Pseudo-stopping times and the hypothesis $(\mathcal{H})$

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Abstract

The main goals of this study is to relate the class of pseudo-stopping times to the hypothesis $(\mathcal{H})$, to provide alternative characterizations of the hypothesis $(\mathcal{H})$ and to study the relationships between pseudo-stopping times, honest times and barrier hitting times. Our main result states that given two filtrations $\mathbb{F} \subset \mathbb{G}$, then every $\mathbb{F}$-martingale is an $\mathbb{G}$-martingale, or equivalently, the hypothesis $(\mathcal{H})$ is satisfied for $\mathbb{F}$ and $\mathbb{G}$, if and only if every $\mathbb{G}$-stopping time is an $\mathbb{F}$-pseudo-stopping time.

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1 Introduction

Based on the example given in Williams [11], the concept of a pseudo-stopping time was formally introduced by Nikeghbali and Yor in [10]. As its name suggest, the class of pseudo-stopping time is larger than the class of stopping times and enjoys stopping times like properties. In this paper, we study the properties of *pseudo-stopping times* (see Definition 2.2) in the context of the theory of enlargement of filtrations.

Let us describe first our setting and results. We work on a filtered probability space $(\Omega, \mathcal{A}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ denotes a filtration satisfying the usual conditions and we set $\mathcal{F}_\infty = \bigvee_{t\geq 0} \mathcal{F}_t$. A process on this space is said to be *raw* if it is not adapted to the filtration $\mathbb{F}$. As convention, for any martingale, we work always with its càdlàg modification, while for any random process $(X_t)_{t\geq 0}$, we set $X_{0-} = 0$ and $X_\infty = \lim_{t\to\infty} X$ a.s, if it exists.

In the theory of enlargement of filtration, we consider another filtration $\mathbb{G} := (\mathcal{G}_t)_{t\geq 0}$ such that $\mathbb{F} \subset \mathbb{G}$, i.e., for each $t \geq 0$, $\mathcal{F}_t \subset \mathcal{G}_t$. Then, one can consider the following hypothesis linked to enlargement of filtration problem, namely:

Definition 1.1. [[3]] The *hypothesis* $(\mathcal{H})$ is satisfied for $\mathbb{F} \subset \mathbb{G}$ if every (bounded) $\mathbb{F}$-martingale is a $\mathbb{G}$-martingale. For ease of language, when the hypothesis $(\mathcal{H})$ is satisfied for $\mathbb{F} \subset \mathbb{G}$, we shall often say $\mathbb{F}$ is *immersed* in $\mathbb{G}$ and write $\mathbb{F} \hookrightarrow \mathbb{G}$.

One special way of enlarging a filtration is the progressive enlargement with a random time. Let τ be a random time, i.e., a measurable mapping $\tau : (\Omega, \mathcal{A}) \to (\bar{\mathbb{R}}^+, \mathcal{B}(\bar{\mathbb{R}}^+))$. Then, the *progressive enlargement* of $\mathbb{F}$ with a random time τ, which we denote by $\mathbb{F}^\tau := (\mathcal{F}^\tau_t)_{t\geq 0}$, is the smallest right-continuous filtration containing $\mathbb{F}$ such that τ is an $\mathbb{F}^\tau$-stopping time. More precisely, for each $t \geq 0$, $\mathcal{F}^\tau_t := \bigcap_{s>t} \mathcal{F}_s \vee \sigma(\tau \wedge s)$.

Our results are as follows. In Nikeghbali and Yor [10], the authors have remarked that given two filtrations $\mathbb{F}$ and $\mathbb{G}$ such that $\mathbb{F}$ is immersed in $\mathbb{G}$, then every $\mathbb{G}$-stopping time is an $\mathbb{F}$-pseudo-stopping time. The main result of the present work is fostered in Theorem 3.3, where we complete this observation by showing that the converse is true and provide alternative characterizations of the hypothesis $(\mathcal{H})$ as done in Brémaud and Yor [3], but based on pseudo-stopping times and (dual) optional projections of processes with finite variation.

