Erratum to "Multipliers and Morrey spaces".

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Abstract

We correct the complex interpolation results for Morrey spaces which is false for the first interpolation functor of Calderón, but is exact for the Calderón's second interpolation functor.

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In my paper "Multipliers and Morrey spaces" [3], there is a slight mistake in Theorem 3, concerning the interpolation of Morrey spaces.

Let $\mathcal{B}$ be the collection of all Euclidean balls B on $\mathbb{R}^d$: $B = B(x_B, r_B) = \{x \in \mathbb{R}^d \ / \ |x - x_B| < R_B$. For $B \in \mathcal{B}$, we write $|B| = \int_B dx = |B(0,1)| r_B^d$. Define, for $1 < p \leq q < +\infty$, the space $\dot{M}^{p,q}(\mathbb{R}^n)$ as the space of locally p-integrable functions f such that

$$\|f\|_{\dot{M}^{p,q}} := \sup_{B \in \mathcal{B}} |B|^{1/q-1/p} \Big(\int_B |f(x)|^p \, dx \Big)^{1/p} < +\infty.$$

It is easy to check that, when $1 < p_0 \leq q_0 < +\infty$ and $1 < p_1 \leq q_1 < +\infty$, and

$$(\frac{1}{p}, \frac{1}{q}) = (1-\theta)(\frac{1}{p_0}, \frac{1}{q_0}) + \theta(\frac{1}{p_1}, \frac{1}{q_1})$$

for some $\theta \in (0,1)$, then $\dot{M}^{p_0,q_0} \cap \dot{M}^{p_1,q_1} \subset \dot{M}^{p,q}$ and

$$\|f\|_{\dot{M}^{p,q}} \leq \|f\|^{1-\theta}_{\dot{M}^{p_0,q_0}} \|f\|^{\theta}_{\dot{M}^{p_1,q_1}}$$

The question I studied was then whether one may find an interpolation functor F of exponent θ such that $F(\dot{M}^{p_0,q_0}, \dot{M}^{p_1,q_1}) = \dot{M}^{p,q}$. If so, one should have the continuous embeddings

$$[\dot{M}^{p_0,q_0}, \dot{M}^{p_1,q_1}]_{\theta,1} \subset \dot{M}^{p,q} \subset [\dot{M}^{p_0,q_0}, \dot{M}^{p_1,q_1}]_{\theta,\infty}.$$

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The inclusion $[A_0, A_1]_{\theta,1} \subset F(A_0, A_1)$ is a direct conclusion from the inequality

$$\|f\|_{F(A_0,A_1)} \leq C\|f\|_{A_0}^{1-\theta}\|f\|_{A_1}^{\theta}$$

(obtained by interpolation inequalities for the operator norms of $\lambda \mapsto \lambda f$ from $\mathbb{R}$ to A_0 and from $\mathbb{R}$ to A_1, hence from $\mathbb{R}$ to $F(A_0, A_1)$). The inclusion $F(A_0, A_1) \subset [A_0, A_1]_{\theta,\infty}$ is proven in [1] under the assumption that $A_0 \cap A_1$ is dense in A_0 and in A_1. This is not the case for Morrey spaces. However, one may easily adapt the proof, as Morrey spaces are dual spaces (see for instance [6]).

If we assume that $A_0 = B_0'$ and $A_1 = B_1'$ and that $B_0 \cap B_1$ is dense in B_0 and B_1; then, for $b \in B_0 \cap B_1$, the linear form $T_b : f \mapsto \langle f|b\rangle$ has a norm less than $\|b\|_{B_0}$ as an operator from A_0 to $\mathbb{R}$ and less than $\|b\|_{B_1}$ as an operator from A_1 to $\mathbb{R}$, hence as a norm less than $C\|b\|_{B_0}^{1-\theta}\|b\|_{B_1}^{\theta}$; thus, $\tilde{T}_f : b \mapsto \langle f|b\rangle$ is a continuous linear form on $[B_0, B_1]_{\theta,1}$. This gives that $f \in ([B_0, B_1]_{\theta,1})' = [B_0', B_1']_{\theta,\infty}$ (since $B_0 \cap B_1$ is dense in B_0 and B_1).

The theorem I proved in [3] is the following one:

Theorem 1
Let $1 < p_0 \leq q_0 < +\infty$ and $1 < p_1 \leq q_1 < +\infty$, and

$$(\frac{1}{p}, \frac{1}{q}) = (1-\theta)(\frac{1}{p_0}, \frac{1}{q_0}) + \theta(\frac{1}{p_1}, \frac{1}{q_1})$$

for some $\theta \in (0, 1)$. Then there exists an interpolation functor F of exponent θ such that $F(\dot{M}^{p_0,q_0}, \dot{M}^{p_1,q_1}) = \dot{M}^{p,q}$ if and only if $p_0/q_0 = p_1/q_1$.

