On an optional semimartingale decomposition and the existence of the deflator in an enlarged filtration

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Abstract In this paper, we present a semimartingale decomposition formula in an enlargement of filtration setting. Given a reference filtration $\mathbb{F}$, we consider the cases where the enlarged filtration $\mathbb{G}$ is constructed from $\mathbb{F}$ in two different ways: progressively and initially. We focus on two specific situations where the hypothesis $(\mathscr{H}')$, i.e., the property that each $\mathbb{F}$-martingale remains an $\mathbb{G}$-semimartingale, is satisfied. In both situations, under suitable conditions, we develop a $\mathbb{G}$-optional semimartingale decomposition for $\mathbb{F}$-local martingales. Our study is then applied to the question of how an arbitrage-free semimartingale model is affected when stopped at a random horizon or when a random variable satisfying Jacod's hypothesis is incorporated. Precisely, we focus on the No-Unbounded-Profit-with-Bounded-Risk (NUPBR) condition. We provide alternative proofs of some results from [2], incorporating a different methodology based on our optional semimartingale decomposition. Moreover, analogous results are formulated for an initial enlargement with a random variable satisfying Jacod's hypothesis.

Introduction

We study two filtrations $\mathbb{F}$ and $\mathbb{G}$ such that $\mathbb{F} \subset \mathbb{G}$. We consider the cases where the enlarged filtration $\mathbb{G}$ is constructed from $\mathbb{F}$ in two different ways: progressively

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and initially. We focus on two specific situations where hypothesis $(\mathscr{H}')$, i.e., the property that each $\mathbb{F}$-martingale remains a $\mathbb{G}$-semimartingale, is satisfied. In both situations, under suitable conditions, we develop a $\mathbb{G}$-optional semimartingale decomposition for $\mathbb{F}$-local martingales (Sections 2.1 and 3.1). An interesting link with absolutely continuous change of measure problem is observed in Section 4.

In Section 2.1, we introduce a new semi-martingale decomposition, which will be useful in the sequel, and we give the link between this new tool and the deflators exhibited in the literature [2]. In Section 2.2 we provide alternative and shorter proofs of some results from [2], using a methodology different from the one used in that paper avoiding the introduction of optional integral, but based on our optional semimartingale decomposition. In Section 3.2 analogous results are formulated for initial enlargement with a random variable satisfying Jacod's hypothesis. The results presented in this last section were obtained in parallel and independently of [1].

1 Preliminaries

Let $(\Omega,\mathscr{G},\mathbb{P})$ be a complete probability space and $\mathbb{F}=(\mathscr{F}_t)_{t\geq 0}$ be a filtration satisfying the usual conditions. We say that a filtration $\mathbb{G}=(\mathscr{G}_t)_{t\geq 0}$ is an enlargement of $\mathbb{F}$ if, for each $t\geq 0$, we have $\mathscr{F}_t\subset\mathscr{G}_t$. For an enlargement of filtration, the hypothesis $(\mathscr{H}')$ is said to be satisfied if each $\mathbb{F}$-local martingale is a $\mathbb{G}$-semimartingale. In this work, we consider two specific situations of progressive and initial enlargement, where hypothesis $(\mathscr{H}')$ is satisfied.

We recall some standard definitions and set some notation. For a filtration $\mathbb{H}$, the optional σ-field on $\Omega\times\mathbb{R}_+$, denoted by $\mathscr{O}(\mathbb{H})$, is the σ-field generated by all càdlàg $\mathbb{H}$-adapted processes and the predictable σ-field on $\Omega\times\mathbb{R}_+$, denoted by $\mathscr{P}(\mathbb{H})$, is the σ-field generated by all left-continuous $\mathbb{H}$-adapted processes. A stochastic set or process is called $\mathbb{H}$-optional (respectively $\mathbb{H}$-predictable) if it is $\mathscr{O}(\mathbb{H})$-measurable (respectively $\mathscr{P}(\mathbb{H})$-measurable).

For an $\mathbb{H}$-semimartingale Y, the set of $\mathbb{H}$-predictable processes integrable with respect to Y is denoted by $L(Y,\mathbb{H})$ and for $H\in L(Y,\mathbb{H})$, we denote by $H\bullet Y$ the stochastic integral $\int_0^{\cdot} H_s dY_s$.

As usual, for a process X and a random time ϑ, we denote by X^{ϑ} the stopped process. For a given semimartingale X, $\mathscr{E}(X)$ stands for the stochastic exponential of X. The continuous local martingale part and the jump process of a càdlàg semimartingale X are denoted respectively by X^c and ΔX.

1.1 Progressively enlarged filtration

Consider a random time τ, i.e., a positive random variable. Then, as in [13], we define several processes associated with τ. Two $\mathbb{F}$-supermartingales are given by

$$Z_t := \mathbb{P}(\tau > t|\mathscr{F}_t) \quad \text{and} \quad \widetilde{Z}_t := \mathbb{P}(\tau \geq t|\mathscr{F}_t).$$

Let A^o be the $\mathbb{F}$-dual optional projection of $A := \mathbb{1}_{[\![\tau,\infty[\![}$; then the process

$$m := Z + A^o \tag{1}$$

is an $\mathbb{F}$-martingale. Furthermore, $\widetilde{Z} = Z_- + \Delta m$.

We denote by $\mathbb{F}^\tau = (\mathscr{F}_t^\tau)_{t\geq 0}$ the right continuous progressively enlarged filtration with the random time τ, i.e.,

$$\mathscr{F}_t^\tau := \bigcap_{s>t} (\mathscr{F}_s \vee \sigma(\tau \wedge s)).$$

The following result from [14] states that any $\mathbb{F}$-local martingale stopped at τ is an $\mathbb{F}^\tau$-semimartingale.

Proposition 1. *Let X be an $\mathbb{F}$-local martingale. Then, X^τ is an $\mathbb{F}^\tau$-semimartingale which can be decomposed as*

$$X_t^\tau = \widehat{X}_t + \int_0^{t\wedge\tau} \frac{1}{Z_{s-}} d\langle X, m\rangle_s^{\mathbb{F}} \tag{2}$$

where $\widehat{X}$ is an $\mathbb{F}^\tau$-local martingale.

In what follows, we will call the equality (2) the predictable decomposition of the semi-martingale X^τ.

Remark 1. Using the $\mathbb{F}$-local martingale N which appears in Kardaras' (optional) multiplicative decomposition [17], i.e.,

$$N := \mathscr{E}\left(\frac{1}{Z_-}\mathbb{1}_{\{Z_->0\}} \cdot m\right), \tag{3}$$

the $\mathbb{F}^\tau$-predictable decomposition can be written as

$$X_t^\tau = \widehat{X}_t + \int_0^{t\wedge\tau} \frac{1}{N_{s-}} d\langle X, N\rangle_s^{\mathbb{F}}. \tag{4}$$

1.2 Initially enlarged filtration

Let ξ be a random variable valued in a Lusin space $(\mathbb{U}, \mathscr{U})$. We assume that the random variable ξ satisfies (absolute continuity) Jacod's hypothesis, i.e., there exists a σ-finite positive measure η on $(\mathbb{U}, \mathscr{U})$ such that for every $t \geq 0$, $\mathbb{P}(\xi \in du|\mathscr{F}_t)(\omega) \ll \eta(du)$ $\mathbb{P}$-a.s. If, in particular, for every $t \geq 0$, we have $\mathbb{P}(\xi \in du|\mathscr{F}_t)(\omega) \sim \eta(du)$, $\mathbb{P}$-a.s., then we say that ξ satisfies equivalence Jacod's hypothesis. As shown by Jacod [12], without loss of generality, η can be taken as the

law of ξ. We do not impose any condition on η, in particular, it is not necessarily atomless.

Let $\mathbb{F}^{\sigma(\xi)} = (\mathscr{F}^{\sigma(\xi)})_{t\geq 0}$ be the right continuous initial enlargement of the filtration $\mathbb{F}$ with the random variable ξ, i.e.,

$$\mathscr{F}_t^{\sigma(\xi)} := \bigcap_{s>t} (\mathscr{F}_s \vee \sigma(\xi)).$$

The next theorem gives, in the case of equivalence Jacod's hypothesis, a particular change of measure making the reference filtration $\mathbb{F}$ and the random variable ξ independent, see [25], [4, Proposition 1.6], [9].

Theorem 1. *Assume that the equivalence Jacod's hypothesis is satisfied. Then*
(a) the process $\frac{1}{q_t^\xi}$ is an $\mathbb{F}^{\sigma(\xi)}$-martingale,
(b) the probability measure $\mathbb{P}^*$, defined as

$$\frac{d\mathbb{P}^*}{d\mathbb{P}}\Big|_{\mathscr{F}_t^{\sigma(\xi)}} = \frac{1}{q_t^\xi}$$

has the following properties:
(i) under $\mathbb{P}^*$, τ is independent from $\mathscr{F}_t$ for any t,
(ii) $\mathbb{P}^*|_{\mathscr{F}_t} = \mathbb{P}|_{\mathscr{F}_t}$,
(iii) $\mathbb{P}^*|_{\sigma(\xi)} = \mathbb{P}|_{\sigma(\xi)}$.

The following result is due to Jacod (see [12, Lemme 1.8]). We recall here the formulation of Amendinger as it provides nice measurability property (see Remark 1 on page 17 [4]).

Proposition 2. *For ξ satisfying absolute continuity Jacod's hypothesis, there exists a positive $\mathscr{O}\otimes\mathscr{U}$-measurable function $(t,\omega,u)\to q_t^u(\omega)$ càdlàg in t such that*
(i) for every u, the process $(q_t^u, t\geq 0)$ is an $\mathbb{F}$-martingale, and if

$$R^u := \inf\{t : q_{t-}^u = 0\} \tag{5}$$

we have $q^u > 0$ and $q_-^u > 0$ on $[\![0,R^u[\![$ and $q^u = 0$ on $[\![R^u,\infty[\![$,
(ii) for every $t\geq 0$, the measure $q_t^u(\omega)\eta(du)$ is a version of $\mathbb{P}(\xi\in du|\mathscr{F}_t)(\omega)$.

In the following lemma, $\mathbb{F}$-predictable and $\mathbb{F}$-optional projections of some $\mathbb{F}^{\sigma(\xi)}$-adapted processes are given. The first part is due to Jacod [12, Lemme 1.10], the second part is found in Amendinger [4, Lemma 1.3].

Lemma 1. *(i) Assume that the map $(t,\omega,u)\to Y_t^u(\omega)$ is $\mathscr{P}(\mathbb{F})\otimes\mathscr{U}$-measurable, positive or bounded. Then, the $\mathbb{F}$-predictable projection of the process $(Y_t^\xi)_{t\geq 0}$ is given by*

$${}^{p,\mathbb{F}}\left(Y^\xi\right)_t = \int_{\mathbb{U}} Y_t^u q_{t-}^u \eta(du) \quad t\geq 0. \tag{6}$$

(ii) Assume that the map $(t,\omega,u)\to Y_t^u(\omega)$ is $\mathscr{O}(\mathbb{F})\otimes\mathscr{U}$-measurable, positive or bounded. Then, the $\mathbb{F}$-optional projection of the process $(Y_t^\xi)_{t\geq 0}$ is given by

$$^{o,\mathbb{F}}\left(Y^{\xi}\right)_t = \int_{\mathbb{U}} Y_t^u q_t^u \eta(du) \quad t \geq 0. \tag{7}$$

As noticed in Jacod [12, Corollary 1.11], Lemma 1 implies in particular that

$$R^{\xi} = \infty \quad \mathbb{P}-\text{a.s.} \tag{8}$$

where R^u is defined through (5), or equivalently $q_t^{\xi} > 0$ and $q_{t-}^{\xi} > 0$, for $t \geq 0$, $\mathbb{P}$-a.s. Therefore, the $\mathbb{F}^{\sigma(\xi)}$-optional process $(\frac{1}{q_t^{\xi}})_{t\geq 0}$ is well-defined.

The $\mathbb{F}^{\sigma(\xi)}$-semimartingale (predictable) decomposition of an $\mathbb{F}$-local martingale is given in [12, Theorem 2.5] in the following way:

Proposition 3. *Let X be an $\mathbb{F}$-local martingale. Then*

$$X_t = \widehat{X}_t + \int_0^t \frac{1}{q_{s-}^{\xi}} d\langle X, q^u\rangle_s|_{u=\xi}$$

where $\widehat{X}$ is an $\mathbb{F}^{\sigma(\xi)}$-local martingale.

