Fractional operators with singular drift: Smoothing properties and Morrey-Campanato spaces

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22/12/2014

Abstract

We investigate some smoothness properties for a transport-diffusion equation involving a class of non-degerate Lévy type operators with singular drift. Our main argument is based on a duality method using the molecular decomposition of Hardy spaces through which we derive some Hölder continuity for the associated parabolic PDE. This property will be fulfilled as far as the singular drift belongs to a suitable Morrey-Campanato space for which the regularizing properties of the Lévy operator suffice to obtain global Hölder continuity.

Keywords: Lévy-type operators, Morrey-Campanato spaces, Hölder regularity, molecular Hardy spaces.

1 Introduction and Main Results

In this article, we are interested in studying some smoothness properties of the real-valued equation

$$\begin{cases} \partial_t \theta(t,x) + \nabla \cdot (v\,\theta)(t,x) + \mathcal{L}\theta(t,x) = 0, \\ \theta(0,x) = \theta_0(x), \qquad \text{for } x \in \mathbb{R}^n,\ n \geq 2, \\ \text{with } \ \nabla \cdot (v) = 0 \ \text{ and } t \in [0,T]. \end{cases} \tag{1}$$

The operator $\mathcal{L}$ is given by the expression

$$\mathcal{L}(f)(x) = \text{v.p.} \int_{\mathbb{R}^n} \big[f(x) - f(x-y)\big]\pi(y)dy, \tag{2}$$

where $\pi(y)dy$ is a non-degenerate and bounded Lévy measure. The first order term is written in divergence form and the velocity field v is meant to be rather *singular*. The divergence free condition of the drift term v is usual in problems arising from fluid mechanics.

When the operator $\mathcal{L}$ is a fractional power of the Laplace operator $(-\Delta)^{\frac{\alpha}{2}}$ with $0 < \alpha < 2$ (given in the Fourier level by $\widehat{(-\Delta)^{\frac{\alpha}{2}} f}(\xi) = c|\xi|^\alpha \widehat{f}(\xi)$), equation (1) can indeed be seen as a simplified version of the quasi-geostrophic equation (denoted by $(QG)_\alpha$) which would correspond to the non-linear velocity field $v = (-R_2\theta, R_1\theta)$ where $R_{1,2}$ denote the Riesz Transforms defined by $\widehat{R_j\theta}(\xi) = -\frac{i\xi_j}{|\xi|}\widehat{\theta}(\xi)$ for $j = 1, 2$. It is worth noting in this quasi-geostrophic setting that there is a competition between the drift term v and the diffusion term $(-\Delta)^{\frac{\alpha}{2}}$ and it is classical to distinguish here three regimes: *super-critical* if $0 < \alpha < 1/2$, *critical* if $\alpha = 1/2$ and *sub-critical* if $1/2 < \alpha < 1$, from which only the two first cases are of interest since in the sub-critical case the regularization effect given by the fractional power of the Laplacian $(-\Delta)^{\frac{\alpha}{2}}$ is "stronger" than the non-linear drift and, as a consequence, there is a natural smoothing effect in the solutions of (1). For the two other cases there is an interesting and rather complex competition between the smoothing term and the drift one and, in particular, in the super-critical case it is still an open problem to understand the regularity of the solutions of this equation, see [2], [8], [6], [7], [17] and the references therein for more details.

Following the work of Kiselev and Nazarov [15], it is possible to study the Hölder regularity of the solutions of the $(QG)_{1/2}$ equation (i.e. the critical case) by a duality-based method where the main idea is to control the deformation of a special class of functions in order to deduce the regularity of the solutions of such equation.

The aim of this article is, in the spirit of [3], to generalize this idea using different tools and to apply it to a wider family of operators. Specifically we will work here with Lévy type operators under some hypotheses that will be stated

in the lines below and we will see that this approach actually turns out to be well adapted to investigate the impact of a *singular* divergence free drifts on the smoothing properties of the operator $\mathcal{L}$.

Thus, one of our objectives is to characterize, for a singular drift, the functional spaces for which a Hölder continuity property holds for the solution of the Cauchy problem (1). Under some non-degeneracy assumption on the Lévy measure π, it will be seen that the natural framework for the drifts is the one of Morrey-Campanato spaces, whose parameters will be related to the operator $\mathcal{L}$ thanks to some homogeneity properties and then, with the useful hypotesis $div(v) = 0$, we will prove that it is possible to obtain a small gain of regularity.

In this paper we will mainly establish existence and uniqueness results as well as Hölder regularity for the solutions of equation (1). We will also obtain, as intermediate results, a maximum and a positivity principle for (1).

Let us start by describing our setting in a general way.

[MC] The divergence free drift (or velocity) term v is assumed to belong to $\big(L^\infty([0,T], M^{q,a}(\mathbb{R}^n))\big)^n$ where $M^{q,a}(\mathbb{R}^n)$ is the Morrey-Campanato space defined for $1 \leq q < +\infty$ and $0 \leq a < +\infty$ as the space of locally integrable functions such that

$$\|f\|_{M^{p,a}} = \sup_{x_0\in\mathbb{R}^n} \sup_{0<r<+\infty} \left(\frac{1}{r^a} \int_{B(x_0,r)} |f(x) - \overline{f}_{B(x_0,r)}|^q dx \right)^{\frac{1}{q}} < +\infty,$$

with $\overline{f}_{B(x_0,r)} = \dfrac{1}{|B(x_0,r)|} \displaystyle\int_{B(x_0,r)} f(x)dx.$

Morrey-Campanato spaces $M^{q,a}$ are closely related to other classical spaces, in particular we have:

- If $a = 0$ then $L^q(\mathbb{R}^n) \simeq M^{q,0}(\mathbb{R}^n)$ (where "$\simeq$" means equivalence of norms),
- If $0 < a < n$ we obtain Morrey spaces $M^{q,a}(\mathbb{R}^n)$,
- If $a = n$ then $M^{q,n}(\mathbb{R}^n) \simeq M^{1,n}(\mathbb{R}^n) \simeq BMO(\mathbb{R}^n)$, the space of bounded mean oscillations (locally integrable) functions,
- If $n < a < n + q$, we have $M^{q,a}(\mathbb{R}^n) \simeq \dot{\mathcal{C}}^\lambda(\mathbb{R}^n)$ where $\dot{\mathcal{C}}^\lambda$ is the classical homogeneous Hölder space with $0 < \lambda = \frac{a-n}{q} < 1$,
- If $n + q \leq a$ then $M^{q,a}(\mathbb{R}^n)$ are reduced to constants.

As we can see, following the values of the parameters q and a we can continuously describe a wide family of functional spaces. We refer to [16], [18] and [25] for more details about Morrey-Campanato spaces.

[ND] Introduced in (2), the operator $\mathcal{L}$ we are going to work with is a Lévy operator for which we assume that the function π is symmetric, i.e. $\pi(y) = \pi(-y)$ for all $y \in \mathbb{R}^n$. Also, the following bounds hold:

$$\overline{c}_1|y|^{-n-\alpha} \leq \quad \pi(y) \quad \leq \overline{c}_2|y|^{-n-\alpha} \qquad \text{over } |y| \leq 1, \tag{3}$$

$$0 \leq \quad \pi(y) \quad \leq \overline{c}_2|y|^{-n-\delta} \qquad \text{over } |y| > 1, \tag{4}$$

where $0 < \overline{c}_1 \leq \overline{c}_2$ are positive constants and where $0 < \delta < \alpha < 2$. In the Fourier level we have $\widehat{\mathcal{L}f}(\xi) = a(\xi)\widehat{f}(\xi)$ where the symbol $a(\cdot)$ is given by the Lévy-Khinchin formula

$$a(\xi) = \int_{\mathbb{R}^n\setminus\{0\}} \big(1 - \cos(\xi \cdot y)\big)\pi(y)dy. \tag{5}$$

We refer to [12], [13] and [19] for additional properties concerning Lévy operators and the Lévy-Khinchin representation formula. See also the lecture notes [14] for interesting applications to the PDEs.

Observe carefully that the properties of the operator $\mathcal{L}$ can be easily read, in the real variable or in the Fourier level, through the properties of the function π. In order to have a better understanding of these properties it is helpful to consider the important example provided by the fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$ defined by the expression

$$(-\Delta)^{\frac{\alpha}{2}} f(x) = \text{v.p.} \int_{\mathbb{R}^n} \frac{f(x) - f(x-y)}{|y|^{n+\alpha}} dy, \quad \text{with } 0 < \alpha < 2.$$

Note that we have here $\pi(y) = |y|^{-n-\alpha}$ and π satisfies (3) and (4) with $\alpha = \delta$. Equivalently, we have a Fourier characterisation by the formula $\widehat{(-\Delta)^{\frac{\alpha}{2}} f}(\xi) = |\xi|^\alpha \widehat{f}(\xi)$ so the function $a(\xi)$ is equal to $|\xi|^\alpha$. With this example we

observe that the lower bound in (3) guarantees a *diffusion or regularization effect*[1] like $(-\Delta)^{\frac{\alpha}{2}}$ for $\mathcal{L}$. Indeed, in some general sense, only the part of the integral (2) near the origin is critical as π satisfies (4). Assumption **[ND]** can therefore be viewed as a kind of *non-degeneracy* condition which roughly means that in terms of regularizing effects (which are induced by the behavior of π near the origin) the operator $\mathcal{L}$ behaves as $(-\Delta)^{\frac{\alpha}{2}}$.

As the case $\delta = \alpha = 1$ was already treated in [3] in a different framework and since the case $\delta = \alpha$ corresponds to the fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$ where the computations are considerably simplified, we will always assume in this article that $0 < \delta < \alpha < 2$.

Presentation of the results

We will from now on assume that assumptions **[MC]** and **[ND]** are in force. Our first result concerns existence and uniqueness to (1).

Theorem 1 (Existence and uniqueness for L^p initial data) *If the initial data θ_0 in equation (1) belongs to $L^p(\mathbb{R}^n)$ with $2 \leq p \leq +\infty$, then (1) has a unique weak solution $\theta \in L^\infty([0,T]; L^p(\mathbb{R}^n))$.*

Our main theorem is the next one. Following the usual terminology for the quasi-geostrophic equation we will say that equation (1) is *super-critical* in the (resp. *sub-critical* case) if $\alpha \in]0,1[$ (resp. $\alpha \in]1,2[$).

Theorem 2 (Hölder property of the solution) *Fix a small time $T_0 > 0$ and let θ_0 be an initial data such that $\theta_0 \in L^\infty(\mathbb{R}^n)$. If $\theta(t,x)$ is a solution for the equation (1) and the velocity field $v(t,x)$ belongs to the space $\left(L^\infty([0,T], M^{q,a}(\mathbb{R}^n))\right)^n$ with $\frac{a-n}{q} = 1-\alpha$, then for all time $T_0 < t < T$, we have that the solution $\theta(t,\cdot)$ belongs to the Hölder space $\mathcal{C}^\gamma(\mathbb{R}^n)$ with*

- $0 < \gamma < \delta < \alpha < 1$ *in the super-critical case ($\alpha \in]0,1[$).*
- $0 < \gamma < \min(\delta, 1)$ *in the sub-critical case ($\alpha \in]1,2[$).*

Observe that the maximal Hölder regularity obtained by this method is controlled by the parameter δ defining the Lévy operator. In this context the most important issue is to obtain *some* Hölder regularity since it should be possible to apply a bootstrap argument as in [2], Section B, in order to obtain higher regularity.

The case $\alpha = 1$ known as the *critical* case is not treated here as it has already been studied in [3] where a $L^\infty(BMO)$ drift was considered. In this particular case, the BMO space corresponds to the Morrey-Campanato space $M^{q,a}$ with $a = n$.

It is worth noting that the Morrey-Campanato space used in this theorem is fixed by the relationship $\frac{a-n}{q} = 1-\alpha$. In the super-critical case, since $0 < \alpha < 1$ we have $n < a < n+q$, thus the Morrey-Campanato space $M^{q,a}$ is equivalent to a classical Hölder space of regularity $1-\alpha$. We have here that the low or super-critical regularization effect of the Lévy-type operator $\mathcal{L}$ driven by the parameter α is exactly compensated by the Hölder regularity of order $1-\alpha$ of the velocity field. In the sub-critical case, since $1 < \alpha < 2$ we have that $0 < a < n$ and thus the higher or sub-critical regularization effect of the Lévy-type operator $\mathcal{L}$ given by the parameter α allows to consider a more irregular velocity field belonging to a *true* Morrey-Campanato space. Observe that in both cases we are able to obtain a smoothing effect and we can prove that the solutions of (1) belong to a Hölder space $\mathcal{C}^\gamma$.

The relationship between the parameter α which rules the regularization effect of the Lévy type operator and the indexes q and a defining the Morrey-Campanato spaces $M^{q,a}$ is actually quite sharp. Indeed, if the identity $\frac{a-n}{q} = 1-\alpha$ is not verified, it is possible to provide counterexamples of Theorem 2 in some particular cases. See [20] for a construction of such counterexamples and see also [7] for similar results in the setting of the quasi-geostrophic equation.

The strategy to derive the previous results is the following. For existence and uniqueness, we first start from a fixed point argument for a modified problem, with mollified drift and an additional viscosity term in Δ which is meant to vanish, and for which a uniform maximum principle is established (see Proposition 2.2) for any L^p initial data with $p \in [1, +\infty[$. For $p \in [2, +\infty[$ the result is then derived through compactness arguments which anyhow require some Besov regularity, yielding the constraint on p (see Theorem 5) and the fact that Morrey-Campanato spaces are dual spaces (see Proposition 2.1 and [25]). The extension of the result to the case $p = +\infty$, which is crucial to derive Theorem 2 with our duality method, is obtained thanks to a positivity principle established in Theorem 6. As a by-product, a global maximum principle is obtained for the limit weak solutions for $p \in [2, +\infty]$ (see Theorem 7).

[1] the term "diffusion" must be taken in the sense of the PDEs considered by analysts.

For the Hölder properties of the solutions, we use the duality between local Hardy spaces and Hölder spaces and the fact that we have a molecular decomposition of local Hardy spaces. Roughly speaking, to derive the smoothness, it suffices, thanks to those two previous features and to a transfer property (see Proposition 4.1), to control the L^1 norm of the adjoint equation to (1) where the initial condition can be any molecule. A molecule ψ at scale $r > 0$, can be viewed as a function satisfying an L^∞ condition, $\|\psi\|_{L^\infty(\mathbb{R}^n)} \leq Cr^{-(n+\gamma)}$, and a concentration condition around its *center* x_0, i.e. $\int_{\mathbb{R}^n} |\psi(x)||x - x_0|^\omega dx \leq Cr^{\omega-\gamma}$, where n is the dimension, γ is the final Hölder index and ω is a technical parameter. We refer to Definition 4.2 for a precise statement.

To control the evolution in time of the L^1 norm of the adjoint equation having a molecule as initial condition, two cases are to be distinguished. If the molecule is *big*, i.e. $r > 1$, the previously established maximum principle readily gives the result. The *small* molecules require a more subtle treatment. The evolution of the L^1 norm of such molecules can be investigated updating in time the previous L^∞ and concentration conditions, this latter being considered around the current *spatial center* in time corresponding to the evolution of the differential system, starting from the initial *center* of the molecule with the *averaged* drift of (1) on a suitable ball. In other words, the evolution of the initial center of the molecule is nothing but its transport by an averaged, less singular, velocity field associated with the initial one. Averaging is a way to regularize, once this choice is made, the functional framework of Morrey-Campanato spaces is indeed very natural since it allows to sharply control the differences between the initial drift and the regularized one.

The article is organized as follows. In Section 2 we study existence and uniqueness of solutions with initial data in L^p with $2 \leq p < +\infty$. We will also prove a maximum principle for the weak solutions of (1). Section 3 is devoted to a positivity principle that is crucial to prove the Hölder regularity. It also allows to extend the previous existence and uniqueness results to the case $\theta_0 \in L^\infty$. In Section 4 we study the Hölder regularity of the solutions of equation (1) by a duality method. This is the core of the paper. Technical computations are postponed to the appendix.

