lambda}(w(t_i, \hat{y}_i) - v_m(t_i, \hat{x}_i)) + \overline{\lambda} \sum_{i=1}^{n-1} \left| C_{n-i}(\hat{t}_i, \hat{x}_i) - C_{n-i}(\hat{t}_i, \hat{y}_i) \right|
\end{aligned}
$$

for some positive constant $C > 0$. From (3.8), we know that

$$
\begin{aligned}
& \gamma_{\hat{t}_i} \int_{\mathbb{R}} \frac{i}{2} (\hat{x}_i^2 \mid g(\hat{x}_i, z) - g(\hat{y}_i, z) \mid^2 + g(\hat{y}_i, z)^2 \mid \hat{x}_i - \hat{y}_i \mid^2 - \mid \hat{x}_i - \hat{y}_i \mid^2) m(dz) \\
\leq \gamma_{\hat{t}_i} & \int_{\mathbb{R}} \frac{i}{2} (\hat{x}_i^2 \mid \hat{x}_i - \hat{y}_i \mid^2 \tilde{g}(z)^2 + (2g(0, z)^2 + 2C^2 \hat{y}_i^2 \hat{g}(z)^2) \mid \hat{x}_i - \hat{y}_i \mid^2 - \mid \hat{x}_i - \hat{y}_i \mid^2) m(dz).
\end{aligned}
$$

Hence, the above integral converges to 0 because of (2.3), (3.8) and the fact that $m(\mathbb{R}) < \infty$. Furthermore, from (3.10), we find the following upper bound for $\sigma^2(\hat{t}_i, \hat{x}_i)M - \sigma^2(\hat{t}_i, \hat{y}_i)M'$:

$$
\begin{aligned}
\sigma^2(\hat{t}_i, \hat{x}_i)M - \sigma^2(\hat{t}_i, \hat{y}_i)M' &= \begin{pmatrix} \sigma(\hat{t}_i, \hat{x}_i) & \sigma(\hat{t}_i, \hat{y}_i) \end{pmatrix} \begin{pmatrix} M & 0 \\ 0 & -M' \end{pmatrix} \begin{pmatrix} \sigma(\hat{t}_i, \hat{x}_i) \\ \sigma(\hat{t}_i, \hat{y}_i) \end{pmatrix} \\
&\leq \begin{pmatrix} \sigma(\hat{t}_i, \hat{x}_i) & \sigma(\hat{t}_i, \hat{y}_i) \end{pmatrix} (A + \epsilon A^2) \begin{pmatrix} \sigma(\hat{t}_i, \hat{x}_i) \\ \sigma(\hat{t}_i, \hat{y}_i) \end{pmatrix} \\
&= i(\sigma(\hat{t}_i, \hat{x}_i) - \sigma(\hat{t}_i, \hat{y}_i))^2 + 2\epsilon i^2 (\sigma(\hat{t}_i, \hat{x}_i) - \sigma(\hat{t}_i, \hat{y}_i))^2
\end{aligned}
$$

by direct calculation. Remembering the Lipschitz condition of σ, we easily deduce that, for some

positive constant C,

$$\sigma^2(\hat{t}_i, \hat{x}_i)M - \sigma^2(\hat{t}_i, \hat{y}_i)M' \quad \leq \quad iC|\hat{x}_i - \hat{y}_i|^2$$

by taking $\epsilon = \frac{1}{i}$. As we have seen before, this term goes to zero. Consequently, Expression (3.11) converges to $-\underline{\lambda}\varrho$ and $\varrho \leq -\frac{\varepsilon}{m\underline{\lambda}}$, a contradiction. As a result, we find that $\varrho \leq 0$. □

Proof of Theorem 3.1:

We know that C_0 is continuous and it is the unique viscosity solution of (3.7). By induction, suppose C_k is the unique continuous viscosity solutions of (3.7) for $k \leq n-1$. By the previous propositions, we then obtain that C_n is the unique viscosity solution of (3.7) and that the comparison result holds. In particular, C_n is continuous. □

4 Numerical Results

We calibrated the model to market data corresponding to four different firms traded on the NYSE exchange through the *ArcaBook* on February 24th, 2011. The data files obtained from NYX-data.com contains all time-stamped limit orders entered, removed, modified, filled or partially filled on the NYSE ArcaBook platform. The firms considered are Google (GOOG), IBM (IBM), Polaris Industries (PII) and WESCO International (WCC). We see from the calibration below that GOOG and IBM are very liquid stocks. Yet a major difference is that the empirical distribution of their bid-ask spreads differ considerably, as seen in Figure 1. This is due to the fact that their stock price differ considerably with GOOG opening at 611.39 and IBM opening at 159.63 on this day, and in percentage IBM has a smaller spread (0.03% of stock price) than GOOG (0.08% of stock price). Since prices are quoted in cents, this offers a large array of values of spreads for GOOG, for which the spread varied from 0.02$ to 2.14$ during the trading day considered. On the other hand, WCC and PII have similar market capitalizations evaluated at 2.354B and 2.497B, respectively on February 24th, 2011. They also present similar liquidity characteristics as can be observed below.

