A GENERAL HJM FRAMEWORK FOR MULTIPLE YIELD CURVE MODELING

CHRISTA CUCHIERO, CLAUDIO FONTANA, AND ALESSANDRO GNOATTO

Abstract. We propose a general framework for modeling multiple yield curves which have emerged after the last financial crisis. In a general semimartingale setting, we provide an HJM approach to model the term structure of multiplicative spreads between (normalized) FRA rates and simply compounded OIS risk-free forward rates. We derive an HJM drift and consistency condition ensuring absence of arbitrage and, in addition, we show how to construct models such that multiplicative spreads are greater than one and ordered with respect to the tenor's length. When the driving semimartingale is specified as an affine process, we obtain a flexible Markovian structure which allows for tractable valuation formulas for most interest rate derivatives. Finally, we show that the proposed framework allows to unify and extend several recent approaches to multiple yield curve modeling.

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1. Introduction

The last financial crisis has profoundly affected fixed income markets. Most notably, significant spreads have emerged between interbank (Libor/Euribor) rates and (risk-free) OIS rates as well as between interbank rates associated to different tenor lengths, mainly due to an increase in credit and liquidity risk. While negligible in the pre-crisis environment, such spreads represent nowadays one of the most striking features of interest rate markets, with the consequence that interbank rates cannot be considered risk-free any longer (see Section 1.1 for more details). From a modeling perspective, this new market situation necessitates a new generation of interest rate models, which are able to represent in a consistent way the evolution of multiple yield curves and allow to value fixed income derivatives.

The present paper aims at providing a coherent and general modeling approach for multiple interest rate curves. We shall adopt an HJM framework driven by general semimartingales in the spirit of [41] in order to model the joint evolution of the *term structure of OIS zero coupon bond prices* and of the *term structure of spreads* between forward rates linked to interbank rates and OIS forward rates. More specifically, we shall model the term structure of multiplicative spreads between (normalized) forward rates implied by market forward rate agreement (FRA) rates, associated to a family of different tenors, and (normalized) simply compounded OIS forward rates. As discussed below, besides admitting a natural economic interpretation in terms of exchange rates, multiplicative spreads provide a particularly convenient parametrization of the term structures of interbank rates. Referring to Section 1.2 for a more detailed discussion of the proposed framework, let us just mention here that, besides the great generality and flexibility, this modeling approach has the advantage of considering as model fundamentals easily observable market quantities and of leading to a clear characterization of order relations between spreads associated to different tenors. Moreover, specifying the driving semimartingale as an affine process leads to a Markovian structure and tractable valuation formulas.

The remaining part of the introduction is structured as follows. Section 1.1 introduces the basic quantities considered in the paper and illustrates the spreads existing among different yield curves. In Section 1.2, we discuss the philosophy behind the proposed HJM framework. Section 1.3 contains a concise overview of the relevant literature and Section 1.4 closes the introduction with an outline of the subsequent sections of the paper.

1.1. The post-crisis interest rate market.

In fixed income markets, the underlying quantities for the vast majority of traded contracts are Libor (or Euribor) rates $L_T(T, T+\delta)$, for some time interval $[T, T+\delta]$, where the *tenor* $\delta > 0$ is typically one day (1D) or several months (typically 1M, 3M, 6M or 12M). While before the last financial crisis rates associated to different tenors were simply related by no-arbitrage arguments, nowadays, for every tenor $\delta \in \{\delta_1, \dots, \delta_m\}$, a specific yield curve is constructed from market instruments that depend on Libor rates corresponding to the specific tenor δ.

The rate for overnight borrowing, denoted by $L_T(T, T+1/360)$, is the Federal Funds rate in the US market and the Eonia (Euro OverNight Index Average) rate in the Euro area. Overnight rates represent the underlying of overnight indexed swaps (OIS). An OIS contract is a swap contract with a fixed leg versus a floating leg, where the floating rate is a geometric average of Eonia rates, and OIS rates are then the market quotes for these swaps (see Section 2.5.1). OIS rates play an important role, being commonly assumed to be the best proxy for risk-free rates, and are also used as collateral rates in collateralized transactions, thus leading to OIS discounting (see Appendix A). By relying on bootstrapping techniques (see, e.g., [1]), the following curves can be obtained from OIS rates, for all $0 \le t \le T$:

- (risk-free) OIS zero coupon bond prices $T \mapsto B(t,T)$;
- instantaneous (risk-free) OIS forward rates $T \mapsto f_t(T) = -\partial_T \log B(t,T)$;
- simply compounded (risk-free) OIS forward rates

$$T \mapsto L_t^D(T, T+\delta) := \frac{1}{\delta}\left(\frac{B(t,T)}{B(t,T+\delta)} - 1\right).$$

In particular, note that $L_t^D(T, T+\delta)$ corresponds to the pre-crisis risk-free forward Libor rate at time t for the interval $[T, T+\delta]$ (the superscript D stands for discounting).

While OIS rates provide a complete picture of the *risk-free* (discounting) yield curve, the underlying quantities of typical fixed income products, such as forward rate agreements (FRAs), swaps, caps/floors and swaptions, are given by Libor rates $L_T(T, T+\delta)$ for some tenor $\delta > 1/360$. Since Libor rates are affected by the credit and liquidity risk of the panel of contributing banks (interbank risk), we shall sometimes refer to Libor rates as *risky*.

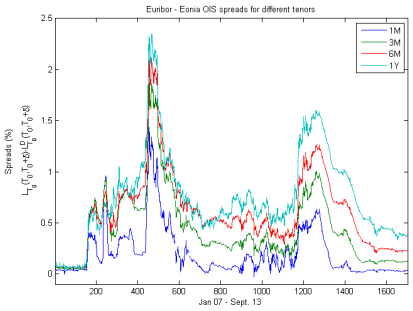


FIGURE 1. Additive Eonia–Euribor spreads from Jan. 2007 to Sept. 2013 for $\delta = 1/12, 3/12, 6/12, 1$

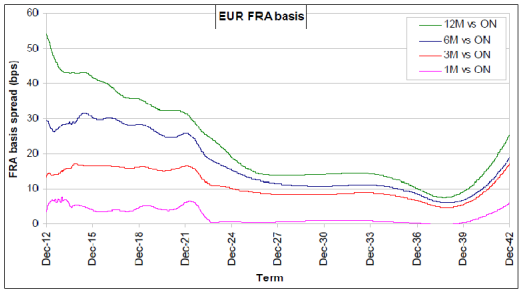


FIGURE 2. Term structure of additive spreads between FRA rates and OIS forward rates, at $T_0 = 11.12.12$ for $\delta = 1/12, 3/12, 6/12, 1$.

Among all financial contracts written on Libor rates, FRAs can be rightfully considered – due to the simplicity of their payoff – as the most fundamental instruments and are also liquidly traded on the derivatives' market, especially for short maturities. Moreover, typical interest rate linear derivatives, like interest rate swaps or basis swaps (see Section 2.5.1) can be represented as portfolios of FRAs. Recall that the FRA rate at time t for the interval $[T, T+\delta]$, denoted by $L_t(T, T+\delta)$, is defined as the rate fixed at time t such that the fair value of a FRA contract is null. Market quotes of FRA rates correspond to *clean prices*, in the sense that they refer to perfectly collateralized transactions, so that counterparty risk becomes negligible. Hence, as shown in Appendix A, assuming absence of arbitrage for the whole family of (risk-free) OIS zero coupon bonds (with the OIS bank account as numéraire, which is also assumed to be the collateral asset), the no-arbitrage value of the FRA rate $L_t(T, T+\delta)$ is given by the following classical expression (which is also in line with current market practice):

$$L_t(T, T+\delta) = \mathbb{E}^{\mathbb{Q}^{T+\delta}}\left[L_T(T, T+\delta) \,\middle|\, \mathcal{F}_t\right], \tag{1.1}$$

where $\mathbb{Q}^{T+\delta}$ denotes the $(T+\delta)$-forward measure with the OIS bond $B(\cdot, T+\delta)$ as numéraire. In particular, $\big(L_t(T, T+\delta)\big)_{t\in[0,T]}$ is a $\mathbb{Q}^{T+\delta}$-martingale, for all $T \geq 0$, which will be the crucial no-arbitrage property that has to be satisfied when setting up a multiple yield curve model.

The spreads mentioned at the very beginning of the present paper arise from the fact that market FRA rates are typically higher than simply compounded OIS forward rates, i.e., $L_t(T, T+\delta) > L_t^D(T, T+\delta)$. This is related to the fact that the Libor panel is periodically updated to include only creditworthy banks. Hence, Libor rates incorporate the risk that the average credit quality of an initial set of banks deteriorates over the term of the loan, while OIS rates reflect the average credit quality of a newly refreshed Libor panel (see, e.g., [26]). Therefore, since the year 2007, we observe positive spreads between FRA and OIS forward rates, as illustrated in Figures 1 and 2. Figure 1 shows the time series of the additive Euribor - Eonia spot spread $L_T(T, T+\delta) - L_T^D(T, T+\delta)$ from January 2007 to September 2013 for different tenors δ, providing a clear evidence that these spreads are positive and also (most of the time) ordered according to the tenor length δ. Figure 2 is taken from the recent paper [1] and displays the term structure of the additive spread $L_{T_0}(T, T+\delta) - L_{T_0}^D(T, T+\delta)$ between FRA and OIS forward rates at $T_0 = 11.12.12$ for different δ. Again, we see that spreads are positive and increasing with respect to the tenor's length δ.

1.2. **Problem formulation and modeling approach.** Motivated by the fixed income market structure, we consider (risk-free) OIS zero coupon bonds and FRA contracts, for different tenors $\{\delta_1, \ldots, \delta_m\}$, as the basic market quantities of a multiple yield curve model. Note that, in the post-crisis interest rate market, FRA contracts must be added to the market composed of all risk-free zero coupon bonds, because they cannot be perfectly replicated by the latter any longer[1]. In other words, the main goal of the present paper consists in solving in a general way the following problem.

Problem 1.1. *Given today's prices of OIS zero coupon bonds $B(0,T)$ and FRA rates $L_0(T, T+\delta)$, for different tenors $\delta \in \{\delta_1, \ldots, \delta_m\}$ and for all maturities $T \geq 0$, model their stochastic evolution so that the market consisting of all OIS zero coupon bonds and all FRA contracts is free of arbitrage.*

[1]Alternatively, artificial untraded bonds associated to Libor rates as defined below could be introduced, as done in a number of recent papers, see e.g. [15, 30, 35]. However, an important feature of our modeling approach is that only observable traded quantities are taken as model fundamentals.

Apart from the presence of FRA contracts, due to the fact that Libor rates are no longer risk-free, this is exactly the problem dealt with in the classical Heath-Jarrow-Morton (HJM) framework [33], which describes the arbitrage-free stochastic evolution of the term structure of risk-free zero coupon bond prices. Our approach consists thus in extending the classical HJM framework in order to include FRA contracts for a finite collection of tenors $\{\delta_1, \dots, \delta_m\}$ and for all maturities. Of course, the crucial question is how to preclude arbitrage in this setting and how to translate such a fundamental requirement into a transparent condition on the model's ingredients. As anticipated in the preceding section (see equation (1.1) and the following discussion), this will follow from the $\mathbb{Q}^{T+\delta}$-martingale property of $\big(L_t(T, T+\delta)\big)_{t\in[0,T]}$, for all $T \geq 0$ and $\delta \in \{\delta_1, \dots, \delta_m\}$.

From a modeling perspective, a first possibility would be to directly specify the stochastic evolution of $\big(L_t(T, T+\delta)\big)_{t\in[0,T]}$, for $\delta \in \{\delta_1, \dots, \delta_m\}$ and $T \geq 0$. However, such an approach is not well suited to capture the positivity and monotonicity of the spreads with respect to the tenor δ, in line with the empirical findings reported above. Therefore, we choose to model spreads. Although additive spreads might seem to be the most natural quantity at first sight, we shall consider *multiplicative* spreads

$$S^\delta(t,T) := \frac{1+\delta L_t(T, T+\delta)}{1+\delta L_t^D(T, T+\delta)}, \tag{1.2}$$

for $\delta \in \{\delta_1, \dots, \delta_m\}$. This corresponds to considering multiplicative spreads between (normalized) FRA rates and (normalized) simply compounded OIS forward rates. Note also that the initial curve of the above spreads can be directly obtained from the OIS and FRA rates observed on the market. As an example, Figure 3 displays the curve of $T \mapsto S^\delta(T_0, T)$ obtained from market data at $T_0 = 04.08.2013$.

Let us now explain the reasoning behind this modeling choice, considering the case of a single tenor δ for simplicity of presentation. As a preliminary, let us illustrate a simple foreign exchange rate analogy (see Appendix B for more details). To the risky Libor rate $L_T(T, T+\delta)$ one can associate an *artificial* risky bond price $B^\delta(t,T)$ so that $1 + \delta L_T(T, T+\delta) = 1/B^\delta(T, T+\delta)$, for all $T \geq 0$, in analogy to the classical single curve risk-free setting. Following for instance the discussion in [59], we can think of such artificial risky bonds as issued by a bank representative of the Libor panel[2]. Therefore, due to equation (1.2), the multiplicative spread (for $t = T$) satisfies $S^\delta(T,T) = B(T, T+\delta)/B^\delta(T, T+\delta)$. Hence, if we interpret risk-free bonds as *domestic* bonds and artificial risky bonds as *foreign* bonds, the term $S^\delta(T,T)$ can then be thought of as an exchange rate which measures the riskiness of foreign bonds with respect to domestic bonds. In that sense, multiplicative spreads represent a rather natural quantity to model in a multiple yield curve framework (see Appendix B for more details).

Our approach consists in formulating an HJM framework in order to model the joint evolution of the term structures of OIS bond prices and of multiplicative spreads $S^\delta(t,T)$. While in the case of OIS bonds the situation is analogous to the classical HJM setting, the modeling of the term structure of multiplicative spreads is much less standard. To this effect, we propose an approach inspired by the HJM philosophy, as put forward in [41, Section 2.1].

In HJM-type models there typically exists a *canonical underlying asset* or a *reference process* which is the underlying of the assets of interest. In our context, the assets of interest are OIS zero coupon bonds and FRA contracts. In the case of OIS bonds, the canonical underlying asset is the (risk-free) OIS bank account. Concerning FRA contracts, the choice is less obvious. Inspired by the foreign exchange rate analogy discussed above, we consider as reference process the quantity

$$Q_T^\delta := S^\delta(T,T) = \frac{B(T, T+\delta)}{B^\delta(T, T+\delta)}, \quad \text{for all } T \geq 0.$$

In order to obtain a convenient parametrization ("codebook") of the term structures, the next step in the formulation of an HJM-type model consists in specifying *simple models* for the evolution of the canonical underlying assets/reference processes.

In the case of OIS bonds, this is done by supposing that the OIS bank account, denoted by $(B_t)_{t\geq 0}$, is simply given by $B_t = \exp\big(\int_0^t r_s\, ds\big)$, where $(r_t)_{t\geq 0}$ is a deterministic short rate. This yields the relation $r_T := -\partial_T \log\big(B(t,T)\big)$. However, market data do not follow such a simple model and, hence, $-\partial_T \log\big(B(t,T)\big)$ yields a parameter manifold which changes randomly over time. This leads to instantaneous forward rates $f_t(T) := -\partial_T \log\big(B(t,T)\big)$, for which a stochastic evolution has to be specified.

[2]We want to emphasize that artificial risky bond prices are only introduced here as an explanatory tool and shall not be considered in the following sections of the paper. We also stress that $B^\delta(T, T+\delta)$ differs, in general, from the OIS zero coupon risk-free bond price $B(T, T+\delta)$.

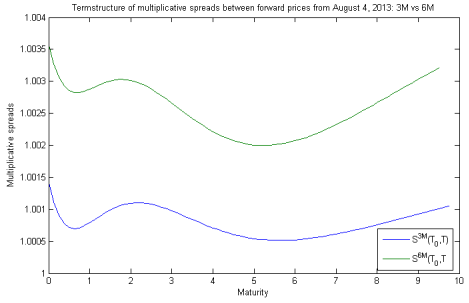


FIGURE 3. Multiplicative spreads $S^\delta(T_0, T)$ at $T_0 = 04.08.13$ for $\delta = 3/12, 6/12$.

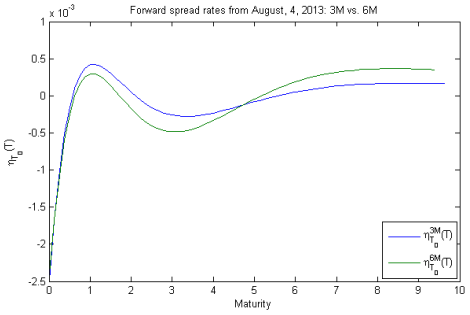


FIGURE 4. Forward spread rates $\eta_{T_0}(T)$ at $T_0 = 04.08.13$ for $\delta = 3/12, 6/12$.

Absence of arbitrage is implied by the requirement that discounted bond prices are martingales, which then yields the well-known HJM *drift condition*. The dynamics of the underlying reference asset, or, equivalently, of the short rate $(r_t)_{t\geq 0}$, are determined via the *consistency condition*, that is, $r_t = f_t(t)$.

In the case of FRA contracts, we keep the simple model for the OIS bonds, assuming a deterministic short rate, and suppose additionally – similarly to [41] – the following simple model for Q_T^δ

$$Q_T^\delta = \exp(Z_T), \quad \text{for all } T \geq 0, \tag{1.3}$$

where $(Z_t)_{t\geq 0}$ is a one-dimensional time-inhomogeneous Lévy process under a given pricing measure $\mathbb{Q}$ (and thus under all forward measures due to the deterministic short rate). Its Lévy exponent is denoted by $\psi(t,u)$, for $(t,u) \in \mathbb{R}_+ \times \mathbb{R}$. In view of equations (1.1)-(1.2) (compare also with Appendix A) and recalling the relation $1 + \delta L_T(T, T+\delta) = 1/B^\delta(T, T+\delta)$, this leads to the following representation of multiplicative spreads $S^\delta(t,T)$:[3]

$$\begin{aligned} S^\delta(t,T) &= \frac{B(t,T+\delta)}{B(t,T)} \mathbb{E}^{\mathbb{Q}^{T+\delta}}\big[1 + \delta L_T(T,T+\delta) \,\big|\, \mathcal{F}_t\big] = \frac{B(t,T+\delta)}{B(t,T)} \mathbb{E}^{\mathbb{Q}^{T+\delta}}\left[\frac{1}{B^\delta(T,T+\delta)} \,\middle|\, \mathcal{F}_t\right] \\ &= \mathbb{E}^{\mathbb{Q}^T}\left[\frac{B(T,T+\delta)}{B^\delta(T,T+\delta)} \,\middle|\, \mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}^T}\left[Q_T^\delta \,|\, \mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}^T}\left[e^{Z_T} \,|\, \mathcal{F}_t\right] \\ &= \exp\Big(Z_t + \int_t^T \psi(s,1)\, ds\Big). \end{aligned} \tag{1.4}$$

In particular, this implies the following relation:

$$\partial_T \log\big(S^\delta(t,T)\big) = \psi(T,1). \tag{1.5}$$

As in the case of the OIS term structure, since market data do not follow such a simple model and $\big(S^\delta(t,T)\big)_{t\in[0,T]}$ evolves randomly over time, we need to put $\psi(T,1)$ "in motion". To this effect, we define an instantaneous forward spread rate via the left-hand side of (1.5), i.e., $\eta_t^\delta(T) := \partial_T \log\big(S^\delta(t,T)\big)$, and specify general stochastic dynamics for $\eta_t^\delta(T)$. A typical shape of the curve $T \mapsto \eta_{T_0}^\delta(T)$ is shown in Figure 4, obtained from market data at $T_0 = 04.08.2013$. From the defining property of FRA rates (see equation (1.1)), which is equivalent to the $\mathbb{Q}^T$-martingale property of $\big(S^\delta(t,T)\big)_{t\in[0,T]}$, for $T \geq 0$, an HJM *drift condition* can be deduced, which then ensures absence of arbitrage. Moreover, the dynamics of the reference process $(Q_t^\delta)_{t\geq 0}$, or, equivalently, those of $(Z_t)_{t\geq 0}$ (assumed to be a general Itô-semimartingale in the following), have to satisfy a suitable *consistency condition*, similar to the requirement $f_t(t) = r_t$ in the case of the OIS term structure.

Summing up, let us highlight the main advantages and novel contributions of the proposed approach:

- The term structure associated to Libor rates for different tenors $\{\delta_1, \ldots, \delta_m\}$ is modeled via the multiplicative spreads $S^\delta(t,T)$, which are in turn directly related to observable market OIS and FRA rates. Modeling multiplicative spreads, rather than additive ones, allows for
  - a natural economic interpretation in terms of exchange rates between prices of risk-free OIS bonds and prices of risky bonds associated to Libor rates (see Appendix B);

[3]Actually, due to the deterministic short rate, it is not necessary to distinguish the expectations with respect to different measures, but we explicitly indicate them for consistency of the exposition with the general setting of the following sections.

  - high tractability (in particular, in the case when the semimartingale driving $f_t(T)$ and $\eta_t^\delta(T)$ is an affine process, as considered in Section 3).
- The modeling of the multiplicative spread $S^\delta(t,T)$ is split into two components: an instantaneous *forward* spread rate $\eta_t^\delta(T)$ and a *spot* component $Q_t^\delta = S^\delta(t,t)$, which is directly observable from market data. In particular, this separation allows for great modeling flexibility.
- By adopting an abstract HJM formulation, inspired from [41] (see Section 2.1), we derive a simple HJM drift and consistency condition ensuring absence of arbitrage in a general semimartingale setting, thus providing a complete and general solution to Problem 1.1.
- By choosing a common $\mathbb{R}^n$-valued semimartingale Z for the spot spreads $S^\delta(t,t)$ corresponding to different tenors $\delta \in \{\delta_1 \dots, \delta_m\}$, such that $S^{\delta_i}(t,t) = \exp(u_i^\top Z_t)$ for $u_i \in \mathbb{R}^n$, the inherent dependence between spreads associated to different tenors (as visible from Figure 1) can be captured. Moreover, complex correlation structures between OIS and FRA rates can be built in through a common driving process for the forward rates $f_t(T)$ and the spread rates $\eta_t^\delta(T)$.
- The desired feature that $S^\delta(t,T) \geq 1$, for all $0 \leq t \leq T$, can be easily achieved in full generality. Moreover, we can easily characterize order relations between spreads associated to different tenors, i.e, when $S^{\delta_j}(t,T) \geq S^{\delta_i}(t,T)$ for $\delta_j \geq \delta_i$ and for all $0 \leq t \leq T$, as is the case in typical market situations.
- Starting from a given tuple of basic building blocks (see Section 2.4), we provide a general construction of multiple yield curve models, satisfying the HJM drift and consistency condition and generating spreads which are ordered and greater than one. To this effect, we prove existence and uniqueness of the SPDEs associated to the forward curves when translated to the Musiela parametrization and show how to guarantee the consistency condition by constructing an appropriate pure jump process whose compensator solves a random generalized moment problem.
- When considering finite-dimensional factor models, we are naturally led to the class of affine processes. In this case, the model for the OIS term structure becomes a classical short rate model driven by a multidimensional affine process, which also determines the dynamics of the multiplicative spreads. In this context, we obtain tractable valuation formulas for derivatives written on Libor rates (see Section 3).
- Most of the multiple curve models proposed in the literature can be easily recovered as suitable specifications of our general framework, thus underlying once more the high flexibility of the proposed approach (see Section 4).

1.3. **Literature overview.** We give a brief overview of the relevant literature on the post-crisis interest rate market, with a particular focus on the existing approaches to multiple yield curve modeling and the contributions that are especially related to the present paper.

Starting with the last financial crisis, the multiple curve phenomenon has attracted significant attention from market practice as well as from the academic literature (see e.g. the recent book [36] and the references therein). To the best of our knowledge, the first paper highlighting the relevance of the multiple curve issue shortly before the beginning of the credit crunch was [34]. A first and basic aspect, with significant practical implications, is represented by the construction of multiple yield curves associated to different tenors. Among the many contributions in this direction, let us mention in particular [1] and [27], which also provides software implementations.

From a modeling perspective, as in the case of classical interest rate models, most of the models proposed so far in the literature can be ascribed to three main mutually related families: short-rate approaches, Libor market models and HJM models. Referring to Section 4 for a detailed comparison of the different approaches, we just mention that multiple curve short rate models have been first introduced in [49], [48], [26] and, more recently, in [60] and [30], while Libor market models have been extended to the multiple curve setting in [57], [56] and, more recently, in [32]. Our approach is closer to the multiple curve HJM-type models proposed in the literature, see in particular [58], [61], [29], [15] and [14]. Note also that the idea of modeling multiplicative spreads goes back to [34] and [35].

We also want to mention that the joint modeling of the risk-free term-structure together with a "risky" term-structure goes back to the earlier contributions [39] and [20]. In these two papers, in a credit-risky setting, a term structure of zero-coupon defaultable bonds (with an additional rating score in [20]) is modeled together with the classical risk-free term structure via an HJM-type approach. More recently, spreads between Libor rates and risk-free OIS rates have also been modeled in [11] by introducing default risk in a HJM framework (see also [31] for a similar interpretation in the case of a Libor market model).

1.4. **Outline of the paper.** The remainder of the paper is organized as follows. In Section 2, we introduce a general HJM-type framework, inspired by [41], which we then apply to define multiple yield curve models as introduced above. In particular, we derive an HJM drift and consistency condition and provide a construction of arbitrage-free models satisfying these necessary requirements and ensuring ordered spreads. We also provide general valuation formulas for typical interest rate derivatives. Section 3 is devoted to a general affine model specification and we derive semi-analytic pricing formulas for caps/floors and an accurate approximation for swaptions. In Section 4, we establish several connections to existing multiple curve models and show how they can be embedded in our framework. Finally, Appendix A contains a review of pricing under collateral and its implication for the definition of fair FRA rates (see equation (1.1) above), Appendix B illustrates a simple foreign exchange analogy of the multiplicative spreads introduced in (1.2), while Appendices C-F contain technical results as well as possible extensions of the framework.

2. The general framework

The purpose of this section is to establish a general HJM-type framework for multiple yield curve modeling as explained in the introduction. We start in Section 2.1 by defining an abstract setting where we consider general families of semimartingales. In Sections 2.2 and 2.3, we then apply this to the term structure modeling of OIS zero coupon bond prices and of the multiplicative spreads defined in (1.2), respectively. In Section 2.4, we provide a general approach to construct arbitrage-free HJM-type multiple yield curve models, while Section 2.5 presents general valuation formulas for interest rate derivatives.

2.1. **Abstract HJM setting.** Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{Q})$ be a stochastic basis supporting several processes introduced in the sequel. We aim to model a family of one-dimensional positive semimartingales

$$\big\{(S(t,T))_{t\in[0,T]}, T \geq 0\big\}$$

such that $(S(t,t))_{t\geq 0}$ is also a (positive) semimartingale. Supposing differentiability of $T \mapsto \log(S(t,T))$ (a.s.), we can represent $S(t,T)$ by

$$S(t,T) = e^{Z_t + \int_t^T \eta_t(u)du}, \tag{2.1}$$

where $Z_t := \log(S(t,t))$ and $\eta_t(T) := \partial_T \log(S(t,T))$. Modeling the family $\{(S(t,T))_{t\in[0,T]}, T \geq 0\}$ is thus equivalent to modeling $(Z_t)_{t\geq 0}$ and $\{(\eta_t(T))_{t\in[0,T]}, T \geq 0\}$. We call Z the *log-spot rate* and $\eta_t(T)$ the *generalized forward rate*.

Remark 2.1. The representation (2.1) is motivated by the HJM philosophy as put forward in Section 1.2. Indeed, suppose that the spot process $S(t,t)$ corresponds to a canonical underlying asset and that $S(t,T) = \mathbb{E}[S(T,T)|\mathcal{F}_t]$. If $S(t,t)$ is modeled as an exponential time-inhomogeneous Lévy process $\exp(Z_t)$, we obtain expression (1.4) for $S(t,T)$ (under the measure $\mathbb{Q}$). Putting the Lévy exponent "in motion" naturally leads to (2.1) with a general semimartingale Z.

Inspired by [41], we define *HJM-type models* as follows.

Definition 2.2. *A quintuple $(Z, \eta_0, \alpha, \sigma, X)$ is called* HJM-type model *for a family of positive semimartingales $\{(S(t,T))_{t\in[0,T]}, T \geq 0\}$ if*

(i) *(X, Z) is an $\mathbb{R}^{d+1}$-valued Itô-semimartingale, i.e., its characteristics are absolutely continuous with respect to the Lebesgue measure,*
(ii) *$\eta_0 \colon \mathbb{R}_+ \to \mathbb{R}$ is measurable and $\int_0^T |\eta_0(u)|du < \infty$ $\mathbb{Q}$-a.s. for all $T \in \mathbb{R}_+$,*
(iii) *$(\omega, t, T) \mapsto \alpha_t(T)(\omega)$ and $(\omega, t, T) \mapsto \sigma_t(T)(\omega)$ are $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}_+)$-measurable $\mathbb{R}$- and $\mathbb{R}^d$-valued processes, respectively, where $\mathcal{P}$ denotes the predictable σ-algebra, and satisfy*
- *$\int_0^t \int_0^T |\alpha_s(u)|duds < \infty$ $\mathbb{Q}$-a.s. for all $t, T \in \mathbb{R}_+$,*
- *$\int_0^T \|\sigma_t(u)\|^2 du < \infty$ $\mathbb{Q}$-a.s. for any $t, T \in \mathbb{R}_+$,*
- *$\big((\int_0^T |\sigma_{t,j}(u)|^2 du)^{\frac{1}{2}}\big)_{t\geq 0} \in L(X_j)$ for all $T \in \mathbb{R}_+$ and $j \in \{1, \dots, d\}$, where $L(X_j)$ denotes the set of processes which are integrable with respect to X_j,*

(iv) *for every $T \in \mathbb{R}_+$, the generalized forward rate $(\eta_t(T))_{t\in[0,T]}$ is given by*

$$\eta_t(T) = \eta_0(T) + \int_0^t \alpha_s(T)ds + \int_0^t \sigma_s(T)dX_s, \tag{2.2}$$

(v) $\{(S(t,T))_{t\in[0,T]}, T \geq 0\}$ *satisfies, for all* $t \leq T$ *and* $T \geq 0$,

$$S(t,T) = e^{Z_t + \int_t^T \eta_t(u)du} \tag{2.3}$$

and, in particular, $S(t,t) = e^{Z_t}$ *for all* $t \geq 0$.

Typically, $(S(t,T))_{t\in[0,T]}$ corresponds to the evolution of the price of a derivative with maturity T and is thus a (local) martingale under some equivalent measure. Supposing that $\mathbb{Q}$ already represents a (local) martingale measure, the (local) martingale property of $(S(t,T))_{t\in[0,T]}$ can be characterized by Theorem 2.5 below. As a preliminary, let us recall the notion of *derivative of the (modified) Laplace cumulant process* (compare [42, Definitions 2.22 and 2.23]) (or, equivalently, *local exponent*[4], see [41, Definition A.6]).

