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Non-Arbitrage up to Random Horizon for Semimartingale Models

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Abstract This paper addresses the question of how an arbitrage-free semimartingale model is affected when stopped at a random horizon. We focus on No-Unbounded-Profit-with-Bounded-Risk (called NUPBR hereafter) concept, which is also known in the literature as the first kind of non-arbitrage. For this non-arbitrage notion, we obtain two principal results. The first result lies in describing the pairs of market model and random times for which the resulting stopped model fulfills NUPBR condition. The second main result characterises the random time models that preserve the NUPBR property after stopping for any market model. These results are elaborated in a general market model, and we pay attention to some particular and practical models. The analysis that drives these results is based on new stochastic developments in semimartingale theory with progressive enlargement. Furthermore, we construct explicit martingale densities (deflators) for some classes of local martingales when stopped at random time.

1 Introduction

From the very beginning of studies in the field of mathematical finance, the economical concept of no arbitrage was considered as a building block and an important property that asset price models must satisfy to develop pricing rules and hedging frameworks. The well known theory of NFLVR [15] leads to the fundamental theorem of asset pricing (FTAP), and relates this concept with the existence of an equivalent martingale measure.

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More recently, the important rôle of numéraire portfolio was recognized by many researchers (see [34], [26], [32] and [37]) in a setting of pricing, hedging and optimisation problems, where the existence of a solution can be obtained under a condition weaker than NFLVR (see [31] , [3], [10], [9], [37], and the references therein), e.g., the existence of a growth-optimal portfolio does not require the NFLVR assumption. The necessary and sufficient condition for existence of the numéraire portfolio is the boundedness in probability of the collection of terminal wealths attainable by trading using capped strategies, called No-Unbounded-Profit-with-Bounded-Risk (called NUPBR hereafter). This condition is strongly related with the existence of an equivalent local martingale measure, also called a deflator (see Takaoka [38]),and implies the existence of a growth-optimal portfolio and the existence of a numéraire portfolio (see [19] and [26]).

On the one hand, the NUPBR property is the non-arbitrage concept that is intimately related to the weakest form of markets' viability (see [9], [32], and [28] for details about this issue, and [2] for examples of market models violating NFLVR and fulfilling NUPBR). It has been proved recently that for a model violating the NUPBR, the optimal portfolio will not exist even locally, and the pricing rules fail as well. As recognized in [38] and in [9], the NUPBR property is mathematically very attractive and possesses the 'dynamic/localization' feature that the NFLVR and other arbitrage concepts lack to possess. By localization feature, we mean that if the property holds locally (i.e. the property holds for the stopped models with a sequence of stopping times that increases to infinity), then it holds globally.

On the other hand, the question of the choice of the relevant information has to be considered. Many papers ([17], [35],[5]) are devoted to the case of insider trading, where the insider has a private information at the beginning of the period, which requires the study of an initial enlargement of filtration, or in a more general enlargement of filtration setting [30]. The case of a progressive enlargement of filtration, as in credit risk modeling, is less investigated and presents an interest, for pricing derivatives or solving optimization problems. The validity of NUPRB condition was studied in the literature, in the particular case of complete market with continuous filtration, when τ avoids $\mathbb{F}$-stopping times in [16]. It is also possible to derive some of our results using a new optional decomposition formula, established recently in [4]. After that a first version of our results has been available, Acciaio et al. [1] proved some of our results, using a different method.

In this paper, we consider a general semimartingale model S satisfying the NUPBR property under the "public information" denoted by $\mathbb{F}$ and an arbitrary random time τ and we answer to the following questions:

For which pairs (S, τ), does NUPBR property hold for S^τ? **(P1)**

and

For which τ, is the NUPBR preserved for any S after stopping at τ? **(P2)**

To deepen our understanding of the precise interplay between the initial market model and the random time model, we address these two principal questions separately in the case of quasi-left-continuous models, and then in the case of thin processes with predictable jumps. Afterwards, we combine the two cases and state the results for the most general framework.
This paper is organized as follows. The next section (Section 2) presents our main results in different contexts, and discusses their meaning and/or their economical interpretations and their consequences as well. Section 3 develops new stochastic results, that are the key mathematical ideas behind the answers to **(P1)**-**(P2)**. Section 4 gives an explicit form for the deflator in the case where S is quasi-left continuous. Section 5 contains the proofs of the main theorems announced, without proofs, in Section 2. The paper concludes with an Appendix, where some classical results on the predictable characteristics of a semimartingales and other related results are recalled. Some technical proofs are also postponed to the Appendix, for the ease of the reader.

2 Main Results and their Interpretations

This section is devoted to the presentation of our main results and their immediate consequences. To this end, we start specifying our mathematical setting and the economical concepts that we will address.

2.1 Notations and Preliminary Results on NUPBR

We consider a stochastic basis $(\Omega, \mathcal{G}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, P)$, where $\mathbb{F}$ is a filtration satisfying the usual hypotheses (i.e., right continuity and completeness), and $\mathcal{F}_\infty \subseteq \mathcal{G}$. Financially speaking, the filtration $\mathbb{F}$ represents the flow of public information through time. On this basis, we consider an arbitrary but fixed d-dimensional càdlàg semimartingale S. This represents the discounted price processes of d-stocks, while the riskless asset's price is assumed to be constant. Beside the initial model $(\Omega, \mathcal{G}, \mathbb{F}, P, S)$, we consider a random time τ, i.e., a non-negative $\mathcal{G}$-measurable random variable. To this random time, we associate the process D and the filtration $\mathbb{G}$ given by

$$D := I_{[\![\tau,+\infty[\![}, \quad \mathbb{G} = (\mathcal{G}_t)_{t\geq 0}\,, \quad \mathcal{G}_t = \bigcap_{s>t}\Big(\mathcal{F}_s \vee \sigma(D_u, u \leq s)\Big).$$

The filtration $\mathbb{G}$ is the smallest right-continuous filtration which contains $\mathbb{F}$ and makes τ a stopping time. In the probabilistic literature, $\mathbb{G}$ is called the progressive enlargement of $\mathbb{F}$ with τ. In addition to $\mathbb{G}$ and D, we associate to τ two important $\mathbb{F}$-supermartingales given by

$$Z_t := P\left(\tau > t \mid \mathcal{F}_t\right) \text{ and } \quad \widetilde{Z}_t := P\left(\tau \geq t \;\middle|\; \mathcal{F}_t\right). \tag{2.1}$$

The supermartingale Z is right-continuous with left limits and coincides with the $\mathbb{F}$-optional projection of $I_{[\![0,\tau[\![}$, while $\widetilde{Z}$ admits right limits and left limits only and is the $\mathbb{F}$-optional projection of $I_{[\![0,\tau]\!]}$. The $\mathbb{F}$-martingale m, given by

$$m := Z + D^{o,\mathbb{F}}, \tag{2.2}$$

where $D^{o,\mathbb{F}}$ is the $\mathbb{F}$-dual optional projection of D (See [22] for more details) will play an important role in what follows.

In what follows, $\mathbb{H}$ is a filtration satisfying the usual hypotheses and Q a probability measure on the filtered probability space $(\Omega, \mathbb{H})$. The set of martingales for the filtration $\mathbb{H}$ under Q is denoted by $\mathcal{M}(\mathbb{H}, Q)$. When $Q = P$, we simply denote $\mathcal{M}(\mathbb{H})$. As usual, $\mathcal{A}^+(\mathbb{H})$ denotes the set of increasing, right-continuous, $\mathbb{H}$-adapted and integrable processes.
If $\mathcal{C}(\mathbb{H})$ is a class of $\mathbb{H}$ adapted processes, we denote by $\mathcal{C}_0(\mathbb{H})$ the set of processes $X \in \mathcal{C}(\mathbb{H})$ with $X_0 = 0$, and by $\mathcal{C}_{loc}$ the set of processes X such that there exists a sequence $(T_n)_{n\geq 1}$ of $\mathbb{H}$-stopping times that increases to $+\infty$ and the stopped processes X^{T_n} belong to $\mathcal{C}(\mathbb{H})$. We put $\mathcal{C}_{0,loc}(\mathbb{H}) = \mathcal{C}_0(\mathbb{H}) \cap \mathcal{C}_{loc}(\mathbb{H})$.
In all the paper, we shall write $A = \emptyset$ (or $X = Y$) as a shortcut for A is a negligible set (resp. $X = Y$ a.s.).
For a process K with $\mathbb{H}$-locally integrable variation, we denote by $K^{o,\mathbb{H}}$ its dual optional projection. The dual predictable projection of K (also called the $\mathbb{H}$-predictable dual projection) is denoted $K^{p,\mathbb{H}}$. For a process X, we denote ${}^{o,\mathbb{H}}X$ (resp. ${}^{p,\mathbb{H}}X$) its optional (resp. predictable) projection with respect to $\mathbb{H}$.
For an $\mathbb{H}$- semi-martingale Y, the set $L(Y, \mathbb{H})$ is the set of $\mathbb{H}$ predictable processes integrable w.r.t. Y and for $H \in L(Y, \mathbb{H})$, we denote $H \bullet Y_t := \int_0^t H_s dY_s$.
As usual, for a process X and a random time ϑ, we denote by X^ϑ the stopped process. To distinguish the effect of filtration, we will denote $\langle ., . \rangle^{\mathbb{F}}$, or $\langle ., . \rangle^{\mathbb{G}}$ the sharp bracket (predictable covariation process) calculated in the filtration $\mathbb{F}$ or $\mathbb{G}$, if confusion may rise. We recall that, for general semi-martingales X and Y, the sharp bracket is (if it exists) the dual predictable projection of the covariation process $[X, Y]$.

We introduce the non-arbitrage notion that will be addressed in this paper.

Definition 2.1 An $\mathbb{H}$-semimartingale X satisfies the *No-Unbounded-Profit-with-Bounded-Risk* condition under $(\mathbb{H}, Q)$ (hereafter called NUPBR$(\mathbb{H}, Q)$) if for any $T \in (0, +\infty)$ the set

$$\mathcal{K}_T(X, \mathbb{H}) := \Big\{ (H \bullet S)_T \;\Big|\; \quad H \in L(X, \mathbb{H}), \text{ and } H \bullet X \geq -1 \Big\}$$

is bounded in probability under Q. When $Q \sim P$, we simply write, with an abuse of language, X satisfies NUPBR$(\mathbb{H})$.

Remark 2.2 (a) It is important to notice that this definition for NUPBR condition appeared first in [27] (up to our knowledge), and it differs when the time

horizon is infinite from that of the literature given in Delbaen and Schachermayer [15], Kabanov [24] and Karatzas and Kardaras[26]. It is obvious that, when the horizon is deterministic and finite, the current NUPBR condition coincides with that of the literature. We could name the current NUPBR as NUPBR$_{loc}$, but for the sake of simplifying notation, we opted for the usual terminology.
(b) In general, when the horizon is infinite, the NUPBR condition of the literature implies the NUPBR condition defined above. However, the reverse implication may not hold in general. In fact if we consider $S_t = \exp(W_t + t),\ t \geq 0$, then it is clear that S satisfies our NUPBR($\mathbb{H}$), while the NUPBR($\mathbb{H}$) of the literature is violated. To see this last claim, it is enough to remark that

$$\lim_{t \longrightarrow +\infty} (S_t - 1) = +\infty \quad P-a.s. \quad S^t - 1 = H \bullet S \geq -1 \quad H := I_{]\!]0,t]\!]}.$$

The following proposition slightly generalizes Takaoka's results obtained for a finite horizon (see Theorem 2.6 in [38]) to our NUPBR context.

Proposition 2.3 *Let X be an $\mathbb{H}$-semimartingale. Then the following assertions are equivalent.*
(a) *X satisfies NUPBR($\mathbb{H}$).*
(b) *There exist a positive $\mathbb{H}$-local martingale Y and an $\mathbb{H}$-predictable process θ satisfying $0 < \theta \leq 1$ and $Y(\theta \bullet X)$ is a local martingale.*

Proof The proof of the implication (b)⇒ (a) is based on [38] and is omitted. Thus, we focus on proving the reverse implication and suppose that assertion (a) holds. Therefore, a direct application of Theorem 2.6 in [38] to each $(S_{t\wedge n})_{t\geq 0}$, we obtain the existence of a positive $\mathbb{H}$-local martingale $Y^{(n)}$ and an $\mathbb{H}$-predictable process θ_n such that $0 < \theta_n \leq 1$ and $Y^{(n)}(\theta_n \bullet S^n)$ is a local martingale. Then, it is obvious that the process

$$N := \sum_{n=1}^{+\infty} I_{]\!]n-1,n]\!]} (Y_-^{(n)})^{-1} \bullet Y^{(n)}$$

is a local martingale and $Y := \mathcal{E}(N) > 0$. On the other hand, the $\mathbb{H}$-predictable process $\theta := \sum_{n\geq 1} I_{]\!]n-1,n]\!]}\theta_n$ satisfies $0 < \theta \leq 1$ and $Y(\theta \bullet S)$ is a local martingale. This ends the proof of the proposition. □

For any $\mathbb{H}$-semimartingale X, the local martingales fulfilling the assertion (b) of Proposition 2.3 are called σ-martingale densities for X. The set of these σ-martingale densities will be denoted throughout the paper by

$$\mathcal{L}(\mathbb{H}, X) := \left\{ Y \in \mathcal{M}_{loc}(\mathbb{H}) \middle|\ Y > 0,\ \exists \theta \in \mathcal{P}(\mathbb{H}),\ 0 < \theta \leq 1,\ Y(\theta \bullet X) \in \mathcal{M}_{loc}(\mathbb{H}) \right\}$$

where, as usual, $\mathcal{P}(\mathbb{H})$ stands for the predictable σ-field on $\Omega \times [0, \infty[$ and by abuse of notation $\theta \in \mathcal{P}(\mathbb{H})$ means that θ is $\mathcal{P}(\mathbb{H})$-measurable. We state, without proof, an obvious lemma.

Lemma 2.4 *For any $\mathbb{H}$-semimartingale X and any $Y \in \mathcal{L}(\mathbb{H}, X)$, one has ${}^{p,\mathbb{H}}(Y|\Delta X|) < \infty$ and ${}^{p,\mathbb{H}}(Y\Delta X) = 0$*

Remark 2.5 Proposition 2.3 implies that for any process X and any **finite** stopping time σ, the two concepts of NUPBR($\mathbb{H}$) (the current concept and the one of the literature) coincide for X^σ.

Below, we prove that, in the case of infinite horizon, the current NUPBR condition is stable under localization, while this is not the case for the NUPBR condition defined in the literature.

Proposition 2.6 *Let X be an $\mathbb{H}$-semimartingale. Then, the following assertions are equivalent.*
(a) *There exists a sequence $(T_n)_{n\geq 1}$ of $\mathbb{H}$-stopping times that increases to $+\infty$, such that for each $n \geq 1$, there exists a probability Q_n on $(\Omega, \mathbb{H}_{T_n})$ such that $Q_n \sim P$ and X^{T_n} satisfies NUPBR($\mathbb{H}$) under Q_n.*
(b) *X satisfies NUPBR($\mathbb{H}$).*
(c) *There exists an $\mathbb{H}$-predictable process ϕ, such that $0 < \phi \leq 1$ and $(\phi \bullet X)$ satisfies NUPBR($\mathbb{H}$).*

Proof The proof for (a)$\Longleftrightarrow$(b) follows from the stability of NUPBR condition for a finite horizon under localization which is due to [38] (see also [7] for further discussion about this issue), and the fact that the NUPBR condition is stable under any equivalent probability change.
The proof of (b)$\Rightarrow$(c) is trivial and is omitted. To prove the reverse, we assume that (c) holds. Then Proposition 2.3 implies the existence of an $\mathbb{H}$-predictable process ψ such that $0 < \psi \leq 1$ and a positive $\mathbb{H}$-local martingale Y such that $Y(\psi\phi \bullet X)$ is a local martingale. Since $\psi\phi$ is predictable and $0 < \psi\phi \leq 1$, we deduce that X satisfies NUPBR($\mathbb{H}$). This ends the proof of the proposition. □

We end this section with a simple, but useful result for predictable process with finite variation.

Lemma 2.7 *Let X be an $\mathbb{H}$-predictable process with finite variation. Then X satisfies NUPBR($\mathbb{H}$) if and only if $X \equiv X_0$ (i.e. the process X is constant).*

Proof It is obvious that if $X \equiv X_0$, then X satisfies NUPBR($\mathbb{H}$). Suppose that X satisfies NUPBR($\mathbb{H}$). Consider a positive $\mathbb{H}$-local martingale Y, and an $\mathbb{H}$-predictable process θ such that $0 < \theta \leq 1$ and $Y(\theta \bullet X)$ is a local martingale. Let $(T_n)_{n\geq 1}$ be a sequence of $\mathbb{H}$-stopping times that increases to $+\infty$ such that Y^{T_n} and $Y^{T_n}(\theta \bullet X)^{T_n}$ are true martingales. Then, for each $n \geq 1$, define $Q_n := (Y_{T_n}/Y_0) \cdot P$. Since X is predictable, then $(\theta \bullet X)^{T_n}$ is also predictable with finite variation and is a Q_n-martingale. Thus, we deduce that $(\theta \bullet X)^{T_n} \equiv 0$ for each $n \geq 1$. Therefore, we deduce that X is constant (since $X^{T_n} - X_0 = \theta^{-1} \bullet (\theta \bullet X)^{T_n} \equiv 0$). This ends the proof of the lemma. □

2.2 The Quasi-Left-Continuous Processes

In this subsection, we present our two main results on the NUPBR condition under stopping at τ for quasi-left-continuous processes[1]. The first result con-

[1] A process X is quasi-left-continuous if it does not jump at predictable jump times

sists of characterizing the pairs (S,τ) of market and random time models, for which S^τ fulfills the NUPBR condition. The second result focuses on determining the models of random times τ such that, for any semi-martingale S enjoying NUPBR($\mathbb{F}$), the stopped process S^τ enjoys NUPBR($\mathbb{G}$) .

We start by recalling some general notation. For any filtration $\mathbb{H}$, we denote

$$\widetilde{\mathcal{O}}(\mathbb{H}) := \mathcal{O}(\mathbb{H}) \otimes \mathcal{B}(\mathbb{R}^d), \qquad \widetilde{\mathcal{P}}(\mathbb{H}) := \mathcal{P}(\mathbb{H}) \otimes \mathcal{B}(\mathbb{R}^d),$$

where $\mathcal{B}(\mathbb{R}^d)$ is the Borel σ-field on $\mathbb{R}^d$. The jump measure of S is denoted by μ, and is given by

$$\mu(dt,dx) = \sum_{u>0} I_{\{\Delta S_u \neq 0\}} \delta_{(u,\Delta S_u)}(dt,dx)\,. \tag{2.3}$$

For a product-measurable functional $W \geq 0$ on $\Omega \times [0,+\infty[\times\mathbb{R}^d$, we denote $W \star \mu$ (or sometimes, with abuse of notation $W(x) \star \mu$) the process

$$(W\star\mu)_t := \int_0^t \int_{\mathbb{R}^d - \{0\}} W(u,x)\mu(du,dx) = \sum_{0<u\leq t} W(u,\Delta S_u) I_{\{\Delta S_u \neq 0\}}. \tag{2.4}$$

Also on $\Omega\times[0,+\infty[\times\mathbb{R}^d$, we define the measure $M_\mu^P := P\otimes\mu$ by $\int W dM_\mu^P := E\left[W\star\mu_\infty\right]$ (when the integrals are well defined). The conditional "expectation" given $\widetilde{\mathcal{P}}(\mathbb{H})$ of a product-measurable functional W, is the unique $\widetilde{\mathcal{P}}(\mathbb{H})$-measurable functional $\widetilde{W}$ satisfying

$$E\left[W I_\Sigma \star \mu_\infty\right] = E\left[\widetilde{W} I_\Sigma \star \mu_\infty\right], \quad \text{for all } \Sigma \in \widetilde{\mathcal{P}}(\mathbb{H}).$$

The following theorem gives a characterization of $\mathbb{F}$-quasi-left continuous processes that satisfy NUPBR($\mathbb{G}$) after stopping with τ. The proof of this theorem will be given in Subsection 5.1, while its statement is based on the following $\mathbb{F}$-semimartingale

$$S^{(0)} := x I_{\{\psi=0<Z_-\}} \star \mu, \qquad \text{where} \qquad \psi := M_\mu^P\left(I_{\{\widetilde{Z}>0\}} \middle| \widetilde{\mathcal{P}}(\mathbb{F})\right). \tag{2.5}$$

Theorem 2.8 *Suppose that S is $\mathbb{F}$-quasi-left-continuous. Then, the following assertions are equivalent.*
(a) *S^τ satisfies NUPBR($\mathbb{G}$).*
(b) *For any $\delta>0$, the process*

$$I_{\{Z_-\geq\delta\}} \cdot \left(S - S^{(0)}\right) \qquad \textit{satisfies NUPBR($\mathbb{F}$).}$$

(c) *For any $n\geq 1$, the process $(S-S^{(0)})^{\sigma_n}$ satisfies NUPBR($\mathbb{F}$), where*

$$\sigma_n := \inf\{t\geq 0 \ : \ Z_t < 1/n\}. \tag{2.6}$$

Remark 2.9 (a) From assertion (c) one can understand that the NUPBR($\mathbb{G}$) property for S^τ can be verified by checking whether $S-S^{(0)}$ satisfies NUPBR($\mathbb{F}$) up to $\sigma_\infty := \sup_n \sigma_n$. This last fact means that $(S-S^{(0)})^T$ satisfies NUPBR($\mathbb{F}$) for any $\mathbb{F}$-stopping time T such that $[\![0,T]\!] \subset \{Z_- > 0\}$. It is important to mention that this property (i.e. NUPBR($\mathbb{F}$) up to σ_∞) may not be equivalent to NUPBR($\mathbb{F}$) of $Z_- \bullet (S - S^{(0)})$, which is equivalent to the NUPBR($\mathbb{F}$) of $I_{\{Z_->0\}} \bullet (S - S^{(0)})$ due to Proposition 2.6. We believe that a counter-example can be found in the same spirit of Remark 2.2–(b).
(b) The functionals ψ and f_m defined as $f_m := M_\mu^P(\widetilde{Z} - Z_-|\widetilde{\mathcal{P}}(\mathbb{F}))$ satisfy

$$\{\psi = 0\} = \{Z_- + f_m = 0\} \subset \{\widetilde{Z} = 0\}, \quad M_\mu^P - a.e. \tag{2.7}$$

Indeed, due to $\widetilde{Z} \leq I_{\{\widetilde{Z}>0\}}$, we have

$$0 \leq Z_- + f_m = M_\mu^P\left(\widetilde{Z}\middle|\ \widetilde{\mathcal{P}}(\mathbb{F})\right) \leq \psi.$$

Thus, we get $\{\psi = 0\} \subset \{Z_- + f_m = 0\} \subset \{\widetilde{Z} = 0\}$ $M_\mu^P - a.e.$ on the one hand. On the other hand, the reverse inclusion follows from

$$0 = M_\mu^P\left(I_{\{Z_-+f_m=0\}} I_{\{\widetilde{Z}>0\}}\right) = M_\mu^P\left(I_{\{Z_-+f_m=0\}}\psi\right).$$

c) As a result of remark (b) above and $\{\widetilde{Z} = 0 < Z_-\} \subset [\![\sigma_\infty]\!]$, we deduce that $S^{(0)}$ is a càdlàg $\mathbb{F}$-adapted process with finite variation with $var(S^{(0)})_\infty \leq |\Delta S_{\sigma_\infty}| I_{\{\sigma_\infty<+\infty\}}$. Furthermore, it can be written as

$$S^{(0)} := \Delta S_{\sigma_\infty} I_{\{\widetilde{Z}_{\sigma_\infty}=0=\psi(\sigma_\infty,\Delta S_{\sigma_\infty})\ \&\ Z_{\sigma_\infty-}>0\}} I_{[\![\sigma_\infty,+\infty[\![}.$$

This proves that $S^{(0)}$ is a semimartingale (this is stated before Theorem 2.8).

