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Non-Arbitrage under a Class of Honest Times

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Abstract This paper addresses the question of non-arbitrage (precisely No-Unbounded-Profit-with-Bounded-Risk, NUPBR hereafter) after a specific random time. This study completes the one of Aksamit et al. [1], devoted to the study before the random time, by elaborating results for the part after the random time under consideration. We restrict our attention to a subclass of honest times, and we characterize the pairs of initial market and honest time for which the resulting model preserves the NUPBR property. Furthermore, we characterize the honest times that preserve the NUPBR property for any semimartingale model. These findings are essentially based on new stochastic results that are interesting in themselves. Furthermore, we construct explicitly local martingale deflators for a large class of processes.

1 Introduction

We consider a stochastic basis $(\Omega, \mathcal{G}, \mathbb{F}, P)$, where $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ is a filtration satisfying the usual hypotheses (i.e., right continuity and completeness), and $\mathcal{F}_\infty \subseteq \mathcal{G}$. Financially speaking, the filtration $\mathbb{F}$ represents the flow of public information through time. Throughout the paper, a financial market will be represented by a d-dimensional semimartingale S and a riskless asset, with null interest rate. In addition to this model, we consider a fixed random time (a non-negative random variable) denoted by τ. If the initial model $(\Omega, \mathbb{F}, S)$ is arbitrage free, then what can we say about the new model $(\Omega, \mathbb{F}, S, \tau)$? This question, after mathematically modeling the *new informational system*, can be formulated by whether the model $(\Omega, \mathbb{G}, S)$ is arbitrage free, where $\mathbb{G}$ is the new flow of information that incorporates the flow $\mathbb{F}$ and τ, as soon as it

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occurs. In Aksamit et al. [1], the authors addressed partially this question, and focused precisely on the arbitrage theory for the sub-model $(\Omega, \mathbb{G}, S^\tau)$. Therefore, in virtue of the obtained results, our original question can be reduced to the arbitrage of the sub-model $(\Omega, \mathbb{G}, S - S^\tau)$. It is known in the literature that a process satisfies the NUPBR property only if this process is a semimartingale. Thus, the first question that arises before any arbitrage inquiry is whether the process $S - S^\tau$ is a $\mathbb{G}$ semimartingale. This is the main reason why we focus on random times that are honest times, when dealing with the "after" τ case, since honest times preserve the semimartingale structures. It is known, since the work of Imkeller [11] (see also Fontana et al. [9]), that for honest times avoiding stopping times in a Brownian filtration, the NUPBR property is violated after τ. Thus, the first challenging question is: Is there any subclass of honest times for which one can hope for the preservation of the NUPBR property for some models? The positive answer to this question constitutes our first original contribution of the paper. This subclass of honest times will be described later on in (2.3). Examples of these honest times, and their investigation is summarized in [3]. Herein, for this class of honest times, we will address two important questions. The first question is concerned with

the pairs (S, τ) such that the NUPBR property holds for $S - S^\tau$ **(Prob.1)**

The answer to this question will lead us to a deep understanding of the precise parts from both the random time and the initial market model that allow arbitrage to occur. The second question focuses on

for which τ, is $(S - S^\tau, \mathbb{G})$ arbitrage-free for any arbitrage-free S? **(Prob.2)**

This will allow us to single out the class of honest times that preserve the non-arbitrage for any arbitrage-free market model.

This paper is organized as follows. In the following section (Section 2), we present our main results, their immediate consequences, and/or their economic and financial interpretations. These results are formulated for particular as well as general framework. Section 3 is devoted to new stochastic developments vital for the proof of the main results, while Section 4 deals with the derivation of explicit local martingale deflators for a class of processes. The last section (Section 5) focuses on proving the main theorems of Section 2.

2 The Main Results

In what follows, $\mathbb{H}$ denotes a filtration satisfying the usual hypotheses. The set of $\mathbb{H}$-martingales is denoted by $\mathcal{M}(\mathbb{H})$. As usual, $\mathcal{A}^+(\mathbb{H})$ denotes the set of increasing, right-continuous, $\mathbb{H}$-adapted and integrable processes.
If $\mathcal{C}(\mathbb{H})$ is a class of $\mathbb{H}$ adapted processes, we denote by $\mathcal{C}_0(\mathbb{H})$ the set of processes $X \in \mathcal{C}(\mathbb{H})$ with $X_0 = 0$, and by $\mathcal{C}_{loc}$ the set of processes X such that there exists a sequence $(T_n)_{n\geq 1}$ of $\mathbb{H}$-stopping times that increases to $+\infty$ and

the stopped processes X^{T_n} belong to $\mathcal{C}(\mathbb{H})$. We put $\mathcal{C}_{0,loc} = \mathcal{C}_0 \cap \mathcal{C}_{loc}$.
For a process K with $\mathbb{H}$-locally integrable variation, we denote by $K^{o,\mathbb{H}}$ its dual optional projection. The dual predictable projection of K is denoted $K^{p,\mathbb{H}}$. For a process X, we denote ${}^{o,\mathbb{H}}X$ (resp. ${}^{p,\mathbb{H}}X$) its optional (resp. predictable) projection with respect to $\mathbb{H}$.
For an $\mathbb{H}$- semi-martingale Y, the set $\mathcal{L}(Y,\mathbb{H})$ is the set of $\mathbb{H}$ predictable processes integrable w.r.t. Y and for $H \in \mathcal{L}(Y,\mathbb{H})$, we denote $H \bullet Y_t := \int_0^t H_s dY_s$.

We recall the notion of non-arbitrage that is addressed in this paper.

Definition 2.1 An $\mathbb{H}$-semimartingale S satisfies the *No-Unbounded-Profit-with-Bounded-Risk* condition under $(\mathbb{H}, Q)$ (hereafter called NUPBR$(\mathbb{H}, Q)$) if for any finite deterministic horizon T', the set

$$\mathcal{K}_{T'}(S) := \Big\{ (H \bullet S)_{T'} \;\Big|\; H \in \mathcal{L}(S,\mathbb{H}), \text{ and } H \bullet S \geq -1 \Big\}$$

is bounded in probability under Q. When $Q \sim P$, we simply write NUPBR$(\mathbb{H})$ and say that S satisfies NUPBR$(\mathbb{H})$ instead of S satisfies the NUPBR$(\mathbb{H})$ property.

For more details about this type of non-arbitrage condition and its relationship to the literature, we refer the reader to Aksamit et al. [1]. The NUPBR property is intimately related to the existence of a σ-martingale density for the process S. Below, we recall the definition of σ-martingale and σ-martingale density for a process.

Definition 2.2 An $\mathbb{H}$-adapted process X is called an $(\mathbb{H},\sigma)$-martingale if there exists an $\mathbb{H}$-predictable process ϕ such that

$$0 < \phi \leq 1, \quad \text{and } \phi \bullet X \text{ is an } \mathbb{H}\text{-martingale.}$$

If S is $\mathbb{H}$-adapted, we call $(\mathbb{H},\sigma)$-martingale density for S, any positive $\mathbb{H}$-local martingale L such that SL is an $(\mathbb{H},\sigma)$-martingale. The set of all $(\mathbb{H},\sigma)$-martingale densities for S is denoted by

$$\mathcal{L}_\sigma(S,\mathbb{H}) := \big\{ L \in \mathcal{M}_{loc}(\mathbb{H}) \big|\; L > 0, \quad LS \text{ is an } (\mathbb{H},\sigma)\text{-martingale} \big\}.$$

The equivalence between NUPBR$(\mathbb{H})$ for a process S and $\mathcal{L}_\sigma(S,\mathbb{H}) \neq \emptyset$ is established in [1] (see Proposition 2.3) when the horizon may be infinite, and in [17] of the case of finite horizon.

Beside the initial model represented by $(\Omega, \mathbb{F}, P, S)$, we consider a finite random time τ, to which we associate the process D and the filtration $\mathbb{G}$ given by

$$D := I_{[\![\tau,+\infty[\![}, \quad \mathbb{G} = (\mathcal{G}_t)_{t\geq 0}\,, \quad \mathcal{G}_t = \bigcap_{s>t} \Big(\mathcal{F}_s \vee \sigma(D_u, u \leq s) \Big).$$

The filtration $\mathbb{G}$ is the smallest right-continuous filtration which contains $\mathbb{F}$ and makes τ a stopping time. In the probabilistic literature, $\mathbb{G}$ is called the

progressive enlargement of $\mathbb{F}$ with τ. In addition to $\mathbb{G}$ and D, we associate to τ two important $\mathbb{F}$-supermartingales given by

$$Z_t := P\left(\tau > t \mid \mathcal{F}_t\right) \text{ and } \quad \widetilde{Z}_t := P\left(\tau \geq t \;\middle|\; \mathcal{F}_t\right). \tag{2.1}$$

The $\mathbb{F}$-supermartingale Z is right-continuous with left limits and coincides with the $\mathbb{F}$-optional projection of $I_{\rrbracket 0,\tau\llbracket}$, while $\widetilde{Z}$ admits right limits and left limits only and is the $\mathbb{F}$-optional projection of $I_{\rrbracket 0,\tau\rrbracket}$. The decomposition of Z leads to an important $\mathbb{F}$-martingale that we denote by m, given by

$$m := Z + D^{o,\mathbb{F}}, \tag{2.2}$$

where $D^{o,\mathbb{F}}$ is the $\mathbb{F}$-dual optional projection of $D = I_{\llbracket \tau,\infty\llbracket}$.

To distinguish the effect of filtration, we will denote $\langle .,. \rangle^{\mathbb{F}}$, or $\langle .,. \rangle^{\mathbb{G}}$ to specify the sharp bracket (predictable covariation process) calculated in the filtration $\mathbb{F}$ or $\mathbb{G}$, if confusion may rise. We recall that, for general semi-martingales X and Y, the sharp bracket is (if it exists) the dual predictable projection of the covariation process $[X, Y]$. For the reader's convenience, we recall the definition of honest time.

Definition 2.3 A random time σ is honest, if for any t, there exists an $\mathcal{F}_t$ measurable r.v. σ_t such that $\sigma I_{\{\sigma<t\}} = \sigma_t I_{\{\sigma<t\}}$.

We refer to Jeulin [14, Chapter 5] and Barlow [4] for more information about honest times. For the case of quasi-left-continuous processes, we restrict our study to the following subclass of honest times.

$$\tau \text{ is an honest time satisfying } \quad Z_\tau < 1, \quad P-a.s. \tag{2.3}$$

It is clear that any $\mathbb{F}$-stopping time satisfies (2.3) (we even have $Z_\tau = 0$), and hence our subclass of honest times is not empty. Furthermore, in [2] the authors provide many examples of random times that are not $\mathbb{F}$-stopping times and satisfy (2.3). In [9], there are examples where $Z_\tau = 1$ and NUPBR fails.

To the process S, we associate its random measure of jumps $\mu(dt, dx) := \sum_{u>0} I_{\{\Delta S_u \neq 0\}} \delta_{(u,\Delta S_u)}(dt, dx)$. For any nonnegative product-measurable functional $H(t, \omega, x)$, we define $H \star \mu$ and $\int H dM_\mu^P$ (a σ-finite measure on the measurable space $\left(\Omega \times \mathbb{R}^+ \times \mathbb{R}^d, \mathcal{F}_\infty \otimes \mathcal{B}(\mathbb{R}^+) \otimes \mathcal{B}(\mathbb{R}^d)\right)$ by

$$H \star \mu_t := \int_0^t \int_{\mathbb{R}^d} H(u, x)\mu(du, dx) \quad \text{ and } \quad M_\mu^P(H) = E\left[H \star \mu_\infty\right]. \tag{2.4}$$

We also recall the following lemma, obtained in [1].

Lemma 2.4 *Let X be an $\mathbb{H}$-predictable process with finite variation. Then X satisfies NUPBR($\mathbb{H}$) if and only if $X \equiv X_0$ (i.e. the process X is constant).*

Another simple lemma will be used many times

Lemma 2.5 *For an $\mathbb{F}$-predictable stopping time, one has*

$$\{Z_{T-}=1\}=\{P(\widetilde{Z}_T<1|\mathcal{F}_{T-})=0\}\subset\{\widetilde{Z}_T=1\}$$

Proof We first note that $\{Z_{T-}=1\}\subset\{P(\widetilde{Z}_T<1|\mathcal{F}_{T-})=0\}$. This inclusion is an immediate consequence of $\{Z_{T-}=1\}\subset\{\widetilde{Z}_T=1\}$ and

$$0=P\left(\{Z_{T-}=1\}\cap\{\widetilde{Z}_T<1\}\right)=E[I_{\{Z_{T-}=1\}}P(\widetilde{Z}_T<1|\mathcal{F}_{T-})]\,.$$

Then, we remark that the equalities

$$E[I_{\{\widetilde{Z}_T<1\}}I_{\{P(\widetilde{Z}_T<1|\mathcal{F}_{T-})=0\}}]=E[P(\widetilde{Z}_T<1|\mathcal{F}_{T-})I_{\{P(\widetilde{Z}_T<1|\mathcal{F}_{T-})=0\}}]=0,$$

and $\quad E[(1-\widetilde{Z}_T)I_{\{P(\widetilde{Z}_T<1|\mathcal{F}_{T-})=0\}}]=E[(1-Z_{T-})I_{\{P(\widetilde{Z}_T<1|\mathcal{F}_{T-})=0\}}]$

imply that $\{P(\widetilde{Z}_T<1|\mathcal{F}_{T-})=0\}\subset\{\widetilde{Z}_T=1\}$ and $\{P(\widetilde{Z}_T<1|\mathcal{F}_{T-})=0\}\subset\{Z_{T-}=1\}$ respectively. This completes the proof. □

2.1 The Case of Quasi-Left-Continuous Processes

In this subsection, we answer the two questions **(Prob.1)** and **(Prob.2)** for the case of $\mathbb{F}$-quasi-left-continuous processes.
In a first step, we answer the question **(Prob.1)** and characterize processes S and honest times τ such that $S-S^\tau$ satisfies NUPBR($\mathbb{G}$). To this end, we consider the following $\mathbb{F}$-semimartingale

$$S^{(1)}:=xf_mI_{\{Z_-<1=Z_-+f_m\}}\star\mu\quad\text{ where }\quad f_m:=M_\mu^P\left(\Delta m\mid\widetilde{\mathcal{P}}(\mathbb{F})\right).\quad(2.5)$$

Here, $\widetilde{\mathcal{P}}(\mathbb{H}):=\mathcal{P}(\mathbb{H})\otimes\mathcal{B}(\mathbb{R}^d)$. We recall that $Z_-+\Delta m=\widetilde{Z}$, so that

$$\{1=Z_-+f_m\}\subset\{\widetilde{Z}=1\},\qquad M_\mu^P-a.e.$$

Theorem 2.6 *Suppose that S is $\mathbb{F}$-quasi-left-continuous, and τ is a finite honest time satisfying (2.3). Then, the following assertions are equivalent.*
(a) *The process $S-S^\tau$ satisfies NUPBR($\mathbb{G}$).*
(b) *For any $\delta>0$, $(1-Z_-)I_{\{1-Z_-\geq\delta\}}\cdot S-I_{\{1-Z_-\geq\delta\}}\cdot S^{(1)}$ satisfies NUPBR($\mathbb{F}$).*

The proof of this theorem will be given in Subsection 5.1.

Remark 2.7 (a) The process $S^{(1)}$ is a càdlàg process with finite (locally integrable if S is a special semimartingale) variation. To see this fact, on the one hand, we remark that $f_m^2\star\mu$ is a càdlàg locally bounded and nondecreasing process. On the other hand, for any $0\leq u<v$, we have

$$\mathrm{Var}(S^{(1)})_v-\mathrm{Var}(S^{(1)})_u\leq\sqrt{[S,S]_v-[S,S]_u}\sqrt{f_m^2I_{]u,v]}\star\mu}.$$

(b) The assertion (b) of Theorem 2.6 is equivalent to the NUPBR($\mathbb{F}$) property of the process $I_{\{1-Z_-\geq\delta\}}\cdot S-I_{\{1-Z_-\geq\delta\}}xI_{\{Z_-<1=Z_-+f_m\}}\star\mu$. It is important to mention that the process $I_{\{1-Z_-\geq\delta\}}xI_{\{Z_-<1=Z_-+f_m\}}\star\mu=(1-$

$Z_-)^{-1} I_{\{1-Z_- \geq \delta\}} \cdot S^{(1)}$ is a well defined process with finite variation, while the process $|x| I_{\{Z_- < 1 = Z_- + f_m\}} \star \mu$ may take infinite values.
(c) It is important to notice that, in the above theorem, we do not assume that S satisfies NUPBR($\mathbb{F}$). The theorem gives a complete characterisation for the NUPBR property under $\mathbb{G}$ of $S - S^\tau$ in terms of the NUPBR property under $\mathbb{F}$ of a transformed process.