Definition 2.3. *Let* X *be an* $\mathbb{R}^d$*-valued Itô-semimartingale and* $\beta = (\beta_t)_{t\geq 0}$ *an* $\mathbb{R}^d$*-valued predictable* X*-integrable process (i.e.,* $\beta \in L(X)$*). A predictable real-valued process* $\left(\Psi_t^X(\beta_t)\right)_{t\geq 0}$ *is called* local exponent *(or derivative of the Laplace cumulant process) of* X *at* β *if*

$$\left(\exp\left(\int_0^t \beta_s dX_s - \int_0^t \Psi_s^X(\beta_s)ds\right)\right)_{t\geq 0}$$

is a local martingale. We denote by $\mathcal{U}^X$ *the set of processes* β *such that the local exponent* $\Psi^X(\beta)$ *exists.*

In other words, $(\int_0^t \Psi_s^X(\beta_s)ds)_{t\geq 0}$ is the *exponential compensator* (compare [42, Definition 2.14]) of $(\int_0^t \beta_s dX_s)_{t\geq 0}$. The following proposition asserts that the local exponent (when it exists) is of Lévy-Khintchine form, where the Lévy triplet is replaced by the differential characteristics of the Itô-semimartingale.

Proposition 2.4. *Let* X *be an* $\mathbb{R}^d$*-valued Itô-semimartingale with differential characteristics* (b,c,K) *with respect to some truncation function* χ*. Let* $\beta \in L(X)$*. Then the following are equivalent:*

(i) $\beta \in \mathcal{U}^X$,
(ii) $(\int_0^t \beta_s dX_s)_{t\geq 0}$ *is an exponentially special semimartingale, that is* $(e^{\int_0^t \beta_s dX_s})_{t\geq 0}$ *is a special semimartingale,*
(iii) $\int_0^t \int_{|\beta_s^\top \xi|>1} e^{\beta_s^\top \xi} K_s(d\xi)ds < \infty$ $\mathbb{Q}$*-a.s for all* $t \geq 0$.

In this case, outside some $d\mathbb{Q}\otimes dt$ *nullset, it holds that*

$$\Psi_t^X(\beta_t) = \beta_t^\top b_t + \frac{1}{2}\beta_t^\top c_t \beta_t + \int \left(e^{\beta_t^\top \xi} - 1 - \beta_t^\top \chi(\xi)\right) K_t(d\xi). \tag{2.4}$$

Proof. For the proof of the equivalence of (i)-(ii)-(iii) see [42, Lemma 2.13]. Representation (2.4) follows from [42, Theorem 2.18, statements 1 and 6, and Theorem 2.19]. □

Using the notion of the local exponent and defining an $\mathbb{R}^d$-valued process $(\Sigma_t(T))_{t\in[0,T]}$ via

$$\Sigma_t(T) := \int_t^T \sigma_t(u)du,$$

for all $t \leq T$ and $T \geq 0$, we are now in a position to state the following theorem, which characterizes the martingale property of the family of semimartingales $\{(S(t,T))_{t\in[0,T]}, T \geq 0\}$.

Theorem 2.5. *For an HJM-type model as of Definition 2.2 the following conditions are equivalent:*

(i) *The process* $(S(t,T))_{t\in[0,T]}$ *is a martingale, for every* $T \geq 0$.
(ii) *For every* $T \geq 0$*, it holds that*

$$\mathbb{E}\left[e^{Z_T}|\mathcal{F}_t\right] = e^{Z_t + \int_t^T \eta_t(u)du}, \qquad \text{for all } t \in [0,T],$$

which is called conditional expectation hypothesis.
(iii) *For every* $T \geq 0$, $\left(1, \Sigma^\top(T)\right)^\top \in \mathcal{U}^{Z,X}$ *and the following conditions are satisfied:*
- *The process*

$$\left(\exp\left(Z_t + \int_0^t \Sigma_s(T)dX_s - \int_0^t \Psi_s^{Z,X}\left(\left(1, \Sigma_s^\top(T)\right)^\top\right)ds\right)\right)_{t\in[0,T]} \tag{2.5}$$

is a martingale, for every $T \geq 0$.

[4]The definition of the local exponent in [41, Definition A.6] is slightly different since it is defined in terms of the exponential compensator of $(\int_0^t \mathrm{i}\beta_s dX_s)_t$ which is due to the fact that complex valued processes are considered.

- *The* consistency condition

$$\Psi_t^Z(1) = \eta_{t-}(t), \qquad \text{for all } t > 0, \tag{2.6}$$

holds.

- *The* HJM drift condition

$$\int_t^T \alpha_t(u)du = \Psi_t^Z(1) - \Psi_t^{Z,X}\left(\left(1, \Sigma_t^\top(T)\right)^\top\right) \tag{2.7}$$

holds for every $t \in [0,T]$ *and* $T \geq 0$.

Moreover, if any (and, hence, all) of conditions (i)-(ii)-(iii) is satisfied, it holds that

$$\begin{aligned} S(t,T) &= \mathbb{E}\left[S(T,T)|\mathcal{F}_t\right] = \mathbb{E}\left[e^{Z_T}|\mathcal{F}_t\right] \\ &= \exp\left(\int_0^T \eta_0(u)du + Z_t + \int_0^t \Sigma_s(T)dX_s - \int_0^t \Psi_s^{Z,X}\left(\left(1, \Sigma_s^\top(T)\right)^\top\right)ds\right). \end{aligned} \tag{2.8}$$

for all $t \leq T$ *and* $T \geq 0$.

Remark 2.6. (i) The above theorem can be weakened to consider only the case of local martingales. Indeed, $(S(t,T))_{t\in[0,T]}$ is a local martingale, for all $T \geq 0$, if and only if $(1, \Sigma^\top(T))^\top \in \mathcal{U}^{Z,X}$ and the HJM drift and consistency conditions are satisfied. By Definition 2.3 and Proposition 2.4, $(1, \Sigma^\top(T))^\top \in \mathcal{U}^{Z,X}$ is equivalent to the local martingale property of (2.5) and to $\left(Z_t + \int_0^t \Sigma_s(T)dX_s\right)_{t\in[0,T]}$ being an exponentially special semimartingale, for all $T \geq 0$.

(ii) General sufficient conditions for (2.5) being a true martingale have been established by Kallsen and Shiryaev in [42, Corollary 3.10] (compare also with an earlier version of Lépingle and Mémin [54]). Condition $I(0,1)$ in their formulation (see [42, Definition 3.1] reads in our case as

$$\int_{|\left(1,\Sigma_t^\top(T)\right)\xi|>1} e^{\left(1,\Sigma_t^\top(T)\right)^\top \xi}\left(1, \Sigma_t^\top(T)\right)^\top \xi\, K_t^{Z,X}(d\xi) \in \mathcal{V},$$

where $\mathcal{V}$ denotes the set of processes with finite variation, together with

$$\begin{aligned} \sup_{t\leq T} \mathbb{E}\Bigg[&\exp\left(\frac{1}{2}\left(1, \Sigma_t^\top(T)\right)^\top c_t^{Z,X}\left(1, \Sigma_t^\top(T)\right)\right) \\ &\times \exp\left(\int \left(e^{\left(1,\Sigma_t^\top(T)\right)^\top \xi}\left(\left(1, \Sigma_t^\top(T)\right)^\top \xi - 1\right) + 1\right) K_t^{Z,X}(d\xi)\right)\Bigg] < \infty, \end{aligned}$$

where $c^{Z,X}$ and $K^{Z,X}$ denote the second and third characteristic of the semimartingale (Z,X).

Proof of Theorem 2.5. In the sequel, let $T > 0$ be fixed.

(i) $\Rightarrow$ (ii): From (2.3) and the martingale property of $(S(t,T))_{t\in[0,T]}$, it follows that

$$e^{Z_t + \int_t^T \eta_t(u)du} = S(t,T) = \mathbb{E}\left[S(T,T)|\mathcal{F}_t\right] = \mathbb{E}\left[e^{Z_T}|\mathcal{F}_t\right], \qquad \text{for all } t \in [0,T],$$

whence (ii).

(ii) $\Rightarrow$ (i): Again by (2.3), it holds that

$$\mathbb{E}\left[S(T,T)|\mathcal{F}_t\right] = \mathbb{E}\left[e^{Z_T}|\mathcal{F}_t\right] = e^{Z_t + \int_t^T \eta_t(u)du} = S(t,T),$$

for all $t \in [0,T]$, and thus (i).

(i) $\Rightarrow$ (iii): Let us define $R_t := Z_t + \int_t^T \eta_t(u)du$. Then the martingale property of $S(\cdot,T) = \exp(R)$ and Definition 2.3 yields $1 \in \mathcal{U}^R$ and $\Psi_t^R(1) = 0$. By applying the classical and the stochastic Fubini theorem [64, Theorem IV.65], which is justified due to the integrability conditions on α and σ in Definition 2.2, we can write

$$\begin{aligned} \int_t^T \eta_t(u)du = &\int_0^T \eta_0(u)du + \int_0^t \int_s^T \alpha_s(u)duds + \int_0^t \Sigma_s(T)dX_s \\ &- \int_0^t \underbrace{\left(\eta_0(u) + \int_0^u \alpha_s(u)ds + \int_0^u \sigma_s(u)dX_s\right)}_{=\eta_u(u)} du. \end{aligned}$$

We thus obtain (e.g., by applying [41, Lemma A.20])

$$0 = \psi_t^R(1) = \psi_t^{Z,X}\Big(\big(1, \Sigma_t^\top(T)\big)^\top\Big) + \int_t^T \alpha_t(u)du - \eta_{t-}(t). \tag{2.9}$$

Setting $T = t$ and noting that $\Sigma_t(t) = 0$ yields (2.6), namely $\eta_{t-}(t) = \psi_t^Z(1)$, for all $t > 0$, which together with (2.9) then implies (2.7). By the drift and consistency conditions, $S(\cdot, T)$ is then of the form (2.8) and since $(S(t,T))_{t\in[0,T]}$ is a martingale, the martingale property holds for (2.5) as well.
(iii) $\Rightarrow$ (i): The martingale property of (2.5) implies the martingale property of $(S(t,T))_{t\in[0,T]}$, since – due to the drift and consistency condition – it is again of the form (2.8). This clearly proves also the last statement of the theorem. □

The following proposition asserts the existence of HJM-type models satisfying condition (iii) of the above theorem.

Proposition 2.7. *Consider a quadruple $(\widetilde{Z}, \eta_0, \sigma, X)$ satisfying the conditions (i)-(iii) of Definition 2.2 such that $\big(1, \Sigma^\top(T)\big)^\top \in \mathcal{U}^{\widetilde{Z},X}$ and*

$$\left(\exp\left(\widetilde{Z}_t + \int_0^t \Sigma_s(T)dX_s - \int_0^t \Psi_s^{\widetilde{Z},X}\Big(\big(1, \Sigma_s^\top(T)\big)^\top\Big)\,ds\right)\right)_{t\in[0,T]},$$

is a martingale, for every $T \geq 0$. Define

$$\alpha_t(T) := -\partial_T \Psi_t^{\widetilde{Z},X}\Big(\big(1, \Sigma_t^\top(T)\big)^\top\Big)$$

and suppose that for each $T \geq 0$ there exists a solution of (2.2). Then there exists a semimartingale Z satisfying

$$\Psi_t^Z(1) = \eta_{t-}(t), \tag{2.10}$$

$$\partial_T \Psi_t^{Z,X}\Big(\big(1, \Sigma_t^\top(T)\big)^\top\Big) = \partial_T \Psi_t^{\widetilde{Z},X}\Big(\big(1, \Sigma_t^\top(T)\big)^\top\Big) \tag{2.11}$$

for all $t \leq T$ and $T \geq 0$ and such that $\big(e^{Z_t + \int_t^T \eta_t(s)ds}\big)_{t\in[0,T]}$ is a martingale, for every $T \geq 0$.

Proof. By [64, Corollary to Theorem IV.63] and the conditions on σ, the process $(\eta_t(t))_{t\geq 0}$ given by

$$\eta_t(t) = \eta_0(t) + \int_0^t \alpha_s(t)ds + \int_0^t \sigma_s(t)dX_s$$

has a càdlàg modification and thus $\eta_{t-}(t)$ is predictable. We can then define a predictable process b^Z by

$$b_t^Z = \eta_{t-}(t) - \frac{1}{2}c_t^{\widetilde{Z}} - \int \big(e^\xi - 1 - \chi(\xi)\big)K_t^{\widetilde{Z}}(d\xi), \quad t > 0,$$

where χ denotes some truncation function. Denoting by $(b^{\widetilde{Z}}, c^{\widetilde{Z}}, K^{\widetilde{Z}})$ the differential characteristics of $\widetilde{Z}$ with respect to χ, let us then define the semimartingale Z via $Z := \widetilde{Z} + \int_0^\cdot (b_t^Z - b_t^{\widetilde{Z}})dt$, so that $c^Z = c^{\widetilde{Z}}$ and $K^Z = K^{\widetilde{Z}}$. With this specification of Z, condition (2.10) is then automatically satisfied and condition (2.11) also holds since

$$\Psi_t^{Z,X}\Big(\big(1, \Sigma_t^\top(T)\big)^\top\Big) = \Psi_t^{\widetilde{Z},X}\Big(\big(1, \Sigma_t^\top(T)\big)^\top\Big) + b_t^Z - b_t^{\widetilde{Z}}.$$

Finally, it also holds that

$$\exp\left(Z_t + \int_0^t \Sigma_s(T)dX_s - \int_0^t \Psi_s^{Z,X}\Big(\big(1, \Sigma_s^\top(T)\big)^\top\Big)\,ds\right) =$$
$$\exp\left(\widetilde{Z}_t + \int_0^t \Sigma_s(T)dX_s - \int_0^t \Psi_s^{\widetilde{Z},X}\Big(\big(1, \Sigma_s^\top(T)\big)^\top\Big)\,ds\right),$$

which implies – due to the assumption on the second term – that the first one is a martingale too. Condition (iii) of Theorem 2.5 is therefore satisfied for the quintuple $(Z, \eta_0, \alpha, \sigma, X)$, thus implying that $\big(e^{Z_t + \int_t^T \eta_t(s)ds}\big)_{t\in[0,T]}$ is a martingale, for every $T \geq 0$. □

Remark 2.8. We want to point out that the above choice of the semimartingale characteristics of Z is of course not unique. Indeed, in the above proof only $b^{\widetilde{Z}}$ is changed so that $\Psi_t^Z(1) = \eta_{t-}(t)$ is satisfied. Alternatively, one could change the diffusion characteristic or the compensator of the jumps. This observation turns out to be crucial for the construction of multiple yield curve models in Section 2.4.

2.2. **Modeling the term structure of OIS zero coupon bond prices.** In this section, we show that the classical HJM approach for risk-free bond prices, which we use for modeling OIS bonds, can be formulated in terms of the above general framework (compare also with [66] by J. Teichmann). We start by defining (OIS) bond price models via a model for the instantaneous (OIS) forward rates and the assumption that an (OIS) bank account B exists, given by

$$B_t = e^{\int_0^t r_s ds},$$

where r denotes the (OIS) short rate process.

Definition 2.9. *A* bond price model *is a quintuple $(B, f_0, \widetilde{\alpha}, \widetilde{\sigma}, X)$, where*

(i) *the bank account B satisfies $B_t = e^{\int_0^t r_s ds}$, for all $t \geq 0$, with short rate $(r_t)_{t\geq 0}$,*
(ii) *X is an $\mathbb{R}^d$-valued Itô-semimartingale,*
(iii) *$f_0 \colon \mathbb{R}_+ \to \mathbb{R}$ is measurable and $\int_0^T |f_0(t)|dt < \infty$ $\mathbb{Q}$-a.s. for all $T \geq 0$,*
(iv) *$(\omega, t, T) \mapsto \widetilde{\alpha}_t(T)(\omega)$ and $(\omega, t, T) \mapsto \widetilde{\sigma}_t(T)(\omega)$ are $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}_+)$ measurable $\mathbb{R}$- and $\mathbb{R}^d$-valued processes and satisfy the integrability conditions of Definition 2.2-(iii),*
(v) *for every $T \in \mathbb{R}_+$, the forward rate $(f_t(T))_{t\in[0,T]}$ is given by*

$$f_t(T) = f_0(T) + \int_0^t \widetilde{\alpha}_s(T)ds + \int_0^t \widetilde{\sigma}_s(T)dX_s,$$

(vi) *the bond prices $\{(B(t,T))_{t\in[0,T]}, T \geq 0\}$ satisfy $B(t,T) = e^{-\int_t^T f_t(s)ds}$, for all $t \leq T$ and $T \geq 0$. In particular, $B(t,t) = 1$ for all $t \geq 0$.*

The following definition is motivated by the fact that, as usual, we take the (OIS) bank account as numéraire and assume that $\mathbb{Q}$ corresponds to the risk neutral measure (see however Remark 2.12). In the following, if the measure is not explicitly indicated (e.g., in expectations), then it is meant to be $\mathbb{Q}$.

Definition 2.10. *A bond price model is said to be* risk neutral *if the discounted bond prices*

$$\left\{\left(B(t,T)/B_t\right)_{t\in[0,T]}, T \geq 0\right\}$$

are martingales.

The following proposition shows that a bond price model can be identified with an HJM-type model. For its formulation, let us introduce the process $\widetilde{\Sigma}(T)$ defined via $\widetilde{\Sigma}_t(T) := \int_t^T \widetilde{\sigma}_t(u)du$, for all $t \leq T$.

Proposition 2.11. *A bond price model can be identified with an HJM-type model $(Z, \eta_0, \alpha, \sigma, X)$ for the family of discounted bond prices $\left\{\left(B(t,T)/B_t\right)_{t\in[0,T]}, T \geq 0\right\}$ by setting $\eta_0 = -f_0$, $\alpha = -\widetilde{\alpha}$, $\sigma = -\widetilde{\sigma}$ (so that $\eta_t(t) = -f_t(T)$) and $Z_t = -\log B_t = -\int_0^t r_s ds$. Moreover, the following are equivalent:*

(i) *The bond price model is risk neutral, i.e., $(B(t,T)/B_t)_{t\in[0,T]}$ is a martingale, for every $T \geq 0$.*
(ii) *For every $T \geq 0$, the conditional expectation hypothesis holds:*

$$\mathbb{E}\left[e^{Z_T}|\mathcal{F}_t\right] = e^{Z_t + \int_t^T \eta_t(u)du} \Leftrightarrow \mathbb{E}\left[B_t/B_T \,|\, \mathcal{F}_t\right] = e^{-\int_t^T f_t(u)du}, \qquad \text{for all } t \in [0,T].$$

(iii) *For every $T \geq 0$, $-\widetilde{\Sigma}(T) \in \mathcal{U}^X$ and the following conditions are satisfied:*
- *The process*

$$\left(\exp\left(-\int_0^t \widetilde{\Sigma}_s(T)dX_s - \int_0^t \Psi_s^X\left(-\widetilde{\Sigma}_s(T)\right)ds\right)\right)_{t\in[0,T]},$$

is a martingale, for every $T \geq 0$.
- *The consistency condition holds, i.e.,*

$$\Psi_t^Z(1) = -r_{t-} = -f_{t-}(t), \qquad \text{for all } t > 0.$$

- *The HJM drift condition*

$$\int_t^T \widetilde{\alpha}_t(u)du = \Psi_t^X\left(-\widetilde{\Sigma}_t(T)\right)$$

holds for every $t \in [0,T]$ and $T \geq 0$.

Proof. The proposition follows from Theorem 2.5 by identifying $S(t,T)$ with $B(t,T)/B_t$, so that $Z_t = \log(S(t,t)) = -\log(B_t)$ and noting that

$$\Psi_t^{Z,X}\left(\left(1, -\widetilde{\Sigma}_t^\top(T)\right)^\top\right) = -r_{t-} + \Psi_t^X\left(-\widetilde{\Sigma}_t(T)\right),$$

which follows for instance from [41, Lemma A.20]. □

Remark 2.12. We want to point out that the assumption of the existence of a bank account is actually not necessary. Indeed, one could also consider the (OIS) bond market for maturities $T \leq T^*$ with respect to a bond $B(\cdot, T^*)$ as numéraire, where T^* denotes some fixed terminal maturity (in this regard, compare the recent paper [50]). In that context, we refer to Appendix C for the derivation of the corresponding HJM drift and consistency conditions.

2.3. **Modeling the term structure of multiplicative spreads.** In this section we introduce the modeling framework for multiple yield curves, where we extend the above considered model for the OIS bonds with an HJM-type model for the multiplicative spreads introduced in (1.2).

2.3.1. *Modeling the term structure of multiplicative spreads.* Let $\mathcal{D} = \{\delta_1, \ldots, \delta_m\}$ denote a family of tenors, with $0 < \delta_1 < \delta_2 < \ldots < \delta_m$, for some $m \in \mathbb{N}$. As argued in the introduction, we aim at modeling the term structure of multiplicative spreads between normalized FRA rates and simply compounded OIS forward rates given by

$$S^{\delta_i}(t,T) = \frac{1 + \delta_i L_t(T, T+\delta_i)}{1 + \delta_i L_t^D(T, T+\delta_i)},$$

for all $i = 1, \ldots, m$. Starting with time-inhomogeneous exponential Lévy models for the multiplicative spot spread (or "exchange rate") process Q^{δ_i} defined in (1.3) and putting the Lévy exponent (evaluated at 1) "in motion", as described in Section 1.2, naturally leads to HJM-type models where

$$S^{\delta_i}(t,T) = e^{Z_t^{\delta_i} + \int_t^T \eta_t^i(u)du}.$$

In particular, this allows to model the observed log-spot spreads $Z_t^{\delta_i} = \log(S^{\delta_i}(t,t))$ and the forward spread rates $\eta_t^i(T) = \partial_T(\log(S^{\delta_i}(t,T)))$ separately. This feature is important in order to capture the dependence structures between different spreads (which are easiest observed on the spot level), while guaranteeing at the same time that $S^{\delta_i}(t,T) \geq 1$ for all maturities. If desired, this approach also allows to easily accommodate monotonicity for all maturities with respect to the tenors δ_i (see Figure 1 and Corollary 2.19).

As in Section 2.2, we assume to have an OIS bank account and we work under a risk neutral measure $\mathbb{Q}$ under which discounted OIS bond prices $B(t,T)/B_t$ are martingales, as required in Definition 2.10. As a consequence of the defining property of the FRA rates (specified in (1.1)), we obtain the following lemma, which is crucial for absence of arbitrage in our setting.

Lemma 2.13. *For every $\delta_i \in \mathcal{D}$ and $T \geq 0$, the process $(S^{\delta_i}(t,T))_{t\in[0,T]}$ is a $\mathbb{Q}^T$-martingale, where $\mathbb{Q}^T$ denotes the T-forward measure whose density process is given by $\frac{d\mathbb{Q}^T}{d\mathbb{Q}}|_{\mathcal{F}_t} = \frac{B(t,T)}{B_t B(0,T)}$, $t \in [0,T]$.*

Proof. For $T \geq 0$, by the generalized Bayes' formula, $(S^{\delta_i}(t,T))_{t\in[0,T]}$ is a $\mathbb{Q}^T$-martingale if and only if

$$M_t^i := S^{\delta_i}(t,T)\frac{B(t,T)}{B_t B(0,T)}$$

is a $\mathbb{Q}$-martingale. By definition of $S^{\delta_i}(t,T)$, the process M^i can be rewritten as

$$M_t^i = (1 + \delta_i L_t(T, T+\delta_i))\frac{B(t,T+\delta_i)}{B_t B(0,T+\delta_i)}\frac{B(0,T+\delta_i)}{B(0,T)},$$

which is – again by Bayes' formula – a $\mathbb{Q}$-martingale, since $(1+\delta_i L_t(T,T+\delta_i))_{t\in[0,T]}$ is a $\mathbb{Q}^{T+\delta_i}$-martingale by (1.1) and $\frac{d\mathbb{Q}^{T+\delta_i}}{d\mathbb{Q}}|_{\mathcal{F}_t} = \frac{B(t,T+\delta_i)}{B_t B(0,T+\delta_i)}$, for all $i = 1, \ldots, m$. □

By relying on the above lemma and referring to the arguments already discussed in the introduction, let us now summarize the modeling requirement on $\{(S^\delta(t,T))_{t\in[0,T]}, T \geq 0, \delta \in \mathcal{D}\}$:

Requirement 2.14. *The family of spreads $\{(S^\delta(t,T))_{t\in[0,T]}, T \geq 0, \delta \in \mathcal{D}\}$ should satisfy*

(i) *$(S^{\delta_i}(t,T))_{t\in[0,T]}$ is a $\mathbb{Q}^T$-martingale, for every $T \geq 0$ and for all $i \in \{1, \ldots, m\}$;*
(ii) *$S^{\delta_i}(t,T) \geq 1$ for all $t \leq T$, $T \geq 0$ and $i \in \{1, \ldots, m\}$.*

In typical market situations, it is additionally desirable to have spreads which are ordered with respect to the different tenors δ_i, that is

$$S^{\delta_1}(t,T) \leq \cdots \leq S^{\delta_m}(t,T)$$

for all $t \leq T$ and $T \geq 0$. Moreover, due to the apparent strong interdependencies between the different spot spreads associated to different tenors δ_i (compare Figure 1), we model Z^{δ_i} via a common lower

dimensional process Y taking values in $\mathbb{R}^n$ (with $n \leq m$) such that

$$Z_t^{\delta_i} := u_i^\top Y_t,$$

where $u_1, \dots, u_m$ are given vectors in $\mathbb{R}^n$. The dimension of Y and the vectors u_i can for instance be obtained by a principal component analysis (PCA).

2.3.2. *Definition and characterization of multiple yield curve models.* We are now in a position to give a formal definition of a multiple yield curve model.

Definition 2.15. *Let the number of different tenors be $m := |\mathcal{D}|$. We call a model consisting of*

- *an $\mathbb{R}^{d+n+1}$-valued Itô-semimartingale (X, Y, B),*
- *vectors $u_1, \dots, u_m$ in $\mathbb{R}^n$,*
- *functions f_0, $\eta_0^1, \dots, \eta_0^m$,*
- *processes $\widetilde{\alpha}, \alpha^1, \dots, \alpha^m$ and $\widetilde{\sigma}, \sigma^1, \dots, \sigma^m$*

an HJM-type multiple yield curve model *for $\{(B(t,T))_{t\in[0,T]}, T \geq 0\}$ and $\{(S^\delta(t,T))_{t\in[0,T]}, T \geq 0, \delta \in \mathcal{D}\}$ if*

(i) *$(B, f_0, \widetilde{\alpha}, \widetilde{\sigma}, X)$ is a bond price model;*
(ii) *for every $i \in \{1, \dots, m\}$, $(u_i^\top Y, \eta_0^i, \alpha^i, \sigma^i, X)$ is an HJM-type model for $\{(S^{\delta_i}(t,T))_{t\in[0,T]}, T \geq 0\}$.*

As before, we write $\Sigma_t^i(T)$ for $\Sigma_t^i(T) = \int_t^T \sigma_t^i(u)du$ for all $t \leq T$ and $i \in \{1, \dots, m\}$. In view of Lemma 2.13, we define the *risk neutrality* of an HJM-type multiple yield curve model as follows.

Definition 2.16. *An HJM-type multiple yield curve model is said to be* risk neutral *if*

(i) *discounted OIS bond prices $\left\{(B(t,T)/B_t)_{t\in[0,T]}, T \geq 0\right\}$ are $\mathbb{Q}$-martingales;*
(ii) *for every $T \geq 0$, $\{(S^\delta(t,T))_{t\in[0,T]}, \delta \in \mathcal{D}\}$ are $\mathbb{Q}^T$-martingales.*

The subsequent theorem follows from Theorem 2.5 and characterizes condition (ii) of the above definition (recall that condition (i) has been already characterized in Proposition 2.11).

Theorem 2.17. *For an HJM-type multiple yield curve model satisfying condition (i) of Definition 2.16, the following are equivalent:*

(i) *Condition (ii) of Definition 2.16 is satisfied.*
(ii) *For every $T \geq 0$ and every $i \in \{1, \dots, m\}$, the following conditional expectation hypothesis holds:*

$$\mathbb{E}^{\mathbb{Q}^T}\left[e^{u_i^\top Y_T} \,\middle|\, \mathcal{F}_t\right] = e^{u_i^\top Y_t + \int_t^T \eta_t^i(u)du}, \qquad \text{for all } t \in [0,T].$$

(iii) *For every $T \geq 0$ and $i \in \{1, \dots, m\}$, $\left(u_i^\top, \Sigma^{i\top} - \widetilde{\Sigma}^\top\right)^\top \in \mathcal{U}^{Y,X}$ and the following conditions are satisfied:*

- *The process*

$$\left(\exp\left(u_i^\top Y_t + \int_0^t \left(\Sigma_s^i(T) - \widetilde{\Sigma}_s(T)\right) dX_s - \int_0^t \Psi_s^{Y,X}\left(\left(u_i^\top, \Sigma_s^{i\top}(T) - \widetilde{\Sigma}_s^\top(T)\right)^\top\right) ds\right)\right)_{t\in[0,T]} \tag{2.12}$$

  is a $\mathbb{Q}$-martingale, for every $T \geq 0$ and $i \in \{1, \dots, m\}$.

- *The following consistency condition holds for every $i \in \{1, \dots, m\}$:*

$$\Psi_t^Y(u_i) = \eta_{t-}^i(t), \qquad \text{for all } t > 0. \tag{2.13}$$

- *The following HJM drift condition*

$$\int_t^T \alpha_t^i(u)du = \Psi_t^Y(u_i) - \Psi_t^{Y,X}\left(\left(u_i^\top, \Sigma_t^{i\top}(T) - \widetilde{\Sigma}_t^\top(T)\right)^\top\right) + \Psi_t^X\left(-\widetilde{\Sigma}_t(T)\right) \tag{2.14}$$

  holds for every $t \in [0,T]$, $T \geq 0$ and $i \in \{1, \dots, m\}$.

Proof. The equivalence between (i) and (ii) can be shown as in the proof of Theorem 2.5. Concerning (iii), note that (i) is – by Bayes' Theorem – equivalent to

$$S^{\delta_i}(t,T)\frac{B(t,T)}{B_t} = e^{u_i^\top Y_t - \int_0^t r_s ds + \int_t^T (\eta_t^i(u) - f_t(u))du} \tag{2.15}$$

being a $\mathbb{Q}$-martingale, for every $T \geq 0$ and $i = 1, \dots, m$. Theorem 2.5 then yields the following consistency condition

$$\Psi_t^{u_i^\top Y - \int_0^\cdot r_s ds}(1) = \eta_{t-}^i(t) - f_{t-}(t), \qquad \text{for all } t > 0.$$

Since $\Psi_t^{u_i^\top Y - \int_0^\cdot r_s ds}(1) = \Psi_t^Y(u_i) - r_{t-}$ and since $r_{t-} = f_{t-}(t)$ by condition (i) of Definition 2.16, condition (2.13) follows. Concerning the drift condition (2.14), we have by Theorem 2.5

$$\int_t^T \big(\alpha_t^i(u) - \widetilde{\alpha}_t(u)\big)du = \Psi_t^{u_i^\top Y - \int_0^\cdot r_s ds}(1) - \Psi_t^{u_i^\top Y - \int_0^\cdot r_s ds, X}\left(\big(1, \Sigma_t^{i\top}(T) - \widetilde{\Sigma}_t^\top(T)\big)^\top\right).$$

As the right hand side is equal to

$$\Psi_t^Y(u_i) - \Psi_t^{Y,X}\left(\big(u_i^\top, \Sigma_t^{i\top}(T) - \widetilde{\Sigma}_t^\top(T)\big)^\top\right)$$

and as $\int_t^T \widetilde{\alpha}_t(u))du = \Psi_t^X(-\widetilde{\Sigma}_t(T))$ (by Proposition 2.11), the asserted drift condition follows. By the drift and consistency condition, (2.15) is actually of the form (2.12) (up to the constant term $\exp(\int_0^T(\eta_0^i(u) - f_0(u))du)$, which finally implies the equivalence between (i) and (iii). □

Remark 2.18. The martingale property of (2.12) can be assured similarly as in Remark 2.6.