2 Existence and uniqueness with L^p initial data and Maximum Principle.

In this section we will study existence and uniqueness for weak solution of equation (1) with initial data $\theta_0 \in L^p(\mathbb{R}^n)$ where $p \geq 1$. We will start by considering *viscosity solutions* with an approximation of the velocity field v, and we will prove existence and uniqueness for this system. To pass to the limit we will need a further step which follows from the maximum principle.

2.1 Viscosity solutions

The point in this section consists in introducing an approximate equation deriving from (1), where we add an additional viscosity contribution in $\varepsilon\Delta$ and suitably mollify the potentially singular drift. Precisely, for $\varepsilon > 0$, we introduce:

$$\begin{cases} \partial_t\theta(t,x) + \nabla \cdot (v_\varepsilon\ \theta)(t,x) + \mathcal{L}\theta(t,x) = \varepsilon\Delta\theta(t,x), \\ \\ \theta(0,x) = \theta_0(x), \\ \\ div(v_\varepsilon) = 0 \quad \text{ and } v_\varepsilon \in L^\infty([0,T]; L^\infty(\mathbb{R}^n)). \end{cases} \tag{6}$$

Above v_ε is defined by $v_\varepsilon = v^\kappa * \omega_\varepsilon$ where ω_ε is a usual mollifying kernel, i.e. $\omega_\varepsilon(x) = \varepsilon^{-n}\omega(x/\varepsilon)$, $\omega \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ is a non-negative function such that $\int_{\mathbb{R}^n} \omega(x)dx = 1$. Also, $(v^\kappa)_{\kappa\in\mathbb{N}}$ is a familiy of L^∞ functions such that $div(v^\kappa) = 0$ and that converges weakly-$*$ towards $v \in M^{q,a}$. This is not very restrictive since we have the following proposition whose proof can be found in [25].

Proposition 2.1 *The Morrey-Campanato spaces are dual spaces. In particular, any element of $M^{q,a}$ can be approximated in the weak-$*$ topology by a sequence of L^∞ functions.*

From this first regularization, the approximate drift is smooth and bounded. The role of the additional viscosity is then clear. We can view the right hand side of (6) as a source term for the *usual* Heat equation. We will prove existence and uniqueness results, see Theorem 3, Remark 2.1, as well as uniform controls with respect to the mollifying parameter/vanishing viscosity that are the preliminary step of our compactness based approach, see Proposition 2.2.

Following [8], the solutions of problem (6) are called *viscosity solutions*.

Note now that the problem (6) admits the following equivalent integral representation:

$$\theta(t,x) = e^{\varepsilon t\Delta}\theta_0(x) - \int_0^t e^{\varepsilon(t-s)\Delta}\nabla\cdot(v_\varepsilon\,\theta)(s,x)ds - \int_0^t e^{\varepsilon(t-s)\Delta}\mathcal{L}\theta(s,x)ds. \tag{7}$$

In order to prove Theorem 1, we will first investigate a local result with the following theorem where we will apply the Banach contraction scheme in the space $L^\infty([0,T];L^p(\mathbb{R}^n))$ with the norm $\|f\|_{L^\infty(L^p)} = \sup_{t\in[0,T]}\|f(t,\cdot)\|_{L^p}$.

Theorem 3 (Local existence for viscosity solutions) *Let $1 \le p < +\infty$. If the initial data satisfies $\|\theta_0\|_{L^p} \le K$ and if T' is a time small enough, then (7) has a unique solution $\theta \in L^\infty([0,T];L^p(\mathbb{R}^n))$ on the closed ball $\overline{B}(0,2K) \subset L^\infty([0,T];L^p(\mathbb{R}^n))$.*

Proof of Theorem 3. We denote by $N_\varepsilon^v(\theta)$ and $L_\varepsilon(\theta)$ the quantities

$$N_\varepsilon^v(\theta)(t,x) = \int_0^t e^{\varepsilon(t-s)\Delta}\nabla\cdot(v_\varepsilon\,\theta)(s,x)ds \quad \text{and} \quad L_\varepsilon(\theta)(t,x) = \int_0^t e^{\varepsilon(t-s)\Delta}\mathcal{L}\theta(s,x)ds.$$

We construct now a sequence of functions in the following way

$$\theta_{n+1}(t,x) = e^{\varepsilon t\Delta}\theta_0(x) - N_\varepsilon^v(\theta_n)(t,x) - L_\varepsilon(\theta_n)(t,x),$$

we take the $L^\infty(L^p)$-norm of this expression to obtain

$$\|\theta_{n+1}\|_{L^\infty(L^p)} \le \|e^{\varepsilon t\Delta}\theta_0\|_{L^\infty(L^p)} + \|N_\varepsilon^v(\theta_n)\|_{L^\infty(L^p)} + \|L_\varepsilon(\theta_n)\|_{L^\infty(L^p)}, \tag{8}$$

and we will treat each one of the terms of the right-hand side separately.

For the first term above we note that, since $e^{\varepsilon t\Delta}$ is a contraction operator, the estimate $\|e^{\varepsilon t\Delta}f\|_{L^p} \le \|f\|_{L^p}$ is valid for all function $f \in L^p(\mathbb{R}^n)$ with $1 \le p \le +\infty$, for all $t > 0$ and all $\varepsilon > 0$. Thus, we have

$$\|e^{\varepsilon t\Delta}f\|_{L^\infty(L^p)} \le \|f\|_{L^p}. \tag{9}$$

For the second term of (8) we have the following inequality: if $f \in L^\infty([0,T];L^p(\mathbb{R}^n))$ and if $v^\kappa \in L^\infty([0,T];L^\infty(\mathbb{R}^n))$, then

$$\|N_\varepsilon^v(f)\|_{L^\infty(L^p)} \le C\sqrt{\frac{T'}{\varepsilon}}\,\|v^\kappa\|_{L^\infty(L^\infty)}\|f\|_{L^\infty(L^p)}. \tag{10}$$

Indeed, since $e^{\varepsilon t\Delta}f = f * h_{\varepsilon t}$, where $h_{\varepsilon t}$ is the associated heat kernel, we write:

$$\begin{aligned}
\|N_\varepsilon^v(f)\|_{L^\infty(L^p)} &= \sup_{0<t<T'}\left\|\int_0^t e^{\varepsilon(t-s)\Delta}\nabla\cdot(v_\varepsilon f)(s,\cdot)ds\right\|_{L^p} = \sup_{0<t<T'}\left\|\int_0^t \nabla\cdot(v_\varepsilon f) * h_{\varepsilon(t-s)}(s,\cdot)ds\right\|_{L^p} \\
&\le \sup_{0<t<T'}\int_0^t \|v_\varepsilon f(s,\cdot)\|_{L^p}\left\|\nabla h_{\varepsilon(t-s)}\right\|_{L^1}ds \le \sup_{0<t<T'}\int_0^t \|v_\varepsilon(s,\cdot)\|_{L^\infty}\|f(s,\cdot)\|_{L^p}\,C(\varepsilon(t-s))^{-1/2}ds \\
&\le \|v^\kappa\|_{L^\infty(L^\infty)}\|f\|_{L^\infty(L^p)}\sup_{0<t<T'}\int_0^t C(\varepsilon(t-s))^{-1/2}ds \le C\sqrt{\frac{T'}{\varepsilon}}\|v^\kappa\|_{L^\infty(L^\infty)}\|f\|_{L^\infty(L^p)}.
\end{aligned}$$

For the last term of (8) we have the following fact: if $f \in L^\infty([0,T];L^p(\mathbb{R}^n))$, then

$$\|L_\varepsilon(f)\|_{L^\infty(L^p)} \le C\left(\frac{T'^{1-\frac{\alpha}{2}}}{\varepsilon^{\frac{\alpha}{2}}} + \frac{T'^{1-\frac{\delta}{2}}}{\varepsilon^{\frac{\delta}{2}}}\right)\|f\|_{L^\infty(L^p)}. \tag{11}$$

Indeed, we write

$$\|L_\varepsilon(f)\|_{L^\infty(L^p)} = \sup_{0<t<T'}\left\|\int_0^t e^{\varepsilon(t-s)\Delta}\mathcal{L}f(s,\cdot)ds\right\|_{L^p} = \sup_{0<t<T'}\left\|\int_0^t \mathcal{L}f * h_{\varepsilon(t-s)}(s,\cdot)ds\right\|_{L^p},$$

where h_t is the heat kernel on $\mathbb{R}^n$. Then by the properties of the Lévy operator $\mathcal{L}$ we can write $\mathcal{L}f * h_{\varepsilon(t-s)} = f * \mathcal{L}h_{\varepsilon(t-s)}$ and we obtain the estimate

$$\|L_\varepsilon(f)\|_{L^\infty(L^p)} \le \sup_{0<t<T'}\int_0^t \|f(s,\cdot)\|_{L^p}\|\mathcal{L}h_{\varepsilon(t-s)}\|_{L^1}ds \le \|f\|_{L^\infty(L^p)}\sup_{0<t<T'}\int_0^t \|\mathcal{L}h_{\varepsilon(t-s)}\|_{L^1}ds.$$

We need now to study the quantity $\|\mathcal{L}h_{\varepsilon(t-s)}\|_{L^1}$, for this we use the following lemma (proved in Appendix A):

Lemma 2.1 *Let $0 < \delta < \alpha < 2$ and let $\mathcal{L}$ be a Lévy-type operator of the form (2) with hypotheses (3) and (4). Let h_t be the heat kernel. Then we have the inequality:*

$$\|\mathcal{L}h_{\varepsilon(t-s)}\|_{L^1} \leq C\left([\varepsilon(t-s)]^{-\frac{\alpha}{2}} + \varepsilon(t-s)]^{-\frac{\delta}{2}}\right).$$

Thus, with this result at hand and after an integration in time we obtain the wished inequality (11).

Now, applying the inequalities (9), (10) and (11) to the right-hand side of (8) we have

$$\|\theta_{n+1}\|_{L^\infty(L^p)} \leq \|\theta_0\|_{L^p} + C\left(\frac{T'^{\frac{1}{2}}}{\varepsilon^{\frac{1}{2}}}\|v^\kappa\|_{L^\infty(L^\infty)} + \frac{T'^{1-\frac{\alpha}{2}}}{\varepsilon^{\frac{\alpha}{2}}} + \frac{T'^{1-\frac{\delta}{2}}}{\varepsilon^{\frac{\delta}{2}}}\right)\|\theta_n\|_{L^\infty(L^p)}.$$

Thus, if $\|\theta_0\|_{L^p} \leq K$ and if we define the time T' to be such that $C\left(\frac{T'^{\frac{1}{2}}}{\varepsilon^{\frac{1}{2}}}\|v^\kappa\|_{L^\infty(L^\infty)} + \frac{T'^{1-\frac{\alpha}{2}}}{\varepsilon^{\frac{\alpha}{2}}} + \frac{T'^{1-\frac{\delta}{2}}}{\varepsilon^{\frac{\delta}{2}}}\right) \leq \frac{1}{2}$, we have by iteration that $\|\theta_{n+1}\|_{L^\infty(L^p)} \leq 2K$: the sequence $(\theta_n)_{n\in\mathbb{N}}$ constructed from initial data θ_0 belongs to the closed ball $\overline{B}(0, 2K)$. In order to finish this proof, let us show that $\theta_n \longrightarrow \theta$ in $L^\infty([0,T'];L^p(\mathbb{R}^n))$. For this we write

$$\|\theta_{n+1} - \theta_n\|_{L^\infty(L^p)} \leq \|N_\varepsilon^v(\theta_n - \theta_{n-1})\|_{L^\infty(L^p)} + \|L_\varepsilon(\theta_n - \theta_{n-1})\|_{L^\infty(L^p)},$$

and using the previous results we have

$$\|\theta_{n+1} - \theta_n\|_{L^\infty(L^p)} \leq C\left(\frac{T'^{\frac{1}{2}}}{\varepsilon^{\frac{1}{2}}}\|v^\kappa\|_{L^\infty(L^\infty)} + \frac{T'^{1-\frac{\alpha}{2}}}{\varepsilon^{\frac{\alpha}{2}}} + \frac{T'^{1-\frac{\delta}{2}}}{\varepsilon^{\frac{\delta}{2}}}\right)\|\theta_n - \theta_{n-1}\|_{L^\infty(L^p)},$$

so, by iteration we obtain

$$\|\theta_{n+1} - \theta_n\|_{L^\infty(L^p)} \leq \left[C\left(\frac{T'^{\frac{1}{2}}}{\varepsilon^{\frac{1}{2}}}\|v^\kappa\|_{L^\infty(L^\infty)} + \frac{T'^{1-\frac{\alpha}{2}}}{\varepsilon^{\frac{\alpha}{2}}} + \frac{T'^{1-\frac{\delta}{2}}}{\varepsilon^{\frac{\delta}{2}}}\right)\right]^n \|\theta_1 - \theta_0\|_{L^\infty(L^p)}.$$

Hence, with the definition of T' we have $\|\theta_{n+1} - \theta_n\|_{L^\infty(L^p)} \leq \left(\frac{1}{2}\right)^n \|\theta_1 - \theta_0\|_{L^\infty(L^p)}$. Finally, if $n \longrightarrow +\infty$, the sequence $(\theta_n)_{n\in\mathbb{N}}$ converges towards θ in $L^\infty([0,T'];L^p(\mathbb{R}^n))$. Since it is a Banach space we deduce uniqueness for the solution θ of problem (7). The proof of Theorem 3 is finished. ■

Corollary 2.1 *The solution constructed above depends continuously on the initial value θ_0.*

Proof. Let $\varphi_0, \theta_0 \in L^p(\mathbb{R}^n)$ be two initial values and let φ and θ be the associated solutions. We write

$$\theta(t,x) - \varphi(t,x) = e^{\varepsilon t\Delta}(\theta_0(x) - \varphi_0(x)) - N_\varepsilon^v(\theta - \varphi)(t,x) - L_\varepsilon(\theta - \varphi)(t,x).$$

Taking $L^\infty L^p$-norm in the above formula and applying the same previous calculations one obtains

$$\|\theta - \varphi\|_{L^\infty(L^p)} \leq \|\theta_0 - \varphi_0\|_{L^p} + C_0\|\theta - \varphi\|_{L^\infty(L^p)}.$$

This shows continuous dependence of the solution since $C_0 = C\left(\frac{T'^{\frac{1}{2}}}{\varepsilon^{\frac{1}{2}}}\|v^\kappa\|_{L^\infty(L^\infty)} + \frac{T'^{1-\frac{\alpha}{2}}}{\varepsilon^{\frac{\alpha}{2}}} + \frac{T'^{1-\frac{\delta}{2}}}{\varepsilon^{\frac{\delta}{2}}}\right) \leq \frac{1}{2}$. ■

Remark 2.1 (From Local to Global) *Once we obtain a local result, global existence easily follows by a simple iteration since problems studied here (equations (1) or (6)) are linear as the velocity v does not depend on θ.*

We now study the regularity of the solutions constructed by this method.

Theorem 4 *Solutions of the approximated problem (6) are smooth.*

Proof. By iteration we will prove that $\theta \in \bigcap_{0<T_0<T_1<t<T_2<T^*} L^\infty([0,t];W^{\frac{k}{2},p}(\mathbb{R}^n))$ for all $k \geq 0$ where we define the Sobolev space $W^{s,p}(\mathbb{R}^n)$ for $s \in \mathbb{R}$ and $1 < p < +\infty$ by $\|f\|_{W^{s,p}} = \|f\|_{L^p} + \|(-\Delta)^{\frac{s}{2}}f\|_{L^p}$. Note that this is true for $k = 0$. So let us assume that it is also true for $k > 0$ and we will show that it is still true for $k + 1$.