Assuming that the filtration $\mathbb{F}$ is immersed in the filtration $\mathbb{G}$, in Proposition 3.5, we re-examine and generalize a known result in the literature (see Gapeev [5] and Jeanblanc [9]), which states that every finite $\mathbb{G}$-stopping time that avoids $\mathbb{F}$-stopping times (see Assumption 2.1, **(A)**) is a $\mathbb{F}$-*barrier hitting time* (see Definition 2.9) with a uniformly distributed barrier, which is independent of $\mathcal{F}_\infty$. As an application of Proposition 3.5, we show in Corollary 3.7 that every $\mathbb{G}$-stopping time can be written as the minimum of two $\mathbb{G}$-stopping times, which are $\mathbb{F}$-barrier hitting time and $\mathbb{F}$-thin time respectively.

To conclude, in Lemma 3.9, we extent Proposition 6 in [10] by removing the assumption that all $\mathbb{F}$-martingales are continuous (see Assumption 2.1, **(C)**) to show that in general, a pseudo-stopping time is a *honest time* (see Definition 2.5) if and only if it is also a stopping time. This result, when combined with Theorem 3.3 gives an alternative proof to the classical result that the hypothesis $(\mathcal{H})$ is not satisfied between $\mathbb{F}$ and the progressive enlargement of $\mathbb{F}$ with a honest time.

2 Tools and background

The main tools used in this study is the (dual) optional projections onto the reference filtration $\mathbb{F}$. We record here some known results from the general theory of stochastic processes, for more detail of the theory the reader is referred to He et al. [6] or Jacod and Shiryaev [7]. For any pre-locally integrable variation process V ([6] section 5.18, 5.19), we denote the $\mathbb{F}$-optional projection of V by ${}^oV$ and the dual $\mathbb{F}$-optional projection of V by V^o. It is known that the process $N^V := {}^oV - V^o$ is an uniformly integrable $\mathbb{F}$-martingale with $N^V_0 = 0$ and ${}^o(\Delta V) = \Delta V^o$.

Specializing to the study of random times, for an arbitrary random time τ, we set $A^\tau := \mathbb{1}_{[\tau,\infty[}$[1] and define

- the supermartingale Z^τ associated with τ, $Z^\tau :=^o (\mathbb{1}_{[0,\tau[}) = 1 - {}^o(A^\tau)$,
- the supermartingale $\widetilde{Z}^\tau$ associated with τ, $\widetilde{Z}^\tau :=^o (\mathbb{1}_{[0,\tau]}) = 1 - {}^o(A^\tau_-)$,
- the martingale $m^\tau := 1 - ({}^o(A^\tau) - (A^\tau)^o)$.

Those processes are linked through the following relationships:

$$Z^\tau = m^\tau - (A^\tau)^o \quad \text{and} \quad \widetilde{Z}^\tau = Z^\tau + \Delta(A^\tau)^o. \tag{1}$$

Typically, one finds in the literature the two following assumptions related to the reference filtration $\mathbb{F}$ and random time τ:

Assumption 2.1.
Assumption **(C)** is satisfied if all $\mathbb{F}$-martingales are continuous,
Assumption **(A)** is satisfied if τ avoids all $\mathbb{F}$-stopping times or equivalently $\Delta(A^\tau)^o = 0$.

In the following, we introduce the classes of random times studied in this work. The main object is the class of pseudo-stopping times introduced by Nikeghbali and Yor in [10] as an extension of William's example in [11]. Pseudo-stopping times provide examples of random times where the supermartingale Z is decreasing, but the hypothesis $(\mathcal{H})$ is not satisfied for the progressive enlargement.

We first recall their definition of pseudo-stopping times, with a slight modification, that is the random time is allowed to take the value infinity.

Definition 2.2. ([11], [10]) A random time τ is an $\mathbb{F}$-*pseudo-stopping time* if for every bounded $\mathbb{F}$-martingale M, we have $\mathbb{E}(M_\tau) = \mathbb{E}(M_0)$.

Theorem 2.3 ([10])**.** *The following conditions are equivalent:*
(i) τ is a finite $\mathbb{F}$-pseudo-stopping time;
(ii) $(A^\tau)^o_\infty = 1$;
(iii) $m^\tau = 1$ or equivalently ${}^o(A^\tau) = (A^\tau)^o$;
(iv) for every $\mathbb{F}$-local martingale M, the process $(M_{t\wedge\tau})_t$ is an $\mathbb{F}^\tau$-local martingale.
*Moreover, if either **(C)** or **(A)** holds then the process Z^τ is decreasing $\mathbb{F}$-predictable process.*

Remark 2.4. We would like to point out that one motivation of this work is to better understand the property that ${}^o(A^\tau) = (A^\tau)^o$, which is somewhat hidden in [10]. In essence, this property says that the optional projection is equal to the dual optional projection, which is not true in general.