Additionally, if one is interested in modeling ordered spot spreads $1 \le S^{\delta_1}(t,t) \le \cdots \le S^{\delta_m}(t,t)$, this can be easily obtained by considering a process Y taking values is some cone $C \subset \mathbb{R}^n$ and vectors $u_i \in C^*$ such that $0 < u_1 \prec u_2 \prec \cdots \prec u_m$, where C^* denotes the dual cone of C and $\prec$ the order relation thereon. In that context, we have the following corollary.

Corollary 2.19. *Consider a risk neutral HJM-type multiple yield curve model such that Y takes values in a cone $C \subset \mathbb{R}^n$ and $u_i \in C^*$, for $i = 1, \dots, m$, where C^* denotes the dual cone of C. Then Requirement 2.14 is satisfied. Moreover, if $u_1 \prec u_2 \prec \cdots \prec u_m$, where $\prec$ denotes the partial order of C^*, then we have $S^{\delta_1}(t,T) \le \cdots \le S^{\delta_m}(t,T)$ for all $t \le T$ and $T > 0$.*

Proof. Condition (i) of Requirement 2.14 is satisfied by definition, while condition (ii) follows from the conditional expectation hypothesis, since $S^{\delta_i}(t,T) = \mathbb{E}^{\mathbb{Q}^T}[e^{u_i^\top Y_T}|\mathcal{F}_t] \ge 1$ for all $t \le T$ and $T \ge 0$, as $u_i^\top Y_T \ge 0$ due to the condition on Y and u_i. Similarly, if $u_i \prec u_j$ for $i < j$, we obtain ordered spreads, since

$$S^{\delta_i}(t,T) = \mathbb{E}^{\mathbb{Q}^T}\left[e^{u_i^\top Y_T} \,\middle|\, \mathcal{F}_t\right] \le \mathbb{E}^{\mathbb{Q}^T}\left[e^{u_j^\top Y_T} \,\middle|\, \mathcal{F}_t\right] = S^{\delta_j}(t,T).$$

□

Remark 2.20. Note that, in the case when Y is not one-dimensional, the ordering of the spreads can vary over time if the vectors u_i, $i \in \{1, \dots, m\}$ are not chosen to be ordered. This means that the model can reproduce market situations where the order relations between spreads associated to different tenors change randomly over time. Nevertheless, $S^{\delta_i}(t,T) \ge 1$ for all $t \le T$ and $T \ge 0$ as long as $u_i \in C^*$.

Remark 2.21. One possibility to specify the process Y is an analogy to the bank account $B = \exp(\int_0^\cdot r_s ds)$. Indeed, let q be an $\mathbb{R}^n$-valued process and set

$$Y := \int_0^\cdot q_s ds.$$

Then,

$$S^{\delta_i}(t,T) = \mathbb{E}^{\mathbb{Q}^T}\left[e^{\int_0^T u_i^\top q_s ds}\middle|\mathcal{F}_t\right] = e^{\int_0^t u_i^\top q_s ds + \int_t^T \eta_t^i(u)du},$$

for all $t \le T$ and $T \ge 0$. The consistency condition $\Psi_t^{\int_0^\cdot q_s ds}(u_i) = \eta_{t-}^i(t)$ is then equivalent to $u_i^\top q_{t-} = \eta_{t-}^i(t)$ since $\Psi_t^{\int_0^\cdot q_s ds}(u_i) = u_i^\top q_{t-}$ and the drift condition becomes

$$\int_t^T \alpha_t^i(u)du = -\Psi_t^X\left(\big(\Sigma_t^{i\top}(T) - \widetilde{\Sigma}_t^\top(T)\big)^\top\right) + \Psi_t^X\left(-\widetilde{\Sigma}_t^\top(T)\right).$$

Remark 2.22. In our model formulation, we have implicitly considered a fixed reference currency. However, referring to Appendix D for more details, the present modeling framework can also be extended to consider the case of several currencies, with multiple yield curves associated to each currency. In a related context, see also [50, Section 8.3].

2.4. **Construction of HJM-type multiple yield curve models.** Theorem 2.17 gives necessary and sufficient conditions for an HJM-type multiple yield curve model to be risk neutral. In this section, we provide a general approach to construct risk neutral HJM-type multiple yield curve models starting from a given tuple of basic building blocks, as precisely defined below (see Definition 2.23).

In particular, we aim at constructing models that satisfy Requirement 2.14 and can potentially generate spreads which are ordered with respect to the tenor's length. To this effect, if the process Y takes values

in some cone $C \subset \mathbb{R}^n$, part (ii) of Requirement 2.14 and ordered spreads can then be easily achieved by relying on Corollary 2.19. Therefore, the crucial issue is to construct a model that satisfies the three conditions in part (iii) of Theorem 2.17, which in particular imply that the model ingredients in Definition 2.15 cannot be chosen arbitrarily. Our construction starts from the following basic building blocks on a given filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{Q})$ (compare also with [41, Definition 4.2]). For simplicity of notation, let us denote $u_0 := 0 \in \mathbb{R}^n$ and $\Sigma^0_\cdot(\cdot) := 0 \in \mathbb{R}^n$.

Definition 2.23. *A tuple* $(X, \hat{Y}, u_1, \dots, u_m, f_0, \eta_0^1, \dots, \eta_0^m, \widetilde{\sigma}, \sigma^1, \dots, \sigma^m)$ *is called* building blocks *for an HJM-type multiple yield curve model if*

(i) $(X, \hat{Y})$ *is an* $(\mathbb{R}^d \times C)$*-valued Itô-semimartingale such that* $\hat{Y}$ *is exponentially special and* $\hat{Y}^{\|} = \hat{Y}$*, with* $\hat{Y}^{\|}$ *denoting the dependent part of* $\hat{Y}$ *relative to* X *(see Appendix E);*
(ii) $u_1, \dots, u_m$ *are vectors in* C^**, with* C^* *denoting the dual cone of* C*;*
(iii) $f_0, \eta_0^1, \dots, \eta_0^m$ *are Borel-measurable functions satisfying condition (ii) of Definition 2.2;*
(iv) $\widetilde{\sigma}, \sigma^1, \dots, \sigma^m$ *are* $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}_+)$*-measurable processes satisfying condition (iii) of Definition 2.2;*
(v) $\big(u_i^\top, \Sigma^{i\top}(T) - \widetilde{\Sigma}^\top(T)\big)^\top \in \mathcal{U}^{\hat{Y},X}$*, for every* $T \geq 0$ *and* $i \in \{0, 1, \dots, m\}$*;*
(vi) *the process*

$$\left(\exp\left(u_i^\top \hat{Y}_t + \int_0^t \big(\Sigma_s^i(T) - \widetilde{\Sigma}_s(T)\big) dX_s - \int_0^t \Psi_s^{\hat{Y},X}\Big(\big(u_i^\top, \Sigma_s^{i\top}(T) - \widetilde{\Sigma}_s^\top(T)\big)^\top\Big) ds\right)\right)_{t\in[0,T]} \tag{2.16}$$

is a $\big(\mathbb{Q}, (\mathcal{F}_t)_{t\geq 0}\big)$*-martingale, for all* $T \geq 0$ *and* $i \in \{0, 1, \dots, m\}$*.*

Note that, if $(X, \hat{Y})$ is chosen to be a Lévy process (as is the case in Section 2.4.1), condition (vi) of the above definition is automatically satisfied if the volatilities $\widetilde{\sigma}, \sigma^1, \dots, \sigma^m$ are deterministic (more generally, the validity of condition (vi) can be established analogously as in Remark 2.6).

As in the classical HJM framework, the drift processes $(\widetilde{\alpha}, \alpha^1, \dots, \alpha^m)$ will be entirely determined by the building blocks (see part (ii) of Definition 2.24). Therefore, in view of Definition 2.23, the main model construction problem becomes finding an $\mathbb{R}^n$-valued Itô-semimartingale Y which, together with the given building blocks, generates a risk neutral HJM-type multiple yield curve model, as formalized in the following definition. In particular, note that the process Y needs to satisfy the crucial consistency condition (2.13).

Definition 2.24. *A* C*-valued Itô-semimartingale* Y *is said to be* compatible *with the building blocks* $(X, \hat{Y}, u_1, \dots, u_m, f_0, \eta_0^1, \dots, \eta_0^m, \widetilde{\sigma}, \sigma^1, \dots, \sigma^m)$ *if the following hold:*

(i) $Y^{\|} = \hat{Y}$*, with* $Y^{\|}$ *denoting the dependent part of* Y *relative to* X*;*
(ii) *the tuple* $(X, Y, \exp(\int_0^\cdot f_s(s)ds), u_1, \dots, u_m, f_0, \eta_0^1, \dots, \eta_0^m, \widetilde{\alpha}, \alpha^1, \dots, \alpha^m, \widetilde{\sigma}, \sigma^1, \dots, \sigma^m)$ *is a risk-neutral HJM-type multiple yield curve model, in the sense of Definitions 2.15-2.16, where*

$$\widetilde{\alpha}_t(T) = \partial_T \Psi_t^X\big(-\widetilde{\Sigma}_t(T)\big), \tag{2.17}$$

$$\alpha_t^i(T) = -\partial_T \Psi_t^{\hat{Y},X}\Big(\big(u_i^\top, \Sigma_t^{i\top}(T) - \widetilde{\Sigma}_t^\top(T)\big)^\top\Big) + \partial_T \Psi_t^X\big(-\widetilde{\Sigma}_t(T)\big), \tag{2.18}$$

for all $t \leq T$*,* $T \geq 0$ *and* $i \in \{1, \dots, m\}$*.*

In other words, starting from given building blocks $(X, \hat{Y}, u_1, \dots, u_m, f_0, \eta_0^1, \dots, \eta_0^m, \widetilde{\sigma}, \sigma^1, \dots, \sigma^m)$ and then searching for a compatible Itô-semimartingale Y amounts to a model construction strategy which proceeds along the three following subsequent steps:

(a) define the drift processes $(\widetilde{\alpha}, \alpha^1, \dots, \alpha^m)$ via the right-hand sides of (2.17)-(2.18);
(b) prove the existence and the uniqueness of the forward processes $(f, \eta^1, \dots, \eta^m)$, given as the solutions to (2.2) with initial values $(f_0, \eta_0^1, \dots, \eta_0^m)$, drift processes $(\widetilde{\alpha}, \alpha^1, \dots, \alpha^m)$ and volatility processes $(\widetilde{\sigma}, \sigma^1, \dots, \sigma^m)$;
(c) construct a C-valued Itô-semimartingale Y satisfying the three following requirements:
  (i) $Y^{\|} = \hat{Y}$;
  (ii) $\Psi_t^Y(u_i) = \eta_{t-}^i(t)$, for all $t > 0$ and $i \in \{1, \dots, m\}$ (*consistency condition*);
  (iii) the process given in (2.12) is an $\big(\mathbb{Q}, (\mathcal{F}_t)_{t\geq 0}\big)$-martingale, for every $T \geq 0$ and $i = 1, \dots, m$.

If steps (b)-(c) can be successfully solved, then a risk-neutral HJM-type multiple yield curve model is given by the tuple $(X, Y, \exp(\int_0^\cdot f_s(s)ds), u_1, \dots, u_m, f_0, \eta_0^1, \dots, \eta_0^m, \widetilde{\alpha}, \alpha^1, \dots, \alpha^m, \widetilde{\sigma}, \sigma^1, \dots, \sigma^m)$. Indeed, in view of Theorem 2.17, the HJM drift condition (2.14) follows from step (a), noting that

$$-\partial_T \Psi_t^{\hat{Y},X}\Big(\big(u_i^\top, \Sigma_t^{i\top}(T) - \widetilde{\Sigma}_t^\top(T)\big)^\top\Big) = -\partial_T \Psi_t^{Y,X}\Big(\big(u_i^\top, \Sigma_t^{i\top}(T) - \widetilde{\Sigma}_t^\top(T)\big)^\top\Big),$$

for all $t \leq T$, $T \geq 0$ and $i \in \{1, \dots, m\}$, since $Y^{\|} = \hat{Y}$ together with Definition E.1 and Lemma E.2 implying local independence of $Y^{\perp} := Y - Y^{\|}$ and $(Y^{\|}, X)$. The consistency condition and the martingale property of the process in (2.12) follow from step (c) of the above procedure. Finally, part (ii) of Requirement 2.14 and ordered spreads can be achieved by taking, for instance, $C = \mathbb{R}_+$, as considered in Section 2.4.2.

From now on, we fix a given tuple of building blocks $(X, \hat{Y}, u_1, \dots, u_m, f_0, \eta_0^1, \dots, \eta_0^m, \widetilde{\sigma}, \sigma^1, \dots, \sigma^m)$. In Section 2.4.1, we prove the existence and the uniqueness of the forward curves $(f, \eta^1, \dots, \eta^m)$, thus solving step (b) above, while, in Section 2.4.2, we present a general procedure to construct a compatible Itô-semimartingale Y, thus solving step (c) above.

2.4.1. *Existence and uniqueness of the forward spread curves.* In order to address the issue of existence and uniqueness of η^i, $i \in \{1, \dots, m\}$, and also of the OIS forward curve f, we shall rely on the results of [25], adapted to the present multiple curve setting. For notational convenience, we shall denote the OIS forward curve by $\eta^0 := f$ and its volatility and drift by $\sigma_t^0(T) := \widetilde{\sigma}_t(T)$ and $\alpha_t^0(T) := \widetilde{\alpha}_t(T)$.

We are interested in volatility structures which depend on the forward (spread) curves $\eta_t(\cdot) := \big(\eta_t^0(\cdot), \eta_t^1(\cdot), \dots, \eta_t^m(\cdot)\big)^\top$ in the following way

$$\sigma_t^i(T) = \begin{cases} \zeta^i(\theta_t)(T-t), & t \leq T, \\ 0, & t > T. \end{cases}, \quad i \in \{0, 1, \dots, m\}, \tag{2.19}$$

where $\theta_t(s) := \eta_t(t+s)$ corresponds to the Musiela parametrization and ζ^i, for $i \in \{0, 1, \dots, m\}$, is a function from some Hilbert space H_{m+1}^λ of forward (spread) curves $h : \mathbb{R}_+ \to \mathbb{R}^{m+1}$ specified below. Note that, for all $i \in \{0, 1, \dots, m\}$, the volatility $\sigma_t^i(T)$ of each individual forward curve η^i is allowed to depend through the function ζ^i on the whole family of forward curves $\eta = (\eta^0, \eta^1, \dots, \eta^m)^\top$. In view of the empirical facts reported in Section 1.1, this is a relevant feature of the model. We therefore switch to the Musiela parametrization and view $(\theta_t)_{t \geq 0}$ as a single stochastic process with values in H_{m+1}^λ.

Until the end of Section 2.4.1, in order to apply the results of [25], we assume that $(X, \hat{Y})$ is a Lévy martingale taking values in $\mathbb{R}^{d+n}$. In particular, the driving semimartingale $(X_t)_{t \geq 0}$ is of the form

$$X_t = \beta_t + \int_0^t \int_{\mathbb{R}^d} \xi\big(\mu(dt, d\xi) - F(d\xi)dt\big),$$

where $(\beta_t)_{t \geq 0}$ is an $\mathbb{R}^d$-valued standard Brownian motion and μ a homogeneous Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}^d$ with compensator $F(d\xi)dt$. The SDE for η^i, for $i \in \{0, 1, \dots, m\}$, thus becomes

$$\begin{aligned} \eta_t^i(T) &= \eta_0^i(T) + \int_0^t \alpha_s^i(T)ds + \int_0^t \sigma_s^i(T)dX_s \\ &= \eta_0^i(T) + \int_0^t \alpha_s^i(T)ds + \int_0^t \sigma_s^i(T)d\beta_s + \int_0^t \int_{\mathbb{R}^d} \big(\sigma_s^i(T)\big)^\top \xi\big(\mu(ds, d\xi) - F(d\xi)ds\big). \end{aligned} \tag{2.20}$$

Note that the processes $\sigma(t, T)$ and $\gamma(t, \xi, T)$ in [25] correspond to $\sigma_t^i(T)$ and $\big(\sigma_t^i(T)\big)^\top \xi$, respectively, for $i \in \{0, 1, \dots, m\}$. Assuming continuity of $T \mapsto \eta_t(T)$, we can transform (2.20) into the following integral equation for θ^i:

$$\begin{aligned} \theta_t^i(x) = S_t\eta_0^i(x) &+ \int_0^t S_{t-s}\alpha_s^i(s+x)ds + \int S_{t-s}\sigma_s^i(s+x)d\beta_s \\ &+ \int_0^t \int_{\mathbb{R}^d} S_{t-s}\big(\sigma_s^i(s+x)\big)^\top \xi\big(\mu(dt, d\xi) - F(d\xi)dt\big), \quad i \in \{0, 1, \dots, m\}, \end{aligned} \tag{2.21}$$

where $(S_t)_{t \geq 0}$ denotes the shift semigroup, that is $S_t h = h(t + \cdot)$. In order to establish existence of solutions to such equations, let us introduce the following spaces of forward curves, in line with [25] (but generalized to the multivariate case). Fix an arbitrary constant $\lambda > 0$ and let H_k^λ be the space of all absolutely continuous functions $h : \mathbb{R}_+ \to \mathbb{R}^k$ such that

$$\|h\|_{\lambda,k} := \left(\|h(0)\|_k^2 + \int_{\mathbb{R}_+} \|\partial_s h(s)\|_k^2 e^{\lambda s} ds \right)^{\frac{1}{2}},$$

where $\|\cdot\|_k$ denotes the norm in $\mathbb{R}^k$, with $k \in \{1, d, m+1\}$. Moreover, let $\mathcal{H}_k^\lambda$ be the space of all absolutely continuous functions $h : \mathbb{R} \to \mathbb{R}^k$ such that

$$\|h\|_{\lambda,k} := \left(\|h(0)\|_k^2 + \int_{\mathbb{R}} \|\partial_s h(s)\|_k^2 e^{\lambda |s|} ds \right)^{\frac{1}{2}}.$$

As stated above, we shall consider drift and volatility structures which are functions of the prevailing forward (spread) curves, i.e.,

$$\alpha_t^i(T) = \begin{cases} \kappa^i(\theta_t)(T-t), & t \le T, \\ 0, & t > T, \end{cases}, \quad \sigma_t^i(T) = \begin{cases} \zeta^i(\theta_t)(T-t), & t \le T, \\ 0, & t > T, \end{cases}, \quad i \in \{0,1,\dots,m\}.$$

In particular, we require $\kappa^i : H_{m+1}^\lambda \to H_1^\lambda$ and $\zeta^i : H_{m+1}^\lambda \to H_d^\lambda$.

Let us denote by $c^{\hat{Y},X}$ and $K^{\hat{Y},X}$ the second and third terms of the Lévy triplet of $(\hat{Y},X)$, so that $c^{\hat{Y},X} \in \mathbb{R}^{n\times d}$ and $K^{\hat{Y},X}$ is a Lévy measure on $\mathbb{R}^{n\times d}$, with X-marginal denoted by $F(d\xi)$. The drift conditions (2.17)-(2.18) then read as

$$\begin{aligned}\alpha_t^i(T) = &-\big(\sigma_t^i(T)-\sigma_t^0(T)\big)^\top c^{\hat{Y},X}u_i - \big(\sigma_t^i(T)-\sigma_t^0(T)\big)^\top\big(\Sigma_t^i(T)-\Sigma_t^0(T)\big)\\ &- \int \big(\sigma_t^i(T)-\sigma_t^0(T)\big)^\top \xi\big(e^{u_i^\top\hat{\xi}+(\Sigma_t^i(T)-\Sigma_t^0(T))^\top\xi}-1\big)K^{\hat{Y},X}(d\hat{\xi},d\xi)\\ &+ \big(\sigma_t^0(T)\big)^\top\Sigma_t^0(T) - \int \sigma_t^0(T)^\top\xi\big(e^{-(\Sigma_t^0(T))^\top\xi}-1\big)F(d\xi), \qquad i\in\{0,1,\dots,m\},\end{aligned}$$

as long as

$$\begin{aligned}&\int \sup_{T\ge t}\Big(\big(\sigma_t^i(T)-\sigma_t^0(T)\big)^\top\xi\big(e^{u_i^\top\hat{\xi}+(\Sigma_t^i(T)-\Sigma_t^0(T))^\top\xi}-1\big)\Big)K^{\hat{Y},X}(d\hat{\xi},d\xi)<\infty,\\ &\int \sup_{T\ge t}\Big(\big(\sigma_t^0(T)\big)^\top\xi\big(e^{-(\Sigma_t^0(T))^\top\xi}-1\big)\Big)F(d\xi)<\infty,\end{aligned}$$

so that we are allowed to differentiate under the integral sign. This translates to κ^i as follows

(2.22)

$$\begin{aligned}\kappa^i(h)(s) = &-\big(\zeta^i(h)(s)-\zeta^0(h)(s)\big)^\top c^{\hat{Y},X}u_i - \big(\zeta^i(h)(s)-\zeta^0(h)(s)\big)^\top\big(Z^i(h)(s)-Z^0(h)(s)\big)\\ &- \int \big(\zeta^i(h)(s)-\zeta^0(h)(s)\big)^\top\xi(e^{u_i^\top\hat{\xi}+(Z^i(h)(s)-Z^0(h)(s))^\top\xi}-1)K^{\hat{Y},X}(d\hat{\xi},d\xi)\\ &+ \big(\zeta^0(h)(s)\big)^\top Z^0(h)(s) - \int \big(\zeta^0(h)(s)\big)^\top\xi(e^{-(Z^0(h)(s))^\top\xi}-1)F(d\xi), \quad i\in\{0,1,\dots,m\},\end{aligned}$$

where $Z^i(h)(s) := \int_0^s \zeta^i(h)(u)du$. In the sequel, for a function $g : H_{m+1}^\lambda \to H_d^\lambda$ and a vector $z\in\mathbb{R}^d$, we shall write $g(h)^\top z$ for $\sum_{j=1}^d z_j g_j(h)$. The above specification leads to forward (spread) rates in (2.21) being a solution of

$$\theta_t^i = S_t\eta_0^i + \int_0^t S_{t-s}\kappa^i(\theta_s)ds + \int S_{t-s}\zeta^i(\theta_s)d\beta_s + \int_0^t\int_{\mathbb{R}^d} S_{t-s}\big(\zeta^i(\theta_{s-})\big)^\top\xi\big(\mu(dt,d\xi)-F(d\xi)dt\big), \tag{2.23}$$

for $i\in\{0,1,\dots,m\}$ and where κ is specified in (2.22). An H_{m+1}^λ-valued process θ satisfying (2.23) is said to be a *mild solution* to the stochastic partial differential equation (for $i\in\{0,1,\dots,m\}$)

$$d\theta_t^i = \left(\frac{d}{ds}\theta_t^i + \kappa^i(\theta_t)\right)dt + \zeta^i(\theta_t)d\beta_t + \int_{\mathbb{R}^d}\big(\zeta^i(\theta_{t-})\big)^\top\xi\big(\mu(dt,d\xi)-F(d\xi)dt\big), \quad \theta_0^i=\eta_0^i. \tag{2.24}$$

We are thus concerned with the question of existence of mild solutions to (2.24). Following [25], such SPDEs can be understood as time-dependent transformations of time-dependent SDEs with infinite dimensional state space. More precisely, on the enlarged space $\mathcal{H}_k^\lambda$ of forward (spread) curves $h:\mathbb{R}\to\mathbb{R}^k$ (for $k\in\{1,d,m+1\}$), which are indexed by the whole real line, equipped with the strongly continuous shift group $(U_t)_{t\in\mathbb{R}}$ (i.e. $U_t h := h(t+\cdot)$ for $t\in\mathbb{R}$), the above SPDE can be associated with the SDE

$$d\eta_t^i = U_{-t}\ell\kappa^i(\pi U_t\eta_t) + U_{-t}\ell\zeta^i(\pi U_t\eta_t)d\beta_t + \int_{\mathbb{R}^d}\big(U_{-t}\ell\zeta^i(\pi U_t\eta_{t-})\big)^\top\xi\big(\mu(dt,d\xi)-F(d\xi)dt\big), \tag{2.25}$$

where $\ell : H_k^\lambda \to \mathcal{H}_k^\lambda$ and $\pi : \mathcal{H}_k^\lambda \to H_k^\lambda$ are defined respectively by

$$\ell(h)(s) = \begin{cases} h(0), & s<0, \\ h(s), & s\ge 0, \end{cases} \text{ for } h\in H_k^\lambda, \qquad \text{and} \qquad \pi(h) = h|_{\mathbb{R}_+^k}, \text{ for } h\in\mathcal{H}_k^\lambda.$$

In order to obtain a mild solution for (2.24), the solution process $(\eta_t)_{t\ge0}$ is transformed by $\theta_t = \pi U_t\eta_t$. Note that (2.25) corresponds to (2.20) where, of course, the forward spread rates $\eta_t(T)$ have no interpretation for $T<t$.

For convenience of notation, let us denote $\zeta^{i0}(h) := \zeta^i(h)-\zeta^0(h)$ and $Z^{i0}(h) := Z^i(h)-Z^0(h)$, for all $i\in\{1,\dots,m\}$ and $h\in H_{m+1}^\lambda$. Moreover, we decompose $\kappa^i(h) = \kappa_1^i(h)+\kappa_2^i(h)+\kappa_3^i(h)+\kappa_4(h)+\kappa_5(h)$,

where

$$\begin{aligned}
\kappa_1^i(h) &= -\big(\zeta^i(h) - \zeta^0(h)\big)^\top c^{\hat{Y},X} u_i = -\big(\zeta^{i0}(h)\big)^\top c^{\hat{Y},X} u_i,\\
\kappa_2^i(h) &= -\big(\zeta^i(h) - \zeta^0(h)\big)^\top (Z^i(h) - Z^0(h)) = -\big(\zeta^{i0}(h)\big)^\top Z^{i0}(h),\\
\kappa_3^i(h) &= -\int \big(\zeta^i(h) - \zeta^0(h)\big)^\top \xi \big(e^{u_i^\top \hat{\xi} + (Z^i(h) - Z^0(h))^\top \xi} - 1\big) K^{\hat{Y},X}(d\hat{\xi}, d\xi)\\
&= -\int \big(\zeta^{i0}(h)\big)^\top \xi \big(e^{u_i^\top \hat{\xi} + (Z^{i0}(h))^\top \xi} - 1\big) K^{\hat{Y},X}(d\hat{\xi}, d\xi),\\
\kappa_4(h) &= \big(\zeta^0(h)\big)^\top Z^0(h),\\
\kappa_5(h) &= -\int \big(\zeta^0(h)\big)^\top \xi \big(e^{-(Z^0(h))^\top \xi} - 1\big) F(d\xi).
\end{aligned}$$

Aiming at establishing existence and uniqueness of the solution to the HJM equation (2.25), let us introduce suitable Lipschitz continuity conditions on the volatility functions ζ^i, for all $i = 0, 1, \dots, m$, as formulated in the following assumption (compare also with [25, Assumption 3.1]).

Assumption 2.25. *$\zeta^i : H^\lambda_{m+1} \to H^{\lambda,0}_d$, for all $i = 0, 1, \dots, m$, where $H^{\lambda,0}_k := \{h \in H^\lambda_k \mid \|h(\infty)\|_k = 0\}$, for $k \in \{1, d\}$. Moreover, for all $i \in \{0, 1, \dots, m\}$, there exist positive constants C_i, L_i, M_i such that*

$$\begin{aligned}
&\|Z^i(h)(s)\|_d \le C_i, && \text{for all } h \in H^\lambda_{m+1}, s \in \mathbb{R}_+,\\
&\|\zeta^i(h_1) - \zeta^i(h_2)\|_{\lambda,d} \le L_i \|h_1 - h_2\|_{\lambda,m+1}, && \text{for all } h_1, h_2 \in H^\lambda_{m+1},\\
&\|\zeta^i(h)\|_{\lambda,d} \le M_i, && \text{for all } h \in H^\lambda_{m+1},
\end{aligned}$$

and constants $K_0 > 0$ and $K_i > 0$ such that

$$\int e^{C_0\|\xi\|_d} \big(\|\xi\|_d^2 \vee \|\xi\|_d^4\big) F(d\xi) \le K_0, \tag{2.26}$$

$$\int e^{\|u_i\|_n \|\hat{\xi}\|_n + (C_0 + C_i)\|\xi\|_d} \big(\|\hat{\xi}\|_n^2 + (\|\xi\|_d^2 \vee \|\xi\|_d^4)\big) K^{\hat{Y},X}(d\hat{\xi}, d\xi) \le K_i, \quad i \in \{1, \dots, m\}. \tag{2.27}$$

Furthermore, we suppose that, for each $h \in H^\lambda_{m+1}$, the maps $\kappa_3^i(h)$ and $\kappa_5(h)$ are absolutely continuous with weak derivatives

$$\begin{aligned}
\frac{d}{ds}\kappa_3^i(h) &= -\int \big((\zeta^{i0}(h))^\top \xi\big)^2 \big(e^{u_i^\top \hat{\xi} + (Z^{i0}(h))^\top \xi}\big) K^{\hat{Y},X}(d\hat{\xi}, d\xi)\\
&\quad - \int \left(\frac{d}{ds}\big(\zeta^{i0}(h)\big)^\top \xi\right) \big(e^{u_i^\top \hat{\xi} + (Z^{i0}(h))^\top \xi} - 1\big) K^{\hat{Y},X}(d\hat{\xi}, d\xi),
\end{aligned} \tag{2.28}$$

$$\frac{d}{ds}\kappa_5(h) = \int \big((\zeta^0(h))^\top \xi\big)^2 e^{-(Z^0(h))^\top \xi} F(d\xi) - \int \frac{d}{ds}\big(\zeta^0(h)\big)^\top \xi \big(e^{-(Z^0(h))^\top \xi} - 1\big) F(d\xi). \tag{2.29}$$

As shown in the next proposition (the rather technical proof of which is postponed to Appendix F), Assumption 2.25 implies that the drift functions κ^i, for all $i = 0, 1, \dots, m$, are also Lipschitz continuous. This property will be crucial in order to establish existence and uniqueness of the solution to (2.25).

Proposition 2.26. *Suppose that Assumption 2.25 is satisfied. Then, for all $i \in \{0, 1, \dots, m\}$, it holds that $\kappa^i(H^\lambda_{m+1}) \subseteq H^{\lambda,0}_1$ and there exist constants $Q_i > 0$ such that*

$$\|\kappa^i(h_1) - \kappa^i(h_2)\|_{\lambda,1} \le Q_i \|h_1 - h_2\|_{\lambda,m+1} \tag{2.30}$$

for all $h_1, h_2 \in H^\lambda_{m+1}$.

We are now in a position to prove the following theorem, which asserts existence and uniqueness of the solution to the HJM equation (2.25) and extends [25, Theorem 3.2] to the present multiple curve setting.