Definition 1. Consider a mapping $X : (t,\omega,u) \to X_t^u(\omega)$ on $\mathbb{R}_+ \times \Omega \times \mathbb{U}$ with values in $\mathbb{R}$. Let $\mathscr{J}$ be a class of $\mathbb{F}$-optional processes, for example the class of $\mathbb{F}$-(local) martingales or the class of $\mathbb{F}$-locally integrable variation processes. Then, $(X^u, u \in \mathbb{U})$ is called a *parametrized* $\mathscr{J}$-process if for each $u \in \mathbb{U}$ the process X^u belongs to $\mathscr{J}$ and if it is measurable with respect to $\mathscr{O}(\mathbb{F}) \otimes \mathscr{U}$.

By [26, Proposition 3], the second condition can be obtained by taking appropriate versions of processes X^u.

To ensure the existence of well measurable versions of dual projections of parametrized processes, we assume from now on that the space $L^1(\Omega, \mathscr{G}, \mathbb{P})$ is separable. Then, we apply [26, Proposition 4].

In the next proposition, we state an analogical result to Proposition 3 for a bigger class of processes: parametrized $\mathbb{F}$-local martingales.

Proposition 4. *Let $(X^u, u \in \mathbb{U})$ be a parametrized $\mathbb{F}$-local martingale. Then*

$$X_t^{\xi} = \widehat{X}_t^{\xi} + \int_0^t \frac{1}{q_{s-}^{\xi}} d\langle X^u, q^u\rangle_s|_{u=\xi}$$

where $\widehat{X}^{\xi}$ is an $\mathbb{F}^{\sigma(\xi)}$-local martingale.

Proof. Let X be of the form $X_t^u(\omega) = g(u)x_t(\omega)$ where x is an $\mathbb{F}$-martingale and g is a Borel function. Then, $X^{\xi} = g(\xi)x$ and, using Jacod's decomposition given in Proposition 3, for $t \geq s$, we have

$$\begin{aligned}\mathbb{E}(X_t^{\xi} - X_s^{\xi}|\mathscr{F}_s^{\sigma(\xi)}) &= g(\xi)\mathbb{E}(x_t - x_s|\mathscr{F}_s^{\sigma(\xi)}) \\ &= g(\xi)\mathbb{E}\left(\int_s^t \frac{1}{q_{v-}^{\xi}} d\langle x, q^u\rangle_v|_{u=\xi}\,\Big|\,\mathscr{F}_s^{\sigma(\xi)}\right) \\ &= \mathbb{E}\left(\int_s^t \frac{1}{q_{v-}^{\xi}} d\langle X^u, q^u\rangle_v|_{u=\xi}\,\Big|\,\mathscr{F}_s^{\sigma(\xi)}\right).\end{aligned}$$

For a general X, we proceed by Monotone Class Theorem. □

1.3 Local martingale deflators and related notions

In this paper, we study the No Unbounded Profit with Bounded Risk (NUPBR) condition of no arbitrage. Let $\mathbb{H}$ be one of the filtrations $\{\mathbb{F}, \mathbb{F}^{\tau}, \mathbb{F}^{\sigma(\xi)}\}$. Then we have the following definition and alternative characterization of NUPBR condition. NUPBR condition is related to other no arbitrage conditions like No Free Lunch with Vanishing Risk or (classical) No Arbitrage, see [6, 15]. However, NUPBR condition is proved to be an appropriate condition to study some financial notions like numéraire portfolio, or market viability (see [5, 10, 16, 18, 19, 20, 23] and references therein).

Definition 2. (a) Let $(X^u, u \in \mathbb{U})$ be a parametrized $\mathbb{H}$-semimartingale. We say that $(X^u, u \in \mathbb{U})$ satisfies No Unbounded Profit with Bounded Risk condition if for each $T < \infty$, $\mathscr{K}_T(X)$ defined as

$$\mathscr{K}_T(X) := \{(H \bullet X^u)_T : H \in L(\mathbb{H}, X^u) \text{ and } H \bullet X^u \geq -1 \text{ for each } u \in \mathbb{U}\}$$

is bounded in probability.
(b) A process Y is called an $\mathbb{H}$-local martingale deflator for $(X^u, u \in \mathbb{U})$ if it is a strictly positive $\mathbb{H}$-local martingale such that $(YX^u, u \in \mathbb{U})$ is a parametrized $\mathbb{H}$-σ-martingale.
(c) A process $\widetilde{Y}$ is called an $\mathbb{H}$-supermartingale deflator for $(X^u, u \in \mathbb{U})$ if it is a strictly positive $\mathbb{H}$-supermartingale such that for each $H \in L(\mathbb{H}, X^u)$ with $H \bullet X^u \geq -1$, the process $(1 + H \bullet X^u)\widetilde{Y}$ is an $\mathbb{H}$-supermartingale.

As proved in [22] in full generality, condition (a) and the existence of a supermartingale deflator in Definition 2 are equivalent. Moreover, as shown in [24], for a process which does not depend on parameter, condition (a) and the existence of a local martingale deflator are equivalent. So, we have the following theorem:

Theorem 2. *For a semimartingale X, NUPBR is equivalent to the existence of a local martingale deflator and to the existence of a supermartingale deflator.*

The following proposition is a parametrized version of Proposition 2.5 from [2].

Proposition 5. *Let $(X^u, u \in \mathbb{U})$ be a parametrized $\mathbb{H}$-adapted semi-martingale. Then, the following assertions are equivalent.*
(a) The process $(X^u, u \in \mathbb{U})$ admits an $\mathbb{H}$-local martingale deflator.
(b) There exist a $\mathscr{P}(\mathbb{H}) \otimes \mathscr{U}$-measurable parametrized process $(\phi^u, u \in \mathbb{U})$ such that $0 < \phi^u \leq 1$ and an $\mathbb{H}$-local martingale deflator for $(\phi^u \bullet X^u, u \in \mathbb{U})$.
(c) There exists a sequence of $\mathbb{H}$-stopping times $(T_n)_{n\geq 1}$ that increases to ∞ such that for each $n \geq 1$, there exist a probability $\mathbb{Q}_n$ on $(\Omega, \mathscr{F}_{T_n})$ such that $\mathbb{Q}_n \sim \mathbb{P}$ and an $\mathbb{H}$-local martingale deflator for $((X^u)^{T_n}, u \in \mathbb{U})$ under $\mathbb{Q}_n$.

2 Progressive Enlargement up to a Random Time

2.1 Optional semimartingale decomposition for progressive enlargement

In this section, we derive an $\mathbb{F}^\tau$-semimartingale decomposition of the $\mathbb{F}$-local martingales stopped at τ different from the one in Proposition 1. Let us start by the definition of a particular $\mathbb{F}$-stopping time R

$$R := \inf\{t \geq 0 \ : \ Z_t = 0\}.$$

Then, the $\mathbb{F}$-stopping times defined as

$$\bar{R} := R_{\{Z_{R-}=0\}} \qquad \widetilde{R} := R_{\{\widetilde{Z}_R=0<Z_{R-}\}} \qquad \widehat{R} := R_{\{\widetilde{Z}_R>0\}}$$

have disjoint graphs and $R = \widehat{R} \wedge \widetilde{R} \wedge \bar{R}$. We note that $\bar{R}$ is an $\mathbb{F}$-predictable stopping time.

We establish an optional decomposition in the following theorem. By abuse of language, we shall refer to this decomposition as "the" optional decomposition even if there is no uniqueness for optional decomposition of a semimartingale.

Theorem 3. *Let X be an $\mathbb{F}$-local martingale. Then the process*

$$\bar{X}_t := X_t^\tau - \int_0^{t\wedge\tau} \frac{1}{\widetilde{Z}_s} d[X,m]_s + \left(\Delta X_{\widetilde{R}} \, \mathbb{1}_{[\![\widetilde{R},\infty[\![}\right)^{p,\mathbb{F}}_{t\wedge\tau} \tag{9}$$

is an $\mathbb{F}^\tau$-local martingale.

Proof. First of all, let us recall that for each $\mathbb{F}$-local martingale, the stopped process X^τ is an $\mathbb{F}^\tau$-semimartingale as stated in Proposition 2.
Let H be an $\mathbb{F}^\tau$-predictable bounded process. Then, there exists an $\mathbb{F}$-predictable process J such that $H\mathbb{1}_{[\![0,\tau]\!]} = J\mathbb{1}_{[\![0,\tau]\!]}$. By [13, Section IV-3 and Lemme (5,17)], the $\mathbb{F}$-martingale m given in (1) satisfies that for each $H^1(\mathbb{F})$ martingale Y, one has $\mathbb{E}(Y_\tau) = \mathbb{E}([m,Y]_\infty)$. Thus, we have

$$\begin{aligned}\mathbb{E}\left((H\bullet X^{\tau})_{\infty}\right)&=\mathbb{E}\left((J\bullet X)_{\tau})\right)=\mathbb{E}\left([J\bullet X,m]_{\infty}\right)\\&=\mathbb{E}\left(\int_0^{\infty}\frac{J_s\widetilde{Z}_s}{\widetilde{Z}_s}\mathbb{1}_{\{\widetilde{Z}_s>0\}}d[X,m]_s\right)+\mathbb{E}\left(\int_0^{\infty}J_s\mathbb{1}_{\{\widetilde{Z}_s=0<Z_{s-}\}}d[X,m]_s\right)\\&=\mathbb{E}\left(\int_0^{\tau}\frac{J_s}{\widetilde{Z}_s}\mathbb{1}_{\{\widetilde{Z}_s>0\}}d[X,m]_s\right)-\mathbb{E}\left(\int_0^{\infty}J_sZ_{s-}d\left(\Delta X_{\widetilde{R}}\,\mathbb{1}_{[\![\widetilde{R},\infty[\![}\right)_s\right)\end{aligned}$$

since $\mathbb{1}_{\{\widetilde{Z}=0<Z_-\}}\bullet[X,m]=\Delta X_{\widetilde{R}}\,\Delta m_{\widetilde{R}}\,\mathbb{1}_{[\![\widetilde{R},\infty[\![}=-Z_{\widetilde{R}-}\Delta X_{\widetilde{R}}\,\mathbb{1}_{[\![\widetilde{R},\infty[\![}$ as $\{\widetilde{Z}_s=0<Z_{s-}\}=[\![\widetilde{R}]\!]$. Then we obtain

$$\begin{aligned}\mathbb{E}\left((H\bullet X^{\tau})_{\infty}\right)&=\mathbb{E}\left(\int_0^{\tau}\frac{J_s}{\widetilde{Z}_s}\mathbb{1}_{\{\widetilde{Z}_s>0\}}d[X,m]_s\right)-\mathbb{E}\left(\int_0^{\infty}J_sZ_{s-}d\left(\Delta X_{\widetilde{R}}\,\mathbb{1}_{[\![\widetilde{R},\infty]\!]}\right)^{p,\mathbb{F}}_s\right)\\&=\mathbb{E}\left(\int_0^{\tau}\frac{H_s}{\widetilde{Z}_s}\mathbb{1}_{\{\widetilde{Z}_s>0\}}d[X,m]_s\right)-\mathbb{E}\left(\int_0^{\tau}H_sd\left(\Delta X_{\widetilde{R}}\,\mathbb{1}_{[\![\widetilde{R},\infty]\!]}\right)^{p,\mathbb{F}}_s\right),\end{aligned}$$

where we have used again the fact that $Z_s=\mathbb{P}(\tau>s|\mathscr{F}_s)$, which ends the proof. □

Remark 2. In [7, Paragraph 77, Chapter XX] an optional semimartingale decomposition is mentioned (without any proof) in the form: given an $\mathbb{F}$-local martingale X, the process

$$\bar{X}_t:=X^{\tau}_t-\int_0^{t\wedge\tau}\frac{1}{\widetilde{Z}_s}d[X,m]_s$$

is an $\mathbb{F}^{\tau}$-local martingale. This decomposition is valid for any $\mathbb{F}$-local martingale if and only if $\widetilde{R}=\infty$. In particular, if all $\mathbb{F}$-martingales are continuous then $\widetilde{R}=\infty$ and the formula is valid.