The following corollary is useful for studying the problem (**P2**), and it describes examples of $\mathbb{F}$-quasi-left-continuous model S that fulfill (2.5) as well.

Corollary 2.10 *Suppose that S is $\mathbb{F}$-quasi-left-continuous and satisfies NUPBR($\mathbb{F}$). Then, the following assertions hold.*
(a) *If $\left(S, S^{(0)}\right)$ satisfies NUPBR($\mathbb{F}$), then S^τ satisfies NUPBR($\mathbb{G}$).*
(b) *If $S^{(0)} \equiv 0$, then the process S^τ satisfies NUPBR($\mathbb{G}$).*
(c) *If $\{\Delta S \neq 0\} \cap \{\widetilde{Z} = 0 < Z_-\} = \emptyset$, then S^τ satisfies NUPBR($\mathbb{G}$).*
(d) *If $\widetilde{Z} > 0$ (equivalently $Z > 0$ or $Z_- > 0$), then S^τ satisfies NUPBR($\mathbb{G}$).*

Proof (a) Suppose that $\left(S, S^{(0)}\right)$ satisfies NUPBR($\mathbb{F}$). Then, it is obvious that $S - S^{(0)}$ satisfies NUPBR($\mathbb{F}$), and assertion (a) follows from Theorem 2.8.
(b) Since S satisfies NUPBR($\mathbb{F}$) and $S^{(0)} \equiv 0$, then $\left(S, S^{(0)}\right) \equiv (S, 0)$ satisfies NUPBR($\mathbb{F}$), and assertion (b) follows from assertion (a).
(c) It is easy to see that $\{\Delta S \neq 0\} \cap \{\widetilde{Z} = 0 < Z_-\} = \emptyset$ implies that $S^{(0)} \equiv 0$ (due to (2.7)). Hence, assertions (c) and (d) follow from assertion (b), and the proof of the corollary is completed. □

Remark 2.11 It is worth mentioning that $X - Y$ may satisfy NUPBR($\mathbb{H}$), while (X, Y) may not satisfy NUPBR($\mathbb{H}$). For a non trivial example, consider $X_t = B_t + \lambda t$ and $Y_t = N_t$ where B is a standard Brownian motion and N is the Poisson process with intensity λ.

We now give an answer to the second problem **(P2)** for the quasi-left-continuous semimartingales. Later on (in Theorem 2.23) we will generalize this result. We recall the definition of thin processes/sets for the reader's convenience

Definition 2.12 A set $A \subset \Omega \times [0, \infty[$ is thin if, for all $\omega \in \Omega$, the set $A(\omega)$ is countable. A process X is called thin if there exists a sequence of random variables ξ_n and an increasing sequence of random times T_n such that $X_t = \sum_{n=1}^{\infty} \xi_n I_{[\![T_n, \infty[\![}$. Its paths vary on a thin set only: $I_{\cup_{n=1}^{\infty} [\![T_n]\!]} \cdot X = \sum_{n=1}^{\infty} I_{[\![T_n]\!]} \cdot X = \sum_{n=1}^{\infty} \Delta X_{T_n} I_{[\![T_n, +\infty[\![}$.

Proposition 2.13 *The following assertions are equivalent:*
(a) *The thin set* $\left\{ \widetilde{Z} = 0 \ \& \ Z_- > 0 \right\}$ *is accessible.*
(b) *For any (bounded)* S *that is* $\mathbb{F}$*-quasi-left-continuous and satisfies NUPBR*($\mathbb{F}$)*, the process* S^τ *satisfies NUPBR*($\mathbb{G}$)*.*

Proof The implication (a)⇒(b) follows from Corollary 2.10–(c), since we have

$$\{\Delta S \neq 0\} \cap \{\widetilde{Z} = 0 < Z_-\} = \emptyset.$$

We now focus on proving the reverse implication. To this end, we suppose that assertion (b) holds, and we consider an $\mathbb{F}$-stopping time σ such that $[\![\sigma]\!] \subset \{\widetilde{Z} = 0 < Z_-\}$. It is known that σ can be decomposed into a totally inaccessible part σ^i and an accessible part σ^a such that $\sigma = \sigma^i \wedge \sigma^a$. Consider the quasi-left-continuous $\mathbb{F}$-martingale

$$M = V - \widetilde{V} \in \mathcal{M}_{0,loc}(\mathbb{F})$$

where $V := I_{[\![\sigma^i, +\infty[\![}$ and $\widetilde{V} := V^{p,\mathbb{F}}$. It is known from [14, paragraph 14, Chapter XX], that

$$\{\widetilde{Z} = 0\} \text{ and } \{Z_- = 0\} \text{ are disjoint from }]\!]0, \tau]\!]. \tag{2.8}$$

This implies that $\tau < \sigma \leq \sigma^i \ \ P - a.s.$. Hence, we get

$$M^\tau = -\widetilde{V}^\tau \quad \text{is } \mathbb{G}\text{-predictable.}$$

By hypothesis (b), M being a quasi-left continuous martingale, M^τ satisfies NUPBR($\mathbb{G}$), then we conclude that this process is null (i.e. $\widetilde{V}^\tau = 0$) due to Lemma 2.7. Thus, we get

$$0 = E\left(\widetilde{V}_\tau\right) = E\left(\int_0^{+\infty} Z_{s-} d\widetilde{V}_s\right) = E\left(Z_{\sigma^i -} I_{\{\sigma^i < +\infty\}}\right),$$

or equivalently $Z_{\sigma^i -} I_{\{\sigma^i < +\infty\}} = 0 \quad P - a.s.$ This is possible only if $\sigma^i = +\infty \ \ P - a.s.$ since on $\{\sigma^i < +\infty\} \subset \{\sigma = \sigma^i < +\infty\}$ we have $Z_{\sigma^i -} = Z_{\sigma -} > 0$. This proves that σ is an accessible stopping time. Since $\{\widetilde{Z} = 0 < Z_-\}$ is an optional thin set, assertion (a) follows immediately. This ends the proof of the proposition. □

2.3 Thin Processes with Predictable Jump Times

In this subsection, we outline the main results on the NUPBR condition for the stopped accessible parts of $\mathbb{F}$-semimartingales with a random time. This boils down to consider thin semimartingales with predictable jump times only. We start by addressing the question **(P1)** in the case of single jump process with predictable jump time.

Theorem 2.14 *Consider an $\mathbb{F}$-predictable stopping time T and an $\mathcal{F}_T$-measurable random variable ξ such that $E(|\xi|\big|\ \mathcal{F}_{T-}) < +\infty\ P-a.s.$.*
Let $S := \xi I_{\{Z_{T-}>0\}} I_{[\![T,+\infty[\![}$. Then the following assertions are equivalent:
(a) *S^τ satisfies NUPBR($\mathbb{G}$),*
(b) *The process $\widetilde{S} := \xi I_{\{\widetilde{Z}_T>0\}} I_{[\![T,+\infty[\![} = I_{\{\widetilde{Z}>0\}} \bullet S$ satisfies NUPBR($\mathbb{F}$).*
(c) *There exists a probability measure on $(\Omega, \mathcal{F}_T)$, denoted by Q_T, such that Q_T is absolutely continuous with respect to P, and S satisfies NUPBR($\mathbb{F}, Q_T$).*

The proof of this theorem is long and requires intermediary results that are interesting in themselves. Thus, this proof will be given later in Section 5.

Remark 2.15 (a) The importance of Theorem 2.14 goes beyond its vital role, as a building block for the more general result. In fact, Theorem 2.14 provides two different characterizations for NUPBR($\mathbb{G}$) of S^τ. The first characterization is expressed in term of NUPBR($\mathbb{F}$) of S under absolute continuous change of measure, while the second characterization uses transformation of S without any change of measure. Furthermore, Theorem 2.14 can be easily extended to the case of countably many ordered predictable jump times $T_0 = 0 \leq T_1 \leq T_2 \leq ...$ with $\sup_n T_n = +\infty\ P-a.s.$.
(b) In Theorem 2.14, the choice of S having the form $S := \xi I_{\{Z_{T-}>0\}} I_{[\![T,+\infty[\![}$ is not restrictive. This can be understood from the fact that any single jump process S can be decomposed as follows

$$S := \xi I_{[\![T,+\infty[\![} = \xi I_{\{Z_{T-}>0\}} I_{[\![T,+\infty[\![} + \xi I_{\{Z_{T-}=0\}} I_{[\![T,+\infty[\![} =: \overline{S} + \widehat{S}.$$

Thanks to $\{T \leq \tau\} \subset \{Z_{T-} > 0\}$, we have $\widehat{S}^\tau = \xi I_{\{Z_{T-}=0\}} I_{\{T\leq\tau\}} I_{[\![T,+\infty[\![} \equiv 0$ is (obviously) a $\mathbb{G}$-martingale. Thus, the only part of S that requires careful attention is $\overline{S} := \xi I_{\{Z_{T-}>0\}} I_{[\![T,+\infty[\![}$.

The following result is a complete answer to **(P2)** in the case of predictable single jump processes.

Proposition 2.16 *Let T be an $\mathbb{F}$-predictable stopping time. Then, the following assertions are equivalent:*
(a) *On $\{T < +\infty\}$, we have*

$$\Big\{\widetilde{Z}_T = 0\Big\} \subset \Big\{Z_{T-} = 0\Big\}. \tag{2.9}$$

(b) *For any $M := \xi I_{[\![T,+\infty[\![}$ where $\xi \in L^\infty(\mathcal{F}_T)$ such that $E(\xi|\mathcal{F}_{T-}) = 0$, M^τ satisfies NUPBR($\mathbb{G}$).*

Proof We start by proving (a) ⇒ (b). Suppose that (2.9) holds; due to Remark 2.15(b), we can restrict our attention to the case $M := \xi I_{\{Z_{T-}>0\}} I_{[\![T,+\infty[\![}$ where $\xi \in L^\infty(\mathcal{F}_T)$ such that $E(\xi|\mathcal{F}_{T-}) = 0$. Since assertion (a) is equivalent to $[\![T]\!] \cap \{\widetilde{Z} = 0 \ \& \ Z_- > 0\} = \emptyset$, we deduce that

$$\widetilde{M} := \xi I_{\{\widetilde{Z}_T>0\}} I_{\{Z_{T-}>0\}} I_{[\![T,+\infty[\![} = M \qquad \text{is an } \mathbb{F}\text{-martingale.}$$

Therefore, a direct application of Theorem 2.14 (to M) allows us to conclude that M^τ satisfies the NUPBR($\mathbb{G}$). This ends the proof of (a)⇒ (b). To prove the reverse implication, we suppose that assertion (b) holds and consider

$$M := \xi I_{[\![T,+\infty[\![}, \qquad \text{where } \ \xi := I_{\{\widetilde{Z}_T=0\}} - P(\widetilde{Z}_T = 0|\mathcal{F}_{T-}).$$

From (2.8), we obtain $\{T \le \tau\} \subset \{\widetilde{Z}_T > 0\} \subset \{Z_{T-} > 0\}$ which implies that

$$M^\tau = -P(\widetilde{Z}_T = 0|\mathcal{F}_{T-}) I_{\{T\le\tau\}} I_{[\![T,+\infty[\![} \ \text{ is } \mathbb{G}\text{-predictable.}$$

Therefore, M^τ satisfies NUPBR($\mathbb{G}$) if and only if it is a constant process equal to $M_0 = 0$ (see Lemma 2.7). This is equivalent to

$$0 = E\Big[P(\widetilde{Z}_T = 0|\mathcal{F}_{T-}) I_{\{T\le\tau\}} I_{[\![T,+\infty[\![}\Big] = E\left(Z_{T-} I_{\{\widetilde{Z}_T=0 \ \&\ T<+\infty\}}\right).$$

It is obvious that this equality is equivalent to (2.9), and assertion (a) follows. This ends the proof of the theorem. □

We now state the following version of Theorem 2.14, which provides, as already said, an answer to **(P1)** in the case where there are countable many arbitrary predictable jumps. The proof of this theorem will be given in Subsection 5.3.

Theorem 2.17 *Let S be a thin process with predictable jump times only and satisfying NUPBR($\mathbb{F}$). Then, the following assertions are equivalent.*
(a) *The process S^τ satisfies NUPBR($\mathbb{G}$).*
(b) *For any $\delta > 0$, there exists a positive $\mathbb{F}$-local martingale, Y, such that ${}^{p,\mathbb{F}}(Y|\Delta S|) < +\infty$ and*

$${}^{p,\mathbb{F}}\left(Y \Delta S I_{\{\widetilde{Z}>0 \ \&\ Z_-\ge\delta\}}\right) = 0.$$

Remark 2.18 (a) Suppose that S is a thin process with predictable jumps only, satisfying NUPBR($\mathbb{F}$), and that $\{\widetilde{Z} = 0 \ \&\ Z_- > 0\} \cap \{\Delta S \neq 0\} = \emptyset$ holds. Then, S^τ satisfies NUPBR($\mathbb{G}$). This follows immediately from Theorem 2.17 by using $Y \in \mathcal{L}(S, \mathbb{F})$ and Lemma 2.4.
(b) Similarly to Proposition 2.13, we can easily prove that the thin set $\{\widetilde{Z} = 0 \ \&\ Z_- > 0\}$ is totally inaccessible if and only if X^τ satisfies NUPBR($\mathbb{G}$) for any thin process X with predictable jumps only satisfying NUPBR($\mathbb{F}$).

2.4 The General Framework

Throughout the paper, with any $\mathbb{H}$-semimartingale, X, we associate a sequence of ($\mathbb{H}$)-predictable stopping times $(T_n^X)_{n\geq 1}$ that exhaust the accessible jump times of X. Furthermore, we can decompose X as follows.

$$X = X^{(qc)} + X^{(a)},\ X^{(a)} := I_{\Gamma_X} \centerdot X,\ X^{(qc)} := X - X^{(a)},\ \Gamma_X := \bigcup_{n=1}^{\infty} [\![T_n^X]\!]. \tag{2.10}$$

The process $X^{(a)}$ (the accessible part of X) is a thin process with predictable jumps only, while $X^{(qc)}$ is a $\mathbb{H}$-quasi-left-continuous process (the quasi-left-continuous part of X).

Lemma 2.19 *Let X be an $\mathbb{H}$-semimartingale. Then X satisfies NUPBR($\mathbb{H}$) if and only if $X^{(a)}$ and $X^{(qc)}$ satisfy NUPBR($\mathbb{H}$).*

Proof Thanks to Proposition 2.3, X satisfies NUPBR($\mathbb{H}$) if and only if there exist an $\mathbb{H}$-predictable real-valued process $\phi > 0$ and a positive $\mathbb{H}$-local martingale Y such that $Y(\phi \centerdot X)$ is an $\mathbb{H}$- local martingale. Then, it is obvious that $Y(\phi I_{\Gamma_X} \centerdot X)$ and $Y(\phi I_{\Gamma_X{}^c} \centerdot X)$ are both $\mathbb{H}$-local martingales. This proves that $X^{(a)}$ and $X^{(qc)}$ both satisfy NUPBR($\mathbb{H}$).
Conversely, if $X^{(a)}$ and $X^{(qc)}$ satisfy NUPBR($\mathbb{H}$), then there exist two $\mathbb{H}$-predictable real-valued processes $\phi_1, \phi_2 > 0$ and two positive $\mathbb{H}$-local martingales $D_1 = \mathcal{E}(N_1), D_2 = \mathcal{E}(N_2)$ such that $D_1(\phi_1 \centerdot (I_{\Gamma_X} \centerdot S))$ and $D_2(\phi_2 \centerdot (I_{\Gamma_X{}^c} \centerdot X))$ are both $\mathbb{H}$-local martingales. Remark that there is no loss of generality in assuming $N_1 = I_{\Gamma_X} \centerdot N_1$ and $N_2 = I_{\Gamma_X{}^c} \centerdot N_2$. Put

$$N := I_{\Gamma_X} \centerdot N_1 + I_{\Gamma_X{}^c} \centerdot N_2 \quad \text{and} \quad \psi := \phi_1 I_{\Gamma_X} + \phi_2 I_{\Gamma_X{}^c}.$$

Obviously, $\mathcal{E}(N) > 0$; furthermore, $\mathcal{E}(N)$ and $\mathcal{E}(N)(\psi \centerdot S)$ are $\mathbb{H}$-local martingales, ψ is $\mathbb{H}$-predictable and $0 < \psi \leq 1$. This ends the proof of the lemma. □

Below, we state the answer to question **(P1)** in this general framework, which, using Lemma 2.19 will be a consequence of Theorems 2.8 and 2.14.

Theorem 2.20 *Suppose that S satisfies NUPBR($\mathbb{F}$). Then, the following assertions are equivalent.*
(a) *The process S^τ satisfies NUPBR($\mathbb{G}$).*
(b) *For any $\delta > 0$, the process*

$$I_{\{Z_- \geq \delta\}} \centerdot (S^{(qc)} - S^{(qc,0)}) := I_{\{Z_- \geq \delta\}} \centerdot (S^{(qc)} - I_{\Gamma^c} \centerdot S^{(0)})$$

satisfies NUPBR($\mathbb{F}$), and there exists a positive $\mathbb{F}$-local martingale, Y, such that ${}^{p,\mathbb{F}}(Y|\Delta S|) < +\infty$ and

$${}^{p,\mathbb{F}}\left(Y \Delta S I_{\{\widetilde{Z}>0 \ \&\ Z_- \geq \delta\}}\right) = 0.$$

Proof Due to Lemma 2.19, it is obvious that S^τ satisfies NUPBR($\mathbb{G}$) if and only if both $(S^{(qc)})^\tau$ and $(S^{(a)})^\tau$ satisfy NUPBR($\mathbb{G}$). Thus, using both Theorems 2.8 and 2.17, we deduce that this last fact is true if and only if for any $\delta > 0$, the process $I_{\{Z_- \geq \delta\}} \bullet (S^{(qc)} - I_{\Gamma^c} \bullet S^{(0)})$ satisfies NUPBR($\mathbb{F}$) and there exists a positive $\mathbb{F}$-local martingale Y such that

$$^{p,\mathbb{F}}(Y|\Delta S|) = {}^{p,\mathbb{F}}\left(Y|\Delta S^{(a)}|\right) < +\infty \quad \text{and}$$

$$^{p,\mathbb{F}}\left(Y\Delta S I_{\{\widetilde{Z}>0,\ Z_- \geq \delta\}}\right) = {}^{p,\mathbb{F}}\left(Y\Delta S^{(a)} I_{\{\widetilde{Z}>0,\ Z_- \geq \delta\}}\right) = 0.$$

This ends the proof of the theorem. □

Corollary 2.21 *The following assertions hold.*
(a) *If either m is continuous or Z is positive (equivalently $\widetilde{Z} > 0$ or $Z_- > 0$), S^τ satisfies NUPBR($\mathbb{G}$) whenever S satisfies NUPBR($\mathbb{F}$).*
(b) *If S satisfies NUPBR($\mathbb{F}$) and $\{\Delta S \neq 0\} \cap \{\widetilde{Z} = 0 < Z_-\} = \emptyset$, then S^τ satisfies NUPBR($\mathbb{G}$).*
(c) *If S is continuous and satisfies NUPBR($\mathbb{F}$), then for any random time τ, S^τ satisfies NUPBR($\mathbb{G}$).*

Proof 1) The proof of the assertion (a) of the corollary follows easily from Theorem 2.20. Indeed, in the two cases, one has $\{\widetilde{Z} = 0 < Z_-\} = \emptyset$ which implies that $\{\widetilde{Z} = 0, \delta \leq Z_-\} = \emptyset$ and $S^{(qc,0)} \equiv 0$ (due to (2.7)). Then, due to Lemma 2.4, It suffices to take $Y \in \mathcal{L}(S, \mathbb{F})$ —since this set is non-empty— and apply Theorem 2.20.
2) It is obvious that assertion (c) follows from assertion (b). To prove this latter, it is enough to remark that $\{\Delta S \neq 0\} \cap \{\widetilde{Z} = 0, \delta \leq Z_-\} = \emptyset$ implies that

$$I_{\{Z_- \geq \delta\}} \bullet S^{(qc,0)} \equiv 0 \quad \text{and} \quad \Delta S I_{\{\widetilde{Z}=0, \delta \leq Z_-\}} = 0.$$

Thus, again, it is enough to take $Y \in \mathcal{L}(S, \mathbb{F})$ and apply Theorem 2.20. This ends the proof of the corollary. □

Remark 2.22 Any of the two assertions (a) or (c) of the above corollary generalizes the main result of Fontana et al.[16], obtained under some restrictive assumptions on the random time τ and the market model as well.

Below, we provide a general answer to question **(P2)** , as a consequence of Theorems 2.8, 2.17 and Proposition 2.13.