Corollary 2.8 *Suppose that S is $\mathbb{F}$-quasi-left-continuous, and τ is a finite honest time satisfying (2.3). Then the following assertions hold:*
(i) *In addition to assertions* (a) *and* (b) *of Theorem 2.6, we consider the following assertions:*
(c) $(1 - Z_-) \cdot S - S^{(1)}$ *satisfies NUPBR*($\mathbb{F}$).
(d) $\big(S, S^{(1)}\big)$ *satisfies NUPBR*($\mathbb{F}$).
Then, we have (d)$\Longrightarrow$ (c) $\Longrightarrow$ (b) $\Longleftrightarrow$ (a).
(ii) *If S satisfies NUPBR*($\mathbb{F}$) *and $S^{(1)} \equiv 0$, then $S - S^\tau$ satisfies NUPBR*($\mathbb{G}$).
(iii) *If S satisfies NUPBR*($\mathbb{F}$) *and $\{\Delta S \neq 0\} \cap \{\widetilde{Z} = 1 > Z_-\} = \emptyset$, then $S - S^\tau$ satisfies NUPBR*($\mathbb{G}$).
(iv) *If S satisfies NUPBR*($\mathbb{F}$) *and is continuous, then $S - S^\tau$ satisfies NUPBR*($\mathbb{G}$).

Proof (i) In fact, the proof of (c) $\Longrightarrow$(b) is obvious, while the proof of (d)$\Longrightarrow$(c) follows directly from $(1 - Z_-) \cdot S - S^{(1)} = (1 - Z_-, -1) \cdot \big(S, S^{(1)}\big)$.
(ii) Suppose that $S^{(1)} \equiv 0$. Then, $(S, S^{(1)})$ satisfies NUPBR($\mathbb{F}$) and (ii) follows directly from assertion (i).
(iii) Suppose that $\{\Delta S \neq 0\} \cap \{\widetilde{Z} = 1 > Z_-\} = \emptyset$. Then, since $\{\Delta S \neq 0\} \cap \{1 = Z_- + f_m\} \subset \{\Delta S \neq 0\} \cap \{\widetilde{Z} = 1\}$, it is easy to see that $S^{(1)} \equiv 0$, and (iii) follows from assertion (ii).
(iv) When S is continuous, then $\{\Delta S \neq 0\} \cap \{\widetilde{Z} = 1 > Z_-\} \subset \{\Delta S \neq 0\} = \emptyset$. Hence assertion (iv) follows from assertion (iii). This ends the proof. □

In the following proposition, we answer **(Prob.2)** for the class of quasi-left continuous processes S.

Proposition 2.9 *The following assertions are equivalent.*
(a) *The thin set $\{\widetilde{Z} = 1 > Z_-\}$ is accessible.*
(b) *For every (bounded) $\mathbb{F}$-quasi-left-continuous martingale X, the process $X - X^\tau$ satisfies NUPBR*($\mathbb{G}$).

Proof We start by proving that (a)⇒(b). Suppose that the thin set $\{\widetilde{Z} = 1 > Z_-\}$ is accessible. Consider an $\mathbb{F}$-quasi-left-continuous martingale X, and let $X^{(1)}$ be the process associated with X as in (2.5). Therefore, since X is quasi-left-continuous, we have $\{\Delta X \neq 0\} \cap \{\widetilde{Z} = 1 > Z_-\} = \emptyset$. Hence, we get $X^{(1)} \equiv 0$ and assertion (a) of Theorem 2.6 holds, and we deduce that $X - X^\tau$ satisfies NUPBR($\mathbb{G}$). This completes the proof of (a)⇒(b).

To prove the reverse, assuming that assertion (b) holds, we consider a sequence of stopping times $(T_n)_{n \geq 1}$ that exhausts the thin set $\{\widetilde{Z} = 1 \ \& \ Z_- < 1\}$ (i.e.,

$\left\{\widetilde{Z}=1 \ \& \ Z_- <1\right\} = \bigcup_{n=1}^{+\infty} [\![T_n]\!]$). Then, each T_n – that we denote by T for the sake of simplicity– can be decomposed into a totally inaccessible part T^i and an accessible part T^a as $T = T^i \wedge T^a$. Consider the following quasi-left-continuous processes

$$V := I_{]\!]T^i,+\infty[\![}, \quad \text{and} \quad M := V - V^{p,\mathbb{F}} =: V - \widetilde{V}.$$

Then, since $\{T^i < +\infty\} \subset \{\widetilde{Z}_{T^i} = 1\}$, we deduce that $\{T^i < +\infty\} \subset \{\tau \geq T^i\}$ and hence

$$I_{]\!]\tau,+\infty[\![} \cdot M = -I_{]\!]\tau,+\infty[\![} \cdot \widetilde{V} \quad \text{ is } \mathbb{G}\text{-predictable.}$$

Then, the finite variation and $\mathbb{G}$-predictable process, $I_{]\!]\tau,+\infty[\![} \cdot M$, satisfies NUPBR($\mathbb{G}$) if and only if it is null, or equivalently

$$0 = E\left(I_{]\!]\tau,+\infty[\![} \cdot \widetilde{V}_\infty\right) = E\left(\int_0^\infty (1 - Z_{s-}) d\widetilde{V}_s\right) = E\left((1 - Z_{T^i-}) I_{\{T^i<+\infty\}}\right).$$

Therefore, we conclude that $T^i = +\infty, \ P-a.s.$, and the stopping time T is an accessible stopping time. This proves the proposition. □

Corollary 2.10 *Suppose that τ satisfies (2.3) and $\mathbb{F}$ is a quasi-left-continuous filtration. Then the following assertions are equivalent.*
(a) *The thin set $\{\widetilde{Z} = 1 > Z_-\}$ is evanescent.*
(b) *For every (bounded) $\mathbb{F}$-martingale X, the process $X - X^\tau$ satisfies NUPBR($\mathbb{G}$).*

Proof The proof of (a) $\Longrightarrow$(b) follows immediately from Proposition 2.9-(a) or Corollary 2.8–(iii). Thus, the remaining part of the proof will focus on proving the reverse sense, where we will use the fact that, when $\mathbb{F}$ is a quasi-left-continuous filtration, any accessible $\mathbb{F}$-stopping time is predictable (see [6] or [10, Th. 4.26]). Suppose that assertion (b) holds. Then, since $\mathbb{F}$ is a quasi-left-continuous filtration, any $\mathbb{F}$-martingale is quasi-left-continuous, and from Proposition 2.9 we deduce that the thin set, $\{\widetilde{Z} = 1 < Z_-\}$, is predictable. Now take any $\mathbb{F}$-predictable stopping time T such that

$$[\![T]\!] \subset \{\widetilde{Z} = 1 > Z_-\}.$$

This implies that $\{T < +\infty\} \subset \{\widetilde{Z}_T = 1\}$. Hence we calculate, using Lemma 2.5

$$E(I_{\{T<+\infty\}}(1 - Z_{T-})) = E(I_{\{T<+\infty\}}(1 - \widetilde{Z}_T)) = 0,$$

which is equivalent to $T = +\infty \ P - a.s$ due to $\{T < +\infty\} \subset \{Z_{T-} < 1\}$. This ends the proof of the corollary. □

2.2 Thin Processes with Predictable Jump Times

This subsection addresses the two questions **(Prob.1)** and **(Prob.2)** for the case where the process S is a single jump process with predictable jump time. The results of this framework can be easily generalized to the case of finite number of jumps. It is important to highlight the fact that, in this subsection, we work with the whole class of honest times and we do **not** assume the condition (2.3) on τ.

Below, we state our first main result in this context, that answers **(Prob.1)** . We give a characterisation of processes S and honest times τ such that $S-S^\tau$ satisfies NUPBR($\mathbb{G}$).

Theorem 2.11 *Suppose that τ is an honest time. Consider an $\mathbb{F}$-predictable stopping time T and an $\mathcal{F}_T$-measurable r.v. ξ such that $E(|\xi|\big|\ \mathcal{F}_{T-})<+\infty$ $P-$a.s.. If $S:=\xi I_{\{Z_{T-}<1\}}I_{[\![T,+\infty[\![}$, then the following assertions are equivalent:*
(a) *$S-S^\tau$ satisfies NUPBR($\mathbb{G}$).*
(b) *$\widetilde{S}:=\xi I_{\{\widetilde{Z}_T<1\}}I_{[\![T,+\infty[\![}$ satisfies NUPBR($\mathbb{F}$).*
(c) *There exists a probability measure Q_T on $(\Omega,\mathcal{F}_T)$, absolutely continuous with respect to P such that S satisfies NUPBR($\mathbb{F},Q_T$).*

The proof of this theorem is long and requires intermediary results. Thus, we postpone the proof to Subsection 5.2.

Remark 2.12 Theorem 2.11 provides two equivalent (and conceptually different) characterisations for the condition that $S-S^\tau$ satisfies NUPBR($\mathbb{G}$). One of these characterisations uses the NUPBR($\mathbb{F}$) property under P for a transformation of S, while the other characterisation is essentially based on the NUPBR($\mathbb{F}$) for S under an absolutely continuous probability measure.

The following theorem answers the question **(Prob.2)** in this framework of single jump martingales.

Theorem 2.13 *Consider an $\mathbb{F}$-predictable stopping time T, and an honest time τ. Then, the following assertions are equivalent:*
(a) *On $\{T<+\infty\}$, we have*

$$\Big\{\widetilde{Z}_T=1\Big\}\subset\Big\{Z_{T-}=1\Big\}. \tag{2.6}$$

(b) *For any $\xi\in L^\infty(\mathcal{F}_T)$ such that $E(\xi\big|\ \mathcal{F}_{T-})=0$, the process $M-M^\tau$ satisfies NUPBR($\mathbb{G}$), where $M:=\xi I_{[\![T,+\infty[\![}$.*

Proof Suppose that assertion (a) holds, and consider $\xi\in L^\infty(\mathcal{F}_T)$ such that $E(\xi\mid\mathcal{F}_{T-})=0,\ \ P-a.s.$ on $\{T<+\infty\}$. Splitting M as

$$M=I_{\{Z_{T-}<1\}}\xi I_{[\![T,+\infty[\![}+I_{\{Z_{T-}=1\}}\xi I_{[\![T,+\infty[\![}:=M^1+M^2,$$

and noting that $M^2 - (M^2)^\tau = 0$, we can restrict our attention to the case where $M = M^1$. Then, it is obvious, from Lemma 2.5 that (2.6) implies $\{\widetilde{Z}_T < 1\} = \{Z_{T-} < 1\}$, and hence

$$\widetilde{M} := I_{\{\widetilde{Z}_T<1\}}M = M \quad \text{is an } \mathbb{F}\text{-martingale.}$$

Thus, assertion (b) follows from a direct application of Theorem 2.11 to M. This ends the proof of (a)⇒ (b).
To prove the converse, we assume that assertion (b) holds, and we consider the $\mathcal{F}_T$-measurable and bounded r.v. $\xi := (I_{\{\widetilde{Z}_T=1\}} - P(\widetilde{Z}_T = 1|\mathcal{F}_{T-}))I_{\{T<+\infty\}}$ and the bounded $\mathbb{F}$-martingale $M := \xi I_{[\![T,+\infty[\![}$. Then, on the one hand, $M - M^\tau$ satisfies NUPBR($\mathbb{G}$). On the other hand, due to $\{T > \tau\} \subset \{\widetilde{Z}_T < 1\}$, the finite variation process

$$M - M^\tau = -P(\widetilde{Z}_T = 1|\mathcal{F}_{T-})I_{\{T>\tau\}}I_{[\![T,+\infty[\![} \text{ is } \mathbb{G} - \text{predictable.}$$

Thus, this finite variation process is null, or equivalently $\{Z_{T-} < 1\} \subset \{\widetilde{Z}_T < 1\}$ $P - a.s.$ on $\{T < +\infty\}$. This proves assertion (a), and the proof of the theorem is completed. □

We now give the main result for the case of thin semimartingales with predictable jumps that answers **(Prob.1)**. This theorem constitutes with Theorem 2.6 the building blocks for the general result that addresses **(Prob.1)**.

Theorem 2.14 *Suppose that τ is an honest time, and that S is a thin semimartingale with predictable jumps only (i.e., $S^{(qc)} \equiv 0$, where $S^{(qc)}$ is defined in (2.8) below). Then the following assertions are equivalent.*
(a) *The process $S - S^\tau$ satisfies NUPBR($\mathbb{G}$).*
(b) *For any $\delta > 0$, there exists a positive $\mathbb{F}$-local martingale Y, such that*

$$^{p,\mathbb{F}}(Y|\Delta S|) < +\infty \quad \text{and} \quad {}^{p,\mathbb{F}}\left(Y\Delta S I_{\{\widetilde{Z}<1\}}\right) I_{\{1-Z_-\geq\delta\}} = 0. \tag{2.7}$$

The proof of this theorem is long and is based on intermediatory results that are interesting in themselves. Thus, this proof is postponed to Subsection 5.3.

2.3 The General Framework

With any càdlàg $\mathbb{F}$-semimartingale, X, we associate a sequence of $\mathbb{F}$-predictable stopping times $(T_X^n)_{n\geq 1}$ that exhausts all the accessible jumps of X. Throughout the paper, we use the following notations

$$\Gamma_X := \bigcup_{n=1}^{+\infty} [\![T_X^n]\!], \qquad X^{(a)} := I_{\Gamma_X} \bullet X, \quad X^{(qc)} := I_{\Gamma_X^c} \bullet X := X - X^{(a)}. \tag{2.8}$$

We now state the first main result of this subsection, where we characterise the pairs (S, τ) such that $S - S^\tau$ satisfies NUPBR ($\mathbb{G}$).

Theorem 2.15 *Suppose that τ satisfies (2.3) and consider $S^{(1)}$ given by (2.5). Then, the following assertions are equivalent.*
(a) *The process $S - S^\tau$ satisfies NUPBR $(\mathbb{G})$.*
(b) *For any $\delta > 0$, the following properties hold, for $\Gamma = \Gamma_S$ defined in (2.8):*
(b.1) $(1 - Z_-)I_{\{1-Z_- \geq \delta\}} \cdot S^{(qc)} - I_{\Gamma^c} I_{\{1-Z_- \geq \delta\}} \cdot S^{(1)}$ *satisfies NUPBR*$(\mathbb{F})$,
(b.2) *there exists a positive $\mathbb{F}$-local martingale Y satisfying*

$$^{p,\mathbb{F}}(Y|\Delta S|) < +\infty \qquad \text{and} \qquad ^{p,\mathbb{F}}\left(Y\Delta S I_{\{\widetilde{Z}<1\}}\right) I_{\{1-Z_- \geq \delta\}} = 0.$$

Proof Remark that assertion (a) holds if and only if both $S^{(qc)} - (S^{(qc)})^\tau$ and $S^{(a)} - (S^{(a)})^\tau$ satisfy NUPBR$(\mathbb{G})$. Therefore, using Theorems 2.6 and 2.14, we deduce that the property (b.1) of the assertion (b) holds, and for any $\delta > 0$, there exists a positive $\mathbb{F}$-martingale, Y, such that on $\{1 - Z_- \geq \delta\}$, we have

$$^{p,\mathbb{F}}(Y|\Delta S|) = {}^{p,\mathbb{F}}\left(Y|\Delta S^{(a)}|I_{\{\widetilde{Z}<1\}}\right) < +\infty$$

and

$$^{p,\mathbb{F}}\left(Y\Delta S I_{\{\widetilde{Z}<1\}}\right) = {}^{p,\mathbb{F}}\left(Y\Delta S^{(a)} I_{\{\widetilde{Z}<1\}}\right) = 0.$$

This is equivalent to the property (b.2) of assertion (b). This ends the proof of the theorem. □

Corollary 2.16 *Suppose that τ satisfies (2.3), and consider $S^{(1)}$ defined in (2.5). Then, the following assertions hold.*
(a) *If S satisfies NUPBR$(\mathbb{F})$ and $\{\widetilde{Z} = 1 > Z_-\} \cap \{\Delta S \neq 0\} = \emptyset$, then $S - S^\tau$ satisfies NUPBR$(\mathbb{G})$.*
(b) *If S satisfies NUPBR$(\mathbb{F})$ and is continuous, then $S - S^\tau$ satisfies NUPBR$(\mathbb{G})$.*

Proof (a) Suppose that S satisfies NUPBR$(\mathbb{F})$ and $\{\widetilde{Z} = 1 > Z_-\} \cap \{\Delta S \neq 0\}$ is evanescent. Thus, we deduce that $S^{(1)} \equiv 0$ and hence assertion (b) of Theorem 2.15 is fulfilled with Y any σ-martingale density for S (which exists since S satisfies NUPBR$(\mathbb{F})$). Hence, the proof of assertion (a) follows immediately from Theorem 2.15.
(b) The proof of assertion (b) follows immediately from assertion (a) since when S is continuous, the set $\{\widetilde{Z} = 1 > Z_-\} \cap \{\Delta S \neq 0\}$ is evanescent. This ends the proof of the corollary. □

We conclude this section with our second main general result that answers **(Prob.2)** in full generality and characterises honest times τ for which the NUPBR property is stable.