The negative result for the case $p_0/q_0 \neq p_1/q_1$ was proven by a generalization of a counterexample by Ruiz and Vega [4] which proves that, in that case, we don't have the embedding of $\dot{M}^{p,q}$ into $[\dot{M}^{p_0,q_0}, \dot{M}^{p_1,q_1}]_{\theta,\infty}$.

The proof for the positive result (on the case $p_0/q_0 = p_1/q_1$) was inexact. I claimed that in that case we have the complex interpolation $\dot{M}^{p,q} = [\dot{M}^{p_0,q_0}, \dot{M}^{p_1,q_1}]_{\theta}$. But this is false as pointed to me by Sickel (who has recently characterized the intermediate space $\dot{M}^{p,q} \subset [\dot{M}^{p_0,q_0}, \dot{M}^{p_1,q_1}]_{\theta}$ in a joint work with Yang and Yuan [5]). Indeed, it is easy to see that, when $p_0/q_0 = p_1/q_1 = p_0/q_0 < 1$ and $p_0 \neq p_1$, $\dot{M}^{p_0,q_0} \cap \dot{M}^{p_1,q_1}$ is not dense in $\dot{M}^{p,q}$, while it is always true that $A_0 \cap A_1$ is dense in $[A_0, A_1]_{\theta}$ (see [1]). Sickel's counterexample is very clear : if $r = \min(p_0, p_1)$ and $s = \max(q_0, q_1)$, we have $\dot{M}^{p,q} \subset \dot{M}^{r,q}$ and $\dot{M}^{p_0,q_0} \cap \dot{M}^{p_1,q_1} \subset \dot{M}^{r,s}$; thus the applications $f \mapsto \rho^{d(1/q-1/r)}1_{B(0,\rho)}f$ are equicontinuous from $\dot{M}^{p,q}$ to L^r; for $f \in \dot{M}^{p_0,q_0} \cap \dot{M}^{p_1,q_1}$, we have $\lim_{\rho\to 0} \rho^{d(1/q-1/r)}\|1_{B(0,\rho)}f\|_r = 0$, while for $f_0 = |x|^{-d/q} \in \dot{M}^{p,q}$, we have $\lim_{\rho\to 0} \rho^{d(1/q-1/r)}\|1_{B(0,\rho)}f_0\|_r > 0$; thus f_0 does not belong to the closure of $\dot{M}^{p_0,q_0} \cap \dot{M}^{p_1,q_1}$

However, a slight modification of the proof of [3] gives the following theorem :

Theorem 2
Let $1 < p_0 \leq q_0 < +\infty$ *and* $1 < p_1 \leq q_1 < +\infty$, *and*

$$(\frac{1}{p}, \frac{1}{q}) = (1-\theta)(\frac{1}{p_0}, \frac{1}{q_0}) + \theta(\frac{1}{p_1}, \frac{1}{q_1})$$

If $p_0/q_0 = p_1/q_1$, *then*

$$\dot{M}^{p,q} = [\dot{M}^{p_0,q_0}, \dot{M}^{p_1,q_1}]^\theta.$$

Let us recall that Calderón [2] defined two complex interpolation functors : $[A_0, A_1]_\theta$ and $[A_0, A_1]^\theta$. We have $[A_0, A_1]_\theta = [A_0, A_1]^\theta$ (with equality when at least one of the two spaces A_0 and A_1 is reflexive).

Proof :
Let us recall the definition of $[A_0, A_1]_\theta$ and $[A_0, A_1]^\theta$.

Let Ω be the open complex strip $\Omega = \{z \in \mathbb{C} \ / \ 0 < \Re z < 1\}$. $\mathcal{F}$ is the space of functions F defined on the closed complex strip $\bar{\Omega}$ such that :

1. F is continuous and bounded from $\bar{\Omega}$ to $A_0 + A_1$
2. F is analytic from Ω to $A_0 + A_1$
3. $t \mapsto F(it)$ is continuous from $\mathbb{R}$ to A_0, and $\lim_{|t|\to+\infty} \|F(it)\|_{A_0} = 0$
4. $t \mapsto F(1+it)$ is continuous from $\mathbb{R}$ to A_1, and $\lim_{|t|\to+\infty} \|F(1+it)\|_{A_0} = 0$

Then

$$f \in [A_0, A_1]_\theta \Leftrightarrow \exists F \in \mathcal{F}, f = F(\theta)$$

and

$$\|f\|_{[A_0,A_1]_\theta} = \inf_{f=F(\theta)} \max(\sup_{t\in\mathbb{R}} \|F(it)\|_{A_0}, \sup_{t\in\mathbb{R}} \|F(1+it)\|_{A_1}).$$