Theorem 2.23 *The following assertions are equivalent:*
(a) *The thin set $\left\{\widetilde{Z} = 0 \ \&\ Z_- > 0\right\}$ is evanescent.*
(b) *For any (bounded) X satisfying NUPBR($\mathbb{F}$), X^τ satisfies NUPBR($\mathbb{G}$).*

Proof Suppose that assertion (a) holds, and consider a process X satisfying NUPBR($\mathbb{F}$). Then, $X^{(qc,0)} := I_{\Gamma_X^c} \bullet X^{(0)} \equiv 0$, where $X^{(0)}$ is defined as in (2.5). Hence $I_{\{Z_- \geq \delta\}} \bullet \left(X^{(qc)} - I_{\Gamma_X^c} \bullet X^{(0)}\right) = I_{\{Z_- \geq \delta\}} \bullet X^{(qc)}$ satisfies NUPBR($\mathbb{F}$)

for any $\delta > 0$, and the NUPBR($\mathbb{G}$) property for $(X^{(qc)})^\tau$ follows immediately from Theorem 2.8 on the one hand. On the other hand, it is easy to see that $X^{(a)}$ fulfills assertion (b) of Theorem 2.17 with $Y \in \mathcal{L}(X, \mathbb{F})$ due to Lemma 2.4. Thus, thanks to Theorem 2.17 (applied to the thin process $X^{(a)}$ satisfying NUPBR($\mathbb{F}$)), we conclude that $(X^{(a)})^\tau$ satisfies NUPBR($\mathbb{G}$). Thus, due to Lemma 2.19, the proof of (a)⇒(b) is completed.
We now suppose that assertion (b) holds. On the one hand, from Proposition 2.13, we deduce that $\{\widetilde{Z} = 0 < Z_-\}$ is accessible and can be covered with the graphs of $\mathbb{F}$-predictable stopping times $(T_n)_{n\geq 1}$. On the other hand, a direct application of Proposition 2.16 to all single predictable jump $\mathbb{F}$-martingales, we obtain $\{\widetilde{Z} = 0 < Z_-\} \cap [\![T]\!] = \emptyset$ for any $\mathbb{F}$-predictable stopping time T. Therefore, we get

$$\{\widetilde{Z} = 0 < Z_-\} = \bigcup_{n=1}^{\infty} \Big(\{\widetilde{Z} = 0 < Z_-\} \cap [\![T_n]\!]\Big) = \emptyset.$$

This proves assertion (a), and the proof of the theorem is completed. □

3 Stochastics from–and–for Informational Non-Arbitrage

In this section, we develop new stochastic results that will play a key role in the proofs and/or the statements of the main results outlined in the previous section. The first subsection compares the $\mathbb{G}$-compensators and the $\mathbb{F}$-compensators, while the second subsection studies a $\mathbb{G}$-martingale that is vital in the explicit construction of deflators. We recall that $Z_- + \Delta m = \widetilde{Z}$ (see [23]).

Lemma 3.1 *Let Z and $\widetilde{Z}$ be the two supermartingales given by (2.1).*
(a) *The three sets $\{\widetilde{Z} = 0\}$, $\{Z = 0\}$ and $\{Z_- = 0\}$ have the same début which is an $\mathbb{F}$-stopping time that we denote by*

$$\widehat{R} := \inf\{t \geq 0 \big|\ Z_{t-} = 0\}.$$

(b) *The following $\mathbb{F}$-stopping times*

$$\widehat{R}_0 := \begin{cases} \widehat{R} & \text{on } \{Z_{\widehat{R}-} = 0\} \\ +\infty & \text{otherwise} \end{cases} \qquad \text{and} \qquad \widetilde{R}_0 := \begin{cases} \widehat{R} & \text{on } \{\widetilde{Z}_{\widehat{R}} = 0\} \\ +\infty & \text{otherwise} \end{cases}$$

are such that $\widehat{R}_0$ is an $\mathbb{F}$-predictable stopping time, and

$$\tau \leq \widehat{R}, \qquad \tau < \widetilde{R}_0, \qquad P - a.s.$$

(c) *The $\mathbb{G}$-predictable process*

$$H_t := (Z_{t-})^{-1} I_{[\![0,\tau]\!]}(t), \tag{3.1}$$

is $\mathbb{G}$-locally bounded.

Proof From [14, paragraph 14, Chapter XX], for any random time τ, the sets $\{\widetilde{Z} = 0\}$ and $\{Z_- = 0\}$ are disjoint from $]\!]0,\tau]\!]$ and have the same lower bound $\widehat{R}$, the smallest $\mathbb{F}$-stopping time greater than τ. Thus, we also conclude that $\{Z = 0\}$ is disjoint from $]\!]0,\tau[\![$. This leads to assertion (a). The process $X := Z^{-1} I_{]\!]0,\tau[\![}$ being a càdlàg $\mathbb{G}$-supermartingale [39], its left limit is locally bounded. Then, due to

$$(Z_-)^{-1} I_{]\!]0,\tau]\!]} = X_-,$$

the local boundedness of H follows. This ends the proof of the lemma. □

3.1 Exact Relationship between Dual Predictable Projections under $\mathbb{G}$ and $\mathbb{F}$

The main results of this subsection are summarized in Lemmas 3.3 and 3.4, where we address the question of how to compute $\mathbb{G}$-dual predictable projections in term of $\mathbb{F}$-dual predictable projections and vice versa. These results are based essentially on the following standard result on progressive enlargement of filtration (we refer the reader to [14,22] for proofs).

Proposition 3.2 *Let M be an $\mathbb{F}$-local martingale. Then, for any random time τ, the process*

$$\widehat{M}_t := M^\tau_t - \int_0^{t\wedge\tau} \frac{d\langle M, m\rangle^{\mathbb{F}}_s}{Z_{s-}} \tag{3.2}$$

is a $\mathbb{G}$-local martingale, where m is defined in (2.2).

In the following lemma, we express the $\mathbb{G}$–dual predictable projection of an $\mathbb{F}$-locally integrable variation process in terms of an $\mathbb{F}$–dual predictable projection, and $\mathbb{G}$-predictable projection in terms of $\mathbb{F}$-predictable projection.

Lemma 3.3 *The following assertions hold.*
(a) *For any $\mathbb{F}$-adapted process V with locally integrable variation, we have*

$$(V^\tau)^{p,\mathbb{G}} = (Z_-)^{-1} I_{]\!]0,\tau]\!]} \bullet \left(\widetilde{Z} \bullet V\right)^{p,\mathbb{F}}. \tag{3.3}$$

(b) *For any $\mathbb{F}$-local martingale M, we have, on $[\![0,\tau]\!]$*

$${}^{p,\mathbb{G}}\left(\frac{\Delta M}{\widetilde{Z}}\right) = \frac{{}^{p,\mathbb{F}}\left(\Delta M I_{\{\widetilde{Z}>0\}}\right)}{Z_-}, \quad \textit{and} \quad {}^{p,\mathbb{G}}\left(\frac{1}{\widetilde{Z}}\right) = \frac{{}^{p,\mathbb{F}}\left(I_{\{\widetilde{Z}>0\}}\right)}{Z_-}. \tag{3.4}$$

(c) *For any quasi-left-continuous $\mathbb{F}$-local martingale M, we have, on $[\![0,\tau]\!]$*

$${}^{p,\mathbb{G}}\left(\frac{\Delta M}{\widetilde{Z}}\right) = 0, \quad \textit{and} \quad {}^{p,\mathbb{G}}\left(\frac{1}{Z_- + \Delta m^{(qc)}}\right) = \frac{1}{Z_-}, \tag{3.5}$$

where $m^{(qc)}$ is the quasi-left-continuous $\mathbb{F}$-martingale defined in (2.10).

Proof (a) Using the notation (3.1), the equality (3.2) takes the form

$$M^\tau = \widehat{M} + HI_{\rrbracket 0,\tau\rrbracket} \centerdot \langle M, m\rangle^{\mathbb{F}}.$$

By taking $M = V - V^{p,\mathbb{F}}$, we obtain

$$V^\tau = I_{\rrbracket 0,\tau\rrbracket}\centerdot V^{p,\mathbb{F}} + \widehat{M} + HI_{\rrbracket 0,\tau\rrbracket}\centerdot\langle V, m\rangle^{\mathbb{F}} = \widehat{M} + I_{\rrbracket 0,\tau\rrbracket}\centerdot V^{p,\mathbb{F}} + \frac{1}{Z_-} I_{\rrbracket 0,\tau\rrbracket}\centerdot(\Delta m \centerdot V)^{p,\mathbb{F}},$$

which, using the fact that $Z_- + \Delta m = \widetilde{Z}$, proves assertion (a).
(b) Let M be an $\mathbb{F}$-local martingale, then, for any positive integers (n,k) the process $V^{(n,k)} := \sum \frac{\Delta M}{\widetilde{Z}} I_{\{|\Delta M|\geq k^{-1},\ \widetilde{Z}\geq n^{-1}\}}$ has a locally integrable variation. Then, by using the known equality ${}^{p,\mathbb{G}}(\Delta V) = \Delta(V^{p,\mathbb{G}})$ (see Theorem 76 in pages 149–150 of [13] or Theorem 5.27 in page 150 of [18]), and applying assertion (a) to the process $V^{(n,k)}$, we get, on $\rrbracket 0,\tau\rrbracket$

$${}^{p,\mathbb{G}}\left(\frac{\Delta M}{\widetilde{Z}} I_{\{|\Delta M|\geq k^{-1},\ \widetilde{Z}\geq n^{-1}\}}\right) = \frac{1}{Z_-}\ {}^{p,\mathbb{F}}\left(\Delta M I_{\{|\Delta M|\geq k^{-1},\ \widetilde{Z}\geq n^{-1}\}}\right).$$

Since M is a local martingale, by stopping we can exchange limits with projections in both sides. Then by letting n and k go to infinity, and using the fact that $\widetilde{Z} > 0$ on $\rrbracket 0,\tau\rrbracket$, we deduce that

$${}^{p,\mathbb{G}}\left(\frac{\Delta M}{\widetilde{Z}}\right) = \frac{1}{Z_-}\ {}^{p,\mathbb{F}}\left(\Delta M I_{\{\widetilde{Z}>0\}}\right).$$

This proves the first equality in (3.4), while the second equality follows from $\widetilde{Z} = \Delta m + Z_-$:

$$\begin{aligned} Z_-\ {}^{p,\mathbb{G}}\left(\widetilde{Z}^{-1}\right) &= {}^{p,\mathbb{G}}\left((\widetilde{Z} - \Delta m)/\widetilde{Z}\right) = 1 - {}^{p,\mathbb{G}}\left(\Delta m/\widetilde{Z}\right) \\ &= 1 - (Z_-)^{-1}\ {}^{p,\mathbb{F}}\left(\Delta m I_{\{\widetilde{Z}>0\}}\right) = 1 - {}^{p,\mathbb{F}}\left(I_{\{\widetilde{Z}=0\}}\right) = {}^{p,\mathbb{F}}\left(I_{\{\widetilde{Z}>0\}}\right). \end{aligned}$$

In the above string of equalities, the third equality follows from the first equality in (3.4), while the fourth equality is due to ${}^{p,\mathbb{F}}(\Delta m) = 0$ and $\Delta m I_{\{\widetilde{Z}=0\}} = -Z_- I_{\{\widetilde{Z}=0\}}$. This ends the proof of assertion (b).

(c) If M is a quasi-left-continuous $\mathbb{F}$-local martingale, then ${}^{p,\mathbb{F}}\left(\Delta M I_{\{\widetilde{Z}>0\}}\right) = 0$, and the first property of the assertion (c) follows. Applying the first property to $M = m^{(qc)}$ and using that, on $\rrbracket 0,\tau\rrbracket$, one has $\Delta m^{(qc)}\left(Z_- + \Delta m\right)^{-1} = \Delta m^{(qc)}\left(Z_- + \Delta m^{(qc)}\right)^{-1}$, we obtain

$$\frac{1}{Z_-}\ {}^{p,\mathbb{G}}\left(\frac{Z_-}{Z_- + \Delta m^{(qc)}}\right) = \frac{1}{Z_-}\left(1 - {}^{p,\mathbb{G}}\left(\frac{\Delta m^{(qc)}}{Z_- + \Delta m^{(qc)}}\right)\right) = \frac{1}{Z_-}.$$

This proves assertion (c), and the proof of the lemma is achieved. □

The next lemma proves that $\widetilde{Z}^{-1}I_{]\!]0,\tau]\!]}$ is Lebesgue-Stieljes-integrable with respect to any process that is $\mathbb{F}$-adapted with $\mathbb{F}$-locally integrable variation. Using this fact, the lemma addresses the question of how an $\mathbb{F}$-compensator stopped at τ can be written in terms of a $\mathbb{G}$-compensator, and constitutes a *sort of* converse result to Lemma 3.3–(a).

Lemma 3.4 *Let V be an $\mathbb{F}$-adapted càdlàg process. Then the following properties hold.*
(a) *If V belongs to $\mathcal{A}^+_{loc}(\mathbb{F})$ (respectively $V \in \mathcal{A}^+(\mathbb{F})$), then the process*

$$U := \widetilde{Z}^{-1}I_{]\!]0,\tau]\!]} \bullet V, \tag{3.6}$$

belongs to $\mathcal{A}^+_{loc}(\mathbb{G})$ (respectively to $\mathcal{A}^+(\mathbb{G})$).
(b) *If V has $\mathbb{F}$-locally integrable variation, then the process U is well defined, its variation is $\mathbb{G}$-locally integrable, and its $\mathbb{G}$-dual predictable projection is given by*

$$U^{p,\mathbb{G}} = \left(\frac{1}{\widetilde{Z}}I_{]\!]0,\tau]\!]} \bullet V\right)^{p,\mathbb{G}} = \frac{1}{Z_-}I_{]\!]0,\tau]\!]} \bullet \left(I_{\{\widetilde{Z}>0\}} \bullet V\right)^{p,\mathbb{F}}. \tag{3.7}$$

In particular, if supp$V \subset \{\widetilde{Z} > 0\}$, then, on $]\!]0,\tau]\!]$, one has $V^{p,\mathbb{F}} = Z_- \bullet U^{p,\mathbb{G}}$.

Proof (a) Suppose that $V \in \mathcal{A}^+_{loc}(\mathbb{F})$. First, remark that, due to the fact that $\widetilde{Z}$ is positive on $]\!]0,\tau]\!]$, U is well defined. Let $(\vartheta_n)_{n\geq 1}$ be a sequence of $\mathbb{F}$-stopping times that increases to $+\infty$ such that $E\left(V_{\vartheta_n}\right) < +\infty$. Then, if $E\left(U_{\vartheta_n}\right) \leq E\left(V_{\vartheta_n}\right)$, assertion (a) follows. Thus, we calculate

$$\begin{aligned} E\left(U_{\vartheta_n}\right) &= E\left(\int_0^{\vartheta_n} I_{\{0<t\leq\tau\}}\frac{1}{\widetilde{Z}_t}dV_t\right) = E\left(\int_0^{\vartheta_n}\frac{P(\tau\geq t|\mathcal{F}_t)}{\widetilde{Z}_t}I_{\{\widetilde{Z}_t>0\}}dV_t\right) \\ &\leq E\left(V_{\vartheta_n}\right). \end{aligned}$$

The last inequality is obtained due to $\widetilde{Z}_t := P(\tau \geq t|\mathcal{F}_t)$. This ends the proof of assertion (a) of the lemma.
(b) Suppose that $V \in \mathcal{A}_{loc}(\mathbb{F})$, and denote by $W := V^+ + V^-$ its variation. Then $W \in \mathcal{A}^+_{loc}(\mathbb{F})$, and a direct application of the first assertion implies that

$$\left(\widetilde{Z}\right)^{-1} I_{]\!]0,\tau]\!]} \bullet W \in \mathcal{A}^+_{loc}(\mathbb{G}).$$

As a result, we deduce that U given by (3.6) for the case of $V = V^+ - V^-$ is well defined and has variation equal to $\left(\widetilde{Z}\right)^{-1} I_{]\!]0,\tau]\!]} \bullet W$ which is $\mathbb{G}$-locally integrable. By setting $U_n := I_{]\!]0,\tau]\!]} \bullet \left(\widetilde{Z}^{-1}I_{\{\widetilde{Z}\geq 1/n\}} \bullet V\right)$, we derive, due to (3.3),

$$(U_n)^{p,\mathbb{G}} = \frac{1}{Z_-}I_{]\!]0,\tau]\!]} \bullet \left(I_{\{\widetilde{Z}\geq 1/n\}} \bullet V\right)^{p,\mathbb{F}}.$$

Hence, since $U^{p,\mathbb{G}} = \lim_{n\longrightarrow+\infty}(U_n)^{p,\mathbb{G}}$, by taking the limit in the above equality, (3.7) follows immediately, and the lemma is proved. □

3.2 An Important $\mathbb{G}$- local martingale

In this subsection, we will introduce a $\mathbb{G}$- local martingale that will be crucial for the construction of the deflator.

Lemma 3.5 *The following nondecreasing process*

$$V_t^{\mathbb{G}} := \sum_{0\leq u\leq t} {}^{p,\mathbb{F}}\left(I_{\{\widetilde{Z}=0\}}\right)_u I_{\{u\leq\tau\}} \tag{3.8}$$

is $\mathbb{G}$-predictable, càdlàg, and locally bounded.

Proof The $\mathbb{G}$-predictability of $V^{\mathbb{G}}$ being obvious, it remains to prove that this process is $\mathbb{G}$-locally bounded. Since $Z_-^{-1}I_{\rrbracket 0,\tau\rrbracket}$ is $\mathbb{G}$-locally bounded, then there exists a sequence of $\mathbb{G}$-stopping times $(\tau_n^{\mathbb{G}})_{n\geq 1}$ increasing to infinity such that

$$\left(\frac{1}{Z_-}I_{\rrbracket 0,\tau\rrbracket}\right)^{\tau_n^{\mathbb{G}}} \leq n+1.$$

Consider a sequence of $\mathbb{F}$-stopping times $(\sigma_n)_{n\geq 1}$ that increases to infinity such that $\langle m,m\rangle_{\sigma_n} \leq n+1$. Then, for any nonnegative $\mathbb{F}$-predictable process H which is bounded by $C>0$, we calculate that

$$\begin{aligned}(H\centerdot V^{\mathbb{G}})_{\sigma_n\wedge\tau_n^{\mathbb{G}}} &= \sum_{0\leq u\leq\sigma_n\wedge\tau_n^{\mathbb{G}}} H_u\ {}^{p,\mathbb{F}}\left(I_{\{\widetilde{Z}=0\}}\right)_u I_{\{u\leq\tau\}}I_{\{Z_{u-}\geq\frac{1}{n+1}\}}\\ &\leq \sum_{0\leq u\leq\sigma_n} H_u\ {}^{p,\mathbb{F}}\left(I_{\{\Delta m\leq-\frac{1}{n+1}\}}\right)_u\\ &\leq (n+1)^2 H\centerdot\langle m,m\rangle_{\sigma_n} \leq C(n+1)^3.\end{aligned}$$

This ends the proof of the proposition. □

The important $\mathbb{G}$-local martingale will result from an optional integral. For the notion of compensated stochastic integral (or optional stochastic integral), we refer the reader to [20] (Chapter III.4.b p. 106-109) and [13] (Chapter VIII.2 sections 32-35 p. 356-361). Below, for the sake of completeness, we give the definition of this integration.

Definition 3.6 (see [20], Definition (3.80)) Let N be an $\mathbb{H}$-local martingale with continuous martingale part N^c, and let H be an $\mathbb{H}$-optional process.
i) The process H is said to be integrable with respect to N if ${}^{p,\mathbb{H}}H$ is N^c integrable, ${}^{p,\mathbb{H}}(|H\Delta N|) < +\infty$ and the process

$$\left(\sum_{s\leq t}\left(H_s\Delta N_s - {}^{p,\mathbb{H}}(H\Delta N)_s\right)^2\right)^{1/2}$$

is locally integrable. The set of integrable processes with respect to N is denoted by ${}^oL^1_{loc}(N,\mathbb{H})$.

ii) For $H \in {}^{o}L^{1}_{loc}(N, \mathbb{H})$, the compensated stochastic integral of H with respect to N, denoted by $H \odot N$, is the unique local martingale M which satisfies

$$M^c = {}^{p,\mathbb{H}}H \bullet N^c \quad \text{and} \quad \Delta M = H\Delta N - {}^{p,\mathbb{H}}(H\Delta N).$$

Among the most useful results of the literature involving this integral is the following

Proposition 3.7 *(see [13])* (a) *The compensated stochastic integral $M = H \odot N$ is the unique $\mathbb{H}$-local martingale such that, for any $\mathbb{H}$-local martingale Y,*

$$\mathbb{E}\left([M,Y]_\infty\right) = \mathbb{E}\left(\int_0^\infty H_s d[N,Y]_s\right).$$

(b) *The process $[M,Y] - H \bullet [N,Y]$ is an $\mathbb{H}$-local martingale. As a result $[M,Y] \in \mathcal{A}_{loc}(\mathbb{H})$ if and only if $H \bullet [N,Y] \in \mathcal{A}_{loc}(\mathbb{H})$ and in this case we have*

$$\langle M,Y\rangle^{\mathbb{H}} = (H \bullet [N,Y])^{p,\mathbb{H}}.$$

Now, we are in the stage of defining the $\mathbb{G}$-local martingale which will play the role of deflator for a class of processes.