Set t such that $0 < T_0 < T_1 < t < T_2 < T^*$ and let us consider the next problem

$$\theta(t,x) = e^{\varepsilon(t-T_0)\Delta}\theta(T_0,x) - \int_{T_0}^t e^{\varepsilon(t-s)\Delta}\nabla\cdot(v_\varepsilon\,\theta)(s,x)ds - \int_{T_0}^t e^{\varepsilon(t-s)\Delta}\mathcal{L}\theta(s,x)ds.$$

We have then the following estimate

$$\begin{aligned}\|\theta\|_{L^\infty(W^{\frac{k+1}{2},p})} &\leq \|e^{\varepsilon(t-T_0)\Delta}\theta(T_0,\cdot)\|_{L^\infty(W^{\frac{k+1}{2},p})} \\ &+ \left\|\int_{T_0}^t e^{\varepsilon(t-s)\Delta}\nabla\cdot(v_\varepsilon\,\theta)(s,\cdot)ds\right\|_{L^\infty(W^{\frac{k+1}{2},p})} + \left\|\int_{T_0}^t e^{\varepsilon(t-s)\Delta}\mathcal{L}\theta(s,\cdot)ds\right\|_{L^\infty(W^{\frac{k+1}{2},p})}.\end{aligned}$$

Now, we will treat separately each of the previous terms.

(i) For the first one we have

$$\begin{aligned}\|e^{\varepsilon(t-T_0)\Delta}\theta(T_0,\cdot)\|_{W^{\frac{k+1}{2},p}} &= \|\theta(T_0,\cdot)*h_{\varepsilon(t-T_0)}\|_{L^p} + \|\theta(T_0,\cdot)*(-\Delta)^{\frac{k+1}{4}}h_{\varepsilon(t-T_0)}\|_{L^p} \\ &\leq \|\theta(T_0,\cdot)\|_{L^p} + \|\theta(T_0,\cdot)\|_{L^p}\|(-\Delta)^{\frac{k+1}{4}}h_{\varepsilon(t-T_0)}\|_{L^1},\end{aligned}$$

so we can write, using the properties of the heat kernel h_t:

$$\|e^{\varepsilon(t-T_0)\Delta}\theta(T_0,\cdot)\|_{L^\infty(W^{\frac{k+1}{2},p})} \leq C\|\theta(T_0,\cdot)\|_{L^p}\max\left\{1;[\varepsilon(t-T_0)]^{-\frac{k+1}{4}}\right\}.$$

(ii) For the second term, one has

$$\begin{aligned}\left\|\int_{T_0}^t e^{\varepsilon(t-s)\Delta}\nabla\cdot(v_\varepsilon\,\theta)(s,\cdot)ds\right\|_{W^{\frac{k+1}{2},p}} &\leq \int_{T_0}^t \|\nabla\cdot(v_\varepsilon\,\theta)*h_{\varepsilon(t-s)}\|_{W^{\frac{k+1}{2},p}}ds \\ &\leq \int_{T_0}^t \|\nabla\cdot(v_\varepsilon\,\theta)*h_{\varepsilon(t-s)}\|_{L^p} + \|(-\Delta)^{\frac{k+1}{4}}\left[\nabla\cdot(v_\varepsilon\,\theta)*h_{\varepsilon(t-s)}\right]\|_{L^p}ds \\ &\leq \int_{T_0}^t \|v_\varepsilon\,\theta\|_{L^p}\|\nabla h_{\varepsilon(t-s)}\|_{L^1} + \|(-\Delta)^{\frac{k}{4}}(v_\varepsilon\,\theta)\|_{L^p}\|(-\Delta)^{\frac{1}{4}}(\nabla h_{\varepsilon(t-s)})\|_{L^1}ds \\ &\leq C\int_{T_0}^t \|v_\varepsilon\,\theta(s,\cdot)\|_{W^{\frac{k}{2},p}}\max\left\{[\varepsilon(t-s)]^{-\frac{1}{2}};[\varepsilon(t-s)]^{-\frac{3}{4}}\right\}ds.\end{aligned}$$

For $N \geq \frac{k}{2}$ we have the estimations below

$$\|v_\varepsilon\theta(s,\cdot)\|_{W^{\frac{k}{2},p}} \leq \|v_\varepsilon(s,\cdot)\|_{\mathcal{C}^N}\|\theta(s,\cdot)\|_{W^{\frac{k}{2},p}} \leq C\varepsilon^{-N}\|v^\kappa(s,\cdot)\|_{L^\infty}\|\theta(s,\cdot)\|_{W^{\frac{k}{2},p}}.$$

Hence, we can write

$$\left\|\int_{T_0}^t e^{\varepsilon(t-s)\Delta}\nabla\cdot(v_\varepsilon\,\theta)(s,\cdot)ds\right\|_{L^\infty(W^{\frac{k+1}{2},p})} \leq C\|v^\kappa\|_{L^\infty(L^\infty)}\|\theta\|_{L^\infty(W^{\frac{k}{2},p})}\sup\int_{T_0}^t \varepsilon^{-N}\max\left\{[\varepsilon(t-s)]^{-\frac{1}{2}};[\varepsilon(t-s)]^{-\frac{3}{4}}\right\}ds.$$

(iii) Finally, for the last term we have

$$\begin{aligned}\left\|\int_{T_0}^t e^{\varepsilon(t-s)\Delta}\mathcal{L}\theta(s,\cdot)ds\right\|_{W^{\frac{k+1}{2},p}} &\leq \int_{T_0}^t \left\|\theta(s,\cdot)*\mathcal{L}h_{\varepsilon(t-s)}\right\|_{L^p} + \left\|(-\Delta)^{\frac{k}{4}}\theta(s,\cdot)*\mathcal{L}(-\Delta)^{\frac{1}{4}}h_{\varepsilon(t-s)}\right\|_{L^p}ds \\ &\leq \int_{T_0}^t \|\theta(s,\cdot)\|_{L^p}\left\|\mathcal{L}h_{\varepsilon(t-s)}\right\|_{L^1} + \left\|(-\Delta)^{\frac{k}{4}}\theta(s,\cdot)\right\|_{L^p}\left\|\mathcal{L}(-\Delta)^{\frac{1}{4}}h_{\varepsilon(t-s)}\right\|_{L^1}ds.\end{aligned}$$

Applying Lemma 2.1 to the function $(-\Delta)^{\frac{1}{4}}h_{\varepsilon(t-s)}$ we obtain by homogeneity that

$$\|\mathcal{L}(-\Delta)^{\frac{1}{4}}h_{\varepsilon(t-s)}\|_{L^1} \leq \left([\varepsilon(t-s)]^{-\frac{1+2\alpha}{4}} + [\varepsilon(t-s)]^{-\frac{1+2\delta}{4}}\right),$$

and then we have

$$\left\|\int_{T_0}^t e^{\varepsilon(t-s)\Delta}\mathcal{L}\theta(s,\cdot)ds\right\|_{L^\infty(W^{\frac{k+1}{2},p})} \leq C\|\theta\|_{L^\infty(W^{\frac{k}{2},p})}$$

$$\times\int_{T_0}^t \max\left\{\left([\varepsilon(t-s)]^{-\frac{\alpha}{2}} + \varepsilon(t-s)]^{-\frac{\delta}{2}}\right);\left([\varepsilon(t-s)]^{-\frac{1+2\alpha}{4}} + [\varepsilon(t-s)]^{-\frac{1+2\delta}{4}}\right)\right\}ds.$$

Now, with formulas (i)-(iii) at our disposal, we have that the norm $\|\theta\|_{L^\infty(W^{\frac{k+1}{2},p})}$ is controlled for all $\varepsilon > 0$: we have proven spatial regularity.

Time regularity follows since we have

$$\frac{\partial^k}{\partial t^k}\theta(t,x) + \nabla\cdot\left(\frac{\partial^k}{\partial t^k}(v_\varepsilon\,\theta)\right)(t,x) + \mathcal{L}\left(\frac{\partial^k}{\partial t^k}\theta\right)(t,x) = \varepsilon\Delta\left(\frac{\partial^k}{\partial t^k}\theta\right)(t,x).$$

■

Remark 2.2 *The solutions $\theta(\cdot,\cdot)$ constructed above depend on ε.*

2.2 Maximum principle for viscosity solutions

The maximum principle we are studying here will be a consequence of few inequalities, some of them are well known. We will start with the solutions obtained in the previous section:

Proposition 2.2 (Maximum Principle for Viscosity Solutions) *Let $\theta_0 \in L^p(\mathbb{R}^n)$ with $1 < p < +\infty$ be an initial data, then the associated solution of the viscosity problem (6) satisfies the following maximum principle for all $t \in [0,T]$: $\|\theta(t,\cdot)\|_{L^p} \leq \|\theta_0\|_{L^p}$.*

Proof. We write for $1 < p < +\infty$:

$$\begin{aligned}\frac{d}{dt}\|\theta(t,\cdot)\|_{L^p}^p &= p\int_{\mathbb{R}^n}|\theta|^{p-2}\theta\Big(\varepsilon\Delta\theta - \nabla\cdot(v_\varepsilon\,\theta) - \mathcal{L}\theta\Big)(t,x)dx\\ &= p\varepsilon\int_{\mathbb{R}^n}|\theta|^{p-2}\theta\Delta\theta(t,x)dx - p\int_{\mathbb{R}^n}|\theta|^{p-1}sgn(\theta)\mathcal{L}\theta(t,x)dx,\end{aligned}$$

where we used the fact that $\nabla\cdot(v_\varepsilon) = 0$. Thus, we have

$$\frac{d}{dt}\|\theta(t,\cdot)\|_{L^p}^p - p\varepsilon\int_{\mathbb{R}^n}|\theta|^{p-2}\theta\Delta\theta(t,x)dx + p\int_{\mathbb{R}^n}|\theta|^{p-1}sgn(\theta)\mathcal{L}\theta(t,x)dx = 0,$$

and integrating in time we obtain

$$\|\theta(t,\cdot)\|_{L^p}^p - p\varepsilon\int_0^t\int_{\mathbb{R}^n}|\theta|^{p-2}\theta\Delta\theta(s,x)dxds + p\int_0^t\int_{\mathbb{R}^n}|\theta|^{p-1}sgn(\theta)\mathcal{L}\theta(s,x)dxds = \|\theta_0\|_{L^p}^p. \tag{12}$$

To finish, we have that the quantities

$$-p\varepsilon\int_{\mathbb{R}^n}|\theta|^{p-2}\theta\Delta\theta(s,x)dx \quad \text{and} \quad \int_0^t\int_{\mathbb{R}^n}|\theta|^{p-1}sgn(\theta)\mathcal{L}\theta(s,x)dxds$$

are both positive. Indeed, for the first expression, since $(e^{\varepsilon u\Delta})_{u\geq 0}$ is a contraction semi-group we have $\|e^{\varepsilon u\Delta}f\|_{L^p} \leq \|f\|_{L^p}$ for all $u > 0$ and all $f \in L^p(\mathbb{R}^n)$. Thus $F(u) = \|e^{\varepsilon u\Delta}f\|_{L^p}$ is decreasing in u; taking the derivative in u and evaluating in $u = 0$ for $f = \theta(s,.)$ we obtain the desired result. The positivity of the second expression above follows immediately from the *Stroock-Varopoulos estimate* for general Lévy-type operators given by the following formula (see Remark 1.23 of [14] for a proof, more details can be found in [22] and [24]):

$$C\langle\mathcal{L}|\theta|^{p/2}, |\theta|^{p/2}\rangle \leq \langle\mathcal{L}\theta, |\theta|^{p-1}sgn(\theta)\rangle, \tag{13}$$

it is enough to note now that $\langle\mathcal{L}|\theta|^{p/2}, |\theta|^{p/2}\rangle = \|\mathcal{L}^{\frac{1}{2}}|\theta|^{p/2}\|_{L^2}^2 \geq 0$, where the operator $\mathcal{L}^{\frac{1}{2}}$ is defined by the formula $(\mathcal{L}^{\frac{1}{2}}f)\widehat{\ }(\xi) = a^{\frac{1}{2}}(\xi)\widehat{f}(\xi)$.

Thus, getting back to (12), we have that all these quantities are bounded and positive and we write for all $1 < p < +\infty$: $\|\theta(t,\cdot)\|_{L^p} \leq \|\theta_0\|_{L^p}$. ■

2.3 Besov Regularity and the limit $\varepsilon \longrightarrow 0$ for viscosity solutions

In order to deal with Theorem 1 we will need some additional results that will allow us to pass to the limit. Indeed, a more detailed study of expression (12) above will lead to a result concerning the regularity of weak solutions.

Lemma 2.2 *If the function π satisfies the conditions (3) and (4), then for the associated operator $\mathcal{L}$ we have the following pointwise estimates on its symbol $a(\cdot)$ for all $\xi \in \mathbb{R}^n$:*

1) $a(\xi) \leq C\left(|\xi|^\alpha + |\xi|^\delta\right)$,

2) $|\xi|^\alpha \leq a(\xi) + C$.

Proof. We use the Lévy-Khinchin formula to obtain $|\xi|^\alpha = \displaystyle\int_{\mathbb{R}^n\setminus\{0\}} \big(1-\cos(y\cdot\xi)\big)|y|^{-n-\alpha}dy$ (see [12] for a proof of this fact). It is enough to apply the hypotheses (3) and (4) to conclude. ■

We state now a useful result for passing to the limit when $\varepsilon \longrightarrow 0$ which is interesting for its own sake:

Theorem 5 (Besov Regularity) *Let $\mathcal{L}$ be a Lévy-type operator of the form (2) with hypotheses (3) and (4) for the function π. Let $2 \leq p < +\infty$ and let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function such that $f \in L^p(\mathbb{R}^n)$ and*

$$\int_{\mathbb{R}^n} |f(x)|^{p-2} f(x)\mathcal{L}f(x)dx < +\infty, \qquad \textit{then} \qquad f \in \dot{B}_p^{\frac{\alpha}{p},p}(\mathbb{R}^n).$$

Proof. We will prove the following estimates valid for a positive function f:

$$\|f\|^p_{\dot{B}_p^{\frac{\alpha}{p},p}} \leq C\|f^{p/2}\|^2_{\dot{B}_2^{\frac{\alpha}{2},2}} \leq \|f^{p/2}\|^2_{L^2} + \int_{\mathbb{R}^n} |f(x)|^{p-2} f(x)\mathcal{L}f(x)dx. \tag{14}$$

The first inequality can be found in [4], so we only need to focus on the right-hand side of the previous estimate. For this, we will start assuming that the function f is positive. Using Plancherel's formula, the characterisation of $\mathcal{L}^{\frac{1}{2}}$ via the symbol $a^{\frac{1}{2}}(\xi)$ and Lemma 2.2 we write

$$\|f^{p/2}\|^2_{\dot{B}_2^{\frac{\alpha}{2},2}} \quad = \quad \|f^{p/2}\|^2_{\dot{H}^{\frac{\alpha}{2}}} = \int_{\mathbb{R}^n} |\xi|^\alpha |\widehat{f^{p/2}}(\xi)|^2 d\xi \leq \int_{\mathbb{R}^n} (a^{\frac{1}{2}}(\xi)+C)^2 |\widehat{f^{p/2}}(\xi)|^2 d\xi \leq c\left(\|f^{p/2}\|^2_{L^2} + \|\mathcal{L}^{\frac{1}{2}} f^{p/2}\|^2_{L^2}\right).$$

Now, using the Stroock-Varopoulos inequality (13) we have

$$\|f^{p/2}\|^2_{L^2} + \|\mathcal{L}^{\frac{1}{2}} f^{p/2}\|^2_{L^2} \leq \|f^{p/2}\|^2_{L^2} + c\int_{\mathbb{R}^n} f^{p-2} f\mathcal{L}f dx.$$