Theorem 2.27. *Suppose that Assumption 2.25 is satisfied. Then, for each initial curve $h_0 \in H^\lambda_{m+1}$, there exists a unique adapted, càdlàg, mean-square continuous $\mathcal{H}^\lambda_{m+1}$-valued solution $(\eta_t)_{t \ge 0}$ to the system of equations (2.25) with $\eta_0 = \ell h_0$ satisfying*

$$\mathbb{E}\left[\sup_{t \in [0,T]} \|\eta_t\|^2_{\lambda,m+1}\right] < \infty, \quad \text{for all } T \in \mathbb{R}_+,$$

and there exists a unique adapted càdlàg, mean-square continuous mild and weak H^{λ}_{m+1}-valued solution $(\theta_t)_{t\geq 0}$ with $\theta_0 = h_0$ satisfying

$$\mathbb{E}\bigg[\sup_{t\in[0,T]} \|\theta_t\|^2_{\lambda,m+1}\bigg] < \infty, \quad \textit{for all } T \in \mathbb{R}_+,$$

which is given by $\theta_t := \pi U_t \eta_t$, for $t \geq 0$.

Proof. By virtue of [25, Theorem 2.1], Assumption 2.25 and Propositon 2.26, [24, Corollary 10.9] applies and yields the claimed existence and uniqueness result. □

Remark 2.28. In view of applications, one is often interested in constructing multiple yield curve models producing positive OIS forward rates f as well as positive forward spread rates η^i, for $i \in \{1,\dots,m\}$. Similarly as in the case of Theorem 2.27, necessary and sufficient conditions for the positiveness of f and η^i, for $i \in \{1,\dots,m\}$, can be obtained by adapting to the present context the results of [25, Section 4].

2.4.2. *Construction of $Y^\perp$.* Until the end of the present section, we shall suppose existence and uniqueness of the forward curves $(f, \eta^1, \dots, \eta^m)$, but we do not necessarily assume that $(X, \hat{Y})$ is a Lévy martingale. We now present a general procedure to construct an Itô-semimartingale Y compatible with the building blocks $(X, \hat{Y}, u_1, \dots, u_m, f_0, \eta^1_0, \dots, \eta^m_0, \widetilde{\sigma}, \sigma^1, \dots, \sigma^m)$, in the sense of Definition 2.24, or, equivalently, satisfying the three requirements in step (c) of the model construction procedure described at the beginning of Section 2.4.

As a preliminary observation, note that, by the definition of local independence (see Definition E.1 and Lemma E.2), constructing a C-valued process Y such that $Y^{\|} = \hat{Y}$ (requirement (i) of step (c)) can be achieved by constructing a C-valued process $Y^\perp$ which is locally independent of $(X, \hat{Y})$ and then letting $Y := \hat{Y} + Y^\perp$. As a consequence, from now on we shall focus on constructing a C-valued process $Y^\perp$, locally independent of $(X, \hat{Y})$. By local independence, the local exponent $\Psi^{Y^\perp}$ must then satisfy the following condition, which amounts to the consistency condition (requirement (ii) of step (c)):

$$\Psi^{Y^\perp}_t(u_i) = \eta^i_{t-}(t) - \Psi^{\hat{Y}}_t(u_i), \qquad \text{for all } t > 0 \text{ and } i \in \{1,\dots,m\}. \tag{2.31}$$

In the easy case where $n = m$, one can then choose arbitrarily the characteristics $c^{Y^\perp}$ and $K^{Y^\perp}$ and then specify the drift characteristic $b^{Y^\perp}$ in such a way that (2.31) holds, exactly as in the proof of Proposition 2.7. However, the case $m = n$ is rather unrealistic, since we aim to model the log-spot spreads for different tenors by means of a lower dimensional process Y in order to capture their interdependencies. In this case, i.e., when $m > n$, the easy idea of Proposition 2.7 does not suffice any more. Note also that, even in the case $n = m$, one has to impose further conditions in order to ensure that $Y^\perp$ lies in C.

For simplicity of presentation, let us consider the case when Y is a one-dimensional process taking values in the cone $C = \mathbb{R}_+$ and $0 < u_1 < u_2 < \ldots < u_m$. We aim at constructing a process $Y^\perp$, locally independent of $(X, \hat{Y})$, such that the consistency condition (2.31) is satisfied and the process given in equation (2.12) is a martingale (requirement (iii) of step (c)). We shall construct the process $Y^\perp$ as a finite activity pure jump process (see however Remark 2.36) on a suitably extended probability space. Note that, since we want $Y^\perp$ to take values in $\mathbb{R}_+$, we need to restrict its jump sizes so that $\Delta Y^\perp \geq -Y^\perp_-$ a.s. Hence, the construction problem amounts to determine the compensating jump measure of $Y^\perp$, which we denote as $K_t\big(\omega, Y^\perp_{t-}(\omega), d\xi\big)dt$ in order to make explicit the dependence of the jumps on $Y^\perp_-$.

The crucial consistency condition (2.31) will be satisfied if the kernel $K_t(\omega, y, d\xi)$ satisfies, for all $\omega \in \Omega$, $y \in \mathbb{R}_+$ and $t > 0$,

$$\int (e^{u_i\xi} - 1) K_t(\omega, y, d\xi) = \eta^i_{t-}(t)(\omega) - \Psi^{\hat{Y}}_t(u_i)(\omega) =: p^i_t(\omega), \qquad \text{for } i = 1,\dots,m. \tag{2.32}$$

Note that the right-hand side of (2.32) is fully determined from the previous steps of the model's construction. In particular, this means that, for every $\omega \in \Omega$, $y \in \mathbb{R}_+$ and $t > 0$, $K_t(\omega, y, d\xi)$ needs to be a measure on $\big(\mathbb{R}, \mathcal{B}(\mathbb{R})\big)$ supported by $[-y, \infty)$. Moreover, in order to ensure the martingale property of (2.12), we also require $K_t(\omega, y, d\xi)$ to satisfy the following integrability condition, for all $\omega \in \Omega$, $y \in \mathbb{R}_+$ and $t \geq 0$:

$$\int \big(|\xi| \vee 1\big) e^{(u_m \vee 1)|\xi|} K_t(\omega, y, d\xi) = p^{m+1}_t(\omega, y), \tag{2.33}$$

with respect to some family $\big\{\big(p^{m+1}_t(\cdot, y)\big)_{t\geq 0} \,|\, y \in \mathbb{R}_+\big\}$ of predictable processes, measurable with respect to y, satisfying $p^{m+1}_t(\omega, y) \leq \bar{H}$ $\mathbb{Q}$-a.s. for all $y \in \mathbb{R}_+$ and $t \geq 0$, for some constant $\bar{H} > 0$.

For fixed $\omega \in \Omega$, $y \in \mathbb{R}_+$ and $t \geq 0$, the question of whether such a measure $K_t(\omega, y, \cdot)$ exists corresponds to the generalized moment problem considered by Krein and Nudelman [52] and puts some restrictions on the possible values of $\big(p_t^1(\omega), \dots, p_t^m(\omega), p_t^{m+1}(\omega, y)\big)$.

Let us briefly recall the formulation of the generalized moment problem. Let $[a, b] \subset \mathbb{R}$ (with b possibly ∞) be some interval and consider a family of linearly independent continuous functions $f_i : [a, b] \to \mathbb{R}$, $i = 1, \dots, m+1$. Let $c \in \mathbb{R}^{m+1}$. Then the generalized moment problem consists in finding a positive measure μ on $\big([a, b], \mathcal{B}([a, b])\big)$ such that

$$\int_a^b f_i(\xi)\mu(d\xi) = c_i, \qquad \text{for all } i = 1, \dots, m+1.$$

Under the condition that there exists some function h being a linear combination of f_i, $i = 1, \dots, m+1$, which is strictly positive on $[a, b]$, the result of Krein and Nudelman [52, Theorem I.3.4, Theorem III 1.1 and p. 175] states that the generalized moment problem admits a solution if and only if c is an element of the closed conic hull $K(U)$ of

$$U = \big\{\big(f_1(\xi), \dots, f_{m+1}(\xi)\big) \,|\, \xi \in [a, b]\big\}.$$

In our context, this directly implies the following lemma. As a preliminary, let us define the family of functions $g_i(\xi) := e^{u_i \xi} - 1$, for $i = 1, \dots, m$, and $g_{m+1}(\xi) := \big(|\xi| \vee 1\big)e^{(u_m \vee 1)|\xi|}$.

Lemma 2.29. *Let $0 < u_1 < \ldots < u_m$. Then, for every $\omega \in \Omega$, $y \in \mathbb{R}_+$ and $t \geq 0$, there exists a non-negative measure $K_t(\omega, y, d\xi)$ on $\big([-y, \infty), \mathcal{B}([-y, \infty))\big)$ satisfying* (2.32)-(2.33) *if and only if*

$$p_t(\omega, y) := \big(p_t^1(\omega), \dots, p_t^m(\omega), p_t^{m+1}(\omega, y)\big) \in K\Big(\big\{\big(g_1(\xi), \dots, g_m(\xi), g_{m+1}(\xi)\big) \,|\, \xi \in [-y, \infty)\big\}\Big). \tag{2.34}$$

Proof. For every fixed $\omega \in \Omega$, $y \in \mathbb{R}_+$ and $t \geq 0$, the claim follows directly from [52, Theorem I.3.4, Theorem III 1.1 and p. 175], noting that the functions g_i, $i = 1, \dots, m+1$, are continuous and linearly independent and that the function g_{m+1} is strictly positive. □

As we are going to show in the remaining part of this section, the construction of a process $Y^\perp$ satisfying all the desired properties will be possible as long as there exists a solution to the generalized moment problem. More precisely, in view of Lemma 2.29, let us formulate the following assumption.

Assumption 2.30. *There exists a family $\big\{\big(p_t^{m+1}(\cdot, y)\big)_{t \geq 0} \,|\, y \in \mathbb{R}_+\big\}$ of predictable processes, measurable with respect to y, satisfying $p_t^{m+1}(\omega, y) \leq \bar{H}$ $\mathbb{Q}$-a.s. for all $y \in \mathbb{R}_+$ and $t \geq 0$, for some constant $\bar{H} > 0$, such that condition* (2.34) *is satisfied, for all $\omega \in \Omega$, $y \in \mathbb{R}_+$ and $t \geq 0$.*

Remark 2.31. If $m = 1$, then for any given $p_t^1(\omega)$ and $y > 0$, we can find some $p_t^2(\omega, y)$ such that $(p_t^1(\omega), p_t^2(\omega, y)) \in K(\{(g_1(\xi), g_2(\xi)) \,|\, \xi \in [-y, \infty)\})$. If $y = 0$, then $p_t^1(\omega)$ has to be nonnegative. If $\omega \mapsto p_t^1(\omega)$ is bounded and nonnegative, Assumption 2.30 is always satisfied. Similarly, for $m = 2$ the conditions $p_t^1(\omega) \geq 0$ and $p_t^2(\omega) \geq \frac{u_2}{u_1} p_t^1(\omega)$ and boundedness (in ω) are sufficient for the validity of Assumption 2.30.

The next proposition establishes the existence of a process $Y^\perp$ with jump measure $K_t\big(\omega, Y_{t-}^\perp(\omega), d\xi\big)dt$ on an extension of the original probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{Q})$. We rely on a constructive proof on a specific stochastic basis which is defined as follows (compare also with [17, Appendix A]):

- $(\widetilde{\Omega}, \mathcal{G}, (\mathcal{G}_t)_{t \geq 0})$ is a filtered space, with $\widetilde{\Omega} := \Omega \times \Omega'$, $\mathcal{G}_t := \mathcal{F}_t \otimes \mathcal{H}_t$ and $\mathcal{G} = \mathcal{F} \otimes \mathcal{H}$. Here, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{Q})$ is the probability space on which we worked so far and $(\Omega', \mathcal{H}, \mathcal{H}_t)$ is precisely defined below. Note that we do not assume to have a measure on $(\widetilde{\Omega}, \mathcal{G})$ for the moment. The generic sample element is denoted by $\widetilde{\omega} := (\omega, \omega') \in \widetilde{\Omega}$.
- $(\Omega', \mathcal{H})$ is the canonical space of real-valued marked point processes (see e.g. [37]), meaning that Ω' consists of all càdlàg piecewise constant functions $\omega' : \big[0, T_\infty(\omega')\big) \to \mathbb{R}$ with $\omega'(0) = 0$ and $T_\infty(\omega') = \lim_{n \to \infty} T_n(\omega') \leq \infty$, where $T_n(\omega')$, defined by $T_0 = 0$ and

$$T_n(\omega') := \inf\big\{t > T_{n-1}(\omega') \,|\, \omega'(t) \neq \omega'(t-)\big\} \wedge \infty, \qquad \text{for } n \geq 1,$$

  are the successive jump times of ω'. We denote by

$$J_t(\widetilde{\omega}) := J_t(\omega') := \omega'(t) \text{ on } \big[0, T_\infty(\omega')\big)$$

  the canonical jump process, and let $(\mathcal{H}_t)_{t \geq 0}$ be its natural filtration, i.e., $\mathcal{H}_t = \sigma(J_s \,|\, s \leq t)$, with $\mathcal{H} = \mathcal{H}_\infty$. Note that T_n are $(\mathcal{H}_t)$- and $(\mathcal{G}_t)$-stopping times if interpreted as $T_n(\widetilde{\omega}) = T_n(\omega')$.

It is also useful to introduce the larger filtration $(\overline{\mathcal{G}}_t)_{t\geq 0}$ defined by $\overline{\mathcal{G}}_t := \mathcal{F}_\infty \otimes \mathcal{H}_t$, for all $t \geq 0$. In particular, observe that $\mathcal{G}_t \subseteq \overline{\mathcal{G}}_t$, for all $t \geq 0$, and $\overline{\mathcal{G}}_0 = \mathcal{F}_\infty \otimes \{\emptyset, \Omega'\}$.

Proposition 2.32. *Suppose that Assumption 2.30 holds and let $(\widetilde{\Omega}, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0})$ and $(\overline{\mathcal{G}}_t)_{t\geq 0}$ be defined as above. Then there exists a probability measure $\widetilde{\mathbb{Q}}$ on $(\widetilde{\Omega}, \mathcal{G})$ satisfying $\widetilde{\mathbb{Q}}|_{\mathcal{F}} = \mathbb{Q}$ and a càdlàg $(\mathcal{G}_t)$-adapted $\mathbb{R}_+$-valued finite activity pure jump process $Y^\perp$ with jump measure $K_t\big(\omega, Y^\perp_{t-}(\omega,\omega'), d\xi\big)dt$ with respect to both $(\mathcal{G}_t)_{t\geq 0}$ and $(\overline{\mathcal{G}}_t)_{t\geq 0}$.*

Proof. For every $(\omega,\omega') \in \widetilde{\Omega}$ and $t \geq 0$, let us define $Y^\perp_t(\omega,\omega') := y_0 + J_t(\omega')$, for some starting value $y_0 \in \mathbb{R}_+$. Clearly, $Y^\perp$ is a pure jump $(\mathcal{G}_t)$-adapted process. In order to prove the existence of a probability measure $\widetilde{\mathbb{Q}}$ such that the jump measure of $Y^\perp$ with respect to the two filtrations $(\mathcal{G}_t)_{t\geq 0}$ and $(\overline{\mathcal{G}}_t)_{t\geq 0}$ is given by $K_t\big(\omega, Y^\perp_{t-}(\omega,\omega'), d\xi\big)dt$ and $\widetilde{\mathbb{Q}}|_{\mathcal{F}} = \mathbb{Q}$ holds true, we shall rely on [37, Theorem 3.6].

For all $\omega \in \Omega$, $y \in \mathbb{R}_+$ and $t \geq 0$, let us first extend the definition of $K_t(\omega, y, d\xi)$ as of (2.32)-(2.33) to $y \in \mathbb{R}_-$ by requiring that it is supported on $[-|y|, \infty)$ and by setting $p_t(y,\omega) = p_t(-y,\omega)$ for $y \in \mathbb{R}_-$. Due to condition (2.34), the measure $K_t(\omega, Y^\perp_{t-}(\omega,\omega'), d\xi)dt$ defined via the moment problem (2.32)-(2.33) is a positive random measure on $\mathbb{R}_+ \times \mathbb{R}$. Since $Y^\perp$ is càdlàg and $(\mathcal{G}_t)$-adapted and since $\big(p_t(\cdot,y)\big)_{t\geq 0}$ is $(\mathcal{F}_t)$-predictable and depends in a measurable way on y, the process $p_t\big(\omega, Y^\perp_{t-}(\omega,\omega')\big)$ is $(\mathcal{G}_t)$-predictable and the same $(\mathcal{G}_t)$-predictability and, hence, $(\overline{\mathcal{G}}_t)$-predictability, is inherited by $K_t\big(\omega, Y^\perp_{t-}(\omega,\omega'), d\xi\big)$. Let us then define the $(\overline{\mathcal{G}}_t)$-predictable random measure ν by

$$\nu(\widetilde{\omega}, dt, d\xi) = \begin{cases} K_t\big(\omega, Y^\perp_{t-}(\omega,\omega'), d\xi\big)dt, & t < T_\infty; \\ 0, & t \geq T_\infty. \end{cases}$$

[37, Theorem 3.6] implies that there exists a unique probability kernel $\mathbb{P}$ from Ω to $\mathcal{H}$, such that ν is the $(\overline{\mathcal{G}}_t)$-compensator of the random measure μ associated with the jumps of J. On $(\widetilde{\Omega}, \mathcal{G})$ we then define the probability measure $\widetilde{\mathbb{Q}}$ by $\widetilde{\mathbb{Q}}(d\widetilde{\omega}) = \mathbb{Q}(d\omega)\mathbb{P}(\omega, d\omega')$, whose restriction to $\mathcal{F}$ is equal to $\mathbb{Q}$. Moreover, since $Y^\perp$ is $(\mathcal{G}_t)$-adapted and $K_t\big(\omega, Y^\perp_{t-}(\omega,\omega'), d\xi\big)$ is $(\mathcal{G}_t)$-predictable, the random measure ν is also the $(\mathcal{G}_t)$-compensator of the jump measure of J.

Since, for every $\omega \in \Omega$, $y \in \mathbb{R}_+$ and $t \geq 0$, the measure $K_t(\omega, y, d\xi)$ is supported by $[-y, \infty)$, the process $Y^\perp$ takes values in $\mathbb{R}_+$. It remains to show that $Y^\perp$ is of finite activity or, equivalently, that $T_\infty = \infty$ $\widetilde{\mathbb{Q}}$-a.s. Since $g_{m+1}(\xi) \geq 1$ for all $\xi \in \mathbb{R}$ and due to condition (2.33), it holds that, for all $T \geq 0$,

$$\begin{aligned}\mathbb{E}^{\widetilde{\mathbb{Q}}}\big[\mu\big([0,T]\times\mathbb{R}\big)\big] &= \mathbb{E}^{\widetilde{\mathbb{Q}}}\big[\nu\big([0,T]\times\mathbb{R}\big)\big] = \mathbb{E}^{\widetilde{\mathbb{Q}}}\left[\int_0^T K_t\big(\omega, Y_{t-}(\omega,\omega'), \mathbb{R}\big)dt\right] \\ &\leq \mathbb{E}^{\widetilde{\mathbb{Q}}}\left[\int_0^T \int g_{m+1}(\xi) K_t\big(\omega, Y_{t-}(\omega,\omega'), d\xi\big)dt\right] = \mathbb{E}^{\widetilde{\mathbb{Q}}}\left[\int_0^T p_t^{m+1}\big(\omega, Y_{t-}(\omega,\omega')\big)dt\right] \leq \bar{H}T,\end{aligned}$$

due to the boundedness of the family $\{\big(p_t^{m+1}(\cdot,y)\big)_{t\geq 0}; y \in \mathbb{R}_+\}$. This implies that $\mu\big([0,T]\times\mathbb{R}\big) < \infty$, $\widetilde{\mathbb{Q}}$-a.s. for all $T \geq 0$ and, hence, $\widetilde{\mathbb{Q}}[T_\infty < \infty] = 0$. □

The next lemma shows that the semimartingale property as well as the semimartingale characteristics of $(X, \hat{Y})$ are not altered when considered in the extended probability space.

Lemma 2.33. *Suppose that Assumption 2.30 holds. Then the couple $(X, \hat{Y})$ is an $\mathbb{R}^{d+1}$-valued semimartingale on the extended filtered probability space $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, \widetilde{\mathbb{Q}})$ with the same semimartingale characteristics as in the original filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{Q})$.*

Proof. For $t \geq 0$, let H be a bounded $\mathcal{H}_t$-measurable random variable, F a bounded $\mathcal{F}_t$-measurable random variable and $A \in \mathcal{F}_\infty$. As can be deduced from the proof of the previous proposition, the conditional law under $\widetilde{\mathbb{Q}}$ of $\big(\omega'(s)\big)_{s\in[0,t]}$ given $\mathcal{F}_\infty$ is $\mathcal{F}_t$-measurable (compare also with [23, part (iv) of Theorem 5.1]). In particular, this means that $\mathbb{E}^{\widetilde{\mathbb{Q}}}[H|\mathcal{F}_\infty] = \mathbb{E}^{\widetilde{\mathbb{Q}}}[H|\mathcal{F}_t]$. In turn, this implies that

$$\mathbb{E}^{\widetilde{\mathbb{Q}}}[FH\mathbf{1}_A] = \mathbb{E}^{\widetilde{\mathbb{Q}}}[F\,\mathbb{E}^{\widetilde{\mathbb{Q}}}[H|\mathcal{F}_\infty]\mathbf{1}_A] = \mathbb{E}^{\widetilde{\mathbb{Q}}}[F\,\mathbb{E}^{\widetilde{\mathbb{Q}}}[H|\mathcal{F}_t]\mathbf{1}_A] = \mathbb{E}^{\widetilde{\mathbb{Q}}}[\mathbb{E}^{\widetilde{\mathbb{Q}}}[FH|\mathcal{F}_t]\mathbf{1}_A].$$

By a monotone class argument, this means that $\mathbb{E}^{\widetilde{\mathbb{Q}}}[G|\mathcal{F}_\infty] = \mathbb{E}^{\widetilde{\mathbb{Q}}}[G|\mathcal{F}_t]$ for every bounded $\mathcal{G}_t$-measurable random variable G. It is well-known (see e.g. [40, Proposition 5.9.1.1]) that the latter property is equivalent to the fact that all $\big(\widetilde{\mathbb{Q}}, (\mathcal{F}_t)_{t\geq 0}\big)$-martingales are also $\big(\widetilde{\mathbb{Q}}, (\mathcal{G}_t)_{t\geq 0}\big)$martingales. Since $\widetilde{\mathbb{Q}}|_{\mathcal{F}_\infty} = \mathbb{Q}$, this implies that all $\big(\mathbb{Q}, (\mathcal{F}_t)_{t\geq 0}\big)$-local martingales are also $\big(\widetilde{\mathbb{Q}}, (\mathcal{G}_t)_{t\geq 0}\big)$-local martingales. As a consequence,

every $(\mathcal{F}_t)$-semimartingale is also a $(\mathcal{G}_t)$-semimartingale and, moreover, since semimartingale characteristics can be characterized in terms of local martingales (see e.g. [38, Theorem II.2.21]), this implies that $(X, \hat{Y})$ is a semimartingale with respect to $(\widetilde{\mathbb{Q}}, (\mathcal{G}_t)_{t\geq 0})$ with unchanged characteristics. □

The following lemma shows that, besides satisfying the consistency condition (2.31), the constructed process $Y^\perp$ is locally independent of $(X, \hat{Y})$ in the extended filtered probability space $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, \widetilde{\mathbb{Q}})$.

Lemma 2.34. *Suppose that Assumption 2.30 holds and let the process $Y^\perp$ be constructed as in Proposition 2.32. Then $Y^\perp$ is locally independent of $(X, \hat{Y})$ on $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, \widetilde{\mathbb{Q}})$. Moreover, for all $i = 1, \dots, m$, the process $\big(\exp(u_i Y_t^\perp - \int_0^t \Psi_s^{Y^\perp}(u_i) ds)\big)_{t\geq 0}$ is a $\big(\widetilde{\mathbb{Q}}, (\overline{\mathcal{G}}_t)_{t\geq 0}\big)$-martingale as well as a $\big(\widetilde{\mathbb{Q}}, (\mathcal{G}_t)_{t\geq 0}\big)$-martingale.*

Proof. Since $Y^\perp$ is a pure jump process, in order to prove its local independence with respect to $(X, \hat{Y})$, it is enough to show that $Y^\perp$ and $(X, \hat{Y})$ do never jump together. Moreover, in view of (E.1), it suffices to show that $\widetilde{\mathbb{Q}}\big(\exists\, t > 0 | \Delta Y_t^\perp \neq 0 \text{ and } \Delta X_t \neq 0\big) = 0$. Let $\mathfrak{T}$ be the set of jump times of X. Since X is càdlàg, the set $\mathfrak{T}$ is countable (see e.g. [38, Proposition I.1.32]) and, similarly as in [41, Theorem 4.7],

$$\widetilde{\mathbb{Q}}\big(\exists\, t > 0 | \Delta Y_t^\perp \neq 0 \text{ and } \Delta X_t \neq 0\big) \leq \mathbb{E}^{\widetilde{\mathbb{Q}}}\left[\sum_{t\in\mathfrak{T}} \mathbf{1}_{\{\Delta Y_t^\perp \neq 0\}}\right] = \mathbb{E}^{\widetilde{\mathbb{Q}}}\left[\sum_{t\in\mathfrak{T}} \mathbb{E}^{\widetilde{\mathbb{Q}}}[\mathbf{1}_{\{\Delta Y_t^\perp \neq 0\}} | \mathcal{F}_\infty]\right] = 0,$$

where the last equality follows from the fact that $\mathbb{E}^{\widetilde{\mathbb{Q}}}[\mathbf{1}_{\{\Delta Y_t^\perp \neq 0\}} | \mathcal{F}_\infty] = 0$ for all $t > 0$, since, due to Proposition 2.32, the jump measure of $Y^\perp$ with respect to the larger filtration $(\overline{\mathcal{G}}_t)_{t\geq 0}$ (which satisfies $\overline{\mathcal{G}}_0 = \mathcal{F}_\infty \otimes \{\emptyset, \Omega'\}$) is absolutely continuous with respect to the Lebesgue measure, so that $Y^\perp$ does not have any fixed time of discontinuity (see e.g. [38, Lemma II.2.54]).

In order to prove the second part of the lemma, note that condition (2.33) implies that condition $I(0,1)$ from [42] is satisfied, since, for all $i = 1, \dots, m$, $y \in \mathbb{R}_+$ and $T \geq 0$,

$$\begin{aligned}
&\sup_{t\in[0,T]} \mathbb{E}^{\widetilde{\mathbb{Q}}}\left[\exp\left(\int_0^t \int \big(e^{u_i\xi}(u_i\xi - 1) + 1\big) K_t\big(\omega, Y_{t-}^\perp(\omega,\omega'), d\xi\big) dt\right)\right] \\
&\leq \sup_{t\in[0,T]} \mathbb{E}^{\widetilde{\mathbb{Q}}}\left[\exp\left((1+u_m)\int_0^t \int g_{m+1}(\xi) K_t\big(\omega, Y_{t-}^\perp(\omega,\omega'), d\xi\big) dt\right)\right] \\
&= \sup_{t\in[0,T]} \mathbb{E}^{\widetilde{\mathbb{Q}}}\left[\exp\left((1+u_m)\int_0^t p_t^{m+1}\big(\omega, Y_{t-}^\perp(\omega,\omega')\big) dt\right)\right] \leq e^{(1+u_m)T\bar{H}} < \infty.
\end{aligned}$$

Moreover, condition (2.33) can be easily shown to imply that $\int_0^T \int |\xi e^{u_i\xi} - \xi| K_t\big(\omega, Y_{t-}^\perp(\omega,\omega'), d\xi\big) dt$ is $\widetilde{\mathbb{Q}}$-a.s. finite for all $T \geq 0$. Hence, [42, Theorem 3.2] implies that $\big(\exp(u_i Y_t^\perp - \int_0^t \Psi_s^{Y^\perp}(u_i) ds)\big)_{t\in[0,T]}$ is a uniformly integrable $\big(\widetilde{\mathbb{Q}}, (\overline{\mathcal{G}}_t)_{t\geq 0}\big)$-martingale, for all $i = 1, \dots, m$. In turn, since $T \geq 0$ is arbitrary, this proves the $\big(\widetilde{\mathbb{Q}}, (\overline{\mathcal{G}}_t)_{t\geq 0}\big)$-martingale property of $\big(\exp(u_i Y_t^\perp - \int_0^t \Psi_s^{Y^\perp}(u_i) ds)\big)_{t\geq 0}$, for all $i = 1, \dots, m$. Finally, since the latter process is $(\mathcal{G}_t)$-adapted, it is also a martingale in the smaller filtration $(\mathcal{G}_t)_{t\geq 0}$. □

We are now in a position to prove the following result, which shows that, on the extended filtered probability space $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, \widetilde{\mathbb{Q}})$, step (c) of the model construction procedure described at the beginning of Section 2.4 can be successfully achieved and, hence, the three requirements of part (iii) of Theorem 2.17 are satisfied.

Theorem 2.35. *Suppose that Assumption 2.30 holds and let the process $Y^\perp$ be constructed as in Proposition 2.32. Then, on the extended filtered probability space $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, \widetilde{\mathbb{Q}})$, the process $Y := \hat{Y} + Y^\perp$ is compatible with the building blocks $(X, \hat{Y}, u_1, \dots, u_m, f_0, \eta_0^1, \dots, \eta_0^m, \widetilde{\sigma}, \sigma^1, \dots, \sigma^m)$, in the sense of Definition 2.24.*

Proof. Due to Lemma 2.33, the local exponent of $\hat{Y}$ with respect to the extended filtered probability space $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, \widetilde{\mathbb{Q}})$ is still given by $\Psi^{\hat{Y}}$ and $Y^\parallel = (\hat{Y})^\parallel = \hat{Y}$. Since $Y^\perp$ and $Y - Y^\perp = \hat{Y}$ are locally independent (see Lemma 2.34), the consistency condition (2.13) directly follows from condition (2.32).

In order to prove the martingale property of the process given in equation (2.12), note first that the $\big(\widetilde{\mathbb{Q}}, (\mathcal{G}_t)_{t\geq 0}\big)$-martingale property of the process $\Big(\exp\big(u_i Y_t^\perp - \int_0^t \Psi_s^{Y^\perp}(u_i) ds\big)\Big)_{t\geq 0}$ (see Lemma 2.34), together with Lemma 2.33, condition (iv) in Definition 2.23 and the local independence of $Y^\perp$ and $(X, \hat{Y})$ on $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, \widetilde{\mathbb{Q}})$, implies that the process given in equation (2.12) is a $\big(\widetilde{\mathbb{Q}}, (\mathcal{G}_t)_{t\geq 0}\big)$-local martingale, for all $i = 1, \dots, m$. Being a non-negative local martingale, it is also a supermartingale by Fatou's lemma.