Remark 3. The $\mathbb{F}^{\tau}$-local martingale introduced in (9) can be expressed in terms of the $\mathbb{F}$-local martingale N defined in (3).

$$\bar{X}_t=X^{\tau}_t-\int_0^{t\wedge\tau}\frac{1}{N_s}d[X,N]_s+\left(\Delta X_{\widetilde{R}}\mathbb{1}_{[\![\widetilde{R},\infty[\![}\right)^{p,\mathbb{F}}_{t\wedge\tau}.$$

Indeed, from equalities $N=N_-\left(\mathbb{1}_{\{Z_->0\}}\frac{\widetilde{Z}}{Z_-}+\mathbb{1}_{\{Z_-=0\}}\right)$ and $N=1+N_-\frac{1}{Z_-}\mathbb{1}_{\{Z_->0\}}\bullet m$ and the fact that $Z_->0$ on $[\![0,\tau]\!]$, it follows that

$$\frac{1}{N}\mathbb{1}_{[\![0,\tau]\!]}\bullet[X,N]=\frac{1}{\widetilde{Z}}\mathbb{1}_{[\![0,\tau]\!]}\bullet[X,m].$$

The next lemma presents another form of the $\mathbb{F}^{\tau}$-local martingale introduced in equation (3.9) in [2, Proposition 3.6] (see the form of this martingale in the following lemma). Here we denote it by L^{pr} where "pr" stands for progressive.

Lemma 2. *(a) The $\mathbb{F}^{\tau}$-predictable process $\frac{1}{Z_-}\mathbb{1}_{[\![0,\tau]\!]}$ is integrable with respect to $\bar{m}$ – the $\mathbb{F}^{\tau}$-martingale part from the optional decomposition of m.*
(b) The $\mathbb{F}^{\tau}$-local martingale L^{pr} defined as

$$L^{\mathrm{pr}} := \frac{Z_-^2}{Z_-^2 + \Delta\langle m\rangle^{\mathbb{F}}} \frac{1}{\widetilde{Z}} \mathbb{1}_{[\![0,\tau]\!]} \odot \widehat{m},$$

where $\widehat{m}$ is the $\mathbb{F}^\tau$-local martingale part from predictable decomposition of m and $\odot$ stands for the optional integral, satisfies $L^{\mathrm{pr}} = \frac{1}{Z_-} \mathbb{1}_{[\![0,\tau]\!]} \cdot \bar{m}$.

Proof. In the proof, we set $L = L^{\mathrm{pr}}$ for simplicity.
(a) The process $\frac{1}{Z_-} \mathbb{1}_{[\![0,\tau]\!]}$ is càglàd so it is locally bounded.
(b) The $\mathbb{F}^\tau$-continuous martingale part and the jump part of $\frac{1}{Z_-} \mathbb{1}_{[\![0,\tau]\!]} \cdot \bar{m}$ are given by

$$\left(\frac{1}{Z_-} \mathbb{1}_{[\![0,\tau]\!]} \cdot \bar{m}\right)^c = \frac{1}{Z_-} \mathbb{1}_{[\![0,\tau]\!]} \cdot \left(m^c - \frac{1}{Z_-} \cdot \langle m^c\rangle^{\mathbb{F}}\right)$$
$$\Delta\left(\frac{1}{Z_-} \mathbb{1}_{[\![0,\tau]\!]} \cdot \bar{m}\right) = \frac{\Delta m}{\widetilde{Z}} \mathbb{1}_{[\![0,\tau]\!]} - {}^{p,\mathbb{F}}\left(\mathbb{1}_{[\![\widetilde{R}]\!]}\right) \mathbb{1}_{[\![0,\tau]\!]}.$$

where m^c is the $\mathbb{F}$-continuous martingale part of m. Let us now compute the $\mathbb{F}^\tau$-continuous martingale part and the jump part of L. By definition of the optional stochastic integral and Lemma 3.1 (b) in [2], we have:

$$\begin{aligned} L^c &= \frac{Z_-^2}{Z_-^2 + \Delta\langle m\rangle^{\mathbb{F}}} \, {}^{p,\mathbb{F}^\tau}\left(\frac{1}{\widetilde{Z}}\right) \mathbb{1}_{[\![0,\tau]\!]} \cdot \widehat{m}^c \\ &= \frac{1}{Z_-} \mathbb{1}_{[\![0,\tau]\!]} \cdot \widehat{m}^c - \frac{{}^{p,\mathbb{F}}(\mathbb{1}_{\{\widetilde{Z}=0<Z_-\}})}{Z_-} \mathbb{1}_{[\![0,\tau]\!]} \cdot \widehat{m}^c. \end{aligned}$$

As $\{\widetilde{Z} = 0 < Z_-\}$ is a thin set, the set $\{{}^{p,\mathbb{F}}(\mathbb{1}_{\{\widetilde{Z}=0<Z_-\}}) \neq 0\}$ is also thin, and from continuity of $\widehat{m}^c$ we conclude that

$$L^c = \frac{1}{Z_-} \mathbb{1}_{[\![0,\tau]\!]} \cdot \widehat{m}^c.$$

In the proof of Proposition 3.6 in [2], it is established that the jump process of L is given by

$$\Delta L = \frac{\Delta m}{\widetilde{Z}} \mathbb{1}_{[\![0,\tau]\!]} - {}^{p,\mathbb{F}}\left(\mathbb{1}_{\{\widetilde{Z}=0<Z_-\}}\right) \mathbb{1}_{[\![0,\tau]\!]}. \tag{10}$$

This completes the proof. □

The connection between the $\mathbb{F}^\tau$-local martingale L^{pr} and the $\mathbb{F}^\tau$-adapted process $\frac{1}{N^\tau}$ is exploited in the next lemma.

Proposition 6. *Let N be defined in* (3).
(a) The process $\frac{1}{N^\tau}$ is an $\mathbb{F}^\tau$-supermartingale of the form

$$\frac{1}{N^\tau} = \mathscr{E}\left(-(L^{\mathrm{pr}})^\tau - \left(\mathbb{1}_{[\![\widetilde{R},\infty[\![}\right)^{p,\mathbb{F}}_{\cdot\wedge\tau}\right).$$

(b) The process $\frac{1}{N^\tau}$ *is an* $\mathbb{F}^\tau$*-local martingale if and only if* $\widetilde{R}=\infty$*. In that case* $\frac{1}{N^\tau}=\mathscr{E}\left(-(L^{\mathrm{pr}})^\tau\right)$.

Proof. (a) By Itô's formula and the obvious equality $dN = N_-\frac{1}{Z_-}\mathbb{1}_{\{Z_->0\}}dm$

$$\begin{aligned}\frac{1}{N_t^\tau} &= 1-\int_0^{t\wedge\tau}\frac{1}{N_{s-}^2}dN_s+\int_0^{t\wedge\tau}\frac{1}{N_{s-}^3}d\langle N^c\rangle_s+\sum_{s\le t\wedge\tau}\left(\frac{1}{N_s}-\frac{1}{N_{s-}}+\frac{1}{N_{s-}^2}\Delta N_s\right)\\ &= 1-\int_0^{t\wedge\tau}\frac{1}{N_{s-}Z_{s-}}dm_s+\int_0^{t\wedge\tau}\frac{1}{N_{s-}Z_{s-}^2}d\langle m^c\rangle_s+\sum_{s\le t\wedge\tau}\frac{(\Delta m_s)^2}{N_{s-}Z_{s-}\widetilde{Z}_s},\end{aligned}$$

where we have used the fact that $\widetilde{Z}=Z_-+\Delta m$. We continue with

$$\begin{aligned}\frac{1}{N_t^\tau} &= 1-\int_0^{t\wedge\tau}\frac{1}{N_{s-}}d\left(\frac{1}{Z_-}\bullet m-\frac{1}{Z_-^2}\bullet\langle m^c\rangle-\sum\frac{(\Delta m)^2}{Z_-\widetilde{Z}}\right)_s\\ &= 1-\int_0^{t\wedge\tau}\frac{1}{N_{s-}}d\left(\frac{1}{Z_-}\bullet\bar{m}+\left(\mathbb{1}_{[\![\widetilde{R},\infty[\![}\right)^{p,\mathbb{F}}\right)_s\end{aligned}$$

where the second equality comes from Theorem 3 applied to the $\mathbb{F}$-martingale m. Finally we conclude that

$$\frac{1}{N^\tau}=\mathscr{E}\left(-\frac{1}{Z_-}\mathbb{1}_{[\![0,\tau]\!]}\bullet\bar{m}-\left(\mathbb{1}_{[\![\widetilde{R},\infty[\![}\right)^{p,\mathbb{F}}_{\cdot\wedge\tau}\right).$$

(b) The process $\frac{1}{N^\tau}$ is an $\mathbb{F}^\tau$-local martingale if and only if $\left(\mathbb{1}_{[\![\widetilde{R},\infty[\![}\right)^{p,\mathbb{F}}_{\cdot\wedge\tau}=0$. The last equality is equivalent to

$$0=\mathbb{E}\left(\left(\mathbb{1}_{[\![\widetilde{R},\infty[\![}\right)^{p,\mathbb{F}}_\tau\right)=\mathbb{E}\left(\int_0^\infty Z_{s-}d\left(\mathbb{1}_{[\![\widetilde{R},\infty[\![}\right)^{p,\mathbb{F}}_s\right)=\mathbb{E}\left(Z_{\widetilde{R}-}\mathbb{1}_{\{\widetilde{R}<\infty\}}\right),$$

which in turn is equivalent to $\widetilde{R}=\infty$, $\mathbb{P}$-a.s. since $Z_{\widetilde{R}-}>0$ on $\{\widetilde{R}<\infty\}$. □

2.2 Deflators for progressive enlargement up to τ

In this section, we give alternative proofs, based on the optional semimartingale decomposition, to Theorem 2.23 and Corollary 2.18 (c) from [2] (or their general version in [3]). Our proof is simpler, since the optional decomposition involves the quadratic covariation, which does not depend on the filtration, whereas the predictable decomposition involves the predictable bracket in $\mathbb{F}$, which is not the predictable bracket in $\mathbb{F}^\tau$. In Proposition 7 (a) we examine the $\mathbb{F}^\tau$-local martingale deflator for $\mathbb{F}$-local martingales. In Proposition 7 (b), an $\mathbb{F}^\tau$-supermartingale deflator for $\mathbb{F}$-local martingales is studied.

We now define an $\mathbb{F}^\tau$-predictable process V^{pr} which is crucial for proofs therein (also used in [2]). Denoting by $\widetilde{R}^a$ the accessible part of the $\mathbb{F}$-stopping time $\widetilde{R}$, we set

$$V_t^{\mathrm{pr}} := \left(\mathbb{1}_{[\![\widetilde{R}^a,\infty[\![} \right)^{p,\mathbb{F}}_{t\wedge\tau}.$$

Using the process V^{pr} we study, in the next proposition, the behaviour of a particular $\mathbb{F}^\tau$-supermartingale which will play the role of a deflator.

Proposition 7. *(a) Let $Y^{\mathrm{pr}} := \mathscr{E}(-L^{\mathrm{pr}})$. If X is a quasi-left continuous $\mathbb{F}$-local martingale and $\Delta X_{\widetilde{R}} = 0$ on $\{\widetilde{R} < \infty\}$, then Y^{pr} is an $\mathbb{F}^\tau$-local martingale deflator for X^τ.*

(b) Let $\widetilde{Y}^{\mathrm{pr}} := \mathscr{E}(-L^{\mathrm{pr}} - V^{\mathrm{pr}})$. If X is an $\mathbb{F}$-local martingale such that $\Delta X_{\widetilde{R}} = 0$ on $\{\widetilde{R} < \infty\}$, then $\widetilde{Y}^{\mathrm{pr}}$ is an $\mathbb{F}^\tau$-supermartingale deflator for X^τ.