So inequality (14) is proven for positive functions. For the general case we write $f(x) = f_+(x) - f_-(x)$ where $f_\pm(x)$ are positive functions with disjoint support and we have:

$$\begin{aligned}\int_{\mathbb{R}^n} |f(x)|^{p-2} f(x)\mathcal{L}f(x)dx \quad &= \quad \int_{\mathbb{R}^n} f_+(x)^{p-2} f_+(x)\mathcal{L}f_+(x)dx + \int_{\mathbb{R}^n} f_-(x)^{p-2} f_-(x)\mathcal{L}f_-(x)dx \\ &- \quad \int_{\mathbb{R}^n} f_+(x)^{p-2} f_+(x)\mathcal{L}f_-(x)dx - \int_{\mathbb{R}^n} f_-(x)^{p-2} f_-(x)\mathcal{L}f_+(x)dx.\end{aligned} \tag{15}$$

We only need to treat the two last integrals, and in fact we just need to study one of them since the other can be treated in a similar way. So, for the third integral we have

$$\begin{aligned}\int_{\mathbb{R}^n} f_+(x)^{p-2} f_+(x)\mathcal{L}f_-(x)dx \quad &= \quad \int_{\mathbb{R}^n} f_+(x)^{p-2} f_+(x) \int_{\mathbb{R}^n} [f_-(x) - f_-(y)]\pi(x-y)dydx \\ &= \quad \int_{\mathbb{R}^n} f_+(x)^{p-2} \int_{\mathbb{R}^n} [f_+(x)f_-(x) - f_+(x)f_-(y)]\pi(x-y)dydx.\end{aligned}$$

However, since f_+ and f_- have disjoint supports we obtain the following estimate:

$$\int_{\mathbb{R}^n} f_+(x)^{p-2} f_+(x)\mathcal{L}f_-(x)dx = -\int_{\mathbb{R}^n} f_+(x)^{p-2} \int_{\mathbb{R}^n} [f_+(x)f_-(y)]\pi(x-y)dydx \leq 0,$$

since π is a positive function and all the terms inside the integral are positive. With this observation we see that the last terms of (15) are positive and we have

$$\int_{\mathbb{R}^n} f_+(x)^{p-2} f_+(x)\mathcal{L}f_+(x)dx + \int_{\mathbb{R}^n} f_-(x)^{p-2} f_-(x)\mathcal{L}f_-(x)dx \leq \int_{\mathbb{R}^n} |f(x)|^{p-2} f(x)\mathcal{L}f(x)dx < +\infty.$$

Then, using the first part of the proof we have $f_\pm \in \dot{B}_p^{\frac{\alpha}{p}p,p}(\mathbb{R}^n)$ and since $f = f_+ - f_-$ we conclude that f belongs to the Besov space $\dot{B}_p^{\frac{\alpha}{p},p}(\mathbb{R}^n)$. ■

Remark 2.3 *The lower bound $p \geq 2$ in Theorem 1 is a consequence of Theorem 5 above. This constraint results from the first inequality in (14).*

Proof of Theorem 1 for $p \in [2, +\infty[$**.** We have obtained with the previous results in Sections 2.1 and 2.2 a family of regular functions $(\theta^{(\varepsilon)})_{\varepsilon>0} \in L^\infty([0,T]; L^p(\mathbb{R}^n))$ which are solutions of (6) and satisfy the uniform bound $\|\theta^{(\varepsilon)}(t,\cdot)\|_{L^p} \leq \|\theta_0\|_{L^p}$; in order to conclude we need to pass to the limit $\varepsilon \longrightarrow 0$.

Since $L^\infty([0,T]; L^p(\mathbb{R}^n)) = \big(L^1([0,T]; L^q(\mathbb{R}^n))\big)'$, with $\frac{1}{p} + \frac{1}{q} = 1$, we can extract from those solutions $\theta^{(\varepsilon)}$ a subsequence $(\theta_k)_{k\in\mathbb{N}}$ which is $*$-weakly convergent to some function θ in the space $L^\infty([0,T]; L^p(\mathbb{R}^n))$, which implies convergence in $\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^n)$. However, this weak convergence is not sufficient to assure the convergence of $(v_\varepsilon\, \theta_k)$ to $v\, \theta$. For this we use the remarks that follow.

First, using the Proposition 2.1 we can consider a sequence $(v^\kappa)_{\kappa\in\mathbb{N}}$ with $v^\kappa \in L^\infty(\mathbb{R}^n)$ such that $v^\kappa \longrightarrow v$ $*$-weakly in $M^{q,a}(\mathbb{R}^n)$. Secondly, combining Proposition 2.2 and Theorem 5 we obtain that solutions θ_k belongs to the space $L^\infty([0,T]; L^p(\mathbb{R}^n)) \cap L^1([0,T]; \dot{B}_p^{\frac{\alpha}{p},p}(\mathbb{R}^n))$ for all $k \in \mathbb{N}$.

To finish, fix a function $\varphi \in \mathcal{C}_0^\infty([0,T] \times \mathbb{R}^n)$. Then we have the fact that $\varphi\theta_k \in L^1([0,T]; \dot{B}_p^{\frac{\alpha}{p},p}(\mathbb{R}^n))$ and $\partial_t\varphi\theta_k \in L^1([0,T]; \dot{B}_p^{-N,p}(\mathbb{R}^n))$. This implies the local inclusion, in space as well as in time, $\varphi\theta_k \in \dot{W}_{t,x}^{\frac{\alpha}{p},p} \subset \dot{W}_{t,x}^{\frac{\alpha}{p},2}$ so we can apply classical results such as the Rellich's theorem to obtain convergence of $v^\kappa\, \theta_k$ to $v\, \theta$.

Thus, we obtain existence and uniqueness of weak solutions for the problem (1) with an initial data in $\theta_0 \in L^p(\mathbb{R}^n)$, $2 \leq p < +\infty$ that satisfy the maximum principle. Moreover, we have that these solutions $\theta(t,x)$ belong to the space $L^\infty([0,T]; L^p(\mathbb{R}^n)) \cap L^p([0,T]; \dot{B}_p^{\frac{\alpha}{p},p}(\mathbb{R}^n))$. ■

3 Positivity principle and Maximum Principle for Weak Solutions

We will first prove in this section the following Theorem.

Theorem 6 (Positivity Principle) *Let $\max(n, \frac{n}{\delta}) < p < +\infty$ and $M > 0$ a constant, if the initial data $\theta_0 \in L^p(\mathbb{R}^n)$ is such that $0 \leq \theta_0 \leq M$ a.e. then the weak solution of equation (1) satisfies $0 \leq \theta(t,x) \leq M$ for all $t \in [0,T]$.*

As a by-product, we will finish the proof of Theorem 1 considering the case $\theta_0 \in L^\infty(\mathbb{R}^n)$. We will then also state a global maximum principle for weak solutions.

3.1 Proof of the Positivity Principle of Theorem 6.

Recall that by hypothesis we have $0 \leq \theta_0 \leq M$ an initial datum for the equation (1) with $\theta_0 \in L^p(\mathbb{R}^n)$, we will assume for a while that $1 \leq p \leq +\infty$: the condition $p > \max(n, \frac{n}{\delta})$ will appear clearly at the end of the proof of this theorem. We will show here that the associated solution $\theta(t,x)$ satisfies the bounds $0 \leq \theta(t,x) \leq M$.

To begin with, we fix two constants, ρ, R such that $R > 2\rho > 0$. Then we set $A_{0,R}(x)$ a function equals to $M/2$ over $|x| \leq 2R$ and equals to $\theta_0(x)$ over $|x| > 2R$ and we write $B_{0,R}(x) = \theta_0(x) - A_{0,R}(x)$, so by construction we have

$$\theta_0(x) = A_{0,R}(x) + B_{0,R}(x),$$

with $\|A_{0,R}\|_{L^\infty} \leq M$ and $\|B_{0,R}\|_{L^\infty} \leq M/2$. Remark that by construction we have $A_{0,R}, B_{0,R} \in L^p(\mathbb{R}^n)$ with $1 \leq p \leq +\infty$.

Now fix $v \in \big(L^\infty([0,T]; M^{q,a}(\mathbb{R}^n))\big)^n$ such that $div(v) = 0$ and consider the equations

$$\begin{cases} \partial_t A_R(t,x) + \nabla\cdot(v\, A_R)(t,x) + \mathcal{L}A_R(t,x) = 0, \\ \\ A_R(0,x) = A_{0,R}(x), \end{cases} \quad \text{and} \quad \begin{cases} \partial_t B_R(t,x) + \nabla\cdot(v\, B_R)(t,x) + \mathcal{L}B_R(t,x) = 0, \\ \\ B_R(0,x) = B_{0,R}(x). \end{cases} \tag{16}$$

Using the maximum principle and by construction we have the following estimates for $t \in [0,T]$:

$$\begin{aligned} \|A_R(t,\cdot)\|_{L^p} &\leq \|A_{0,R}\|_{L^p} \leq \|\theta_0\|_{L^p} + CMR^{\frac{n}{p}} \quad (1 \leq p < +\infty), \\ \\ \|A_R(t,\cdot)\|_{L^\infty} &\leq \|A_{0,R}\|_{L^\infty} \leq M, \\ \\ \|B_R(t,\cdot)\|_{L^\infty} &\leq \|B_{0,R}\|_{L^\infty} \leq \frac{M}{2}. \end{aligned} \tag{17}$$

where $A_R(t,x)$ and $B_R(t,x)$ are the weak solutions of the systems (16). Indeed, since $A_{0,R}, B_{0,R} \in L^p(\mathbb{R}^n)$ for $p \leq +\infty$, we can perform the limit $\|\cdot\|_{L^p} \underset{p\to+\infty}{\longrightarrow} \|\cdot\|_{L^\infty}$ to obtain the previous inequalities.

We can see now that the function $\theta(t,x) = A_R(t,x) + B_R(t,x)$ is the unique solution for the problem

$$\begin{cases} \partial_t\theta(t,x) + \nabla\cdot(v\,\theta)(t,x) + \mathcal{L}\theta(t,x) = 0, \\ \\ \theta(0,x) = A_{0,R}(x) + B_{0,R}(x). \end{cases} \tag{18}$$

Indeed, using hypothesis for $A_R(t,x)$ and $B_R(t,x)$ and the linearity of equation (18) we have that the function $\theta_R(t,x) = A_R(t,x) + B_R(t,x)$ is a solution for this equation. Uniqueness is assured by the maximum principle and by the continuous dependence from initial data given in Corollary 2.1, thus we can write $\theta(t,x) = \theta_R(t,x)$.

To continue, we will need an auxiliary function $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ such that $\phi(x) = 0$ for $|x| \geq 1$ and $\phi(x) = 1$ if $|x| \leq 1/2$ and we set $\varphi(x) = \phi(x/R)$. Now, we will estimate the L^p-norm of $\varphi(x)\big(A_R(t,x) - \frac{M}{2}\big)$. We write:

$$\begin{aligned} \partial_t\left\|\varphi(\cdot)\big(A_R(t,\cdot) - \frac{M}{2}\big)\right\|_{L^p}^p &= p\int_{\mathbb{R}^n}\left|\varphi(x)\big(A_R(t,x) - \frac{M}{2}\big)\right|^{p-2}\Big(\varphi(x)\big(A_R(t,x) - \frac{M}{2}\big)\Big) \\ &\quad\times \partial_t\Big(\varphi(x)\big(A_R(t,x) - \frac{M}{2}\big)\Big)dx. \end{aligned} \tag{19}$$

We observe that we have the following identity for the last term above

$$\begin{aligned} \partial_t\Big(\varphi(x)\big(A_R(t,x) - \frac{M}{2}\big)\Big) &= -\nabla\cdot\Big(\varphi(x)\,v\big(A_R(t,x) - \frac{M}{2}\big)\Big) - \mathcal{L}\Big(\varphi(x)\big(A_R(t,x) - \frac{M}{2}\big)\Big) \\ &+ \big(A_R(t,x) - \frac{M}{2}\big)v\cdot\nabla\varphi(x) + [\mathcal{L},\varphi]A_R(t,x) - \frac{M}{2}\mathcal{L}\varphi(x), \end{aligned}$$

where we noted $[\mathcal{L},\varphi]$ the commutator between $\mathcal{L}$ and φ. Thus, using this identity in (19) and the fact that $div(v) = 0$ we have

$$\begin{aligned} \partial_t\left\|\varphi(\cdot)(A_R(t,\cdot) - \frac{M}{2})\right\|_{L^p}^p &= -p\int_{\mathbb{R}^n}\left|\varphi(x)\big(A_R(t,x) - \frac{M}{2}\big)\right|^{p-2}\Big(\varphi(x)\big(A_R(t,x) - \frac{M}{2}\big)\Big) \\ &\quad\times \mathcal{L}\Big(\varphi(x)\big(A_R(t,x) - \frac{M}{2}\big)\Big)dx \\ &+ p\int_{\mathbb{R}^n}\left|\varphi(x)\big(A_R(t,x) - \frac{M}{2}\big)\right|^{p-2}\Big(\varphi(x)\big(A_R(t,x) - \frac{M}{2}\big)\Big) \\ &\quad\times \left([\mathcal{L},\varphi]A_R(t,x) - \frac{M}{2}\mathcal{L}\varphi(x)\right)dx. \end{aligned} \tag{20}$$

Remark that the integral (20) is positive by (13) so one has

$$\begin{aligned} \partial_t\left\|\varphi(\cdot)(A_R(t,\cdot) - \frac{M}{2})\right\|_{L^p}^p &\leq p\int_{\mathbb{R}^n}\left|\varphi(x)\big(A_R(t,x) - \frac{M}{2}\big)\right|^{p-2}\Big(\varphi(x)\big(A_R(t,x) - \frac{M}{2}\big)\Big) \\ &\quad\times \left([\mathcal{L},\varphi]A_R(t,x) - \frac{M}{2}\mathcal{L}\varphi(x)\right)dx. \end{aligned}$$

Using Hölder's inequality and integrating in time the previous expression we have

$$\begin{aligned} \left\|\varphi(\cdot)\left(A_R(t,\cdot) - \frac{M}{2}\right)\right\|_{L^p}^p &\leq \left\|\varphi(\cdot)\big(A_R(0,\cdot) - \frac{M}{2}\big)\right\|_{L^p}^p \\ &+ p2^{1-1/p}\int_0^t\left(\|[\mathcal{L},\varphi]A_R(s,\cdot)\|_{L^p} + \left\|\frac{M}{2}\mathcal{L}\varphi\right\|_{L^p}\right)\left\|\varphi(\cdot)\left(A_R(s,\cdot) - \frac{M}{2}\right)\right\|_{L^p}^{p-1}ds. \end{aligned}$$

The first term of the right side is null since on the support of φ we have $A_R(0,x)=\frac{M}{2}$. Use now Young's inequality and Gronwall's lemma to derive:

$$\begin{aligned}\left\|\varphi(\cdot)\left(A_R(t,\cdot)-\frac{M}{2}\right)\right\|_{L^p}^p &\leq (p-1)2^{1-1/p}\int_0^t\left\|\varphi(\cdot)\left(A_R(s,\cdot)-\frac{M}{2}\right)\right\|_{L^p}^p ds\\ &\quad +2^{p+1/p}\int_0^t\|[\mathcal{L},\varphi]A_R(s,\cdot)\|_{L^p}^p\,ds+2^{-1/p}M^pt\|\mathcal{L}\varphi\|_{L^p}^p\\ &\leq \exp((p-1)2^{1-1/p}t)2^{p+1/p}\left\{\int_0^t\|[\mathcal{L},\varphi]A_R(s,\cdot)\|_{L^p}^p\,ds+2^{-1/p}M^pt\|\mathcal{L}\varphi\|_{L^p}^p\right\}. \quad (21)\end{aligned}$$

For the term $\|[\mathcal{L},\varphi]A_R(s,\cdot)\|_{L^p}$ we will need the following lemma (see the proof in the Appendix A):