Hence, to establish the true martingale property, it suffices to observe that, since $\overline{\mathcal{G}}_0 = \mathcal{F}_\infty \otimes \{\emptyset, \Omega'\}$, it holds that, for all $T \geq 0$ and $i = 1, \dots, m$,

$$
\begin{aligned}
&\mathbb{E}^{\widetilde{\mathbb{Q}}}\left[\exp\left(u_i Y_T + \int_0^T \big(\Sigma_s^i(T) - \widetilde{\Sigma}_s(T)\big) dX_s - \int_0^T \Psi_s^{Y,X}\left(\big(u_i, \Sigma_s^{i\top}(T) - \widetilde{\Sigma}_s^\top(T)\big)^\top\right) ds\right)\right] \\
&= \mathbb{E}^{\widetilde{\mathbb{Q}}}\left[\exp\left(u_i \hat{Y}_T + \int_0^T \big(\Sigma_s^i(T) - \widetilde{\Sigma}_s(T)\big) dX_s - \int_0^T \Psi_s^{\hat{Y},X}\left(\big(u_i, \Sigma_s^{i\top}(T) - \widetilde{\Sigma}_s^\top(T)\big)^\top\right) ds\right)\right. \\
&\qquad \left.\exp\left(u_i Y_T^\perp - \int_0^T \Psi_s^{Y^\perp}(u_i)\, ds\right)\right] \\
&= \mathbb{E}^{\widetilde{\mathbb{Q}}}\left[\exp\left(u_i \hat{Y}_T + \int_0^T \big(\Sigma_s^i(T) - \widetilde{\Sigma}_s(T)\big) dX_s - \int_0^T \Psi_s^{\hat{Y},X}\left(\big(u_i, \Sigma_s^{i\top}(T) - \widetilde{\Sigma}_s^\top(T)\big)^\top\right) ds\right)\right. \\
&\qquad \left.\mathbb{E}^{\widetilde{\mathbb{Q}}}\left[\exp\left(u_i Y_T^\perp - \int_0^T \Psi_s^{Y^\perp}(u_i)\, ds\right) \Big| \overline{\mathcal{G}}_0\right]\right] \\
&= \mathbb{E}^{\widetilde{\mathbb{Q}}}\left[\exp\left(u_i \hat{Y}_T + \int_0^T \big(\Sigma_s^i(T) - \widetilde{\Sigma}_s(T)\big) dX_s - \int_0^T \Psi_s^{\hat{Y},X}\left(\big(u_i, \Sigma_s^{i\top}(T) - \widetilde{\Sigma}_s^\top(T)\big)^\top\right) ds\right)\right] e^{u_i y_0} \\
&= \mathbb{E}^{\mathbb{Q}}\left[\exp\left(u_i \hat{Y}_T + \int_0^T \big(\Sigma_s^i(T) - \widetilde{\Sigma}_s(T)\big) dX_s - \int_0^T \Psi_s^{\hat{Y},X}\left(\big(u_i, \Sigma_s^{i\top}(T) - \widetilde{\Sigma}_s^\top(T)\big)^\top\right) ds\right)\right] e^{u_i y_0} \\
&= \mathbb{E}^{\mathbb{Q}}\big[\exp(u_i Y_0)\big],
\end{aligned}
$$

where in the last two equalities we have used the fact that $\widetilde{\mathbb{Q}}|_{\mathcal{F}} = \mathbb{Q}$ and the $\big(\mathbb{Q}, (\mathcal{F}_t)_{t\geq 0}\big)$-martingale property of (2.16). □

Remark 2.36. We want to point out that the present construction can be rather easily extended in order to include non-null drift and diffusion components in the process $Y^\perp$, by adapting the proof of Proposition 2.32 in the spirit of [17, Theorem A.4], under suitable hypotheses on the diffusion component so that the local independence as well as the martingale property are ensured. In a similar way, the requirement that the process $Y^\perp$ be of finite activity can also be relaxed.

2.5. **General pricing formulae.** In this section, we present general pricing formulas for typical interest rate derivatives. As we are going to show, the quantity $S^\delta(t,T)$ plays a pivotal role in the valuation of interest rate products. We here derive clean prices in the spirit of Appendix A assuming perfect collateralization and the collateral rate equal to the OIS rate.

2.5.1. *Linear products.* The prices of linear interest rate products (i.e., without optionality features) can be directly expressed in terms of the basic quantities $B(t,T)$ and $S^\delta(t,T)$.

Forward rate agreement.
A forward rate agreement (FRA) starting at T, with maturity $T+\delta$, fixed rate K and notional N is a contract which pays at time $T+\delta$ the following amount

$$\Pi^{FRA}(T+\delta; T, T+\delta, K, N) = N\delta\big(L_T(T, T+\delta) - K\big).$$

The value of such a claim at time $t \leq T$ is

$$
\begin{aligned}
\Pi^{FRA}(t; T, T+\delta, K, N) &= NB(t, T+\delta)\delta\, \mathbb{E}^{\mathbb{Q}^{T+\delta}}\left[L_T(T, T+\delta) - K \,|\, \mathcal{F}_t\right] \\
&= NB(t, T+\delta)\mathbb{E}^{\mathbb{Q}^{T+\delta}}\left[1 + \delta L_T(T, T+\delta) - (1+\delta K) \,|\, \mathcal{F}_t\right] \\
&= N\big(B(t,T)S^\delta(t,T) - B(t, T+\delta)(1+\delta K)\big).
\end{aligned} \tag{2.35}
$$

In particular, observe that the time t value of the Libor rate will differ from its pre-crisis replication value in terms of risk-free zero-coupon bonds, i.e.,

$$L_t(T, T+\delta) = \frac{1}{\delta}\left(\frac{B(t,T)}{B(t,T+\delta)} S^\delta(t,T) - 1\right) \neq \frac{1}{\delta}\left(\frac{B(t,T)}{B(t,T+\delta)} - 1\right).$$

Overnight indexed swap.
An overnight indexed swap (OIS) is a contract where two counterparties exchange two streams of payments: the first one is computed with respect to a fixed rate K, whereas the second one is indexed by

an overnight rate (EONIA). Let us denote by $T_1, \dots, T_n$ the payment dates, with $T_{i+1} - T_i = \delta$ for all $i = 1, \dots, n-1$. The swap is initiated at time $T_0 \in [0, T_1)$. The value of the OIS at time $t \leq T_0$, with notional N, can be expressed as (see e.g. [26, Section 2.5])

$$\Pi^{OIS}(t; T_1, T_n, K, N) = N \left(B(t, T_0) - B(t, T_n) - K\delta \sum_{i=1}^{n} B(t, T_i) \right).$$

Therefore, the OIS rate K^{OIS}, which is by definition the value for K such that the OIS contract has zero value at inception, is given by

$$K^{OIS}(T_1, T_n) = \frac{B(t, T_0) - B(t, T_n)}{\delta \sum_{k=1}^{n} B(t, T_k)}.$$

Interest rate swap.
In an interest rate swap (IRS), two streams of payments are exchanged between two counterparties: the first cash flow is computed with respect to a fixed rate K, whereas the second one is indexed by the prevailing Libor rate. Using the same tenor structure as in the previous section, at each time T_i the value for the receiver of the floating rate is

$$\Pi^{IRS}(T_i; T_{i-1}, T_i, K, N) = N\delta\big(L_{T_{i-1}}(T_{i-1}, T_i) - K\big), \qquad \text{for } i = 1, \dots, n.$$

The value of the IRS at time $t \leq T_0$, where T_0 denotes the inception time, is given by

$$\begin{aligned} \Pi^{IRS}(t; T_1, T_n, K, N) &= N \sum_{i=1}^{n} B(t, T_i)\delta\, \mathbb{E}^{\mathbb{Q}^{T_i}} \left[L_{T_{i-1}}(T_{i-1}, T_i) - K \middle| \mathcal{F}_t \right] \\ &= N \sum_{i=1}^{n} \left(B(t, T_{i-1}) S^{\delta}(t, T_{i-1}) - B(t, T_i)(1 + \delta K) \right). \end{aligned}$$

The swap rate K^{IRS}, which is by definition the value for K such that the contract has zero value at inception, is given by

$$K^{IRS}(T_1, T_n, \delta) = \frac{\sum_{i=1}^{n} \big(B(t, T_{i-1}) S^{\delta}(t, T_{i-1}) - B(t, T_i) \big)}{\delta \sum_{i=1}^{n} B(t, T_i)} = \frac{\sum_{i=1}^{n} B(t, T_i) L_t(T_{i-1}, T_1)}{\sum_{i=1}^{n} B(t, T_i)}.$$

Basis swap.
A basis swap is a special type of interest rate swap where two cash flows related to Libor rates associated to different tenors are exchanged between two counterparties. For instance, a typical basis swap may involve the exchange of the 3-month against the 6-month Libor rate. Following the standard convention for the definition of a basis swap in the Euro market (see [1]), the basis swap is equivalent to a long/short position on two different interest rate swaps which share the same fixed leg. Let $\mathcal{T}^1 = \left\{ T_0^1, \cdots, T_{n_1}^1 \right\}$, $\mathcal{T}^2 = \left\{ T_0^2, \cdots T_{n_2}^2 \right\}$ and $\mathcal{T}^3 = \left\{ T_0^3, \cdots, T_{n_3}^3 \right\}$, with $T_{n_1}^1 = T_{n_2}^2 = T_{n_3}^2$, $\mathcal{T}^1 \subset \mathcal{T}^2$, $n_1 < n_2$ and corresponding tenor lengths $\delta_1 > \delta_2$, with no constraints on δ_3. The first two tenor structures on the one side and the third on the other are associated to the two floating and to the single fixed leg, respectively. As usual, we denote by N the notional of the swap, which is initiated at time $T_0^1 = T_0^2 = T_0^3$. The value at time

$t \le T_0^1$ can be computed as

$$\begin{aligned}
&\Pi^{BSW}(t;\mathcal{T}^1,\mathcal{T}^2,\mathcal{T}^3,N)\\
&= N\left(\sum_{i=1}^{n_1}\delta^1 B(t,T_i^1)\mathbb{E}^{\mathbb{Q}^{T_i^1}}\left[L_{T_{i-1}^1}(T_{i-1}^1,T_i^1)\Big|\mathcal{F}_t\right]\right.\\
&\quad\left.-\sum_{j=1}^{n_2}\delta^2 B(t,T_j^2)\mathbb{E}^{\mathbb{Q}^{T_j^2}}\left[L_{T_{j-1}^2}(T_{j-1}^2,T_j^2)\Big|\mathcal{F}_t\right]-K\sum_{\ell=1}^{n_3}\delta^3 B(t,T_\ell^3)\right)\\
&= N\left(\sum_{i=1}^{n_1}\big(B(t,T_{i-1}^1)S^{\delta^1}(t,T_{i-1}^1)-B(t,T_i^1)\big)\right.\\
&\quad\left.-\sum_{j=1}^{n_2}\big(B(t,T_{j-1}^2)S^{\delta^2}(t,T_{j-1}^2)-B(t,T_j^2)\big)-K\sum_{\ell=1}^{n_3}\delta^3 B(t,T_\ell^3)\right)
\end{aligned}$$

The value K^{BSW} (called *basis swap spread*) such that the value of the contract at initiation is zero is then given by

$$K^{BSW}(\mathcal{T}^1,\mathcal{T}^2,\mathcal{T}^3)=\frac{\sum_{i=1}^{n_1}\big(B(t,T_{i-1}^1)S^{\delta^1}(t,T_{i-1}^1)-B(t,T_i^1)\big)-\sum_{j=1}^{n_2}\big(B(t,T_{j-1}^2)S^{\delta^2}(t,T_{j-1}^2)-B(t,T_j^2)\big)}{\delta_3\sum_{\ell=1}^{n_3}B(t,T_\ell^3)}.$$

It is interesting to observe that, prior to the financial crisis, the value of K^{BSW} used to be (approximately) zero.

2.5.2. *Products with optionality features.* In this section, we report general valuation formulas for plain vanilla interest rate products such as European caps/swaptions. As such products are typically used as calibration instruments, the necessity of tractable valuation formulas is essential. In Section 3, by relying on affine processes, we will tackle this issue.

Caplet.
The price at time t of a caplet with strike price K, maturity T, settled in arrears at $T+\delta$, is given by

$$\Pi^{CPLT}(t;T,T+\delta,K,N)=NB(t)\delta\,\mathbb{E}\left[\frac{1}{B(T+\delta)}\Big(L_T(T,T+\delta)-K\Big)^+\Big|\mathcal{F}_t\right]$$

This valuation formula above admits the following representation in terms of $S^\delta(t,T)$.

$$\begin{aligned}
\Pi^{CPLT}(t;T,T+\delta,K,N)&=N\mathbb{E}\left[\frac{B(t)}{B(T+\delta)}\left(\frac{S^\delta(T,T)}{B(T,T+\delta)}-(1+\delta K)\right)^+\Big|\mathcal{F}_t\right]\\
&=N\mathbb{E}\left[\frac{B(t)}{B(T)}\Big(S^\delta(T,T)-(1+\delta K)B(T,T+\delta)\Big)^+\Big|\mathcal{F}_t\right] \qquad (2.36)
\end{aligned}$$

Remark 2.37. Note that the valuation formula (2.36) in the classical single curve setting (i.e., under the assumption that $S^\delta(T,T)$ is identically equal to one), reduces to the classical relationship between a caplet and a put option on a zero-coupon bond with strike $1/(1+\delta K)$.

Swaption.
We consider two types of swaptions. The first one is a standard European payer swaption with maturity T, written on a (payer) interest rate swap starting at $T_0=T$ and payment dates $T_1,...,T_n$, with $T_{i+1}-T_i=\delta$ for all $i=1,\dots,n-1$, with notional N. The value of such a claim at time t is given by

(2.37)

$$\begin{aligned}
\Pi^{SWPTN}(t;T_1,T_n,K,N)&=N\mathbb{E}\left[\frac{B(t)}{B(T)}\left(\sum_{i=1}^{n}\delta B(T,T_i)\big(L_T(T_{i-1},T_i)-K\big)\right)^+\Big|\mathcal{F}_t\right]\\
&=N\mathbb{E}\left[\frac{B(t)}{B(T)}\left(\sum_{i=1}^{n}B(T,T_{i-1})S^\delta(T,T_{i-1})-(1+\delta K)B(T,T_i)\right)^+\Big|\mathcal{F}_t\right].
\end{aligned}$$

In a similar way, European *basis swaptions* can also be introduced. The time t price of a European basis swaption with maturity T written on a basis swap starting at $T = T_0^1 = T_0^2 = T_0^3$ can be written as

$$\begin{aligned}
&\Pi^{BSWPTN}(t;\mathcal{T}^1,\mathcal{T}^2,\mathcal{T}^3,K,N)\\
&= N\mathbb{E}\left[\frac{B(t)}{B(T)}\left(\sum_{i=1}^{n_1}\big(B(t,T_{i-1}^1)S^{\delta^1}(t,T_{i-1}^1)-B(t,T_i^1)\big)\right.\right.\\
&\left.\left.-\sum_{j=1}^{n_2}\big(B(t,T_{j-1}^2)S^{\delta^2}(t,T_{j-1}^2)-B(t,T_j^2)\big)-K\sum_{\ell=1}^{n_3}\delta^3 B(t,T_\ell^3)\right)^+\Big|\mathcal{F}_t\right]
\end{aligned} \tag{2.38}$$

3. Models based on affine processes

In this section, we propose a flexible and tractable specification of the general framework of Section 2 based on affine processes. We start by recalling some general results on affine processes in Section 3.1, by relying mainly on [21], [17] and [44]. In Section 3.2, we present the affine model specification. Then, in Section 3.3, we derive a general semi-closed valuation formula for caplets and an approximation formula for the price of a swaption. Section 3.4 briefly presents a possible deterministic shift extension and Section 3.5 contains two basic examples of the affine model specification.

3.1. Preliminaries on affine processes.

Let V be a real Euclidean vector space with associated scalar product $\langle\cdot,\cdot\rangle$. We denote by D a closed convex subset of V which will serve as state space for a stochastic process, endowed with the Borel sigma algebra $\mathcal{B}(D)$. For the set D and for $d \in \mathbb{N}$ we may choose e.g., $D \in \{\mathbb{R}_+^d, S_+^d\}$ or combinations thereof, where S_+^d denotes the cone of symmetric $d \times d$ positive semidefinite matrices. Depending on whether $D = \mathbb{R}_+^d$ or $D = S_+^d$, the scalar product will be given by

$$\langle x, y\rangle = \begin{cases} x^\top y & \text{if } x, y \in \mathbb{R}_+^d,\\ \mathrm{Tr}[xy] & \text{if } x, y \in S_+^d, \end{cases}$$

where Tr denotes the trace operator.

Let $\mathbf{T}$ be a fixed time horizon. On a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le \mathbf{T}}, \mathbb{Q})$, we introduce a stochastic process $\mathcal{X} = (\mathcal{X}_t)_{0\le t\le \mathbf{T}}$ with initial state $\mathcal{X}_0 = x$, which is assumed to be a càdlàg, adapted, time-homogeneous and conservative Markov process. We associate to the state space D the set

$$\mathfrak{U} := \big\{ u \in V + \mathrm{i}V \,|\, \mathbb{E}\big[e^{\langle u, \mathcal{X}_t\rangle}\big] < \infty, \forall t \in [0, \mathbf{T}]\big\}$$

and we define the class the process $\mathcal{X}$ belongs to as follows.

Definition 3.1. *(Affine Markov Process) The Markov Process $\mathcal{X}$ is called affine if*

(i) *it is stochastically continuous, i.e., the transition kernels satisfy, for every $t \ge 0$ and $x \in D$, $\lim_{s\to t} p_s(x,\cdot) = p_t(x,\cdot)$ weakly on D,*

(ii) *its Fourier-Laplace transform has exponential-affine dependence on the initial state i.e., there exists functions $\phi : \mathbb{R}_+ \times \mathfrak{U} \to \mathbb{C}$ and $\phi : \mathbb{R}_+ \times \mathfrak{U} \to V + \mathrm{i}V$ such that, $\forall x \in D$ and $(t,u) \in \mathbb{R}_+ \times \mathfrak{U}$*

$$\mathbb{E}\big[e^{\langle u, \mathcal{X}_t\rangle}\big] = \int_D e^{\langle u,\xi\rangle} p_t(x, d\xi) = e^{\phi(t,u) + \langle \psi(t,u), x\rangle}. \tag{3.1}$$

Remark 3.2. The above definition slightly differs from the usual definition of affine processes where the exponential affine form is only assumed to hold on the set of bounded exponentials. The question when it can be extended to the set $\mathfrak{U}$ in the case of convex state spaces is treated in [44]. Note also that, in view of [44, Lemma 4.2], as long as $\mathcal{X}$ is an affine process, it holds that $\mathfrak{U} = \big\{u \in V + \mathrm{i}V \,|\, \mathbb{E}\big[e^{\langle u, \mathcal{X}_{\mathbf{T}}\rangle}\big] < \infty\big\}$.

The affine property (3.1) in conjunction with the Chapman-Kolmogorov equation implies that the functions ϕ, ψ enjoy the *semiflow property*, i.e., for $u \in \mathfrak{U}$ and $t, s \in \mathbb{R}_+$,

$$\begin{aligned}
\phi(t+s,u) &= \phi(t,u) + \phi\big(s, \psi(t,u)\big),\\
\psi(t+s,u) &= \psi\big(s, \psi(t,u)\big).
\end{aligned} \tag{3.2}$$

The stochastic continuity of $\mathcal{X}$, according to [46], [17] (see also [18] for general state spaces) implies that the process is also *regular*, in the sense that the derivatives

$$F(u) = \left.\frac{\partial \phi(t,u)}{\partial t}\right|_{t=0+} \quad \text{and} \quad R(u) = \left.\frac{\partial \psi(t,u)}{\partial t}\right|_{t=0+}$$

exist and are continuous at $u = 0$. Regularity implies that one can differentiate the semiflow relations (3.2), thus obtaining the following system of *generalized Riccati ODEs*

$$\frac{\partial \phi(t,u)}{\partial t} = F\big(\psi(t,u)\big), \quad \phi(0,u) = 0, \tag{3.3}$$

$$\frac{\partial \psi(t,u)}{\partial t} = R\big(\psi(t,u)\big), \quad \psi(0,u) = u. \tag{3.4}$$

Let us now relate the class of affine processes with general semimartingales. According to [16, Theorem 1.4.8 and 1.5.4], the affine process $\mathcal{X}$ is a semimartingale with characteristics (with respect to some truncation function χ)

$$B_t = \int_0^t b(\mathcal{X}_{s-})ds, \tag{3.5a}$$

$$C_t = \int_0^t c(\mathcal{X}_{s-})ds, \tag{3.5b}$$

$$\nu(\omega, dt, d\xi) = K(\mathcal{X}_{t-}(\omega), d\xi)dt, \tag{3.5c}$$

where $b(x)$, $c(x)$ and $K(x, d\xi)$ are affine functions of the form

$$b(x) = b + B(x), \tag{3.6a}$$

$$c(x) = c + C(x), \tag{3.6b}$$

$$K(x, d\xi) = m(d\xi) + M(x, d\xi). \tag{3.6c}$$

Hence, recalling Definition 2.3 and Proposition 2.4, we can restate the analogous of [44, Proposition 2.8].

Proposition 3.3. *Let $\mathcal{X}$ be an affine process with state space D, then the local exponent $\Psi^{\mathcal{X}}$ admits the following representation*

$$\Psi_s^{\mathcal{X}}(u) = F(u) + \langle R(u), \mathcal{X}_s\rangle \tag{3.7}$$

for all $x \in D$, $s \geq 0$ and $u \in \mathfrak{U}$. Moreover, for some truncation function χ, the functions F and R are of the form

$$F(u) = \frac{1}{2}\langle u, cu\rangle + \langle b, u\rangle + \int_{D\setminus\{0\}} \Big(e^{\langle u,\xi\rangle} - 1 - \langle u, \chi(\xi)\rangle\Big)\, m(d\xi), \tag{3.8}$$

$$\langle R(u), x\rangle = \frac{1}{2}\langle u, C(x)u\rangle + \langle B(x), u\rangle + \int_{D\setminus\{0\}} \Big(e^{\langle u,\xi\rangle} - 1 - \langle u, \chi(\xi)\rangle\Big)\, M(x, d\xi). \tag{3.9}$$

To shed more light on the structure of the linear maps B, C, M in (3.6a)-(3.6c), we further specialize the function R.

- For $D = \mathbb{R}^d_+$, we have $R(u) = (R_1(u), ..., R_d(u))^\top \in \mathbb{C}^d$ and
$$R_i(u) = \frac{1}{2}\langle u, \alpha_i u\rangle + \langle \beta_i, u\rangle + \int_{\mathbb{R}^d_+\setminus\{0\}} \Big(e^{\langle u,\xi\rangle} - 1 - \langle u, \chi(\xi)\rangle\Big)\, \mu_i(d\xi)$$
- For $D = S^d_+$, we have $R(u) \in S^d \times \mathrm{i}S^d$ and
$$R(u) = 2u\alpha u + B^\top(u) + \int_{S^d_+\setminus\{0\}} \Big(e^{\langle u,\xi\rangle} - 1 - \langle u, \chi(\xi)\rangle\Big)\, \mu(d\xi)$$
with $B(x) = \sum_{i,j} \beta^{ij} x_{ij}$.

The family of parameters $(a, \alpha, b, \beta, m, \mu)$ is termed *admissible parameter set*. For the specific form of the parameters and necessary and sufficient restrictions for $D \in \{\mathbb{R}^d_+, S^d_+\}$ see [21], [17] and [55].

Before we proceed to present the affine specification of the multi-curve model, we report the following result (see [43, Thm. 4.10]), which will be useful in the sequel.

Lemma 3.4. *Consider an affine process $\mathcal{X} = \big(\mathcal{X}^1, \mathcal{X}^2\big)$ with initial state $\mathcal{X} = \big(x^1, x^2\big)$ on some mixed state space $D = D^1 \times D^2$ such that the characteristics of $\mathcal{X}^2$ only depend on $\mathcal{X}^1$. Then, for $u = (u_1, u_2) \in \mathfrak{U}$*

$$\mathbb{E}\left[e^{\langle u_1, \mathcal{X}^1_t\rangle + \langle u_2, \mathcal{X}^2_t\rangle}\right] = e^{\phi(t,u_1,u_2) + \langle \psi(t,u_1,u_2), x^1\rangle + \langle u_2, x^2\rangle}.$$

Typical examples where affine processes such as those from Lemma 3.4 are employed, are provided by affine stochastic volatility models, where the characteristics of the log-stock price only depend on those of the instantaneous variance, and short rate models where we consider integrated processes, see e.g. [16].

3.2. **An affine model specification.** We consider an affine process $\mathcal{X} = (X, Y, Z)$, with state space $D = D_X \times \mathbb{R}^{n+1}$, i.e., X takes values on D_X and (Y, Z) in $\mathbb{R}^{n+1}$ (or a suitable subspace thereof). We also require the additional property that the characteristics of (Y, Z) only depend on X, in line with Lemma 3.4. Intuitively, the process X will represent the general driving process from Section 2, whereas the process Y and Z will be related to the log-spot spreads and to the risk-free bank account, respectively. By Lemma 3.4, the joint characteristic function of $\mathcal{X} = (X, Y, Z)$, for $(v, u, w) \in \mathfrak{U}$ will be given by

$$\mathbb{E}\left[e^{\langle v, X_T\rangle + \langle u, Y_T\rangle + wZ_T}\middle|\mathcal{F}_t\right] = e^{\phi(T-t,v,u,w) + \langle \psi(T-t,v,u,w), X_t\rangle + \langle u, Y_t\rangle + wZ_t}.$$

Definition 3.5. *An* affine multiple yield curve model *is defined via*

(i) *an affine process $\mathcal{X} = (X, Y, Z)$, with X and (Y, Z) taking values in D_X and in $\mathbb{R}^{n+1}$, respectively, such that the characteristics of (Y, Z) only depend on X. In particular, we let*

$$Z := -\int_0^{\cdot} r_s ds := -\int_0^{\cdot} \big(l + \langle \lambda, X_s\rangle\big) ds, \tag{3.10}$$

for some $l \in \mathbb{R}$ and $\lambda \in \mathbb{R}^n$. Moreover, for some vectors $0 = u_0, u_1, \dots, u_m \in \mathbb{R}^n$ we assume $(0, u_i, 1) \in \mathfrak{U}$, for all $i = 0, \dots, m$,

(ii) *a risk-free bank account, satisfying $B(t) = e^{-Z_t} = e^{\int_0^t r_s ds}$, for all $t \geq 0$,*

(iii) *a family $\big\{\big(B(t,T)\big)_{t\in[0,T]}, T \geq 0\big\}$ of risk-free zero-coupon bond price processes satisfying*

$$B(t,T) = \mathbb{E}\left[B(t)/B(T)|\mathcal{F}_t\right] = e^{\phi(T-t,0,0,1) + \langle \psi(T-t,0,0,1), X_t\rangle}, \qquad \text{for all } t \leq T \text{ and } T \geq 0, \tag{3.11}$$

(iv) *a family $\big\{\big(S^{\delta_i}(t,T)\big)_{t\in[0,T]}, T \geq 0, \delta_i \in \mathcal{D}\big\}$ of multiplicative spreads satisfying*

$$S^{\delta_i}(t,T) := \frac{\mathbb{E}\left[e^{Z_T + u_i^\top Y_T}\middle|\mathcal{F}_t\right]}{\mathbb{E}\left[e^{Z_T}|\,\mathcal{F}_t\right]} = e^{u_i^\top Y_t + \phi(T-t,0,u_i,1) - \phi(T-t,0,0,1) + \langle \psi(T-t,0,u_i,1) - \psi(T-t,0,0,1), X_t\rangle}, \tag{3.12}$$

for all $t \leq T$, $T \geq 0$ and $i = 1, \dots, m$, for the vectors $u_1, \dots, u_m \in \mathbb{R}^n$.

Remark 3.6. In line with Remark 2.21 and in analogy to the bank account, Y can be specified as $Y = \int_0^{\cdot} q(X_s) ds$, where $q : D_X \to \mathbb{R}^n$ here denotes an affine function.

In particular, note that if Y lies in some cone $C \subset \mathbb{R}^n$ and $u_1, \dots u_m$ satisfy the requirements of Corollary 2.19, the multiplicative spreads are greater than 1 and ordered with respect to the tenor's length. Besides the modeling flexibility and tractability ensured by affine processes, this represents one of the main advantages of the specification (3.12) of spreads.

The above definition of affine multiple yield curve model can be directly mapped into the general setup of HJM-type multiple yield curve models from Section 2.3 via the following proposition, which also shows that the risk neutral property (see Definition 2.16) is satisfied by construction.

Proposition 3.7. *Every affine multiple yield curve model is a risk neutral HJM-type multiple yield curve model where*

(i) *the driving process is X,*

(ii) *the bank account is given by $B(t) = e^{-Z_t}$, for all $t \geq 0$,*

(iii) *the log-spot spread is given by $\log S^{\delta_i}(t,t) = u_i^\top Y_t$, for all $i = 1, \dots, m$ and $t \geq 0$,*

(iv) *the forward rate $f_t(T)$ and the forward spread rates $\eta_t^i(T)$ are given by*

$$f_t(T) = -F\big(\psi(T-t,0,0,1),0,1\big) - \big\langle R\big(\psi(T-t,0,0,1),0,1\big), X_t\big\rangle, \tag{3.13}$$

$$\begin{aligned}\eta_t^i(T) = {} & F\big(\psi(T-t,0,u_i,1),u_i,1\big) - F\big(\psi(T-t,0,0,1),0,1\big) \\ & + \big\langle R\big(\psi(T-t,0,u_i,1),u_i,1\big) - R\big(\psi(T-t,0,0,1),0,1\big), X_t\big\rangle,\end{aligned} \tag{3.14}$$

for all $t \leq T$, $T \geq 0$ and $i = 1, \dots, m$.

Proof. The first three claims follow upon direct inspection. The expressions for $f_t(T)$ and $\eta_t^i(T)$ are obtained from (3.11) and (3.12) by simply noting that

$$\begin{aligned}-\int_t^T f_t(s) ds &= \phi(T-t,0,0,1) + \langle \psi(T-t,0,0,1), X_t\rangle, \\ \int_t^T \eta_t(s) ds &= \phi(T-t,0,u_i,1) - \phi(T-t,0,0,1) + \langle \psi(T-t,0,u_i,1) - \psi(T-t,0,0,1), X_t\rangle\end{aligned}$$

and by differentiating both sides. The risk neutral property is a direct consequence of the definition of $B(t,T)$ and $S^{\delta_i}(t,T)$. □

3.3. **Pricing of interest rate derivatives.** We now show that affine multiple yield curve models, in the sense of Definition 3.5, lead to tractable general valuation formulae for caplets and swaptions. For simplicity of presentation, we shall consider a fixed tenor δ and a fixed maturity $T > 0$.

3.3.1. *Caplets.* In the present affine setting, caplets can be easily priced by means of Fourier techniques. As a preliminary, let us introduce the stochastic process $(\mathcal{Y}_t)_{t\in[0,T+\delta]}$ defined by

$$\mathcal{Y}_t := u_i^\top Y_t - \phi(T+\delta-t,0,0,1) - \langle \psi(T+\delta-t,0,0,1), X_t\rangle, \qquad \text{for } 0 \le t \le T+\delta.$$

We also denote by $\mathcal{A}_{\mathcal{Y}_T}$ the interior of the set

$$\left\{\nu \in \mathbb{R} : \ \mathbb{E}\left[\frac{B(T,T+\delta)}{B(T)} e^{\nu \mathcal{Y}_T}\middle| \mathcal{F}_t\right] < \infty\right\},$$

and introduce the strip of complex numbers $\Lambda_{\mathcal{Y}_T} = \{\zeta \in \mathbb{C} : \ -\mathrm{Im}(\zeta) \in \mathcal{A}_{\mathcal{Y}_T}\}$. For $\zeta \in \Lambda_{\mathcal{Y}_T}$, we can easily compute the following expectation, which may be informally interpreted as the discounted characteristic function of $\mathcal{Y}_T$:

$$\begin{aligned}\varphi_{\mathcal{Y}_T}(\zeta) &:= \mathbb{E}\left[\frac{B(T,T+\delta)}{B(T)} e^{\mathsf{i}\zeta \mathcal{Y}_T}\middle| \mathcal{F}_t\right] \\ &= e^{(1-\mathsf{i}\zeta)\phi(\delta,0,0,1)} \mathbb{E}\left[e^{Z_T + \mathsf{i}\zeta u_i^\top Y_T + (1-\mathsf{i}\zeta)\langle \psi(\delta,0,0,1), X_T\rangle}\middle| \mathcal{F}_t\right] \\ &= e^{(1-\mathsf{i}\zeta)\phi(\delta,0,0,1) + Z_t + \mathsf{i}\zeta u_i^\top Y_t} e^{\phi(T-t,(1-\mathsf{i}\zeta)\psi(\delta,0,0,1),\mathsf{i}\zeta u_i,1) + \langle \psi(T-t,(1-\mathsf{i}\zeta)\psi(\delta,0,0,1),\mathsf{i}\zeta u_i,1), X_t\rangle}.\end{aligned} \tag{3.15}$$

Note that the sets $\mathcal{A}_{\mathcal{Y}_T}, \Lambda_{\mathcal{Y}_T}$ depend on the specific choice of the driving process X. Since we are not adopting a specific choice for the driving process at the moment, we adapt to our setting the result of [53, Theorem 5.1], where a general pricing formula for a call option is established, with respect to different choices of the contour of integration. In particular, the next result highlights the tractability of the general affine specification: caplets can be priced by means of univariate Fourier integrals. In particular, this implies that a calibration may be obtained via a reasonable amount of computational effort (which may be further reduced by means of an application of an FFT algorithm to perform the numerical integration).