Proof. (a) In the proof, we set $Y = Y^{\mathrm{pr}}$ and $L = L^{\mathrm{pr}}$ for simplicity. Using integration by parts and the optional decomposition (9) given in Theorem 3 for X and then for m, we obtain:

$$\begin{aligned}
Y\ X^\tau &= X^\tau_- \bullet Y + Y_- \bullet X^\tau + [Y, X^\tau] \\
&= X^\tau_- \bullet Y + Y_- \bullet \bar{X} + Y_- \frac{1}{\widetilde{Z}} \mathbb{1}_{[\![0,\tau]\!]} \bullet [m, X] - Y_- \mathbb{1}_{[\![0,\tau]\!]} \bullet (\Delta X_{\widetilde{R}} \mathbb{1}_{[\![\widetilde{R},\infty[\![})^{p,\mathbb{F}} - Y_- \mathbb{1}_{[\![0,\tau]\!]} \bullet [L, X] \\
&= X^\tau_- \bullet Y + Y_- \bullet \bar{X} + Y_- \frac{1}{\widetilde{Z}} \mathbb{1}_{[\![0,\tau]\!]} \bullet [\bar{m}, X] + Y_- \frac{1}{\widetilde{Z}^2} \mathbb{1}_{[\![0,\tau]\!]} \bullet [[m], X] \\
&\quad - Y_- \frac{1}{\widetilde{Z}} \mathbb{1}_{[\![0,\tau]\!]} \bullet [(\Delta m_{\widetilde{R}} \mathbb{1}_{[\![\widetilde{R},\infty[\![})^{p,\mathbb{F}}, X] - Y_- \mathbb{1}_{[\![0,\tau]\!]} \bullet (\Delta X_{\widetilde{R}} \mathbb{1}_{[\![\widetilde{R},\infty[\![})^{p,\mathbb{F}} - \frac{Y_-}{Z_-} \mathbb{1}_{[\![0,\tau]\!]} \bullet [\bar{m}, X] \\
&=: I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7.
\end{aligned}$$

We look closer at the sum of third and seventh term of the last expression

$$\begin{aligned}
I_3 + I_7 &= Y_- \left(\frac{1}{\widetilde{Z}} - \frac{1}{Z_-} \right) \mathbb{1}_{[\![0,\tau]\!]} \bullet [\bar{m}, X] = -Y_- \frac{\Delta m}{\widetilde{Z} Z_-} \mathbb{1}_{[\![0,\tau]\!]} \bullet [\bar{m}, X] \\
&= -\sum Y_- \frac{\Delta m}{\widetilde{Z} Z_-} \mathbb{1}_{[\![0,\tau]\!]} \Delta \bar{m} \Delta X,
\end{aligned}$$

where the third equality comes from the fact that $\{\Delta m \neq 0\}$ is a thin set. We add the term I_4 to the previous two

$$\begin{aligned}
I_4 + (I_3 + I_7) &= \sum Y_- \frac{1}{\widetilde{Z}^2} \mathbb{1}_{[\![0,\tau]\!]} (\Delta m)^2 \Delta X - \sum Y_- \frac{\Delta m}{\widetilde{Z} Z_-} \mathbb{1}_{[\![0,\tau]\!]} \Delta \bar{m} \Delta X \\
&= -\sum Y_- \frac{\Delta m}{\widetilde{Z}} \Delta X \mathbb{1}_{[\![0,\tau]\!]} \left(\frac{1}{Z_-} \Delta \bar{m} - \frac{1}{\widetilde{Z}} \Delta m \right) \\
&= \sum Y_- \frac{\Delta m}{\widetilde{Z}} \Delta X \mathbb{1}_{[\![0,\tau]\!]}{}^{p,\mathbb{F}} \left(\mathbb{1}_{[\![\widetilde{R}]\!]} \right),
\end{aligned}$$

where the last equality comes from (10). Note that, by Yoeurp's lemma and properties of dual predictable projection, the fifth term in the expression for YX^τ is equal

to

$$I_5 = -Y_- \frac{1}{\widetilde{Z}} \mathbb{1}_{[\![0,\tau]\!]} \cdot [(\Delta m_{\widetilde{R}} \mathbb{1}_{[\![\widetilde{R},\infty[\![})^{p,\mathbb{F}}, X] = -Y_- \frac{1}{\widetilde{Z}} \mathbb{1}_{[\![0,\tau]\!]}{}^{p,\mathbb{F}}(\Delta m_{\widetilde{R}} \mathbb{1}_{[\![\widetilde{R}]\!]}) \cdot X$$
$$= \sum Y_- \frac{Z_-}{\widetilde{Z}} \mathbb{1}_{[\![0,\tau]\!]}{}^{p,\mathbb{F}}(\mathbb{1}_{[\![\widetilde{R}]\!]}) \Delta X.$$

Finally, using the fact that $Z_- + \Delta m = \widetilde{Z}$, we get

$$I_5 + (I_4 + I_3 + I_7) = \sum Y_- \frac{Z_-}{\widetilde{Z}} \mathbb{1}_{[\![0,\tau]\!]}{}^{p,\mathbb{F}}(\mathbb{1}_{[\![\widetilde{R}]\!]}) \Delta X + \sum Y_- \frac{\Delta m}{\widetilde{Z}} \Delta X \mathbb{1}_{[\![0,\tau]\!]}{}^{p,\mathbb{F}} \left(\mathbb{1}_{[\![\widetilde{R}]\!]} \right)$$
$$= \sum Y_- \mathbb{1}_{[\![0,\tau]\!]}{}^{p,\mathbb{F}}(\mathbb{1}_{[\![\widetilde{R}]\!]}) \Delta X.$$

Summing up we have that

$$YX^\tau = X^\tau_- \cdot Y + Y_- \cdot \bar{X} + \sum Y_- \mathbb{1}_{[\![0,\tau]\!]}{}^{p,\mathbb{F}}(\mathbb{1}_{[\![\widetilde{R}]\!]}) \Delta X - Y_- \mathbb{1}_{[\![0,\tau]\!]} \cdot \left(\Delta X_{\widetilde{R}} \mathbb{1}_{[\![\widetilde{R},\infty[\![}} \right)^{p,\mathbb{F}}.$$

Since X is an $\mathbb{F}$-quasi-left continuous martingale and $\Delta X_{\widetilde{R}} = 0$ on $\{\widetilde{R} < \infty\}$, then Y^{pr} is an $\mathbb{F}^\tau$-local martingale deflator for X^τ.
(b) In the proof, we set $\widetilde{Y} = \widetilde{Y}^{\mathrm{pr}}$, $L = L^{\mathrm{pr}}$ and $V = V^{\mathrm{pr}}$ for simplicity. Let H be an $\mathbb{F}^\tau$-predictable process such that $H \cdot X \geq -1$. By integration by parts, we get

$$(1 + H \cdot X^\tau)\, \widetilde{Y} = (1 + H \cdot X^\tau)_- \cdot \widetilde{Y} + H\, \widetilde{Y}_- \cdot X^\tau - H\, \widetilde{Y}_- \cdot [X^\tau, L] - H\, \widetilde{Y}_- \cdot [X^\tau, V].$$

Note that

$$H\widetilde{Y}_- \cdot [X^\tau, V] = \sum H\, \widetilde{Y}_-\, \mathbb{1}_{[\![0,\tau]\!]}{}^{p,\mathbb{F}}(\mathbb{1}_{[\![\widetilde{R}]\!]}) \Delta X.$$

Then, using the same arguments as in the proof of (a), we get

$$(1 + H \cdot X^\tau)\widetilde{Y} = (1 + H \cdot X^\tau)_- \cdot \widetilde{Y} + H\widetilde{Y}_- \cdot \bar{X} - H\widetilde{Y}_- \mathbb{1}_{[\![0,\tau]\!]} \cdot \left(\Delta X_{\widetilde{R}} \mathbb{1}_{[\![\widetilde{R},\infty[\![}} \right)^{p,\mathbb{F}}.$$

In particular, if $\Delta X_{\widetilde{R}} = 0$ on $\{\widetilde{R} < \infty\}$, then $\widetilde{Y}^{\mathrm{pr}}$ is an $\mathbb{F}^\tau$-supermartingale deflator for X^τ and X^τ satisfies NUPBR($\mathbb{F}^\tau$). This ends the proof of the proposition. □

Proposition 8. *Let X be a process such that $\Delta X_{\widetilde{R}} = 0$ on $\{\widetilde{R} < \infty\}$ and admitting an $\mathbb{F}$-local martingale deflator. Then X^τ admits an $\mathbb{F}^\tau$-local martingale deflator.*

Proof. There exist a real-valued $\mathbb{F}$-predictable process ϕ and a positive $\mathbb{F}$-local martingale K such that

$$0 < \phi \leq 1 \quad \text{and} \quad K(\phi \cdot X) \quad \text{is an } \mathbb{F}\text{-local martingale.}$$

Then there exists a sequence of $\mathbb{F}$-stopping times $(\nu_n)_n$ that increases to infinity such that the stopped process K^{ν_n} is an $\mathbb{F}$-martingale. Put $\mathbb{Q}_n := K_{\nu_n} \cdot \mathbb{P} \sim \mathbb{P}$. Then, by applying Proposition 7 to $(\phi \cdot X)^{\nu_n}$ under $\mathbb{Q}_n$, we conclude that $(\phi \cdot X)^{\nu_n \wedge \tau}$ satisfies NUPBR($\mathbb{F}^\tau$) under $\mathbb{Q}_n$. Thanks to Proposition 5, NUPBR($\mathbb{F}^\tau$) under $\mathbb{P}$ of X^τ follows immediately. □

The next result provides explicit $\mathbb{F}^\tau$-local martingale deflators for $\mathbb{F}$-local martingales. The proof differs from the one of Theorem [2, Theorem 2.23] and is based on the optional semimartingale decomposition and direct computations.

Theorem 4. *The following conditions are equivalent.*
(a) The thin set $\left\{\widetilde{Z} = 0 < Z_-\right\}$ *is evanescent.*
(b) The $\mathbb{F}$*-stopping time* $\widetilde{R}$ *is infinite* ($\widetilde{R} = \infty$).
(c) For any $\mathbb{F}$*-local martingale* X*, the process* X^τ *admits* Y^{pr} *as* $\mathbb{F}^\tau$*-local martingale deflator (hence, satisfies NUPBR*$(\mathbb{F}^\tau)$*).*
(d) For any (bounded) process X *satisfying NUPBR*$(\mathbb{F})$*, the process* X^τ *satisfies NUPBR*$(\mathbb{F}^\tau)$.

Proof. The equivalence between (a) and (b) is obvious from definition of $\widetilde{R}$.
The implication (b)⇒(c) follows from Proposition 7. To prove (c)⇒(b) (and (d)⇒(b)), we consider the $\mathbb{F}$-martingale

$$X = \mathbb{1}_{[\![\widetilde{R},\infty[\![} - \left(\mathbb{1}_{[\![\widetilde{R},\infty[\![}\right)^{p,\mathbb{F}}.$$

Note that $\mathbb{P}(\tau = \widetilde{R}) = \mathbb{E}(\Delta A^o_{\widetilde{R}}) = \mathbb{E}(\widetilde{Z}_{\widetilde{R}} - Z_{\widetilde{R}}) = 0$ (since $0 = \widetilde{Z}_{\widetilde{R}} \geq Z_{\widetilde{R}} \geq 0$). This implies that $\tau < \widetilde{R}$ and

$$X^\tau = -\left(\mathbb{1}_{[\![\widetilde{R},\infty[\![}\right)^{p,\mathbb{F}}_{\cdot\wedge\tau}$$

is a decreasing process. Thus, X^τ satisfies NUPBR$(\mathbb{F}^\tau)$ if and only if it is a null process. Then, we conclude that $\widetilde{R}$ is infinite using the same argument as in the proof of Lemma 6 (b). The implication (c) ⇒ (d) follows from Proposition 8. □

3 Initial Enlargement under Jacod's Hypothesis

In this section, we study initial enlargement of filtration and NUPBR under absolute continuity Jacod's hypothesis. We extend some results of Amendinger [4] which were established under equivalence Jacod's hypothesis and based on Theorem 1.

3.1 Optional semimartingale decomposition for initial enlargement

In this section we develop the $\mathbb{F}^{\sigma(\xi)}$-optional semimartingale decomposition of parametrized $\mathbb{F}$-local martingales. In order to find it, we decompose the $\mathbb{F}$-stopping time R^u, introduced in (5), as $R^u = \widetilde{R}^u \wedge \bar{R}^u$ with

$$\widetilde{R}^u = R^u_{\{q^u_{R^u_-}>0\}} \quad \text{and} \quad \bar{R}^u = R^u_{\{q^u_{R^u_-}=0\}}. \tag{11}$$

Clearly $\bar{R}^u$ is an $\mathbb{F}$-predictable stopping time and $\{R^u = \infty\} \subset \{\bar{R}^u = \infty\}$ so

$$\left(\mathbb{1}_{[\![R^u,\infty[\![}\right)^{p,\mathbb{F}}|_{u=\xi} = \left(\mathbb{1}_{[\![\widetilde{R}^u,\infty[\![}\right)^{p,\mathbb{F}}|_{u=\xi}.$$

In the following Lemma, we express the $\mathbb{F}^{\sigma(\xi)}$-dual predictable projection in terms of the $\mathbb{F}$-dual predictable projection. This is a result analogous to Lemma 3.1 (a) and Lemma 3.2 from [2] for initial enlargement case.