Lemma 3.1 *For $1\leq p\leq+\infty$ we have the following inequality*

- *if $0<\delta<\alpha<1$: $\|[\mathcal{L},\varphi]A_R(s,\cdot)\|_{L^p}\leq C(R^{-\alpha}+R^{-\delta})\|A_{0,R}\|_{L^p}$.*
- *if $0<\delta<1$ and $\alpha=1$: $\|[\mathcal{L},\varphi]A_R(s,\cdot)\|_{L^p}\leq C(R^{-1}+R^{-\delta})\|A_{0,R}\|_{L^p}$.*
- *if $1<\delta<\alpha<2$: $\|[\mathcal{L},\varphi]A_R(s,\cdot)\|_{L^p}\leq C\big(\|A_{0,R}\|_{L^\infty}R^{-\alpha+n}+\|A_{0,R}\|_{L^1}R^{-1}\big)^{\frac{1}{p}}\big((R^{-\alpha}+R^{-1})\|A_{0,R}\|_{L^\infty}\big)^{1-\frac{1}{p}}$.*

Now, getting back to the last term of (21) we have by the definition of φ and the properties of the operator $\mathcal{L}$ the estimate:

$$\|\mathcal{L}\varphi\|_{L^p}\leq CR^{\frac{n}{p}}(R^{-\alpha}+R^{-\delta}).$$

We thus have the following inequalities for $0<\delta<\alpha<1$:

$$\left\|\varphi(\cdot)\left(A_R(t,\cdot)-\frac{M}{2}\right)\right\|_{L^p}^p\leq C\Big[(R^{-\alpha p}+R^{-\delta p})\|A_{0,R}\|_{L^p}^p+M^p(R^{n-\alpha p}+R^{n-\delta p})\Big],\ C:=C(p,t),$$

or, if $1<\delta<\alpha<2$:

$$\begin{aligned}\left\|\varphi(\cdot)\left(A_R(t,\cdot)-\frac{M}{2}\right)\right\|_{L^p}^p &\leq C\big[\big(\|A_{0,R}\|_{L^\infty}R^{-\alpha+n}+\|A_{0,R}\|_{L^1}R^{-1}\big)\big((R^{-\alpha}+R^{-1})\|A_{0,R}\|_{L^\infty}\big)^{p-1}\\ &\quad +M^p(R^{n-p\alpha}+R^{n-p\delta})\big].\end{aligned}$$

Observe that we have at our disposal the estimates (17), so we can write

$$\left\|\varphi(\cdot)\left(A_R(t,\cdot)-\frac{M}{2}\right)\right\|_{L^p}^p\leq C\Big[(R^{-\alpha p}+R^{-\delta p})\left(\|\theta_0\|_{L^p}^p+M^pR^n\right)+M^p[R^{n-\alpha p}+R^{n-\delta p}]\Big]\quad\text{if } 0<\delta<\alpha<1,$$

and if $1<\delta<\alpha<2$

$$\left\|\varphi(\cdot)\left(A_R(t,\cdot)-\frac{M}{2}\right)\right\|_{L^p}^p\leq C\left[\big(MR^{-\alpha+n}+(\|\theta_0\|_{L^1}+MR^n)R^{-1}\big)\big((R^{-\alpha}+R^{-1})M\big)^{p-1}+M^p(R^{n-\alpha p}+R^{n-\delta p})\right].$$

Then, using again the definition of φ we have that the left-hand side above is greater than $\displaystyle\int_{B(0,\rho)}\left|A_R(t,\cdot)-\frac{M}{2}\right|^p dx$. Now, if $R\longrightarrow+\infty$ and since $p>\max(n,\frac{n}{\delta})$, this latter quantity is null in any case and we have $A_R(t,x)=\frac{M}{2}$ over $B(0,\rho)$.

Hence, by construction we have $\theta(t,x)=A_R(t,x)+B_R(t,x)$ where θ is a solution of (18) with initial data $\theta_0=A_{0,R}+B_{0,R}$, but, since over $B(0,\rho)$ we have $A_R(t,x)=\frac{M}{2}$ and $\|B_R(t,\cdot)\|_{L^\infty}\leq\frac{M}{2}$, one finally has the desired estimate $0\leq\theta(t,x)\leq M$. ■

3.2 Proof of Theorem 1 for an L^∞ initial data

The proof given before for the positivity principle allows us to obtain the existence of solutions for the fractional diffusion transport equation (1) when the initial data θ_0 belongs to the space $L^\infty(\mathbb{R}^n)$. This extension is crucial for our duality method to work in order to establish Theorem 2, see next Section.

Let us fix $\theta_0^R = \theta_0 \mathbb{1}_{B(0,R)}$ with $R > 0$ so we have $\theta_0^R \in L^p(\mathbb{R}^n)$ for all $1 \leq p \leq +\infty$. Following Section 2, there is a unique solution θ^R for the problem

$$\begin{cases} \partial_t \theta^R + \nabla \cdot (v\theta^R) + \mathcal{L}\theta^R = 0, \\ \theta^R(0,x) = \theta_0^R(x), \\ \nabla \cdot (v) = 0 \quad \text{and } v \in \left(L^\infty([0,T]; M^{q,a}(\mathbb{R}^n))\right)^n, \end{cases}$$

such that $\theta^R \in L^\infty([0,T]; L^p(\mathbb{R}^n))$. By the maximum principle we have $\|\theta^R(t,\cdot)\|_{L^p} \leq \|\theta_0^R\|_{L^p} \leq v_n \|\theta_0\|_{L^\infty} R^{\frac{n}{p}}$ for $1 < p < +\infty$. Taking the limit $p \longrightarrow +\infty$ and making $R \longrightarrow +\infty$ we finally get

$$\|\theta(t,\cdot)\|_{L^\infty} \leq C\|\theta_0\|_{L^\infty}.$$

This shows that for an initial data $\theta_0 \in L^\infty(\mathbb{R}^n)$ there exists an associated solution $\theta \in L^\infty([0,T]; L^\infty(\mathbb{R}^n))$.

3.3 Maximum Principle for Weak Solutions

From the previous paragraphs, the end of the proof of Theorem 1 for $p \in [2, +\infty[$ and Proposition 2.2, we eventually derive the following theorem.

Theorem 7 (Maximum Principle) *Let $\theta_0 \in L^p(\mathbb{R}^n)$ with $2 \leq p \leq +\infty$ then the weak solution of equation (1) satisfies the following maximum principle for all $t \in [0,T]$: $\|\theta(t,\cdot)\|_{L^p} \leq \|\theta_0\|_{L^p}$.*

4 Hölder Regularity

We will now study Hölder regularity by duality using Hardy spaces. These spaces have several equivalent characterizations (see [5], [10] and [21] for a detailed treatment). In this paper we are interested mainly in the molecular approach that defines *local* Hardy spaces.

Definition 4.1 (Local Hardy spaces h^σ) *Let $0 < \sigma < 1$. The local Hardy space $h^\sigma(\mathbb{R}^n)$ is the set of distributions f that admits the following molecular decomposition:*

$$f = \sum_{j \in \mathbb{N}} \lambda_j \psi_j, \tag{22}$$

where $(\lambda_j)_{j\in\mathbb{N}}$ is a sequence of complex numbers such that $\sum_{j\in\mathbb{N}} |\lambda_j|^\sigma < +\infty$ and $(\psi_j)_{j\in\mathbb{N}}$ is a family of r-molecules in the sense of Definition 4.2 below. The h^σ-norm[2] is then fixed by the formula

$$\|f\|_{h^\sigma} = \inf\left\{ \left(\sum_{j\in\mathbb{N}} |\lambda_j|^\sigma \right)^{1/\sigma} : \ f = \sum_{j\in\mathbb{N}} \lambda_j \psi_j \right\},$$

where the infimum runs over all possible decompositions (22).

Local Hardy spaces have many remarkable properties and we will only stress here, before passing to duality results concerning h^σ spaces, the fact that Schwartz class $\mathcal{S}(\mathbb{R}^n)$ is dense in $h^\sigma(\mathbb{R}^n)$, this fact is of course very useful for approximation procedures.

Now, let us take a closer look at the dual space of the local Hardy spaces. In [10], D. Goldberg proved the following important theorem:

Theorem 8 (Hardy-Hölder duality) *Let $\frac{n}{n+1} < \sigma < 1$ and fix $\gamma = n(\frac{1}{\sigma} - 1)$. Then the dual of local Hardy space $h^\sigma(\mathbb{R}^n)$ is the Hölder space $\mathcal{C}^\gamma(\mathbb{R}^n)$ fixed by the norm*

$$\|f\|_{\mathcal{C}^\gamma} = \|f\|_{L^\infty} + \sup_{x\neq y} \frac{|f(x) - f(y)|}{|x-y|^\gamma}.$$

This result allows us to study the Hölder regularity of functions in terms of Hardy spaces and it will be applied to the solutions of the equation (1).

[2] it is not actually a *norm* since $0 < \sigma < 1$. More details can be found in [10] and [21].

Remark 4.1 *Since* $\frac{n}{n+1} < \sigma < 1$, *we have* $\sum_{j\in\mathbb{N}} |\lambda_j| \leq \left(\sum_{j\in\mathbb{N}} |\lambda_j|^\sigma\right)^{1/\sigma}$ *thus for testing Hölder continuity of a function* f *it is enough to study the quantities* $|\langle f, \psi_j\rangle|$ *where* ψ_j *is an* r*-molecule.*

Since we are going to work with local Hardy spaces, we will introduce a size treshold in order to distinguish *small* molecules from *big* ones in the following way:

Definition 4.2 (r-molecules) *Set* $\frac{n}{n+1} < \sigma < 1$, *define* $\gamma = n(\frac{1}{\sigma} - 1)$ *and fix a real number* ω *such that* $0 < \gamma < \omega < 1$. *An integrable function* ψ *is an* r*-molecule if we have*

- *Small molecules* $(0 < r < 1)$*:*

$$\int_{\mathbb{R}^n} |\psi(x)||x - x_0|^\omega dx \leq (\zeta^\beta r)^{\omega-\gamma}, \text{ for } x_0 \in \mathbb{R}^n \qquad \text{(concentration condition)}, \tag{23}$$

$$\|\psi\|_{L^\infty} \leq \frac{1}{(\zeta^\beta r)^{n+\gamma}} \qquad \text{(height condition)}, \tag{24}$$

$$\int_{\mathbb{R}^n} \psi(x)dx = 0 \qquad \text{(moment condition)}. \tag{25}$$

  In the above conditions ζ *and* β *denote positive constants that depend on* γ, ω, α *and other parameters to be specified later on.*

- *Big molecules* $(1 \leq r < +\infty)$*:*

  In this case we only require conditions (23) and (24) for the r*-molecule* ψ *while the moment condition (25) is dropped.*

Remark 4.2

1) Note that the point $x_0 \in \mathbb{R}^n$ *can be considered as the "center" of the molecule.*

2) Conditions (23) and (24) imply the estimate $\|\psi\|_{L^1} \leq C\,(\zeta^\beta r)^{-\gamma}$ *thus every* r*-molecule belongs to* $L^p(\mathbb{R}^n)$ *with* $1 < p < +\infty$. *In particular we have for any small molecule and for* $1 < p < +\infty$

$$\|\psi\|_{L^p} \leq C(\zeta^\beta r)^{-n+\frac{n}{p}-\gamma}. \tag{26}$$

3) In this definition, we find more convenient to show explicitely the dependence on the Hölder parameter γ *instead of* σ.

For a more concise definition of molecules see [21], Chapter III, 5.7. See also [23], Chapter XIV, 6.6 or [15] for a similar characterization.

The main interest of using molecules relies in the possibility of *transfering* the regularity problem to the evolution of such molecules. This idea is borrowed from [15].

Proposition 4.1 (Transfer property) *Let* $t \in [0, T]$ *be fixed and* ψ *be a solution of the following backward problem for* $s \in [0, t]$*:*

$$\begin{cases} \partial_s\psi(s,x) = & -\nabla\cdot[v(t-s,x)\psi(s,x)] - \mathcal{L}\psi(s,x), \\ \psi(0,x) = & \theta_0(x) \in L^1 \cap L^\infty(\mathbb{R}^n), \\ div(v) = & 0 \quad \text{and} \quad v \in \left(L^\infty([0,t]; M^{q,a}(\mathbb{R}^n))\right)^n. \end{cases} \tag{27}$$

If $\theta(t,.)$ *is a solution of (1) at time* t *with* $\theta_0 \in L^\infty(\mathbb{R}^n)$ *then we have the identity*

$$\int_{\mathbb{R}^n} \theta(t,x)\psi(0,x)dx = \int_{\mathbb{R}^n} \theta(0,x)\psi(t,x)dx.$$

Proof. We first consider the expression

$$\partial_s \int_{\mathbb{R}^n} \theta(t-s,x)\psi(s,x)dx = \int_{\mathbb{R}^n} -\partial_s\theta(t-s,x)\psi(s,x) + \partial_s\psi(s,x)\theta(t-s,x)dx.$$

Using equations (1) and (27) we obtain

$$\partial_s \int_{\mathbb{R}^n} \theta(t-s,x)\psi(s,x)dx = \int_{\mathbb{R}^n} \Big\{ -\nabla\cdot[(v(t-s,x)\theta(t-s,x)]\,\psi(s,x) + \mathcal{L}\theta(t-s,x)\psi(s,x)$$

$$- \nabla\cdot[(v(t-s,x)\psi(s,x))]\,\theta(t-s,x) - \mathcal{L}\psi(s,x)\theta(t-s,x)\Big\}dx.$$

Now, using the fact that v is divergence free and the symmetry of the operator $\mathcal{L}$ we have that the expression above is equal to zero, so the quantity

$$\int_{\mathbb{R}^n} \theta(t-s,x)\psi(s,x)dx,$$

remains constant in time. We only have to set $s=0$ and $s=t$ to conclude. ■

This proposition says, that in order to control $\langle\theta(t,\cdot),\psi_0\rangle$, it is enough (and much simpler) to study the bracket $\langle\theta_0,\psi(t,\cdot)\rangle$.

Proof of Theorem 2. Once we have the transfer property proven above, the proof of the Theorem 2 is quite direct and it reduces to an L^1 estimate for molecules. Indeed, assume that for *all* molecular initial data ψ_0 we have an L^1 control for $\psi(t,\cdot)$ a solution of (27), then Theorem 2 follows easily: applying Proposition 4.1 with the fact that $\theta_0 \in L^\infty(\mathbb{R}^n)$ we have

$$|\langle\theta(t,\cdot),\psi_0\rangle| = \left|\int_{\mathbb{R}^n}\theta(t,x)\psi_0(x)dx\right| = \left|\int_{\mathbb{R}^n}\theta(0,x)\psi(t,x)dx\right| \le \|\theta_0\|_{L^\infty}\|\psi(t,\cdot)\|_{L^1} < +\infty. \tag{28}$$

From this, we obtain that $\theta(t,\cdot)$ belongs to the Hölder space $\mathcal{C}^\gamma(\mathbb{R}^n)$.

Now we need to study the control of the L^1 norm of $\psi(t,\cdot)$ and we divide our proof in two steps following the molecule's size. For the initial big molecules, *i.e.* if $r \ge 1$, the needed control is straightforward: apply the maximum principle and the Remark 4.2-2) above to obtain

$$\|\theta_0\|_{L^\infty}\|\psi(t,\cdot)\|_{L^1} \le \|\theta_0\|_{L^\infty}\|\psi_0\|_{L^1} \le C\frac{1}{r^\gamma}\|\theta_0\|_{L^\infty},$$

but, since $r \ge 1$, we have that $|\langle\theta(t,\cdot),\psi_0\rangle| < +\infty$ for all *big* molecules.