On the other hand, $\mathcal{G}$ is the space of functions G defined on the closed complex strip $\bar{\Omega}$ such that :

1. $\frac{1}{1+|z|}G$ is continuous and bounded from $\bar{\Omega}$ to $A_0 + A_1$
2. G is analytic from Ω to $A_0 + A_1$
3. $t \mapsto G(it) - G(0)$ is Lipschitz from $\mathbb{R}$ to A_0

4. $t \mapsto G(1+it) - G(1)$ is Lipschitz from $\mathbb{R}$ to A_1

Then

$$f \in [A_0, A_1]^\theta \Leftrightarrow \exists G \in \mathcal{G}, f = G'(\theta)$$

and

$$\|f\|_{[A_0,A_1]^\theta} = \inf_{f=G'(\theta)} \max(\sup_{t_1,t_2\in\mathbb{R}} \|\frac{G(it_2)-G(it_1)}{t_2-t_1}\|_{A_0}, \sup_{t_1,t_2\in\mathbb{R}} \|\frac{G(1+it_2)-G(1+it_1)}{t_2-t_1}\|_{A_1}).$$

Let us remark that, for continuous functions, (strong) analyticity is equivalent to weak analyticiy or even *-weak analyticity when A_0 and A_1 are dual spaces of B_0 and B_1 with $B_0 \cap B_1$ dense in B_0 and B_1. Indeed, analyticity is equivalent to the fact that, whenever the closed ball $\bar{B}(z_0, r)$ is contained in Ω and $|w - z_0| < r$, then $F(w) = \frac{1}{2i\pi} \int_{|z-z_0|=r} F(z) \frac{dz}{z-w}$. As F is continuous, we have, for $b \in B_0 \cap B_1$,

$$\langle b| \frac{1}{2i\pi} \int_{|z-z_0|=r} F(z) \frac{dz}{z-w} \rangle_{B_0\cap B_1, A_0+A_1} = \frac{1}{2i\pi} \int_{|z-z_0|=r} \langle b|F(z)\rangle_{B_0\cap B_1, A_0+A_1} \frac{dz}{z-w}$$

The equvalence remains true for *-weakly continuous functions.

However, of course, there is no equivalence between (strong) continuity and *-weak continuity. In the original proof of [3], one made two remaks :

1. Let $1 < p_0 \leq q_0 < +\infty$ and $1 < p_1 \leq q_1 < +\infty$, and

$$(\frac{1}{p}, \frac{1}{q}) = (1-\theta)(\frac{1}{p_0}, \frac{1}{q_0}) + \theta(\frac{1}{p_1}, \frac{1}{q_1}).$$

   If F is an interpolation functor of exponent θ that satisfies $F(L^{p_0}, L^{p_1}) = L^p$, then $F(\dot{M}^{p_0,q_0}, \dot{M}^{p_1,q_1}) \subset \dot{M}^{p,q}$. Thus, we have the embeddings of $[\dot{M}^{p_0,q_0}, \dot{M}^{p_1,q_1}]_{\theta,p}$, $[\dot{M}^{p_0,q_0}, \dot{M}^{p_1,q_1}]_\theta$ and $[\dot{M}^{p_0,q_0}, \dot{M}^{p_1,q_1}]^\theta$ into $\dot{M}^{p,q}$.

2. When moreover $p_0/q_0 = p_1/q_1 = p/q$ we may define for $f \in \dot{M}^{p,q}$ the function $F(z) = \frac{f}{|f|} |f|^{(1-z)\frac{p}{p_0} + z\frac{p}{p_1}}$. This is a bounded *-weakly continuous function of $z = x+iy$ (for $0 \leq x \leq 1$) with values in $\dot{M}^{p_0,q_0} + \dot{M}^{p_1,q_1}$, holomorphic on the strip $0 < x < 1$, with $\sup_{\in\mathbb{R}} \|F(iy)\|_{\dot{M}^{p_0,q_0}} < +\infty$, $\sup_{\in\mathbb{R}} \|F(1+iy)\|_{\dot{M}^{p_1,q_1}} < +\infty$, and $F(\theta) = f$.

If F was strongly continuous, we would find that $f = F(\theta)$ would belong to $[\dot{M}^{p_0,q_0}, \dot{M}^{p_1,q_1}]_\theta$. But F is only *-weakly continuous. We may define $G(z) = \int_0^z F(w)\, dw$. Then we have $G \in \mathcal{G}$, and $G'(\theta) = f$; thus f belongs to $[\dot{M}^{p_0,q_0}, \dot{M}^{p_1,q_1}]^\theta$ $\diamond$

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