Theorem 2.17 *Suppose that τ satisfies (2.3). Then, the following assertions are equivalent.*
(a) *For any S satisfying NUPBR$(\mathbb{F})$, the process $S - S^\tau$ satisfies NUPBR$(\mathbb{G})$.*
(b) *The thin set $\{\widetilde{Z} = 1 \ \& \ Z_- < 1\}$ is evanescent.*

Proof The implication (b) ⇒ (a) is a trivial consequence of Corollary 2.16-(a). To prove the reverse, we assume that assertion (a) holds, and consider $V := I_{[\![T,+\infty[\![}$ and $M := V - V^{p,\mathbb{F}}$, where T is an $\mathbb{F}$-stopping time such that $[\![T]\!] \subset \{\widetilde{Z} = 1 > Z_-\}$. Since $\{T > \tau\} \subset \{\widetilde{Z}_T < 1\}$ and $\{T < +\infty\} \subset \{\widetilde{Z}_T = 1\}$, we derive

$$M - M^\tau = I_{]\!]\tau,+\infty[\![} \cdot V^{p,\mathbb{F}} \qquad \text{which is } \mathbb{G} - \text{predictable.}$$

Thus, $M - M^\tau$ satisfies NUPBR($\mathbb{G}$) if and only if it is a null process. This is equivalent to

$$0 = E\left((1 - Z_-) \cdot V^{p,\mathbb{F}}_\infty\right) = E\left((1 - Z_{T-})I_{\{T<+\infty\}}\right).$$

Since $\{T < +\infty\} \subset \{Z_{T-} < 1\}$, we conclude that $T = +\infty,\ P - a.s.$ This proves that $[\![T]\!] \subset \{\widetilde{Z} = 1 > Z_-\}$ implies that $[\![T]\!] = \emptyset$. Therefore, we obtain

$$\{\widetilde{Z} = 1 > Z_-\} = \bigcup_{i=1}^{+\infty} [\![T_i]\!] = \emptyset,$$

since $\{\widetilde{Z} = 1 > Z_-\} \subset \{\Delta m \neq 0\}$ is a thin set. This ends the proof of the theorem. □

3 New Stochastic Developments for the Part after τ

This section provides new stochastic results that constitute the key stochastic tools for the proof of the main results outlined in the previous section. These results complete those elaborated in Aksamit et al.[1], where the authors addressed the same problems for the part up to an arbitrary random time τ. This section contains two subsections. The first subsection gives the relationship between the dual predictable projections under $\mathbb{F}$ and $\mathbb{G}$, while the second subsection recalls the notion of optional integral and introduces an important $\mathbb{G}$-martingale, which will play the role of a deflator.
The results of this section are based on the following standard result on progressive enlargement of filtration (see [4,8,14]).

Lemma 3.1 *Let M be an $\mathbb{F}$-local martingale, and τ be an honest time. Then the process $\widehat{M}$, defined as*

$$\widehat{M}_t := M_t - M_{t\wedge\tau} + \int_{\tau\wedge t}^{t} \frac{d\langle M, m\rangle^{\mathbb{F}}_s}{1 - Z_{s-}}, \tag{3.1}$$

is a $\mathbb{G}$-local martingale.

3.1 Comparing Dual Predictable Projections under $\mathbb{G}$ and $\mathbb{F}$

In the following, we start our study by writing the $\mathbb{G}$-compensators/projections in terms of $\mathbb{F}$-compensators/projections respectively.

Lemma 3.2 *Suppose that (2.3) holds, and denote*

$$J := 1 - Z_- - \Delta m = 1 - \widetilde{Z}\,. \tag{3.2}$$

Then the following assertions hold.
(a) *The $\mathbb{G}$-predictable process $(1-Z_-)^{-1} I_{[\![\tau,+\infty[\![}$ is $\mathbb{G}$-locally bounded.*
(b) *For any $\mathbb{F}$-adapted process V, with locally integrable variation we have*

$$I_{[\![\tau,+\infty[\![} \cdot V^{p,\mathbb{G}} = I_{[\![\tau,+\infty[\![} \frac{1}{1-Z_-} \cdot (J \cdot V)^{p,\mathbb{F}}\,, \tag{3.3}$$

and on $]\!]\tau,+\infty[\![$

$$^{p,\mathbb{G}}(\Delta V) = \frac{1}{1-Z_-}\ {}^{p,\mathbb{F}}(J\Delta V)\,. \tag{3.4}$$

(d) *For any $\mathbb{F}$-local martingale M, one has, on $]\!]\tau,+\infty[\![$*

$$^{p,\mathbb{G}}\left(\frac{\Delta M}{1-\widetilde{Z}}\right) = \frac{^{p,\mathbb{F}}\left(\Delta M I_{\{\widetilde{Z}<1\}}\right)}{1-Z_-}, \quad \textit{and} \quad ^{p,\mathbb{G}}\left(\frac{1}{J}\right) = \frac{^{p,\mathbb{F}}\left(I_{\{\widetilde{Z}<1\}}\right)}{1-Z_-}. \tag{3.5}$$

(e) *For any quasi-left-continuous $\mathbb{F}$-local martingale M, one has, on $]\!]\tau,+\infty[\![$*

$$^{p,\mathbb{G}}\left(\frac{\Delta M}{J}\right) = 0, \quad \textit{and} \quad ^{p,\mathbb{G}}\left(\frac{1}{1-Z_- - \Delta m^{(qc)}}\right) = \frac{1}{1-Z_-}, \tag{3.6}$$

where $m^{(qc)}$ is the quasi-left-continuous part (see (2.8)) of m, defined in (2.2).

Proof The proof of the lemma will be achieved in four steps.
1) Herein we prove assertion (a). It is known [8, Chapter XX] that $Z = \widetilde{Z}$ on $]\!]\tau,+\infty[\![$, and

$$]\!]\tau,+\infty[\![\subset \{Z_- < 1\} \cap \{\widetilde{Z} < 1\} = \{Z_- < 1\} \cap \{Z < 1\}\,.$$

Then due to our specific assumption (2.3), we deduce that $[\![\tau,+\infty[\![\subset \{Z<1\}$, and hence the process

$$X := (1-Z)^{-1} I_{[\![\tau,+\infty[\![},$$

is càdlàg $\mathbb{G}$-adapted. Combining these with $]\!]\tau,+\infty[\![\subset \{Z_- < 1\}$, we easily prove that $\sup_{0\le u\le t} X_u < +\infty$, $P-a.s.$ This is equivalent to say that X is pre-locally bounded and hence its left limit process $X_- = (1-Z_-)^{-1} I_{[\![\tau,+\infty[\![}$ is locally bounded. This proves assertion (a).

2) This part proves assertion (b). From Lemma 3.1

$$I_{[\![\tau,+\infty[\![} \cdot V - I_{[\![\tau,+\infty[\![} \cdot V^{p,\mathbb{F}} + I_{[\![\tau,+\infty[\![}(1-Z_-)^{-1} \cdot \langle V, m\rangle^{\mathbb{F}}$$

is a $\mathbb{G}$-local martingale, hence

$$\begin{aligned}\left(I_{]\!]\tau,+\infty[\![}\,.\,V\right)^{p,\mathbb{G}} &= I_{]\!]\tau,+\infty[\![}\,.\,V^{p,\mathbb{F}} - I_{]\!]\tau,+\infty[\![}(1-Z_-)^{-1}\,.\,\langle V,m\rangle^{\mathbb{F}},\\ &= I_{]\!]\tau,+\infty[\![}\,.\,V^{p,\mathbb{F}} - I_{]\!]\tau,+\infty[\![}(1-Z_-)^{-1}\,.\,(\Delta m\,.\,V)^{p,\mathbb{F}},\\ &= I_{]\!]\tau,+\infty[\![}(1-Z_-)^{-1}\,.\,\Big((1-Z_--\Delta m)\,.\,V\Big)^{p,\mathbb{F}},\end{aligned}$$

where the second equality follows from Yoeurp's lemma. This ends the proof of (3.3). The equality (3.4) follows immediately from (3.3) by taking the jumps in both sides.

3) Now, we prove assertion (c). By applying (3.3) for $V_{\epsilon,\delta}\in\mathcal{A}_{loc}(\mathbb{F})$ given by

$$V_{\epsilon,\delta} := \sum \frac{\Delta M}{J} I_{\{|\Delta M|\geq\epsilon,\ J\geq\delta\}},$$

we get, on $]\!]\tau,+\infty[\![$,

$${}^{p,\mathbb{G}}\left(\frac{\Delta M}{J} I_{\{|\Delta M|\geq\epsilon,\ J\geq\delta\}}\right) = (1-Z_-)^{-1}\ {}^{p,\mathbb{F}}\left(\Delta M\, I_{\{|\Delta M|\geq\epsilon,\ J\geq\delta\}}\right).$$

Then, by letting ϵ and δ go to zero, on $]\!]\tau,+\infty[\![$ we obtain

$${}^{p,\mathbb{G}}\left(\frac{\Delta M}{J}\right) = (1-Z_-)^{-1}\ {}^{p,\mathbb{F}}\left(\Delta M\, I_{\{J>0\}}\right).$$

This proves the first equality in (3.5). To prove the remaining equality in (3.5), we write that, on $]\!]\tau,+\infty[\![$,

$$\begin{aligned}{}^{p,\mathbb{G}}\left(\frac{1}{J}\right) &= (1-Z_-)^{-1} + (1-Z_-)^{-1}\ {}^{p,\mathbb{G}}\left(\frac{\Delta m}{J}\right)\\ &= (1-Z_-)^{-1} + (1-Z_-)^{-2}\ {}^{p,\mathbb{F}}\left(\Delta m I_{\{J>0\}}\right)\\ &= (1-Z_-)^{-1} - (1-Z_-)^{-1}\ {}^{p,\mathbb{F}}\left(I_{\{J=0\}}\right) = (1-Z_-)^{-1}\ {}^{p,\mathbb{F}}\left(I_{\{\widetilde{Z}<1\}}\right).\end{aligned}$$

where the second equality follows from the first equality in (3.5) and the third equality from the fact that ${}^{p,\mathbb{F}}(\Delta m)=0$. This ends the proof of assertion (c).

4) The proof of the first equality in (3.6) follows immediately from assertion (c) and the fact that the thin process ${}^{p,\mathbb{F}}\left(\Delta M\, I_{\{J>0\}}\right)$ may take nonzero values on countably many predictable stopping times only, on which ΔM already vanishes. A direct application of this first equality implies that, on $\{Z_-<1\}$, we have

$${}^{p,\mathbb{F}}\left(\frac{1}{1-Z_--\Delta m^{(qc)}}\right) = \frac{1}{1-Z_-} + \frac{1}{1-Z_-}\ {}^{p,\mathbb{F}}\left(\frac{\Delta m^{(qc)}}{1-Z_--\Delta m^{(qc)}}\right) = \frac{1}{1-Z_-}.$$

This completes the proof of (3.6) as well as the proof of the lemma. □

The next lemma focuses on the integrability of the process $J^{-1}I_{\rrbracket\tau,+\infty\llbracket}$ with respect to any process with $\mathbb{F}$-locally integrable variation. As a result, we complete our comparison of $\mathbb{G}$ and $\mathbb{F}$ compensators.

Lemma 3.3 *Let V be a càdlàg $\mathbb{F}$-adapted process and*

$$U := (1-Z)^{-1} I_{\rrbracket\tau,+\infty\llbracket} \cdot V. \tag{3.7}$$

Then, the following assertions hold.
(a) *If V is nondecreasing and locally integrable (respectively integrable), then U is $\mathbb{G}$-locally integrable (respectively $\mathbb{G}$-integrable).*
(b) *If $V \in \mathcal{A}_{loc}(\mathbb{F})$, then U is a well defined process, $U \in \mathcal{A}_{loc}(\mathbb{G})$ and*

$$U^{p,\mathbb{G}} = I_{\rrbracket\tau,+\infty\llbracket}(1-Z_-)^{-1} \cdot (I_{\{\widetilde{Z}<1\}} \cdot V)^{p,\mathbb{F}}. \tag{3.8}$$

Proof (a) Let $(\vartheta_n)_{n\geq 1}$ be a sequence of $\mathbb{F}$-stopping times that increases to $+\infty$ such that

$$E\left(V_{\vartheta_n}\right) < +\infty.$$

Then assertion (a) follows immediately if we prove $E\left(U_{\vartheta_n}\right) \leq E\left(V_{\vartheta_n}\right)$. To this end, we write

$$\begin{aligned} E\left(U_{\vartheta_n}\right) &= E\left(\int_0^{\vartheta_n} I_{\{t>\tau\}} \frac{1}{1-Z_t} dV_t\right) = E\left(\int_0^{\vartheta_n} \frac{P(\tau<t|\mathcal{F}_t)}{1-Z_t} I_{\{Z_t<1\}} dV_t\right) \\ &= E\left(\int_0^{\vartheta_n} I_{\{Z_s<1\}} dV_s\right) \leq E\left(V_{\vartheta_n}\right). \end{aligned}$$

Since ϑ_n increases to $+\infty$, we get $U \in \mathcal{A}^+_{loc}(\mathbb{G})$. This proves assertion (a).
(b) Suppose that $V \in \mathcal{A}_{loc}(\mathbb{F})$. Then, $\mathrm{Var}(V) = V^+ + V^- \in \mathcal{A}^+_{loc}(\mathbb{F})$, and hence due to assertion (a), we deduce that U has a $\mathbb{G}$-local integrable variation that coincides with

$$\mathrm{Var}(U) = (1-Z)^{-1} I_{\rrbracket\tau,+\infty\llbracket} \cdot \mathrm{Var}(V).$$

For any $n \geq 1$, introduce

$$U_n := (1-Z)^{-1} I_{\rrbracket\tau,+\infty\llbracket} I_{\{\widetilde{Z}\leq 1-\frac{1}{n}\}} \cdot V = \left(1-\widetilde{Z}\right)^{-1} I_{\rrbracket\tau,+\infty\llbracket} I_{\{\widetilde{Z}\leq 1-\frac{1}{n}\}} \cdot V.$$

Then, thanks to (3.3), we derive

$$\begin{aligned} U^{p,\mathbb{G}} &= \lim_{n\longrightarrow+\infty} (U_n)^{p,\mathbb{G}} \\ &= \lim_{n\longrightarrow+\infty} (1-Z_-)^{-1} I_{\rrbracket\tau,+\infty\llbracket} \cdot \left(I_{\{\widetilde{Z}\leq 1-\frac{1}{n}\}} \cdot V\right)^{p,\mathbb{F}}. \end{aligned}$$

This clearly implies (3.8), and the proof of the lemma is completed. □

We end this subsection with a lemma that we shall use later and that is interesting in itself. This lemma establishes a complete connection between $\mathbb{G}$-local integrability and $\mathbb{F}$-local integrability with respect to a class of nondecreasing and $\mathbb{F}$-predictable processes.

Lemma 3.4 *Suppose that τ is a finite honest time satisfying (2.3). Let V be an $\mathbb{F}$-predictable process that belongs to $\mathcal{A}^+_{loc}(\mathbb{F})$ (in particular, it is càdlàg with finite values), and ϕ be a nonnegative and $\mathbb{F}$-predictable process. Then, $\phi I_{]\!]\tau,+\infty[\![} \bullet V \in \mathcal{A}^+_{loc}(\mathbb{G})$ if and only if $I_{\{1-Z_-\geq\delta\}}\phi \bullet V \in \mathcal{A}^+_{loc}(\mathbb{F})$ for any $\delta > 0$.*

Proof 1) The proof of $I_{\{1-Z_-\geq\delta\}}\phi \bullet V \in \mathcal{A}^+_{loc}(\mathbb{F}) \Longrightarrow I_{\{1-Z_-\geq\delta\}}\phi I_{]\!]\tau,+\infty[\![} \bullet V \in \mathcal{A}^+_{loc}(\mathbb{G})$ is obvious. Then $\phi I_{]\!]\tau,+\infty[\![} \bullet V \in \mathcal{A}^+_{loc}(\mathbb{G})$ follows from a combination of $\left(\mathcal{A}^+_{loc}(\mathbb{G})\right)_{loc} = \mathcal{A}^+_{loc}(\mathbb{G})$ and the fact that for any $\delta > 0$, there exists a $\mathbb{G}$-stopping time τ_δ such that τ_δ increases to infinity when δ decreases to zero and

$$]\!]\tau,+\infty[\![\cap]\!]0,\tau_\delta]\!] \subset]\!]\tau,+\infty[\![\cap\{1-Z_-\geq\delta\}.$$

2) Herein, we will concentrate on proving the reverse sense. Thus, we suppose that $\phi I_{]\!]\tau,+\infty[\![} \bullet V \in \mathcal{A}^+_{loc}(\mathbb{G})$. This guarantees the existence of a sequence of $\mathbb{G}$-stopping times $(T_k)_{k\geq 1}$ that increases to infinity, $T_k > \tau$ $P-a.s.$, and

$$\left(\phi I_{]\!]\tau,+\infty[\![} \bullet V\right)^{T_k} \in \mathcal{A}^+(\mathbb{G}).$$

This implies that the $\mathbb{F}$-predictable process

$$\phi(Z^{(k)}_- - Z_-) \bullet V \quad \text{ is càdlàg with finite value,} \tag{3.9}$$

where

$$Z^{(k)}_t := {}^{o,\mathbb{F}}\left(I_{[\![0,T_k]\!]}\right)_t = P\left(T_k > t \middle| \mathcal{F}_t\right).$$

In the following we will prove that (a subsequence of) the $\mathbb{F}$-predictable processes $Z^{(k)}_-$ converge to 1, almost surely and uniformly on each compact interval $[0,t]$. To this end, we fix t, and we calculate

$$\sup_{0\leq u\leq t}(1-Z^{(k)}_u) = \sup_{0\leq u\leq t} P\left(T_k \leq u|\mathcal{F}_u\right) \leq \sup_{0\leq u\leq t} P\left(T_k \leq t|\mathcal{F}_u\right)$$

Then, using Doob's inequality (for $p=2$), we obtain

$$E\left[\sup_{0\leq u\leq t} P(T_k\leq t|\mathcal{F}_u)^2\right] \leq 4P\left(T_k\leq t\right) \longrightarrow 0 \text{ as } k \text{ goes to infinity.}$$

Thus, for any $\delta > 0$, and for almost all $\omega \in \Omega$, there exists k_0 such that for any $u \in [0,t]$, we have for $k \geq k_0$

$$Z^{(k)}_{u-} \geq 1 - \frac{\delta}{2}.$$

Thus, by combining all these, we deduce that for any $\delta > 0$, for any $t \in (0, +\infty)$, and for almost all $\omega \in \Omega$, there exists k_0 such that for any $k \geq k_0$, and for any $s \in [0, t]$

$$\begin{aligned}(Z_-^{(k)} - Z_-)\phi \bullet V_s &\geq (Z_-^{(k)} - Z_-)\phi I_{\{1-Z_- \geq \delta\}} \bullet V_s \\ &\geq (1 - Z_- - \frac{\delta}{2})\phi I_{\{1-Z_- \geq \delta\}} \bullet V_s \geq \frac{\delta}{2}\phi I_{\{1-Z_- \geq \delta\}} \bullet V_s.\end{aligned}$$

Thus, a combination of this inequality and (3.9) allows us to conclude that for any $\delta > 0$, the $\mathbb{F}$-predictable process $\phi I_{\{1-Z_- \geq \delta\}} \bullet V$, is càdlàg with finite values almost surely. This ends the proof of the lemma. □

3.2 An Important $\mathbb{G}$-Local Martingale

In this subsection, we introduce a $\mathbb{G}$-local martingale that will play the role of deflator for a class of $\mathbb{F}$-local martingales. The construction of this local martingale relies on the following proposition.