In order to finish the proof of this theorem, it only remains to treat the L^1 control for *small* molecules. This is the most complex part of the proof and it is treated in the following theorem:

Theorem 9 *For all small r-molecules (i.e. $0 < r < 1$), there exists a time $T_0 > 0$ such that we have the following control of the L^1-norm.*

$$\|\psi(t,\cdot)\|_{L^1} \le CT_0^{-\gamma} \qquad (T_0 < t < T),$$

with $0 < \gamma < \min(\delta, 1)$.

Accepting for a while this result, we have then a good control over the quantity $\|\psi(t,\cdot)\|_{L^1}$ for all $0 < r < 1$ and getting back to (28) we obtain that $|\langle\theta(t,\cdot),\psi_0\rangle|$ is always bounded for $T_0 < t < T$ and for any molecule ψ_0: we have proven Theorem 2 by a duality argument. ■

Let us now briefly explain the main steps to prove Theorem 9. We need to construct a suitable control in time for the L^1-norm of the solutions $\psi(t,\cdot)$ of the backward problem (27) where the inital data ψ_0 is a *small* r-molecule. This will be achieved by iteration in two different steps:

- The first step explains the molecules' deformation after a very small time $s_0 > 0$, which is related to the size r by the bounds $0 < s_0 \le \epsilon r$ with ϵ a small constant. This will be done in Section 4.1.
- In order to obtain a control of the L^1 norm for larger times we need to perform a second step which takes as a starting point the results of the first step and gives us the deformation for another small time s_1, which is also related to the original size r. This part is treated in Section 4.2.

To conclude it is enough to iterate the second step as many times as necessary to get rid of the dependence of the times $s_0, s_1, ...$ from the molecule's size. This way we obtain the L^1 control needed for all time $T_0 < t < T$.

4.1 Small time molecule's evolution: First step

The following theorem shows how the molecular properties are deformed with the evolution for a small time s_0.

Theorem 10 *Set σ, γ and ω three real numbers such that $\frac{n}{n+1} < \sigma < 1$, $\gamma = n(\frac{1}{\sigma} - 1)$. Let $\psi(s_0, x)$ be a solution of the problem*

$$\left\{ \begin{array}{rcl} \partial_s \psi(s,x) = & -\nabla \cdot (v\,\psi)(s,x) - \mathcal{L}\psi(s,x), \ s \in [0,T], \\ \psi(0,x) = & \psi_0(x), \\ div(v) = & 0 \quad and \ \ v \in \left(L^\infty([0,T]; M^{q,a}(\mathbb{R}^n))\right)^n \quad with \ \sup\limits_{s\in[0,T]} \|v(s,\cdot)\|_{M^{q,a}} \leq \mu. \end{array} \right. \tag{29}$$

There exist positive constants K and ϵ small enough such that if ψ_0 is a small r-molecule in the sense of Definition 4.2 for the local Hardy space $h^\sigma(\mathbb{R}^n)$, then for all $0 < s_0 \leq \epsilon r^\alpha$, we have the following estimates:

$$\int_{\mathbb{R}^n} |\psi(s_0,x)||x - x(s_0)|^\omega dx \leq ((\zeta^\beta r)^\alpha + K s_0)^{\frac{\omega-\gamma}{\alpha}}, \tag{30}$$

$$\|\psi(s_0,\cdot)\|_{L^\infty} \leq \frac{1}{\left((\zeta^\beta r)^\alpha + K s_0\right)^{\frac{n+\gamma}{\alpha}}}, \tag{31}$$

$$\|\psi(s_0,\cdot)\|_{L^1} \leq \frac{2 v_n^{\frac{\omega}{n+\omega}}}{\left((\zeta^\beta r)^\alpha + K s_0\right)^{\frac{\gamma}{\alpha}}}, \tag{32}$$

where v_n denotes the volume of the n-dimensional unit ball.

The new molecule's center $x(s_0)$ used in formula (30) is given by the evolution of the differential system

$$\left\{ \begin{array}{rcl} x'(s) = & \overline{v}_{B(x(s),\zeta r)} = \frac{1}{|B(x(s),\zeta r)|} \displaystyle\int_{B(x(s),\zeta r)} v(s,y)dy, \quad s \in [0,s_0], \\ x(0) = & x_0. \end{array} \right. \tag{33}$$

In the previous controls and in the dynamics for the evolution of the center the parameters $\zeta := \zeta(\alpha,\omega,\gamma,\mu)$, $\beta := \beta(\alpha,\omega)$, to be specified later on, are the same as in Definition 4.2.

Remark 4.3

1) *The definition of the point $x(s_0)$ given by (33) reflects the molecule's center transport using velocity v.*
2) *Remark that it is enough to treat the case $0 < ((\zeta^\beta r)^\alpha + K s_0) < 1$ since s_0 is small: otherwise the L^1 control will be trivial for time s_0 and beyond: we only need to apply the maximum principle.*
3) *The parameter ζ was introduced in the definition of the molecules in order to absorb the Morrey-Campanato norm of the velocity field which is denoted by μ. This fact will appear clearly later on with the formula (36). Now since ζ can be a rather large quantity, in order to obtain $((\zeta^\beta r)^\alpha + K s_0) < 1$ we need r to be very small and this fact is compatible with our interest in small molecules.*

Proof of the Theorem 10. We will follow the next scheme: first we prove the small Concentration condition (30) and then we prove the Height condition (31). Once we have these two conditions, the L^1 estimate (32) will follow easily.

1) Small time Concentration condition

Let us write for $s \in [0,s_0]$, $\Omega_s(x) = |x - x(s)|^\omega$ and $\psi(x) = \psi_+(x) - \psi_-(x)$ where the functions $\psi_\pm(x) \geq 0$ have disjoint support. We will denote $\psi_\pm(s_0,x)$ two solutions of (29) at time s_0 with $\psi_\pm(0,x) = \psi_\pm(x)$. At this point, we use the positivity principle, thus by linearity we have

$$|\psi(s_0,x)| = |\psi_+(s_0,x) - \psi_-(s_0,x)| \leq \psi_+(s_0,x) + \psi_-(s_0,x),$$

and we can write

$$\int_{\mathbb{R}^n} |\psi(s_0,x)|\Omega_{s_0}(x)dx \leq \int_{\mathbb{R}^n} \psi_+(s_0,x)\Omega_{s_0}(x)dx + \int_{\mathbb{R}^n} \psi_-(s_0,x)\Omega_{s_0}(x)dx,$$

so we only have to treat one of the integrals on the right hand side above. We have for all $s \in [0, s_0]$:

$$\begin{aligned} I_s &= \left| \partial_s \int_{\mathbb{R}^n} \Omega_s(x)\psi_+(s,x)dx \right| \\ &= \left| \int_{\mathbb{R}^n} \partial_s\Omega_s(x)\psi_+(s,x) + \Omega_s(x)\left[-\nabla\cdot(v\,\psi_+(s,x)) - \mathcal{L}\psi_+(s,x)\right]dx \right| \\ &= \left| \int_{\mathbb{R}^n} -\nabla\Omega_s(x)\cdot x'(s)\psi_+(s,x) + \Omega_s(x)\left[-\nabla\cdot(v\,\psi_+(s,x)) - \mathcal{L}\psi_+(s,x)\right]dx \right|. \end{aligned}$$

Using the fact that v is divergence free, we obtain

$$I_s = \left| \int_{\mathbb{R}^n} \nabla\Omega_s(x)\cdot(v - x'(s))\psi_+(s,x) - \Omega_s(x)\mathcal{L}\psi_+(s,x)dx \right|.$$

Since the operator $\mathcal{L}$ is symmetric and using the definition of $x'(s)$ given in (33) we have

$$I_s \le c\underbrace{\int_{\mathbb{R}^n} |x - x(s)|^{\omega-1}|v - \overline{v}_{B_{s,\zeta r}}||\psi_+(s,x)|dx}_{I_{s,1}} + c\underbrace{\int_{\mathbb{R}^n} \left|\mathcal{L}\Omega_s(x)\right||\psi_+(s,x)|dx}_{I_{s,2}}, \tag{34}$$

denoting $B_{s,\zeta r} := B(x(s), \zeta r)$. We will study separately each of the integrals $I_{s,1}$ and $I_{s,2}$ by two lemmas that will be proven in Appendix B in a more general way.

Lemma 4.1 *For the integral $I_{s,1}$ above we have the estimate*

$$I_{s,1} \le C\|v(s,\cdot)\|_{M^{q,a}}(\zeta r)^{\omega-1}\left((\zeta r)^{\frac{n}{p}+\frac{a}{q}}\|\psi(s,\cdot)\|_{L^z} + (\zeta r)^{\frac{a}{q}}\|\psi(s,\cdot)\|_{L^{q'}} + (\zeta r)^{\frac{a-n}{q}+\frac{n}{\bar q}}\|\psi(s,\cdot)\|_{L^{\bar p}}\right),$$

where $\frac{1}{p}+\frac{1}{q}+\frac{1}{z} = 1$, $1 < p < \frac{n}{1-\omega}$, $\frac{1}{q}+\frac{1}{q'} = 1$, $\frac{1}{\bar p}+\frac{1}{\bar q} = 1$ and we assume that $\omega - 1 + \frac{a}{q} < 0$, $\omega - \alpha + \frac{n}{\bar q} < 0$.

Lemma 4.2 *For the integral $I_{s,2}$ in (34) we have the inequality for $0 < \delta < \alpha < 2$:*

$$I_{s,2} \le C(\zeta r)^{\omega-\alpha}\left((\zeta r)^{\frac{n}{2}}\|\psi(s,\cdot)\|_{L^2} + (\zeta r)^{\frac{n}{\bar q}}\|\psi(s,\cdot)\|_{L^{\bar p}}\right),$$

for $\frac{1}{\bar p}+\frac{1}{\bar q} = 1$ and $\omega - \delta + \frac{n}{\bar q} < 0$.

Using these lemmas and getting back to estimate (34) we have

$$\begin{aligned} I_s \le C\|v(s,\cdot)\|_{M^{q,a}}(\zeta r)^{\omega-1}&\left((\zeta r)^{\frac{n}{p}+\frac{a}{q}}\|\psi(s,\cdot)\|_{L^z} + (\zeta r)^{\frac{a}{q}}\|\psi(s,\cdot)\|_{L^{q'}} + (\zeta r)^{\frac{a-n}{q}+\frac{n}{\bar q}}\|\psi(s,\cdot)\|_{L^{\bar p}}\right) \\ &+C(\zeta r)^{\omega-\alpha}\left((\zeta r)^{n/2}\|\psi(s,\cdot)\|_{L^2} + (\zeta r)^{\frac{n}{\bar q}}\|\psi(s,\cdot)\|_{L^{\bar p}}\right). \end{aligned}$$

Then, since $\sup_{0<s<T}\|v(s,\cdot)\|_{M^{q,a}} \le \mu$ and by the maximum principle we write

$$I_s \le C\mu\,(\zeta r)^{\omega-1}\left((\zeta r)^{\frac{n}{p}+\frac{a}{q}}\|\psi_0\|_{L^z} + (\zeta r)^{\frac{a}{q}}\|\psi_0\|_{L^{q'}} + (\zeta r)^{\frac{a-n}{q}+\frac{n}{\bar q}}\|\psi_0\|_{L^{\bar p}}\right) + C(\zeta r)^{\omega-\alpha}\left((\zeta r)^{\frac{n}{2}}\|\psi_0\|_{L^2} + (\zeta r)^{\frac{n}{\bar q}}\|\psi_0\|_{L^{\bar p}}\right).$$

At this point we use the fact that ψ_0 satisfies the molecular condition (24) and the inequality (26):

$$\begin{aligned} I_s \le C\mu\,(\zeta r)^{\omega-1}&\left((\zeta r)^{\frac{n}{p}+\frac{a}{q}} \times (\zeta^\beta r)^{-n+\frac{n}{z}-\gamma} + (\zeta r)^{\frac{a}{q}} \times (\zeta^\beta r)^{-n+\frac{n}{q'}-\gamma} + (\zeta r)^{\frac{a-n}{q}+\frac{n}{\bar q}} \times (\zeta^\beta r)^{-n+\frac{n}{\bar p}-\gamma}\right) \\ &+C(\zeta r)^{\omega-\alpha}\left((\zeta r)^{\frac{n}{2}} \times (\zeta^\beta r)^{-n+\frac{n}{2}-\gamma} + (\zeta r)^{\frac{n}{\bar q}} \times (\zeta^\beta r)^{-n+\frac{n}{\bar p}-\gamma}\right). \end{aligned}$$

Now, since $\frac{1}{p}+\frac{1}{q}+\frac{1}{z} = 1$, $\frac{1}{q}+\frac{1}{q'} = 1$, $\frac{1}{\bar p}+\frac{1}{\bar q} = 1$ and since the parameters that define the Morrey-Campanato space $M^{q,a}$ are related to the regularization properties of the Lévy-type operator $\mathcal{L}$ by the relationship $\frac{a-n}{q} = 1 - \alpha$ we obtain

$$\begin{aligned} I_s &\le C(\mu+1)\,r^{\omega-\gamma-\alpha} \\ &\times\left(\zeta^{\omega-1+\frac{n}{p}+\frac{a}{q}+\beta(-n+\frac{n}{z}-\gamma)} + \zeta^{\omega-1+\frac{a}{q}+\beta(-n+\frac{n}{q'}-\gamma)} + \zeta^{\omega-\alpha-\beta\frac{n}{\bar q}-\beta\gamma} + \zeta^{\omega-\alpha+\frac{n}{2}(1-\beta)-\beta\gamma}\right). \end{aligned} \tag{35}$$

In order to compensate the constants we want to take ζ large enough and choose $\beta > 1$ so that all the exponents in (35) remain negative. Let us rewrite from the above conjugacy relations:

$$I_s \leq C(\mu+1)\, r^{\omega-\gamma-\alpha} \times \left(\zeta^{\omega-\alpha-\beta\gamma+(\frac{n}{p}+\frac{n}{q})(1-\beta)} + \zeta^{\omega-\alpha-\beta\gamma+\frac{n}{q}(1-\beta)} + \zeta^{\omega-\alpha-\beta\frac{n}{q}-\beta\gamma} + \zeta^{\omega-\alpha-\beta\gamma+\frac{n}{2}(1-\beta)}\right).$$

Since we have $1 < p < \frac{n}{1-\omega}$ and $\omega - \alpha + \frac{n}{q} < 0,\ \omega - \alpha + \frac{n}{\bar{q}} < 0$, we can see that the largest constant above is $\zeta^{\omega-\alpha-\beta\gamma+\frac{n}{2}(1-\beta)}$, thus denoting by $-\mathcal{M} = \omega - \alpha - \beta\gamma + \frac{n}{2}(1-\beta)$ we have

$$I_s \leq 4C(\mu+1)\, r^{\omega-\gamma-\alpha}\zeta^{-\mathcal{M}}.$$

Now, for any $\eta \in]0,1[$, taking

$$\zeta := \left(\frac{4C(\mu+1)}{\eta}\right)^{1/\mathcal{M}} > 1, \tag{36}$$

we will obtain the following inequality

$$\left|\partial_s \int_{\mathbb{R}^n} \Omega_s(x)\psi_+(s,x)dx\right| = I_s \leq \eta r^{\omega-\gamma-\alpha}.$$

This estimation, associated with the initial concentration condition (23), now gives:

$$\begin{aligned}\int_{\mathbb{R}^n} |x - x(s_0)|^\omega \psi_+(s_0,x)dx &\leq (\zeta^\beta r)^{\omega-\gamma} + \eta r^{\omega-\gamma-\alpha}s_0 \\ &\leq (\zeta^\beta r)^{\omega-\gamma}(1 + \eta\frac{s_0}{\zeta^{\beta(\omega-\gamma)}r^\alpha}).\end{aligned}$$

Now, since $0 \leq s_0 \leq \epsilon r^\alpha$ and that in all cases $1/\zeta^{\beta(\omega-\gamma)} \leq 1$, we can choose $\eta := \eta(\alpha,\omega,\gamma)$ small enough to have:

$$\begin{aligned}\int_{\mathbb{R}^n} |x - x(s_0)|^\omega \psi_+(s_0,x)dx &\leq (\zeta^\beta r)^{\omega-\gamma}(1 + 2\frac{\alpha}{\omega-\gamma}\eta\frac{s_0}{\zeta^{\beta(\omega-\gamma)}r^\alpha})^{\frac{\omega-\gamma}{\alpha}} \\ &\leq ((\zeta^\beta r)^\alpha + 2\frac{\alpha}{\omega-\gamma}\eta\frac{s_0}{\zeta^{\beta(\omega-\gamma-\alpha)}})^{\frac{\omega-\gamma}{\alpha}}.\end{aligned} \tag{37}$$

At this point we want to make the quantity $2\frac{\alpha}{\omega-\gamma}\eta\frac{1}{\zeta^{\beta(\omega-\gamma-\alpha)}}$ very small. Using formula (36) that defines ζ and recalling that $0 < \gamma < \omega < \alpha$ we obtain that

$$\eta\frac{1}{\zeta^{\beta(\omega-\gamma-\alpha)}} = \eta\left(\frac{4C(\mu+1)}{\eta}\right)^{\frac{\beta(\alpha+\gamma-\omega)}{\mathcal{M}}} = \eta^{1-\frac{\beta(\alpha+\gamma-\omega)}{\mathcal{M}}}\big(4C(\mu+1)\big)^{\frac{\beta(\alpha+\gamma-\omega)}{\mathcal{M}}}.$$

Since $\beta > 1$, the exponent of η in the previous control is positive and, since we can choose η very small, we can absorb the Morrey-Campanato norm represented here by the quantity μ. Thus the inequality (37) is compatible with the estimate (30) for every $K \geq 2\frac{\alpha}{\omega-\gamma}\eta\frac{1}{\zeta^{\beta(\omega-\gamma-\alpha)}}$. Observe anyhow that at this stage we can still choose η arbitrarily small, meaning that the choice of the averaging of the drift can indeed be chosen in order to have a really concentrated molecule at time s_0. Of course, for the machinery to work, we would like to take the largest possible K, to minimize the number of iterations to obtain a "big molecule" for which we can conclude directly with the maximum principle. The choice of this constant is actually guided by the evolution of the L^∞ norm.