Following the methodology of Blais and Protter [6], Eq. (3.2), we consider a linear model for the liquidity cost of trading:

$$L(t, x, n) = nx + n^2 m. \tag{4.12}$$

Stock	$\mathbf{E}(m_t)(\times 10^{-5})$	$\mathbf{E}(\xi_t)(\times 10^{-9})$
GOOG	101.01	2.75
IBM	4.55	1.78
WCC	10.94	35.71
PII	23.98	44.33

Table 1: Statistics of liquidity costs for the 4 firms studied on 02/24/2012.

Blais and Protter [6] compute at every point in time (e.g., every second of the day) the liquidity cost per share, as a function of the number of shares sold, as

$$\frac{L(t, x_t, n)}{n} = x_t + nm_t.$$

They then perform a linear regression with the observed liquidity cost per share as a function of n, to determine the value of x_t and m_t, at each point in time during the day. We define m in (4.12) as the average observed value of m_t over the trading day of February 24th, 2011, for each stock studied. However, in order to better compare liquidity across different stocks we also compute the liquidity cost per dollars invested squared:

$$\xi_t = \frac{m_t}{S_t^2},$$

with S_t the bid price at time t. This is the liquidity cost coming from the second term in (4.12) normalized in terms of the number of dollars invested in the stock. The fitted values are given in Table (1). From the values of ξ, we see that GOOG and IBM are more liquid than PII and WCC, as expected from their market capitalization.

We consider a linear model for λ:

$$\lambda(\alpha) = \lambda_0 + \lambda_1 \alpha.$$

Estimated parameters are presented in Table 3 and Figure 3. In Figures 4 to 7, we have represented the control strategy as a function of time and current spread.

We see in Figures 1 and 2 that a diffusion spread model and a order book density is more appropriate for GOOG and PII, whereas a discrete model may be more fitted to the spread dynamics of IBM and WCC due to the restricted number of realized values of spread.

The order book density function that corresponds to (4.12) is simply the constant $\frac{1}{2m}$. Indeed, then I is given by $I(x,n) = 2mn$ and

$$L(x,n) = nx + \int_0^{I(x,n)} \frac{1}{2m} y dy = nx + mn^2.$$

The discrete limit order book which gives a liquidity cost of this form (4.12) is constructed as follows. Let

$$x_{-j} = x + mn'(2j-1)$$

and

$$b(x) = 0, b(x_{-j}) = n', j \geq 1.$$

The quantity $b(y)$ denotes the number of shares in the order book with a price equal to the best ask price minus y at time t, and $\mathcal{B}_0 = \{x, x_{-1}, x_{-2}, x_{-3}, \dots\}$ is the support of b. For instance, $b(x)$ is the number of shares in the order book at the best bid price when x equals the bid-ask spread. Define $B(x_{-i}, x) = b(x) + \sum_{k=1}^{i} b(x_{-k})$, the number of shares offered at prices no more than x_{-i} dollars below the best ask price when the bid-ask spread equals x. The relation between L and b is then given inductively as follows:

$$L(x,n) = \begin{cases} nx, & \text{for } n \leq b(x); \\ L(t,x,b_t(x)) + (n - b(x))x_{-1}, & \text{for } b(x) < n \leq B(x_{-1},x); \\ L(t,x,B(x_{-1},x)) + (n - B(x_{-1},x))x_{-2}, & \text{for } B(x_1) < n \leq B(x_{-2},x); \\ L(t,x,B(x_{-2},x)) + (n - B(x_{-2},x))x_{-3}, & \text{for } B(x_2) < n \leq B(x_{-3},x); \\ \dots \end{cases} \quad (4.13)$$

Note that L does not depend on the current time t. Furthermore, the extra spread function is given by

$$I(x,n) = x - \inf\{x_{-i} : n < B(x_{-i},x)\}. \quad (4.14)$$

We assume that the investor's trade size is a multiple of n'.