The present paper is devoted mostly to the study of pseudo-stopping times. However, we explore also the relationship between pseudo-stopping times and other classes of random times, namely honest times, thin times and barrier hitting times.

Definition 2.5. ([8] ch. 5, p. 73) A random time τ is an $\mathbb{F}$-honest time if for every $t > 0$ there exists an $\mathcal{F}_t$-measurable random variable τ_t such that $\tau = \tau_t$ on $\{\tau < t\}$.

Proposition 2.6 ([8] Proposition (5,1) p.73)**.** *Let τ be a random time. Then, the following conditions are equivalent:*
(i) τ is an honest time;
(ii) there exists an optional set Γ such that $\tau(\omega) = \sup\{t : (\omega, t) \in \Gamma\}$ on $\{\tau < \infty\}$;
(iii) $\tau = \sup\{t : \widetilde{Z}^\tau_t = 1\}$ a.s. on $\{\tau < \infty\}$;

[1] We shall not make use of stopped processes, hence no confusion of notation can take place.

Definition 2.7. ([1]) A random time τ is an $\mathbb{F}$-*thin time* if its graph $[\![\tau]\!]$ is contained in a thin set, i.e., if there exists a sequence of $\mathbb{F}$-stopping times $(T_n)_{n=1}^{\infty}$ with disjoint graphs such that $[\![\tau]\!] \subset \bigcup_n [\![T_n]\!]$. We say that such a sequence $(T_n)_n$ *exhausts* the thin random time τ or that $(T_n)_n$ is an *exhausting sequence* of the thin random time τ.

Lemma 2.8. *Any random time τ can be written as $\tau^c \wedge \tau^d$, where τ^c is a random time that avoids finite $\mathbb{F}$-stopping times and τ^d is an $\mathbb{F}$-thin time. More precisely $\tau^c = \tau \mathbb{1}_{\{\Delta(A^\tau)^o_\tau = 0\}} + \infty \mathbb{1}_{\{\Delta(A^\tau)^o_\tau > 0\}}$ and $\tau^d = \tau \mathbb{1}_{\{\Delta(A^\tau)^o_\tau > 0\}} + \infty \mathbb{1}_{\{\Delta(A^\tau)^o_\tau = 0\}}$.*

Definition 2.9. A random time τ is an $\mathbb{F}$-barrier hitting time if it is of the form

$$\tau = \inf\{s : X_s \geq U\}$$

where X is an $\mathbb{F}$-adapted non-decreasing process and U is a random barrier that is $\mathcal{A}$ measurable.

If the random barrier U is uniformly distributed on $[0,1]$ and independent of $\mathcal{F}_\infty$, then we are in the case of the Cox construction which is often used in credit risk modeling (see Bielecki et al. [2]).

3 Pseudo-stopping times and hypothesis $(\mathcal{H})$

Before proceeding to the main results of the paper, we give an auxiliary lemma which characterises the main property of our interest, that is, given a process of finite variation, when is its optional projection equal to the dual optional projection.

Lemma 3.1. *Given an raw increasing process V, the following properties (i) ${}^o(V_-)$ is a càglàd increasing process or ${}^o(V_-) = V^o_-$, (ii) ${}^o(V_-) = {}^oV_-$ and (iii) ${}^oV = V^o$ are equivalent.*

Proof. For any raw increasing process V, from classic theory we know that the process $N^V := {}^oV - V^o$ is an uniformly integrable martingale with $N^V_0 = 0$ and ${}^o(\Delta V) = \Delta V^o$. The combination of these two properties gives $N^V_0 = 0$,

$$N^V = {}^o(V_-) - V^o_- \quad \text{and} \quad N^V_- = {}^oV_- - V^o_-. \tag{2}$$

If ${}^o(V_-)$ is a càglàd increasing process, then from (2), we see that N^V is a predictable martingale of finite variation, therefore is constant and equal to zero, since predictable martingales are continuous. This shows $(i) \implies (iii)$, while for the converse, it is enough to use the definition of N^V. Since N^V is càdlàg, we know that $N^V = 0$ if and only if $N^V_- = 0$. This fact combined with (2) gives the equivalence between (i) and (ii). □

In the following, we extent Theorem 2.3 due to Nikeghbali and Yor to non-finite pseudo-stopping times and remove **(C)** and **(A)** from the last statement in Theorem 2.3.