Proposition 3.5 *Let τ be an honest time satisfying (2.3). Then, the following assertions hold.*
(a) *The nondecreasing process*

$$V_t^{\mathbb{G}} := \sum_{0<u\leq t} {}^{p,\mathbb{F}}\left(I_{\{\widetilde{Z}=1\}}\right)_u I_{\{u>\tau\}}, \tag{3.10}$$

is $\mathbb{G}$-predictable, càdlàg, and locally bounded.
(b) *The nonnegative and $\mathbb{G}$-predictable process $\left(1 - \Delta V^{\mathbb{G}}\right)^{-1}$ is locally bounded.*

Proof a) Since under assumption (2.3) the process $(1 - Z_-)^{-1} I_{\rrbracket \tau, +\infty \rrbracket}$ is $\mathbb{G}$-locally bounded, then there exists a sequence of $\mathbb{G}$-stopping times $(\tau_n)_{n\geq 1}$ that increases to infinity such that

$$\left((1 - Z_-)^{-1} I_{\rrbracket \tau, +\infty \rrbracket}\right)^{\tau_n} \leq n.$$

Consider a sequence of $\mathbb{F}$-stopping times, $(\sigma_n)_n$, that increases to infinity and $\langle m, m\rangle_{\sigma_n} \leq n$. Then, for any nonnegative $\mathbb{F}$-predictable process H which is bounded by $C > 0$, we derive

$$\begin{aligned}(H \bullet V^{\mathbb{G}})_{\sigma_n \wedge \tau_n} &\leq \sum_{0\leq u\leq \sigma_n} H_u \, {}^{p,\mathbb{F}}\left(I_{\{\widetilde{Z}=1>Z_-\}}\right)_u I_{\{1-Z_{u-} \geq \frac{1}{n}\}} \\ &= \sum_{0\leq u\leq \sigma_n} H_u \, {}^{p,\mathbb{F}}\left(I_{\{\Delta m = 1-Z_- \geq \frac{1}{n}\}} I_{\{Z_- \leq 1-\frac{1}{n}\}}\right)_u \\ &\leq \sum_{0\leq u\leq \sigma_n} H_u \, {}^{p,\mathbb{F}}\left(I_{\{\Delta m \geq \frac{1}{n}\}}\right)_u \leq n^2 \, H \bullet \langle m, m\rangle_{\sigma_n} \leq Cn^3.\end{aligned}$$

This ends the proof of the assertion (a).
(b) We first calculate

$$1-\Delta V^{\mathbb{G}} = 1- {}^{p,\mathbb{F}}\left(I_{\{\widetilde{Z}=1\}}\right) I_{]\!]\tau,+\infty[\![} \\ = I_{]\!]0,\tau]\!]} + {}^{p,\mathbb{F}}\left(I_{\{\widetilde{Z}<1\}}\right) I_{]\!]\tau,+\infty[\![} \geq I_{]\!]0,\tau]\!]} + (1-Z_-) I_{]\!]\tau,+\infty[\![}.$$

The last inequality follows from $1-\widetilde{Z} \leq I_{\{\widetilde{Z}<1\}}$. Thus, we deduce that

$$(1-\Delta V^{\mathbb{G}})^{-1} \leq I_{]\!]0,\tau]\!]} + (1-Z_-)^{-1} I_{]\!]\tau,+\infty[\![},$$

which is locally bounded due to Lemma 3.2–(a). This proves assertion (b), and the proof of the proposition is completed. □

For the construction of the local martingale deflator, we will use, as in [1], the optional integral, defined in [12] (Chapter III.4.b p. 106-109) and studied in [7] (Chapter VIII.2 sections 32-35 p. 356-361).

Definition 3.6 (see [12], Definition (3.80)) Let N be an $\mathbb{H}$-local martingale with continuous martingale part N^c, and let H be an $\mathbb{H}$-optional process.
i) The process H is said to be integrable with respect to N if ${}^{p,\mathbb{H}}H$ is N^c integrable and the process

$$\left(\sum_{s\leq t}\left(H_s\Delta N_s - {}^{p,\mathbb{H}}(H\Delta N)_s\right)^2\right)^{1/2}$$

is locally integrable. The set of integrable processes with respect to N is denoted by ${}^{o}L^1_{loc}(N,\mathbb{H})$.
ii) For $H \in {}^{o}L^1_{loc}(N,\mathbb{H})$, the compensated stochastic integral of H with respect to N, denoted by $H\odot N$, is the unique local martingale M satisfies

$$M^c = {}^{p,\mathbb{H}}H \bullet N^c \quad \text{and} \quad \Delta M = H\Delta N - {}^{p,\mathbb{H}}(H\Delta N). \tag{3.11}$$

We recall some important properties of the compensated stochastic integral:

Proposition 3.7 *(see [7])* (a) *The compensated stochastic integral $M = H\odot N$ is the unique $\mathbb{H}$-local martingale such that, for any $\mathbb{H}$-local martingale Y,*

$$\mathbb{E}\left([M,Y]_\infty\right) = \mathbb{E}\left(\int_0^\infty H_s d[N,Y]_s\right).$$

(b) *The process $[M,Y] - H\bullet[N,Y]$ is an $\mathbb{H}$-local martingale. As a result $[M,Y]\in\mathcal{A}_{loc}(\mathbb{H})$ if and only if $H\bullet[N,Y]\in\mathcal{A}_{loc}(\mathbb{H})$ and in this case we have*

$$\langle M,Y\rangle^{\mathbb{H}} = (H\bullet[N,Y])^{p,\mathbb{H}}.$$

Now, we are in the stage of presenting our second main result of this subsection. We shall denote, as usual, $\mathcal{E}(Y)$ the Doléans-Dade exponential of Y.

Proposition 3.8 *Suppose that τ satisfies (2.3) and consider*

$$\widehat{m} := I_{]\!]\tau,+\infty[\![} \cdot m + \frac{1}{(1-Z_-)} I_{]\!]\tau,+\infty[\![} \cdot \langle m \rangle^{\mathbb{F}}, \tag{3.12}$$

and the space ${}^{o}L^1_{loc}(\widehat{m},\mathbb{G})$ defined in Definition 3.6. Then, we have

$$K := \frac{(1-Z_-)^2}{(1-Z_-)^2 + \Delta\langle m\rangle^{\mathbb{F}}} \frac{1}{1-\widetilde{Z}} I_{]\!]\tau,+\infty[\![} \in {}^{o}L^1_{loc}(\widehat{m},\mathbb{G}). \tag{3.13}$$

Furthermore, the associated $\mathbb{G}$-local martingale,

$$L^{\mathbb{G}} := K \odot \widehat{m}, \tag{3.14}$$

satisfies the following
(a) $\mathcal{E}(L^{\mathbb{G}}) > 0$ *(or equivalently* $1 + \Delta L^{\mathbb{G}} > 0$*) and* $I_{]\!]0,\tau]\!]} \cdot L^{\mathbb{G}} = 0$.
(b) *For any $M \in \mathcal{M}_{0,loc}(\mathbb{F})$, we have*

$$[L^{\mathbb{G}}, \widehat{M}] \in \mathcal{A}_{loc}(\mathbb{G}) \quad \Big(\textit{i.e. } \langle L^{\mathbb{G}}, \widehat{M}\rangle^{\mathbb{G}} \quad \textit{exists}\Big), \tag{3.15}$$

where $\widehat{M}$ is defined in (3.1).

Proof 1) The rather technical proof of (3.13) will be given in Appendix B.
2) Assertion (a) follows from $I_{]\!]0,\tau]\!]} \cdot \widehat{m} = 0$ and (B.2) which implies that

$$1 + \Delta L^{\mathbb{G}} = 1 + \frac{\Delta m}{1-\widetilde{Z}} I_{]\!]\tau,+\infty[\![} + {}^{p,\mathbb{F}}\left(I_{\{\widetilde{Z}=1\}}\right) I_{]\!]\tau,+\infty[\![} > 0\,,$$

where we used the fact that $\Delta m = \widetilde{Z} - Z_-$. We now prove (3.15). Due to Proposition 3.7 and the $\mathbb{G}$-local boundedness of $(1-Z_-)^{-2} I_{]\!]\tau,+\infty[\![}$ (see Lemma 3.2(a)), we deduce that $[L^{\mathbb{G}}, \widehat{M}] \in \mathcal{A}_{loc}(\mathbb{G})$ if and only if $J^{-1}\cdot[\widehat{m}, \widehat{M}] \in \mathcal{A}_{loc}(\mathbb{G})$, where we recall that $J := 1-\widetilde{Z} = 1-Z$ on $]\!]\tau,+\infty[\![$. We calculate this quantity

$$J^{-1} \cdot [\widehat{m}, \widehat{M}] = J^{-1} I_{]\!]\tau,+\infty[\![} \cdot [m, M] + J^{-1}(1-Z_-)^{-1}\Delta m I_{]\!]\tau,+\infty[\![} \cdot \langle m, M\rangle^{\mathbb{F}}$$

$$+ \frac{\Delta M}{J(1-Z_-)} I_{]\!]\tau,+\infty[\![} \cdot \langle m\rangle^{\mathbb{F}} + \frac{\Delta\langle m\rangle^{\mathbb{F}}}{J(1-Z_-)^2} I_{]\!]\tau,+\infty[\![} \cdot \langle m, M\rangle^{\mathbb{F}}.$$

On the one hand,

$$(1-Z_-)^{-1}\Delta m I_{]\!]\tau,+\infty[\![},\ \Delta\langle m\rangle^{\mathbb{F}}(1-Z_-)^{-2} I_{]\!]\tau,+\infty[\![} \text{ and } (1-Z_-)^{-1} I_{]\!]\tau,+\infty[\![}$$

are $\mathbb{G}$-locally bounded. On the other hand, since $[m, M]$, $\Delta M \cdot \langle m\rangle^{\mathbb{F}}$ and $\langle m, M\rangle^{\mathbb{F}}$ belong to $\mathcal{A}_{loc}(\mathbb{F})$, thanks to Lemma 3.3(a) we conclude that

$$J^{-1} I_{]\!]\tau,+\infty[\![} \cdot [m, M],\ J^{-1} I_{]\!]\tau,+\infty[\![} \cdot \left(\Delta M \cdot \langle m\rangle^{\mathbb{F}}\right) \text{ and } J^{-1} I_{]\!]\tau,+\infty[\![} \cdot \langle m, M\rangle^{\mathbb{F}}$$

belong to $\mathcal{A}_{loc}(\mathbb{G})$. Therefore, this achieves the proof of (3.15), and the proof of the proposition is completed. □

4 Explicit Deflators

We start this section by constructing explicitly local martingale densities under $\mathbb{G}$ for a class of $\mathbb{F}$-local martingales. This will be achieved in Subsections 4.1 and 4.2, where the cases of quasi-left-continuous and thin with predictable jumps are addressed respectively.

4.1 The Quasi-Left-Continuous Case

Theorem 4.1 *Let τ be an honest time satisfying (2.3), and $L^{\mathbb{G}}$ be the $\mathbb{G}$-local martingale defined by (3.14). Let M be a quasi-left-continuous $\mathbb{F}$-local martingale such that $\{\widetilde{Z} = 1 > Z_- \ \& \ \Delta M \neq 0\}$ is evanescent. Then, the following two equivalent assertions hold.*
(a) $\mathcal{E}(L^{\mathbb{G}})\,(M - M^{\tau})$ *is a $\mathbb{G}$-local martingale.*
(b) $\mathcal{E}\left(I_{\{\widetilde{Z}=1>Z_-\}} \odot m\right) M$ *is an $\mathbb{F}$-local martingale.*

Proof Suppose that $\{\widetilde{Z} = 1 > Z_- \ \& \ \Delta M \neq 0\}$ is evanescent. Then

$$\sum I_{\{\widetilde{Z}=1>Z_-\}} \Delta M (1 - Z_-) = I_{\{\widetilde{Z}=1>Z_-\}} \centerdot [M, m] \equiv 0,$$

and due to the quasi-left-continuity of M and Proposition 3.7-(b), we derive

$$\langle I_{\{\widetilde{Z}=1>Z_-\}} \odot m, M\rangle^{\mathbb{F}} = \ {}^{p,\mathbb{F}}\left(I_{\{\widetilde{Z}=1>Z_-\}} \centerdot [M, m]\right) = 0,$$

which is equivalent to assertion (b). To achieve the proof of the theorem, we will prove that both assertions (a) and (b) are equivalent.
Notice that the quasi-left-continuity of M implies $[\langle m\rangle, M] = [m, \langle m, M\rangle] = [\langle m\rangle, \langle m, M\rangle] = 0$. Hence, taking K as defined in (3.13), we obtain

$$K \centerdot [\widehat{m}, \widehat{M}] = K \centerdot [m, M] = \frac{1}{1 - \widetilde{Z}} I_{]\!]\tau, +\infty[\![} \centerdot [m, M]\,. \tag{4.1}$$

Then, by combining (4.1), Proposition 3.7-(b) and (3.3), we calculate

$$\begin{aligned}
\langle L^{\mathbb{G}}, \widehat{M}\rangle^{\mathbb{G}} &= \left(K \centerdot [\widehat{m}, \widehat{M}]\right)^{p,\mathbb{G}} = \left((1 - \widetilde{Z})^{-1} I_{]\!]\tau, +\infty[\![} \centerdot [m, M]\right)^{p,\mathbb{G}} \\
&= \frac{I_{]\!]\tau, +\infty[\![}}{1 - Z_-} \centerdot \langle m, M\rangle^{\mathbb{F}} - \frac{I_{]\!]\tau, +\infty[\![}}{1 - Z_-} \centerdot \left(I_{\{\widetilde{Z}=1>Z_-\}} \centerdot [M, m]\right)^{p,\mathbb{F}} \\
&= \frac{I_{]\!]\tau, +\infty[\![}}{1 - Z_-} \centerdot \langle m, M\rangle^{\mathbb{F}} - \frac{I_{]\!]\tau, +\infty[\![}}{1 - Z_-} \centerdot \langle I_{\{\widetilde{Z}=1>Z_-\}} \odot m, M\rangle^{\mathbb{F}}.
\end{aligned}$$

Then, from the above equality, the equivalence between the two assertions (a) and (b) follows immediately. □

As an immediate consequence of this theorem, we describe a class of $\mathbb{F}$-quasi-left-continuous processes for which the NUPBR property is preserved for the part after τ.