Lemma 3. *Let $(V^u, u\in\mathbb{U})$ be a parametrized $\mathbb{F}$-adapted càdlàg process with locally integrable variation ($V\in\mathscr{A}_{loc}(\mathbb{F})$). Then the following properties hold.*
(a) The $\mathbb{F}^{\sigma(\xi)}$-dual predictable projection of V^ξ is

$$(V^\xi)^{p,\mathbb{F}^{\sigma(\xi)}} = \frac{1}{q^\xi_-}\cdot\left(q^u\cdot V^u\right)^{p,\mathbb{F}}|_{u=\xi}. \tag{12}$$

(b) If $(V^u, u\in\mathbb{U})$ belongs to $\mathscr{A}^+_{loc}(\mathbb{F})$ (respectively $V\in\mathscr{A}^+(\mathbb{F})$), then the process $(U^u, u\in\mathbb{U})$ with

$$U^u := \frac{1}{q^\xi}\cdot V^u, \tag{13}$$

belongs to $\mathscr{A}^+_{loc}(\mathbb{F}^{\sigma(\xi)})$ (respectively to $\mathscr{A}^+(\mathbb{F}^{\sigma(\xi)})$).
(c) The process $(U^u, u\in\mathbb{U})$ is well defined, its variation is $\mathbb{F}^{\sigma(\xi)}$-locally integrable, and the $\mathbb{F}^{\sigma(\xi)}$-dual predictable projection of U^ξ is given by

$$(U^\xi)^{p,\mathbb{F}^{\sigma(\xi)}} = \frac{1}{q^\xi_-}\cdot\left(\mathbb{1}_{\{q^u>0\}}\cdot V^u\right)^{p,\mathbb{F}}|_{u=\xi}.$$

Proof. (a) We apply the predictable semimartingale decomposition given in Proposition 4 to the parametrized $\mathbb{F}$-local martingale $(X^u, u\in\mathbb{U}) = (V^u - (V^u)^{p,\mathbb{F}}, u\in\mathbb{U})$, obtaining

$$\begin{aligned}
V^\xi &= \widehat{X}^\xi + (V^u)^{p,\mathbb{F}}|_{u=\xi} + \frac{1}{q^\xi_-}\cdot\langle V^u, q^u\rangle^{\mathbb{F}}|_{u=\xi}\\
&= \widehat{X}^\xi + \left(\mathbb{1}_{\{q^u>0\}}\cdot V^u\right)^{p,\mathbb{F}}|_{u=\xi} + \left(\frac{\Delta q^u}{q^u_-}\mathbb{1}_{\{q^u>0\}}\cdot V^u\right)^{p,\mathbb{F}}\\
&= \widehat{X}^\xi + \frac{1}{q^\xi_-}\cdot(q^u\cdot V^u)^{p,\mathbb{F}}|_{u=\xi},
\end{aligned}$$

which proves assertion (a).
(b) Suppose that $(V^u, u\in\mathbb{U})\in\mathscr{A}^+_{loc}(\mathbb{F})$. For fixed u, let $(\vartheta_n)_{n\geq 1}$ be a sequence of $\mathbb{F}$-stopping times that increases to infinity such that $\mathbb{E}\left(V^u_{\vartheta_n}\right)<\infty$. Then, $\mathbb{E}\left(U^u_{\vartheta_n}\right)<\infty$ since

$$\mathbb{E}\left(U^u_{\vartheta_n}\right) = \mathbb{E}\left(\int_0^{\vartheta_n}\frac{1}{q^\xi_t}dV^u_t\right) = \mathbb{E}\left(\int_0^{\vartheta_n}\int_{\mathbb{R}}\mathbb{1}_{\{q^y_t>0\}}\eta(dy)\,dV^u_t\right)\leq\mathbb{E}\left(V^u_{\vartheta_n}\right)<\infty,$$

where the last equality comes from (7) applied to $\frac{1}{q_t^u}\mathbb{1}_{\{q_t^u>0\}}$.
(c) Suppose that $(V^u, u\in\mathbb{U})\in\mathscr{A}_{loc}(\mathbb{F})$, and denote by $W:=V^+ + V^-$ its variation. Then $(W^u, u\in\mathbb{U})\in\mathscr{A}^+_{loc}(\mathbb{F})$, and a direct application of (b) implies that

$$\left(\frac{1}{q^\xi}\centerdot W^u, u\in\mathbb{U}\right)\in\mathscr{A}^+_{loc}(\mathbb{F}^\tau).$$

As a result, we deduce that U given by (13) for the case of $V=V^+-V^-$ is well defined and has variation equal to $\frac{1}{q^\xi}\centerdot W$ which is $\mathbb{F}^{\sigma(\xi)}$-locally integrable. For each $n\geq 1$, let us consider the parametrized process $(U^u_n, u\in\mathbb{U})$ with

$$U^u_n := \frac{1}{q^u}\mathbb{1}_{\{q^u\geq\frac{1}{n}\}}\centerdot V^u.$$

Due to (12), we derive

$$\left(U^\xi_n\right)^{p,\mathbb{F}^{\sigma(\xi)}} = \frac{1}{q^\xi_-}\centerdot\left(\mathbb{1}_{\{q^u\geq 1/n\}}\centerdot V^u\right)^{p,\mathbb{F}}|_{u=\xi}.$$

Hence, since $(U^\xi)^{p,\mathbb{F}^{\sigma(\xi)}} = \lim_{n\to\infty}\left(U^\xi_n\right)^{p,\mathbb{F}^{\sigma(\xi)}}$ by taking the limit in the above equality, we get

$$(U^\xi)^{p,\mathbb{F}^{\sigma(\xi)}} = \frac{1}{q^\xi_-}\centerdot\left(\mathbb{1}_{\{q^u>0\}}\centerdot V^u\right)^{p,\mathbb{F}}|_{u=\xi}.$$

This ends the proof. □

We are now ready to state, in the next theorem, the main result of this section which is based on Lemma 3.

Theorem 5. *Let $(X^u, u\in\mathbb{U})$ be a parametrized $\mathbb{F}$-local martingale. Then,*

$$\bar{X}^\xi_t := X^\xi_t - \int_0^t \frac{1}{q^\xi_s} d[X^\xi, q^\xi]_s + \left(\Delta X^u_{\widetilde{R}^u}\mathbb{1}_{[\![\widetilde{R}^u,\infty[\![}\right)^{p,\mathbb{F}}_t|_{u=\xi} \tag{14}$$

is an $\mathbb{F}^{\sigma(\xi)}$-local martingale. Here, $\widetilde{R}^u$ is defined in (11).

Proof. From the predictable decomposition given in Proposition 4 and Lemma 3 (c) we develop the decomposition of X^ξ as

$$\begin{aligned}
X^\xi &= \widehat{X}^\xi + \frac{1}{q^\xi_-}\centerdot\left(\mathbb{1}_{\{q^u>0\}}\centerdot[X^u,q^u]\right)^{p,\mathbb{F}}|_{u=\xi} + \frac{1}{q^\xi_-}\centerdot\left(\mathbb{1}_{\{q^u=0\}}\centerdot[X^u,q^u]\right)^{p,\mathbb{F}}|_{u=\xi}\\
&= \widehat{X}^\xi + \left(\frac{1}{q^\xi}\centerdot[X^\xi,q^\xi]\right)^{p,\mathbb{F}^{\sigma(\xi)}} + \frac{1}{q^\xi_-}\centerdot\left(\Delta X^u_{\widetilde{R}^u}\Delta q_{\widetilde{R}^u}(x)\mathbb{1}_{[\![\widetilde{R}^u,\infty[\![}\right)^{p,\mathbb{F}}|_{u=\xi}\\
&= \bar{X}^\xi + \frac{1}{q^\xi}\centerdot[X^\xi,q^\xi] - \left(\Delta X^u_{\widetilde{R}^u}\mathbb{1}_{[\![\widetilde{R}^u,\infty[\![}\right)^{p,\mathbb{F}}|_{u=\xi}
\end{aligned}$$

where

$$\bar{X}^\xi = \widehat{X}^\xi - \frac{1}{q^\xi} \bullet [X^\xi, q^\xi] + \left(\frac{1}{q^\xi} \bullet [X^\xi, q^\xi] \right)^{p,\mathbb{F}^{\sigma(\xi)}}$$

is an $\mathbb{F}^{\sigma(\xi)}$-local martingale. □

In the next two lemmas, we study the properties of the process q^ξ. In Lemma 4 we define particular $\mathbb{F}^{\sigma(\xi)}$-local martingales based on q^ξ. Then, in Lemma 5, we focus on the process $\frac{1}{q^\xi}$, which is proved to be an $\mathbb{F}^{\sigma(\xi)}$-supermartingale, and we give its semimartingale decomposition. We give a necessary and sufficient condition such that $\frac{1}{q^\xi}$ is an $\mathbb{F}^{\sigma(\xi)}$-local martingale. In [4], the process $\frac{1}{q^\xi}$ was studied in the case of a random variable ξ satisfying equivalence Jacod's hypothesis, and was proved to be a $\mathbb{F}^{\sigma(\xi)}$-local martingale. Here we work under a more general assumption.

Lemma 4. *Let $\bar{q}^\xi$ be the $\mathbb{F}^{\sigma(\xi)}$-local martingale part of q^ξ given by* (14)*, i.e.,*

$$\bar{q}^\xi := q^\xi - \frac{1}{q^\xi} \bullet [q^\xi] - q^\xi_- \bullet \left(\mathbb{1}_{[\![\widetilde{R}^u, \infty[\![} \right)^{p,\mathbb{F}} |_{u=\xi}.$$

(a) The $\mathbb{F}^{\sigma(\xi)}$-predictable process $\frac{1}{q^\xi_-}$ is integrable with respect to $\bar{q}^\xi$.

(b) The $\mathbb{F}^{\sigma(\xi)}$-local martingale

$$L^{\mathrm{i}} := \frac{1}{q^\xi_-} \bullet \bar{q}^\xi \tag{15}$$

*is such that $1 - \Delta L^{\mathrm{i}} > 0$. Here, the superscript "*i*" stands for initial.*

Proof. (a) The process $\frac{1}{q^\xi_-}$ is càglàd so it is locally bounded.

(b) The definition of $\bar{q}^\xi$ implies

$$\begin{aligned} 1 - \Delta L^{\mathrm{i}} &= 1 - \frac{1}{q^\xi_-} \left(\Delta q^\xi - \frac{1}{q^\xi} (\Delta q^\xi)^2 \right) + \Delta \left(\mathbb{1}_{[\![\widetilde{R}^u, \infty[\![} \right)^{p,\mathbb{F}} |_{u=\xi} \\ &= 1 - \frac{\Delta q^\xi}{q^\xi} + {}^{p,\mathbb{F}} \left(\mathbb{1}_{[\![\widetilde{R}^u]\!]} \right) |_{u=\xi} \\ &= \frac{q^\xi_-}{q^\xi} + {}^{p,\mathbb{F}} \left(\mathbb{1}_{[\![\widetilde{R}^u]\!]} \right) |_{u=\xi} > 0, \end{aligned}$$

which completes the proof. □

Under equivalent Jacod's hypothesis, as stated in Theorem 1, the process $\frac{1}{q^\xi}$ is true $\mathbb{F}^{\sigma(\xi)}$-martingale and provides an interesting change of probability. In the following lemma we prove that under absolute continuity Jacod's hypothesis, the process $\frac{1}{q^\xi}$ is an $\mathbb{F}^{\sigma(\xi)}$-supermartingale and we give its decomposition. We give also conditions so that it is a martingale.

Lemma 5. *(a) The process $\frac{1}{q^\xi}$ is an $\mathbb{F}^{\sigma(\xi)}$-supermartingale with decomposition*

$$\frac{1}{q^\xi} = 1 - \frac{1}{(q^\xi)^2_-} \cdot \bar{q}^\xi - \frac{1}{q^\xi_-} \cdot \left(\mathbb{1}_{[\![\widetilde{R}^u,\infty[\![} \right)^{p,\mathbb{F}} |_{u=\xi}. \tag{16}$$

Or equivalently, it can be written as a stochastic exponential of the form

$$\frac{1}{q^\xi} = \mathscr{E}\left(-L^{\mathrm{i}} - \left(\mathbb{1}_{[\![\widetilde{R}^u,\infty[\![} \right)^{p,\mathbb{F}} |_{u=\xi} \right).$$

(b) The process $\frac{1}{q^\xi}$ is an $\mathbb{F}^{\sigma(\xi)}$-local martingale if and only if $\widetilde{R}^u = \infty$ $\mathbb{P}\otimes\eta$-a.s. Then $\frac{1}{q^\xi} = \mathscr{E}(-L^{\mathrm{i}})$.
(c) In particular, the process $\frac{1}{q^\xi}$ is a true $\mathbb{F}^{\sigma(\xi)}$-martingale if and only if $R^u = \infty$ $\mathbb{P}\otimes\eta$-a.s. (i.e., under equivalent Jacod's hypothesis).