Theorem 3.2. *The following conditions are equivalent:*
(i) τ is an $\mathbb{F}$-pseudo-stopping time;
(ii) $(A^\tau)^o_\infty = \mathbb{P}(\tau < \infty \,|\, \mathcal{F}_\infty)$;
(iii) $m^\tau = 1$ or equivalently ${}^o(A^\tau) = (A^\tau)^o$;
(iv) for every $\mathbb{F}$-local martingale M, the process $(M_{t\wedge\tau})_t$ is an $\mathbb{F}^\tau$-local martingale;
(v) the process $\widetilde{Z}^\tau$ is a càglàd decreasing $\mathbb{F}$-adapted process.

Proof. To see that (i) are equivalent (ii), suppose τ is an $\mathbb{F}$-pseudo-stopping time then, by properties of optional and dual optional projection, we have

$$\mathbb{E}(M_\tau \mathbb{1}_{\{\tau<\infty\}}) = \mathbb{E}(\int_{[0,\infty[} M_s d(A^\tau)^o_s) = \mathbb{E}(M_\infty (A^\tau)^o_\infty).$$

Therefore, the equality, $\mathbb{E}(M_\tau) = \mathbb{E}(M_\infty)$ holds true for every bounded $\mathbb{F}$-martingale M if and only if $(A^\tau)^o_\infty = \mathbb{P}(\tau < \infty \,|\, \mathcal{F}_\infty)$, since $\mathbb{E}(M_\tau) = \mathbb{E}(M_\infty((A^\tau)^o_\infty + \mathbb{1}_{\{\tau=\infty\}}))$.

On the other hand, ${}^o(A^\tau)_\infty = \lim_{s\to\infty} \mathbb{P}(\tau \le s \,|\, \mathcal{F}_s) = \mathbb{P}(\tau < \infty \,|\, \mathcal{F}_\infty)$ a.s, and from the definition of m^τ, we note that (ii) holds if and only if (iii) holds, that is $m^\tau = 1$ or equivalently ${}^o(A^\tau) = (A^\tau)^o$. The equivalence of (iii) and (v) follows directly from Lemma 3.1.

To see the equivalence between (i) and (iv) also holds in the case of non-finite pseudo stopping times. For any $\mathbb{G}$-stopping time ν, from page 186 of Dellacherie et al. [4], we know there exists an $\mathbb{F}$-stopping time σ such that $\rho\wedge\nu = \rho\wedge\sigma$. Therefore, together with the definition of pseudo-stopping time, we have

$$\mathbb{E}(M_{\rho\wedge\nu}) = \mathbb{E}(M_{\rho\wedge\sigma}) = \mathbb{E}(M_0),$$

which is an uniformly integrable $\mathbb{G}$-martingale by Theorem 1.42 [7]. □

Theorem 3.3. *Given two filtrations $\mathbb{F}$ and $\mathbb{G}$ such that $\mathbb{F} \subset \mathbb{G}$, the following conditions are equivalent*
(i) every bounded $\mathbb{F}$-martingale is a $\mathbb{G}$-martingale (i.e. the hypothesis $(\mathcal{H})$ is satisfied for $\mathbb{F} \subset \mathbb{G}$),
(ii) every $\mathbb{G}$-stopping time is an $\mathbb{F}$-pseudo-stopping time,
(iii) the $\mathbb{F}$-dual optional projection of any $\mathbb{G}$-optional process of integrable variation is equal to its $\mathbb{F}$-optional projection.

Proof. To show $(i) \implies (ii)$, let M be any bounded $\mathbb{F}$-martingale and ν an $\mathbb{G}$-stopping time. Then, from hypothesis $(\mathcal{H})$, M is a $\mathbb{G}$-martingale and $\mathbb{E}(M_\nu) = \mathbb{E}(M_0)$, which implies ν is an $\mathbb{F}$-pseudo-stopping time.