Proposition 3.8. *Let $\zeta \in \mathbb{C}$, $\alpha \in \mathbb{R}$, and $\bar{K} := 1 + \delta K$. Assume that $\{1, 1+\alpha\} \in \mathcal{A}_{\mathcal{Y}_T}$ Then the price of a caplet with notional N, maturity T, settled in arrears at $T+\delta$, at time t is given by*

$$\Pi^{CPLT}(t;T,T+\delta,K,N) = NB(t)\left(R\left(\mathcal{Y},\bar{K},\alpha\right) + \frac{1}{\pi}\int_{0-\mathsf{i}\alpha}^{\infty-\mathsf{i}\alpha} \mathrm{Re}\left(e^{-\mathsf{i}\zeta\log(\bar{K})}\frac{\varphi_{\mathcal{Y}_T}(\zeta-\mathsf{i})}{-\zeta(\zeta-\mathsf{i})}\right) d\zeta\right), \tag{3.16}$$

where $R\left(\mathcal{Y},\bar{K},\alpha\right)$ is given by

$$R\left(\mathcal{Y},\bar{K},\alpha\right) = \begin{cases} \varphi_{\mathcal{Y}_T}(-\mathsf{i}) - \bar{K}\varphi_{\mathcal{Y}_T}(0), & \text{if } \alpha < -1, \\ \varphi_{\mathcal{Y}_T}(-\mathsf{i}) - \frac{\bar{K}}{2}\varphi_{\mathcal{Y}_T}(0), & \text{if } \alpha = -1, \\ \varphi_{\mathcal{Y}_T}(-\mathsf{i}) & \text{if } -1 < \alpha < 0, \\ \frac{1}{2}\varphi_{\mathcal{Y}_T}(-\mathsf{i}) & \text{if } \alpha = 0, \\ 0 & \text{if } \alpha > 0. \end{cases} \tag{3.17}$$

Proof. As shown in Section 2.5.2, the price of the caplet can be expressed as

$$\begin{aligned}\Pi^{CPLT}(t;T,T+\delta,K,N) &= N\,\mathbb{E}\left[\frac{B(t)}{B(T)}\left(e^{u_i^\top Y_T} - (1+\delta K)B(T,T+\delta)\right)^+\middle| \mathcal{F}_t\right] \\ &= N\,\mathbb{E}\left[\frac{B(t)}{B(T)}B(T,T+\delta)\left(e^{u_i^\top Y_T}B(T,T+\delta)^{-1} - (1+\delta K)\right)^+\middle| \mathcal{F}_t\right] \\ &= N\,\mathbb{E}\left[\frac{B(t)}{B(T)}B(T,T+\delta)\left(e^{u_i^\top Y_T - \phi(\delta,0,0,1) - \langle\psi(\delta,0,0,1),X_T\rangle} - (1+\delta K)\right)^+\middle| \mathcal{F}_t\right].\end{aligned}$$

In the last line we recognize $\mathcal{Y}_T = u_i^\top Y_T - \phi(\delta,0,0,1) - \langle\psi(\delta,0,0,1),X_T\rangle$. The characteristic function of $\mathcal{Y}_T$ may be explicitly computed as in (3.15). We have thus reduced the pricing of a caplet to the pricing of a call option on an asset whose characteristic function can be explicitly computed. At this point, a direct application of [53, Theorem 5.1] gives then the result. □

3.3.2. *Swaptions.* In the present general setting, swaptions can not be in general priced in closed form. However, since characteristic functions can be explicitly computed, many well-known approximation

procedures proposed in the literature may be employed, see e.g., [65] or [12]. In the following, we shall adapt to our setting the recently introduced approach of [10] (see also [8]), which provides an accurate lower bound for the price of a swaption. To this end, let us introduce

- the weight vectors

$$w_i = \begin{cases} 1, & i = 1, \dots, n, \\ -(1+\delta K), & i = n+1, \dots, 2n, \end{cases} \qquad w_i^\star = -\frac{w_i}{\left|\sum_{i=1}^{2n} w_i\right|}, \tag{3.18}$$

- the log-asset prices

$$\mathcal{L}_T^i := \begin{cases} Z_T + u_i^\top Y_T + \phi(T_{i-1}-T, 0, u_i, 1) + \langle \psi(T_{i-1}-T, 0, u_i, 1), X_T \rangle, & i = 1, \dots, n, \\ Z_T + \phi(T_i - T, 0, 0, 1) + \langle \psi(T_i - T, 0, 0, 1), X_T \rangle, & i = n+1, \dots, 2n, \end{cases} \tag{3.19}$$

- the geometric and log-geometric averages

$$G_n(T) := \prod_{i=1}^{2n} \left(e^{\mathcal{L}_T^i}\right)^{w_i^\star} \quad \text{and} \quad Y_n(T) := \log\big(G_n(T)\big).$$

- and, finally, the joint characteristic functions

$$\Phi_T(\gamma_0, \boldsymbol{\gamma}, \mathbf{w}^\star, Y_n(t)) := \mathbb{E}\left[e^{\mathsf{i}\sum_{i=1}^{2n}\gamma_i \mathcal{L}^i(T) + \mathsf{i}\gamma_0 Y_n(T)} \middle| \mathcal{F}_t \right] = e^{\mathsf{i}\gamma_0 Y_n(t)} \mathbb{E}\left[e^{\mathsf{i}\sum_{i=1}^{2n}(\gamma_i + \gamma_0 w_i^*)\mathcal{L}^i(T)} \middle| \mathcal{F}_t \right]$$

We denote by $\mathcal{A}_{\mathcal{L}_T}$ the interior of the set

$$\left\{ \nu \in \mathbb{R}^{2n} : \ \mathbb{E}\left[e^{\langle \nu, \mathcal{L}_T \rangle} \middle| \mathcal{F}_t \right] < \infty \right\}.$$

Proposition 3.9. *Let $\alpha > 0$ and assume $\mathbf{e}_i, \alpha\mathbf{e}_i + \mathbf{w}^\star \in \mathcal{A}_{\mathcal{L}_T}$ for every $i = 1, .., 2n$. A lower bound for the price of a swaption with maturity T notional N, strike price K, written on a swap with payment dates $T_1, \cdots, T_n$ with $T \le T_1$, is given by*

$$\Pi_{LB}^{SWPTN}(t; T_1, T_n, K, N) = NB(t)\left|\sum_{i=1}^{2n} w_i\right| \left(-\sum_{i=1}^{2n} w_i^\star \mathbb{E}\left[e^{\mathcal{L}_T^i} | \mathcal{F}_t \right] + \max_\kappa \left(\frac{e^{-\alpha\kappa}}{\pi} \int_0^{+\infty} \mathrm{Re}\left(e^{-\mathsf{i}\gamma\kappa} \Psi(\gamma; \alpha) \right) d\gamma \right)^+ \right),$$

with

$$\Psi(\gamma; \alpha) = \frac{1}{i\gamma + \alpha} \sum_{i=1}^{2n} w_i^\star \Phi_T(\gamma - \mathsf{i}\alpha, -\mathsf{i}\mathbf{e}_i, \mathbf{w}^\star, Y_n(t))$$

where Φ_T denotes the joint characteristic function of $\{\mathcal{L}_T^1, \dots, \mathcal{L}_T^{2n}\}$ and $Y_n(T)$.

Proof. Let us first recall the general risk-neutral valuation formula for an European swaption:

$$\Pi^{SWPTN}(t; T_1, T_n, K, N) = NB(t)\mathbb{E}\left[\left(\sum_{i=1}^{n} \frac{B(T, T_{i-1})}{B(T)} S^\delta(T, T_{i-1}) - \sum_{i=1}^{n} (1+\delta K) \frac{B(T, T_i)}{B(T)} \right)^+ \middle| \mathcal{F}_t \right].$$

Recall also that the quantities appearing in the above expression are of the form

$$\frac{B(T, T_{i-1})}{B(T)} S^\delta(T, T_{i-1}) = \exp\left\{ Z_T + u_i^\top Y_T + \phi(T_{i-1} - T, 0, u_i, 1) + \langle \psi(T_{i-1} - T, 0, u_i, 1), X_T \rangle \right\},$$

$$\frac{B(T, T_i)}{B(T)} = \exp\left\{ Z_T + \phi(T_i - T, 0, 0, 1) + \langle \psi(T_i - T, 0, 0, 1), X_T \rangle \right\}.$$

Hence, we can interpret then the swaption as a basket/spread option written on $2n$ assets with weights w_i, $i = 1, .., 2n$ as in (3.18). The associated log-asset prices are given by (3.19). As a consequence, the valuation of the swaption corresponds to the valuation of the claim

$$\Pi^{SWPTN}(t; T_1, T_n, K, N) = NB(t)\mathbb{E}\left[\left(\sum_{1=1}^{2n} w_i e^{\mathcal{L}_T^i} \right)^+ \middle| \mathcal{F}_t \right].$$

Hence, we can write

$$\begin{aligned}\Pi^{SWPTN}(t;T_1,T_n,K,N) &= NB(t)\left|\sum_{i=1}^{2n} w_i\right| \mathbb{E}\left[\left(-\sum_{i=1}^{2n} w_i^\star e^{\mathcal{L}_T^i}\right)^+ \middle| \mathcal{F}_t\right] \\ &= NB(t)\left|\sum_{i=1}^{2n} w_i\right| \left\{-\sum_{i=1}^{2n} w_i^\star \mathbb{E}\left[e^{\mathcal{L}_T^i}\middle|\mathcal{F}_t\right] + \mathbb{E}\left[\left(\sum_{i=1}^{2n} w_i^\star e^{\mathcal{L}_T^i}\right)^+ \middle| \mathcal{F}_t\right]\right\}.\end{aligned}$$

Now, let us observe that the above expression consists of two summands. The first one can be thought of as the forward price of the basket and, given the affine structure of the model, it can be immediately evaluated. Concerning the second term, it represents the value of a basket/spread call option with zero strike, where the (real valued) weights are summing to one. On the second conditional expectation we apply the approach of [10] which directly yields the formula for the lower bound. The approximating payoff may be directly associated to the function $G_3(x,k)$ in [53]: using the notation therein we have $b_0 = \mathbf{w}^\star$ and $b_1 = \mathbf{e}_i$ so that [53, Theorem 5.1] yields the moment conditions. □

For an in-depth analysis of the accuracy of the proposed lower bound for the price of a swaption, we refer the reader to [10] or to the forthcoming paper [9]. Note that the result of the above proposition can also be applied to obtain a lower bound for the price of a basis swaption (see (2.38)), via a straightforward generalization (we omit the details, taking also into account that basis swaptions are not liquidly traded).

3.4. **A deterministic shift extension.** In view of practical implementations, a relevant issue is represented by the capability of the model to provide an exact fit to the initially observed term structures of risk-free bonds and spreads. To this effect, a first possibility would be to specify rich multi-dimensional dynamics and then look for a vector of parameters such that both the observed term structures and the implied volatilities of derivatives are fitted by the model in a satisfactory way, thus leading to a joint calibration problem with respect to term structures and traded derivatives' prices. A more common and easier procedure involves the specification of a model in such a way that initially observed term structures are automatically fitted by the model, in so far as they represent initial values for reference quantities, while model parameters allow to fit implied volatilities associated to interest rate derivatives.

Our affine specification can be easily extended in order to ensure an exact fit to the initially observed term structures. Let us denote by $\{B^M(0,T), T \geq 0\}$ and $\{S^{\delta_i,M}(0,T), T \geq 0, i = 1,\dots,m\}$, the term structures of risk-free bonds and multiplicative spreads observed on the market at the initial date $t = 0$. By following [6], we can easily extend the affine specification as follows.

Definition 3.10. *A* shifted affine multiple yield curve model *is defined by introducing the following specifications for OIS bonds and multiplicative spreads (compare with Definition 3.5):*

$$\begin{aligned}B(t,T) &= \frac{B^M(0,T)}{B^M(0,t)} \frac{e^{\phi(t,0,0,1)+\langle\psi(t,0,0,1),X_0\rangle}}{e^{\phi(T,0,0,1)+\langle\psi(T,0,0,1),X_0\rangle}} e^{\phi(T-t,0,0,1)+\langle\psi(T-t,0,0,1),X_t\rangle} \\ S^{\delta_i}(t,T) &= \frac{S^{\delta_i,M}(0,T)}{S^{\delta_i,M}(0,t)} \frac{e^{u_i^\top Y_0+\phi(t,0,u_i,1)-\phi(t,0,0,1)+\langle\psi(t,0,u_i,1)-\psi(t,0,0,1),X_0\rangle}}{e^{u_i^\top Y_0+\phi(T,0,u_i,1)-\phi(T,0,0,1)+\langle\psi(T,0,u_i,1)-\psi(T,0,0,1),X_0\rangle}} \\ &\quad \times e^{u_i^\top Y_t+\phi(T-t,0,u_i,1)-\phi(T-t,0,0,1)+\langle\psi(T-t,0,u_i,1)-\psi(T-t,0,0,1),X_t\rangle}.\end{aligned}$$

With the above specification, the term structures obtained by bootstrapping market data can be regarded as inputs of the model, while model parameters are employed in order to fit volatility surfaces of derivatives like caplets or swaptions. As shown in [6], the above generalization is also equivalent to a Hull-White extension for affine short-rate models, which in general affects the state independent characteristics of the driving affine process. We also notice that it is not a priori guaranteed that $S^\delta(t,T) > 1$. Deterministic shift extensions have been recently employed also by [30].

3.5. **Examples.** In this section, we propose two simple examples of affine multiple yield curve models. In the first example, the state space is chosen as $D_X = \mathbb{R}^d_+$, while in the second example we have $D_X = S^d_+$.

3.5.1. *Affine processes on* $\mathbb{R}^d_+$. Assume that $D_X = \mathbb{R}^d_+$ and let $W = (W_t)_{t\geq 0}$ be an $\mathbb{R}^d$-valued Brownian motion. Let $b \in \mathbb{R}^d_+$ and $\beta, \sigma \in \mathbb{R}^{d\times d}$ be diagonal matrices. We assume that the driving process X satisfies

$$dX_t = (b + \beta X_t)\,dt + \sigma\sqrt{X_t}dW_t, \qquad X_0 = x_0,$$

where $\sqrt{X_t}$ is a diagonal matrix whose jj-th component is given by $\sqrt{X_t^j}$. Given the dynamics for X, the process Z is defined as in (3.10), for some $l \in \mathbb{R}_+$ and $\lambda \in \mathbb{R}_+^d$, while the process Y is supposed to be of the simple form

$$Y_t := g(X_t) + Y_t^{\perp} := g(X_t) + \int_0^t \int_{\mathbb{R}_+^n \setminus \{0\}} \xi \, \mu^{Y^{\perp}}(d\xi, ds),$$

where $g : \mathbb{R}_+^d \to \mathbb{R}_+^n$ is an affine function, μ^Y is a Poisson random measure with associated Lévy measure ν^Y with support on $\mathbb{R}_+^n \setminus \{0\}$ and satisfying

$$\int_{\mathbb{R}_+^n \setminus \{0\}} (|\xi| \wedge 1) \, \nu^Y(d\xi) < \infty.$$

In particular, using the notation introduced in Appendix E, it holds that $Y^{\|} = g(X)$ and $Y^{\perp}$ is a pure jump process (compare also with Section 2.4.2). More specifically, the process $Y^{\perp}$ is a Lévy subordinator.

3.5.2. *Matrix variate affine processes.* We now suppose that $D_X = S_+^d$. We let $W = (W_t)_{t \geq 0}$ be a matrix Brownian motion, i.e., a $d \times d$ matrix whose components are standard independent Brownian motions. Let us introduce a matrix $Q \in GL(d)$ so that $\alpha = Q^\top Q$ is an admissible diffusion coefficient. We let also $M \in M_d$, the set of real valued $d \times d$ matrices and let $b \succeq (d-1)\alpha$. We assume that the main driving process X is a Wishart process, see [7]

$$dX_t = \left(b + MX_t + X_t M^\top\right) dt + \sqrt{X_t} dW_t Q + Q^\top dW_t^\top \sqrt{X_t}, \; X_0 = x_0,$$

where $\sqrt{X_t}$ denotes the matrix square root. In line with the previous example, we let Y be of the form

$$Y_t = g(X_t) + \int_0^t \int_{\mathbb{R}_+^n \setminus \{0\}} \xi \mu^Y(d\xi, ds).$$

where $g : S_+^d \to \mathbb{R}_+^n$ is an affine function and the second term is a jump process which is specified as in Example 3.5.1.

4. Relations with other multiple yield curve modeling approaches

In the present section, we study how our HJM-type framework relates to several multiple yield curve models that have been recently proposed in the literature. In particular, it will be shown that most of the existing modeling approaches can be recovered from our general setting. For consistency of exposition, we shall adapt to our notation the original notation used in the papers mentioned below.

4.1. **HJM models.** Let us start by showing how the multiple curve HJM models recently proposed in [15, 14] can be easily recovered within our general HJM-type framework.

The model proposed in [15] adopts a classical HJM setup for describing the term structure of risk-free interest rates. More specifically, risk-free discounted bond prices are specified as $B(t,T)/B_t = \exp\big(-\int_0^t r_s \, ds - \int_t^T f_t(s) \, ds\big)$, where r_t and $f_t(T)$ denote the risk-free short rate and T-forward rate, respectively. Forward rates are specified as

$$f_t(T) = -\frac{\partial}{\partial T} \log B(0,T) + \int_0^t \widetilde{\alpha}_s(T) \, ds + \int_0^t \widetilde{\sigma}_s(T) \, dX_s,$$

where X is a time-inhomogeneous Lévy process (with finite exponential moments) and $\int_t^T \widetilde{\alpha}_t(s) \, ds = \Psi_t^X\big(-\int_t^T \widetilde{\sigma}_t(s) \, ds\big)$, for all $t \leq T$ and $T \geq 0$, where Ψ^X denotes the Lévy exponent of X. In particular, the latter condition amounts to the drift condition given in Proposition 2.11. Note that in [15], the martingale property of $B(\cdot,T)/B$, for every $T \geq 0$, is ensured by assuming that $\int_t^T \widetilde{\sigma}_t(s) ds$ is uniformly bounded.

Contrary to our approach, [15] do not directly model market FRA rates or spreads, but instead introduce a family of artificial bond prices $B^\delta(t,T)$, for $t \leq T$ and $T \geq 0$, which do not represent real traded assets and such that Libor rates are given as $L_T(T, T+\delta) = \big(1/B^\delta(T, T+\delta) - 1\big)/\delta$, for all $T \geq 0$, by analogy to the classical single-curve setting. The quantities $B^\delta(t,T)$ are then modeled as

$$B^\delta(t,T) = B^\delta(0,T) \exp\left(\int_0^t \big(r_s^* - \bar{A}_s^*(T)\big) \, ds - \int_0^t \bar{\Sigma}_s^*(T) \, dX_s\right),$$

where $(r_t^*)_{t \geq 0}$ is an artificial risky short rate and $\bar{A}_t^*(T) := \widetilde{A}_t(T) + A_t^*(T) := \int_t^T \big(\widetilde{\alpha}_t(s) + a_t^*(s)\big) ds$ and $\bar{\Sigma}_t^*(T) := \widetilde{\Sigma}_t(T) + \Sigma_t^*(T) := \int_t^T \big(\widetilde{\sigma}_t(s) + \sigma_t^*(s)\big) ds$. In order to derive explicit valuation formulas, [15]

assume that drift and volatilities and given by deterministic functions. Then, as can be deduced from equations (13) and (52)-(54) of [15], the multiplicative spread $S^\delta(t,T)$ is given by[5]

$$\begin{aligned}S^\delta(t,T) &= \frac{B(0,T+\delta)B^\delta(0,T)}{B(0,T)B^\delta(0,T+\delta)}\exp\bigg(\int_t^T \big(\Psi_s^{T+\delta,X}\big(\bar\Sigma_s^*(T+\delta)-\bar\Sigma_s^*(T)\big)+\bar A_s^*(T+\delta)-\bar A_s^*(T)\big)ds\\ &\quad+\int_0^t\big(A_s^*(T+\delta)-A_s^*(T)\big)ds+\int_0^t\big(\Sigma_s^*(T+\delta)-\Sigma_s^*(T)\big)dX_s\bigg)\\ &= S^\delta(0,T)\exp\bigg(-\int_0^t\Psi_s^{T,X}\big(\Sigma_s^*(T+\delta)-\Sigma_s^*(T)\big)ds+\int_0^t\big(\Sigma_s^*(T+\delta)-\Sigma_s^*(T)\big)dX_s\bigg),\end{aligned}$$

where $\Psi^{T+\delta,X}$ denotes the local exponent of X under the measure $\mathbb{Q}^{T+\delta}$ (analogously for $\Psi^{T,X}$) and where we have used the fact that $\widetilde A_t(T)=\Psi_t^X\big(-\widetilde\Sigma_t(T)\big)$. By means of standard computations, it can be readily checked that the quantity $S^\delta(t,T)$ admits the representation $S^\delta(t,T)=\exp\big(Z_t^\delta+\int_t^T\eta_t^\delta(s)ds\big)$, with $Z_t^\delta:=\log S^\delta(t,t)$ and $\eta_t^\delta(T):=\frac{\partial}{\partial T}\log S^\delta(t,T)$, for $t\in[0,T]$. More explicitly (assuming enough regularity of Σ^*),

$$\begin{aligned}Z_t^\delta &= \log\bigg(\frac{B(0,t+\delta)B^\delta(0,t)}{B(0,t)B^\delta(0,t+\delta)}\bigg)+\int_0^t\big(A_s^*(t+\delta)-A_s^*(t)\big)ds+\int_0^t\big(\Sigma_s^*(t+\delta)-\Sigma_s^*(t)\big)dX_s,\\ \eta_t^\delta(T) &= \frac{\partial}{\partial T}\log S^\delta(0,T)-\int_0^t\frac{\partial}{\partial T}\Psi_t^{T,X}\big(\Sigma_s^*(T+\delta)-\Sigma_s^*(T)\big)ds+\int_0^t\big(\sigma_s^*(T+\delta)-\sigma_s^*(T)\big)dX_s.\end{aligned}$$

Moreover, according to the notation of [15], observe that $Z_t^\delta=\int_t^{t+\delta}g_t(s)\,ds$, for any tenor length $\delta>0$, with $g_t(T)$ representing the spread between risky and risk-free instantaneous T-forward rates.

The model proposed in [14] is closer to our philosophy since it directly models FRA rates according to an HJM methodology, rather than introducing artificial risky bond prices, while the risk-free term structure is modeled as in the classical HJM setup. More specifically, the discounted price $B(t,T)/B_t$ of a risk-free bond is specified as $B(t,T)/B_t=B(0,T)\exp\big(-\int_0^t\widetilde A(s,T)ds-\int_0^t\widetilde\Sigma(s,T)dX_s\big)$, for all $t\le T$ and $T\ge0$, where $\widetilde A$ and $\widetilde\Sigma$ are deterministic functions, continuously differentiable with respect to the second argument, and X is a multivariate Lévy process (with finite exponential moments). As before, one can easily obtain the representation $B(t,T)/B_t=\exp\big(-\int_0^t r_s ds-\int_t^T f_t(s)ds\big)$, for all $t\le T$ and $T\ge0$, where the instantaneous risk-free forward rate $f_t(T)$ satisfies $f_t(t)=r_t$ and is given by

$$f_t(T)=-\frac{\partial}{\partial T}\log B(0,T)+\int_0^t\frac{\partial}{\partial T}\widetilde A(s,T)ds+\int_0^t\frac{\partial}{\partial T}\widetilde\Sigma(s,T)dX_s.$$

Moreover, in order to ensure the martingale property of $B(\cdot,T)/B$, for every $T\ge0$, the drift condition $\widetilde A(t,T)=\Psi_t^X\big(-\widetilde\Sigma(t,T)\big)$ is assumed to hold, for all $0\le t\le T$, with Ψ^X denoting the local exponent of X, and the volatility function $\widetilde\Sigma$ is assumed to be uniformly bounded.

In the spirit of [58], in [14] the authors adopt an HJM model for market FRA rates. According to their notation, their model essentially amounts to adopting the following specification of $S^\delta(t,T)$:

$$\begin{aligned}S^\delta(t,T) &= S^\delta(0,T)\exp\bigg(\int_0^t\big(\alpha(s,T,T+\delta)-\widetilde A(s,T+\delta)+\widetilde A(s,T)\big)ds\\ &\quad+\int_0^t\big(\varsigma(s,T,T+\delta)-\widetilde\Sigma(s,T+\delta)+\widetilde\Sigma(s,T)\big)dX_s\bigg)\\ &= S^\delta(0,T)\exp\bigg(-\int_0^t\Psi^{T,X}\big(\Sigma^\delta(s,T)\big)ds+\int_0^t\Sigma^\delta(s,T)dX_s\bigg)\end{aligned}$$

where $\Sigma^\delta(t,T):=\varsigma(t,T,T+\delta)-\widetilde\Sigma(t,T+\delta)+\widetilde\Sigma(t,T)$ and $\Psi^{T,X}$ denotes the local exponent of X under the T-forward measure $\mathbb{Q}^T$ and where we have used the drift condition (12) of [14], which ensures the $\mathbb{Q}^T$-martingale property of $S^\delta(\cdot,T)$, for all $T\ge0$. In particular, by means of standard computations, one can obtain the representation $S^\delta(t,T)=\exp\big(Z_t^\delta+\int_t^T\eta_t^\delta(s)ds\big)$, for all $t\le T$ and $T\ge0$, where

$$\begin{aligned}Z_t^\delta &:= \log S^\delta(t,t)=\log S^\delta(0,t)-\int_0^t\Psi^{t,X}\big(\Sigma^\delta(s,t)\big)ds+\int_0^t\Sigma^\delta(s,t)dX_s,\\ \eta_t^\delta(T) &:= \frac{\partial}{\partial T}\log S^\delta(t,T)=\frac{\partial}{\partial T}\log S^\delta(0,T)-\int_0^t\frac{\partial}{\partial T}\Psi^{T,X}\big(\Sigma^\delta(s,T)\big)ds+\int_0^t\frac{\partial}{\partial T}\Sigma^\delta(s,T)dX_s.\end{aligned}$$

[5]Note that $S^\delta(t,T)$ corresponds to the quantity $v_t^{T,T+\delta}B_t(T+\delta)/B_t(T)$, according to the notation of [15].

As a special case, let us consider the Lévy Hull-White specification proposed in Section 2.3 of [14], where X is a two-dimensional Lévy process whose components X^1 and X^2 are independent NIG processes and where the volatility structure is given by

$$\widetilde{\Sigma}(t,T) = \Big(\frac{\sigma}{a}\big(1-e^{-a(T-t)}\big), 0\Big)^\top,$$
$$\varsigma(t,T,T+\delta) = \left(\frac{\sigma}{a}\big(e^{-a(T-t)} - e^{-a(T+\delta-t)}\big), \frac{\sigma^*(T,T+\delta)}{a^*}\big(e^{-a^*(T-t)} - e^{-a^*(T+\delta-t)}\big)\right)^\top,$$

where $\sigma, \sigma^*(T,T+\delta) > 0$ and $a, a^* \neq 0$ are constants. Assuming for simplicity that $\sigma^*(T,T+\delta) = \sigma^*_\delta$, for all $T \geq 0$, we then get

$$\Sigma^\delta(t,T) = \left(0, \frac{\sigma^*_\delta}{a^*}\big(e^{-a^*(T-t)} - e^{-a^*(T+\delta-t)}\big)\right)^\top,$$

so that, in view of the independence of X^1 and X^2, it holds that $\Psi^{T,X}\big(\Sigma^\delta(t,T)\big) = \Psi^{X^2}\big(\Sigma^{\delta,2}(t,T)\big)$, with $\Sigma^{\delta,2}(t,T) := (e^{-a^*(T-t)} - e^{-a^*(T+\delta-t)})\sigma^*_\delta/a^*$. Hence:

$$Z^\delta_t := \log S^\delta(0,t) - \int_0^t \Psi^{X^2}\big(\Sigma^{\delta,2}(s,t)\big)ds + \frac{\sigma^*_\delta}{a^*}\left(1-e^{-a^*\delta}\right)q_t,$$
$$\eta^\delta_t(T) := \frac{\partial}{\partial T}\log S^\delta(0,T) - \int_0^t \frac{\partial}{\partial T}\Psi^{X^2}\big(\Sigma^{\delta,2}(s,T)\big)ds - \sigma^*_\delta(1-e^{-a^*\delta})e^{-a^*(T-t)}q_t,$$

where the process $(q_t)_{t\geq 0}$ is defined by $q_t := \int_0^t e^{-a^*(t-s)}dX^2_s$, for all $t \geq 0$.

Remark 4.1. We want to point out that, by relying on arguments analogous to those above, one can readily check that the multi-curve HJM model proposed in [58] can also be recovered from our framework.

4.2. **Short rate models.** As mentioned in the introduction, models based on a short rate approach have also been proposed for modeling multiple curves, see in particular [48, 49, 60]. As we show in the present section, such a short rate modeling approach can be easily embedded within our general framework.