For the two stocks for which a discretization of the order book and prices is more realistic, we consider a pure jump model for the dynamics of the spread. Given the availability of high frequency

Stock	κ_x	$\bar{x}$	σ_x	γ
GOOG	0.1646	0.4630	0.1263	0.6038
IBM	—	—	—	0.8379
PII	0.1588	0.1555	0.0337	0.6799
WCC	—	—	—	0.6931

Table 2: Fitted parameters for spread dynamics (4.16)

data, we can accurately estimate the conditional probability of increments of X_t conditioned on the prior value X_{t-}. Consequently, we consider the following model for X

$$dX_t = \int_{\mathbb{R}} X_{t-} g(X_{t^-}, z) \tilde{M}(dt, dz), \tag{4.15}$$

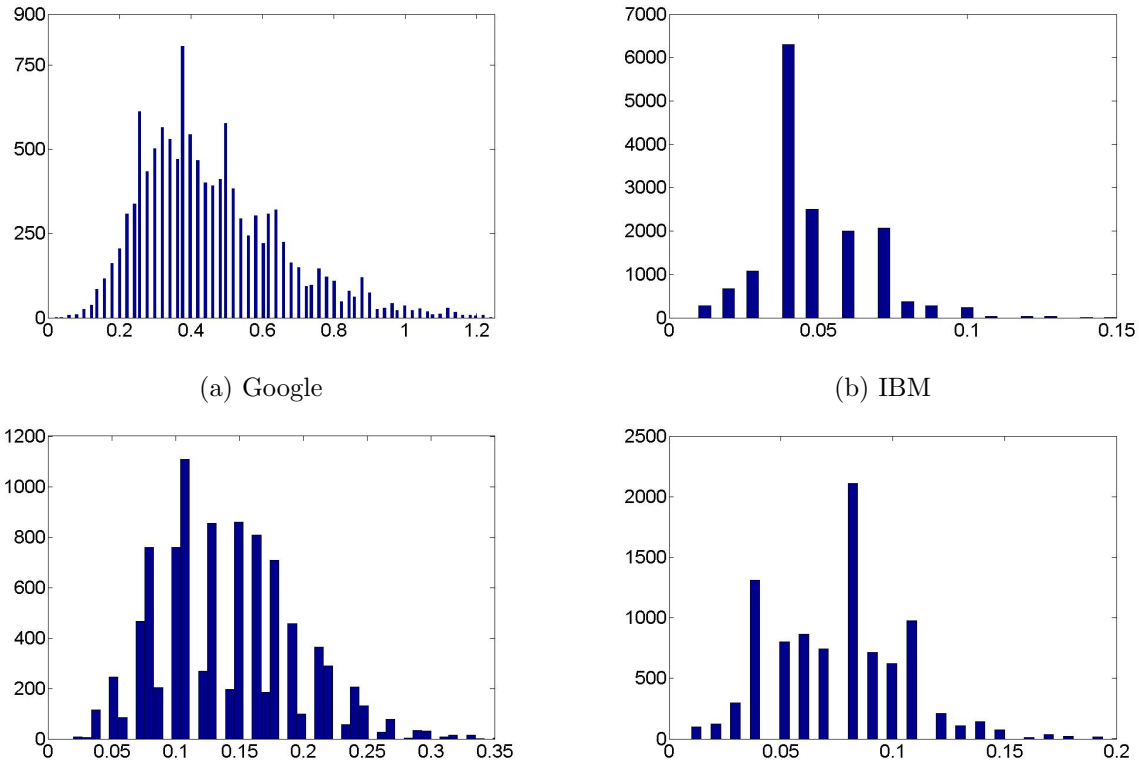
in which we estimate the function g by the empirical distribution of X_t given X_{t-}. Recall that M is a random Poisson measure with mean measure $\gamma_t dt m(dz)$. Here we take γ as constant.

The dynamic of the spread X_t for the other two stocks is assumed to be a jump-diffusion. We assume the following dynamics:

$$dX_t = \kappa_x(\bar{x} - X_{t^-})dt + \sigma_x dW_t + \int_{\mathbb{R}} (z - X_{t^-}) \tilde{M}(dt, dz), \tag{4.16}$$

In other words, when the process X jumps at time t, its value X_t is distributed according to a fixed distribution. Here again we take γ as constant and we calibrate m as the empirical stationary distribution of the bid-ask spread, as represented in Figure 1. The fitted parameters are given in Table 2.