Proposition 3.8 *Consider the following $\mathbb{G}$-local martingale*

$$\widehat{m} := I_{\rrbracket 0,\tau\rrbracket} \bullet m - \frac{1}{Z_-} I_{\rrbracket 0,\tau\rrbracket} \bullet \langle m\rangle^{\mathbb{F}},$$

and the process

$$K := \frac{Z_-^2}{Z_-^2 + \Delta\langle m\rangle^{\mathbb{F}}} \frac{1}{\widetilde{Z}} I_{\rrbracket 0,\tau\rrbracket}. \tag{3.9}$$

Then, K belongs to the space ${}^{o}L^{1}_{loc}(\widehat{m}, \mathbb{G})$ defined in 3.6. Furthermore, the $\mathbb{G}$-local martingale

$$L := -K \odot \widehat{m}, \tag{3.10}$$

satisfies the following
(a) $\mathcal{E}(L) > 0$ *(or equivalently $1 + \Delta L > 0$).*
(b) *For any $M \in \mathcal{M}_{0,loc}(\mathbb{F})$, setting $\widehat{M} := M^\tau - Z_-^{-1} I_{\rrbracket 0,\tau\rrbracket} \bullet \langle M, m\rangle^{\mathbb{F}}$, we have*

$$[L, \widehat{M}] \in \mathcal{A}_{loc}(\mathbb{G}) \quad \left(\textit{i.e. } \langle L, \widehat{M}\rangle^{\mathbb{G}} \ \textit{ exists}\right). \tag{3.11}$$

Proof We shall prove that $K \in {}^{o}L^{1}_{loc}(\widehat{m}, \mathbb{G})$ in the appendix B. For the sake of simplicity in notations, throughout this proof, we will use $\kappa := Z_-^2 + \Delta\langle m\rangle^{\mathbb{F}}$. We now prove assertions (a) and (b). Due to (B.1), we have, on $\rrbracket 0, \tau\rrbracket$,

$$-\Delta L = K\Delta\widehat{m} - {}^{p,\mathbb{G}}(K\Delta\widehat{m}) = 1 - Z_-\left(\widetilde{Z}\right)^{-1} - {}^{p,\mathbb{F}}\left(I_{\{\widetilde{Z}=0\}}\right).$$

Thus, we deduce that $1+\Delta L>0$, and assertion (a) is proved. In the rest of this proof, we will prove (3.11). To this end, let $M\in\mathcal{M}_{0,loc}(\mathbb{F})$. Thanks to Proposition 3.7, (3.11) is equivalent to

$$K\bullet[\widehat{m},\widehat{M}]\in\mathcal{A}_{loc}(\mathbb{G}) \text{ (or equivalently } \frac{1}{\widetilde{Z}}I_{[\![0,\tau]\!]}\bullet[\widehat{m},\widehat{M}]\in\mathcal{A}_{loc}(\mathbb{G})),$$

for any $M\in\mathcal{M}_{0,loc}(\mathbb{F})$. Then, it is easy to check that

$$\begin{aligned}\frac{1}{\widetilde{Z}}I_{[\![0,\tau]\!]}\bullet[\widehat{m},\widehat{M}] &= \frac{1}{\widetilde{Z}}I_{[\![0,\tau]\!]}\bullet[m,\widehat{M}]-\frac{1}{Z_-\widetilde{Z}}I_{[\![0,\tau]\!]}\bullet[\langle m\rangle^{\mathbb{F}},\widehat{M}]\\ &= \frac{1}{\widetilde{Z}}I_{[\![0,\tau]\!]}\bullet[m,M]-\frac{1}{Z_-\widetilde{Z}}I_{[\![0,\tau]\!]}\bullet[m,\langle M,m\rangle^{\mathbb{F}}]\\ &\quad -\frac{1}{Z_-\widetilde{Z}}I_{[\![0,\tau]\!]}\bullet[\langle m\rangle^{\mathbb{F}},M]+\frac{1}{Z_-^2\widetilde{Z}}I_{[\![0,\tau]\!]}\bullet[\langle m\rangle^{\mathbb{F}},\langle M,m\rangle^{\mathbb{F}}].\end{aligned}$$

Since m is an $\mathbb{F}$-locally bounded local martingale, all the processes

$$[m,M],\ [m,\langle M,m\rangle^{\mathbb{F}}],\ [\langle m\rangle^{\mathbb{F}},M],\ \text{and } [\langle m\rangle^{\mathbb{F}},\langle M,m\rangle^{\mathbb{F}}]$$

belong to $\mathcal{A}_{loc}(\mathbb{F})$. Thus, by combining this fact with Lemma 3.4 and the $\mathbb{G}$-local boundedness of $Z_-^{-p}I_{[\![0,\tau]\!]}$ for any $p>0$, the result follows. This ends the proof of the proposition. □

4 Explicit Deflators

This section describes some classes of $\mathbb{F}$-quasi-left-continuous local martingales for which the NUPBR is preserved after stopping at τ. For these stopped processes, we describe explicitly their local martingale densities in Theorems 4.1–4.3 with an increasing degree of generality. We recall that $m^{(qc)}$ is defined in (2.10) and L is defined in Proposition 3.8.

Theorem 4.1 *Suppose S is a quasi-left-continuous. Then the following hold.*
(1)If S is an $\mathbb{F}$-local martingale such that (S,τ) satisfies

$$\{\Delta S\neq 0\}\cap\{Z_->0\}\cap\{\widetilde{Z}=0\}=\emptyset, \tag{4.1}$$

then the following equivalent assertions hold
(a) $\mathcal{E}(L)S^{\tau}$ *is a $\mathbb{G}$-local martingale.*
(b) $\mathcal{E}\left(I_{\{\widetilde{Z}=0<Z_-\}}\odot m^{(qc)}\right)S$ *is an $\mathbb{F}$-local martingale.*
(2) If S satisfies NUPBR($\mathbb{F}$) and (4.1) holds, then S^{τ} satisfies NUPBR($\mathbb{G}$)

Proof We start by giving some useful observations. Since S is $\mathbb{F}$-quasi-left-continuous, on the one hand we deduce that (Γ_m is defined in (2.10))

$$\langle S,m\rangle^{\mathbb{F}}=\langle S,m^{(qc)}\rangle^{\mathbb{F}}=\langle S,I_{\Gamma_m^c}\bullet m\rangle^{\mathbb{F}}. \tag{4.2}$$

On the other hand, we note that assertion (a) is equivalent to $\mathcal{E}(L^{(qc)})S^\tau$ is a $\mathbb{G}$-local martingale, where $L^{(qc)}$ is the quasi-left-continuous local martingale part of L given by $L^{(qc)} := I_{\Gamma_m^c} \cdot L = -K \odot \widehat{m}^{(qc)}$. Here K is given in Proposition 3.8 and

$$\widehat{m}^{(qc)} := I_{\rrbracket 0,\tau \rrbracket} \cdot m^{(qc)} - (Z_-) I_{\rrbracket 0,\tau \rrbracket} \cdot \langle m^{(qc)} \rangle^{\mathbb{F}}.$$

It is easy to check that (4.1) is equivalent to

$$I_{\{Z_- > 0 \ \& \ \widetilde{Z}=0\}} \cdot [S,m] = 0. \tag{4.3}$$

We now compute $-\langle L^{(qc)}, \widehat{S} \rangle^{\mathbb{G}}$, where $\widehat{S}$ is the $\mathbb{G}$-local martingale given by

$$\widehat{S} := S^\tau - (Z_-)^{-1} I_{\rrbracket 0,\tau \rrbracket} \cdot \langle S,m \rangle^{\mathbb{F}}.$$

Due to the quasi-left continuity of S and that of $m^{(qc)}$, the two processes $\langle S,m \rangle^{\mathbb{F}}$ and $\langle m^{(qc)} \rangle^{\mathbb{F}}$ are continuous and $[S, m^{(qc)}] = [S,m]$. Hence, we obtain

$$\begin{aligned} K \cdot [\widehat{S}, \widehat{m}^{(qc)}] &= K \cdot [S, \widehat{m}^{(qc)}] - K \Delta \widehat{m}^{(qc)} (Z_-)^{-1} \cdot \langle S,m \rangle^{\mathbb{F}} \\ &= (\widetilde{Z})^{-1} I_{\rrbracket 0,\tau \rrbracket} \cdot [S, m^{(qc)}] = (\widetilde{Z})^{-1} I_{\rrbracket 0,\tau \rrbracket} \cdot [S,m]. \end{aligned}$$

It follows that

$$\begin{aligned} -\langle L^{(qc)}, \widehat{S} \rangle^{\mathbb{G}} &= \Big(K \cdot [\widehat{S}, \widehat{m}^{(qc)}] \Big)^{p,\mathbb{G}} = \Big((\widetilde{Z})^{-1} I_{\rrbracket 0,\tau \rrbracket} \cdot [S,m] \Big)^{p,\mathbb{G}} \\ &= (Z_-)^{-1} I_{\rrbracket 0,\tau \rrbracket} \cdot \Big(I_{\{\widetilde{Z}>0\}} \cdot [S,m] \Big)^{p,\mathbb{F}} \\ &= (Z_-)^{-1} I_{\rrbracket 0,\tau \rrbracket} \cdot \langle S,m \rangle^{\mathbb{F}} - (Z_-)^{-1} I_{\rrbracket 0,\tau \rrbracket} \cdot \Big(I_{\{\widetilde{Z}=0<Z_-\}} \cdot [S,m] \Big)^{p,\mathbb{F}} \\ &= (Z_-)^{-1} I_{\rrbracket 0,\tau \rrbracket} \cdot \langle S,m \rangle^{\mathbb{F}} + (Z_-)^{-1} I_{\rrbracket 0,\tau \rrbracket} \cdot \langle S, -I_{\{\widetilde{Z}=0<Z_-\}} \odot m^{(qc)} \rangle^{\mathbb{F}}. \end{aligned} \tag{4.4}$$

The first and the last equality follow from Proposition 3.7 applied to $L^{(qc)}$ and $-I_{\{\widetilde{Z}=0<Z_-\}} \odot m^{(qc)}$ respectively. The second and the third equalities are due to (4.2) and (3.7) respectively.

Now, we prove the theorem. Thanks to (4.4), it is obvious that assertion (a) is equivalent to $\langle S, -I_{\{\widetilde{Z}=0<Z_-\}} \odot m^{(qc)} \rangle^{\mathbb{F}} \equiv 0$ which in turn is equivalent to assertion (b). This ends the proof of the equivalence between (a) and (b).
It is also clear that the condition (4.1) or equivalently (4.3) implies assertion (b), due to $\langle I_{\{\widetilde{Z}=0<Z_-\}} \odot m^{(qc)}, S \rangle^{\mathbb{F}} = \Big(I_{\{\widetilde{Z}=0<Z_-\}} \cdot [m,S] \Big)^{p,\mathbb{F}} \equiv 0$.
This ends the proof of assertion (1).
The proof of assertion (2) follows from combining Proposition 2.6, assertion (1), and the fact that, for any probability measure Q equivalent to P, we have

$$\{Z_- > 0\} \cap \{\widetilde{Z} = 0\} = \{Z_-^Q > 0\} \cap \{\widetilde{Z}^Q = 0\}.$$

Here $Z_t^Q = Q(\tau > t | \mathcal{F}_t)$ and $\widetilde{Z}_t^Q = Q(\tau \geq t | \mathcal{F}_t)$. This last claim is a direct application of the optional and predictable selection measurable theorems, see Theorems 84 and 85 (or apply Theorem 86 directly) in [13]. □

□

In order to generalize the previous result, we need to introduce more notations and recall others notations and some results that are delegated in the Appendix. We associate to the random measure μ defined in (2.4), its predictable compensator random measure ν. A direct application of Theorem A.1 (in Appendix), to the martingale m, leads to the existence of a local martingale $m^{\perp}$ as well as a $\widetilde{\mathcal{P}}(\mathbb{F})$-measurable functional f_m, a process $\beta_m \in L(S^c, \mathbb{F})$ and an $\widetilde{\mathcal{O}}(\mathbb{F})$-measurable functional g_m such that $f_m \in \mathcal{G}^1_{loc}(\mu, \mathbb{F})$, $g_m \in \mathcal{H}^1_{loc}(\mu, \mathbb{F})$ and $\beta_m \in L(S^c)$ such that

$$m = \beta_m \bullet S^c + f_m \star (\mu - \nu) + g_m \star \mu + m^{\perp}. \tag{4.5}$$

Definition 4.2 For a quasi-left-continuous process, $\mathcal{G}^1_{loc}(\mu, \mathbb{F})$ (respectively $\mathcal{H}^1_{loc}(\mu, \mathbb{F})$) is the set of all $\widetilde{\mathcal{P}}(\mathbb{F})$-measurable functions (respectively all $\widetilde{\mathcal{O}}(\mathbb{F})$-measurable functions) W such that

$$\sqrt{W^2 \star \mu} \in \mathcal{A}^+_{loc}(\mathbb{H}).$$

we introduce $\mu^{\mathbb{G}} := I_{[\![0,\tau]\!]} \star \mu$ and its $\mathbb{G}$ compensated measure

$$\nu^{\mathbb{G}}(dt, dx) := (1 + f_m(x)/Z_{t-}) I_{[\![0,\tau]\!]}(t) \nu(dt, dx). \tag{4.6}$$

Below, we state our general result that extend the previous theorem.

Theorem 4.3 *Suppose that S is an $\mathbb{F}$-quasi-left-continuous local martingale. Consider $S^{(0)}$, ψ, and L defined in (2.5) and (3.10) respectively. If $\left(S, S^{(0)}\right)$ is an $\mathbb{F}$-local martingale, then $\mathcal{E}\left(L + L^{(1)}\right) S^{\tau}$ is a $\mathbb{G}$-local martingale, where*

$$L^{(1)} := g_1 \star (\mu^{\mathbb{G}} - \nu^{\mathbb{G}}), \quad \textit{and} \quad g_1 := \frac{1 - \psi}{1 + f_m/Z_-} I_{\{\psi > 0\}}.$$

Proof We start by recalling from (2.7) that $\{\psi = 0\} = \{Z_- + f_m = 0\}$ M^P_μ − *a.e.*. Thus the functional g_1 is a well defined non-negative $\widetilde{\mathcal{P}}(\mathbb{F})-$measurable functional. The proof of the theorem will be completed in two steps. In the first step we prove that the process $L^{(1)}$ is a well defined local martingale, while in the second step we prove the main statement of the theorem.
1) Herein, we prove that the integral $g_1 \star \left(\mu^{\mathbb{G}} - \nu^{\mathbb{G}}\right)$ is well-defined. To this end, it is enough to prove that $g_1 \star \mu^{\mathbb{G}} \in \mathcal{A}^+(\mathbb{G})$. Therefore, remark that

$$(1 - \psi) I_{\{0 < Z_-\}} = M^P_\mu \left(I_{\{\widetilde{Z} = 0 < Z_-\}} | \widetilde{\mathcal{P}}(\mathbb{F}) \right) = M^P_\mu \left(I_{[\![\widetilde{R}_0]\!]} | \widetilde{\mathcal{P}}(\mathbb{F}) \right) I_{\{0 < Z_-\}},$$

where $\widetilde{R}_0$ is defined in Lemma 3.1(b) and calculate

$$\begin{aligned} E\left(g_1 \star \mu^{\mathbb{G}}(\infty)\right) &= E\left(g_1 \widetilde{Z} \star \mu(\infty)\right) \\ &\leq E\left(I_{[\![\widetilde{R}_0]\!]} \star \mu(\infty)\right) = P\left(\Delta S_{\widetilde{R}_0} \neq 0 \ \& \ \widetilde{R}_0 < +\infty\right) \leq 1. \end{aligned}$$

Thus, the process $L^{(1)}$ is a well defined $\mathbb{G}$-martingale.
2) In this part, we prove that $\mathcal{E}\left(L+L^{(1)}\right)S^\tau$ is a $\mathbb{G}$-local martingale. To this end, it is enough to prove that $\langle S^\tau, L+L^{(1)}\rangle^{\mathbb{G}}$ exists and

$$S^\tau + \left\langle S^\tau, L + g_1 \star \left(\mu^{\mathbb{G}} - \nu^{\mathbb{G}}\right)\right\rangle^{\mathbb{G}} \quad \text{is a } \mathbb{G}\text{-local martingale.} \tag{4.7}$$

Recall that

$$L = -\frac{Z_-^2}{Z_-^2 + \Delta\langle m\rangle^{\mathbb{F}}}\,\frac{1}{\widetilde{Z}}\, I_{\rrbracket 0,\tau\rrbracket} \odot \widehat{m},$$

and hence $\langle S^\tau, L\rangle^{\mathbb{G}}$ exists due to Proposition 3.8–(b). By stopping, there is no loss of generality in assuming that S is a true martingale. Then, using similar calculation as in the first part, we can easily prove that

$$E\left[|x| g_1 \star \mu^{\mathbb{G}}(\infty)\right] \leq E\left(|\Delta S_{\widetilde{R}_0}| I_{\{\widetilde{R}_0 < +\infty\}}\right) < +\infty.$$

This proves that $\left\langle S^\tau, L + L^{(1)}\right\rangle^{\mathbb{G}}$ exists. Now, we calculate and simplify the expression in (4.7) as follows.

$$\begin{aligned}
S^\tau + \left\langle S^\tau, L + g_1 \star \left(\mu^{\mathbb{G}} - \nu^{\mathbb{G}}\right)\right\rangle^{\mathbb{G}} &= \widehat{S} + \frac{1}{Z_-} I_{\rrbracket 0,\tau\rrbracket} \centerdot \langle S, m\rangle^{\mathbb{F}} + \langle S^\tau, L\rangle^{\mathbb{G}} + x g_1 \star \nu^{\mathbb{G}} \\
&= \widehat{S} + \frac{1}{Z_-} I_{\rrbracket 0,\tau\rrbracket} \centerdot \langle S, m\rangle - \frac{1}{Z_-} I_{\rrbracket 0,\tau\rrbracket} \cdot \left(I_{\{\widetilde{Z}>0\}} \cdot [S,m]\right)^{p,\mathbb{F}} + x g_1 \star \nu^{\mathbb{G}} \\
&= \widehat{S} + \frac{1}{Z_-} I_{\rrbracket 0,\tau\rrbracket} \centerdot \left(I_{\{\widetilde{Z}=0\}} \centerdot [S,m]\right)^{p,\mathbb{F}} + x M_\mu^P\left(I_{\{\widetilde{Z}=0<Z_-\}} | \widetilde{\mathcal{P}}(\mathbb{F})\right) I_{\{Z_- + f_m > 0\}} I_{\rrbracket 0,\tau\rrbracket} \star \nu \\
&= \widehat{S} - x M_\mu^P\left(I_{\{\widetilde{Z}=0<Z_-\}} | \widetilde{\mathcal{P}}(\mathbb{F})\right) I_{\{\psi=0\}} I_{\rrbracket 0,\tau\rrbracket} \star \nu = \widehat{S} \in \mathcal{M}_{loc}(\mathbb{G}).
\end{aligned}$$

The second equality is due to (4.4), while the last equality follows directly form the fact that $S^{(0)}$ is an $\mathbb{F}$-local martingale (which is equivalent to $x I_{\{\psi=0<Z_-\}} \star \nu \equiv 0$) and $M_\mu^P\left(I_{\{\widetilde{Z}=0<Z_-\}} | \widetilde{\mathcal{P}}(\mathbb{F})\right) = I_{\{0<Z_-\}}(1-\psi)$. This ends the proof of the theorem. □

Remark 4.4 (a) Both Theorems 4.1-4.3 provide methods that build-up explicitly σ-martingale density for X^τ, whenever X is an $\mathbb{F}$-quasi-left-continuous that is a local martingale under a locally equivalent probability measure, and fulfilling the assumptions of the theorems respectively.
(b) The extension of Theorem 4.1 to the general case where S is an $\mathbb{F}$-local martingale (not necessarily quasi-left-continuous) boils down to find a predictable process Φ such that Φ is locally bounded, $\Phi \geq 1$, $\{\Phi > 1\}$ is thin and $Y^{(1)} := \mathcal{E}(\Phi \centerdot L)$ will be the martingale density for S^τ. Finding the process Φ will be easy to guess when we will address the case of thin semimartingale. However the proof of $Y^{(1)}$ is a local martingale density for S^τ is very technical. The extension of Theorem 4.3 to the case of arbitrary $\mathbb{F}$-local martingale S requires additional careful modification of the functional g_1 so that $1 + \Phi(\Delta L) + \Delta L^{(1)}$ remains positive. While both extensions remain feasible, we opted to not overload the paper with technicalities.

5 Proofs of Main Theorems

This section is devoted to the proofs of Theorems 2.8, 2.14 and 2.17. They are quite long, since some integrability results have to be proved. For the reader's convenience, we recall the canonical decomposition of S,

$$S = S_0 + S^c + h \star (\mu - \nu) + b \cdot A + (x - h) \star \mu, \tag{5.1}$$

where h defined as $h(x) := xI_{\{|x|\leq 1\}}$ is the truncation function. The canonical decomposition of S^τ under $\mathbb{G}$ is given by

$$S^\tau = S_0 + \widehat{S^c} + h \star (\mu^{\mathbb{G}} - \nu^{\mathbb{G}}) + \tfrac{c\beta_m}{Z_-} I_{[\![0,\tau]\!]} \cdot A + h\tfrac{f_m}{Z_-} I_{[\![0,\tau]\!]} \star \nu + b \bullet A^\tau + (x - h) \star \mu^{\mathbb{G}}$$

where $\mu^{\mathbb{G}}$ and $\nu^{\mathbb{G}}$ and (β_m, f_m) are given in (4.6) and (4.5) respectively and

$$\widehat{S^c} := I_{[\![0,\tau]\!]} \bullet S^c - \frac{1}{Z_-} I_{[\![0,\tau]\!]} \bullet \langle m, S^c \rangle.$$

5.1 Proof of Theorem 2.8

The proof of Theorem 2.8 will be completed in four steps. The first step provides an equivalent formulation to assertion (a) using the filtration $\mathbb{F}$ instead. In the second step, we prove (a)⇒(b), while the reverse implication is proved in the third step. The proof of (b)⟺ (c) is given in the last step.

Step 1: Formulation of assertion (a): Thanks to Proposition 2.3, S^τ satisfies NUPBR($\mathbb{G}$) if and only if there exist a $\mathbb{G}$-local martingale $N^{\mathbb{G}}$ with $1 + \Delta N^{\mathbb{G}} > 0$ and a $\mathbb{G}$-predictable process $\phi^{\mathbb{G}}$ such that $0 < \phi^{\mathbb{G}} \leq 1$ and $\mathcal{E}\left(N^{\mathbb{G}}\right)\left(\phi^{\mathbb{G}} \bullet S^\tau\right)$ is a $\mathbb{G}$-local martingale. We can reduce our attention to processes $N^{\mathbb{G}}$ such that (see Theorem A.4 in the Appendix)

$$N^{\mathbb{G}} = \beta^{\mathbb{G}} \bullet \widehat{S^c} + (f^{\mathbb{G}} - 1) \star (\mu^{\mathbb{G}} - \nu^{\mathbb{G}})$$

where $\beta^{\mathbb{G}} \in L(\widehat{S^c}, \mathbb{G})$ and $f^{\mathbb{G}}$ is positive and such that $(f^{\mathbb{G}} - 1) \in \mathcal{G}^1_{loc}(\mu^{\mathbb{G}}, \mathbb{G})$. Then, one notes that $\mathcal{E}\left(N^{\mathbb{G}}\right)\left(\phi^{\mathbb{G}} \bullet S^\tau\right)$ is a $\mathbb{G}$-local martingale if and only if $\phi^{\mathbb{G}} \bullet S^\tau + [\phi^{\mathbb{G}} \bullet S^\tau, N^{\mathbb{G}}]$ is a $\mathbb{G}$-local martingale, which in turn, is equivalent to

$$\phi^{\mathbb{G}} |x f^{\mathbb{G}}(x) - h(x)| \left(1 + \frac{f_m(x)}{Z_-}\right) I_{[\![0,\tau]\!]} \star \nu \in \mathcal{A}^+_{loc}(\mathbb{G}), \tag{5.2}$$

and $P \otimes A - a.e.$ on $[\![0, \tau]\!]$ (using the kernel F defined in the Appendix A)

$$b + c(\frac{\beta_m}{Z_-} + \beta^{\mathbb{G}}) + \int \left[(x f^{\mathbb{G}}(x) - h(x)) \left(1 + \frac{f_m(x)}{Z_-}\right) + h(x) \frac{f_m(x)}{Z_-}\right] F(dx) = 0. \tag{5.3}$$

From Lemma C.1, there exist $\phi^{\mathbb{F}}$ and $\beta^{\mathbb{F}}$ two $\mathbb{F}$-predictable processes and a positive $\widetilde{\mathcal{P}}(\mathbb{F})$-measurable functional, $f^{\mathbb{F}}$, such that $0 < \phi^{\mathbb{F}} \leq 1$,

$$\beta^{\mathbb{F}} = \beta^{\mathbb{G}},\ \phi^{\mathbb{F}} = \phi^{\mathbb{G}},\ f^{\mathbb{F}} = f^{\mathbb{G}} \text{ on } [\![0, \tau]\!]. \tag{5.4}$$

In virtue of these and taking into account integrability conditions given in Proposition C.3, we deduce that (5.2)–(5.3) imply that, on $\{Z_- \geq \delta\}$, we have

$$W^{\mathbb{F}} := \int |(xf^{\mathbb{F}}(x) - h(x))| \left(1 + \frac{f_m(x)}{Z_-}\right) F(dx) < +\infty \quad P \otimes A - a.e, \quad (5.5)$$

and $P \otimes A$-a.e. on $\{Z_- \geq \delta\}$, we have

$$b + c\left(\beta^{\mathbb{F}} + \frac{\beta_m}{Z_-}\right) - \int h(x) I_{\{\psi=0\}} F(dx) + \int \left[xf^{\mathbb{F}}(x)(1 + \frac{f_m(x)}{Z_-}) - h(x)\right] I_{\{\psi>0\}} F(dx) = 0. \quad (5.6)$$

Due to (5.5), this latter equality follows due to $\{\psi = 0\} = \{Z_- + f_m = 0\}$ (see 2.7) and by taking the $\mathbb{F}$-predictable projection of (5.3) after inserting (5.4).