Corollary 4.2 *Suppose that τ satisfies (2.3), and that S is $\mathbb{F}$-quasi-left-continuous and satisfies NUPBR($\mathbb{F}$) and $\{\Delta S \neq 0\} \cap \{\widetilde{Z} = 1 > Z_-\}$ is evanescent. Then, $S - S^\tau$ satisfies NUPBR($\mathbb{G}$).*

Proof The proof follows immediately from a combination of Theorem 4.1, Proposition A.1 (see the appendix), and the fact that

$$\{\widetilde{Z} = 1 > Z_-\} = \{\widetilde{Z}^Q = 1 > Z_-^Q\} \quad \text{for any } Q \sim P, \tag{4.2}$$

where $\widetilde{Z}_t^Q := Q(\tau \geq t|\mathcal{F}_t)$ and $Z_t^Q := Q(\tau > t|\mathcal{F}_t)$. This last fact is an immediate application of Theorem 86 of [7] by taking $X = I_{\{\widetilde{Z}=0\}}$ & $Y = I_{\{\widetilde{Z}^Q=0\}}$ and $X = I_{\{Z_-=0\}}$ & $Y = I_{\{Z_-^Q=0\}}$. □

4.2 The Case of Thin Processes with Predictable Jumps

The construction of deflators for thin $\mathbb{F}$-local martingale requires the following result that is interesting in itself.

Theorem 4.3 *Let τ be an honest time. Consider an $\mathbb{F}$-predictable stopping time T and an $\mathcal{F}_T$-measurable r.v. ξ such that $E[|\xi||\mathcal{F}_{T-}] < +\infty, P$-a.s. Define $M := \xi I_{\{Z_{T-}<1\}} I_{[\![T,+\infty[\![}$,*

$$\begin{aligned}
\frac{dQ_T^{\mathbb{F}}}{dP} := D^{\mathbb{F}} &:= \frac{I_{\{\widetilde{Z}_T<1 \ \&\ P(\widetilde{Z}_T<1|\mathcal{F}_{T-})>0\}}}{P(\widetilde{Z}_T < 1|\mathcal{F}_{T-})} + I_{\{P(\widetilde{Z}_T<1|\mathcal{F}_{T-})=0\}}, \quad \textit{and} \\
\frac{dQ_T^{\mathbb{G}}}{dP} := D^{\mathbb{G}} &:= \frac{1 - Z_{T-}}{(1-\widetilde{Z}_T)P(\widetilde{Z}_T < 1|\mathcal{F}_{T-})} I_{\{T>\tau\}} + I_{\{T\leq\tau\}}.
\end{aligned} \tag{4.3}$$

Then the following assertions are equivalent.
(a) *M is a $(Q_T^{\mathbb{F}}, \mathbb{F})$-martingale.*
(b) *On $\{Z_{T-} < 1\}$, we have*

$$E\left(\xi I_{\{\widetilde{Z}_T<1\}} \mid \mathcal{F}_{T-}\right) = 0, \quad P - a.s. \tag{4.4}$$

(c) *$(M - M^\tau)$ is a $(Q_T^{\mathbb{G}}, \mathbb{G})$-martingale.*

Proof For the sake of simplicity, throughout the proof we will use the notations $Q_1 := Q_T^{\mathbb{F}}$ and $Q_2 := Q_T^{\mathbb{G}}$. Thanks to $E[D^{\mathbb{F}}|\mathcal{F}_{T-}] = 1$ and Lemma (2.5), we obtain

$$E^{Q_1}[\xi I_{\{Z_{T-}<1\}}|\mathcal{F}_{T-}] = E\left[D^{\mathbb{F}} \xi I_{\{Z_{T-}<1\}}|\mathcal{F}_{T-}\right] = \frac{E\left[\xi I_{\{\widetilde{Z}_T<1\}}|\mathcal{F}_{T-}\right]}{P(\widetilde{Z}_T < 1|\mathcal{F}_{T-})} I_{\{Z_{T-}<1\}}.$$

Therefore, the equivalence between assertions (a) and (b) follows from a combination of this equality and the fact that M is a $(Q_1, \mathbb{F})$-martingale if and only if $E^{Q_1}(M_T \mid \mathcal{F}_{T-}) = 0$ on $\{T < +\infty\}$.

In the remaining part of this proof we will prove (b)$\Longleftrightarrow$ (c).
To this end, we first notice that $M - M^\tau$ is $(Q_2, \mathbb{G})$-martingale if and only if $E^{Q_2}[\xi I_{\{Z_{T-}<1 \ \& \ T>\tau\}}|\mathcal{G}_{T-}] = 0$ on $\{T < +\infty\}$. Furthermore, due to (2.5) and $E[D^{\mathbb{G}}|\mathcal{G}_{T-}] = 1$ we have

$$\begin{aligned} &E^{Q_2}[\xi I_{\{Z_{T-}<1 \ \& \ T>\tau\}}|\mathcal{G}_{T-}] = E\left[D^{\mathbb{G}}\xi I_{\{Z_{T-}<1\}}I_{\{T>\tau\}}|\mathcal{G}_{T-}\right] \\ &= E\left[\frac{\xi I_{\{T>\tau\}}}{1-\widetilde{Z}_T}\Big|\mathcal{G}_{T-}\right]\frac{1-Z_{T-}}{P(\widetilde{Z}_T<1|\mathcal{F}_{T-})}I_{\{Z_{T-}<1 \ \& \ T>\tau\}} \\ &= \frac{E\left[\xi I_{\{\widetilde{Z}_T<1\}}|\mathcal{F}_{T-}\right]}{P(\widetilde{Z}_T<1|\mathcal{F}_{T-})}I_{\{Z_{T-}<1\}}I_{\{T>\tau\}}, \end{aligned} \tag{4.5}$$

where the last equality in (4.5) follows from the fact that

$$E\left(H \mid \mathcal{G}_{T-}\right)I_{\{T>\tau\}} = E\left(H(1-\widetilde{Z}_T) \mid \mathcal{F}_{T-}\right)\frac{I_{\{T>\tau\}}}{1-Z_{T-}}.$$

for any $\mathcal{F}_T$-measurable random variable $H$for which the above conditional expectations exist (see Proposition 5.3 of [14]). Therefore, if assertion (b) holds, then assertion (c) follows immediately from (4.5). Conversely, if assertion (c) holds, then $E^{Q_2}[\xi I_{\{Z_{T-}<1\}}I_{\{T>\tau\}}|\mathcal{G}_{T-}] = 0$. Thus, a combination of this with (4.5) leads to $E\left[\xi I_{\{\widetilde{Z}_T<1\}}|\mathcal{F}_{T-}\right](1-Z_{T-}) = 0$. This proves assertion (ii), and the proof of the theorem is completed. □

Remark 4.4 (a) It is important to note that (recall that $V^{\mathbb{G}}$ is defined in (3.10))

$$\begin{aligned} D^{\mathbb{G}}(T) :&= \frac{1-Z_{T-}}{1-\widetilde{Z}_T}\frac{I_{\{T>\tau\}}}{P(\widetilde{Z}_T<1|\mathcal{F}_{T-})} + I_{\{T\leq\tau\}} = \left(1+\frac{\Delta m_T}{1-\widetilde{Z}_T}\right)I_{\{T>\tau\}} + I_{\{T\leq\tau\}} \\ &= \frac{1+\Delta L^{\mathbb{G}} - \Delta V^{\mathbb{G}}}{1-\Delta V^{\mathbb{G}}} \neq 1+\Delta L^{\mathbb{G}} = \frac{\mathcal{E}_T\left(L^{\mathbb{G}}\right)}{\mathcal{E}_{T-}\left(L^{\mathbb{G}}\right)}. \end{aligned}$$

This explains one of the main difficulties that we will face when dealing with countable many predictable jumps that might not be ordered on the one hand. On the other hand, it also gives us a hint about the candidate for martingale measures when dealing with countably many jumps. This point will be detailed below.
(b) By combining the first remark above with Proposition 3.5–(b), we deduce that the local martingale $(1-\Delta V^{\mathbb{G}})^{-1} \bullet L^{\mathbb{G}}$ is well defined.

Theorem 4.5 *Let M be a thin $\mathbb{F}$-local martingale (that is $M^{(qc)} \equiv 0$) such that $\{\Delta M \neq 0\} \cap \{\widetilde{Z} = 1 > Z_-\} = \emptyset$ and $V^{\mathbb{G}}$ defined in (3.10). Then*

$$\mathcal{E}\left((1-\Delta V^{\mathbb{G}})^{-1} \bullet L^{\mathbb{G}}\right)(M - M^\tau) \quad \textit{is a } \mathbb{G}\textit{-local martingale.} \tag{4.6}$$

Proof Thanks to Itô's formula, it is immediate that (4.6) is equivalent to

$$X^{\mathbb{G}} := M - M^\tau + [(1-\Delta V^{\mathbb{G}})^{-1} \bullet L^{\mathbb{G}}, M - M^\tau] \text{ is a } \mathbb{G}\text{-local martingale.} \tag{4.7}$$

Since $X^{\mathbb{G}}$ is a $\mathbb{G}$-special semimartingale, hence it is enough to prove that $X^{\mathbb{G}}$ is a σ-martingale under $\mathbb{G}$. Let $(T_n)_n$ be a sequence of predictable stopping times that exhaust the jumps of M. Since $\{\Delta M \neq 0\} \cap \{\widetilde{Z} = 1 > Z_-\} = \emptyset$, we deduce that the martingale $M^{(n)} := \Delta M_{T_n} I_{[\![T_n,+\infty[\![}$ satisfies assertion (b) of Theorem 4.3. Thus, by combining this theorem and Remark 4.4-(a), we conclude that for any $n \geq 1$,

$$X^{(n)} := \Delta M_{T_n} \left(1 + \frac{\Delta L^{\mathbb{G}}_{T_n}}{1 - \Delta V^{\mathbb{G}}_{T_n}}\right) I_{[\![T_n,+\infty[\![} \quad \text{is a } \mathbb{G}\text{-martingale.} \tag{4.8}$$

Consider the following $\mathbb{G}$-predictable processes

$$\Phi := \sum_{n\geq 1} \frac{2^{-n}}{1+\xi_n} I_{[\![T_n]\!]} + I_{(\cup [\![T_n]\!])^c},\ \xi_n := E\left[|\Delta M_{T_n}|(1 + \frac{|\Delta L^{\mathbb{G}}_{T_n}|}{1 - \Delta V^{\mathbb{G}}_{T_n}})\big|\ \mathcal{G}_{T_n -} \right] I_{\{T_n < +\infty\}}.$$

Then, it is easy to check that $0 < \Phi \leq 1$, and that $\Phi \bullet X^{\mathbb{G}}$ has an integrable variation. Furthermore, due to (4.8), we calculate

$$\left(\Phi \bullet X^{\mathbb{G}}\right)^{p,\mathbb{G}} = \sum_{n\geq 1} \left(\Phi \bullet X^{(n)}\right)^{p,\mathbb{G}} \equiv 0.$$

This proves that $\Phi \bullet X^{\mathbb{G}}$ is a $\mathbb{G}$-martingale, and hence $X^{\mathbb{G}}$ is a $\mathbb{G}$-local martingale. This ends the proof of the theorem. □

Corollary 4.6 *Suppose that S is thin, $\{\Delta S \neq 0\} \cap \{\widetilde{Z} = 1 > Z_-\} = \emptyset$, and S satisfies NUPBR($\mathbb{F}$). Then $S - S^\tau$ satisfies NUPBR($\mathbb{G}$).*

Proof Since S satisfies NUPBR($\mathbb{F}$), then there exist an $\mathbb{F}$-predictable process ϕ, a sequence of $\mathbb{F}$-stopping times $(T_n)_{n\geq 1}$ that increases to infinity, and a probability measure $Q_n \sim P$ on $(\Omega, \mathcal{F}_{T_n})$ such that

$$0 < \phi \leq 1, \quad \phi \bullet S^{T_n} \in \mathcal{M}_{0,loc}(Q,\mathbb{F}).$$

Then, a combination of $\{\Delta S \neq 0\} \cap \{\widetilde{Z} = 1 > Z_-\} = \emptyset$ and (4.2) leads to

$$\{\Delta(\phi \bullet S^{T_n}) \neq 0\} \cap \{\widetilde{Z}^{Q_n} = 1 > Z^{Q_n}_-\} = \emptyset.$$

Therefore, by applying directly Theorem 4.5 to $\phi \bullet S^{T_n}$ under Q_n, we conclude that $\phi \bullet S^{T_n} - (\phi \bullet S^{T_n})^\tau$ has a local martingale density under $(Q_n, \mathbb{G})$. Thus, this implies that $S^{T_n} - S^{T_n \wedge \tau}$ satisfies NUPBR($\mathbb{G}, Q_n$). Hence, the corollary follows immediately from Proposition A.1. This ends the proof of the corollary. □

5 Proofs of Three Main Theorems

In this section, we will prove the main theorems that are not proved in Section 2, namely Theorems 2.6, 2.11 and 2.15. To this end, we introduce some notations that will be used throughout the rest of the paper. Recall that μ is the random measure of the jumps of S, its random measure compensator ν and the functional f_m are defined in (2.4) and (2.5) respectively. Put

$$\mu_{\mathbb{G}}(dt,dx) := I_{]\!]\tau,+\infty[\![}(t)\mu(dt,dx),$$
$$\nu_{\mathbb{G}}(dt,dx) := I_{]\!]\tau,+\infty[\![}(t)\left(1-\frac{f_m(x,t)}{1-Z_{t-}}\right)\nu(dt,dx).$$

It is easy to check that $\nu_{\mathbb{G}}$ is the random measure compensator under $\mathbb{G}$ of $\mu_{\mathbb{G}}$. The canonical decomposition of $S-S^\tau$ under $\mathbb{G}$ is given by

$$S-S^\tau = \widehat{S^c} + h\star(\mu_{\mathbb{G}}-\nu_{\mathbb{G}}) + bI_{]\!]\tau,+\infty[\![}\cdot A - \frac{c\beta_m}{1-Z_-}I_{]\!]\tau,+\infty[\![}\cdot A$$
$$-h\frac{f_m}{1-Z_-}I_{]\!]\tau,+\infty[\![}\star\nu + (x-h)\star\mu_{\mathbb{G}},$$

where $\widehat{S^c}$ is defined by (3.1).

The proof of Theorem 2.6 is based on the following

Lemma 5.1 *$(S-S^\tau)$ satisfies NUPBR($\mathbb{G}$) if and only if there exist a $\mathbb{F}$-predictable process $\beta^{\mathbb{F}}$, and a positive $\widetilde{\mathcal{P}}(\mathbb{F})$-functional $f^{\mathbb{F}}$, such that*

$$(\beta^{\mathbb{F}})^{tr}c\beta^{\mathbb{F}}I_{]\!]\tau,+\infty[\![}\cdot A \text{ and } \sqrt{(f^{\mathbb{F}}-1)^2\star\mu_{\mathbb{G}}} \text{ belong to } \mathcal{A}^+_{loc}(\mathbb{G}). \tag{5.1}$$

and $P\otimes A-a.e.$ on $\{Z_-<1\}$, we have

$$\varphi^{\mathbb{F}} := \int |xf^{\mathbb{F}}(1-Z_--f_m)-h(s)|F(dx) < +\infty\,, \tag{5.2}$$

and

$$b+c\left(\beta^{\mathbb{F}}-\frac{\beta_m}{1-Z_-}\right)+\int\left[xf^{\mathbb{F}}(x)(1-\frac{f_m(x)}{1-Z_-})-h(x)\right]F(dx)\equiv 0. \tag{5.3}$$

Proof $(S-S^\tau)$ satisfies NUPBR($\mathbb{G}$) if and only if there exist a $\mathbb{G}$-predictable process $\Phi^{\mathbb{G}}$, and a $\mathbb{G}$-local martingale $N^{\mathbb{G}}$ that can be chosen as

$$N^{\mathbb{G}} := \beta^{\mathbb{G}}\cdot\widehat{S^c} + (f^{\mathbb{G}}-1)\star(\mu^{\mathbb{G}}-\nu^{\mathbb{G}}),$$

such that $0<\Phi^{\mathbb{G}}\le 1$ and $\mathcal{E}(N^{\mathbb{G}})>0$ and $\left[\Phi^{\mathbb{G}}\cdot(S-S^\tau)\right]\mathcal{E}(N^{\mathbb{G}})$ is a $\mathbb{G}$-local martingale, which is equivalent to

$$\Phi^{\mathbb{G}}\cdot(S-S^\tau)+[N^{\mathbb{G}},\Phi^{\mathbb{G}}\cdot(S-S^\tau)] \text{ is a } \mathbb{G}\text{-local martingale.}$$

Thanks to Itô's formula, this is equivalent to

$$\Phi^{\mathbb{G}}|xf^{\mathbb{G}}(x)-h(x)|\star\mu_{\mathbb{G}}\in\mathcal{A}^{+}_{loc}(\mathbb{G}), \tag{5.4}$$

and $P\otimes A$-a.e. on $]\!]\tau,+\infty[\![$ we have

$$0\equiv b+c\left(\beta^{\mathbb{G}}-\frac{\beta_m}{1-Z_-}\right)+\int\left[xf^{\mathbb{G}}(x)(1-\frac{f_m(x)}{1-Z_-})-h(x)\right]F(dx) \tag{5.5}$$

Then, Lemma A.2 guarantees the existence of two $\mathbb{F}$-predictable processes $\Phi^{\mathbb{F}}$ and $\beta^{\mathbb{F}}$, and a $\widetilde{\mathcal{P}}(\mathbb{F})$-functional $f^{\mathbb{F}}$, such that $0<\Phi^{\mathbb{F}}\leq 1$, $0<f^{\mathbb{F}}$ and

$$\Phi^{\mathbb{F}}=\Phi^{\mathbb{G}},\quad \beta^{\mathbb{F}}=\beta^{\mathbb{G}},\quad f^{\mathbb{F}}=f^{\mathbb{G}}\quad \text{on }]\!]\tau,+\infty[\![.$$

Then, (5.4) becomes $\Phi^{\mathbb{F}}|xf^{\mathbb{F}}(x)-h(x)|\star\mu_{\mathbb{G}}\in\mathcal{A}^{+}_{loc}(\mathbb{G})$, and from a direct application of Proposition C.1–(b), we deduce that (5.2) follows immediately. Furthermore, (5.3) follows from a combination of this property and (5.5) in which we substitute $\beta^{\mathbb{F}}$ and $f^{\mathbb{F}}$ to $\beta^{\mathbb{G}}$ and $f^{\mathbb{G}}$ respectively and we take the $\mathbb{F}$-predictable projection of the resulting equation afterwards. This ends the proof of the lemma. □

5.1 Proof of Theorem 2.6

The proof of the theorem will be achieved in two steps where we prove (a) $\Longrightarrow$ (b) and the reverse sense.