To show $(ii) \implies (i)$, suppose that M is a bounded $\mathbb{F}$-martingale and ν is any $\mathbb{G}$-stopping time. Since every $\mathbb{G}$-stopping time is an $\mathbb{F}$-pseudo-stopping time, we have $\mathbb{E}(M_\nu) = \mathbb{E}(M_0)$ for every $\mathbb{G}$-stopping time ν, which by Theorem 1.42 in [7], implies that M is a uniformly integrable $\mathbb{G}$-martingale.

The implication $(iii) \implies (ii)$ follows directly from Theorem 2.3 (iii), therefore we show only the implication $(i) \implies (iii)$. Under the hypothesis $(\mathcal{H})$, the $\mathbb{F}$-optional projection of any bounded process is equal to its optional projection on to the constant filtration $\mathcal{F}_\infty$ (see Bremaud and Yor [3]). More explicitly, for any given increasing $\mathbb{G}$-adapted process V, we have ${}^o(V_{\sigma-}) = \mathbb{E}\left(V_{\sigma-} \,|\, \mathcal{F}_\infty\right)$ for all $\mathbb{F}$-stopping time σ. From this we see that the process ${}^o(V_-)$ is increasing càglàd and (iii) follows from Lemma 3.1. □

Example 3.4. For any $\mathbb{F}$-stopping time σ, one can shrink $\mathbb{F}$ to $\mathbb{F}_\sigma := (\mathcal{F}_{\sigma\wedge t})_{t\ge 0}$. It can be shown that $\mathbb{F}_\sigma$ is immersed in $\mathbb{F}$ and therefore every $\mathbb{F}$-stopping time is an $\mathbb{F}_\sigma$-pseudo-stopping time.

We suppose that the hypothesis $(\mathcal{H})$ is satisfied for $\mathbb{F} \subset \mathbb{G}$. We characterize all $\mathbb{G}$-stopping times in terms of $\mathbb{F}$-barrier hitting times and $\mathbb{F}$-thin pseudo-stopping times in a similar way to Lemma 2.8. We first present an auxiliary result in Proposition 3.5, which is to some extent known in the current literature (see Remark 3.2 in Gapeev [5] and Jeanblanc [9]) for finite random times, however the exact assumptions on the invertibility of the supermartingale Z is unclear to us. Therefore, we will re-examine the result in the general framework and give a short and concise proof.

Proposition 3.5. *If the hypothesis $(\mathcal{H})$ is satisfied for $\mathbb{F} \subset \mathbb{G}$ and ν is a $\mathbb{G}$-stopping time that avoids all finite $\mathbb{F}$-stopping times then*
(i) the $\mathcal{F}_\infty$-conditional distribution of $(A^\nu)^o_\nu$ is uniform on the interval $[0, (A^\nu)^o_\infty)$, with an atom of size $1 - (A^\nu)^o_\infty$ at $(A^\nu)^o_\infty$.
(ii) the $\mathbb{G}$-stopping time ν is an $\mathbb{F}$-barrier hitting time, that is $\nu = \inf\{t > 0 : (A^\nu)^o_t \ge (A^\nu)^o_\nu\}$.

Proof. To show (i), we compute the $\mathcal{F}_\infty$-conditional distribution of $(A^\nu)^o_\nu$, that is

$$\mathbb{E}_{\mathbb{P}}\left(\mathbb{1}_{\{(A^\nu)^o_\nu \le u\}} \,\middle|\, \mathcal{F}_\infty\right) = \mathbb{E}_{\mathbb{P}}\left(\mathbb{1}_{\{(A^\nu)^o_\nu \le u\}} \,\middle|\, \mathcal{F}_\infty\right) \mathbb{1}_{\{u<(A^\nu)^o_\infty\}} + \mathbb{1}_{\{u\ge(A^\nu)^o_\infty\}}$$

Let us set C to be the right inverse of $(A^\nu)^o$, then the first term in the right hand side above is