Step 2: Proof of (a) $\Rightarrow$ (b). Suppose that S^τ satisfies NUPBR($\mathbb{G}$), hence (5.5)–(5.6) hold. To prove that $I_{\{Z \geq \delta\}} \bullet (S - S^{(0)})$ satisfies NUPBR($\mathbb{F}$), we consider $\Sigma_0 := \{\psi > 0 \ \& \ Z_- \geq \delta\}$ and

$$\beta := \left(\frac{\beta_m}{Z_-} + \beta^{\mathbb{F}}\right) I_{\{Z_- \geq \delta\}} \text{ and } \ f = f^{\mathbb{F}} \left(1 + \frac{f_m}{Z_-}\right) I_{\Sigma_0} + I_{\Sigma_0^c}.$$

If $\beta \in L(S^c, \mathbb{F})$ and $(f-1) \in \mathcal{G}^1_{loc}(\mu, \mathbb{F})$, we conclude that

$$N := \beta \bullet S^c + (f-1) \star (\mu - \nu).$$

is a well defined $\mathbb{F}$-local martingale.Choosing $\phi = \left(1 + W^{\mathbb{F}} I_{\{Z_- \geq \delta\}}\right)^{-1}$, using (5.6), and applying Itô's formula for $\mathcal{E}(N)\left(\phi I_{\{Z_- \geq \delta\}} \cdot (S - S^{(0)})\right)$, we deduce that this process is a local martingale. Hence, $I_{\{Z_- \geq \delta\}} \cdot (S - S^{(0)})$ satisfies NUPBR($\mathbb{F}$), and the proof of (a)$\Rightarrow$(b) is completed.
Now, we focus on proving $\beta \in L(S^c)$ and $(f-1) \in \mathcal{G}^1_{loc}(\mu, \mathbb{F})$ (or equivalently $\sqrt{(f-1)^2 \star \mu} \in \mathcal{A}^+_{loc}(\mathbb{F})$). Since $\beta_m \in L(S^c)$, then it is obvious that $\frac{\beta_m}{Z_-} I_{\{Z_- \geq \delta\}} \in L(S^c)$ on the one hand. On the other hand, $(\beta^{\mathbb{F}})^T c \beta^{\mathbb{F}} I_{\{0 \leq Z_- < \delta\}} \bullet A \in \mathcal{A}^+_{loc}(\mathbb{F})$ due to $(\beta^{\mathbb{F}})^T c \beta^{\mathbb{F}} \bullet A^\tau = (\beta^{\mathbb{G}})^T c \beta^{\mathbb{G}} \bullet A^\tau \in \mathcal{A}^+_{loc}(\mathbb{G})$ and Proposition C.3–(c). This completes the proof of $\beta \in L(S^c)$.

Now, we focus on proving $(f-1) \in \mathcal{G}^1_{loc}(\mu, \mathbb{F})$. Since S is quasi-left-continuous, this is equivalent to prove $\sqrt{(f-1)^2 \star \mu} \in \mathcal{A}^+_{loc}(\mathbb{F})$. Thanks to Proposition C.3 and $\sqrt{(\ f^{\mathbb{F}} - 1)^2 \star \mu^{\mathbb{G}}} = \sqrt{(\ f^{\mathbb{G}} - 1)^2 \star \mu^{\mathbb{G}}} \in \mathcal{A}^+_{loc}(\mathbb{G})$, we deduce that

$$(\ f^{\mathbb{F}} - 1)^2 I_{\{|f^{\mathbb{F}} - 1| \leq \alpha\}} \widetilde{Z} I_{\{Z_- \geq \delta\}} \star \mu \text{ and } \ |f^{\mathbb{F}} - 1| I_{\{|f^{\mathbb{F}} - 1| > \alpha\}} \widetilde{Z} I_{\{Z_- \geq \delta\}} \star \mu \in \mathcal{A}^+_{loc}(\mathbb{F}). \quad (5.7)$$

By stopping, there is no loss of generality in assuming that these two processes and $[m, m]$ are integrable. Then we get

$$f - 1 = \left(f^{\mathbb{F}} - 1\right) \left(1 + \frac{f_m}{Z_-}\right) I_{\Sigma_0} + \frac{f_m}{Z_-} I_{\Sigma_0} =: h_1 + h_2.$$

Therefore, we derive that

$$E\left[h_1^2 I_{\{|f^{\mathbb{F}}-1|\leq\alpha\}}\star\mu_\infty\right] \leq \delta^{-2}E\left[\left(f^{\mathbb{F}}-1\right)^2(Z_-+f_m)^2 I_{\{|f^{\mathbb{F}}-1|\leq\alpha\}}I_{\{Z_-\geq\delta\}}\star\mu_\infty\right]$$
$$\leq \delta^{-2}E\left[\left(f^{\mathbb{F}}-1\right)^2\widetilde{Z}I_{\{|f^{\mathbb{F}}-1|\leq\alpha\}}I_{\{Z_-\geq\delta\}}\star\mu_\infty\right] < +\infty,$$

and

$$E\left[|h_1|I_{\{|f^{\mathbb{F}}-1|>\alpha\}}\star\mu_\infty\right] \leq \delta^{-1}E\left[|f^{\mathbb{F}}-1|\,|Z_-+f_m|I_{\{|f^{\mathbb{F}}-1|>\alpha\}}I_{\{Z_-\geq\delta\}}\star\mu_\infty\right]$$
$$= \delta^{-1}E\left[|f^{\mathbb{F}}-1|\widetilde{Z}I_{\{|f^{\mathbb{F}}-1|>\alpha\}}I_{\{Z_-\geq\delta\}}\star\mu_\infty\right] < +\infty.$$

By combining the above two inequalities, we conclude that $\left(h_1^2\star\mu\right)^{1/2}\in\mathcal{A}^+_{loc}(\mathbb{F})$. It is easy to see that $\left(h_2^2\star\mu\right)^{1/2}\in\mathcal{A}^+_{loc}(\mathbb{F})$ follows from

$$E\left[h_2^2\star\mu_\infty\right]\leq\delta^{-2}E\left[f_m^2\star\mu_\infty\right]\leq\delta^2E\left[(\Delta m)^2\star\mu_\infty\right]\leq\delta^{-2}E\left[m,m\right]_\infty<+\infty.$$

Step 3: Proof of (b) $\Rightarrow$ (a). If for any $\delta>0$, the process $I_{\{Z_-\geq\delta\}}\cdot\left(S-S^{(0)}\right)$ satisfies NUPBR($\mathbb{F}$), then, there exist an $\mathbb{F}$-local martingale $N^{\mathbb{F}}$ and an $\mathbb{F}$-predictable process ϕ such that $0<\phi\leq 1$ and $\mathcal{E}\left(N^{\mathbb{F}}\right)\left[\phi I_{\{Z_-\geq\delta\}}\bullet\left(S-S^{(0)}\right)\right]$ is an $\mathbb{F}$-local martingale. Again, thanks to Theorem A.4, we can restrict our attention to the case

$$N^{\mathbb{F}}:=\beta^{\mathbb{F}}\bullet S^c+(f^{\mathbb{F}}-1)\star(\mu-\nu),$$

where $\beta^{\mathbb{F}}\in L(S^c)$ and $f^{\mathbb{F}}$ is positive such that $(f^{\mathbb{F}}-1)\in\mathcal{G}^1_{loc}(\mu,\mathbb{F})$.
Thanks to Itô's formula, the fact that $\mathcal{E}\left(N^{\mathbb{F}}\right)\left[\phi I_{\{Z_-\geq\delta\}}\bullet\left(S-S^{(0)}\right)\right]$ is an $\mathbb{F}$-local martingale implies that on $\{Z_-\geq\delta\}$

$$k^{\mathbb{F}}:=\int|xf^{\mathbb{F}}(x)I_{\{\psi(x)>0\}}-h(x)|F(dx)<+\infty\qquad P\otimes A-a.e.$$

and $P\otimes A$-a.e. on $\{Z_-\geq\delta\}$, we have

$$b-\int h(x)I_{\{\psi=0\}}F(dx)+c\beta^{\mathbb{F}}+\int\left[xf^{\mathbb{F}}(x)-h(x)\right]I_{\{\psi>0\}}F(dx)=0. \quad (5.8)$$

Recall that $\{\psi=0\}=\{Z_-+f_m=0\}$ and define

$$\beta^{\mathbb{G}}:=\left(\beta^{\mathbb{F}}-\frac{\beta_m}{Z_-}\right)I_{[\![0,\tau]\!]}\quad\text{and}\quad f^{\mathbb{G}}:=\frac{f^{\mathbb{F}}}{1+f_m/Z_-}I_{\Sigma_0}I_{[\![0,\tau]\!]}+I_{\Sigma_0^c\cup]\!]\tau,+\infty[\![}.$$

If we assume for a while that

$$\beta^{\mathbb{G}}\in L(\widehat{S^c})\quad\text{and}\quad(f^{\mathbb{G}}-1)\in\mathcal{G}^1_{loc}(\mu^{\mathbb{G}}), \quad (5.9)$$

then, necessarily $N^{\mathbb{G}}:=\beta^{\mathbb{G}}\bullet\widehat{S^c}+(f^{\mathbb{G}}-1)\star(\mu^{\mathbb{G}}-\nu^{\mathbb{G}})$ is a well defined $\mathbb{G}$-local martingale satisfying $\mathcal{E}(N^{\mathbb{G}})>0$. Furthermore, due to (5.8) and to $\{\psi=0\}=\{Z_-+f_m=0\}$ (see(2.7)), on $[\![0,\tau]\!]$ we obtain

$$b+c\left(\beta^{\mathbb{G}}+\frac{\beta_m}{Z_-}\right)+\int\left(xf^{\mathbb{G}}\left(1+\frac{f_m}{Z_-}\right)-h(x)\right)F(dx)=0\,. \quad (5.10)$$

By taking $\phi^{\mathbb{G}} := \big(1 + k^{\mathbb{F}} I_{\{Z_- \geq \delta\}}\big)^{-1}$, and applying Itô to $(\phi^{\mathbb{G}} I_{\{Z_- \geq \delta\}} \bullet S^\tau)\mathcal{E}(N^{\mathbb{G}})$, we conclude that this process is a $\mathbb{G}$-local martingale due to (5.10). Thus, $I_{\{Z_- \geq \delta\}} \bullet S^\tau$ satisfies NUPBR($\mathbb{G}$) as long as (5.9) is fulfilled.
Since $Z_-^{-1} I_{[\![0,\tau]\!]}$ is $\mathbb{G}$-locally bounded, then there exists a family of $\mathbb{G}$-stopping times $(\tau_\delta)_{\delta>0}$ such that $[\![0, \tau_\delta]\!] \subset \{Z_- \geq \delta\}$ (which implies that $I_{\{Z_- \geq \delta\}} \bullet S^{\tau\wedge\tau_\delta} = S^{\tau\wedge\tau_\delta}$) and τ_δ increases to infinity when δ goes to zero. Thus, using Proposition 2.6, we deduce that S^τ satisfies NUPBR($\mathbb{G}$). This achieves the proof of (b)⇒(a) under (5.9).

To prove that (5.9) holds true, we remark that $Z_-^{-1} I_{[\![0,\tau]\!]}$ is $\mathbb{G}$-locally bounded and both β_m and $\beta^{\mathbb{F}}$ belong to $L(S^c)$. This, easily, implies that $\beta^{\mathbb{G}} \in L(\widehat{S^c})$. Now, we prove that $\sqrt{(f^{\mathbb{G}} - 1)^2 \star \mu^{\mathbb{G}}} \in \mathcal{A}^+_{loc}(\mathbb{G})$. Since $\sqrt{(f^{\mathbb{F}} - 1)^2 \star \mu} \in \mathcal{A}^+_{loc}(\mathbb{F})$, Proposition C.3 allows us again to deduce that

$$(f^{\mathbb{F}} - 1)^2 I_{\{|f^{\mathbb{F}} - 1| \leq \alpha\}} \star \mu \in \mathcal{A}^+_{loc}(\mathbb{F}) \text{ and } |f^{\mathbb{F}} - 1| I_{\{|f^{\mathbb{F}} - 1| > \alpha\}} \star \mu \in \mathcal{A}^+_{loc}(\mathbb{F}).$$

Without loss of generality, we can assume that these two processes and $[m, m]$ are integrable. Put

$$f^{\mathbb{G}} - 1 = I_{\Sigma_0} I_{]\!]0,\tau]\!]} \frac{Z_-(f^{\mathbb{F}} - 1)}{f_m + Z_-} - I_{\Sigma_0} I_{]\!]0,\tau]\!]} \frac{f_m}{f_m + Z_-} := f_1 + f_2.$$

Then, setting $\Sigma_\delta := \{0 < Z_- + f_m \leq \delta/2\} \cap \{Z_- \geq \delta\} = \Sigma_0 \cap \{Z_- + f_m \leq \delta/2\}$, we calculate

$$E\left(f_1^2 I_{\Sigma_\delta^c \cap \{|f^{\mathbb{F}} - 1| \leq \alpha\}} \star \mu^{\mathbb{G}}_\infty\right) \leq \left(\tfrac{2}{\delta}\right)^2 E\big[(f^{\mathbb{F}} - 1)^2 I_{\{|f^{\mathbb{F}} - 1| \leq \alpha\}} \star \mu_\infty\big] < +\infty.$$

and

$$\begin{aligned} E\sqrt{f_1^2 I_{\Sigma_\delta \cap \{|f^{\mathbb{F}} - 1| \leq \alpha\}} \star \mu^{\mathbb{G}}_\infty} &\leq \alpha E\big(I_{\Sigma_\delta}(Z_- + f_m)^{-1} \star \mu^{\mathbb{G}}(\infty)\big) \\ &\leq E\big(I_{\{|f_m| \geq \delta/2\}} \star \mu_\infty\big) \leq \frac{4\alpha}{\delta^2} E[m, m]_\infty < +\infty. \end{aligned}$$

This proves that $\sqrt{f_1^2 I_{\{|f^{\mathbb{F}} - 1| \leq \alpha\}} \star \mu^{\mathbb{G}}} \in \mathcal{A}^+_{loc}(\mathbb{G})$. Similarly, we calculate

$$\begin{aligned} E\sqrt{f_1^2 I_{\{|f^{\mathbb{F}} - 1| > \alpha\}} \star \mu^{\mathbb{G}}_\infty} &\leq E\big(|f_1| I_{\{|f^{\mathbb{F}} - 1| > \alpha\}} \star \mu^{\mathbb{G}}_\infty\big) \leq E\big(\frac{|f^{\mathbb{F}} - 1|}{1 + f_m/Z_-} I_{\{|f^{\mathbb{F}} - 1| > \alpha\}} \star \mu^{\mathbb{G}}_\infty\big) \\ &\leq E\big(|f^{\mathbb{F}} - 1| I_{\{|f^{\mathbb{F}} - 1| > \alpha\}} \star \mu_\infty\big) < +\infty. \end{aligned}$$

Thus, by combining all the remarks obtained above, we conclude that $\sqrt{f_1^2 \star \mu^{\mathbb{G}}}$ is $\mathbb{G}$-locally integrable. For the functional f_2, we proceed by calculating

$$E\big(f_2^2 I_{\Sigma_\delta^c} \star \mu^{\mathbb{G}}_\infty\big) \leq (2/\delta)^2 E\big(f_m^2 \star \mu_\infty\big) \leq (2/\delta)^2 E[m, m]_\infty < +\infty,$$

and

$$\begin{aligned} E\sqrt{f_2^2 I_{\Sigma_\delta} \star \mu^{\mathbb{G}}_\infty} &\leq E\big(|f_m| I_{\{|f_m| \geq \delta/2\}} \star \mu(\infty)\big) \\ &\leq (2/\delta) E\big(f_m^2 \star \mu(\infty)\big) \leq (2/\delta) E[m, m]_\infty < +\infty. \end{aligned}$$

This proves that $\sqrt{f_2^2 \star \mu^{\mathbb{G}}}$ is $\mathbb{G}$-locally integrable. Therefore, we conclude that (5.9) is valid, and the proof of (b)⇒(a) is completed.

Step 4: Proof of (b) $\Longleftrightarrow$ (c). For any $\delta > 0$, we denote

$$\sigma_\infty := \inf\{t \geq 0 : \quad Z_t = 0\}, \qquad \tau_\delta := \sup\{t : Z_{t-} \geq \delta\}.$$

Then, due to $]\!]\sigma_\infty, +\infty[\![\subset \{Z_- = 0\} \subset \{Z_- < \delta\}$, we deduce

$$\sigma_{1/\delta} \leq \tau_\delta \leq \sigma_\infty \quad \text{ and } \quad Z_{\tau_\delta -} \geq \delta > 0 \qquad P-a.s. \text{ on } \{\tau_\delta < \infty\}.$$

Here, $\sigma_{1/\delta}$ is defined by (2.6) ($k = 1/\delta$). Setting $\Sigma := \bigcap_{n\geq 1}(\sigma_n < \sigma_\infty)$, we have

$$\text{on } \Sigma \cap \{\sigma_\infty < \infty\} \quad Z_{\sigma_\infty -} = 0, \text{ and } \tau_\delta < \sigma_\infty \quad P-a.s.$$

We introduce the semimartingale $X := S - S^{(0)}$. For any $\delta > 0$, and any H predictable such that $H_\delta := H I_{\{Z_- \geq \delta\}} \in L(X)$ and $H_\delta \cdot X \geq -1$, due to Theorem 23 of [13] (page 346 in the French version),

$$(H_\delta \cdot X)_T = (H_\delta \cdot X)_{T\wedge\tau_\delta}, \quad \text{and on } \{\theta \geq \tau_\delta\} \;\; (H_\delta \cdot X)_T = (H_\delta \cdot X)_{T\wedge\theta}.$$

Then, for any $T \in (0, +\infty)$, we calculate the following

$$\begin{aligned} P((H_\delta \cdot X)_T > c) &= P((H_\delta \cdot X)_T > c \;\&\; \sigma_n \geq \tau_\delta) + P((H_\delta \cdot X)_T > c \;\&\; \sigma_n < \tau_\delta) \\ &\leq 2 \sup_{\phi \in L(X^{\sigma_n}) : \phi_\bullet X^{\sigma^n} \geq -1} P((\phi \cdot X)_{\sigma_n \wedge T} > c) + P(\sigma_n < \tau_\delta \wedge T). \end{aligned} \tag{5.11}$$

It is easy to prove that $P(\sigma_n < \tau_\delta \wedge T) \longrightarrow 0$ as n goes to infinity. This can be seen due to the fact that on Σ, we have, on the one hand, $\tau_\delta \wedge T < \sigma_\infty$ (by differentiating the two cases whether σ_∞ is finite or not). On the other hand, the event $(\sigma_n < \sigma_\infty)$ increases to Σ with n. Thus, by combining these, we obtain the following

$$\begin{aligned} P(\sigma_n < \tau_\delta \wedge T) &= P((\sigma_n < \tau_\delta \wedge T) \cap \Sigma) + P((\sigma_n < \tau_\delta \wedge T) \cap \Sigma^c) \\ &\leq P(\sigma_n < \tau_\delta \wedge T < \sigma_\infty) + P((\sigma_n < \sigma_\infty) \cap \Sigma^c) \longrightarrow 0. \end{aligned} \tag{5.12}$$

Now suppose that for each $n \geq 1$, the process $(S - S^{(0)})^{\sigma_n}$ satisfies NUPBR($\mathbb{F}$). Then a combination of (5.11) and (5.12) implies that for any $\delta > 0$, the process $I_{\{Z_- \geq \delta\}} \bullet X := I_{\{Z_- \geq \delta\}} \bullet (S - S^{(0)})$ satisfies NUPBR($\mathbb{F}$), and the proof of (c)⇒(b) is completed. The proof of the reverse implication is obvious due to the fact that

$$[\![0, \sigma_n]\!] \subset \{Z_- \geq 1/n\} \subset \{Z_- \geq \delta\}, \quad \text{for} \quad n \leq \delta^{-1},$$

which implies that $(I_{\{Z_- \geq \delta\}} \cdot X)^{\sigma_n} = X^{\sigma_n}$. This ends the proof of (b) $\Longleftrightarrow$(c), and the proof of the theorem is achieved. □

5.2 Intermediate Result

The proofs of Theorems 2.14 and 2.17 rely on the following intermediary result about single jump $\mathbb{F}$-martingales, which is interesting in itself.