Proof. (a) $(q^u, u \in \mathbb{U})$ is a parametrized $\mathbb{F}$-martingale, then by Proposition 4, q^ξ is a $\mathbb{F}^{\sigma(\xi)}$-semimartingale. By (8), q^ξ is strictly positive. Itô's lemma implies that $\frac{1}{q^\xi}$ is an $\mathbb{F}^{\sigma(\xi)}$-semimartingale which can be written as

$$\begin{aligned}
\frac{1}{q^\xi} &= 1 - \frac{1}{(q^\xi)^2_-} \cdot q^\xi + \frac{1}{(q^\xi)^3_-} \cdot \langle (q^\xi)^c \rangle^{\mathbb{F}^{\sigma(\xi)}} + \sum \left(\frac{1}{q^\xi} - \frac{1}{q^\xi_-} + \frac{1}{(q^\xi)^2_-} \Delta q^\xi \right) \\
&\quad 1 - \frac{1}{(q^\xi)^2_-} \cdot q^\xi + \frac{1}{(q^\xi)^3_-} \cdot \langle (q^\xi)^c \rangle^{\mathbb{F}^{\sigma(\xi)}} + \sum \left(\frac{1}{(q^\xi)^2_- q^\xi} (\Delta q^\xi)^2 \right) \\
&= 1 - \frac{1}{(q^\xi)^2_-} \cdot q^\xi + \frac{1}{(q^\xi)^2_- q^\xi} \cdot [q^\xi].
\end{aligned} \tag{17}$$

Applying (14), we finally get that

$$\frac{1}{q^\xi} = 1 - \frac{1}{(q^\xi)^2_-} \cdot \bar{q}^\xi - \frac{1}{q^\xi_-} \cdot \left(\mathbb{1}_{[\![\widetilde{R}^u,\infty[\![} \right)^{p,\mathbb{F}} |_{u=\xi},$$

where $\frac{1}{q^\xi_-} \cdot \left(\mathbb{1}_{[\![\widetilde{R}^u,\infty[\![} \right)^{p,\mathbb{F}} |_{u=\xi}$ is an $\mathbb{F}^{\sigma(\xi)}$-predictable increasing process. The exponential form immediately follows.
(b) The process $\frac{1}{q^\xi}$ is an $\mathbb{F}^{\sigma(\xi)}$-local martingale if and only if $\left(\mathbb{1}_{[\![\widetilde{R}^u,\infty[\![} \right)^{p,\mathbb{F}} |_{u=\xi} = 0$. The last condition is equivalent to have that, for each t

$$\begin{aligned}
0 = \mathbb{E}\left((\mathbb{1}_{[\![\widetilde{R}^u,\infty[\![})^{p,\mathbb{F}}_t |_{u=\xi} \right) &= \mathbb{E}\left({}^{o,\mathbb{F}}\left((\mathbb{1}_{[\![\widetilde{R}^u,\infty[\![})^{p,\mathbb{F}} |_{u=\xi} \right)_t \right) \\
&= \mathbb{E}\left(\int_{\mathbb{R}} (\mathbb{1}_{[\![\widetilde{R}^u,\infty[\![})^{p,\mathbb{F}}_t q^u_t \eta(du) \right) \\
&= \int_{\mathbb{R}} \mathbb{E}\left((\mathbb{1}_{[\![\widetilde{R}^u,\infty[\![})^{p,\mathbb{F}}_t q^u_t \right) \eta(du),
\end{aligned}$$

where the second equality comes from (7). Next, by Yoeurp's lemma we conclude that, for each t

$$\begin{aligned} 0 &= \int_{\mathbb{R}} \mathbb{E}\left(\int_0^t q^u_{s-} d\left((\mathbb{1}_{[\![\widetilde{R}^u,\infty[\![})^{p,\mathbb{F}} \right)_s \right) \eta(du) \\ &= \int_{\mathbb{R}} \mathbb{E}\left(q^u_{\widetilde{R}^u-} \mathbb{1}_{\{\widetilde{R}^u \le t\}} \right) \eta(du) \end{aligned}$$

which in turn is equivalent to $\widetilde{R}^u > t$, $\mathbb{P}\otimes\eta$-a.s. for each t since $q^u_{\widetilde{R}^u-} > 0$. Thus, $\frac{1}{q^\xi}$ is an $\mathbb{F}^{\sigma(\xi)}$-local martingale if and only if $\widetilde{R}^u$ is infinite $\mathbb{P}\otimes\eta$-a.s.
(c) The "if" part is shown in Theorem 1. We show "only if" part here. Assume that the process $\frac{1}{q^\xi}$ is a true $\mathbb{F}^{\sigma(\xi)}$-martingale. Then, for each $t \ge 0$, we have $\mathbb{E}(\frac{1}{q^\xi_t}) = 1$. On the other hand, using Lemma 1 (ii), we have that

$$\mathbb{E}\left(\frac{1}{q^\xi_t}\right) = \mathbb{E}\left(\int_{\mathbb{U}} \frac{1}{q^u_t} \mathbb{1}_{\{q^u_t>0\}} q^u_t \eta(du)\right) = \int_{\mathbb{U}} \mathbb{P}(q^u_t > 0)\eta(du) = \int_{\mathbb{U}} \mathbb{P}(R^u > t)\eta(du),$$

which shows that $R^u = \infty$, $\mathbb{P}\otimes\eta$-a.s.. □

In the following proposition, we investigate the $\mathbb{F}^{\sigma(\xi)}$-semimartingale decomposition of a parametrized $\mathbb{F}$-local martingale X when ξ is plugged in and when multiplied by $\frac{1}{q^\xi}$ from previous lemma.

Proposition 9. *Let $(X^u, u \in \mathbb{U})$ be a parametrized $\mathbb{F}$-local martingale. Then $\frac{X^\xi}{q^\xi}$ is an $\mathbb{F}^{\sigma(\xi)}$-semimartingale with $\mathbb{F}^{\sigma(\xi)}$-local martingale part equal to*

$$X^\xi_0 - \frac{X^\xi_-}{(q^\xi)^2_-} \bullet \bar{q}^\xi + \frac{1}{q^\xi_-} \bullet \bar{X}^\xi,$$

and $\mathbb{F}^{\sigma(\xi)}$-predictable finite variation part equal to

$$-\frac{1}{q^\xi_-} \bullet \left(X^u_{\widetilde{R}^u} \mathbb{1}_{[\![\widetilde{R}^u,\infty[\![} \right)^{p,\mathbb{F}} |_{u=\xi}.$$

Proof. We compute, applying integration by parts formula:

$$\begin{aligned}\frac{X^\xi}{q^\xi} &= X_0^\xi + X_-^\xi \centerdot \frac{1}{q^\xi} + \frac{1}{q_-^\xi} \centerdot X^\xi + \left[X^\xi, \frac{1}{q^\xi}\right] \\ &= X_0^\xi - \frac{X_-^\xi}{(q^\xi)_-^2} \centerdot \bar{q}^\xi - \frac{X_-^\xi}{q_-^\xi} \centerdot \left(\mathbb{1}_{[\![\tilde{R}^u,\infty[\![}\right)^{p,\mathbb{F}}|_{u=\xi} \\ &+ \frac{1}{q_-^\xi} \centerdot \bar{X}^\xi + \frac{1}{q_-^\xi q^\xi} \centerdot [X^\xi, q^\xi] - \frac{1}{q_-^\xi} \centerdot \left(\Delta X^u_{\tilde{R}^u} \mathbb{1}_{[\![\tilde{R}^u,\infty[\![}\right)^{p,\mathbb{F}}|_{u=\xi} \\ &- \frac{1}{(q^\xi)_-^2} \centerdot [X^\xi, q^\xi] + \frac{1}{(q^\xi)_-^2 q^\xi} \centerdot \left[X^\xi, [q^\xi]\right],\end{aligned}$$

where the second equality comes from (16), (17). It follows that

$$\begin{aligned}\frac{X^\xi}{q^\xi} &= X_0^\xi - \frac{X_-^\xi}{(q^\xi)_-^2} \centerdot \bar{q}^\xi + \frac{1}{q_-^\xi} \centerdot \bar{X}^\xi \\ &- \frac{1}{q_-^\xi} \centerdot \left(X^u_{\tilde{R}^u} \mathbb{1}_{[\![\tilde{R}^u,\infty[\![}\right)^{p,\mathbb{F}}|_{u=\xi} - \frac{\Delta q^\xi}{p_(\xi)^2 q^\xi} \centerdot [X^\xi, q^\xi] + \frac{\Delta X^\xi}{(q^\xi)_-^2 q^\xi} \centerdot [q^\xi] \\ &= X_0^\xi - \frac{X_-^\xi}{(q^\xi)_-^2} \centerdot \bar{q}^\xi + \frac{1}{q_-^\xi} \centerdot \bar{X}^\xi - \frac{1}{q_-^\xi} \centerdot \left(X^u_{\tilde{R}^u} \mathbb{1}_{[\![\tilde{R}^u,\infty[\![}\right)^{p,\mathbb{F}}|_{u=\xi}.\end{aligned}$$

Therefore the martingale property established in [4] under equivalence Jacod's hypothesis fails in the general case. □

As a corollary, from Proposition 9, we recover [11, Proposition 5.2] on universal supermartingale density.

Corollary 1. *If X is a positive $\mathbb{F}$-supermartingale, then, $\frac{X}{q^\xi}$ is an $\mathbb{F}^{\sigma(\xi)}$-supermartingale.*

Proof. Let X be decomposed as $X = M^X - V^X$ where M^X is a positive $\mathbb{F}$-local martingale and V^X is an increasing $\mathbb{F}$-predictable process. Then, $\frac{M^X}{q^\xi}$ is an $\mathbb{F}^{\sigma(\xi)}$-supermartingale since from the positiveness of M^X, by Proposition 9, we get that $\frac{1}{q_-^\xi} \centerdot \left(M^X_{\tilde{R}^u} \mathbb{1}_{[\![\tilde{R}^u,\infty[\![}\right)^{p,\mathbb{F}}|_{u=\xi}$ is an $\mathbb{F}^{\sigma(\xi)}$-predictable increasing process. Moreover, as $\frac{1}{q^\xi}$ is an $\mathbb{F}^{\sigma(\xi)}$-supermartingale and V^X is predictable and increasing, the process $-\frac{V^X}{q^\xi}$ is as well an $\mathbb{F}^{\sigma(\xi)}$-supermartingale which ends the proof. □

3.2 NUPBR for initial enlargement

In this section, we focus on the NUPBR condition. Using simple arguments based on the optional semimartingale decomposition, we prove the NUPBR condition stability with respect to an initial enlargement of filtration under Jacod's hypoth-

esis. In Proposition 10, we study the $\mathbb{F}^{\sigma(\xi)}$-local martingale deflators for quasi left-continuous parametrized $\mathbb{F}$-local martingales and the $\mathbb{F}^{\sigma(\xi)}$-supermartingale deflators for parametrized $\mathbb{F}$-local martingales. In Theorem 6 we formulate a necessary and sufficient condition such that each parametrized $\mathbb{F}$-local martingale satisfies NUPBR($\mathbb{F}^{\sigma(\xi)}$). We close this section by giving two particular examples of initial enlargements under Jacod's hypothesis. We also address the reader to [1] for similar study to the one contained in this section using fully different methodology.

Let us denote by $\widetilde{R}^{u,a}$ the accessible part of the $\mathbb{F}$-stopping time $\widetilde{R}^u$. Define the process V^{i} as

$$V^{\mathrm{i}}_t := \sum_{0\le s\le t} {}^{p,\mathbb{F}}\left(\mathbb{1}_{[\![\widetilde{R}^u]\!]}\right)_s|_{u=\xi} = \left(\mathbb{1}_{[\![\widetilde{R}^{u,a},\infty[\![}\right)^{p,\mathbb{F}}|_{u=\xi}. \tag{18}$$

Proposition 10. *(a) Let $Y^{\mathrm{i}} := \mathscr{E}\left(-L^{\mathrm{i}}\right)$. Let $(X^u, u\in\mathbb{U})$ be quasi-left continuous and such that $\Delta X^u_{\widetilde{R}^u} = 0$ on $\{\widetilde{R}^u<\infty\}$ $\mathbb{P}\otimes\eta$-a.s., then Y^{i} is an $\mathbb{F}^{\sigma(\xi)}$-local martingale deflator for X^ξ.*
(b) Let $\widetilde{Y}^{\mathrm{i}} = \mathscr{E}(-L^{\mathrm{i}} - V^{\mathrm{i}})$. Let $(X^u, u\in\mathbb{U})$ be a parametrized $\mathbb{F}$-local martingale such that $\Delta X^u_{\widetilde{R}^u} = 0$ on $\{\widetilde{R}^u<\infty\}$ $\mathbb{P}\otimes\eta$-a.s., then $\widetilde{Y}^{\mathrm{i}}$ is an $\mathbb{F}^{\sigma(\xi)}$-supermartingale deflator for X^ξ.