2) Small time Height condition

We treat now the Height condition (31) and for this we will give a sligthly different proof of the maximum principle of A. Córdoba & D. Córdoba. Indeed, the following proof only relies on the Concentration condition.

Assume that molecules we are working with are smooth enough and in particular continuous. Following an idea of [8] (section 4 p.522-523) (see also [12] p. 346), we will denote for $s \in [0,s_0]$ by $\overline{x}_s$ the point of $\mathbb{R}^n$ such that $\psi(s,\overline{x}_s) = \|\psi(s,\cdot)\|_{L^\infty}$. Thus we can write, by the properties (3)-(4) of the function π :

$$\frac{d}{ds}\|\psi(s,\cdot)\|_{L^\infty} \leq -\int_{\mathbb{R}^n}[\psi(s,\overline{x}_s) - \psi(s,\overline{x}_s - y)]\pi(y)dy \leq -\overline{c}_1\int_{\{|\overline{x}_s-y|<1\}}\frac{\psi(s,\overline{x}_s)-\psi(s,y)}{|\overline{x}_s-y|^{n+\alpha}}dy \leq 0. \tag{38}$$

To establish the control of the theorem we aim at proving that:

$$\frac{d}{ds}\|\psi(s,\cdot)\|_{L^\infty} \leq -K\left(\frac{n+\gamma}{\alpha}\right)((\zeta^\beta r)^\alpha + Ks)^{-\frac{(\omega-\gamma)}{n+\omega}}\|\psi(s,\cdot)\|_{L^\infty}^{1+\frac{\alpha}{n+\omega}}. \tag{39}$$

Indeed, integrating (39) yields:

$$\begin{aligned}\int_0^{s_0} \frac{d}{ds}\left(\|\psi(s,\cdot)\|_{L^\infty}^{-\frac{\alpha}{n+\omega}}\right) ds &\geq \int_0^{s_0} \frac{d}{ds}\left([(\zeta^\beta r)^\alpha + Ks]^{\frac{n+\gamma}{n+\omega}}\right) ds \\ \|\psi(s_0,\cdot)\|_{L^\infty}^{-\frac{\alpha}{n+\omega}} &\geq [(\zeta^\beta r)^\alpha + Ks_0]^{\frac{n+\gamma}{n+\omega}} + \left(\|\psi(0,\cdot)\|_{L^\infty}^{-\frac{\alpha}{n+\omega}} - [(\zeta^\beta r)^\alpha]^{\frac{n+\gamma}{n+\omega}}\right) \\ &\geq [(\zeta^\beta r)^\alpha + Ks_0]^{\frac{n+\gamma}{n+\omega}},\end{aligned}$$

recalling the initial height condition $\|\psi(0,\cdot)\|_{L^\infty} \leq (\zeta^\beta r)^{-(n+\gamma)}$ for the last inequality, we therefore derive

$$\|\psi(s_0,\cdot)\|_{L^\infty} \leq ((\zeta^\beta r)^\alpha + Ks_0)^{-\frac{n+\gamma}{\alpha}},$$

which is the required control.

To establish the differential inequality (39) for $s \in [0, s_0]$, let us first consider a corona centered in $\bar{x}_s$ defined by $\mathcal{C}(R, 2R) = \{y \in \mathbb{R}^n : R \leq |\overline{x}_s - y| \leq 2R\}$, where the parameter $R > 0$ to be specified later on is such that $R \leq \frac{1}{2}$. Then:

$$\int_{\{|\overline{x}_s - y| < 1\}} \frac{\psi(s,\overline{x}_s) - \psi(s,y)}{|\overline{x}_s - y|^{n+\alpha}} dy \geq \int_{\mathcal{C}(R,2R)} \frac{\psi(s,\overline{x}_s) - \psi(s,y)}{|\overline{x}_s - y|^{n+\alpha}} dy.$$

Define the sets B_1 and B_2 by $B_1 = \{y \in \mathcal{C}(R,2R) : \psi(s,\overline{x}_s) - \psi(s,y) \geq \frac{1}{2}\psi(s,\overline{x}_s)\}$ and $B_2 = \{y \in \mathcal{C}(R,2R) : \psi(s,\overline{x}_s) - \psi(s,y) < \frac{1}{2}\psi(s,\overline{x}_s)\}$ such that $\mathcal{C}(R,2R) = B_1 \cup B_2$. We obtain the inequalities

$$\begin{aligned}\int_{\mathcal{C}(R,2R)} \frac{\psi(s,\overline{x}_s) - \psi(s,y)}{|\overline{x}_s - y|^{n+\alpha}} dy &\geq \int_{B_1} \frac{\psi(s,\overline{x}_s) - \psi(s,y)}{|\overline{x}_s - y|^{n+\alpha}} dy \geq \frac{\psi(s,\overline{x}_s)}{2(2R)^{n+\alpha}} |B_1| = \frac{\psi(s,\overline{x}_s)}{2(2R)^{n+\alpha}} \left(|\mathcal{C}(R,2R)| - |B_2|\right) \\ &\geq \frac{\psi(s,\overline{x}_s)}{2(2R)^{n+\alpha}} \Big(v_n(2^n - 1)R^n - |B_2|\Big), \qquad (40)\end{aligned}$$

where v_n denotes the volume of the n-dimensional unit ball.

We now want to establish that if R is *large enough* then $|B_2| \leq \frac{1}{2}|\mathcal{C}(R,2R)|$. This fact can be established through the concentration condition (which holds for $s \in [0, s_0]$). Let us assume that for all $x \in \mathcal{C}(R,2R)$, $|x - x(s)| \geq \Gamma R$, for a parameter $\Gamma > 0$ to be fixed later on and that $|B_2| > \frac{1}{2}\mathcal{C}(R,2R)$, then by the concentration condition:

$$\begin{aligned}((\zeta^\beta r)^\alpha + Ks)^{\frac{\omega-\gamma}{\alpha}} &\geq \int_{\mathbb{R}^n} \psi(s,x)|x - x(s)|^\omega dx \geq \int_{|B_2|} \psi(s,x)|x - x(s)|^\omega dx \\ &> \frac{\psi(s,\bar{x}_s)}{4} R^{n+\omega} \Gamma^\omega v_n (2^n - 1).\end{aligned}$$

Taking $R \geq \left\{ \frac{4((\zeta^\beta r)^\alpha + Ks)^{\frac{\omega-\gamma}{\alpha}}}{\Gamma^\omega \psi(s,\bar{x}_s) v_n (2^n - 1)} \right\}^{\frac{1}{n+\omega}}$ yields to a contradiction. Hence for such parameters R we get in (40):

$$\int_{\mathcal{C}(R,2R)} \frac{\psi(s,\overline{x}_s) - \psi(s,y)}{|\overline{x}_s - y|^{n+\alpha}} dy \geq \frac{\psi(s,\overline{x}_s)}{2(2R)^{n+\alpha}} \frac{v_n}{2}(2^n - 1)R^n \geq \frac{\psi(s,\overline{x}_s)}{2^{n+2+\alpha} R^\alpha} v_n (2^n - 1) \geq \frac{\psi(s,\overline{x}_s)}{2^{3+\alpha} R^\alpha} v_n.$$

Thus, we will obtain (39) provided that:

$$\frac{\psi(s,\overline{x}_s)}{2^{3+\alpha} R^\alpha} v_n \geq K \left(\frac{n+\gamma}{\alpha}\right) ((\zeta^\beta r)^\alpha + Ks)^{-\frac{(\omega-\gamma)}{n+\omega}} \|\psi(s,\cdot)\|_{L^\infty}^{1+\frac{\alpha}{n+\omega}}.$$

Hence, the two constraints needed to have on the one hand that $|B_2| \leq \frac{1}{2}|\mathcal{C}(R,2R)|$ and on the other hand that the differential inequality (39) is fulfilled can be summarized as follows:

$$\Phi_s \left\{ \frac{v_n \alpha}{2^{3+\alpha} K (n+\gamma)} \right\}^{\frac{1}{\alpha}} \geq R \geq \Phi_s \left\{ \frac{4}{\Gamma^\omega v_n (2^n - 1)} \right\}^{\frac{1}{n+\omega}}, \qquad (41)$$

denoting by $\Phi_s := \frac{((\zeta^\beta r)^\alpha + Ks)^{\frac{\omega-\gamma}{\alpha(n+\omega)}}}{\psi(s,\overline{x}_s)^{\frac{1}{n+\omega}}}$ the quantity that can be viewed as the characteristic radius at time s which has typical order the size of the current molecule. These bounds can be achieved for a fixed Γ provided K is sufficiently small (recall indeed that up to now K is a free parameter). Thus, choosing a suitable K in (41) yields inequality (39) provided that $\inf_{x \in \mathcal{C}(R,2R)} |x - x(s)| \geq \Gamma R$ which is for instance the case if $|\bar{x}_s - x(s)| \geq (\Gamma + 2)R$.

It thus remains to handle the case $|\bar{x}_s - x(s)| \leq (\Gamma+2)R$ for which we use a slightly different construction. Namely we consider the corona $\mathcal{C}((1+\Gamma)2R,(2+3\Gamma)R)$ which guarantees that for every $x \in \mathcal{C}((1+\Gamma)2R,(2+3\Gamma)R)$, since $|\bar{x}_s - x(s)| \leq (\Gamma+2)R$, the condition $|x-x(s)| \geq \Gamma R$ holds. Now, setting as above $B_2 = \{y \in \mathcal{C}(2(1+\Gamma)R,(3\Gamma+2)R) : \psi(s,\overline{x}_s) - \psi(s,y) < \frac{1}{2}\psi(s,\overline{x}_s)\}$, $B_1 := \mathcal{C}((1+\Gamma)2R,(2+3\Gamma)R)\backslash B_2$, we derive:

$$\int_{\mathcal{C}(2(\Gamma+1)R,(3\Gamma+2)R)} \frac{\psi(s,\overline{x}_s)-\psi(s,y)}{|\overline{x}_s - y|^{n+\alpha}}dy \geq \int_{B_1} \frac{\psi(s,\overline{x}_s)-\psi(s,y)}{|\overline{x}_s - y|^{n+\alpha}}dy \geq \frac{\psi(s,\overline{x}_s)}{2((3\Gamma+2)R)^{n+\alpha}}|B_1|$$
$$= \frac{\psi(s,\overline{x}_s)}{2((3\Gamma+2)R)^{n+\alpha}}\left(|\mathcal{C}(2(1+\Gamma)R,(3\Gamma+2)R)| - |B_2|\right) = \frac{\psi(s,\overline{x}_s)}{2((3\Gamma+2)R)^{n+\alpha}}\left(v_n R^n[(3\Gamma+2)^n - 2^n(\Gamma+1)^n] - |B_2|\right).$$

Now, assuming that $|B_2| > \frac{1}{2}|\mathcal{C}(2(\Gamma+1)R,(3\Gamma+2)R)|$ we still have from the concentration condition:

$$((\zeta^\beta r)^\alpha + Ks)^{\frac{\omega-\gamma}{\alpha}} \geq \int_{B_2} |x-x(s)|^\omega \psi(s,x)dx > \frac{\psi(s,\bar{x}_s)}{4}\Gamma^\omega R^{n+\omega} v_n[(3\Gamma+2)^n - 2^n(\Gamma+1)^n].$$

Hence, taking $R \geq \left\{\frac{4((\zeta^\beta r)^\alpha + Ks)^{\frac{\omega-\gamma}{\alpha}}}{\Gamma^\omega \psi(s,\bar{x}_s) v_n((3\Gamma+2)^n - 2^n(\Gamma+1)^n)}\right\}^{\frac{1}{n+\omega}}$ yields:

$$\int_{\mathcal{C}(2(\Gamma+1)R,(3\Gamma+2)R)} \frac{\psi(s,\overline{x}_s)-\psi(s,y)}{|\overline{x}_s - y|^{n+\alpha}}dy \geq \frac{\psi(s,\overline{x}_s)}{4((3\Gamma+2)R)^{n+\alpha}}\left(v_n R^n[(3\Gamma+2)^n - 2^n(\Gamma+1)^n]\right).$$

For (39) to be fulfilled we must thus have (with the notations introduced in (41)):

$$\Phi_s\left\{\frac{v_n\alpha}{(3\Gamma+2)^\alpha 4K(n+\gamma)}\left(1-\left[\frac{2(\Gamma+1)}{3\Gamma+2}\right]^n\right)\right\}^{\frac{1}{\alpha}} \geq R \geq \Phi_s\left\{\frac{4}{\Gamma^\omega v_n((3\Gamma+2)^n - 2^n(\Gamma+1)^n)}\right\}^{\frac{1}{n+\omega}}. \quad (42)$$

From (41), (42) we have that for Γ sufficiently large, both constraints will be fulfilled provided that:

$$\Phi_s\left\{\frac{v_n\alpha}{(3\Gamma+2)^\alpha 4K(n+\gamma)}\left(1-\left[\frac{2(\Gamma+1)}{3\Gamma+2}\right]^n\right)\right\}^{\frac{1}{\alpha}} \geq R \geq \Phi_s\left\{\frac{4}{\Gamma^\omega v_n(2^n-1)}\right\}^{\frac{1}{n+\omega}}.$$

In particular, this will be the case setting

$$K := \left(\frac{v_n}{4}\right)^{1+\frac{\alpha}{n+\omega}} \frac{\alpha(2^n-1)^{\frac{\alpha}{n+\omega}}\Gamma^{\frac{\alpha\omega}{n+\omega}}}{(3\Gamma+2)^\alpha(n+\gamma)} \times \left(1-\left[\frac{2(\Gamma+1)}{3\Gamma+2}\right]^n\right).$$

This fixes K and therefore the normalization parameter ζ through the controls of the concentration condition.