For a comparative analysis, we liquidate a number N of shares corresponding to 0.001% of market capitalization for GOOG and IBM, and 0.01% for WCC and PII. Shares are sold in packets of size n', which we take as the average number of shares posted in the limit order book at each price. This corresponds to 639 packets of 5 shares each for GOOG, 115 packets of 110 shares each for IBM, 91 packets of 46 shares for WCC, and 162 packets of 21 shares for PII.



(a) Google

(b) IBM

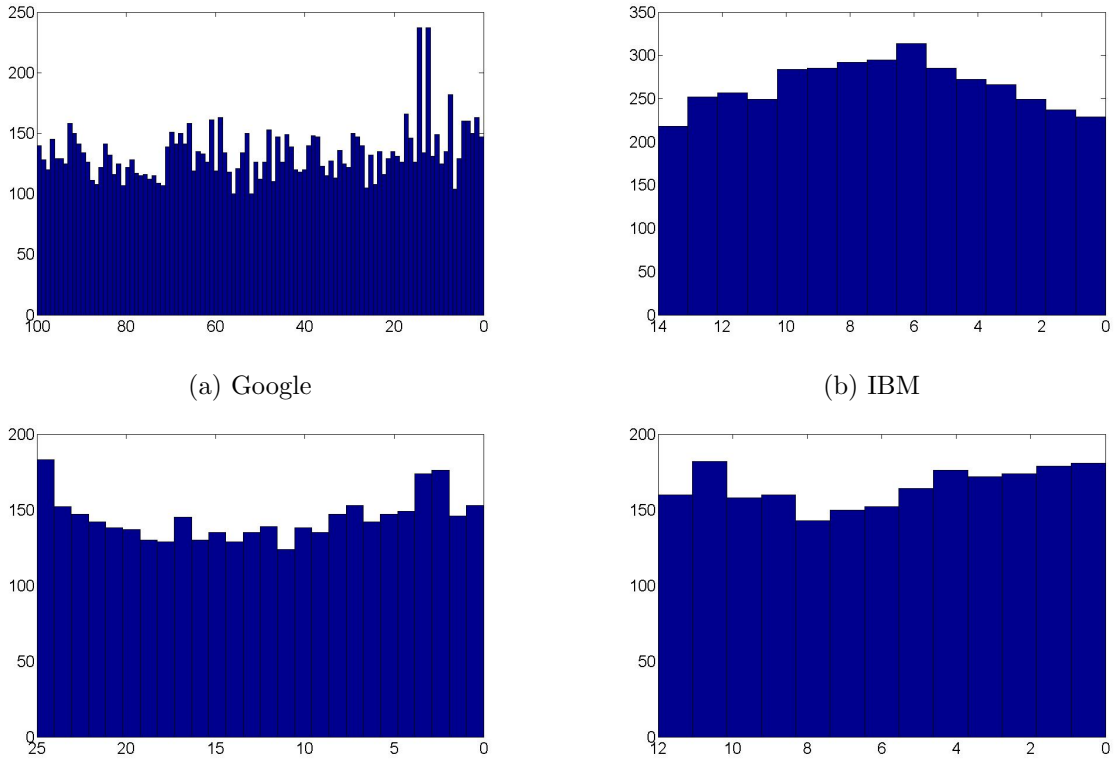
(c) Polaris Industries

(d) WESCO International

Figure 1: Distribution of bid-ask spread

Stock	λ_0	λ_1
GOOG	0.1145	-0.1223
IBM	0.1251	-1.1741
PII	0.0497	-0.1433
WCC	0.0379	-0.1985

Table 3: Estimated parameters for linear model of λ.