Proposition 5.1 *Let M be an $\mathbb{F}$-martingale with $M_0 = 0$ given by $M := \xi I_{[\![T,+\infty[\![}$, where T is an $\mathbb{F}$-predictable stopping time, and ξ is an $\mathcal{F}_T$-measurable random variable. Then the following assertions are equivalent.*
(a) *M is an $\mathbb{F}$-martingale under Q_T given by*

$$\frac{dQ_T}{dP} := \frac{I_{\{\widetilde{Z}_T>0\}\cap\Gamma(T)}}{P(\widetilde{Z}_T > 0\big|\ \mathcal{F}_{T-})} + I_{\Gamma^c(T)}, \quad \Gamma(T) := \{P(\widetilde{Z}_T > 0|\mathcal{F}_{T-}) > 0\}. \tag{5.13}$$

(b) *On the set $\{T < +\infty\}$, we have*

$$E\left(M_T I_{\{\widetilde{Z}_T=0\}}\big|\ \mathcal{F}_{T-}\right) = 0, \quad P-a.s. \tag{5.14}$$

(c) *M^τ is a $\mathbb{G}$-martingale under $Q_T^{\mathbb{G}} := \left(U^{\mathbb{G}}(T)/E(U^{\mathbb{G}}(T)\big|\ \mathcal{G}_{T-})\right)\cdot P$ where*

$$U^{\mathbb{G}}(T) := I_{\{T>\tau\}} + I_{\{T\le\tau\}}\frac{Z_{T-}}{\widetilde{Z}_T} > 0. \tag{5.15}$$

Proof The proof will be achieved in two steps.
Step 1. Here, we prove the equivalence between assertions (a) and (b). For simplicity we denote by $Q := Q_T$, where Q_T is defined in (5.13), and remark that on $\{Z_{T-} = 0\}$, Q coincides with P and (5.14) holds, due to $\{Z_{T-} = 0\} \subset \{\widetilde{Z}_T = 0\}$ and $\{Z_{T-} > 0\} = \Gamma(T)$. Thus, it is enough to prove the equivalence between (a) and (b) on the set $\{T < +\infty\ \&\ Z_{T-} > 0\}$. On this set, due to $E(\xi|\mathcal{F}_{T-}) = 0$ (which comes from the martingale property of M), we derive

$$\begin{aligned} E^Q(\xi\big|\mathcal{F}_{T-}) &= E(\xi I_{\{\widetilde{Z}_T>0\}}\big|\mathcal{F}_{T-})\Big(P(\widetilde{Z}_T > 0\big|\mathcal{F}_{T-})\Big)^{-1} \\ &= -E(\xi I_{\{\widetilde{Z}_T=0\}}\big|\mathcal{F}_{T-})\left(P(\widetilde{Z}_T > 0\big|\mathcal{F}_{T-})\right)^{-1}. \end{aligned}$$

Therefore, we conclude that assertion (a) (or equivalently $E^Q(\xi|\mathcal{F}_{T-}) = 0$) is equivalent to (5.14). This ends the proof of (a) $\Longleftrightarrow$ (b).
Step 2. To prove (a)$\Longleftrightarrow$(c), we first notice that due to $(T \le \tau) \subset (\widetilde{Z}_T > 0) \subset (Z_{T-} > 0)$, on $\{T \le \tau\}$ we have

$$\begin{aligned} P\left(\widetilde{Z}_T > 0\big|\mathcal{F}_{T-}\right) E^{Q_T^{\mathbb{G}}}(\xi|\mathcal{G}_{T-}) &= E\left(\frac{Z_{T-}}{\widetilde{Z}_T}\xi I_{\{T\le\tau\}}|\mathcal{G}_{T-}\right) = E\left(\xi I_{\{\widetilde{Z}_T>0\}}|\mathcal{F}_{T-}\right) \\ &= E^Q(\xi|\mathcal{F}_{T-})\,P\left(\widetilde{Z}_T > 0\big|\mathcal{F}_{T-}\right). \end{aligned}$$

This equality proves that $M^\tau \in \mathcal{M}(Q^{\mathbb{G}}, \mathbb{G})$ if and only if $M \in \mathcal{M}(Q, \mathbb{F})$, and the proof of (a)$\Longleftrightarrow$(c) is completed. This ends the proof of the proposition.
□

5.3 Proof of Theorem 2.14

For the reader convenience, in order to prove Theorem 2.14, we state a more precise version of the theorem, in which we describe explicitly some possible choices for the probability measure Q_T.

Theorem 5.2 *Suppose that the assumptions of Theorem 2.14 are in force. Then, the assertions* (a) *and* (b) *of Theorem 2.14 are equivalent to one of the following assertions.*
(d) *S satisfies NUPBR$(\mathbb{F}, \widetilde{Q}_T)$, where $\widetilde{Q}_T$ is*

$$\widetilde{Q}_T := \left(\frac{\widetilde{Z}_T}{Z_{T-}} I_{\{Z_{T-}>0\}} + I_{\{Z_{T-}=0\}} \right) \cdot P, \tag{5.16}$$

(e) *S satisfies NUPBR$(\mathbb{F}, Q_T)$, where Q_T is defined in (5.13).*

Proof The proof of this theorem will be achieved by proving (d) $\Longleftrightarrow$ (e) $\Longleftrightarrow$ (b) and (b) $\Rightarrow$ (a) $\Rightarrow$ (d). These will be carried out in four steps.

Step 1: In this step, we prove (d) $\Longleftrightarrow$ (e). Since S is a single jump process with predictable jump time T, then it is easy to see that S satisfies NUPBR under some probability R is equivalent to the fact that $I_A S$ and $I_{A^c} S$ satisfies NUPBR(R) for any $\mathcal{F}_{T-}$-measurable event A. Hence, it is enough to prove the equivalence between the assertions (d) and (e) separately on the events $\{Z_{T-} = 0\}$ and $\{Z_{T-} > 0\}$. Since $\{Z_{T-} = 0\} \subset \{\widetilde{Z}_T = 0\}$ and $E(\widetilde{Z}_T|\mathcal{F}_{T-}) = Z_{T-}$ on $\{T < +\infty\}$, by putting $\Gamma_0 := \left\{ P(\widetilde{Z}_T > 0 \big| \mathcal{F}_{T-}) = 0 \right\}$, we derive

$$E\left(Z_{T-} I_{\Gamma_0 \cap \{T<+\infty\}}\right) = E\left(\widetilde{Z}_T I_{\Gamma_0 \cap \{T<+\infty\}}\right) = 0,$$

and

$$\begin{aligned} 0 &= P\left(\{Z_{T-} = 0\} \cap \{\widetilde{Z}_T > 0\} \cap \{T < +\infty\}\right) \\ &= E\left(I_{\{Z_{T-}=0\} \cap \{T<+\infty\}} P\left(\widetilde{Z}_T > 0 | \mathcal{F}_{T-}\right)\right). \end{aligned}$$

These equalities imply that on $\{T < +\infty\}$, $P - a.s.$, we have

$$\{Z_{T-} = 0\} = \Gamma_0 \subset \{\widetilde{Z}_T = 0\}. \tag{5.17}$$

Thus, on the set $\{T < +\infty\} \cap \Gamma_0$, the three probabilities P, Q_T and $\widetilde{Q}_T$ coincide, and the equivalence between assertions (d) and (e) is obvious. On the set $\{T < +\infty \ \& \ P[\widetilde{Z}_T > 0|\mathcal{F}_{T-}] > 0\}$, one has $\widetilde{Q}_T \sim Q_T$, and the equivalence between (d) and (e) is also obvious. This achieves this first step.

Step 2: This step proves (e)$\Longleftrightarrow$ (b). Again thanks to (5.17), we deduce that on $\{Z_{T-} = 0\}$, $\widetilde{S} \equiv S \equiv 0$ and Q_T coincides with P as well. Hence, the equivalence between assertions (e) and (b) is obvious for this case. Thus, it is enough

to prove the equivalence between these assertions on $\{T < +\infty \ \& \ P(\widetilde{Z}_T > 0|\mathcal{F}_{T-}) > 0\}$.
Assume that (e) holds. Then, there exists an $\mathcal{F}_T$-measurable random variable, Y, such that $Y > 0$ $Q_T - a.s.$ and on $\{T < +\infty\}$, we have

$$E^{Q_T}(Y|\mathcal{F}_{T-}) = 1, \quad E^{Q_T}(Y|\xi||\mathcal{F}_{T-}) < +\infty, \quad E^{Q_T}(Y\xi|\mathcal{F}_{T-}) = 0. \quad (5.18)$$

Since $Y > 0$ on $\{\widetilde{Z}_T > 0\}$, by putting

$$Y_1 := YI_{\{\widetilde{Z}_T>0\}} + I_{\{\widetilde{Z}_T=0\}} \ \text{ and } \ \widetilde{Y}_1 := \frac{Y_1}{E[Y_1|\mathcal{F}_{T-}]},$$

it is easy to check that $Y_1 > 0$, $\widetilde{Y}_1 > 0$ and due to (5.18),

$$E\left[\widetilde{Y}_1|\mathcal{F}_{T-}\right] = 1 \text{ and } E\left[\widetilde{Y}_1\xi I_{\{\widetilde{Z}_T>0\}}|\mathcal{F}_{T-}\right] = \frac{E\left[Y\xi I_{\{\widetilde{Z}_T>0\}}|\mathcal{F}_{T-}\right]}{E[Y_1|\mathcal{F}_{T-}]} = 0.$$

Therefore, $\widetilde{S}$ is a martingale under $R := \widetilde{Y}_1 \cdot P \sim P$, and hence $\widetilde{S}$ satisfies NUPBR($\mathbb{F}$). This ends the proof of (e)$\Rightarrow$(b).
To prove the reverse sense, we suppose that assertion (b) holds. Then, there exists $0 < Y \in L^0(\mathcal{F}_T)$, such that $E[Y|\xi|I_{\{\widetilde{Z}_T>0\}}|\mathcal{F}_{T-}] < +\infty$, $E[Y|\mathcal{F}_{T-}] = 1$ and $E[Y\xi I_{\{\widetilde{Z}_T>0\}}|\mathcal{F}_{T-}] = 0$ on $\{Z_{T-} > 0\}$. Then, consider

$$Y_2 := \frac{YI_{\{\widetilde{Z}_T>0\}}P(\widetilde{Z}_T > 0|\mathcal{F}_{T-})}{E[YI_{\{\widetilde{Z}_T>0\}}|\mathcal{F}_{T-}]} + I_{\{\widetilde{Z}_T=0\}}$$

Then it is easy to verify that $Y_2 > 0 \ \ Q_T - a.s.$,

$$E^{Q_T}(Y_2|\mathcal{F}_{T-}) = 1, \qquad \text{and} \qquad E^{Q_T}(Y_2\xi|\mathcal{F}_{T-}) = \frac{E\left[Y\xi I_{\{\widetilde{Z}_T>0\}}|\mathcal{F}_{T-}\right]}{E[YI_{\{\widetilde{Z}_T>0\}}|\mathcal{F}_{T-}]} = 0.$$

This proves assertion (e), and the proof of (e)$\Longleftrightarrow$(b) is achieved.

Step 3: Herein, we prove (a) $\Rightarrow$ (d). For the sake of simplicity we denote $\widetilde{Q} := \widetilde{Q}_T$ (where $\widetilde{Q}_T$ is defined in (5.16)). Suppose that S^τ satisfies NUPBR($\mathbb{G}$). Then there exists a positive $\mathcal{G}_T$-measurable random variable $Y^{\mathbb{G}}$ such that on $\{T < +\infty\}$, we have

$$E[Y^{\mathbb{G}}I_{\{T\leq\tau\}}|\mathcal{G}_{T-}] = I_{\{T\leq\tau\}} \quad \text{and} \quad E[\xi Y^{\mathbb{G}}I_{\{T\leq\tau\}}|\mathcal{G}_{T-}] = 0. \qquad (5.19)$$

Due to Lemma C.1–(e), we deduce the existence of two positive $\mathcal{F}_T$-measurable variable $Y^{(1)}$ and $Y^{(2)}$ satisfying (C.3). Thus by putting

$$\widetilde{Y} := \left[\frac{Z_T}{\widetilde{Z}_T}Y^{(1)} + (1 - \frac{Z_T}{\widetilde{Z}_T})Y^{(2)}\right] I_{\{\widetilde{Z}_T>0\}} + I_{\{Z_{T-}=0\}},$$

and inserting (C.3) into the first equation in (5.19), on $\{Z_{T-} > 0 \ \& \ T < +\infty\}$, we derive

$$\begin{aligned} E^{\widetilde{Q}}(\widetilde{Y}|\mathcal{F}_{T-}) &= E(Z_T Y^{(1)} + (\widetilde{Z}_T - Z_t)Y^{(2)}|\mathcal{F}_{T-})Z_{T-}^{-1} \\ &= E\left(E(Y^{\mathbb{G}}|\mathcal{G}_{T-})I_{\{T\leq\tau\}}\big|\mathcal{F}_{T-}\right) = 1. \end{aligned} \tag{5.20}$$

The second last equality follows from the predictable splitting formula (C.1) (which implies $E(h|\mathcal{G}_{T-})I_{\{T\leq\tau\}} = E(hI_{\{T\leq\tau\}}|\mathcal{F}_{T-})Z_{T-}^{-1}I_{\{T\leq\tau\}}$, for any h for which the conditional expectations exist). Similarly, by plugging (C.3) into the second equation of (5.19), on $\{Z_{T-} > 0 \ \& \ T < +\infty\}$, we obtain

$$\begin{aligned} E^{\widetilde{Q}}(\xi\widetilde{Y}|\mathcal{F}_{T-}) &= E\left([Z_T Y^{(1)} + (\widetilde{Z}_T - Z_t)Y^{(2)}]\xi\big|\mathcal{F}_{T-}\right) Z_{T-} \\ &= E\left(E(Y^{\mathbb{G}}\xi|\mathcal{G}_{T-})I_{\{T\leq\tau\}}\big|\mathcal{F}_{T-}\right) = 0. \end{aligned}$$

Then, by combining this equality, (5.20), and the fact that $\widetilde{Y} > 0 \quad \widetilde{Q} - a.s.$, we conclude that S satisfies the NUPBR$(\widetilde{Q}, \mathbb{F})$. This ends the proof of (a)⇒(d).

Step 4: This last step proves (b)⇒(a). Suppose that $\widetilde{S}$ satisfies NUPBR$(\mathbb{F})$. Then, there exists $Y \in L^1(\mathcal{F}_T)$ such that on $\{T < +\infty\}$ we have

$$E[Y|\mathcal{F}_{T-}] = 1, \quad Y > 0, \qquad E[Y|\xi|I_{\{\widetilde{Z}_T>0\}}|\mathcal{F}_{T-}] < +\infty, \quad P - a.s.$$

and $E[Y\xi I_{\{\widetilde{Z}_T>0\}}|\mathcal{F}_{T-}] = 0$. Then by considering $R := Y \cdot P \sim P$, we get

$$E^R\big[\widetilde{S}_T\big|\mathcal{F}_{T-}\big] = E\big[Y\xi I_{\{\widetilde{Z}_T>0\}}\big|\mathcal{F}_{T-}\big] = 0.$$

Therefore, assertion (a) follows directly from Proposition 5.1 applied to $M = \widetilde{S}$ under $R \sim P$ (it is easy to see that (5.14) holds for $(\widetilde{S}, R)$, i.e. $E^R(\widetilde{S}_T I_{\{\widetilde{Z}_T=0\}}|\mathcal{F}_{T-}) = 0$ on $\{T < +\infty\}$). This ends the fourth step and the proof of the theorem is completed. □

5.4 Proof of Theorem 2.17

To highlight the precise difficulty in proving Theorem 2.17, we remark that on $\{T < +\infty\}$,

$$\frac{U^{\mathbb{G}}(T)}{E(U^{\mathbb{G}}(T)\big|\ \mathcal{G}_{T-})} = 1 + \frac{\Delta L_T}{1 - \Delta V_T^{\mathbb{G}}} \neq 1 + \Delta L_T = \frac{\mathcal{E}(L)_T}{\mathcal{E}(L)_{T-}}.$$

where $U^{\mathbb{G}}(T)$ is defined in (5.15). This highlights one of the main difficulties that we will face when we will formulate the results for possible many predictable jumps that might not be ordered on the one hand. On the other hand, this remark explains the two different methods that one can develop in order to prove (b)⟹(a) of the theorem. The first method is based on optional stochastic integral and Remark 4.4 with $\Phi = (1 - \Delta V^{\mathbb{G}})^{-1}$. One need

to prove that this process is $\mathbb{G}$-locally bounded, and the resulting positive $\mathbb{G}$-local martingale $\mathcal{E}((1-\Delta V^{\mathbb{G}})^{-1}\bullet L)$ is a positive local martingale deflator for S^{τ}, under an equivalent probability, when assertion (b) holds. This approach is very technical due to the predictable jumps. Thus, we opted herein for the second method (which is less technical) and is based on the connection between the NUPBR with the existence of a positive supermartingale (instead) that is a deflator for the market model under consideration.

Definition 5.3 Consider an $\mathbb{H}$-semimartingale X. Then, X is said to admit an $\mathbb{H}$-deflator if there exists a positive $\mathbb{H}$-supermartingale Y such that $Y(\theta\bullet X)$ is a supermartingale, for any $\theta\in L(X,\mathbb{H})$ such that $\theta\bullet X\geq -1$.

For supermartingale deflators, we refer the reader to Rokhlin [36]. Again, the above definition differs from that of the literature when the horizon is infinite, while it is the same as the one of the literature when the horizon is finite (even random). Below, we slightly generalize [36] to our context.

Lemma 5.4 *Let X be an $\mathbb{H}$-semimartingale. Then, the following assertions are equivalent.*
(a) *X admits an $\mathbb{H}$-deflator.*
(b) *X satisfies NUPBR($\mathbb{H}$).*

Proof The proof of this lemma is straightforward, and is omitted. □

Now, we start giving the proof of Theorem 2.17.

Proof of Theorem 2.17. The proof of the theorem will given in two steps, where we prove (b)⇒(a) and the reverse implication respectively. For the sake of simplifying the overall proof of the theorem, we remark that

$$\{\widetilde{Z}_T^Q=0\}=\{\widetilde{Z}_T=0\},\quad \text{for any } Q\sim P \text{ and any } \mathbb{F}\text{-stopping time } T, \tag{5.21}$$

where $\widetilde{Z}_t^Q:=Q[\tau\geq t|\mathcal{F}_t]$. This equality follows from

$$E\left[\widetilde{Z}_T I_{\{\widetilde{Z}_T^Q=0\}}\right]=E\left[I_{\{\tau\geq T\}}I_{\{\widetilde{Z}_T^Q=0\}}\right]=0,$$

(which implies $\{\widetilde{Z}_T^Q=0\}\subset\{\widetilde{Z}_T=0\}$) and the symmetric role of Q and P.
Step 1: Here, we prove (b)⇒ (a). Suppose that assertion (b) holds, and consider a sequence of $\mathbb{F}$-stopping times $(\tau_n)_n$ that increases to infinity such that Y^{τ_n} is an $\mathbb{F}$-martingale. Then, setting $Q_n:=Y_{\tau_n}/Y_0\cdot P$, and using (5.21) and Proposition 2.6, we deduce that there is no loss of generality in assuming $Y\equiv 1$. Condition (5.14) in Proposition 5.1 holds for $T=T_n$ $M_T:=\Delta S_{T_n}I_{\{\widetilde{Z}_{T_n}>0\}}$ and $M:=\Delta S_{T_n}I_{\{\widetilde{Z}_{T_n}>0\}}I_{[\![T_n,+\infty[\![}$,. Therefore, using the notation $V^{\mathbb{G}}$ and L defined in (3.8) and (3.10), for each n, $(1+\Delta L_{T_n}-\Delta V^{\mathbb{G}}_{T_n})\Delta S_{T_n}I_{\{T_n\leq\tau\}}I_{[\![T_n,+\infty[\![}$ is a $\mathbb{G}$-martingale. Then, a direct application of Yor's exponential formula, we get that, for any $\theta\in L(S^{\tau},\mathbb{G})$

$$\mathcal{E}\left(I_{\Gamma}\bullet L-I_{\Gamma}\bullet V^{\mathbb{G}}\right)\mathcal{E}\left(\theta I_{\Gamma}\bullet S^{\tau}\right)=\mathcal{E}\left(X\right)$$

where $\Gamma = \cup_{n\geq 1}[\![T_n]\!]$ and

$$X := I_\Gamma \cdot L - I_\Gamma \cdot V^{\mathbb{G}} + \sum_{n\geq 1} \theta_{T_n} \left(1 + \Delta L_{T_n} - \Delta V^{\mathbb{G}}_{T_n}\right) \Delta S_{T_n} I_{\{T_n\leq\tau\}} I_{[\![T_n,+\infty[\![}.$$

Consider now the $\mathbb{G}$-predictable process

$$\phi = \sum_{n\geq 1} \xi_n I_{[\![T_n]\!]\cap[\![0,\tau]\!]} + I_{\Gamma^c\cup]\!]\tau+\infty[\![}, \qquad \text{where}$$

$$\xi_n := \frac{2^{-n}\left(1+\mathcal{E}(X)_{T_n-}\right)^{-1}}{\left(1 + E\left[|\Delta L_{T_n}|\,\middle|\,\mathcal{G}_{T_n-}\right] + \Delta V^{\mathbb{G}}_{T_n-} + E\left[|\theta_{T_n}\frac{Z_{T_n-}}{\widetilde{Z}_{T_n}} I_{\{T_n\leq\tau\}}\Delta S_{T_n}|\,\middle|\mathcal{G}_{T_n-}\right]\right)}.$$

Then, it is easy to verify that $0 < \phi \leq 1$ and $E\left(|\phi \cdot \mathcal{E}(X)|_{var}(+\infty)\right) \leq \sum_{n\geq 1} 2^{-n} = 1$. Hence, $\phi \cdot \mathcal{E}(X) \in \mathcal{A}(\mathbb{G})$. Since, $\Delta L_{T_n} I_{[\![T_n,+\infty[\![}$ and $(1+\Delta L_{T_n} - \Delta V^{\mathbb{G}}_{T_n})\Delta S_{T_n} I_{\{T_n\leq\tau\}} I_{[\![T_n,+\infty[\![}$ are $\mathbb{G}$-martingales, we derive

$$(\phi \cdot \mathcal{E}(X))^{p,\mathbb{G}} = \sum_{n\geq 1} \phi_{T_n}\mathcal{E}_{T_n-}(X) E(\Delta X_{T_n}|\mathcal{G}_{T_n-}) I_{[\![T_n,+\infty[\![} = -\phi\mathcal{E}_-(X)\cdot V^{\mathbb{G}} \leq 0.$$

This proves that $\mathcal{E}(X)$ is a positive σ-supermartingale[2] as long as $\theta\Delta S^\tau \geq -1$. Thus, thanks to Kallsen [25], we conclude that it is a supermartingale and $\left(I_{\{Z_-\geq\delta\}} \cdot S\right)^\tau$ admits a $\mathbb{G}$-deflator. Then, thanks to Lemma 5.4, we deduce that $\left(I_{\{Z_-\geq\delta\}} \cdot S\right)^\tau$ satisfies NUPBR($\mathbb{G}$). Remark that, due to the $\mathbb{G}$-local boundedness of $(Z_-)^{-1} I_{[\![0,\tau]\!]}$, there exists a family of $\mathbb{G}$-stopping times τ_δ, $\delta > 0$ such that τ_δ converges almost surely to infinity when δ goes zero and

$$[\![0, \tau\wedge\tau_\delta]\!] \subset \{Z_- \geq \delta\}.$$

This implies that $S^{\tau\wedge\tau_\delta}$ satisfies NUPBR($\mathbb{G}$), and the assertion (a) follows from Proposition 2.6 (by taking $Q_n = P$ for all $n \geq 1$). This ends the proof of (b)⇒(a).