Proof. (a) Using the optional decomposition (14) (Theorem 5) of X^ξ and q^ξ we obtain

$$\begin{aligned}
Y^{\mathrm{i}}X^\xi &= X^\xi_- \bullet Y^{\mathrm{i}} + Y^{\mathrm{i}}_- \bullet X^\xi + [Y^{\mathrm{i}}, X^\xi]\\
&= X^\xi_- \bullet Y^{\mathrm{i}} + Y^{\mathrm{i}}_- \bullet \bar{X}^\xi + Y^{\mathrm{i}}_- \frac{1}{q^\xi} \bullet [X^\xi, q^\xi] - Y^{\mathrm{i}}_- \bullet (\Delta X^u_{\widetilde{R}^u})^{p,\mathbb{F}}|_{u=\xi} - Y^{\mathrm{i}}_- \bullet [L^{\mathrm{i}}, X^\xi]\\
&= X^\xi_- \bullet Y^{\mathrm{i}} + Y^{\mathrm{i}}_- \bullet \bar{X}^\xi + Y^{\mathrm{i}}_- \frac{1}{q^\xi} \bullet [X^\xi, \bar{q}^\xi] + Y^{\mathrm{i}}_- \frac{1}{(q^\xi)^2} \bullet [[q^\xi], X^\xi]\\
&\quad - Y^{\mathrm{i}}_- \frac{1}{q^\xi} \bullet [(\Delta q^u_{\widetilde{R}^u-} \mathbb{1}_{[\![\widetilde{R}^u,\infty[\![})^{p,\mathbb{F}}|_{u=\xi}, X^\xi] - Y^{\mathrm{i}}_- \bullet (\Delta X^u_{\widetilde{R}^u} \mathbb{1}_{[\![\widetilde{R}^u,\infty[\![})^{p,\mathbb{F}}|_{u=\xi} - \frac{Y^{\mathrm{i}}_-}{q^\xi_-} \bullet [\bar{q}^\xi, X^\xi].
\end{aligned}$$

We continue as

$$\begin{aligned}
Y^{\mathrm{i}}X^\xi &= X^\xi_- \bullet Y^{\mathrm{i}} + Y^{\mathrm{i}}_- \bullet \bar{X}^\xi - \sum \frac{Y^{\mathrm{i}}_- \Delta q^\xi}{q^\xi q^\xi_-} \Delta X^\xi \Delta \bar{q}^\xi + \sum \frac{Y^{\mathrm{i}}_-}{(q^\xi)^2} (\Delta q^\xi)^2 \Delta X^\xi\\
&\quad + \sum \frac{Y^{\mathrm{i}}_- q^\xi_-}{q^\xi} {}^{p,\mathbb{F}}(\mathbb{1}_{[\![\widetilde{R}^u]\!]})|_{u=\xi} \Delta X^\xi - Y^{\mathrm{i}}_- \bullet \left(\Delta X^u_{\widetilde{R}^u} \mathbb{1}_{[\![\widetilde{R}^u,\infty[\![}\right)^{p,\mathbb{F}}|_{u=\xi}\\
&= X^\xi_- \bullet Y^{\mathrm{i}} + Y^{\mathrm{i}}_- \bullet \bar{X}^\xi + \sum Y^{\mathrm{i}}_- {}^{p,\mathbb{F}}(\mathbb{1}_{[\![\widetilde{R}^u]\!]})|_{u=\xi} \Delta X^\xi - Y^{\mathrm{i}}_- \bullet \left(\Delta X^u_{\widetilde{R}^u} \mathbb{1}_{[\![\widetilde{R}^u,\infty[\![}\right)^{p,\mathbb{F}}|_{u=\xi}\\
&= X^\xi_- \bullet Y^{\mathrm{i}} + Y^{\mathrm{i}}_- \bullet \bar{X}^\xi + \sum Y^{\mathrm{i}}_- {}^{p,\mathbb{F}}(\mathbb{1}_{[\![\widetilde{R}^u]\!]})|_{u=\xi} \Delta X^\xi + \frac{Y^{\mathrm{i}}_-}{q^\xi_-} \bullet \left(\mathbb{1}_{[\![\widetilde{R}^u]\!]} \bullet [X^u, q^u]\right)^{p,\mathbb{F}}|_{u=\xi}.
\end{aligned}$$

Since $(X^u, u \in \mathbb{U})$ is an $\mathbb{F}$-quasi-left continuous martingale and $\Delta X^u_{R^u} = 0$ on $\{R^u < \infty\}$, the two last terms are null. Thus Y^{i} is an $\mathbb{F}^{\sigma(\xi)}$-local martingale deflator for X^ξ.
(b) Let H be an $\mathbb{F}^{\sigma(\xi)}$-predictable process such that $H \bullet X^\xi \geq -1$. Then, by integration by parts, we get

$$(1+H\bullet X^\xi)\,\widetilde{Y}^{\mathrm{i}} = (1+H\bullet X^\xi)_-\bullet \widetilde{Y}^{\mathrm{i}} + H\,\widetilde{Y}^{\mathrm{i}}_-\bullet X^\xi - H\,\widetilde{Y}^{\mathrm{i}}_-\bullet [X^\xi, L^{\mathrm{i}}] - H\,\widetilde{Y}^{\mathrm{i}}_-\bullet [X^\xi, V^{\mathrm{i}}].$$

Note that

$$H\widetilde{Y}^{\mathrm{i}}_-\bullet [X^\xi, V^{\mathrm{i}}] = \sum H\widetilde{Y}^{\mathrm{i}}_-{}^{\,p,\mathbb{F}}(\mathbb{1}_{[\![\widetilde{R}^u]\!]})|_{u=\xi}\Delta X^\xi.$$

Then, using the same arguments as in the proof of Theorem 10 we get

$$(1+H\bullet X^\xi)\widetilde{Y}^{\mathrm{i}} = (1+H\bullet X^\xi)_-\bullet \widetilde{Y}^{\mathrm{i}} + H\,\widetilde{Y}^{\mathrm{i}}_-\bullet \bar{X}^\xi - H\,\widetilde{Y}^{\mathrm{i}}_-\left(\Delta X^u_{\widetilde{R}^u}\mathbb{1}_{[\![\widetilde{R}^u,\infty[\![}\right)^{p,\mathbb{F}}|_{u=\xi}$$

and the assertion is proved. □

Proposition 11. *Let $(X^u, u \in \mathbb{U})$ be a parametrized process admitting an $\mathbb{F}$-local martingale deflator such that $\Delta X^u_{\widetilde{R}^u} = 0$ on $\{\widetilde{R}^u < \infty\}$, η-a.e. Then X^ξ admits an $\mathbb{F}^{\sigma(\xi)}$-local martingale deflator.*

Proof. let $(X^u, u \in \mathbb{U})$ be a parametrized $\mathbb{F}$-semimartingale admitting an $\mathbb{F}$-local martingale deflator, i.e., there exist a real-valued parametrized predictable process $(\phi^u, u \in \mathbb{U})$ and a positive $\mathbb{F}$-local martingale L such that

$$0 < \phi \leq 1 \quad \text{and} \quad (L(\phi^u \bullet X^u), u \in \mathbb{U}) \quad \text{is a parametrized } \mathbb{F}\text{-local martingale.}$$

Then, there exists a sequence of $\mathbb{F}$-stopping times that increases to infinity $(T_n)_n$ such that L^{T_n} is a martingale. Put $\mathbb{Q}_n := L_{T_n} \bullet P \sim P$. Then, by applying Proposition 10 to $((\phi^u \bullet X^u)^{T_n}, u \in \mathbb{U})$ under $\mathbb{Q}_n$, we conclude that $(\phi^\xi \bullet X^\xi)^{T_n}$ satisfies NUPBR$(\mathbb{F}^{\sigma(\xi)})$ under $\mathbb{Q}_n$. Thanks to Proposition 5, NUPBR$(\mathbb{F}^{\sigma(\xi)})$ under $\mathbb{P}$ of X^ξ follows immediately. □

Theorem 6. *The following conditions are equivalent.*
(a) The thin set $\left\{q^u = 0 < q^u_-\right\}$ is evanescent for η-a.e.
(b) The $\mathbb{F}$-stopping time $\widetilde{R}^u$ is infinite $\mathbb{P}\otimes\eta$-a.s.
(c) For any parametrized $\mathbb{F}$-local martingale $(X^u, u \in \mathbb{U})$, the process X^ξ admits an $\mathbb{F}^{\sigma(\xi)}$-local martingale deflator $\frac{1}{q^\xi}$ (and satisfies NUPBR$(\mathbb{F}^{\sigma(\xi)})$).
(d) For any parametrized (bounded) process $(X^u, u \in \mathbb{U})$ admitting an $\mathbb{F}$-local martingale deflator, the process X^ξ admits an $\mathbb{F}^{\sigma(\xi)}$-local martingale deflator (and satisfies NUPBR$(\mathbb{F}^{\sigma(\xi)})$).

Proof. The equivalence between (a) and (b) is obvious from the definition of $\widetilde{R}^u$.
The implication (b) ⇒(c) follows from Proposition 10. To prove (c)⇒(b), we consider a parametrized $\mathbb{F}$-martingale $(M^u, u \in \mathbb{U})$ with

$$M^u := \mathbb{1}_{[\![\widetilde{R}^u,\infty[\![} - (\mathbb{1}_{[\![\widetilde{R}^u,\infty[\![})^{p,\mathbb{F}}.$$

Then, due to the equality $R^{\xi} = \infty$ established in (8), it is clear that

$$M^{\xi} = -\left(\mathbb{1}_{[\![\widetilde{R}^u,\infty[\![}\right)^{p,\mathbb{F}}|_{u=\xi}$$

is decreasing. Thus, M^{ξ} satisfies NUPBR($\mathbb{F}^{\sigma(\xi)}$) if and only if it is a null process. Then, we conclude that $\widetilde{R}$ is infinite using the same argument as in the proof of Lemma 5 (b). The implication (c) $\Rightarrow$ (d) follows from Proposition 11. $\square$

In the two following examples we look at two extreme situations.

Example 1. Let $\mathbb{F}$ be a filtration such that each $\mathbb{F}$-martingale is continuous. Then, the NUPBR condition is preserved in an initially enlarged filtration for any parametrized $\mathbb{F}$-local martingale from the reference filtration.

Example 2. Let B be a $\mathcal{G}$-measurable set such that $\mathbb{P}(B) = \frac{1}{2}$ and consider the filtration $\mathbb{F} = (\mathscr{F}_t)_{t\geq 0}$ defined as

$$\mathscr{F}_t = \{\emptyset, \Omega\} \text{ for } t \in [0,1[\quad \text{and} \quad \mathscr{F}_t := \{\emptyset, B, B^c, \Omega\} \text{ for } t \in [1,\infty[.$$

Define a random variable ξ as $\xi := \mathbb{1}_B + 2\cdot\mathbb{1}_{B^c}$. The random variable ξ satisfies Jacod's hypothesis with density $(q^u, u \in \mathbb{U})$ equal to

$$\begin{aligned} q^1 &= \mathbb{1}_{[\![0,1[\![} + 2\cdot\mathbb{1}_{\{\xi=1\}}\mathbb{1}_{[\![1,\infty[\![}, \\ q^2 &= \mathbb{1}_{[\![0,1[\![} + 2\cdot\mathbb{1}_{\{\xi=2\}}\mathbb{1}_{[\![1,\infty[\![}. \end{aligned}$$

Let the filtration $\mathbb{F}^{\sigma(\xi)} = (\mathscr{F}_t^{\sigma(\xi)})_{t\geq 0}$ be an initial enlargement of the filtration $\mathbb{F}$ with ξ, i.e.,

$$\mathscr{F}_t^{\sigma(\xi)} := \{\emptyset, B, B^c, \Omega\} \quad \text{for} \quad t \in [0,\infty[.$$

Let X be an $\mathbb{F}$-martingale defined as

$$X := \left(\mathbb{1}_{\{\xi=1\}} - \frac{1}{2}\right)\mathbb{1}_{[\![1,\infty[\![}.$$

Then, X is an $\mathbb{F}^{\sigma(\xi)}$-predictable process. Thus, by [2, Lemma 2.6] it does not satisfy NUPBR($\mathbb{F}^{\sigma(\xi)}$).