Proof of (a) $\Longrightarrow$ (b): Suppose that $S-S^{\tau}$ satisfies NUPBR($\mathbb{G}$). Then, thanks to Lemma 5.1, we deduce the existence of the $\mathbb{F}$-predictable pair $(\beta^{\mathbb{F}},f^{\mathbb{F}})$ satisfying $f^{\mathbb{F}}>0$, and (5.1), (5.2) and (5.3) hold. Then, fix $\delta\in(0,1)$, and put

$$\overline{S}:=(1-Z_-)\centerdot S-S^{(1)}\qquad\text{and}\qquad \Gamma:=\{Z_-\leq 1-\delta\ \&\ 1=Z_-+f_m\}.$$

Now, we put $\psi:=1-Z_-+f_m:=1-M^P_{\mu}(\widetilde{Z}|\widetilde{\mathcal{P}}(\mathbb{F}))$, and consider the following

$$\begin{aligned}\beta&:=(\beta^{\mathbb{F}}-\frac{\beta_m}{1-Z_-})I_{\{Z_-\leq 1-\delta\}},\\ f&:=f^{\mathbb{F}}(x)\left(1-\frac{f_m(x)}{1-Z_-}\right)I_{\{\psi>0\ \&\ Z_-\leq 1-\delta\}}+I_{\{\psi=0\ \text{or}\ Z_-\geq 1-\delta\}},\end{aligned}$$

and we assume for a while that

$$\beta\in L(S^c,\mathbb{F})\ \text{ and }\quad (f-1)\in\mathcal{G}^1_{loc}(\mu,\mathbb{F}). \tag{5.6}$$

Then, the process $N:=\beta\centerdot S^c+(f-1)\star(\mu-\nu)$ is a well defined $\mathbb{F}$-local martingale, and it is easy to check, using Itô's formula, that $(\phi I_{\{Z_-\leq 1-\delta\}}\centerdot\overline{S})\mathcal{E}(N)$ is a local martingale due to (5.2) and (5.3), where $\phi:=(1+\varphi^{\mathbb{F}}I_{\{Z_-\leq 1-\delta\}})^{-1}$ and $\varphi^{\mathbb{F}}$ is given by (5.2). This proves that $I_{\{Z_-\leq 1-\delta\}}\centerdot\overline{S}$ satisfies NUPBR($\mathbb{F}$) as long as (5.6) holds. The remaining proof in this part will focus on proving

this assumption.
Since $\beta_m^{tr} c \beta_m \cdot A \in \mathcal{A}^+_{loc}(\mathbb{F})$ and $(\beta^{\mathbb{F}})^{tr} c \beta^{\mathbb{F}} I_{]\!]\tau,+\infty[\![} \cdot A \in \mathcal{A}^+_{loc}(\mathbb{G})$, then Lemma 3.4 leads to $\beta^{tr} c \beta \cdot A \in \mathcal{A}^+_{loc}(\mathbb{F})$ or equivalently $\beta \in L(S^c)$.
By putting $\Sigma_1 := \{\psi > 0 \ \& \ Z_- \leq 1-\delta\}$, we calculate

$$f - 1 = (f^{\mathbb{F}} - 1)\left(1 - \frac{f_m(x)}{1-Z_-}\right) I_{\Sigma_1} - \frac{f_m(x)}{1-Z_-} I_{\Sigma_1} =: W_1 + W_2.$$

Then, we put $\bar{\mu} := (1 - \widetilde{Z}) \cdot \mu$ and we obtain, in the one hand

$$W_1^2 I_{\{|f^{\mathbb{F}}-1| \leq \alpha\}} \star \mu \leq \delta^{-2} (f^{\mathbb{F}} - 1)^2 (1 - Z_- - f_m) I_{\{Z_- \leq 1-\delta\}} I_{\{|f^{\mathbb{F}}-1| \leq \alpha\}} \star \mu$$

where $(f^{\mathbb{F}} - 1)^2 (1 - Z_- - f_m) I_{\{|f^{\mathbb{F}}-1| \leq \alpha\}} I_{\{Z_- \leq 1-\delta\}} \star \mu \in \mathcal{A}^+_{loc}(\mathbb{F})$ (which is equivalent to $(f^{\mathbb{F}} - 1)^2 I_{\{|f^{\mathbb{F}}-1| \leq \alpha\}} I_{\{Z_- \leq 1-\delta\}} \star \bar{\mu} \in \mathcal{A}^+_{loc}(\mathbb{F})$). In the other hand,

$$|W_1| I_{\{|f^{\mathbb{F}}-1| > \alpha\}} \star \mu = \delta^{-1} |f^{\mathbb{F}} - 1| (1 - Z_- - f_m) I_{\{Z_- \leq 1-\delta\}} I_{\{|f^{\mathbb{F}}-1| > \alpha\}} \star \mu$$

where $|f^{\mathbb{F}} - 1|(1 - Z_- - f_m) I_{\{|f^{\mathbb{F}}-1| > \alpha\}} I_{\{Z_- \leq 1-\delta\}} \star \mu \in \mathcal{A}^+_{loc}(\mathbb{F})$ (this is equivalent to $|f^{\mathbb{F}} - 1| I_{\{|f^{\mathbb{F}}-1| > \alpha\}} I_{\{Z_- \leq 1-\delta\}} \star \bar{\mu} \in \mathcal{A}^+_{loc}(\mathbb{F})$). Thus by combining these two remarks and

$$\sqrt{W_1^2 \star \mu} \leq \sqrt{W_1^2 I_{\{|f^{\mathbb{F}}-1| \leq \alpha\}} \star \mu} + |W_1| I_{\{|f^{\mathbb{F}}-1| > \alpha\}} \star \mu,$$

we conclude that $\sqrt{W_1^2 \star \mu} \in \mathcal{A}^+_{loc}(\mathbb{F})$. Similarly we notice that

$$W_2^2 \star \mu \leq \delta^{-2} W_3 \in \mathcal{A}^+_{loc}(\mathbb{F}),$$

where $W_3 := f_m^2 \star \mu \in \mathcal{A}^+_{loc}(\mathbb{F})$ satisfies $E W_3(\sigma) \leq E[m,m]_\sigma$ for any stopping time σ. This proves that $(f-1) \in \mathcal{G}^1_{loc}(\mu, \mathbb{F})$, and the proof of (a)$\Longrightarrow$(b) is completed.

Proof of (b)$\Longrightarrow$(a): Suppose that for any $\delta \in (0,1)$, $I_{\{Z_- \leq 1-\delta\}} \cdot \overline{S}$ satisfies NUPBR($\mathbb{F}$). Then, there exists a pair (β, f) satisfying $f > 0$,

$$\beta^{tr} c \beta \cdot A \in \mathcal{A}^+_{loc}(\mathbb{F}), \qquad \sqrt{(f-1)^2 \star \mu} \in \mathcal{A}^+_{loc}(\mathbb{F}),$$

and

$$\int |x f(x) - h(x)| F(dx) < +\infty \quad P \otimes A - -\text{on} \ \ \{1 - Z_- \geq \delta\}, \tag{5.7}$$

and $P \otimes A$-a.e. on $\{1 - Z_- \geq \delta\}$ (recall that $\psi = 1 - Z_- - f_m$)

$$b + c\beta + \int \left[x f(x) I_{\{\psi > 0\}} - h(x)\right] F(dx) = 0 \tag{5.8}$$

Now we start constructing a σ-martingale density for $I_{\{Z_- \leq 1-\delta\}} \cdot (S - S^\tau)$ as follows. Consider

$$\beta^{\mathbb{G}} := \left(\beta + \frac{\beta_m}{1 - Z_-}\right) I_{]\!]\tau,+\infty[\![}, \qquad f^{\mathbb{G}} := \frac{f}{1 - f_m(x)/(1-Z_-)} I_{]\!]\tau,+\infty[\![} + I_{[\![0,\tau]\!]},$$

and assume for a while that

$$\beta^{\mathbb{G}} \in L(\widehat{S^c}, \mathbb{G}) \quad \text{and} \quad (f^{\mathbb{G}} - 1) \in \mathcal{G}^1_{loc}(\mu_{\mathbb{G}}, \mathbb{G}). \tag{5.9}$$

Then, using Itô and (5.7)—(5.8) afterwards, we can easily prove that the process $\left(\phi^{\mathbb{G}} \bullet (S - S^\tau)\right) \mathcal{E}(N^{\mathbb{G}})$ is a $\mathbb{G}$-local martingale, where

$$\phi^{\mathbb{G}} := \left(1 + \int |xf(x) I_{\{Z_- + f_m < 1\}} - h(x)|F(dx) I_{]\!]\tau, +\infty[\![}\right)^{-1}.$$

Thus assertion (a) follows immediately once (5.9) is proved. This will be the main goal of the rest of the proof of this part.
Since $\beta^T c \beta \bullet A$ and $\beta_m^T c \beta_m \bullet A$ belong to $\mathcal{A}^+_{loc}(\mathbb{F}) \subset \mathcal{A}^+_{loc}(\mathbb{G})$ and $(1-Z_-)^{-1} I_{]\!]\tau,+\infty[\![}$ is $\mathbb{G}$-locally bounded, we deduce that $\beta^{\mathbb{G}} \in L(\widehat{S^c}, \mathbb{G})$. To prove the second property in (5.9), we start by calculating on $]\!]\tau, +\infty[\![$,

$$f^{\mathbb{G}} - 1 = \frac{f-1}{1 - f_m/(1-Z_-)} + \frac{f_m}{1 - Z_- - f_m} =: g_1 + g_2.$$

Since $\sqrt{(f-1)^2 \star \mu} \in \mathcal{A}^+_{loc}(\mathbb{F})$, we deduce that —due to Proposition C.1–(a)— that

$$\left[(f-1)^2 I_{\{|f-1| \le \alpha\}} + |f-1| I_{\{|f-1| > \alpha\}}\right] \star \mu \in \mathcal{A}^+_{loc}(\mathbb{F}).$$

Without loss of generality we assume that this process and $f_m^2 \star \mu$ are integrable. Thanks to Lemma 3.2–(a), there exists a sequence of $\mathbb{G}$-stopping times $(\tau_n)_{n\ge1}$ that increase to infinity and $Z_-^{\tau_n} \le 1 - 1/n$ on $]\!]\tau, +\infty[\![$. Then, setting $\Gamma_n(\alpha) := \{|f-1| \le \alpha \ \& \ 1 - Z_- - f_m \ge 1/(2n)\}$, we calculate

$$\begin{aligned} E\left[g_1^2 I_{\Gamma_n(\alpha)} \star \mu_{\mathbb{G}}(\infty)\right] &= E\left[g_1^2 I_{\Gamma_n(\alpha)} \star \nu_{\mathbb{G}}(\infty)\right] \\ &\le 2nE\left[(f-1)^2 I_{\{|f-1| \le \alpha\}} \star \nu(\infty)\right] < +\infty \end{aligned}$$

and

$$E\left[\sqrt{g_1^2 I_{\{|f-1| \le \alpha\}} I_{\Gamma_n(\alpha)^c} \star \mu_{\mathbb{G}}(\tau_n)}\right] \le E\left[|g_1| I_{\{|f-1| \le \alpha\}} I_{\Gamma_n(\alpha)} \star \nu_{\mathbb{G}}(\tau_n)\right]$$

$$\le \alpha E\left[I_{\{|f_m| > \frac{1}{2n}\}} \star \nu(\infty)\right] \le 4n^2 \alpha E\left[f_m^2 \star \nu(\infty)\right] < +\infty.$$

Also we calculate

$$\begin{aligned} E\sqrt{g_1^2 I_{\{|f-1| > \alpha\}} \star \mu_{\mathbb{G}}(\infty)} &\le E\left[|g_1| I_{\{|f-1| > \alpha\}} \star \nu_{\mathbb{G}}(\infty)\right] \\ &\le \alpha E\left[|f-1| I_{\{|f-1| > \alpha\}} \star \nu(\infty)\right] < +\infty. \end{aligned}$$

Therefore, $\sqrt{g_1^2 \star \mu_{\mathbb{G}}} \in \mathcal{A}^+_{loc}(\mathbb{G})$. Similarly, the fact that $\sqrt{g_2^2 \star \mu_{\mathbb{G}}} \in \mathcal{A}^+_{loc}(\mathbb{G})$ follows from

$$\begin{aligned} E\left[g_2^2 I_{\{1 - Z_- - f_m \ge \frac{1}{2n}\}} \star \mu_{\mathbb{G}}(\infty)\right] &= E\left[g_2^2 I_{\{1 - Z_- - f_m \ge \frac{1}{2n}\}} \star \nu_{\mathbb{G}}(\infty)\right] \\ &\le 2nE\left[(f_m)^2 \star \nu(\infty)\right] < +\infty \end{aligned}$$

and

$$E\sqrt{g_2^2 I_{\{1-Z_- - f_m < \frac{1}{2n}\}} \star \mu_{\mathbb{G}}(\tau_n)} \leq E\left[|g_2| I_{\{1-Z_- - f_m < \frac{1}{2n}\}} \star \mu_{\mathbb{G}}(\tau_n)\right]$$
$$\leq E\left[|f_m| I_{\{|f_m| > \frac{1}{2n}\}} \star \nu(\infty)\right] \leq 2nE\left[f_m^2 \star \nu(\infty)\right] < +\infty.$$

This ends the proof of this part, and the proof of the theorem is completed. □

5.2 Proof of Theorem 2.11

For the reader's convenience, we state and prove a very detailed version of Theorem 2.11, where we provide explicit forms for the absolute continuous probability measure Q_T mentioned in Theorem 2.11–(c).

Theorem 5.2 *Under the assumptions of Theorem 2.11, the assertions* (a) *and* (b) *of Theorem 2.11 are equivalent to the following assertions:*
(d) *S satisfies NUPBR$(\mathbb{F}, \widetilde{Q}_T^{\mathbb{F}})$, where*

$$\widetilde{Q}_T^{\mathbb{F}} := \Big(\frac{1-\widetilde{Z}_T}{1-Z_{T-}} I_{\{Z_{T-}<1\}} + I_{\{Z_{T-}=1\}}\Big) \cdot P. \tag{5.10}$$

(e) *S satisfies NUPBR$(\mathbb{F}, Q_T^{\mathbb{F}})$, where $Q_T^{\mathbb{F}}$ is given by (4.3).*

Proof The proof consists of four steps. The first step addresses (d)⇔(e), the second step focuses on (e)⇒(b), while the third and the fourth steps deal with (b)⇒ (a) and (a)⇒ (d) respectively.

Step 1: We prove (d)⇔(e). Due to $\{Z_{T-} = 1\} = \{P(\widetilde{Z}_T < 1|\mathcal{F}_{T-}) = 0\} \subset \{\widetilde{Z}_T = 1\}$ (see Lemma 2.5), it is easy to see that $\widetilde{Q}_T^{\mathbb{F}} \sim Q_T^{\mathbb{F}} \ll P$. Therefore, the equivalence between (d) and (e) follows immediately.