Step 2: In this step, we focus on (a)⇒(b). Suppose that S^τ satisfies NUPBR($\mathbb{G}$). Then, there exists a σ-martingale density under $\mathbb{G}$, for $I_{\{Z_-\geq\delta\}} \cdot S^\tau$, ($\delta > 0$), that we denote by $D^{\mathbb{G}}$. Then, from a direct application of Theorem A.1 and Theorem A.4, we deduce the existence of a positive $\widetilde{\mathcal{P}}(\mathbb{G})$-measurable functional, $f^{\mathbb{G}}$, such that $D^{\mathbb{G}} := \mathcal{E}(N^{\mathbb{G}}) > 0$, with

$$N^{\mathbb{G}} := W^{\mathbb{G}} \star (\mu^{\mathbb{G}} - \nu^{\mathbb{G}}),\ W^{\mathbb{G}} := f^{\mathbb{G}} - 1 + \frac{\widehat{f}^{\mathbb{G}} - a^{\mathbb{G}}}{1-a^{\mathbb{G}}} I_{\{a^{\mathbb{G}}<1\}},$$

where $\nu^{\mathbb{G}}$ was defined in (4.6), and, introducing f_m defined in (4.5)

$$x f^{\mathbb{G}} I_{\{Z_-\geq\delta\}} \star \nu^{\mathbb{G}} = x f^{\mathbb{G}} \left(1 + \frac{f_m}{Z_-}\right) I_{]\!]0,\tau]\!]} I_{\{Z_-\geq\delta\}} \star \nu \equiv 0. \tag{5.22}$$

[2] Recall that a process X is said to be a σ-supermartingale if it is a semimartingale and there exists a predictable process ϕ such that $0 < \phi \leq 1$ and $\phi \cdot X$ is a supermartingale

Thanks to Lemma C.1, we conclude to the existence of a positive $\widetilde{\mathcal{P}}(\mathbb{F})$-measurable functional, f, such that $f^{\mathbb{G}} I_{\rrbracket 0,\tau\rrbracket} = f I_{\rrbracket 0,\tau\rrbracket}$. Thus (5.22) becomes

$$U^{\mathbb{G}} := xf\left(1+\frac{f_m}{Z_-}\right) I_{\rrbracket 0,\tau\rrbracket} I_{\{Z_->\delta\}} \star \nu \equiv 0.$$

Introduce the following notations

$$\begin{aligned} \mu_0 &:= I_{\{\widetilde{Z}>0 \ \&\ Z_-\geq\delta\}} \cdot \mu, \quad \nu_0 := h_0 I_{\{Z_-\geq\delta\}} \cdot \nu, \ h_0 := M_\mu^P\left(I_{\{\widetilde{Z}>0\}}|\widetilde{\mathcal{P}}\right), \\ g &:= \frac{f(1+\frac{f_m}{Z_-})}{h_0} I_{\{h_0>0\}} + I_{\{h_0=0\}}, \quad a_0(t) := \nu_0(\{t\},\mathbb{R}^d), \end{aligned} \tag{5.23}$$

and assume that

$$\sqrt{(g-1)^2 \star \mu_0} \in \mathcal{A}^+_{loc}(\mathbb{F}). \tag{5.24}$$

Then, thanks to Lemma A.5, we deduce that $W := (g-1)/(1-a^0+\widehat{g}) \in \mathcal{G}^1_{loc}(\mu_0,\mathbb{F})$, and the local martingales

$$N^0 := \frac{g-1}{1-a^0+\widehat{g}} \star (\mu_0-\nu_0), \quad Y^0 := \mathcal{E}(N^0),$$

are well defined satisfying $1+\Delta N^0 > 0$, $[N^0,S] \in \mathcal{A}(\mathbb{F})$, and on $\{Z_->\delta\}$ we have

$$\begin{aligned} \frac{{}^{p,\mathbb{F}}\left(Y^0 \Delta S I_{\{\widetilde{Z}>0\}}\right)}{Y^0_-} &= {}^{p,\mathbb{F}}\left((1+\Delta N^0)\Delta S I_{\{\widetilde{Z}>0\}}\right) = {}^{p,\mathbb{F}}\left(\frac{g}{1-a^0+\widehat{g}}\Delta S I_{\{\widetilde{Z}>0\}}\right) \\ &= \Delta \frac{gxh_0}{1-a^0+\widehat{g}} \star \nu = \Delta \frac{xf(1+f_m/Z_-)}{1-a^0+\widehat{g}} \star \nu = Z_-^{-1} \frac{{}^{p,\mathbb{F}}\left(\Delta U^{\mathbb{G}}\right)}{1-a^0+\widehat{g}} \equiv 0. \end{aligned}$$

This proves that assertion (b) holds under the assumption (5.24).

The remaining part of the proof will show that this assumption always holds. To this end, we start by noticing that on the set $\{h_0>0\}$,

$$\begin{aligned} g-1 &= \frac{f(1+\frac{f_m}{Z_-})}{h_0} - 1 = \frac{(f-1)(1+\frac{f_m}{Z_-})}{h_0} + \frac{f_m}{Z_- h_0} + \frac{M_\mu^P\left(I_{\{\widetilde{Z}=0\}}|\widetilde{\mathcal{P}}\right)}{h_0} \\ &:= g_1+g_2+g_3. \end{aligned}$$

Since $\left((f-1)^2 I_{\rrbracket 0,\tau\rrbracket} \star \mu\right)^{1/2} \in \mathcal{A}^+_{loc}(\mathbb{G})$, then due to Proposition C.3–(e)

$$\sqrt{(f-1)^2 I_{\{Z_-\geq\delta\}} \star (\widetilde{Z}\cdot\mu)} \in \mathcal{A}^+_{loc}(\mathbb{F}), \quad \text{for any} \ \ \delta>0.$$

Then, a direct application of Proposition C.3–(a), for any $\delta>0$, we have

$$(f-1)^2 I_{\{|f-1|\leq\alpha \ \&\ Z_-\geq\delta\}} \star (\widetilde{Z}\cdot\mu), \quad |f-1| I_{\{|f-1|>\alpha \ \&\ Z_-\geq\delta\}} \star (\widetilde{Z}\cdot\mu) \in \mathcal{A}^+_{loc}(\mathbb{F}).$$

By stopping, without loss of generality, we assume these two processes and $[m,m]$ belong to $\mathcal{A}^+(\mathbb{F})$. Remark that $Z_-+f_m = M_\mu^P\left(\widetilde{Z}|\widetilde{\mathcal{P}}\right) \leq M_\mu^P\left(I_{\{\widetilde{Z}>0\}}|\widetilde{\mathcal{P}}\right) = h_0$ that follows from $\widetilde{Z} \leq I_{\{\widetilde{Z}>0\}}$. Therefore, we derive

$$\begin{aligned}
E\left[g_1^2 I_{\{|f-1|\leq\alpha\}} \star \mu_0(\infty)\right] &= E\left[\frac{(f-1)^2(1+\frac{f_m}{Z_-})^2}{h_0^2} I_{\{|f-1|\leq\alpha\}} \star \mu_0(\infty)\right] \\
&= E\left[\frac{(f-1)^2(1+\frac{f_m}{Z_-})^2}{h_0^2} I_{\{|f-1|\leq\alpha\}} \star \nu_0(\infty)\right] \\
&\leq \delta^{-2} E\left[(f-1)^2(Z_-+f_m) I_{\{|f-1|\leq\alpha \ \&\ Z_-\geq\delta\}} \star \nu(\infty)\right] \\
&= \delta^{-2} E\left[(f-1)^2 I_{\{|f-1|\leq\alpha\}} \star (\widetilde{Z} I_{\{Z_-\geq\delta\}} \cdot \mu)(\infty)\right] < +\infty,
\end{aligned}$$

and

$$\begin{aligned}
E\left[g_1 I_{\{|f-1|>\alpha\}} \star \mu_0(\infty)\right] &= E\left[\frac{|f-1|(1+\frac{f_m}{Z_-})}{h_0} I_{\{|f-1|>\alpha\}} \star \mu_0(\infty)\right] \\
&= E\left[|f-1|(1+\frac{f_m}{Z_-}) I_{\{|f-1|>\alpha\}} I_{\{Z_-\geq\delta\}} \star \nu(\infty)\right] \\
&\leq \delta^{-1} E\left[|f-1| I_{\{|f-1|>\alpha\}} \star (\widetilde{Z} I_{\{Z_-\geq\delta\}} \cdot \mu)(\infty)\right] < +\infty.
\end{aligned}$$

Here μ_0 and ν_0 are defined in (5.23). Therefore, again by Proposition C.3–(a), we conclude that $\sqrt{g_1^2 \star \mu_0} \in \mathcal{A}^+_{loc}(\mathbb{F})$.

Notice that $g_2 + g_3 = \frac{M_\mu^P\left(\Delta m I_{\{\widetilde{Z}>0\}}|\widetilde{\mathcal{P}}\right)}{Z_- h_0}$, and due to Lemma A.2, we derive

$$\begin{aligned}
E\left[(g_2+g_3)^2 \star \mu_0(\infty)\right] &= E\left[\frac{M_\mu^P\left(\Delta m I_{\{\widetilde{Z}>0\}}|\widetilde{\mathcal{P}}\right)^2}{Z_-^2 h_0^2} \star \mu_0(\infty)\right] \\
&\leq E\left[\frac{M_\mu^P\left((\Delta m)^2|\widetilde{\mathcal{P}}\right) M_\mu^P\left(I_{\{\widetilde{Z}>0\}}|\widetilde{\mathcal{P}}\right)}{Z_-^2 h_0^2} \star \mu_0(\infty)\right] \\
&= E\left[\frac{M_\mu^P\left((\Delta m)^2|\widetilde{\mathcal{P}}\right)}{Z_-^2} I_{\{Z_-\geq\delta\}} \star \mu(\infty)\right] \\
&\leq \delta^{-2} E\left[[m,m]_\infty\right] < +\infty.
\end{aligned}$$

Hence, we conclude that $\sqrt{(g-1)^2 \star \mu_0} \in \mathcal{A}^+_{loc}(\mathbb{F})$. This ends the proof of (5.24), and the proof of the theorem is completed.

□

APPENDIX

A Representation of Local Martingales

This section recall an important result on representation of local martingales. This result relies on the continuous local martingale part and the jump random measure of a given semimartingale. Thus, throughout this section, we suppose given a d-dimensional semimartingale, $S = (S_t)_{0\leq t\leq T}$. To this semimartingale, we associate its predictable characteristics that we will present below (for more details about these and other related issues, we refer the reader to Section II.2 of [21]). The random measure μ associated to the jumps of S is defined in 2.3 by The continuous local martingale part of S is denoted by S^c. This leads to the following decomposition, called "*the canonical representation*" (see Theorem 2.34, Section II.2 of [21]), namely the decomposition 5.1. For the matrix C with entries $C^{ij} := \langle S^{c,i}, S^{c,j}\rangle$, the triple $(b \centerdot A,\ C,\ \nu)$ is called *predictable characteristics* of S. Furthermore, we can find a version of the characteristics triple satisfying

$$C = c \centerdot A \ \text{ and } \ \nu(\omega,\ dt,\ dx) = dA_t(\omega)F_t(\omega,\ dx). \tag{A.1}$$

Here A is an increasing and predictable process which is continuous if and only if S is quasi-left continuous, b and c are predictable processes, $F_t(\omega,\ dx)$ is a predictable kernel, $b_t(\omega)$ is a vector in $\mathbb{R}^d$ and $c_t(\omega)$ is a symmetric $d \times d$-matrix , for all $(\omega,\ t) \in \Omega \times [0,\ T]$. In the sequel we will often drop ω and t and write, for instance, $F(dx)$ as a shorthand for $F_t(\omega, dx)$.
The characteristics, $B = b \centerdot A,\ C$, and ν, satisfy

$$F_t(\omega,\ \{0\}) = 0, \qquad \int (|x|^2 \wedge 1)F_t(\omega,\ dx) \leq 1,$$

$$\Delta B_t = b\Delta A = \int h(x)\nu(\{t\}, dx), \qquad \text{and} \qquad c = 0 \quad \text{on} \ \ \{\Delta A \neq 0\}.$$

We set

$$\nu_t(dx) := \nu(\{t\}, dx), \quad a_t := \nu_t(\mathbb{R}^d) = \Delta A_t F_t(\mathbb{R}^d) \leq 1.$$

For the following representation theorem, we refer to [20, Theorem 3.75, page 103] and to [21, Lemma 4.24, Chap III].

Theorem A.1 *Let $N \in \mathcal{M}_{0,loc}$. Then, there exist a predictable S^c-integrable process β, $N^\perp \in \mathcal{M}_{0,loc}$ with $N^\perp$ and S orthogonal and functionals $f \in \widetilde{\mathcal{P}}$ and $g \in \widetilde{\mathcal{O}}$ such that*
(a) $\left(\sum_{s\leq t} f_s(\Delta S_s)^2 I_{\{\Delta S_s\neq 0\}}\right)^{1/2}$ *and* $\left(\sum_{s\leq t} g_s(\Delta S_s)^2 I_{\{\Delta S_s\neq 0\}}\right)^{1/2}$ *belong to* $\mathcal{A}^+_{loc}$.
(b) $M^P_\mu(g \mid \widetilde{\mathcal{P}}) = 0, \quad M^P_\mu - a.e.,$ *where* $M^P_\mu := P \otimes \mu$.
(c) *The process N satisfies*

$$N = \beta \centerdot S^c + W \star (\mu - \nu) + g \star \mu + N^\perp, \quad \text{where } W = f + \frac{\widehat{f}}{1-a} I_{\{a<1\}}. \tag{A.2}$$

Here $\widehat{f}_t = \int f_t(x)\nu(\{t\}, dx)$ *and* f *has a version such that* $\{a = 1\} \subset \{\widehat{f} = 0\}$. *Moreover*

$$\Delta N_t = \Big(f_t(\Delta S_t) + g_t(\Delta S_t)\Big) I_{\{\Delta S_t \neq 0\}} - \frac{\widehat{f}_t}{1 - a_t} I_{\{\Delta S_t = 0\}} + \Delta N_t^{\perp}. \tag{A.3}$$

The quadruplet $\left(\beta, f, g, N^{\perp}\right)$ *are called the Jacod's parameters of the local martingale* N *with respect to* S.

The following is a simple but useful result on the conditional expectation with respect to M_μ^P.

Lemma A.2 *Consider a filtration* $\mathbb{H}$ *satisfying the usual conditions. Let* f *and* g *two nonnegative* $\widetilde{\mathcal{O}}(\mathbb{H})$*-measurable functionals. Then we have*

$$M_\mu^P \left(fg \middle| \widetilde{\mathcal{P}}\right)^2 \leq M_\mu^P \left(f^2 \middle| \widetilde{\mathcal{P}}\right) M_\mu^P \left(g^2 \middle| \widetilde{\mathcal{P}}\right), \qquad M_\mu^P\text{–}a.e. \tag{A.4}$$

Proof The proof is the same as the one of the regular Cauchy-Schwarz formula, by putting $\bar{f} := f / \left(M_\mu^P \left(f^2 \middle| \widetilde{\mathcal{P}}\right)\right)^{1/2}$ and $\bar{g} := g / \left(M_\mu^P \left(g^2 \middle| \widetilde{\mathcal{P}}\right)\right)^{1/2}$ and using the simple inequality $xy \leq (x^2 + y^2)/2$. This ends the proof of the lemma. □

The following lemma is borrowed from Jacod's Theorem 3.75 in [20] (see also Proposition 2.2 in [7]).

Lemma A.3 *Let* $\mathcal{E}(N)$ *be a positive local martingale and* (β, f, g, N') *be the Jacod's parameters of* N. *Then* $\mathcal{E}(N) > 0$ *(or equivalently* $1 + \Delta N > 0$*) implies that*

$$f > 0, \quad M_\mu^P - a.e.$$

Theorem A.4 *Let* S *be a semi-martingale with predictable characteristic triplet* $(b, c, \nu = A \otimes F)$, N *be a local martingale such that* $\mathcal{E}(N) > 0$, *and* (β, f, g, N') *be its Jacod's parameters. Then the following assertions hold.*
1) $\mathcal{E}(N)$ *is a* σ*-martingale density of* S *if and only if the following two properties hold:*

$$\int |x - h(x) + xf(x)| F(dx) < +\infty, \quad P \otimes A - a.e. \tag{A.5}$$

$$b + c\beta + \int \Big(x - h(x) + xf(x)\Big) F(dx) = 0, \qquad P \otimes A - a.e. \tag{A.6}$$

2) In particular, we have

$$\int x(1 + f_t(x))\nu(\{t\}, dx) = \int x(1 + f_t(x)) F_t(dx) \Delta A_t = 0, \qquad P - a.e. \tag{A.7}$$

Proof The proof can be found in Choulli et al. [6, Lemma 2.4], and also Choulli and Schweizer [7]. □

Lemma A.5 (see Choulli and Schweizer [7]): *Consider a filtration $\mathbb{H}$ satisfying the usual conditions. Let f be a $\widetilde{\mathcal{P}}(\mathbb{H})$-measurable functional such that $f > 0$ and*

$$\Big[(f-1)^2 \star \mu\Big]^{1/2} \in \mathcal{A}^+_{loc}(\mathbb{H}). \tag{A.8}$$

Then, the $\mathbb{H}$-predictable process $\left(1 - a^{\mathbb{H}} + \widehat{f}^{\mathbb{H}}\right)^{-1}$ is locally bounded, and hence

$$W_t(x) := \frac{f_t(x) - 1}{1 - a^{\mathbb{H}}_t + \widehat{f}^{\mathbb{H}}_t} \in \mathcal{G}^1_{loc}(\mu, \mathbb{H}). \tag{A.9}$$

Here, $a^{\mathbb{H}}_t := \nu^{\mathbb{H}}(\{t\}, \mathbb{R}^d)$, $\widehat{f}^{\mathbb{H}}_t := \int f_t(x)\nu^{\mathbb{H}}(\{t\}, dx)$ and $\nu^{\mathbb{H}}$ is the $\mathbb{H}$-predictable random measure compensator of μ under $\mathbb{H}$.

B Proof of $K \in {}^{o}L^1_{loc}(\widehat{m}, \mathbb{G})$

We start by calculating on $]\!]0, \tau]\!]$, making use of Lemma 3.3. We recall that $\kappa := Z^2_- + \Delta\langle m\rangle^{\mathbb{F}}$.

$$\begin{aligned} K\Delta\widehat{m} - {}^{p,\mathbb{G}}(K\Delta\widehat{m}) &= \frac{I_{]\!]0,\tau]\!]} Z^2_- \Delta\widehat{m}}{\kappa\widetilde{Z}} - {}^{p,\mathbb{G}}\left(\frac{I_{]\!]0,\tau]\!]} Z^2_-}{\kappa\widetilde{Z}}\,\Delta\widehat{m}\right) \\ &= \frac{(Z^2_-\Delta m - Z_-\Delta\langle m\rangle^{\mathbb{F}})}{\kappa\,\widetilde{Z}} + \frac{{}^{p,\mathbb{F}}(I_{\{\widetilde{Z}>0\}}\,\Delta\langle m\rangle^{\mathbb{F}})}{\kappa} - \frac{{}^{p,\mathbb{F}}(\Delta m I_{\{\widetilde{Z}>0\}})\,Z_-}{\kappa} \\ &= \frac{\Delta m}{\widetilde{Z}} I_{]\!]0,\tau]\!]} - {}^{p,\mathbb{F}}\left(I_{\{\widetilde{Z}=0\}}\right) I_{]\!]0,\tau]\!]} =: \Delta V - \Delta V^{\mathbb{G}}. \end{aligned} \tag{B.1}$$

Here, $V^{\mathbb{G}}$, defined in (3.8), is nondecreasing, càdlàg and $\mathbb{G}$-locally bounded (see Proposition 3.5). Hence, we immediately deduce that $\sum(\Delta V^{\mathbb{G}})^2 = \Delta V^{\mathbb{G}} \boldsymbol{.} V^{\mathbb{G}}$ is locally bounded, and in the rest of this part we focus on proving $\sqrt{\sum(\Delta V)^2} \in \mathcal{A}^+_{loc}(\mathbb{G})$. To this end, we consider $\delta \in (0,1)$, and define $C := \{\Delta m < -\delta Z_-\}$ and C^c its complement in $\Omega \otimes [0, +\infty[$. Then we obtain

$$\begin{aligned} \sqrt{\sum(\Delta V)^2} &\le \left(\sum \frac{(\Delta m)^2}{\widetilde{Z}^2} I_C I_{]\!]0,\tau]\!]}\right)^{1/2} + \left(\sum \frac{(\Delta m)^2}{\widetilde{Z}^2} I_{C^c} I_{]\!]0,\tau]\!]}\right)^{1/2} \\ &\le \sum \frac{|\Delta m|}{\widetilde{Z}} I_C I_{]\!]0,\tau]\!]} + \frac{1}{1-\delta}\left(I_{]\!]0,\tau]\!]} \frac{1}{Z^2_-} \boldsymbol{.} [m]\right)^{1/2} =: V_1 + V_2. \end{aligned}$$

The last inequality above is due to $\sqrt{\sum(\Delta X)^2} \le \sum|\Delta X|$ and $\widetilde{Z} \ge Z_-(1-\delta)$ on C^c. Using the fact that $(Z_-)^{-1} I_{]\!]0,\tau]\!]}$ is $\mathbb{G}$-locally bounded and that m is an $\mathbb{F}$-locally bounded martingale, it follows that V_2 is $\mathbb{G}$-locally bounded. Hence, we focus on proving the $\mathbb{G}$-local integrability of V_1.

Consider a sequence of $\mathbb{G}$-stopping times $(\vartheta_n)_n$ that increases to $+\infty$ and

$$\left((Z_-)^{-1} I_{\rrbracket 0,\tau \rrbracket}\right)^{\vartheta_n} \leq n.$$

Also consider an $\mathbb{F}$-localizing sequence of stopping times, $(\tau_n)_n$, for the process $V_3 := \sum \frac{(\Delta m)^2}{1+|\Delta m|}$. Then, it is easy to prove

$$U_n := \sum |\Delta m| I_{\{\Delta m < -\delta/n\}} \leq \frac{n+\delta}{\delta} V_3,$$

and conclude that $(U_n)^{\tau_n} \in \mathcal{A}^+(\mathbb{F})$. Therefore, due to

$$\begin{aligned} C \cap \rrbracket 0,\tau \rrbracket \cap \llbracket 0,\vartheta_n \rrbracket &= \{\Delta m < -\delta Z_-\} \cap \rrbracket 0,\vartheta_n \rrbracket \cap \rrbracket 0,\tau \rrbracket \\ &\subset \rrbracket 0,\tau \rrbracket \cap \rrbracket 0,\vartheta_n \rrbracket \cap \{\Delta m < -\tfrac{\delta}{n}\}, \end{aligned}$$

we derive

$$(V_1)^{\vartheta_n \wedge \tau_n} \leq \left(\widetilde{Z}\right)^{-1} I_{\rrbracket 0,\tau \rrbracket} \bullet (U_n)^{\tau_n}.$$

Since $(U_n)^{\tau_n}$ is $\mathbb{F}$-adapted, nondecreasing and integrable, then due to Lemma 3.4, we deduce that the process $V_1^{\vartheta_n \wedge \tau_n}$ is nondecreasing, $\mathbb{G}$-adapted and integrable. Since $\vartheta_n \wedge \tau_n$ increases to $+\infty$, we conclude that the process V_1 is $\mathbb{G}$-locally integrable. This completes the proof of $K \in {}^o L^1_{loc}(\widehat{m}, \mathbb{G})$, and the process L (given via (3.10) and Definition 3.6) is a $\mathbb{G}$-local martingale. □

C $\mathbb{G}$-Localization versus $\mathbb{F}$-Localization

We now present results that are important for the proofs of Subsection 5.1, and are the most innovative results of the appendix.