4 Connection to Absolutely Continuous Change of Measure

In this section we look at the relationship between new optional semimartingale decompositions in progressive and initial enlargement of filtration cases and the optional semimartingale decomposition in absolutely continuous change of measure set-up. First let us recall [21, Theorem 42, Chapter III].

Theorem 7. *Let X be a $\mathbb{P}$-local martingale with $X_0 = 0$. Let $\mathbb{Q}$ be a probability measure absolutely continuous with respect to $\mathbb{P}$, and let $\zeta_t := \mathbb{E}_{\mathbb{P}}(\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathscr{F}_t)$. Let $r := \inf\{t > 0 : \zeta_t = 0\}$ and $\widetilde{r} := r_{\{\zeta_{r-}>0\}}$. Then*

$$\bar{X} := X - \frac{1}{\zeta} \bullet [X, \zeta] + \left(\Delta X_{\widetilde{r}} \, \mathbb{1}_{[\![\widetilde{r},\infty[\![}\right)^{p,\mathbb{P}} \tag{19}$$

is a $\mathbb{Q}$-local martingale.

It is clear that Theorem 7 implies the same type of decompositions as the two decompositions stated in Sections 2.1 and 3.1.

Up to random time τ:	$X^\tau = \bar{X} + \frac{1}{N^\tau} \bullet [X^\tau, N^\tau] - \left(\Delta X_{\widetilde{R}} \mathbb{1}_{[\![\widetilde{R},\infty[\![}\right)^{p,\mathbb{F}}_{\bullet\wedge\tau}$
Under Jacod's hypothesis:	$X^\xi = \bar{X} + \frac{1}{q^\xi} \bullet [X^\xi, q^\xi] - \left(\Delta X^u_{\widetilde{R}^u} \mathbb{1}_{[\![\widetilde{R}^u,\infty[\![}\right)^{p,\mathbb{F}}\big\vert_{u=\xi}$
Under measure $\mathbb{Q}$:	$X^{\mathbb{Q}} = \bar{X} + \frac{1}{\zeta^{\mathbb{Q}}} \bullet [X^{\mathbb{Q}}, \zeta^{\mathbb{Q}}] - \left(\left(\Delta X_{\widetilde{r}} \mathbb{1}_{[\![\widetilde{r},\infty[\![}\right)^{p,\mathbb{P}}\right)^{\mathbb{Q}}$

In the table above, the superscript $^{\mathbb{Q}}$ means that we consider the given process under the measure $\mathbb{Q}$. In each of the three cases, there is a different mechanism to ensure the strict positivity of N^τ, q^ξ and $\zeta^{\mathbb{Q}}$. In the case of progressive enlargement up to a random time, we stop at τ. In the case of initial enlargement with random variable satisfying Jacod's hypothesis, we plug ξ. In the case of absolutely continuous change of measure, we look through measure $\mathbb{Q}$.

The optional decomposition in the change of measure case can be used in the same way to obtain similar result on stability of the NUPBR condition with respect to absolutely continuous change of measure.

We remark here that the set introduced in Definition 2 may become bigger under absolutely continuous change of measure as under the new measure the condition $H \bullet X \geq -1$ is more likely satisfied. This somehow explains that the NUPBR condition may not be stable under only absolutely continuous change of measure which at the first sight may seem counter-intuitive.

Let $\bar{\zeta}$ given by (19), and define a $\mathbb{Q}$-local martingale L^a by

$$L^a := \frac{1}{\zeta} \bullet \bar{\zeta}.$$

Let us denote by $\widetilde{r}^a$ the accessible part of the stopping time $\widetilde{r}$, and set

$$V^a_t := \left(\mathbb{1}_{[\![\widetilde{r}^a,\infty[\![}\right)^{p,\mathbb{P}}$$

Using the processes L^a and V^a we study, in the next lemma, the behaviour of particular $\mathbb{Q}$-deflators.

Proposition 12. *(a) Let $Y^a := \mathscr{E}(-L^a)$. If X is a quasi-left continuous local martingale and $\Delta X_{\widetilde{r}} = 0$ on $\{\widetilde{r} < \infty\}$ $\mathbb{P}$-a.s., then Y^a is a $\mathbb{Q}$-local martingale deflator for X.*

(b) Let $\widetilde{Y}^a := \mathscr{E}(-L^a - V^a)$. Let X be a $\mathbb{P}$-local martingale such that $\Delta X_{\widetilde{r}} = 0$ on $\{\widetilde{r} < \infty\}$ $\mathbb{P}$-a.s., then $\widetilde{Y}^a$ is a $\mathbb{Q}$-supermartingale deflator for X.

Proof. (a) Using integration by parts and the optional decomposition (19) given in Theorem 7 for X and then for ζ, we obtain

$$\begin{aligned}
Y^a X &= X_- \bullet Y^a + Y^a_- \bullet X + [Y^a, X] \\
&= X_- \bullet Y^a + Y^a_- \bullet \bar{X} + Y^a_- \frac{1}{\zeta} \bullet [\zeta, X] - Y^a_- \bullet (\Delta X_{\widetilde{r}} \mathbb{1}_{[\![\widetilde{r},\infty[\![})^{p,\mathbb{P}} - Y^a_- \bullet [L^a, X] \\
&= X_- \bullet Y^a + Y^a_- \bullet \bar{X} + Y^a_- \frac{1}{\zeta} \bullet [\bar{\xi}, X] + Y^a_- \frac{1}{\zeta^2} [[\zeta], X] \\
&- Y^a_- \frac{1}{\zeta} \bullet [(\Delta \zeta_{\widetilde{r}} \mathbb{1}_{[\![\widetilde{r},\infty[\![})^{p,\mathbb{P}}, X] - Y^a_- \bullet (\Delta X_{\widetilde{r}} \mathbb{1}_{[\![\widetilde{r},\infty[\![})^{p,\mathbb{P}} - \frac{Y^a_-}{\zeta_-} [\bar{\xi}, X].
\end{aligned}$$

We continue as

$$\begin{aligned}
Y^a X &= X_- \bullet Y^a + Y^a_- \bullet \bar{X} - \sum \frac{Y^a_- \Delta \zeta}{\zeta \zeta_-} \Delta X \Delta \bar{\xi} + \sum \frac{Y^a_-}{\zeta^2} (\Delta \zeta)^2 \Delta X \\
&+ \sum \frac{Y^a_- \zeta_-}{\zeta} {}^{p,\mathbb{P}}(\mathbb{1}_{[\![\widetilde{r}]\!]}) \Delta X - Y^a_- \bullet (\Delta X_{\widetilde{r}} \mathbb{1}_{[\![\widetilde{r},\infty[\![})^{p,\mathbb{P}} \\
&= X_- \bullet Y^a + Y^a_- \bullet \bar{X} + \sum Y^a_- {}^{p,\mathbb{P}}(\mathbb{1}_{[\![\widetilde{r}]\!]}) \Delta X - Y^a_- \bullet \left(\Delta X_{\widetilde{r}} \mathbb{1}_{[\![\widetilde{r},\infty[\![}\right)^{p,\mathbb{P}} \qquad (20)
\end{aligned}$$

Since for any $\mathbb{P}$-quasi-left continuous martingale X, the process ${}^{p,\mathbb{P}}(\mathbb{1}_{[\![\widetilde{r}]\!]}) \Delta X$ is null, Y^a is a $\mathbb{Q}$-local martingale deflator for X.
(b) Let H be a predictable process such that $H \bullet X \geq -1$. Then, by integration by parts, we get

$$(1 + H \bullet X)\, \widetilde{Y}^a = (1 + H \bullet X)_- \bullet \widetilde{Y}^a + H\, \widetilde{Y}^a_- \bullet X - H\, \widetilde{Y}^a_- \bullet [X, L^a] - H\, \widetilde{Y}^a_- \bullet [X, V^a].$$

Note that

$$H \widetilde{Y}^a_- \bullet [X, V^a] = \sum H\, \widetilde{Y}^a_-\ {}^{p,\mathbb{P}}(\mathbb{1}_{[\![\widetilde{r}]\!]}) \Delta X.$$

Then, using the same arguments as in the proof of Theorem 12 to derive (20), we get

$$(1 + H \bullet X) \widetilde{Y}^a = (1 + H \bullet X)_- \bullet \widetilde{Y}^a + H \widetilde{Y}^a_- \bullet \bar{X} - H \widetilde{Y}^a_- \bullet \left(\Delta X_{\widetilde{r}} \mathbb{1}_{[\![\widetilde{S},\infty[\![}\right)^{p,\mathbb{P}}$$

and the assertion is proved. □

Proposition 13. *Let X be a process admitting a $\mathbb{P}$-local martingale deflator such that $\Delta X_{\widetilde{\zeta}} = 0$ on $\{\widetilde{\zeta} < \infty\}$. Then X admits a $\mathbb{Q}$-local martingale deflator.*

Proof. Let X be an $\mathbb{P}$-semimartingale satisfying NUPBR($\mathbb{P}$). Thanks to Proposition 5 and Theorem 2, we deduce the existence of a real-valued predictable process ϕ and a positive $\mathbb{P}$-local martingale K such that

$$0 < \phi \leq 1 \quad \text{and} \quad K(\phi \bullet X) \quad \text{is a } \mathbb{P}\text{-local martingale.}$$

Then there exists a sequence of stopping times $(\nu_n)_n$ that increases to infinity such that the stopped process K^{ν_n} is a $\mathbb{P}$-martingale. Put $\mathbb{P}_n := K_{\nu_n} \bullet \mathbb{P} \sim \mathbb{P}$ and

$$\mathbb{Q}_n := \frac{K_{\nu_n}}{\mathbb{E}_{\mathbb{P}}(\zeta_{\nu_n} K_{\nu_n})} \bullet \mathbb{Q} = \frac{\zeta_{\nu_n}}{\mathbb{E}_{\mathbb{P}}(\zeta_{\nu_n} K_{\nu_n})} \bullet \mathbb{P}_n \ll \mathbb{P}_n.$$

Define $\zeta_t^n := \mathbb{E}_{\mathbb{P}_n}\left(\frac{\zeta_{\nu_n}}{\mathbb{E}_{\mathbb{P}_n}(\zeta_{\nu_n})} | \mathscr{F}_t\right)$ and note that the condition that $\{\zeta = 0 < \zeta_-\}$ is $\mathbb{P}$-evanescent implies that $\left\{\zeta^n = 0 < \zeta_-^n\right\}$ is $\mathbb{P}_n$-evanescent. Then, by applying Proposition 12 to $(\phi \bullet X)^{\nu_n}$ under $\mathbb{P}_n$, we conclude that $(\phi \bullet X)^{\nu_n}$ satisfies NUPBR$(\mathbb{Q}_n)$. Thanks to Proposition 5, since $\mathbb{Q}_n \sim \mathbb{Q}$, NUPBR$(\mathbb{Q})$ property of X immediately follows. □

We recover [8, Theorem 5.3] and [8, Proposition 5.7] with alternative proof in the next result.

Theorem 8. *The following conditions are equivalent.*
(a) The thin set $\{\zeta = 0 < \zeta_-\}$ is $\mathbb{P}$-evanescent.
(b) The stopping time $\widetilde{r}$ is infinite $\mathbb{P}$-a.s.
(c) Any $\mathbb{P}$-local martingale X admits Y^a as a $\mathbb{Q}$-local martingale deflator, so X satisfies NUPBR$(\mathbb{Q})$.
(d) Any (bounded) process X satisfying NUPBR$(\mathbb{P})$ satisfies NUPBR$(\mathbb{Q})$.

Proof. The equivalence between (a) and (b) is obvious from the definition of $\widetilde{r}$.
The implication (b)⇒(c) follows from Proposition 12. To prove (c)⇒(b) (and (d)⇒(b)), we consider the $\mathbb{P}$-martingale

$$X = \mathbb{1}_{[\![\widetilde{r},\infty[\![} - \left(\mathbb{1}_{[\![\widetilde{r},\infty[\![}\right)^{p,\mathbb{P}}.$$

Then, due to $\widetilde{r} = \infty$ $\mathbb{Q}$-a.s. we have that, under $\mathbb{Q}$,

$$X = -\left(\mathbb{1}_{[\![\widetilde{r},\infty[\![}\right)^{p,\mathbb{P}}$$

is a predictable decreasing process. Thus, X satisfies NUPBR$(\mathbb{Q})$ if and only if it is a null process. Then, we conclude that $\widetilde{S}$ is infinite using the same argument as in the proof of Lemma 6 (b). The implication (c) ⇒ (d) follows from Proposition 13. □

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