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}\left(\mathbb{1}_{\{(A^\nu)^o_\nu \le u\}}\mathbb{1}_{\{C_u<\infty\}} \,|\, \mathcal{F}_\infty\right) &= \mathbb{E}_{\mathbb{P}}\left(\mathbb{1}_{\{\nu\le C_u\}}\mathbb{1}_{\{C_u<\infty\}} \,|\, \mathcal{F}_{C_u}\right)\\ &= {}^o(A^\nu)_{C_u}\mathbb{1}_{\{C_u<\infty\}}\\ &= (A^\nu)^o_{C_u}\mathbb{1}_{\{C_u<\infty\}}\\ &= u\mathbb{1}_{\{u<(A^\nu)^o_\infty\}}\end{aligned}$$

where we apply Theorem 3.3 in the third equality, while last equality follows from the fact that $(A^\nu)^o_{C_u} = u$, since $(A^\nu)^o$ is continuous except perhaps at infinity. This implies that the $\mathcal{F}_\infty$-conditional distribution of $A^{\nu,o}_\nu$ is uniform on $[0,(A^\nu)^o_\infty)$.

To show (ii), we first define another random time ν^* by setting

$$\nu^* := \inf\{t>0 : (A^\nu)^o_t \ge (A^\nu)^o_\nu\}.$$

To see that $\nu^* = \nu$ (it is obvious that $\nu^* \le \nu$), we use Lemma 4.2 of [8] which states that the left-support of the measure dA^ν, i.e.,

$$\{(\omega,t) : \forall \varepsilon>0 \quad A^\nu_t(\omega) > A^\nu_{t-\varepsilon}(\omega)\} = [\![\nu]\!]$$

belongs to the left-support of $(A^\nu)^o$, i.e., to the set $\{(\omega,t) : \forall\varepsilon>0 \quad (A^\nu)^o_t(\omega) > (A^\nu)^o_{t-\varepsilon}(\omega)\}$. □

Remark 3.6. In [9] the process Z was assumed to be continuous and strictly increasing (i.e invertible), therefore $\nu^* = \nu$ holds without the result of Jeulin [8]. While in Remark 3.2 in [5], the equality $\nu^* = \nu$ appears to be obvious from continuity and the author do not refer to other results.

As a special case, if the hypothesis $(\mathcal{H})$ is satisfied for filtration $\mathbb{F} \subset \mathbb{G}$ and that ν is a finite $\mathbb{G}$-stopping time, then $(A^\nu)^o_\infty = 1$ and $(A^\nu)^o_\nu$ is independent of $\mathbb{F}_\infty$ and uniformly distributed on the interval $[0,1]$. In this case, the stopping time ν is a random time constructed from the Cox construction.

Corollary 3.7. *If the hypothesis $(\mathcal{H})$ is satisfied for filtration $\mathbb{F} \subset \mathbb{G}$ and ν is a $\mathbb{G}$-stopping time, then it can be written as $\nu^c \wedge \nu^d$, where ν^c is a $\mathbb{G}$-stopping time which is an $\mathbb{F}$-barrier hitting time avoiding $\mathbb{F}$-stopping times and ν^d is a $\mathbb{G}$-stopping time whose graph is contained in the graphs of a sequence of $\mathbb{F}$-stopping times.*

Proof. By Lemma 2.8, it has the representation $\nu = \nu^d \wedge \nu^c$. We only show that ν^c is a $\mathbb{G}$-stopping time as for ν^d the proof is analogical. Note that $\{\nu^c \le t\} = \{\nu \le t\} \cap \{\Delta(A^\nu)^o_\nu = 0\}$, thus ν^c is $\mathbb{G}$-stopping time if and only if $\{\Delta(A^\nu)^o_\nu = 0\} \in \mathcal{G}_\nu$. By Corollary 3.23 in [6], $\Delta(A^\nu)^o_\nu\mathbb{1}_{\{\nu<\infty\}}$ is $\mathcal{G}_\nu$-measurable as $\Delta(A^\nu)^o$ is $\mathbb{G}$-optional process and the assertion follows. The $\mathbb{G}$-stopping time ν^c avoids all $\mathbb{F}$-stopping times and we conclude by applying Proposition 3.5. □

Unlike stopping times, the minimum and maximum of two $\mathbb{F}$-pseudo-stopping times is in general not an $\mathbb{F}$-pseudo-stopping time. In the following, we explore extensions to Proposition 4. in [10], which states that the minimum of a pseudo-stopping time ρ with an $\mathbb{F}^\rho$-stopping time is again a pseudo-stopping time.