Step 2: This step focuses on proving (e)⇒(b). Suppose that (e) holds. Then, there exists an $\mathcal{F}_T$-measurable random variable $Y_T > 0$, $Q_T^{\mathbb{F}}$-a.s. such that $E^{Q_T^{\mathbb{F}}}[S_T Y_T|\mathcal{F}_{T-}] = 0$, or equivalently

$$E[\xi Y_T I_{\{\widetilde{Z}_T<1\}}|\mathcal{F}_{T-}] I_{\{Z_{T-}<1\}} = 0 \ \text{ and } \ E[\xi Y_T|\mathcal{F}_{T-}] I_{\{Z_{T-}=1\}} = 0. \tag{5.11}$$

Since, on the set $\{Z_{T-} = 1\}$, $\widetilde{S} \equiv 0$, it is enough to focus on the part corresponding to $\{Z_{T-} < 1\}$. Put

$$\widetilde{Y}_T := Y_T I_{\{\widetilde{Z}_T<1\}} + I_{\{\widetilde{Z}_T=1\}} \quad \text{ and } \ Q_1 := \widetilde{Y}_T / E(\widetilde{Y}_T|\mathcal{F}_{T-}) \cdot P \sim P.$$

Then, from (5.11), we derive that $E^{Q_1}[\widetilde{S}_T|\mathcal{F}_{T-}] = E^{Q_1}[\xi I_{\{\widetilde{Z}_T<1\}}|\mathcal{F}_{T-}] = 0$. Therefore, $\widetilde{S}$ is a $(Q_1, \mathbb{F})$-martingale and assertion (b) follows.

Step 3: This step proves (b)⇒ (a). Suppose that $\widetilde{S}$ satisfies NUPBR$(\mathbb{F})$. Then, there exists an $\mathcal{F}_T$-measurable $Y_3 > 0$ such that $E[Y_3 \xi I_{\{\widetilde{Z}_T<1\}}|\mathcal{F}_{T-}] = 0$. Put

$Q_3 := Y_3/E(Y_3|\mathcal{F}_{T-}) \cdot P \sim P$ and remark that $\{\widetilde{Z}_T < 1\} = \{\widetilde{Z}_T^{Q_3} < 1\}$, where $\widetilde{Z}_t^{Q_3} := Q_3(\tau \geq t|\mathcal{F}_t)$. Therefore, a direct application of Theorem 4.3 under Q_3 proves that $S - S^\tau = \widetilde{S} - \widetilde{S}^\tau$ satisfies NUPBR($\mathbb{G}$), and assertion (a) holds.

Step 4: This last step will prove (a)⇒ (d). Suppose that $S - S^\tau$ satisfies NUPBR($\mathbb{G}$). There exists a positive $\mathcal{G}_T$-measurable random variable $Y^{\mathbb{G}}$ satisfying

$$E[Y^{\mathbb{G}} I_{\{T>\tau\}}|\mathcal{G}_{T-}] = I_{\{T>\tau\}} \quad \& \quad E[\xi Y^{\mathbb{G}} I_{\{T>\tau\}}|\mathcal{G}_{T-}] = 0. \tag{5.12}$$

Then, thanks to Jeulin [14, Proposition 5.3], we deduce the existence of a positive $\mathcal{F}_T$-measurable random variable $Y^{\mathbb{F}}$ such that $Y^{\mathbb{G}} I_{\{T>\tau\}} = Y^{\mathbb{F}} I_{\{T>\tau\}}$. Then, by inserting this equality in (5.12), we obtain $E(Y^{\mathbb{F}}(1 - \widetilde{Z}_T)|\mathcal{F}_{T-}) = 1 - Z_{T-}$ $P - a.s.$ and

$$\begin{aligned} 0 = E[\xi Y^{\mathbb{G}} I_{\{T>\tau\}}|\mathcal{G}_{T-}] &= E[\xi Y^{\mathbb{F}}(1 - \widetilde{Z}_T)|\mathcal{F}_{T-}] \frac{I_{\{T>\tau\}}}{1 - Z_{T-}} \\ &= E^{\widetilde{Q}_T^{\mathbb{F}}}\left(\xi Y^{\mathbb{F}}\middle|\ \mathcal{F}_{T-}\right) I_{\{T>\tau\}} = E^{\widetilde{Q}_T^{\mathbb{F}}}\left(\xi \widetilde{Y}^{\mathbb{F}}\middle|\ \mathcal{F}_{T-}\right) I_{\{T>\tau\}}, \end{aligned}$$

where $\widetilde{Y}^{\mathbb{F}} := Y^{\mathbb{F}} I_{\{Z_{T-}<1\}} + I_{\{Z_{T-}=1\}}$. Therefore, by taking conditional expectation and using the fact that $S_T I_{\{Z_{T-}=0\}} = 0$, we obtain $E^{\widetilde{Q}_T^{\mathbb{F}}}\left(\widetilde{Y}^{\mathbb{F}}\middle|\ \mathcal{F}_{T-}\right) = 1$ and

$$(1 - Z_{T-}) E^{\widetilde{Q}_T^{\mathbb{F}}}[\xi \widetilde{Y}^{\mathbb{F}}\big|\mathcal{F}_{T-}] = 0, \text{ or equivalently } \quad E^{\widetilde{Q}_T^{\mathbb{F}}}[S_T \widetilde{Y}^{\mathbb{F}}\big|\mathcal{F}_{T-}] = 0 \ \ P - a.s.$$

This proves that S satisfies NUPBR($\mathbb{F}, \widetilde{Q}_T^{\mathbb{F}}$) and the proof of the theorem is achieved. □

5.3 Proof of Theorem 2.15

We start the proof of this theorem by outlining a number of remarks that simplify tremendously the proof. On the set $\{T < +\infty\}$, we have

$$\{\widetilde{Z}_T^Q = 1\} = \{\widetilde{Z}_T = 1\} \quad \text{for any } Q \sim P \text{ and any } \mathbb{F}-\text{stopping time } T, \tag{5.13}$$

where $\widetilde{Z}_t^Q := E^Q[I_{\{\tau \geq t\}}|\mathcal{F}_t]$. Indeed, due to

$$E\left[(1 - \widetilde{Z}_T) I_{\{\widetilde{Z}_T^Q = 1\}}\right] = E\left[I_{\{\tau<T\}} I_{\{\widetilde{Z}_T^Q = 1\}}\right] = 0,$$

the inclusion $\{\widetilde{Z}_T^Q = 1\} \subset \{\widetilde{Z}_T = 1\}$ follows, while the reverse inclusion follows by symmetry. This proves (5.13).
The remaining part of this proof contains two parts 1) and 2) where we prove (b)⇒ (a) and (a)⇒ (b) respectively.

Part 1) Herein we prove (b)⇒ (a). Suppose that assertion (b) holds, and consider a sequence of $\mathbb{F}$-stopping times $(\sigma_n)_n$ such that Y^{σ_n} is a martingale, and put $Q_n := (Y_{\sigma_n}/Y_0)\cdot P$. Therefore, thanks to Proposition A.1, it is enough to prove that assertion (a) holds true under Q_n for S^{σ_n}. Therefore, without loss of generality, we assume $Y \equiv 1$. Put

$$X_n := \Delta S_{T_n} I_{\{\widetilde{Z}_{T_n}<1\}} \quad \text{and} \quad M_n := X_n I_{[\![T_n,+\infty[\![}.$$

Hence, the condition (4.4) in Theorem 4.3 is trivially satisfied for X_n and M_n. Thus, we deduce that for each n, for $L^{\mathbb{G}}$ and $V^{\mathbb{G}}$ defined in (3.14) and (3.10)

$$(1+\Delta L^{\mathbb{G}}_{T_n} - \Delta V^{\mathbb{G}}_{T_n})\Delta S_{T_n} I_{\{T_n>\tau\}} I_{[\![T_n,+\infty[\![},$$

is a $\mathbb{G}$-martingale. Then, for each $\theta \in L(S-S^\tau)$ such that $\theta^T \Delta S > -1$, Yor's formula implies that

$$\mathcal{E}\left(I_\Gamma \bullet L^{\mathbb{G}} - I_\Gamma \cdot V^{\mathbb{G}}\right) \mathcal{E}\left(\theta I_\Gamma \bullet (S-S^\tau)\right) = \mathcal{E}\left(W\right),$$

where $\Gamma : \Gamma_S$ defined via (2.8) and

$$W := I_\Gamma \bullet (L^{\mathbb{G}} - V^{\mathbb{G}}) + \sum_{n\geq 1} \theta_{T_n}\left(1+\Delta L^{\mathbb{G}}_{T_n} - \Delta V^{\mathbb{G}}_{T_n}\right)\Delta S_{T_n} I_{\{T_n>\tau\}} I_{[\![T_n,+\infty[\![}\,.$$

Set $\phi = \sum_{n\geq 1} \dfrac{2^{-n}\left(1+\mathcal{E}(W)_{T_n-}\right)^{-1}}{(1+\Xi_n)} I_{]\!]\tau,+\infty[\![\cap[\![T_n]\!]} + I_{\Gamma^c\cup[\![0,\tau]\!]}$ and

$$\Xi_n := E\left[|\Delta L^{\mathbb{G}}_{T_n}|\,\Big|\mathcal{G}_{T_n-}\right] + \Delta V^{\mathbb{G}}_{T_n} + E\left[\frac{|\theta_{T_n}\Delta S_{T_n}|(1-Z_{T_n-})}{1-\widetilde{Z}_{T_n}}\Big|\mathcal{G}_{T_n-}\right] I_{\{T_n<+\infty\}}.$$

Then, it is easy to check that ϕ is $\mathbb{G}$-predictable, $0<\phi\leq 1$, and $E\left(|\phi\bullet\mathcal{E}(W)|_{var}(+\infty)\right) \leq \sum_{n\geq 1} 2^{-n} = 1$. Hence $\phi\bullet\mathcal{E}(W) \in \mathcal{A}(\mathbb{G})$, and using the fact that $\Delta L^{\mathbb{G}}_{T_n} I_{[\![T_n,+\infty[\![}$ and $(1+\Delta L^{\mathbb{G}}_{T_n} - \Delta V^{\mathbb{G}}_{T_n})\Delta S_{T_n} I_{\{T_n>\tau\}} I_{[\![T_n,+\infty[\![}$ are $\mathbb{G}$-martingales, we get that

$$(\phi\bullet\mathcal{E}(W))^{p,\mathbb{G}} = \sum_{n\geq 1} \phi_{T_n} E\left(\Delta W_{T_n}|\mathcal{G}_{T_n-}\right) I_{[\![T_n,+\infty[\![} = -\phi\mathcal{E}_-(W)\bullet V^{\mathbb{G}}$$

is a non-increasing process. This proves that $\mathcal{E}(W)$ is a positive σ-supermartingale, and hence it is a supermartingale due to Kallsen [15]. Thus, $S-S^\tau$ admits a deflator, and the proof of (b)⇒(a) follows immediately from Lemma 5.3 bellow.

Lemma 5.3 *Let X be an $\mathbb{H}$-semimartingale. Then, the following assertions are equivalent.*
(a) *X admits an $\mathbb{H}$-deflator.*
(b) *X satisfies NUPBR($\mathbb{H}$).*

The proof of this lemma can be found in [1], also see Rokhlin [16].

Part 2) Here, we will prove (a)⇒(b). Suppose that $S-S^\tau$ satisfies NUPBR($\mathbb{G}$). Then, there exists a σ-martingale density $D^{\mathbb{G}} := \mathcal{E}(N^{\mathbb{G}})$. This implies the existence of a $\widetilde{\mathcal{P}}(\mathbb{G})$-measurable functional $f^{\mathbb{G}}$ such that $f^{\mathbb{G}} > 0$,

$$N^{\mathbb{G}} := W^{\mathbb{G}} \star (\mu_{\mathbb{G}} - \nu_{\mathbb{G}}), \quad W^{\mathbb{G}} := f^{\mathbb{G}} - 1 + \frac{\widehat{f^{\mathbb{G}}} - a^{\mathbb{G}}}{1 - a^{\mathbb{G}}} I_{\{a^{\mathbb{G}}<1\}},$$

$$xf^{\mathbb{G}} \star \nu_{\mathbb{G}} \;=\; xf^{\mathbb{G}} \left(1 - \frac{f_m}{1 - Z_-}\right) I_{]\!]\tau,+\infty[\![} \star \nu \equiv 0. \tag{5.14}$$

Here

$$\mu_{\mathbb{G}} := I_{]\!]\tau,+\infty[\![} \cdot \mu, \; \nu_{\mathbb{G}} = \left(1 - \frac{f_m}{1 - Z_-}\right) I_{]\!]\tau,+\infty[\![} \cdot \nu, \quad f_m := M_\mu^P(\Delta m | \widetilde{\mathcal{P}}(\mathbb{F})).$$

Thanks to Lemma A.2, there exists a positive and $\widetilde{\mathcal{P}}(\mathbb{F})$-measurable functional f such that $f^{\mathbb{G}} = f$ on the stochastic interval $]\!]\tau, +\infty[\![$, and (5.14) becomes

$$xf \left(1 - \frac{f_m}{1 - Z_-}\right) I_{]\!]\tau,+\infty[\![} \star \nu \equiv 0. \tag{5.15}$$

Since $\left((f-1)^2 I_{]\!]\tau,+\infty[\![} \star \mu\right)^{1/2} \in \mathcal{A}^+_{loc}(\mathbb{G})$, thanks to Proposition C.1 (both assertions (a) and (c)), we deduce the existence of a sequence of $\mathbb{F}$-stopping times $(\tau_n)_n$ that increases to infinity such that the processes $[m,m]^{\tau_n}$,

$$\Big(\frac{(f-1)^2}{(1-Z_-)^2} I_{\{Z_- \le 1-\delta\}} I_{\{|f-1| \le \alpha\}} \star \bar{\mu}\Big)^{\tau_n} \quad \& \quad \Big(\frac{|f-1|}{(1-Z_-)^2} I_{\{Z_- \le 1-\delta\}} I_{\{|f-1| > \alpha\}} \star \bar{\mu}\Big)^{\tau_n}$$

are integrable, where $\bar{\mu} := (1 - \widetilde{Z}) \cdot \mu$. Consider

$$\mu_1 := I_{\{\widetilde{Z}<1\}} \cdot \mu, \quad \nu_1 := M_\mu^P \left(I_{\{\widetilde{Z}<1\}} | \widetilde{\mathcal{P}}\right) \cdot \nu, \; h_1 := M_\mu^P \left(I_{\{\widetilde{Z}<1\}} | \widetilde{\mathcal{P}}\right),$$

$$g := \frac{f(1 - \frac{f_m}{1-Z_-})}{h_1} I_{\{h_1>0 \;\&\; Z_- \le 1-\delta\}} + I_{\{h_1=0 \text{ or } Z_- \ge 1-\delta\}},$$

and suppose that

$$W^{(1)}(t,x) := \frac{g_t(x) - 1}{1 - a_t^{(1)} + \widehat{g}_t} \in \mathcal{G}^1_{loc}(\mu_1, \mathbb{F}), \tag{5.16}$$

where $a_t^{(1)} := \nu_1(\{t\}, \mathbb{R}^d)$ and $\widehat{g}_t := \int g_t(x) \nu_1(\{t\}, dx)$.
Then, we can easily prove that assertion (b) holds. In fact, we take

$$N^{(1)} := \frac{g-1}{1 - a^{(1)} + \widehat{g}} \star (\mu_1 - \nu_1) \quad \text{and} \quad Y := \mathcal{E}(N^{(1)}).$$

Then, it is clear that

$$1 + \Delta N^{(1)} = \frac{1}{1 - a^{(1)} + \widehat{g}} I_{\{\Delta S = 0 \text{ or } \widetilde{Z}=1\}} + \frac{g(\Delta S)}{1 - a^{(1)} + \widehat{g}} I_{\{\Delta S \neq 0 \;\&\; \widetilde{Z}<1\}} > 0,$$

and on $\{Z_- \leq 1-\delta\}$ we get

$$
\begin{aligned}
{}^{p,\mathbb{F}}\left(Y\Delta S I_{\{\widetilde{Z}<1\}}\right)_t &= Y_{t-}\ {}^{p,\mathbb{F}}\left((1+\Delta N^{(1)})\Delta S I_{\{\widetilde{Z}<1\}}\right)_t = \frac{Y_{t-}\ {}^{p,\mathbb{F}}\left(g(\Delta S)\Delta S I_{\{\widetilde{Z}<1\}}\right)_t}{1-a_t^{(1)}+\widehat{g}_t} \\
&= \frac{Y_{t-}}{1-a_t^{(1)}+\widehat{g}_t}\int g_t(x)xh_1(t,x)\nu(\{t\},dx) \\
&= \frac{Y_{t-}}{1-a_t^{(1)}+\widehat{g}_t}\int x f_t(x)\left(1-\frac{f_m(t,x)}{1-Z_-}\right)\nu(\{t\},dx)\equiv 0.
\end{aligned}
$$

The last equality in the above string of equalities follows directly from (5.15). Therefore, assertion (b) will follow immediately as long as we prove (5.16). To this end, on $\{h_1>0\ \&\ Z_-\leq 1-\delta\}$ we calculate

$$
\begin{aligned}
g-1 &= \frac{f(1-Z_--f_m)}{h_1(1-Z_-)}-1 = \frac{(f-1)(1-Z_--f_m)}{h_1(1-Z_-)} - \frac{M_\mu^P\left(\Delta m I_{\{\widetilde{Z}<1\}}|\widetilde{\mathcal{P}}\right)}{(1-Z_-)h_1} \\
&=: g_1+g_2,
\end{aligned}
$$

and remark that $\{1-Z_--f_m>0\}\subset\{h_1>0\}$ which is due to

$$
1-Z_--f_m = 1-M_\mu^P\left(\widetilde{Z}|\widetilde{\mathcal{P}}\right)\leq M_\mu^P\left(I_{\{\widetilde{Z}<1\}}|\widetilde{\mathcal{P}}\right)=h_1,
$$