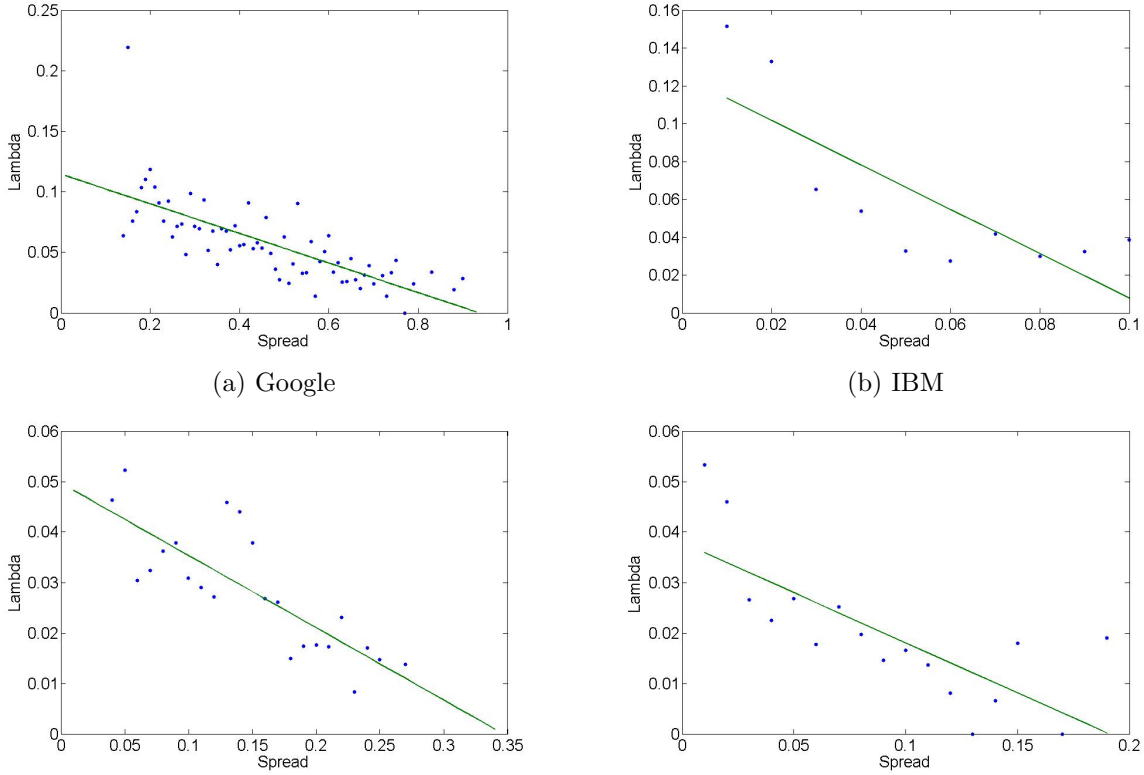
(a) Google

(b) IBM

(c) Polaris Industries

(d) WESCO International

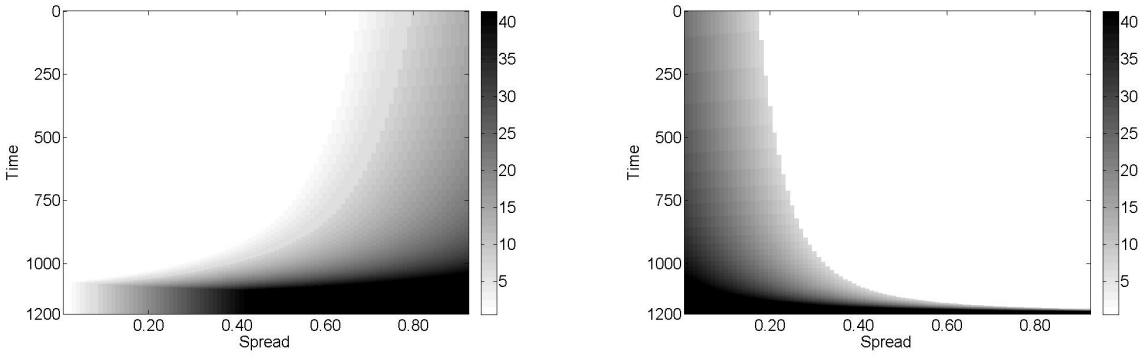
Figure 2: Average observed limit order book density on the bid side, in cents below the best bid price. Number of shares offered at each price below the best bid price is recorded at every second during the day and averaged out over the whole day.



(a) Google (b) IBM

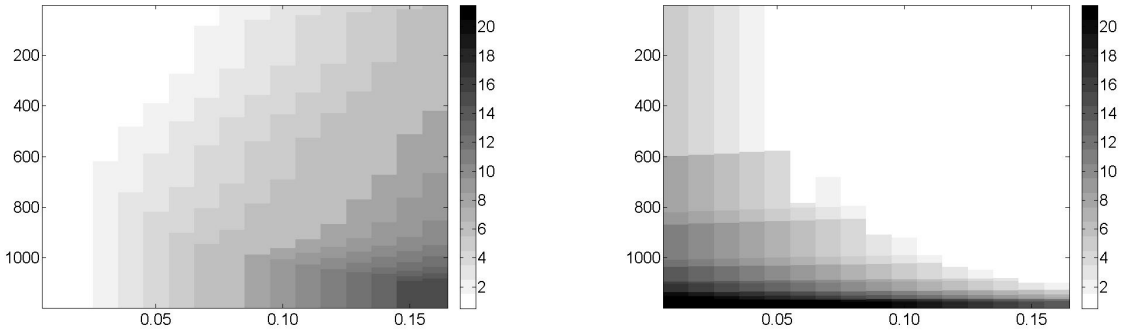
(c) Polaris Industries (d) WESCO International

Figure 3: Arrival intensity (λ) of market orders as function of the current value of spread (time units are seconds).