Lemma 3.8. *(i) Let ρ be a $\mathcal{F}_\infty$-measurable $\mathbb{F}$-pseudo-stopping time and τ be a random time such that $\mathbb{F}$ is immersed in $\mathbb{F}^\tau$, then $\tau\wedge\rho$ is again an $\mathbb{F}$-pseudo-stopping time.*
(ii) Let ρ be an $\mathbb{F}$-pseudo-stopping time and if τ is an $\mathbb{F}^\rho$-pseudo-stopping time, then $\tau\wedge\rho$ is again an $\mathbb{F}$-pseudo-stopping time.

Proof. (i) To compute the $\widetilde{Z}^{\tau\wedge\rho}$, we note that from the fact that ρ is $\mathcal{F}_\infty$ measurable and hypothesis $(\mathcal{H})$ holds between $\mathbb{F}$ and $\mathbb{F}^\tau$, we have

$$\mathbb{P}(\tau\ge t, \rho\ge t \,|\, \mathcal{F}_\infty) = \mathbb{1}_{\{\rho\ge t\}}\mathbb{P}(\tau\ge t \,|\, \mathcal{F}_\infty) = \mathbb{1}_{\{\rho\ge t\}}\mathbb{P}(\tau\ge t \,|\, \mathcal{F}_t).$$

This shows that $\widetilde{Z}^{\tau\wedge\rho} = \widetilde{Z}^{\tau}\widetilde{Z}^{\rho}$ and is decreasing càglàd. We conclude by using Theorem 3.2 (v).

(ii) Since τ is an $\mathbb{F}^{\rho}$-pseudo-stopping time and $(M_{\rho\wedge t})_{t\geq 0}$ is an $\mathbb{F}^{\rho}$ martingale, we have $\mathbb{E}(M_{\rho\wedge\tau}) = \mathbb{E}(M_0)$ from the definition of pseudo-stopping time. Therefore $\tau\wedge\rho$ is an $\mathbb{F}$-pseudo-stopping time. □

Finally, we relate pseudo-stopping times with honest times. Under **(C)**, a result of similar spirit was presented in Proposition 6 in [10], where distributional argument were given. Here, we use sample path properties to show that the same kind of result holds in full generality.

Lemma 3.9. *Let τ be a random time. Then, the following conditions are equivalent*
(i) τ is equal to an $\mathbb{F}$-stopping time on $\{\tau < \infty\}$;
(ii) τ is an $\mathbb{F}$-pseudo-stopping time and an $\mathbb{F}$-honest time.

Proof. The implication $(i) \Longrightarrow (ii)$ holds by Theorem 3.2 (iii) and Proposition 2.6 (iii). To show $(ii) \Longrightarrow (i)$, let us note that the honest time property of τ implies $\tau = \sup\{t : \widetilde{Z}^{\tau}_t = 1\}$ on $\{\tau < \infty\}$ by Proposition 2.6 (iii), and by Theorem 3.2 (v), the pseudo-stopping time property of τ implies that $\widetilde{Z}^{\tau} = 1 - (A^{\tau})^{o}_{-}$. Therefore, on $\{\tau < \infty\}$,

$$\tau = \sup\{t : \widetilde{Z}^{\tau}_t = 1\} = \sup\{t : (A^{\tau})^{o}_{t-} = 0\} = \inf\{t : (A^{\tau})^{o}_{t} > 0\},$$

so, τ is equal to an $\mathbb{F}$-stopping time on $\{\tau < \infty\}$. □

As a simple consequence of Lemma 3.9 and Theorem 3.3, we recover the following classical result found in Jeulin [8], where the result follows from $\mathbb{G}$-semimartingale decompositions of $\mathbb{F}$-martingales.

Corollary 3.10. *If τ is an $\mathbb{F}$-honest time which is not equal to an $\mathbb{F}$-stopping time on $\{\tau < \infty\}$, then the hypothesis $(\mathcal{H})$ is not satisfied for $\mathbb{F} \subset \mathbb{F}^{\tau}$.*

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