In the paper [49], the authors formulate a simple short rate model that allows for a consistent pricing of fixed income products related to different curves. For simplicity, let us consider the case of two interest rate curves: the discounting curve, denoted by D and used in the pricing formulas for discounting cash flows, and the Libor curve, denoted by L and used for calculating Libor rates. To the two curves D and L, [49] associate the short rate processes $(r^D_t)_{t\geq 0}$ and $(r^L_t)_{t\geq 0}$, with corresponding bond prices

$$B(t,T) := \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r^D_u du}\big|\mathcal{F}_t\right] \qquad \text{and} \qquad B^\delta(t,T) := \mathbb{E}^{\mathbb{Q}_L}\left[e^{-\int_t^T r^L_u du}\big|\mathcal{F}_t\right], \tag{4.1}$$

for all $t \leq T$ and $T \geq 0$, where the measure $\mathbb{Q}_L \sim \mathbb{Q}$ represents a risk neutral measure with respect to the L savings account (see [49], Section 2). The Libor rate with tenor δ is then given by $L_T(T,T+\delta) = \big(1/B^\delta(T,T+\delta) - 1\big)/\delta$, for all $T \geq 0$. Hence, according to our notation[6], for all $0 \leq t \leq T$ and $\delta > 0$,

$$\begin{aligned} S^\delta(t,T) &= \mathbb{E}^{\mathbb{Q}^{T+\delta}}\big[1+\delta L_T(T,T+\delta)|\mathcal{F}_t\big]\frac{B(t,T+\delta)}{B(t,T)} = \mathbb{E}^{\mathbb{Q}^{T+\delta}}\left[\frac{1}{B^\delta(T,T+\delta)}\big|\mathcal{F}_t\right]\frac{B(t,T+\delta)}{B(t,T)} \\ &= \frac{\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r^D_u du}\frac{B(T,T+\delta)}{B^\delta(T,T+\delta)}\big|\mathcal{F}_t\right]}{\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r^D_u du}|\mathcal{F}_t\right]} = \frac{\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r^D_u du}\frac{1}{H^L(T,T+\delta)}\big|\mathcal{F}_t\right]}{\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r^D_u du}|\mathcal{F}_t\right]}, \end{aligned} \tag{4.2}$$

with $H^L(t,T) := B^\delta(t,T)/B(t,T)$. Hence, by letting $Z_t := -\int_0^t r^D_u du$ and $Y_t := -\log H^L(t,t+\delta)$, for all $t \geq 0$, we see that the above expression for $S^\delta(t,T)$ is of the same form as (3.12).

More specifically, in Section 4.1 of [49], the short rate $(r^D_t)_{t\geq 0}$ is modeled as a quadratic Gaussian process, i.e., $r^D_t = (\alpha + \beta t + y_t)^2$, with $\alpha, \beta \in \mathbb{R}$ and where the process $(y_t)_{t\geq 0}$ has Ornstein-Uhlenbeck dynamics (see [62]). This implies that $\log B(t,T)$ is a second-order polynomial of y_t, with coefficients that are functions of the time to maturity $T-t$. The short rate $(r^L_t)_{t\geq 0}$ is then modeled as $r^L_t = r^D_t + h^L_t$, where $(h^L_t)_{t\geq 0}$ represents a spread process with Ornstein-Uhlenbeck dynamics (under the measure $\mathbb{Q}_L$), independent of $(r^D_t)_{t\geq 0}$. In particular, it holds that $H^L(t,T) = \mathbb{E}^{\mathbb{Q}_L}\big[\exp\big(-\int_t^T h^L_u du\big)|\mathcal{F}_t\big]$, where the conditional expectation can be explicitly computed as an exponentially affine function of h^L (see [49], eqn.s (23)-(24)). In turn, this implies that Y is given by a linear function (with time varying coefficients) of the affine process $(h^L_t)_{t\geq 0}$. In Section 4.2 of [49], another version of this short rate model is proposed,

[6]Note that our multiplicative spread $S^\delta(t,T)$ corresponds to $K_L(t,T,T+\delta)$, according to the notation adopted in [49].

with the risk-free short rate $(r_t^D)_{t\geq 0}$ being simply modeled as in the classical Vasicek model, with a possibly non-zero correlation with the spread process $(h_t^L)_{t\geq 0}$.

The short rate model proposed in [48] is rather similar to the approach of [49], the main difference being that the short rate process $(r_t^L)_{t\geq 0}$ ($(r_{f_\Delta}(t))_{t\geq 0}$ in the notation of [48]) is assumed to be specific for each tenor $\delta > 0$. Denoting by $(r_t^\delta)_{t\geq 0}$, the short rate associated to Libor rates with tenor δ, risk-free and Libor-related bond prices are then modeled as in (4.1), thus leading to an expression analogous to (4.2) for $S^\delta(t,T)$, with $H^{L_\delta}(t,T) := B^\delta(t,T)/B(t,T)$. In the paper [48], the author adopts multiple Hull-White models for $(r_t^D)_{t\geq 0}$ and $(r_t^\delta)_{t\geq 0}$, with possibly non-zero correlation. It is also assumed that the change of measure from $\mathbb{Q}$ to $\mathbb{Q}^{L_\delta}$ does not affect the dynamics of the process $(r_t^\delta)_{t\geq 0}$. As in the case of [49], letting $Z_t := -\int_0^t r_u^D du$ and $Y_t^\delta := -\log H^{L_\delta}(t,t+\delta)$ and relying on the affine property of $(r_t^D)_{t\geq 0}$ and $(r^\delta)_{t\geq 0}$, it can be checked that the short rate approach of [48] can be also recovered from our general framework (in particular, compare with the general affine specification of Section 3).

Finally, we want to point out that the short rate model recently proposed in [60] can be also embedded in our framework. In the latter paper, FRA rates are defined in terms of artificial "risky" bond prices. Risk-free and risky bond prices are modeled similarly as in (4.1), but under a common risk-neutral measure $\mathbb{Q}$, and with a "risky" short rate obtained by adding to the risk-free rate $(r_t)_{t\geq 0}$ a stochastic spread $(s_t)_{t\geq 0}$. Both r and s are then modeled as linear transformations of an $\mathbb{R}^3$-valued affine diffusion process $(\Psi_t)_{t\geq 0}$, in order to introduce (negative) correlation between risk-free rates and spreads. By inspection, it can be readily verified that the model of [60] can be embedded in our general affine framework of Section 3, with $X = \Psi$, $Z = -\int_0^\cdot r_u du$ and Y given by an affine function (with time-dependent coefficients) of Ψ. In that context, for a given tenor length $\delta > 0$ and for any $0 \leq t \leq T$, our multiplicative spread $S^\delta(t,T)$ corresponds to the ratio $\bar{\nu}_{t,T}/\nu_{t,T}$, according to the notation used in [60].

4.3. **Affine Libor models.** Within our general framework, one can rather easily recover a generalization of the model recently proposed in [32], which extends to a multiple-curve setting the affine Libor model first introduced in [45] (see also [19]). To this effect, let $X = (X_t)_{0\leq t\leq T^*}$ be a regular affine process taking values in $\mathbb{R}^d_+$, with $T^* \in (0,\infty)$ representing a fixed final horizon, and denote by $M^u = (M_t^u)_{0\leq t\leq T^*}$ the martingale defined by $M_t^u := \mathbb{E}^{\mathbb{Q}}\left[\exp(\langle u, X_{T^*}\rangle)|\mathcal{F}_t\right]$, for $t \in [0,T^*]$, where $\mathbb{Q}$ is a fixed risk neutral measure and u is an arbitrary vector in $\mathbb{R}^d$ such that the expectation is finite.

In [32], for each tenor length $\delta > 0$ present in the market, the authors consider a finite collection of ordered maturities $\mathcal{T}^\delta := \{0 = T_0^\delta, T_1^\delta, \ldots, T_{N^\delta}^\delta = T^*\}$. Following the approach of [45], the OIS (risk-free) forward rate $L_t^D(T_{k-1}^\delta, T_k^\delta)$ at time t for the interval $[T_{k-1}^\delta, T_k^\delta]$ is then defined as

$$1 + \delta L_t^D\left(T_{k-1}^\delta, T_k^\delta\right) = \frac{B(t,T_{k-1}^\delta)}{B(t,T_k^\delta)} = \frac{M_t^{u_{k-1}^\delta}}{M_t^{u_k^\delta}}, \tag{4.3}$$

for all $k \in \{1,\ldots,N^\delta\}$ and $t \in [0,T_{k-1}^\delta]$, where $\{u_1^\delta,\ldots,u_{N^\delta}^\delta\}$ is a sequence in $\mathbb{R}^d$ such that $u_k^\delta \geq u_{k+1}^\delta$, for all k, in order to ensure non-negative forward rates, for all $\delta > 0$. In a similar way, for each tenor length $\delta > 0$, the market FRA rate $L_t(T_{k-1}^\delta, T_k^\delta)$ at time t for the interval $[T_{k-1}^\delta, T_k^\delta]$ is defined as

$$1 + \delta L_t(T_{k-1}^\delta, T_k^\delta) = \frac{M_t^{v_{k-1}^\delta}}{M_t^{u_k^\delta}}, \tag{4.4}$$

for all $k \in \{1,\ldots,N^\delta\}$ and $t \in [0,T_{k-1}^\delta]$, where $\{v_1^\delta,\ldots,v_{N^\delta}^\delta\}$ is a sequence in $\mathbb{R}^d$ such that $v_k^\delta \geq u_k^\delta$, for all k, in order to ensure positive spreads between actual FRA rates and theoretical risk-free FRA rates, for all $\delta > 0$ (see below). According to our notation, from (4.3)-(4.4) it follows that, for any tenor length $\delta > 0$, $k \in \{0,\ldots,N^\delta - 1\}$ and $t \in [0,T_k^\delta]$,

$$S^\delta(t,T_k^\delta) = \frac{1+\delta L_t(T_k^\delta, T_{k+1}^\delta)}{1+\delta L_t^D(T_k^\delta, T_{k+1}^\delta)} = \frac{M_t^{v_k^\delta}}{M_t^{u_k^\delta}}. \tag{4.5}$$

By its own nature, the model proposed in [32] only considers a finite collection of maturities. However, as we are now going to show, a continuous extension of such a model can be easily recovered within our general framework. To this effect, let us consider a function $u : [0,T^*] \to \mathbb{R}^d$ such that $u(T_k^\delta) = u_k^\delta$ and a function $v : [0,T^*]\times[0,T^*] \to \mathbb{R}^d$ such that $v(T_k^\delta,\delta) = v_k^\delta$, for all k and $\delta > 0$. In analogy to equation (4.3) above (as well as to Section 6 of [45]), define risk-free bond prices as follows, for all $0 \leq t \leq T \leq T^*$,

$$\frac{B(t,T)}{B(t,T^*)} = M_t^{u(T)}, \tag{4.6}$$

with $M_t^{u(T)} = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(\langle u(T), X_{T^*}\rangle\right)|\mathcal{F}_t\right]$, where we implicitly assume that the function u is such that the expectation is finite, for all $T \in [0, T^*]$. Note that, in order to ensure that risk-free bond prices are decreasing with respect to time to maturity, it suffices to require the function u to be decreasing. In line with equation (4.5), define then the quantity $S^\delta(t,T)$ as follows, for all $\delta > 0$ and $0 \le t \le T \le T^*$,

$$S^\delta(t,T) = \frac{M_t^{v(T,\delta)}}{M_t^{u(T)}}, \tag{4.7}$$

with $M_t^{v(T,\delta)} = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(\langle v(T,\delta), X_{T^*}\rangle\right)|\mathcal{F}_t\right]$, where we implicitly assume that the function v is such that the expectation is finite, for all $(T,\delta) \in [0,T^*] \times [0,T^*]$.

Due to the affine property of X, equation (4.6) leads to the following representation of risk-free bond prices, for all $0 \le t \le T \le T^*$,

$$B(t,T) = \exp\Big(\Phi\big(T^*-t, u(T)\big) + \big\langle \Psi\big(T^*-t, u(T)\big), X_t\big\rangle\Big) B(t,T^*), \tag{4.8}$$

where $\Phi(\cdot,\cdot)$ and $\Psi(\cdot,\cdot)$ are the solutions to the generalized Riccati ODEs associated with the affine process X. In particular, taking $t = T$ in (4.8), we get

$$B(t,T^*) = \exp\Big(-\Phi\big(T^*-t, u(t)\big) - \big\langle \Psi\big(T^*-t, u(t)\big), X_t\big\rangle\Big). \tag{4.9}$$

Hence, from (4.8) and (4.9), assuming enough regularity of the functions involved (compare e.g. [47], Lemma 2.10), we can obtain the representation

$$B(t,T) = \exp\left(-\int_t^T f_t(s)\,ds\right),$$

for all $0 \le t \le T \le T^*$, where

$$f_t(T) = -\frac{\partial}{\partial T}\Phi\big(T^*-t, u(T)\big) - \frac{\partial}{\partial T}\big\langle \Psi\big(T^*-t, u(T)\big), X_t\big\rangle.$$

Analogously, equation (4.7) leads to the following representation, for all $\delta > 0$ and $0 \le t \le T \le T^*$:

$$S^\delta(t,T) = \exp\left(\int_t^T \eta_t^\delta(s)\,ds + Z_t^\delta\right),$$

where

$$\eta_t^\delta = \frac{\partial}{\partial T}\Big(\Phi\big(T^*-t, v(T,\delta)\big) - \Phi\big(T^*-t, u(T)\big)\Big) + \frac{\partial}{\partial T}\Big\langle \Psi\big(T^*-t, v(T,\delta)\big) - \Psi\big(T^*-t, u(T)\big), X_t\Big\rangle$$

and

$$Z_t^\delta = \Phi\big(T^*-t, v(t,\delta)\big) - \Phi\big(T^*-t, u(t)\big) + \Big\langle \Psi\big(T^*-t, v(t,\delta)\big) - \Psi\big(T^*-t, u(t)\big), X_t\Big\rangle. \tag{4.10}$$

We have thus shown that a natural continuous extension of the affine Libor multiple-curve model recently proposed in [32] can be embedded in our framework. Note that in order to ensure that $S^\delta(t,T) \ge 1$, for all $0 \le t \le T \le T^*$ and $\delta > 0$, it suffices to require that the functions $u(\cdot)$ and $v(\cdot,\cdot)$ satisfy $v(T,\delta) \ge u(T)$ for all $T \in [0,T^*]$ and $\delta > 0$. Moreover, order relations among multiplicative spreads $S^\delta(t,T)$ associated to different tenor lengths δ can also be obtained by requiring suitable properties on the function $\delta \mapsto v(T,\delta)$.

Finally, observe that the continuous extension of the affine Libor multiple-curve model discussed in this section can also be seen as a special case of the general affine setup presented in Section 3. Indeed, in view of equation (4.7), it is easy to see that the multiplicative spread $S^\delta(t,T)$ admits a representation of the form (3.12), with a process $(Z_t)_{0\le t\le T^*}$ given by

$$Z_t := \Phi\big(T^*-t, u(t)\big) + \Big\langle \Psi\big(T^*-t, u(t)\big), X_t\Big\rangle,$$

and where the process $(Y_t)_{0\le t\le T^*}$ (considering for simplicity the case of a single tenor δ) is given as in (4.10). In particular, observe that the (time-dependent) characteristics of (Y,Z) only depend on the affine process X, in line with Definition 3.5.

4.4. **Lognormal Libor market models.** Similarly as in the original article by Brace, Gatarek and Musiela [3], we can also obtain a lognormal Libor market model for $L_t(T, T+\delta)$ within the above framework. Let δ be fixed and consider the setting of Remark 2.21 with Y one-dimensional, given by $Y_t = \int_0^t q_s ds$, and $u = 1$. Assume furthermore that the driving process X in our multiple yield curve model is a standard d-dimensional Brownian motion W. Let us now postulate that the dynamics of

$L_t(T,T+\delta)$ (under $\mathbb{Q}$) are governed by the following SDE

$$dL_t(T,T+\delta) = L_t(T,T+\delta)\beta_t(T)^\top \widetilde{\Sigma}_t(T+\delta)dt + L_t(T,T+\delta)\beta_t(T)dW_t,$$

or equivalently under $\mathbb{Q}^{T+\delta}$ by

$$dL_t(T,T+\delta) = L_t(T,T+\delta)\beta_t(T)dW_t^{T+\delta},$$

where $\beta_t(T)$ is a $\mathbb{R}^d$-valued bounded deterministic function and $W^{T+\delta}$ denotes a $\mathbb{Q}^{T+\delta}$ Brownian motion. Recalling that

$$1+\delta L_t(T,T+\delta) = S^\delta(t,T)\big(1+\delta L_t^D(T,T+\delta)\big) = e^{\int_0^t q_s ds + \int_t^T \eta_t^\delta(u)du + \int_T^{T+\delta} f_t(u)du},$$

applying Itô's formula to both sides and comparing the diffusion coefficients, we obtain

$$\frac{\delta L_t(T,T+\delta)}{1+\delta L_t(T,T+\delta)}\beta_t(T) = \Sigma_t(T) + \big(\widetilde{\Sigma}_t(T+\delta) - \widetilde{\Sigma}_t(T)\big),$$

so that

$$\Sigma_t(T) = \frac{\delta L_t(T,T+\delta)}{1+\delta L_t(T,T+\delta)}\beta_t(T) - \big(\widetilde{\Sigma}_t(T+\delta) - \widetilde{\Sigma}_t(T)\big).$$

Supposing differentiability of $T\mapsto\beta_t(T)$, we can derive an expression for $\sigma_t(T)$

$$\begin{aligned}\sigma_t(T) = {} & e^{-\int_0^t q_s ds - \int_t^T \eta_t^\delta(u)du - \int_T^{T+\delta} f_t(u)du}\big((\eta_t^\delta(T) + f_t(T+\delta) - f_t(T))\beta_t(T) - \partial_T\beta_t(T)\big) + \partial_T\beta_t(T)\\ & - \widetilde{\sigma}_t(T+\delta) + \widetilde{\sigma}_t(T).\end{aligned}$$

In order to study the existence of a solution to the S(P)DE for η^δ corresponding to this volatility structure, we switch to the Musiela parametrization by setting $\theta_t^0(x) = f_t(t+x)$ and $\theta_t^1(x) = \eta_t^\delta(t+x)$. Moreover, we let β be independent of t, set $\widetilde{\beta}(x) = \beta(t+x)$ and suppose $\widetilde{\sigma}_t(T) = \zeta^0(\theta_t)(T-t)$ for some function ζ^0 with values in a Hilbert space of $\mathbb{R}^d$-valued functions. For θ in some Hilbert space of $\mathbb{R}^2$-valued forward (spread) curves as in Section 2.4.1, define a function ζ with values in a Hilbert space of $\mathbb{R}^d$-valued functions via

$$\zeta(\theta) = e^{-\int_0^\cdot \theta^1(s)ds - \int_\cdot^{\cdot+\delta}\theta^0(s)ds}(\theta^1 + \theta^0(\cdot+\delta) - \theta^0)\widetilde{\beta} - \partial_x\widetilde{\beta}.$$

Setting

$$\zeta_t^1(\theta) = e^{-\int_0^t q_s ds}\zeta(\theta) - \zeta^0(\theta)(\cdot+\delta) + \zeta^0(\theta),$$

we see that $\sigma_t(T) = \zeta_t^1(\theta_t)$. Under appropriate assumption on β, q and ζ^0, we can obtain similarly as in Theorem 2.27 existence for η. This approach thus provides a theoretical justification in the multiple yield curve setting for the market practice to price caplets by means of Black's formula.

Appendix A. FRA rates and pricing under collateral

Forward rate agreements are OTC contracts in which two counterparties agree to exchange two cash flows, typically tied to a floating Libor rate $L_T(T,T+\delta)$ with tenor δ, fixed at time T, and spanning the time interval $[T,T+\delta]$, versus a fixed rate K spanning the same time interval. The standard (textbook) payoff of a payer FRA at time $T+\delta$ is given by[7]

$$\delta(L_T(T,T+\delta) - K). \tag{A.1}$$

Since FRAs are traded OTC between perfectly collateralized counterparties, their prices need to be computed by taking into account, not only the cashflows generated by the contract itself, but also the cash flows generated by the margination mechanism defined by the collateral agreement (in reality regulated under the Credit Support Annex (CSA) of the ISDA Standard Master Agreement).

Let us here briefly review pricing under perfect collateralization for general derivatives which we then apply to the pricing of FRAs. For a more detailed discussion on general valuation with collateralization and funding costs, we refer to the growing literature on this topic, e.g., [1], [2], [13], [28], [63] (with a correction provided in [4]). We here follow closely [26, Section 2.2].

[7]The "market" FRA rates actually quoted are characterized by a slightly different payoff given by

$$\frac{\delta(L_T(T,T+\delta)-K)}{1+\delta L_T(T,T+\delta)}.$$

We will however stick to the standard textbook definition and neglect the convexity adjustment needed when deriving the price of the "market" FRAs (compare [57, Appendix A] and [1, Appendix C1, equation (125) and (126)]).

Throughout let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space, where $\mathbb{P}$ stands for the statistical probability measure. We consider OIS zero coupon bonds as basic traded instruments, which play the role of risk-free zero coupon bonds in the classical setting. In order to guarantee no arbitrage we assume:

(i) There exists an OIS bank account denoted by $(B_t)_{t\geq 0}$ such that $B_t = \exp(\int_0^t r_s ds)$, where r denotes the OIS short rate.
(ii) There exists an equivalent probability measure $\mathbb{Q}$ such that the OIS bonds for all maturities are $\mathbb{Q}$-martingales when denominated in units of the OIS bank account.[8]

Let now X be an $\mathcal{F}_T$-measurable payoff of some derivative security. We assume here a perfect collateral agreement where the two parties in the contract agree on posting a cash collateral on a continuous marking-to-market basis and at any time $t < T$, the posted amount of the collateral equals 100% of the derivative's present value V_t. At time t the receiver of the collateral (we refer to receiver whenever the collateral is positive) can invest it at risk-free rate r, corresponding to the OIS short rate, and has to pay an agreed *collateral rate* r^c_t to the poster of the collateral, where we assume that this rate is the same for both counterparties. Applying risk neutral pricing, we obtain the following expression for the present value of the collateralized derivative transaction

$$V_t = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds} X + \int_t^T e^{-\int_t^s r_u du}(r_s - r^c_s) V_s ds \,\middle|\, \mathcal{F}_t \right].$$

As shown in [26, Appendix A], this formula is equivalent to

$$V_t = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r^c_s ds} X \,\middle|\, \mathcal{F}_t \right].$$

Assuming that the collateral rate r^c corresponds to the OIS short rate r, which is usually the case, we obtain the classical risk neutral evaluation formula.

Since market quotes of FRAs correspond to perfectly collateralized contracts, where the *collateral rate r^c is assumed to be the OIS short rate r*, the above pricing approach is applied for the definition of FRA rates. As in classical interest rate theory, the FRA rate, denoted by $L_t(T, T+\delta)$, is the rate K fixed at time t such that the value of the FRA contract, whose payoff at time $T+\delta$ is given by (A.1) has value 0. Therefore, it holds that, for all $t \in [0, T]$ and $T \geq 0$,

$$\mathbb{E}^{\mathbb{Q}}\left[e^{\int_t^{T+\delta} r_s ds}(L_T(T, T+\delta) - K) \,\middle|\, \mathcal{F}_t \right] \stackrel{!}{=} 0.$$

Hence by Bayes formula,

$$L_t(T, T+\delta) = \mathbb{E}^{\mathbb{Q}^{T+\delta}}\left[L_T(T, T+\delta) \,\middle|\, \mathcal{F}_t \right],$$

where $\mathbb{Q}^{T+\delta}$ denotes the $(T+\delta)$-forward measure associated with the numéraire $B(\cdot, T+\delta)$ and density process $\frac{d\mathbb{Q}^{T+\delta}}{d\mathbb{Q}}|_{\mathcal{F}_t} = \frac{B(t,T+\delta)}{B_t B(0,T+\delta)}$. In particular, this provides a rigorous justification for the market practice of taking expression (1.1) as the definition of fair FRA rates.

Appendix B. Foreign exchange rate analogy

For simplicity of presentation, let us consider a fixed tenor δ and define artificial "risky" bond prices $B^\delta(t,T)$ at time t and maturity T for the tenor δ by the following relation

$$L_t(T, T+\delta) =: \frac{1}{\delta}\left(\frac{B^\delta(t,T)}{B^\delta(t, T+\delta)} - 1 \right),$$

for all $t \leq T$ and $T \geq 0$. In particular, $B^\delta(T,T) = 1$ so that that

$$L_T(T, T+\delta) = \frac{1}{\delta}\left(\frac{1}{B^\delta(T, T+\delta)} - 1 \right).$$

Interpreting the artificial "risky" bonds as foreign bonds (as opposed to the OIS bonds, which we now consider as the domestic bonds), we can define for each maturity $T \geq t$, a forward exchange rate[9] $R^\delta(t,T)$

[8]Let us remark that in the presence of funding costs which is for example considered in [1], [2], [5], [13], [63], absence of arbitrage is implied by the existence of an equivalent measure under which the risky assets S present in the market are (local) martingales when discounted with their corresponding funding rate r^f. This can be embedded in the classical framework where $\mathbb{Q}$ is a risk neutral measure with the risk-free (OIS) bank account as numéraire, by treating $e^{\int_0^\cdot -r^f_s + r_s} S$ as traded asset.

[9]Note that $R^\delta(T, T+\delta)$ corresponds to Q^δ_T considered in the introduction.

at time $t \leq T$ by

$$R^\delta(t,T) := \frac{B(t,T)}{B^\delta(t,T)},$$

so that, for $t \leq S \leq T$, it holds that

$$B_t \mathbb{E}^{\mathbb{Q}} \left[R^\delta(S,T) \frac{B^\delta(S,T)}{B_S} \,\middle|\, \mathcal{F}_t \right] = B(t,T) = R^\delta(t,T) B^\delta(t,T).$$

In particular, note that the multiplicative spread $S^\delta(t,T)$ introduced in (1.2) corresponds to

$$S^\delta(t,T) = \frac{R^\delta(t,T+\delta)}{R^\delta(t,T)},$$

for all $t \leq T$ and $T \geq 0$, and $S^\delta(T,T) = R^\delta(T,T+\delta)$ for all $T \geq 0$.

Since Libor rates reflect the overall credit risk of the Libor panel, we can think of the forward exchange rate $R^\delta(t,T)$ as a measure of how much the foreign economy is perceived by the market (at time t) to be riskier than the domestic economy on the time period $[t,T]$. According to this interpretation, the quantity $S^\delta(t,T) = R^\delta(t,T+\delta)/R^\delta(t,T)$ can then be seen as a relative measure of the riskiness of the foreign economy on the future time period $[T,T+\delta]$, as seen from the market at time t. For instance, a large value of $S^\delta(t,T)$ means that the market expects a worsening of the credit quality of the Libor panel on $[T,T+\delta]$ compared to the credit quality on $[t,T]$ as seen at time t.

In this sense, the multiplicative spread $S^\delta(t,T)$ is a rather natural quantity to model in a multiple curve setting, because it represents the market expectation at time t (since it is computed from traded instruments at t) of the average credit quality of the Libor panel on $[T,T+\delta]$.

Appendix C. Bond price models with a terminal bond as numéraire

We consider here the (OIS) bond market for maturities $T \leq T^*$ with respect to a terminal bond $B(\cdot,T^*)$ as numéraire, where T^* denotes some fixed terminal maturity. Let us denote by $\mathbb{Q}^{T^*}$ the T^*-forward measure, under which the processes

$$(B(t,T)/B(t,T^*))_{t\in[0,T]}$$

should be (local) martingales for all $T \leq T^*$. Using the same definition of a bond price model without the introduction of a bank account and identifying $S(t,T)$ with $B(t,T)/B(t,T^*)$ so that $Z_t = -\log(B(t,T^*))$ yields a similar assertion as the one of Proposition 2.11. In this case condition (iii) reads as follows, denoting $\widetilde{\Sigma}_t(S,T) := \int_S^T \widetilde{\sigma}_t(u)du)$ for $t \leq S \leq T$:

- The processes

$$\left(\exp\left(\int_0^t \widetilde{\Sigma}_s(T,T^*) dX_s - \int_0^t \Psi_s^{T^*,X}\big(\widetilde{\Sigma}_s(T,T^*)\big) ds \right) \right)_{t\in[0,T]} \tag{C.1}$$

  is a $\mathbb{Q}^{T^*}$-martingale, for every $T \leq T^*$.
- The consistency condition holds:

$$\int_t^{T^*} \widetilde{\alpha}_t(u)du = -\Psi_t^{T^*,X}\big(\widetilde{\Sigma}_t(t,T^*)\big), \quad \text{for all } t \geq 0.$$

- The HJM drift condition

$$\int_T^{T^*} \widetilde{\alpha}_t(u)du = -\Psi_t^{T^*,X}\big(\widetilde{\Sigma}_t(T,T^*)\big)$$

  holds for every $t \in [0,T]$ and $T \leq T^*$.

Here, $\Psi_t^{T^*,X}$ denotes the local exponent of X under $\mathbb{Q}^{T^*}$. A slightly stronger version of Condition $I(0,1)$ of [42] guarantees again the martingale property of (C.1) under $\mathbb{Q}^{T^*}$. It reads as

$$\int_{|\widetilde{\Sigma}_t^\top(T,T^*)\xi|>1} \left(e^{\widetilde{\Sigma}_t^\top(T,T^*)\xi} \widetilde{\Sigma}_t^\top(T,T^*)\xi \right) K_t^{T^*,X}(d\xi) \in \mathcal{V},$$

and

$$\sup_{t\leq T, T\leq T^*} \mathbb{E}^{\mathbb{Q}^{T^*}} \left[\exp\left(\frac{1}{2}\widetilde{\Sigma}_t^\top(T,T^*) c_t^X \widetilde{\Sigma}_t(T,T^*) + \int \left(e^{\widetilde{\Sigma}_t^\top(T,T^*)\xi} \left(\widetilde{\Sigma}_t^\top(T,T^*)\xi - 1 \right) + 1 \right) K_t^{T^*,X}(d\xi) \right) \right] < \infty.$$

Here, $K_t^{T^*,X}$ is the compensator of the jump measure of X under $\mathbb{Q}^{T^*}$. If there is a bank account $(B_t)_{t\in[0,T^*]}$, which qualifies as numéraire, where the latter requirement is equivalent to

$$\frac{d\mathbb{Q}}{d\mathbb{Q}^{T^*}}\Big|_{\mathcal{F}_t} := \frac{B(0,T^*)B_t}{B(t,T^*)} = \exp\left(\int_0^t \widetilde{\Sigma}_s(s,T^*)dX_s - \int_0^t \Psi_s^{T^*,X}\left(\widetilde{\Sigma}_s(s,T^*)\right)ds - \int_0^t f_s(s)ds\right)B_t$$

being a true $\mathbb{Q}^{T^*}$-martingale, we can change to the equivalent measure $\mathbb{Q}$ with the bank account as numéraire. The above conditions also imply (by choosing $T=t$) that

$$\left(\exp\left(\int_0^t \widetilde{\Sigma}_s(s,T^*)dX_s - \int_0^t \Psi_s^{T^*,X}\left(\widetilde{\Sigma}_s(s,T^*)\right)ds\right)\right)_{t\in[0,T^*]}$$

is a $\mathbb{Q}^{T^*}$-martingale and by taking $B_t = \exp(\int_0^t f_s(s)ds)$, it follows that $\frac{B(0,T^*)B_t}{B(t,T^*)}$ is a true $\mathbb{Q}^{T^*}$-martingale. Thus, under $\mathbb{Q}$ we would obtain an HJM bond price model for the family of discounted bond prices $\left\{(B(t,T)/B_t)_{t\in[0,T]}, T\le T^*\right\}$ and the above conditions translate to the conditions of Proposition 2.11. In particular, we have $f_t(t) = r_t$.

Appendix D. A multi-currency extension

The bond price model from Definition 2.9 can be easily extended to a setting with multiple currencies/economies, where each economy features multiple yield curves. Let $\mathcal{N}$ denote the number of currencies/economies.