(due to $I_{\{\widetilde{Z}=1\}}\leq\widetilde{Z}$). Setting $\Sigma:=\{h_1>0\ \&\ Z_-\leq 1-\delta\}$, we derive

$$
\begin{aligned}
&E\left[g_1^2 I_{\{|f-1|\leq\alpha\}}I_\Sigma\star\mu_1(\tau_n)\right] \\
&\quad= E\left[\frac{(f-1)^2(1-Z_--f_m)^2}{h_1^2(1-Z_-)^2}I_{\{|f-1|\leq\alpha\}}I_\Sigma I_{\{\widetilde{Z}<1\}}\star\mu(\tau_n)\right] \\
&\quad= E\left[\frac{(f-1)^2(1-Z_--f_m)^2}{h_1(1-Z_-)^2}I_{\{|f-1|\leq\alpha\}}I_\Sigma\star\nu(\tau_n)\right] \\
&\quad\leq E\left[(f-1)^2\frac{1-Z_--f_m}{(1-Z_-)^2}I_{\{|f-1|\leq\alpha\}}I_{\{Z_-\leq 1-\delta\}}\star\nu(\tau_n)\right] \\
&\quad\leq \delta^{-2}E\left[(f-1)^2 I_{\{|f-1|\leq\alpha\}}I_{\{Z_-\leq 1-\delta\}}\star((1-\widetilde{Z})\cdot\mu)(\tau_n)\right]<+\infty.
\end{aligned}
$$

and

$$
\begin{aligned}
&E\left[g_1 I_\Sigma I_{\{|f-1|>\alpha\}}\star\mu_1(\tau_n)\right] \\
&\quad= E\left[\frac{|f-1|(1-Z_--f_m)}{h_1(1-Z_-)}I_{\{|f-1|>\alpha\}}I_{\{\widetilde{Z}<1\}}I_\Sigma\star\mu(\tau_n)\right] \\
&\quad= E\left[|f-1|\frac{1-Z_--f_m}{1-Z_-}I_{\{|f-1|>\alpha\}}I_{\{Z_-\leq 1-\delta\}}I_\Sigma\star\nu(\tau_n)\right] \\
&\quad\leq \delta^{-1}E\left[|f-1|I_{\{|f-1|>\alpha\}}I_{\{Z_-\leq 1-\delta\}}\star((1-\widetilde{Z})\cdot\mu)(\tau_n)\right]<+\infty.
\end{aligned}
$$

From a combination of these inequalities and

$$\sqrt{g_1^2 \star \mu_1} \leq \sqrt{2}\sqrt{g_1^2 I_{\{|f-1|>\alpha\}} \star \mu_1} + \sqrt{2}\left(g_1 I_{\{|f-1|>\alpha\}} \star \mu_1\right),$$

we deduce that $\sqrt{g_1^2 \star \mu_1} \in \mathcal{A}^+_{loc}(\mathbb{F})$. To prove $\sqrt{g_2^2 \star \mu_1} \in \mathcal{A}^+_{loc}(\mathbb{F})$, we derive

$$\begin{aligned}
E\left[(g_2)^2 I_\Sigma \star \mu_1(\tau_n)\right] &= E\left[\frac{M_\mu^P\left(\Delta m I_{\{\widetilde{Z}<1\}}|\widetilde{\mathcal{P}}\right)^2}{(1-Z_-)^2 h_1^2} I_{\{\widetilde{Z}<1\}} I_\Sigma \star \mu(\tau_n)\right] \\
&\leq E\left[\frac{M_\mu^P\left((\Delta m)^2|\widetilde{\mathcal{P}}\right) M_\mu^P\left(I_{\{\widetilde{Z}<1\}}|\widetilde{\mathcal{P}}\right)^2}{(1-Z_-)^2 h_1^2} I_\Sigma \star \nu(\tau_n)\right] \\
&\leq E\left[\frac{M_\mu^P\left((\Delta m)^2|\widetilde{\mathcal{P}}\right)}{(1-Z_-)^2} I_{\{Z_-\leq 1-\delta\}} \star \nu(\tau_n)\right] \leq \delta^{-2} E\left[[m,m]_{\tau_n}\right] < +\infty.
\end{aligned}$$

Hence, $\sqrt{(g-1)^2 \star \mu_1} \in \mathcal{A}^+_{loc}(\mathbb{F})$ follows. Thanks to Lemma A.4 (see Choulli and Schweizer(2012) [5]), (5.16) follows immediately. This ends the proof of the theorem. □

APPENDIX

A Some Useful Technical Results

Proposition A.1 *Let X be an $\mathbb{H}$-adapted process. Then, the following assertions are equivalent.*
(a) *There exists a sequence $(T_n)_{n\geq 1}$ of $\mathbb{H}$-stopping times that increases to $+\infty$, such that for each $n \geq 1$, there exists a probability Q_n on $(\Omega, \mathbb{H}_{T_n})$ such that $Q_n \sim P$ and X^{T_n} satisfies NUPBR($\mathbb{H}$) under Q_n.*
(b) *X satisfies NUPBR($\mathbb{H}$).*
(c) *There exists an $\mathbb{H}$-predictable process ϕ, such that $0 < \phi \leq 1$ and $(\phi \bullet X)$ satisfies NUPBR($\mathbb{H}$).*

The proof of this proposition can be found in Aksamit et al. [1].

Lemma A.2 *Let $H^{\mathbb{G}}$ be an $\widetilde{\mathcal{P}}(\mathbb{G})$-measurable functional. Then the following assertions hold.*
(a) *There exist two $\widetilde{\mathcal{P}}(\mathbb{F})$-measurable functional $H^{\mathbb{F}}$ and $K^{\mathbb{F}}$ such that*

$$H^{\mathbb{G}}(\omega,t,x) = H^{\mathbb{F}}(\omega,t,x) I_{]\!]0,\tau]\!]} + K^{\mathbb{F}}(\omega,t,x) I_{]\!]\tau,+\infty[\![}. \tag{A.1}$$

(b) *If furthermore $H^{\mathbb{G}} > 0$ (respectively $H^{\mathbb{G}} \leq 1$), then we can choose $K^{\mathbb{F}} > 0$ (respectively $K^{\mathbb{F}} \leq 1$) such that*

$$H^{\mathbb{G}}(\omega,t,x) I_{]\!]\tau,+\infty]\!]} = K^{\mathbb{F}}(\omega,t,x) I_{]\!]\tau,+\infty[\![}.$$

Proof The proof of the lemma follows from mimicking Jeulin's proof [14, Proposition 5,3].

Lemma A.3 *Let V be an $\mathbb{H}$-adapted nondecreasing process with $V_0 = 0$. If $V \in \mathcal{A}^+_{loc}(\mathbb{H})$, then there exists an $\mathbb{H}$-predictable process ϕ such that $0 < \phi \leq 1$ and $\phi \bullet V \in \mathcal{A}^+(\mathbb{H})$.*

Proof The proof of this lemma follows immediately by taking $\phi := \exp(-V^{p,\mathbb{F}})$.

Lemma A.4 (see Choulli and Schweizer [5]): *Consider a filtration $\mathbb{H}$ satisfying the usual conditions. Let f be a $\widetilde{\mathcal{P}}(\mathbb{H})$-measurable functional such that $f > 0$ and*

$$\left[(f-1)^2 \star \mu\right]^{1/2} \in \mathcal{A}^+_{loc}(\mathbb{H}). \tag{A.2}$$

Then, the $\mathbb{H}$-predictable process $\left(1 - a^{\mathbb{H}} + \widehat{f}^{\mathbb{H}}\right)^{-1}$ is locally bounded, and hence

$$W_t(x) := \frac{f_t(x) - 1}{1 - a^{\mathbb{H}}_t + \widehat{f}^{\mathbb{H}}_t} \in \mathcal{G}^1_{loc}(\mu, \mathbb{H}). \tag{A.3}$$

Here, $a^{\mathbb{H}}_t := \nu^{\mathbb{H}}(\{t\}, \mathbb{R}^d)$, $\widehat{f}^{\mathbb{H}}_t := \int f_t(x)\nu^{\mathbb{H}}(\{t\}, dx)$ and $\nu^{\mathbb{H}}$ is the $\mathbb{H}$-predictable random measure compensator of μ under $\mathbb{H}$.

B Proof of $K \in {}^{o}L(L^{\mathbb{G}}, \mathbb{G})$

Thanks to (3.5), we derive

$$0 \leq {}^{p,\mathbb{G}}(K) \leq \frac{(1 - Z_-)}{(1 - Z_-)^2 + \Delta\langle m\rangle^{\mathbb{F}}} I_{]\!]\tau, +\infty[\![} \leq \frac{1}{1 - Z_-} I_{]\!]\tau, +\infty[\![}.$$

Thus, we conclude that $K \in {}^{o}L^1_{loc}(\widehat{m}, \mathbb{G})$ if and only if

$$\left(\sum \left(\frac{1}{J(1 - Z_-)}\Delta\widehat{m} - {}^{p,\mathbb{G}}\left(\frac{1}{J(1 - Z_-)}\Delta\widehat{m}\right)\right)^2\right)^{1/2} \in \mathcal{A}^+_{loc}(\mathbb{G}). \tag{B.1}$$

To prove this fact, we set $\kappa := \frac{(1-Z_-)^2}{(1-Z_-)^2 + \langle m\rangle^{\mathbb{F}}}$, and on $]\!]\tau, +\infty[\![$ we calculate, making use of Lemma 3.2

$$\begin{aligned}
\kappa\frac{\Delta\widehat{m}}{J(1 - Z_-)} - \kappa\, {}^{p,\mathbb{G}}\left(\frac{\Delta\widehat{m}}{J(1 - Z_-)}\right) &= \kappa\frac{\Delta m}{(1 - Z_-)J} + \kappa(1 - Z_-)^{-2}\frac{1}{J}\Delta\langle m\rangle^{\mathbb{F}} \\
&\quad -\kappa(1 - Z_-)^{-3}\,\Delta\langle m\rangle^{\mathbb{F}}\, {}^{p,\mathbb{F}}(I_{\{J>0\}}) - \kappa(1 - Z_-)^{-2}\, {}^{p,\mathbb{F}}(\Delta m I_{\{J>0\}}) \\
&= \frac{\Delta m}{1 - \widetilde{Z}} + {}^{p,\mathbb{F}}\left(I_{\{\widetilde{Z}=1\}}\right).
\end{aligned} \tag{B.2}$$

Therefore, in virtue of Proposition 3.5, the proof of (B.1) follows from

$$\sqrt{\sum \left(\frac{\Delta m}{1-\widetilde{Z}}\right)^2 I_{]\!]\tau,+\infty[\![}} \in \mathcal{A}^+_{loc}(\mathbb{G}). \tag{B.3}$$

To prove this property, we set

$$\Sigma := \{\Delta m \leq (1-\delta)(1-Z_-)\} \quad \text{and} \quad W := \sum \left(\frac{\Delta m}{1-\widetilde{Z}}\right)^2 I_{]\!]\tau,+\infty[\![}.$$

Remark that $I_\Sigma \cdot W \leq \delta^{-2}(1-Z_-)^{-1} I_{]\!]\tau,+\infty[\![} \cdot [m,m] \in \mathcal{A}^+_{loc}(\mathbb{G})$ and

$$\sqrt{I_{\Sigma^c} I_{\{Z_- \leq \alpha\}} \cdot W} \leq \sum \frac{|\Delta m|}{1-\widetilde{Z}} I_{]\!]\tau,+\infty[\![} I_{\{\Delta m > (1-\delta)\alpha\}} \in \mathcal{A}^+_{loc}(\mathbb{G}),$$

for any $\alpha \in (0,1)$. Thus, for any $\alpha \in (0,1)$, the process $\sqrt{I_{\{Z_- \leq \alpha\}} \cdot W} \in \mathcal{A}^+_{loc}(\mathbb{G})$. Since the process $(1-Z_-)^{-1} I_{]\!]\tau,+\infty[\![}$ is $\mathbb{G}$-locally bounded, then there exists a sequence of $\mathbb{G}$-stopping times, $(\sigma_n)_{n\geq 1}$, that increases to infinity such that on $]\!]\tau,+\infty[\![\cap [\![0,\sigma_n]\!]$, we have $Z_- \leq 1 - 1/(n+1)$. This implies that for each n, we have $\sqrt{W^{\sigma_n}} \in \mathcal{A}^+_{loc}(\mathbb{G})$, and hence $\sqrt{W} \in \left(\mathcal{A}^+_{loc}(\mathbb{G})\right)_{loc} = \mathcal{A}^+_{loc}(\mathbb{G})$. For this set equality, we refer the reader to Lemma 1.35 in [13]. This proves (B.3) and hence the proof of (3.13) is completed.

C $\mathbb{G}$-localisation versus $\mathbb{F}$-localisation

This subsection connects the $\mathbb{G}$-localisation and the $\mathbb{F}$-localisation for the part after τ. This completes the analysis of [1], where the part up to τ is fully discussed. There is a major difference between the current results and those of [1], which lies in the fact that for the case up to τ we loose information after an $\mathbb{F}$-stopping when we pass to $\mathbb{F}$. However, for the part after τ, as long as τ is finite, we pass from $\mathbb{G}$-localisation to $\mathbb{F}$-localisation without any loss of information.

Proposition C.1 *Suppose that τ is a finite honest time satisfying (2.3). Then, the following properties hold.*
(a) *Let $\alpha > 0$ and f be a $\widetilde{\mathcal{P}}(\mathbb{H})$-measurable functional. Then, $\sqrt{(f-1)^2 \star \mu} \in \mathcal{A}^+_{loc}(\mathbb{H})$ if and only if*

$$\left[(f-1)^2 I_{\{|f-1|\leq\alpha\}} + |f-1| I_{\{|f-1|>\alpha\}}\right] \star \mu \in \mathcal{A}^+_{loc}(\mathbb{H}).$$

(b) *Let $\Phi^{\mathbb{G}}$ a $\mathbb{G}$-predictable process and k a nonnegative and $\widetilde{\mathcal{P}}(\mathbb{F})$-measurable functional such that $0 < \Phi^{\mathbb{G}} \leq 1$ and $\Phi^{\mathbb{G}} k \star \mu_{\mathbb{G}} \in \mathcal{A}^+_{loc}(\mathbb{G})$. Then, $P \otimes A$-a.e.*

$$\int k(x)\,(1 - Z_- - f_m(x))\,F(dx) < +\infty \quad \text{on} \quad \{Z_- < 1\}. \tag{C.1}$$

(c) *Let f be a $\widetilde{\mathcal{P}}(\mathbb{F})$-measurable and positive functional, and $\overline{\mu} := (1-\widetilde{Z})\cdot\mu$. Then $\sqrt{(f-1)^2 I_{]\!]\tau,+\infty]\!]} \star \mu} \in \mathcal{A}^+_{loc}(\mathbb{G})$ if and only if $\sqrt{(f-1)^2 I_{\{1-Z_-\geq\delta\}} \star \overline{\mu}} \in \mathcal{A}^+_{loc}(\mathbb{F})$ for any $\delta > 0$.*

Proof (a) Assertion (a) is borrowed from [1] (see Proposition C.3–(a)).
(b) This assertion follows directly from a combination of Lemmas A.3 and A.2.
(c) Thanks to assertion (a), we have $\sqrt{(f-1)^2 I_{]\!]\tau,+\infty]\!]} \star \mu} \in \mathcal{A}^+_{loc}(\mathbb{G})$ if and only if

$$W^{\mathbb{G}} := \left[(f-1)^2 I_{\{|f-1|\leq\alpha\}} + |f-1| I_{\{|f-1|>\alpha\}}\right] \star \mu_{\mathbb{G}} \in \mathcal{A}^+_{loc}(\mathbb{G}).$$

This is equivalent to $\left[(f-1)^2 I_{\{|f-1|\leq\alpha\}} + |f-1| I_{\{|f-1|>\alpha\}}\right] \star \nu_{\mathbb{G}} \in \mathcal{A}^+_{loc}(\mathbb{G})$, which, in turn, it is equivalent to

$$\varphi I_{]\!]\tau,+\infty]\!]} \bullet A \in \mathcal{A}^+_{loc}(\mathbb{G}),$$
$$\varphi := \int \left[(f-1)^2 I_{\{|f-1|\leq\alpha\}} + |f-1| I_{\{|f-1|>\alpha\}}\right] (1 - Z_- - f_m) F(dx).$$

Then, due to Lemma 3.4, we deduce that for any $\delta > 0$, $\varphi I_{\{1-Z_-\geq\delta\}} \bullet A \in \mathcal{A}^+_{loc}(\mathbb{F})$. Then, again thanks to assertion (a), we conclude that

$$\sqrt{(f-1) I_{\{1-Z_-\geq\delta\}} \star \bar{\mu}} \in \mathcal{A}^+_{loc}(\mathbb{F}),$$

where $\bar{\mu} := (1-\widetilde{Z})\cdot\mu$. This ends the proof of assertion (c), and the proposition is proved. □

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