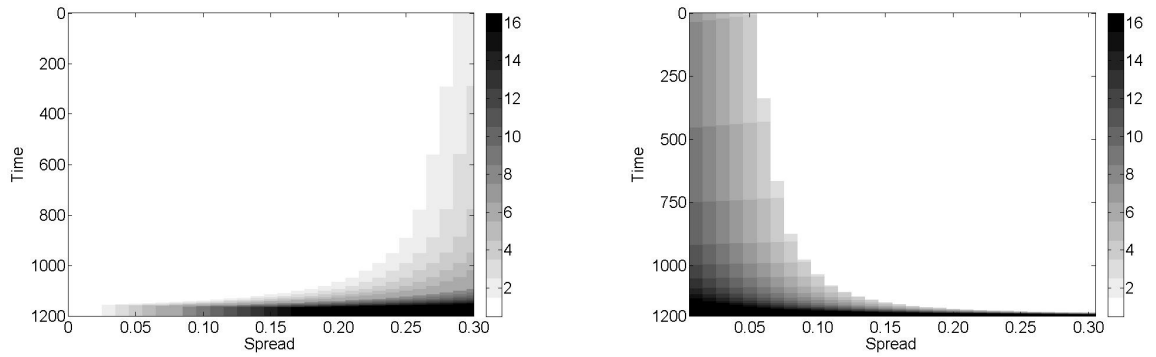
(a) Optimal limit prices (b) Size of optimal market orders

Figure 4: Graphical representation of optimal limit price and market order control policy as a function of current spread and time for Google. Units in (4a) are in cents below the current best ask price. Trade sizes in (4b) are expressed in packets of 10 shares. Liquidation is performed over a period of 75 minutes (4500 seconds).



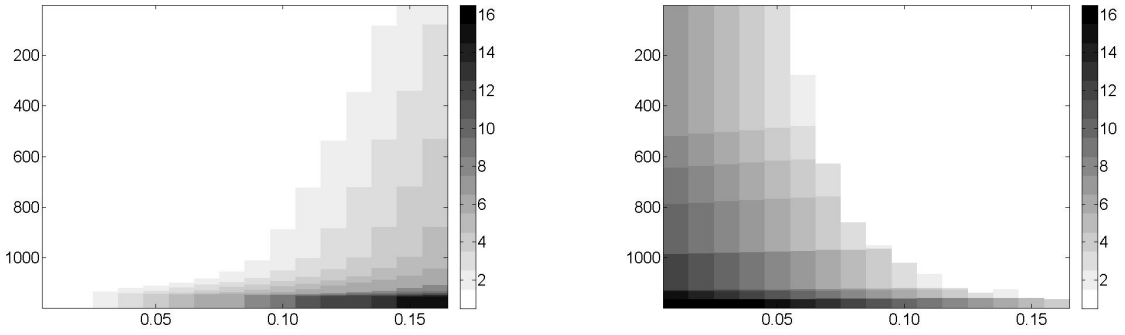
(a) Limit Orders (b) Market Orders

Figure 5: Graphical representation of optimal limit price and market order control policy as a function of current spread and time for IBM. Units in (5a) are in cents below the current best ask price. Trade sizes in (5b) are expressed in packets of 110 shares. Liquidation is performed over a period of 20 minutes (1200 seconds).



(a) Limit Orders (b) Market Orders

Figure 6: Graphical representation of optimal limit price and market order control policy as a function of current spread and time for Polaris Industries. Units in (6a) are in cents below the current best ask price. Trade sizes in (6b) are expressed in packets of 21 shares. Liquidation is performed over a period of 20 minutes (1200 seconds).



(a) Limit Orders (b) Market Orders

Figure 7: Graphical representation of optimal limit price and market order control policy as a function of current spread and time for WESCO International. Units in (6a) are in cents below the current best ask price. Trade sizes in (7b) are expressed in packets of 46 shares. Liquidation is performed over a period of 20 minutes (1200 seconds).

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