Lemma C.1 *The following assertions hold.*
(a) If $H^{\mathbb{G}}$ is a $\widetilde{\mathcal{P}}(\mathbb{G})$-measurable functional, then there exist an $\widetilde{\mathcal{P}}(\mathbb{F})$-measurable functional $H^{\mathbb{F}}$ and a $\mathcal{B}(\mathbb{R}_+) \otimes \widetilde{\mathcal{P}}(\mathbb{F})$-measurable functionals $K^{\mathbb{F}} : \mathbb{R}_+ \times \mathbb{R}_+ \times \Omega \times \mathbb{R}^d \to \mathbb{R}$ such that

$$H^{\mathbb{G}}(\omega,t,x) = H^{\mathbb{F}}(\omega,t,x) I_{\rrbracket 0,\tau \rrbracket} + K^{\mathbb{F}}(\tau(\omega),t,\omega,x) I_{\rrbracket \tau,+\infty \rrbracket}. \tag{C.1}$$

(b) If furthermore $H^{\mathbb{G}} > 0$ (respectively $H^{\mathbb{G}} \leq 1$), then we can choose $H^{\mathbb{F}} > 0$ (respectively $H^{\mathbb{F}} \leq 1$) such that

$$H^{\mathbb{G}}(\omega,t,x) I_{\rrbracket 0,\tau \rrbracket} = H^{\mathbb{F}}(\omega,t,x) I_{\rrbracket 0,\tau \rrbracket}.$$

(d) If $L^{\mathbb{G}}$ is an $\widetilde{\mathcal{O}}(\mathbb{G})$-measurable functional, then there exist a $\widetilde{\mathcal{O}}(\mathbb{G})$-measurable functional $L^{(1)}(t,\omega,x)$, a $\widetilde{\mathcal{P}}_{prog}(\mathbb{F})$-measurable functional $L^{(2)}(t,\omega,x)$ and $\widetilde{\mathcal{P}}_{prog}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^d)$-measurable functional, $L^{(3)}(t,\omega,x,v)$, such that

$$L^{\mathbb{G}}(t,\omega,x) = L^{(1)}(t,\omega,x) I_{\rrbracket 0,\tau \llbracket} + L^{(1)}(t,\omega,x) I_{\llbracket \tau \rrbracket} + L^{(3)}(t,\omega,x,\tau) I_{\rrbracket \tau,+\infty \rrbracket} \tag{C.2}$$

where $\mathcal{P}_{prog}(\mathbb{F})$ is the $\mathbb{F}$-progressive σ-field on $\Omega\times\mathbb{R}^+$, and $\widetilde{\mathcal{P}}_{prog}(\mathbb{F}) := \mathcal{P}_{prog}(\mathbb{F})\otimes \mathcal{B}(\mathbb{R}^d)$. If furthermore, $0 < L^{\mathbb{G}}$ (respectively $L^{\mathbb{G}} \leq 1$), then all $L^{(i)}$ can be chosen such that $0 < L^{(i)}$ (respectively $L^{(i)} \leq 1$), $i = 1, 2, 3$.
(e) For any $\mathbb{F}$-stopping time, T, and any positive $\mathcal{G}_T$-measurable random variable $Y^{\mathbb{G}}$, there exist two positive $\mathcal{F}_T$-measurable random variables, $Y^{(1)}$ and $Y^{(2)}$, satisfying

$$Y^{\mathbb{G}} I_{\{T\leq\tau\}} = Y^{(1)} I_{\{T<\tau\}} + Y^{(2)} I_{\{\tau=T\}}. \tag{C.3}$$

Proof 1) Here we will prove the assertions (a), (b), and (c). The proof of assertion (a) mimics exactly the approach of Jeulin [22], and will be omitted. To prove positivity of $H^{\mathbb{F}}$ when $H^{\mathbb{G}} > 0$ holds, we consider

$$\overline{H}^{\mathbb{F}} := (H^{\mathbb{F}})^+ + I_{\{H^{\mathbb{F}}=0\}} > 0,$$

and we remark that due to (C.1), we have $]\!]0,\tau]\!] \subset \{H^{\mathbb{G}} = H^{\mathbb{F}}\} \subset \{H^{\mathbb{F}} > 0\}$. Thus, we get $H^{\mathbb{G}} I_{]\!]0,\tau]\!]} = \overline{H}^{\mathbb{F}} I_{]\!]0,\tau]\!]}$. Similarly, we consider $H^{\mathbb{F}} \wedge 1$, and we deduce that if $H^{\mathbb{G}}$ is upper-bounded by one, the process $H^{\mathbb{F}}$ can also be chosen to not exceed one.
2) Here we will prove assertion (d), which is the most innovative part of this lemma. For its proof we mimic Jeulin (see [22]) in his proof of Proposition (5,3)–assertion b). In fact, it is clear that the σ-field $\widetilde{\mathcal{O}}(\mathbb{H})$ is generated by the functionals $H((t,\omega,x) := g(x) I_{]\!]\sigma_1,\sigma_2]\!]}$ where σ_i are $\mathbb{G}$-stopping times such that $\sigma_1 < \sigma_2$ on $\{\sigma_1 < +\infty\}$. Then, it is easy to remark that

$$H(t,\omega,x) = \lim_{s\longrightarrow t} \sup \overline{H}(s,\omega,x), \qquad \text{where } \overline{H}(s,\omega,x) := g(x) I_{]\!]\sigma_1,\sigma_2]\!]}(s).$$

Thus a direct application of assertion (a) to the functional $\overline{H}$, we obtain the existence of an $\widetilde{\mathcal{P}}(\mathbb{F})$-measurable functional, $J(t,\omega,x)$, and a $\mathcal{B}(\mathbb{R}_+)\otimes\widetilde{\mathcal{P}}(\mathbb{F})$-measurable functional, $K(t,\omega,x,v)$, such that

$$\overline{H}(s,\omega,x) = J(t,\omega,x) I_{]\!]0,\tau]\!]} + K(t,\omega,x,\tau) I_{]\!]\tau,+\infty]\!]}.$$

Put

$$\overline{J}(t,\omega,x) := \lim_{s\longrightarrow t} \sup J(s,\omega,x), \quad \overline{K}(t,\omega,x,v) := \lim_{s\longrightarrow t} \sup K(s,\omega,x,v),$$

and

$$W(t,\omega,x,v) := \sup \left\{ \lim_{\substack{s\longrightarrow t \\ s<t}} \sup J(s,\omega,x), \lim_{\substack{s\longrightarrow t \\ s>t}} \sup K(s,\omega,x,v) \right\}.$$

Then, due to [11] (see also Lemma (4,1) in Jeulin [22]), $\overline{J}$ is $\widetilde{\mathcal{O}}(\mathbb{F})$-measurable, $\overline{K}$ is $\widetilde{\mathcal{O}}(\mathbb{F})\otimes\mathcal{B}(\mathbb{R}^+)$-measurable and $W(t,\omega,x,v)$ is $\widetilde{\mathcal{P}}_{prog}(\mathbb{F})\otimes\mathcal{B}(\mathbb{R}^+)$-measurable.

As a result, we deduce (due to Lemma C.2 below), that $\overline{W}(t,\omega,x) = W(t,\omega,x,t)$ is $\widetilde{\mathcal{P}}_{prog}(\mathbb{F})$-measurable, and we have

$$H(t,\omega,x) = \overline{J}(t,\omega,x)I_{]\!]0,\tau[\![} + \overline{W}(t,\omega,x)I_{[\![\tau]\!]} + \overline{K}(t,\omega,x,\tau)I_{]\!]\tau,+\infty]\!]}.$$

Thus, the proof of the first part of the assertion (d) follows from the class monotone theorem. The proof of the positivity and the upper boundedness follows the same arguments as in the proof of assertion (b).
3) The proof of assertion (e) is a direct application of assertion (d) combined with the fact that for any $\mathbb{F}$-progressively measurable process Y and any $\mathbb{F}$-stopping time T, we have $Y_T I_{\{T<+\infty\}}$ is $\mathcal{F}_T$-measurable. For this last fact, we refer the reader to Theorem 64 in [12]. This ends the proof of the lemma. □

Below, we state a simple but very useful lemma that generalizes a version elaborated in [8]

Lemma C.2 *If $X(t,\omega,x,v)$ is a $\widetilde{\mathcal{O}}(\mathbb{H}) \otimes \mathcal{B}(\mathbb{R}^+)$-measurable functional, then $\overline{X}(t,\omega,x,t)$ is $\widetilde{\mathcal{O}}(\mathbb{H})$-measurable.*

Proof The proof of this lemma is immediate from a combination of the class monotone theorem, and the proof of the lemma for the generators of $\widetilde{\mathcal{O}}(\mathbb{H}) \otimes \mathcal{B}(\mathbb{R}^+)$ having the form of $X(t,\omega,x,v) = H(t,\omega,x)k(v)$. Here H is $\widetilde{\mathcal{O}}(\mathbb{H})$-measurable and k is $\mathcal{B}(\mathbb{R}^+)$-measurable. For these generators, we have $\overline{X}(t,\omega,x) = H(t,\omega,x)k(t)$ which is obviously $\widetilde{\mathcal{O}}(\mathbb{H})$-measurable

Proposition C.3 *For any $\alpha > 0$, the following assertions hold:*
(a) *Let h be a $\widetilde{\mathcal{P}}(\mathbb{H})$-measurable functional. Then, $\sqrt{(h-1)^2 \star \mu} \in \mathcal{A}^+_{loc}(\mathbb{H})$ iff*

$$(h-1)^2 I_{\{|h-1|\le\alpha\}} \star \mu \text{ and } |h-1|I_{\{|h-1|>\alpha\}} \star \mu \text{ belong to } \mathcal{A}^+_{loc}(\mathbb{H}).$$

(b) *Let $(\sigma^{\mathbb{G}}_n)_n$ be a sequence of $\mathbb{G}$-stopping times that increases to infinity. Then, there exists a nondecreasing sequence of $\mathbb{F}$-stopping times, $(\sigma^{\mathbb{F}}_n)_{n\ge1}$, satisfying the following properties*

$$\sigma^{\mathbb{G}}_n \wedge \tau = \sigma^{\mathbb{F}}_n \wedge \tau, \quad \sigma_\infty := \sup_n \sigma^{\mathbb{F}}_n \ge \widehat{R} \quad P-a.s., \tag{C.4}$$

$$\text{and} \qquad Z_{\sigma_\infty -} = 0 \quad P-a.s. \quad \text{on} \quad \Sigma \cap (\sigma_\infty < +\infty), \tag{C.5}$$

where $\Sigma := \bigcap_{n\ge1} (\sigma^{\mathbb{F}}_n < \sigma_\infty)$.
(c) *Let V be an $\mathbb{F}$-predictable and non-decreasing process. Then, $V^\tau \in \mathcal{A}^+_{loc}(\mathbb{G})$ if and only if $I_{\{Z_-\ge\delta\}} \cdot V \in \mathcal{A}^+_{loc}(\mathbb{F})$ for any $\delta > 0$.*
(d) *Let h be a nonnegative and $\widetilde{\mathcal{P}}(\mathbb{F})$-measurable functional. Then, $hI_{]\!]0,\tau]\!]} \star \mu \in \mathcal{A}^+_{loc}(\mathbb{G})$ if and only if for all $\delta > 0$, $hI_{\{Z_-\ge\delta\}} \star \mu^1 \in \mathcal{A}^+_{loc}(\mathbb{F})$, where $\mu^1 := \widetilde{Z} \cdot \mu$.*
(e) *Let f be positive and $\widetilde{\mathcal{P}}(\mathbb{F})$-measurable, and $\mu^1 := \widetilde{Z} \cdot \mu$. Then $\sqrt{(f-1)^2 I_{]\!]0,\tau]\!]} \star \mu} \in \mathcal{A}^+_{loc}(\mathbb{G})$ iff $\sqrt{(f-1)^2 I_{\{Z_-\ge\delta\}} \star \mu^1} \in \mathcal{A}^+_{loc}(\mathbb{F})$, for all $\delta > 0$.*

Proof (a) Put $W := (h-1)^2 \star \mu = W_1 + W_2$, where $W_1 := (h-1)^2 I_{\{|h-1|\leq\alpha\}} \star \mu$, $W_2 := (h-1)^2 I_{\{|h-1|>\alpha\}} \star \mu$ and $W_2' := |h-1| I_{\{|h-1|>\alpha\}} \star \mu$. Note that

$$\sqrt{W} = \sqrt{W_1 + W_2} \leq \sqrt{W_1} + \sqrt{W_2} \leq \sqrt{W_1} + W_2'.$$

Therefore $\sqrt{W_1}, W_2' \in \mathcal{A}^+_{loc}$ imply that $\sqrt{W}$ is locally integrable.
Conversely, if $\sqrt{W} \in \mathcal{A}^+_{loc}$, then $\sqrt{W_1}$ and $\sqrt{W_2}$ are both locally integrable. Since W_1 is locally bounded and has finite variation, W_1 is locally integrable. In the following, we focus on the proof of the local integrability of W_2'. Denote

$$\tau_n := \inf\{t \geq 0 : \ V_t > n\}, \quad V := W_2.$$

It is easy to see that τ_n increases to infinity and $V_- \leq n$ on the set $]\!]0, \tau_n]\!]$. On the set $\{\Delta V > 0\}$, we have $\Delta V \geq \alpha^2$. By using the elementary inequality $\sqrt{1 + \frac{n}{\alpha^2}} - \sqrt{\frac{n}{\alpha^2}} \leq \sqrt{1+x} - \sqrt{x} \leq 1$, when $0 \leq x \leq \frac{n}{\alpha^2}$, we have

$$\sqrt{V_- + \Delta V} - \sqrt{V_-} \geq \beta_n \sqrt{\Delta V} \ \text{ on }]\!]0, \tau_n]\!], \text{ where } \ \beta_n := \sqrt{1 + \frac{n}{\alpha^2}} - \sqrt{\frac{n}{\alpha^2}},$$

and

$$(W_2')^{\tau_n} = \Big(\sum \sqrt{\Delta V}\Big)^{\tau_n} \leq \frac{1}{\beta_n} \Big(\sum \Delta \sqrt{V}\Big)^{\tau_n} = \frac{1}{\beta_n} \Big(\sqrt{W_2}\Big)^{\tau_n} \in \mathcal{A}^+_{loc}(\mathbb{H})$$

Therefore $W_2' \in (\mathcal{A}^+_{loc}(\mathbb{H}))_{loc} = \mathcal{A}^+_{loc}(\mathbb{H})$.

(b) Due to Jeulin [22], there exists a sequence of $\mathbb{F}$-stopping times $(\sigma_n^{\mathbb{F}})_n$ such that

$$\sigma_n^{\mathbb{G}} \wedge \tau = \sigma_n^{\mathbb{F}} \wedge \tau. \tag{C.6}$$

By putting $\sigma_n := \sup_{k\leq n} \sigma_k^{\mathbb{F}}$, we shall prove that

$$\sigma_n^{\mathbb{G}} \wedge \tau = \sigma_n \wedge \tau, \tag{C.7}$$

or equivalently $\{\sigma_n^{\mathbb{F}} \wedge \tau < \sigma_n \wedge \tau\}$ is negligible. Due to (C.6) and $\sigma_n^{\mathbb{G}}$ is nondecreasing, we derive

$$\{\sigma_n^{\mathbb{F}} < \tau\} = \{\sigma_n^{\mathbb{G}} < \tau\} \subset \bigcap_{i=1}^{n} \{\sigma_i^{\mathbb{G}} = \sigma_i^{\mathbb{F}}\} \subset \{\sigma_n^{\mathbb{F}} = \sigma_n\}.$$

This implies that,

$$\{\sigma_n^{\mathbb{F}} \wedge \tau < \sigma_n \wedge \tau\} = \{\sigma_n^{\mathbb{F}} < \tau, \ \& \ \sigma_n^{\mathbb{F}} < \sigma_n\} = \emptyset,$$

and the proof of (C.7) is completed. Without loss of generality we assume that the sequence $\sigma_n^{\mathbb{F}}$ is nondecreasing. By taking limit in (C.6), we obtain $\tau = \sigma_\infty \wedge \tau, P-$a.s. which is equivalent to $\sigma_\infty \geq \tau, P-$a.s. Since $\widehat{R}$ is the smallest $\mathbb{F}$-stopping time greater or equal than τ almost surely, we obtain,

$\sigma_\infty \geq \widehat{R} \geq \tau$ $P-a.s.$. This achieves the proof of (C.4).
On the set Σ, it is easy to show that

$$I_{[\![0,\sigma_n^{\mathbb{F}}]\!]} \longrightarrow I_{[\![0,\sigma_\infty^{\mathbb{F}}[\![}, \quad \text{when } n \text{ goes to } +\infty.$$

Then, thanks again to (C.6) (by taking $\mathbb{F}$-predictable projection and let n go to infinity afterwards), we obtain

$$Z_- = Z_- I_{[\![0,\sigma_\infty^{\mathbb{F}}[\![}, \qquad \text{on } \ \Sigma. \tag{C.8}$$

Hence, (C.5) follows immediately, and the proof of assertion (b) is completed.

(c) Suppose that $hI_{[\![0,\tau]\!]} \star \mu \in \mathcal{A}^+_{loc}(\mathbb{G})$. Then, there exists a sequence of $\mathbb{G}$-stopping times $(\sigma_n^{\mathbb{G}})$ increasing to infinity such that $hI_{[\![0,\tau]\!]} \star \mu^{\sigma_n^{\mathbb{G}}}$ is integrable. Consider (σ_n) a sequence of $\mathbb{F}$-stopping times satisfying (C.4)–(C.5) (its existence is guaranteed by assertion (b)). Therefore, for any fixed $\delta > 0$

$$W^n := M_\mu^P \left(\widetilde{Z}|\widetilde{\mathcal{P}}\right) I_{\{Z_- \geq \delta\}} h \star \nu^{\sigma_n} \in \mathcal{A}^+(\mathbb{F}), \tag{C.9}$$

or equivalently, this process is càdlàg predictable with finite values. Thus, it is obvious that the proof of assertion (iii) will follow immediately if we prove that the $\mathbb{F}$-predictable and nondecreasing process

$$W := M_\mu^P \left(\widetilde{Z}|\widetilde{\mathcal{P}}\right) I_{\{Z_- \geq \delta\}} h \star \nu \quad \text{is càdlàg with finite values.} \tag{C.10}$$

To prove this last fact, we consider the random time τ^δ defined by

$$\tau^\delta := \sup\{t \geq 0 : Z_{t-} \geq \delta\}.$$

Then, it is clear that $I_{]\!]\tau^\delta,+\infty[\![} \cdot W \equiv 0$ and

$$\tau^\delta \leq \widehat{R} \leq \sigma_\infty \quad \text{and} \quad Z_{\tau^\delta -} \geq \delta \quad P\text{–}a.s. \qquad \text{on } \{\tau^\delta < +\infty\}.$$

The proof of (C.10) will be achieved by considering three sets, namely $\{\sigma_\infty = \infty\}$, $\Sigma \cap \{\sigma_\infty < +\infty\}$, and $\Sigma^c \cap \{\sigma_\infty < +\infty\}$. It is obvious that (C.10) holds on $\{\sigma_\infty = \infty\}$. Due to (C.5), we deduce that $\tau^\delta < \sigma_\infty, P-$a.s. on $\Sigma \cap \{\sigma_\infty < +\infty\}$. Since W is supported on $[\![0, \tau^\delta]\!]$, then (C.10) follows immediately on the set $\Sigma \cap \{\sigma_\infty < +\infty\}$. Finally, on the set

$$\Sigma^c \cap \{\sigma_\infty < +\infty\} = \left(\bigcup_{n \geq 1} \{\sigma_n = \sigma_\infty\}\right) \cap \{\sigma_\infty < +\infty\},$$

the sequence σ_n increases stationarily to σ_∞, and thus (C.10) holds on this set. This completes the proof of (C.10), and hence $hI_{\{Z_- \geq \delta\}} \star (\widetilde{Z} \cdot \mu)$ is locally integrable, for any $\delta > 0$.

Conversely, if $hI_{\{Z_-\geq\delta\}}\widetilde{Z}\star\mu\in\mathcal{A}^+_{loc}(\mathbb{F})$, there exists a sequence of $\mathbb{F}$-stopping times $(\tau_n)_{n\geq1}$ that increases to infinity and $\left(hI_{\{Z_-\geq\delta\}}\widetilde{Z}\star\mu\right)^{\tau_n}\in\mathcal{A}^+(\mathbb{F})$. Then, we have

$$E\left[hI_{\{Z_-\geq\delta\}}I_{[\![0,\tau]\!]}\star\mu(\tau_n)\right]=E\left[hI_{\{Z_-\geq\delta\}}\widetilde{Z}\star\mu(\tau_n)\right]<+\infty. \quad \text{(C.11)}$$

This proves that $hI_{\{Z_-\geq\delta\}}I_{[\![0,\tau]\!]}\star\mu$ is $\mathbb{G}$-locally integrable, for any $\delta>0$. Since $(Z_-)^{-1}I_{[\![0,\tau]\!]}$ is $\mathbb{G}$-locally bounded, then there exists a family of $\mathbb{G}$-stopping times $(\tau_\delta)_{\delta>0}$ that increases to infinity when δ decreases to zero, and

$$[\![0,\tau\wedge\tau_\delta]\!]\subset\{Z_-\geq\delta\}.$$

This implies that the process $\left(hI_{[\![0,\tau]\!]}\star\mu\right)^{\tau_\delta}$ is $\mathbb{G}$-locally integrable, and hence the assertion (c) follows immediately.

(d) The proof of assertion (d) follows from combining assertions (a) and (b). This ends the proof of the proposition. □

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