Definition D.1. *An $\mathcal{N}$-economies bond price model is a family of indexed quintuples $(B^p, f_0^p, \widetilde{\alpha}^p, \widetilde{\sigma}^p, X)$, for $p = 1,..,\mathcal{N}$, and a family of exchange rate processes $(FX_t^{p,q})_{t\ge0}$, for $p,q = 1,\dots,\mathcal{N}, p\neq q$, such that*

(i) *$(B^p, f_0^p, \widetilde{\alpha}^p, \widetilde{\sigma}^p, X)$ is a bond price model, in the sense of Definition 2.9, with associated forward rate f^p and bond prices $\{(B^p(t,T))_{t\in[0,T]},\ T\ge0\}$, for all $p\in\{1,\dots,\mathcal{N}\}$,*

(ii) *the family of exchange rate processes $\{(FX_t^{p,q})_{t\ge0}; p,q=1,\dots,\mathcal{N}, p\neq q\}$ is defined by*

$$\frac{FX_t^{p,q}B_t^q}{B_t^p} = FX_0^{p,q}\exp\left(\int_0^t \hat{\alpha}_s^{p,q}ds + \int_0^t \hat{\sigma}_s^{p,q}dX_s\right), \tag{D.1}$$

where $\hat{\alpha}^{p,q}$ and $\hat{\sigma}^{p,q}$ for $p,q=1,\dots,\mathcal{N}$, $p\neq q$, are predictable $\mathbb{R}$- and $\mathbb{R}^d$-valued processes such that $\int_0^t|\hat{\alpha}_s^{p,q}|ds<\infty$ $\mathbb{Q}$-a.s. for all $t\ge0$ and $\hat{\sigma}^{p,q}\in L(X)$.

For each economy $p\in\{1,\dots,\mathcal{N}\}$, let us introduce the risk neutral measure $\mathbb{Q}^p$ with associated numéraire B^p. We can extend Definition 2.10 as follows.

Definition D.2. *An $\mathcal{N}$-economies bond price model is called* risk neutral *if*

(i) *each individual bond price model $(B^p, f_0^p, \widetilde{\alpha}^p, \widetilde{\sigma}^p, X)$ is risk neutral, i.e., the process*

$$\left(B^p(t,T)/B_t^p\right)_{t\in[0,T]}$$

is a $\mathbb{Q}^p$-martingale, for every $T\ge0$, for all $p\in\{1,\dots,\mathcal{N}\}$;

(ii) *the process*

$$\left(FX_t^{p,q}B_t^q/B_t^p\right)_{t\in[0,T]}$$

is a $\mathbb{Q}^p$-martingale, for every $T\ge0$, for all $p,q\in\{1,\dots,\mathcal{N}\}$, $p\neq q$.

Proposition 2.11 shows that each country-specific bond price model $(B^p, f_0^p, \widetilde{\alpha}^p, \widetilde{\sigma}^p, X)$ can be identified with an HJM-type model and, moreover, provides a characterization of its risk neutrality. As a corollary, we immediately obtain the following proposition.

Proposition D.3. *Let $\{(B^p, f_0^p, \widetilde{\alpha}^p, \widetilde{\sigma}^p, X); p=1,\dots,\mathcal{N}\}$ and $\{(FX_t^{p,q})_{t\ge0}; p,q=1,\dots,\mathcal{N}, p\neq q\}$ be an $\mathcal{N}$-economies bond price model. Then the following are equivalent*

(i) *The $\mathcal{N}$-economies bond price model is risk neutral.*

(ii) *The following conditions hold*

- *The process*

$$\left(\exp\left(\int_0^t \hat{\sigma}_s^{p,q}dX_s - \int_0^t \Psi_s^X\left(\hat{\sigma}_s^{p,q}\right)ds\right)\right)_{t\in[0,T]}$$

is a $\mathbb{Q}^p$-martingale, for every $T\ge0$ and for all $p,q\in\{1,\dots,\mathcal{N}\}$ with $p\neq q$.

- *$\hat{\alpha}_t^{p,q} = \Psi_t^{p,X}\left(\hat{\sigma}_t^{p,q}\right)$, for all $t\ge0$ and for all $p,q\in\{1,\dots,\mathcal{N}\}$ and $p\neq q$* (FX drift condition)*, where $\Psi^{p,X}$ denotes the local exponent of X under the measure $\mathbb{Q}^p$.*

- *Conditions (iii) from Proposition 2.11 hold for the bond price model* $(B^p, f_0^p, \widetilde{\alpha}^p, \widetilde{\sigma}^p, X)$ *with respect to the measure* $\mathbb{Q}^p$*, for all* $p \in \{1, \dots, \mathcal{N}\}$*.*

The semimartingale characteristics of X under each measure $\mathbb{Q}^p$ can be computed as in [51, Sect. 2.1 and 2.2]. We now proceed to define an $\mathcal{N}$-economies HJM-type multiple yield curve model in line with Definition 2.15. For the setup to be more realistic, we let the number of possible tenors be different across different countries.

Definition D.4. *Let* $\mathcal{N}$ *denote the number of countries and let the number of different tenors be* $m(p) := |\mathcal{D}^p|$*, for* $p = 1, ..., \mathcal{N}$*. We call a model consisting of*

- *an indexed family of* $\mathbb{R}^{d+n+1+\mathcal{N}-1}$*-valued semimartingales* $\left(X, Y, B^p, (FX^{p,q})_{q \neq p}\right)$*,*
- *vectors* $u_1^p, \dots, u_{m(p)}^p$ *in* $\mathbb{R}^n$*,*
- *functions* f_0^p*,* $\eta_0^{1,p}, \dots, \eta_0^{m(p),p}$*,*
- *processes* $\widetilde{\alpha}^p, \alpha^{1,p}, \dots, \alpha^{m(p),p}$ *and* $\widetilde{\sigma}^p, \sigma^{1,p}, \dots, \sigma^{m(p),p}$*,*

an $\mathcal{N}$-economies HJM-type multiple yield curve model *for* $\{(B^p(t,T))_{t\in[0,T]}, T \geq 0, p = 1, \dots, \mathcal{N}\}$ *and* $\{(S^{\delta,p}(t,T))_{t\in[0,T]}, T \geq 0, \delta \in \mathcal{D}^p, p = 1, \dots, \mathcal{N}\}$ *if*

(i) $\{(B^p, f_0^p, \widetilde{\alpha}^p, \widetilde{\sigma}^p, X); p = 1, \dots, \mathcal{N}\}$ *and* $\{(FX_t^{p,q})_{t\geq 0}; p, q = 1, \dots, \mathcal{N}, p \neq q\}$ *is an* $\mathcal{N}$*-economies bond price model,*

(ii) *for every* $i \in \{1, \dots, m(p)\}$*,* $(u_i^{p\top} Y, \eta_0^{i,p}, \alpha^{i,p}, \sigma^{i,p}, X)$ *is an HJM-type model for the family of semimartingales* $\{(S^{\delta_i,p}(t,T)), T \geq 0, i = 1, \dots, m(p)\}$*, for all* $p \in \{1, \dots, \mathcal{N}\}$*.*

For each $p \in \{1, \dots, \mathcal{N}\}$, let us denote by $\mathbb{Q}_p^T$ the T-forward measure associated to the numéraire $B^p(\cdot, T)$. In view of Definition 2.15, we can define as follows the notion of risk neutrality for an $\mathcal{N}$-economies HJM-type multiple yield curve model.

Definition D.5. *An* $\mathcal{N}$*-economies HJM-type multiple yield curve model is called* risk neutral *if*

- *The* $\mathcal{N}$*-economies bond price model is risk neutral,*
- $(S^{\delta_i,p}(t,T))_{t\in[0,T]}$ *is a* $\mathbb{Q}_p^T$*-martingale, for every* $T \geq 0$*,* $i \in \{1, \dots, m(p)\}$ *and* $p = 1, .., \mathcal{N}$*.*

Clearly, the risk neutrality of an $\mathcal{N}$-economies HJM-type multiple yield curve model can be characterized analogously as in Theorem 2.17.

Appendix E. Local independence of semimartingales and semimartingale decomposition

In this section, we let (X, Y) be a general Itô-semimartingale taking values in $\mathbb{R}^{d\times n}$ and denote by $\Psi^{X,Y}$ its local exponent and by Ψ^X and Ψ^Y the local exponents of X and Y, respectively, and let $\mathcal{U}^{X,Y}$ be defined as in Definition 2.3. In view of [41, Lemma A.11], the following definition is equivalent to the notion of local independence as given in [41, Definition A.10].

Definition E.1. *We say that* X *and* Y *are* locally independent *if, outside a* $d\mathbb{Q} \otimes dt$*-null set, it holds that*

$$\Psi_t^{X,Y}(u, v)(\omega) = \Psi_t^X(u)(\omega) + \Psi_t^Y(v)(\omega), \qquad \text{for all } (u, v) \in \mathcal{U}^{(X,Y)}.$$

Following [41, Appendix A.3], let us recall the notion of semimartingale decomposition of Y relative to X. We denote by $c^{Y,X}$ and c^X the second local characteristic of (Y, X) and X, respectively, and by $K^{Y,X}$ and K^X the third local characteristic of (Y, X) and X, respectively. Denote also by $\mu^{Y,X}$ the random jump measure of (Y, X). Supposing that $1 \in \mathcal{U}^Y$ (i.e., Y is exponentially special, see Proposition 2.4), let

(E.1)
$$Y^{\|,i} := \log \mathcal{E}\left(\int_0^\cdot \left(c_t^{Y^i,X}(c_t^X)^{-1}\right) dX_t^c + \int_0^\cdot \int (e^{y^i} - 1)\mathbf{1}_{\{x\neq 0\}}\left(\mu^{Y,X}(dy, dx, dt) - K_t^{Y,X}(dy, dx)dt\right)\right),$$

for $i = 1, \dots, n$, where $(c^X)^{-1}$ denotes the pseudoinverse of the matrix c^X and X^c is the continuous local martingale part of X (see [38, Proposition I.4.27]). We call $Y^\| := (Y^{\|,1}, \dots, Y^{\|,n})^\top$ the *dependent part of* Y *relative to* X and $Y^\perp := Y - Y^\|$ the *independent part of* Y *relative to* X. The following lemma corresponds to [41, Lemma A.22 and Lemma A.23].

Lemma E.2. *Let* (X, Y) *be an* $\mathbb{R}^{d\times n}$*-valued Itô-semimartingale such that* $1 \in \mathcal{U}^Y$*. Then the following hold:*

(i) $Y \mapsto Y^\|$ *is a projection, in the sense that* $(Y^\|)^\| = Y^\|$*;*

(ii) *if* Z *is an Itô-semimartingale locally independent of* X*, then it holds that* $(Z + Y)^\| = Y^\|$*;*

(iii) $\exp(Y^{\|,i})$ *is a local martingale, for all* $i = 1, \dots, n$;
(iv) $Y^\perp$ *and* $(Y^\|, X)$ *are locally independent semimartingales.*

Appendix F. Proof of Proposition 2.26

Let us fix any $i \in \{0, 1, \dots, m\}$. By the same argument as in [22, Corollary 5.12], it can be shown that $\kappa_4(H^\lambda_{m+1}) \subseteq H^{\lambda,0}_1$ and $\kappa^i_j(H^\lambda_{m+1}) \subseteq H^{\lambda,0}_1$ for $j = 1, 2$. In the sequel, C will always denote a positive constant which can vary from line to line. The following estimates can be derived analogously as in the proof of [25, Proposition 3.2] (to which we refer the reader for more details), by relying on Assumption 2.25, the Hölder inequality and on [25, Theorem 2.1].

Concerning $\kappa^i_3(h)$ and $\kappa^i_5(h)$, we have that, for all $h \in H^\lambda_{m+1}$, $\xi \in \mathbb{R}^d$ and $s \in \mathbb{R}_+$,

$$|(\zeta^{i0}(h)(s))^\top \xi| \le C\|\zeta^{i0}(h)\|_{\lambda,d}\|\xi\|_d \qquad \text{and} \qquad |(\zeta^0(h)(s))^\top \xi| \le C\|\zeta^0(h)\|_{\lambda,d}\|\xi\|_d,$$

and, for all $\xi \in \mathbb{R}^d$, $\hat\xi \in \mathbb{R}^n$ and $s \in \mathbb{R}_+$,

$$\begin{aligned}\left|e^{u_i^\top \hat\xi + (Z^{i0}(h)(s))^\top \xi} - 1\right| &\le Ce^{\|u_i\|_n\|\hat\xi\|_n + (C_0+C_i)\|\xi\|_d}(\|u_i\|_n\|\hat\xi\|_n + \|\zeta^{i0}(h)\|_{\lambda,d}\|\xi\|_d),\\ \left|e^{-(Z^0(h)(s))^\top \xi} - 1\right| &\le Ce^{C_0\|\xi\|_d}\|\zeta^0(h)\|_{\lambda,d}\|\xi\|_d.\end{aligned}$$

These estimates show together with (2.26) and (2.27) that $\lim_{s\to\infty}\kappa^i_3(h)(s) = 0$ and $\lim_{s\to\infty}\kappa^i_5(h)(s) = 0$. Moreover, we have

$$\begin{aligned}&\int_{\mathbb{R}_+}\left(\int \big((\zeta^{i0}(h)(s))^\top\xi\big)^2\big(e^{u_i^\top\hat\xi + (Z^{i0}(h)(s))^\top\xi}\big)K^{\hat Y,X}(d\hat\xi, d\xi)\right)^2 e^{\lambda s}ds\\ &\quad\le C(M_0+M_i)^2K_i\int e^{\|u_i\|_n\|\hat\xi\|_n + (C_0+C_i)\|\xi\|_d}\|\zeta^{i0}(h)\|^2_{\lambda,d}\|\xi\|^2_d K^{\hat Y,X}(d\hat\xi, d\xi) \le C(M_0+M_i)^4K_i^2,\\ &\int_{\mathbb{R}_+}\left(\int \big((\zeta^0(h)(s))^\top\xi\big)^2 e^{-(Z^0(h)(s))^\top\xi}F(d\xi)\right)^2 e^{\lambda s}ds \le CM_0^2K_0\int e^{C_0\|\xi\|_d}\|\zeta^0(h)\|^2_{\lambda,d}\|\xi\|^2_d F(d\xi) \le CM_0^4K_0^2,\end{aligned}$$

and

$$\begin{aligned}&\int_{\mathbb{R}_+}\left(\int \frac{d}{ds}\big((\zeta^{i0}(h)(s))^\top\xi\big)\big(e^{u_i^\top\hat\xi + (Z^{i0}(h)(s))^\top\xi} - 1\big)K^{\hat Y,X}(d\hat\xi, d\xi)\right)^2 e^{\lambda s}ds\\ &\quad\le C\int e^{\|u_i\|_n\|\hat\xi\|_n + (C_0+C_i)\|\xi\|_d}\|\zeta^{i0}(h)\|^2_{\lambda,d}\|\xi\|^2_d K^{\hat Y,X}(d\hat\xi, d\xi) \le C(M_0+M_i)^2K_i,\\ &\int_{\mathbb{R}_+}\left(\int \frac{d}{ds}\big(\zeta^0(h)\big)^\top\xi\big(e^{-(Z^0(h))^\top\xi} - 1\big)F(d\xi)\right)^2 e^{\lambda s}ds \le CK_0M_0^2\int e^{C_0\|\xi\|_d}\|\zeta^0(h)\|^2_{\lambda,d}\|\xi\|^2_d F(d\xi) \le CM_0^4K_0^2.\end{aligned}$$

In view of (2.28)-(2.29), this implies that $\kappa^i_j(H^\lambda_{m+1}) \subseteq H^{\lambda,0}_1$ for $j = 3, 5$. We have thus shown that $\kappa^i(H^\lambda_{m+1}) \subseteq H^{\lambda,0}_1$.

For $h_1, h_2 \in H^\lambda_{m+1}$ we obtain

$$\begin{aligned}\|\kappa^i_1(h_1) - \kappa^i_1(h_2)\|_{\lambda,1} &\le C(L_0+L_i)\|h_1 - h_2\|_{\lambda,m+1},\\ \|\kappa^i_2(h_1) - \kappa^i_2(h_2)\|_{\lambda,1} &\le C\big(\|\zeta^i(h_1)\|_{\lambda,d} + \|\zeta^i(h_2)\|_{\lambda,d} + \|\zeta^0(h_1)\|_{\lambda,d} + \|\zeta^0(h_2)\|_{\lambda,d}\big)\\ &\quad\times\big(\|\zeta^i(h_1) - \zeta^i(h_2)\|_{\lambda,d} + \|\zeta^0(h_1) - \zeta^0(h_2)\|_{\lambda,d}\big)\\ &\le C(2M_i + 2M_0)(L_0+L_i)\|h_1 - h_2\|_{\lambda,m+1},\\ \|\kappa_4(h_1) - \kappa_4(h_2)\|_{\lambda,1} &\le C\big(\|\zeta^0(h_1)\|_{\lambda,d} + \|\zeta^0(h_2)\|_{\lambda,d}\big)\|\zeta^0(h_1) - \zeta^0(h_2)\|_{\lambda,d}\\ &\le C2M_0L_0\|h_1 - h_2\|_{\lambda,m+1}.\end{aligned}$$

Furthermore, due to (2.28) and (2.29), we can estimate

$$\begin{aligned}\|\kappa^i_3(h_1) - \kappa^i_3(h_2)\|^2_{\lambda,1} &\le 4(I^i_1 + I^i_2 + I^i_3 + I^i_4),\\ \|\kappa_5(h_1) - \kappa_5(h_2)\|^2_{\lambda,1} &\le 4(J_1 + J_2 + J_3 + J_4),\end{aligned}$$

where

$$I^i_1 = \int_{\mathbb{R}_+}\left(\int \big((\zeta^{i0}(h_1)(s))^\top\xi\big)^2 e^{u_i^\top\hat\xi}\big(e^{(Z^{i0}(h_1)(s))^\top\xi} - e^{(Z^{i0}(h_2)(s))^\top\xi}\big)K^{\hat Y,X}(d\hat\xi, d\xi)\right)^2 e^{\lambda s}ds,$$

$$I_2^i = \int_{\mathbb{R}_+} \left(\int e^{u_i^\top \hat{\xi} + (Z^{i0}(h_2)(s))^\top \xi} \Big(\big((\zeta^{i0}(h_1)(s))^\top \xi\big)^2 - \big((\zeta^{i0}(h_2)(s))^\top \xi\big)^2 \Big) K^{\hat{Y},X}(d\hat{\xi}, d\xi) \right)^2 e^{\lambda s} ds,$$

$$I_3^i = \int_{\mathbb{R}_+} \left(\int \frac{d}{ds} \big(\zeta^{i0}(h_1)(s)\big)^\top \xi e^{u_i^\top \hat{\xi}} \big(e^{(Z^{i0}(h_1)(s))^\top \xi} - e^{(Z^{i0}(h_2)(s))^\top \xi} \big) K^{\hat{Y},X}(d\hat{\xi}, d\xi) \right) e^{\lambda s} ds,$$

$$I_4^i = \int_{\mathbb{R}_+} \left(\int \Big(e^{u_i^\top \hat{\xi} + (Z^{i0}(h_2)(s))^\top \xi} - 1 \Big) \left(\frac{d}{ds} \big(\zeta^{i0}(h_1)(s)\big)^\top \xi - \frac{d}{ds} (\zeta^{i0}(h_2)(s))^\top \xi \right) K^{\hat{Y},X}(d\hat{\xi}, d\xi) \right)^2 e^{\lambda s} ds,$$

$$J_1 = \int_{\mathbb{R}_+} \left(\int \big((\zeta^0(h_1)(s))^\top \xi\big)^2 \big(e^{-(Z^0(h_1)(s))^\top \xi} - e^{-(Z^0(h_2)(s))^\top \xi} \big) F(d\xi) \right)^2 e^{\lambda s} ds,$$

$$J_2 = \int_{\mathbb{R}_+} \left(\int e^{-(Z^0(h_2)(s))^\top \xi} \Big(\big((\zeta^0(h_1)(s))^\top \xi\big)^2 - \big((\zeta^0(h_2)(s))^\top \xi\big)^2 \Big) F(d\xi) \right)^2 e^{\lambda s} ds$$

$$J_3 = \int_{\mathbb{R}_+} \left(\int \frac{d}{ds} \big(\zeta^0(h_1)(s)\big)^\top \xi \big(e^{-(Z^0(h_1)(s))^\top \xi} - e^{-(Z^0(h_2)(s))^\top \xi} \big) F(d\xi) \right)^2 e^{\lambda s} ds,$$

$$J_4 = \int_{\mathbb{R}_+} \left(\int \big(e^{-(Z^0(h_2)(s))^\top \xi} - 1 \big) \left(\frac{d}{ds} \big(\zeta^0(h_1)(s)\big)^\top \xi - \frac{d}{ds} \big(\zeta^0(h_2)(s)\big)^\top \xi \right) F(d\xi) \right)^2 e^{\lambda s} ds,$$

We get for all $\xi \in \mathbb{R}^d$, $s \in \mathbb{R}_+$

$$\big| e^{(Z^{i0}(h_1)(s))^\top \xi} - e^{(Z^{i0}(h_2(s)))^\top \xi} \big| \le C e^{(C_0 + C_i)\|\xi\|_d} \big(\|\zeta^i(h_1) - \zeta^i(h_2)\|_{\lambda,d} + \|\zeta^0(h_1) - \zeta^0(h_2)\|_{\lambda,d} \big) \|\xi\|_d,$$

$$\big| e^{-(Z^0(h_1))^\top \xi} - e^{-(Z^0(h_2))^\top \xi} \big| \le C e^{C_0 \|\xi\|_d} \|\zeta^0(h_1) - \zeta^0(h_2)\|_{\lambda,d} \|\xi\|_d.$$

Therefore,

$$\begin{aligned}
I_1^i &\le C \int_{\mathbb{R}_+} \left(\int \big((\zeta^{i0}(h_1)(s))^\top \xi\big)^2 e^{\|u_i\|_n \|\hat{\xi}\|_n + (C_0 + C_i)\|\xi\|_d} \right. \\
&\quad \left. \times \big(\|\zeta^i(h_1) - \zeta^i(h_2)\|_{\delta,d} + \|\zeta^0(h_1) - \zeta^0(h_2)\|_{\lambda,d} \big) \|\xi\|_d K^{\hat{Y},X}(d\hat{\xi}, d\xi) \right)^2 e^{\lambda s} ds, \\
&\le C K_i (L_i^2 + L_0^2) \|h_1 - h_2\|_{\lambda,m+1}^2 \int e^{\|u_i\|_n \|\hat{\xi}\|_n + (C_0 + C_i)\|\xi\|_d} \int_{\mathbb{R}_+} \big((\zeta^{i0}(h_1)(s))^\top \xi\big)^4 e^{\lambda s} ds K^{\hat{Y},X}(d\hat{\xi}, d\xi) \\
&\le C K_i (L_i^2 + L_0^2) \|h_1 - h_2\|_{\lambda,m+1}^2 \int e^{\|u_i\|_n \|\hat{\xi}\|_n + (C_0 + C_i)\|\xi\|_d} \|(\zeta^{i0}(h_1)\|_{\lambda,d}^4 \|\xi\|_d^4 K^{\hat{Y},X}(d\hat{\xi}, d\xi) \\
&\le C K_i^2 (M_i^4 + M_0^4)(L_i^2 + L_0^2) \|h_1 - h_2\|_{\lambda,m+1}^2, \\
J_1 &\le C \int_{\mathbb{R}_+} \left(\int \big((\zeta^0(h_1)(s))^\top \xi\big)^2 e^{C_0 \|\xi\|_d} \|\zeta^0(h_1) - \zeta^0(h_2)\|_{\lambda,d} \|\xi\|_d F(d\xi) \right)^2 e^{\lambda s} ds \\
&\le C K_0 L_0^2 \|h_1 - h_2\|_{\lambda,m+1}^2 \int e^{C_0 \|\xi\|_d} \int_{\mathbb{R}_+} \big((\zeta^0(h_1)(s))^\top \xi\big)^4 e^{\lambda s} ds F(d\xi) \\
&\le C K_0 L_0^2 \|h_1 - h_2\|_{\lambda,m+1}^2 \int e^{C_0 \|\xi\|_d} \|\zeta^0(h_1)\|_{\lambda,d}^4 \|\xi\|_d^4 F(d\xi) \le C K_0^2 M_0^4 L_0^2 \|h_1 - h_2\|_{\lambda,m+1}^2.
\end{aligned}$$

Moreover, for every $s \in \mathbb{R}_+$, we obtain

$$\int e^{\|u_i\|_n \|\hat{\xi}\|_n + (C_0 + C_i)\|\xi\|_d} \big((\zeta^{i0}(h_1)(s))^\top \xi + (\zeta^{i0}(h_2)(s))^\top \xi \big)^2 K^{\hat{Y},X}(d\hat{\xi}, d\xi) \le 2C(M_i^2 + M_0^2) K_i,$$

$$\int e^{C_0 \|\xi\|_d} \big((\zeta^0(h_1)(s))^\top \xi + (\zeta^0(h_2)(s))^\top \xi \big)^2 F(d\xi) \le 2C M_0^2 K_0.$$

Hence,

$$\begin{aligned}
I_2^i &\le 2C(M_i + M_0) K_i \int e^{\|u_i\|_n \|\hat{\xi}\|_n + (C_0 + C_i)\|\xi\|_d} \\
&\qquad \times \int_{\mathbb{R}_+} \Big(\big(\zeta^i(h_1)(s) - \zeta^i(h_2)(s)\big)^\top \xi + \big(\zeta^0(h_2)(s) - \zeta^0(h_1)(s)\big)^\top \xi \Big)^2 e^{\lambda s} ds K^{\hat{Y},X}(d\hat{\xi}, d\xi) \\
&\le 2C(M_i + M_0) K_i^2 (L_0^2 + L_i^2) \|h_1 - h_2\|_{\lambda,m+1}^2, \\
J_2 &\le 2C M_0 K_0 \int e^{C_0 \|\xi\|_d} \int_{\mathbb{R}_+} \Big(\big(\zeta^0(h_2)(s) - \zeta^0(h_1)(s)\big)^\top \xi \Big)^2 e^{\lambda s} ds F(d\xi) \\
&\le 2C M_0 K_0^2 L_0^2 \|h_1 - h_2\|_{\lambda,m+1}^2.
\end{aligned}$$

Moreover,

$$\begin{aligned}
I_3^i &\le CK_i(L_i^2+L_0^2)\|h_1-h_2\|^2_{\lambda,m+1}\int e^{\|u_i\|_n\|\hat\xi\|_n+(C_0+C_i)\|\xi\|_d}\int_{\mathbb{R}_+}\left(\frac{d}{ds}\big(\zeta^{i0}(h_1)(s)\big)^\top\xi\right)^2 e^{\lambda s}ds K^{\hat Y,X}(d\hat\xi,d\xi)\\
&\le CK_i(L_i^2+L_0^2)\|h_1-h_2\|^2_{\lambda,m+1}\int e^{\|u_i\|_n\|\hat\xi\|_n+(C_0+C_i)\|\xi\|_d}\big(\|\zeta^i(h_1)\|^2_{\lambda,m+1}+\|\zeta^0(h_1)\|^2_{\lambda,d}\big)\|\xi\|_d^2K^{\hat Y,X}(d\hat\xi,d\xi)\\
&\le CK_i^2(L_i^2+L_0^2)(M_0+M_i)\|h_1-h_2\|^2_{\lambda,m+1},
\end{aligned}$$

$$\begin{aligned}
J_3 &\le CK_0L_0^2\|h_1-h_2\|^2_{\lambda,m+1}\int e^{C_0\|\xi\|_d}\int_{\mathbb{R}_+}\left(\frac{d}{ds}\big(\zeta^0(h_1)(s)\big)^\top\xi\right)^2 e^{\lambda s}ds F(d\xi)\\
&\le CK_0L_0^2\|h_1-h_2\|^2_{\lambda,m+1}\int e^{C_0\|\xi\|_d}\|\zeta^0(h_1)\|^2_{\lambda,d}\|\xi\|_d^2F(d\xi)\\
&\le CK_0^2L_0^2M_0\|h_1-h_2\|^2_{\lambda,m+1}.
\end{aligned}$$

Finally, we have

$$\begin{aligned}
I_4^i &\le C\int_{\mathbb{R}_+}\Bigg(\int e^{\|u_i\|_n\|\hat\xi\|_n+(C_0+C_i)\|\xi\|_d}\big(\|u_i\|_n\|\hat\xi\|_n+\|\zeta^{i0}(h)\|_{\lambda,d}\|\xi\|_d\big)\\
&\quad\times\left(\frac{d}{ds}\big(\zeta^{i0}(h_1)(s)\big)^\top\xi-\frac{d}{ds}\big(\zeta^{i0}(h_2)(s)\big)^\top\xi\right)K^{\hat Y,X}(d\hat\xi,d\xi)\Bigg)^2e^{\lambda s}ds\\
&\le C(u_i+M_0+M_i)^2K_i\int e^{\|u_i\|_n\|\hat\xi\|_n+(C_0+C_i)\|\xi\|_d}\\
&\quad\times\int_{\mathbb{R}_+}\left(\frac{d}{ds}\big(\zeta^{i0}(h_1)(s)\big)^\top\xi-\frac{d}{ds}\big(\zeta^{i0}(h_2)(s)\big)^\top\xi\right)^2e^{\delta s}ds K^{\hat Y,X}(d\hat\xi,d\xi)\\
&\le C(u_i+M_0+M_i)^2K_i\\
&\quad\times\int e^{\|u_i\|_n\|\hat\xi\|_n+(C_0+C_i)\|\xi\|_d}\big(\|\zeta^i(h_1)-\zeta^i(h_2)\|^2_{\lambda,d}+\|\zeta^0(h_1)-\zeta^0(h_2)\|^2_{\lambda,d}\big)\|\xi\|_d^2K^{\hat Y,X}(d\hat\xi,d\xi)\\
&\le C(u_i+M_0+M_i)^2K_i^2(L_0+L_i)^2\|h_1-h_2\|^2_{\lambda,m+1},\\
J_4 &\le C\int_{\mathbb{R}_+}\left(\int e^{C_0\|\xi\|_d}\|\zeta^0(h)\|_{\lambda,d}\|\xi\|_d\left(\frac{d}{ds}\big(\zeta^0(h_1)(s)\big)^\top\xi-\frac{d}{ds}\big(\zeta^0(h_2)(s)\big)^\top\xi\right)F(d\xi)\right)^2e^{\lambda s}ds,\\
&\le CM_0^2K_0\int e^{C_0\|\xi\|_d}\int_{\mathbb{R}_+}\left(\frac{d}{ds}(\zeta^0(h_1)(s))^\top\xi-\frac{d}{ds}(\zeta^0(h_2)(s))^\top\xi\right)^2e^{\lambda s}ds F(d\xi)\\
&\le CM_0^2K_0\int e^{C_0\|\xi\|_d}\|\zeta^0(h_1)-\zeta^0(h_2)\|^2_{\lambda,d}\|\xi\|_d^2F(d\xi)\\
&\le CM_0^2K_0^2L_0^2\|h_1-h_2\|^2_{\lambda,m+1}.
\end{aligned}$$

Summing up, we have shown that there exist constants $Q_i>0$ such that condition (2.30) is satisfied for all $h_1,h_2\in H^\lambda_{m+1}$.

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(Christa Cuchiero) Vienna University of Technology
Financial and Actuarial Mathematics,
Wiedner Hauptstrasse 8-10/105-1 1040 Wien, Austria
E-mail address, Christa Cuchiero: `cuchiero@fam.tuwien.ac.at`

(Claudio Fontana) Laboratoire de Mathématiques et Modélisation, Université d'Évry Val d'Essonne,
23 boulevard de France, F-91037 Évry Cedex, France
E-mail address, Claudio Fontana: `claudio.fontana@univ-evry.fr`

(Alessandro Gnoatto) Mathematisches Institut der LMU München,
Theresienstrasse, 39 D-80333 München
E-mail address, Alessandro Gnoatto: `gnoatto@mathematik.uni-muenchen.de`