The proof of the Height condition is finished for regular molecules. In order to obtain the global result, remark that, for viscosity solutions studied in Section 2.1, we have $\Delta\psi(s_0,\overline{x}) \leq 0$ at the points $\overline{x}$ where $\psi(s_0,\cdot)$ reaches its maximum value so we only need to study the term $\mathcal{L}\psi(s_0,\overline{x})$ as it was done here. See [8] for more details.

Remark 4.4 *The constants obtained here do not depend on the molecule's size but only on the dimension n and on parameters ω, γ and α.*

Remark 4.5 *The above computations amend the ones performed in [3].*

3) Small time L^1 estimate

This last condition is an easy consequence of the previous computations. Indeed: we write

$$\begin{aligned}\int_{\mathbb{R}^n} |\psi(s_0,x)|dx &= \int_{\{|x-x(s_0)|<D\}} |\psi(s_0,x)|dx + \int_{\{|x-x(s_0)|\geq D\}} |\psi(s_0,x)|dx \\ &\leq v_n D^n \|\psi(s_0,\cdot)\|_{L^\infty} + D^{-\omega}\int_{\mathbb{R}} |\psi(s_0,x)||x-x(s_0)|^\omega dx.\end{aligned}$$

Now using the Concentration condition and the Height condition one has:

$$\int_{\mathbb{R}^n} |\psi(s_0,x)|dx \leq v_n \frac{D^n}{((\zeta^\beta r)^\alpha + Ks_0)^{\frac{n+\gamma}{\alpha}}} + D^{-\omega}((\zeta^\beta r)^\alpha + Ks_0)^{\frac{\omega-\gamma}{\alpha}},$$

where v_n denotes the volume of the unit ball. An optimization over the real parameter D yields:

$$\|\psi(s_0,\cdot)\|_{L^1} \leq \frac{2v_n^{\frac{\omega}{n+\omega}}}{\left((\zeta^\beta r)^\alpha + Ks_0\right)^{\frac{\gamma}{\alpha}}}.$$

Theorem 10 is now completely proven. ■

4.2 Molecule's evolution: Iteration

In the previous section we have quantified the deformation of molecules after a very small time s_0. The next theorem shows us how to obtain similar profiles in the inputs and the outputs in order to perform an iteration in time.

Theorem 11 *Set γ and ω two real numbers such that $0 < \gamma < \omega < \min(\delta, 1)$. For $i \in \mathbb{N}^*$ and a given $0 < s_{i-1} < s_i \leq T$, let $\psi(s,x),\ s \in [s_{i-1}, T]$ be a solution of the problem*

$$\left\{\begin{array}{rcl} \partial_s\psi(s,x) = & -\nabla\cdot(v\,\psi)(s,x) - \mathcal{L}\psi(s,x),\ s\in[s_{i-1},T], \\ \psi(s_{i-1},x) = & \psi(s_{i-1},x), \\ div(v) = & 0 \quad and \quad v \in \left(L^\infty([0,T];M^{q,a}(\mathbb{R}^n))\right)^n \quad with \sup\limits_{s\in[s_{i-1},T]} \|v(s,\cdot)\|_{M^{q,a}} \leq \mu. \end{array}\right. \tag{43}$$

If $\psi(s_{i-1},x)$ satisfies the three following conditions

$$\int_{\mathbb{R}^n} |\psi(s_{i-1},x)||x-x(s_{i-1})|^\omega dx \leq ((\zeta^\beta r)^\alpha + Ks_{i-1})^{\frac{\omega-\gamma}{\alpha}}; \quad \|\psi(s_{i-1},\cdot)\|_{L^\infty} \leq \frac{1}{((\zeta^\beta r)^\alpha + Ks_{i-1})^{\frac{n+\gamma}{\alpha}}};$$

$$\|\psi(s_{i-1},\cdot)\|_{L^1} \leq \frac{2v_n^{\frac{\omega}{n+\omega}}}{\left((\zeta^\beta r)^\alpha + Ks_{i-1}\right)^{\frac{\gamma}{\alpha}}},$$

where $K = K(\mu)$ is as in Theorem 10 and s_{i-1} is such that $((\zeta^\beta r)^\alpha + Ks_{i-1}) < 1$. Then for all $0 < s_i - s_{i-1} \leq \epsilon r^\alpha$, where ϵ is small, we have the following estimates

$$\int_{\mathbb{R}^n} |\psi(s_i,x)||x-x(s_i)|^\omega dx \leq ((\zeta^\beta r)^\alpha + Ks_i)^{\frac{\omega-\gamma}{\alpha}}, \tag{44}$$

$$\|\psi(s_i,\cdot)\|_{L^\infty} \leq \frac{1}{((\zeta^\beta r)^\alpha + Ks_i)^{\frac{n+\gamma}{\alpha}}}, \tag{45}$$

$$\|\psi(s_i,\cdot)\|_{L^1} \leq \frac{2v_n^{\frac{\omega}{n+\omega}}}{\left((\zeta^\beta r)^\alpha + Ks_i\right)^{\frac{\gamma}{\alpha}}}. \tag{46}$$

Remark 4.6

1) *Since $s_i - s_{i-1}$ is small and $((\zeta^\beta r)^\alpha + Ks_{i-1}) < 1$, we can without loss of generality assume that $((\zeta^\beta r)^\alpha + Ks_i) < 1$: otherwise, by the maximum principle there is nothing to prove.*

2) *The new molecule's center $x(s_i)$ used in formula (44) is fixed by the evolution of the following differential system:*

$$\left\{\begin{array}{rcl} x'(s) = & \overline{v}_{B(x(s),\zeta\rho_i)} = \frac{1}{|B(x(s),\zeta\rho_i)|}\displaystyle\int_{B(x(s),\zeta\rho_i)} v(s,y)dy,\ s\in[s_{i-1},s_i], \\ x(s_{i-1}) = & x(s_{i-1}), \end{array}\right. \tag{47}$$

where

$$\rho_i = \left(r^\alpha + \frac{K}{\zeta^{\beta\alpha}} s_{i-1}\right)^{\frac{1}{\alpha}}. \tag{48}$$

Note that by Point 1) above we have $0 < \rho_i < 1/\zeta^\beta$.

3) *We have in particular that the hypotheses on the initial data can be rewriten as follows*

$$\begin{array}{c} \|\psi(s_{i-1},\cdot)\|_{L^\infty} \leq (\zeta^\beta\rho_i)^{-(n+\gamma)}; \quad \|\psi(s_{i-1},\cdot)\|_{L^1} \leq 2v_n^{\frac{\omega}{n+\omega}}\ (\zeta^\beta\rho_i)^{-\gamma}; \quad and \\ \|\psi(s_{i-1},\cdot)\|_{L^p} \leq C\ (\zeta^\beta\rho_i)^{-n+\frac{n}{p}-\gamma} \quad (1 < p < +\infty). \end{array} \tag{49}$$

Proof of the Theorem 11. The proof follows the same lines as the one of Theorem 10. Indeed, the concentration condition (44) can be established similarly to (30) replacing r by ρ_i. The height condition (45) is again proved similarly to (31) replacing $\zeta^\beta r$ by $\zeta^\beta\rho_i$ and s by $s - s_{i-1}$. The condition (46) is eventually derived exactly as (32) from the controls (44) and (45).

End of the proof of Theorem 9

We see with Theorem 10 that it is possible to control the L^1 behavior of the molecules ψ from 0 to a small time s_0. Theorem 11 extends the control from time s_0 to time s_N. We recall that we have $s_i - s_{i-1} \sim \epsilon r^\alpha$ for all $0 \leq i \leq N$ (with $s_{-1} = 0$), so the bound obtained in (46) depends mainly on the size of the molecule r and the number of iterations N.

We observe now that the smallness of r and of the time increments $s_0, s_1 - s_0, ..., s_N - s_{N-1}$ can be compensated by the number of iterations N in the following sense: fix a small $0 < r < 1$ and iterate as explained before. Since each small time increment $s_0, s_1 - s_0, ..., s_N - s_{N-1}$ has order ϵr^α, we have $s_N \sim N\epsilon r^\alpha$. Thus, we will stop the iterations as soon as $N\epsilon r^\alpha \geq T_0$.

Of course, the number of iterations $N = N(r)$ will depend on the smallness of the molecule's size r, and more specifically it is enough to set $N(r) \sim \frac{T_0}{\epsilon r^\alpha}$ in order to obtain this lower bound for $N(r)$.

Proceeding this way we will obtain $\|\psi(s_N, \cdot)\|_{L^1} \leq CT_0^{-\gamma} < +\infty$, for all molecules of size r. Note in particular that, once this estimate is available, for bigger times it is enough to apply the maximum principle.

Finally, and for all $r > 0$, we obtain after a time T_0 a L^1 control for small molecules and we finish the proof of the Theorem 9. ■

Appendix A

We first introduce a measure decomposition that will be frequently used in this appendix. The key idea consists in rewriting the density π of the initial Lévy measure satisfying condition **[ND]** as:

$$\forall y \in \mathbb{R}^n,\ \pi(y) = \widetilde{\pi} + \underline{\pi}, \tag{50}$$

where the function $\widetilde{\pi}$ is defined over $\mathbb{R}^n$ by

$$\begin{cases} \widetilde{\pi}(y) = \pi(y) \quad \text{if } |y| \leq 1,\ \tilde{\pi}(y) = \pi(y/|y|)\frac{1}{|y|^{n+\alpha}} \quad \text{if } |y| \geq 1, \text{ so that} \\ \\ \overline{c}_1|y|^{-n-\alpha} \leq \widetilde{\pi}(y) \leq \overline{c}_2|y|^{-n-\alpha} \quad \text{if } |y| > 1. \end{cases} \tag{51}$$

Remark that for all $y \in \mathbb{R}^n$ we have $\overline{c}_1|y|^{-n-\alpha} \leq \widetilde{\pi}(y) \leq \overline{c}_2|y|^{-n-\alpha}$ and thus the Lévy-type operator $\widetilde{\mathcal{L}}$ associated to the function $\widetilde{\pi}$ is *equivalent* to the fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$. On the other hand the support of $\underline{\pi}$ is included in $B(0,1)^C := \{y \in \mathbb{R}^n : |y| \geq 1\}$ and:

$$|\underline{\pi}(y)| \leq C\{|y|^{-n+\delta} + |y|^{-n+\alpha}\}. \tag{52}$$

It is worth noting that the equivalence of the operator $\widetilde{\mathcal{L}}$ with the action of the fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$ is only valid in a L^p-sense with $1 < p < +\infty$. However, in some very specific cases, it is possible to obtain a similar behavior in a L^1-sense.

Proof of Lemma 2.1. We recall here that we assume the parameter $t > 0$ to be small since Lemma 2.1 is needed to investigate the local existence of solutions. If $0 < \delta < \alpha < 1$, using (3) and (4) we obtain for the heat kernel h_t the inequalities

$$\begin{aligned} \|\mathcal{L}h_t\|_{L^1} &\leq C\Big\{ \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|h_t(x) - h_t(x-y)|}{|y|^{n+\alpha}}dydx + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|h_t(x) - h_t(x-y)|}{|y|^{n+\delta}}dydx \Big\} = C\Big\{ \|h_t\|_{\dot{B}_1^{\alpha,1}} + \|h_t\|_{\dot{B}_1^{\delta,1}} \Big\} \\ &\leq C\big(t^{\frac{-\alpha}{2}} + t^{\frac{-\delta}{2}}\big). \end{aligned}$$

If $1 < \delta < \alpha < 2$, we consider the previous decomposition (50) and the controls (51), (52) to obtain:

$$\begin{aligned} \|\mathcal{L}h_t\|_{L^1} &\leq \int_{\mathbb{R}^n} \Big| \text{v.p.} \int_{\mathbb{R}^n} \big[h_t(x) - h_t(x-y)\big]\widetilde{\pi}(y)dy \Big| dx + C\Big\{ \int_{\mathbb{R}^n}\int_{\{|y|\geq 1\}} \frac{|h_t(x) - h_t(x-y)|}{|y|^{n+\delta}}dxdy \\ &\quad + \int_{\mathbb{R}^n}\int_{\{|y|\geq 1\}} \frac{|h_t(x) - h_t(x-y)|}{|y|^{n+\alpha}}dydx \Big\}. \end{aligned}$$

Since $h_t(x) = \frac{1}{(4\pi t)^{\frac{n}{2}}}e^{-\frac{|x|^2}{4t}}$, by homogeneity we have

$$\begin{aligned} \|\mathcal{L}h_t\|_{L^1} &\leq C\Big\{ t^{-\frac{\alpha}{2}} + t^{-\frac{\delta}{2}} \int_{\mathbb{R}^n}\int_{\{|y|\geq t^{-\frac{1}{2}}\}} \frac{(4\pi t)^{-\frac{n}{2}}\Big|h_1\big(\frac{x}{2t^{\frac{1}{2}}}\big) - h_1\big(\frac{x}{2t^{\frac{1}{2}}} - y\big)\Big|}{|y|^{n+\delta}}dydx \\ &\quad + t^{-\frac{\alpha}{2}} \int_{\mathbb{R}^n}\int_{\{|y|\geq t^{-\frac{1}{2}}\}} \frac{(4\pi t)^{-\frac{n}{2}}\Big|h_1\big(\frac{x}{2t^{\frac{1}{2}}}\big) - h_1\big(\frac{x}{2t^{\frac{1}{2}}} - y\big)\Big|}{|y|^{n+\alpha}}dydx \Big\}. \end{aligned}$$

The first term is the right hand side can be derived observing:

$$T_1 := \int_{\mathbb{R}^n} \left| \text{v.p.} \int_{\mathbb{R}^n} \big[h_t(x) - h_t(x-y)\big]\widetilde{\pi}(y)dy \right| dx = \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \big\{h_t(x+y) - h_t(x) - \nabla h(x)\cdot y \mathbb{1}_{|y|\leq\varepsilon}\big\}\widetilde{\pi}(y)dy \right| dx,$$

for an arbitrary $\varepsilon > 0$ using the symmetry of the measure $\widetilde{\pi}$. Hence:

$$T_1 \leq Ct^{-(n+\alpha)/2} \int_{\mathbb{R}^n} \left\{ \int_{\mathbb{R}^n} \left| h_1\big(\frac{x}{t^{1/2}} + y\big) - h_1\big(\frac{x}{t^{1/2}}\big) - \nabla h_1\big(\frac{x}{t^{1/2}}\big)\cdot y \mathbb{1}_{|y|\leq\frac{\varepsilon}{t^{1/2}}} \right| \frac{dy}{|y|^{n+\alpha}} \right\} dx.$$

Choosing now, $\varepsilon = t^{1/2}$ we get:

$$\begin{aligned} T_1 &\leq Ct^{-\alpha/2} \left\{ \int_{\mathbb{R}^n} \left(\int_{|y|\geq 1} \{h_1\big(x+y\big) + h_1(x)\} \frac{dy}{|y|^{n+\alpha}} \right) dx \right. \\ &\quad \left. + \int_{\mathbb{R}^n} \left(\int_{|y|\leq 1} \exp\big(-C^{-1}(|x|^2/8 - |y|^2/4)\big)|y|^2 \frac{dy}{|y|^{n+\alpha}} \right) dx \right\} \\ &\leq Ct^{-\alpha/2}, \end{aligned}$$

using the usual convexity inequality $|x+y|^2 \geq \frac{1}{2}|x|^2 - |y|^2$ for the last but one inequality and the Fubini theorem for the first term to get the stated upper bound up to a modification of C. Now, since t is a small time as we are working in a local in time framework we have $t^{-\frac{1}{2}} > 1$ and then

$$\begin{aligned} \|\mathcal{L}h_t\|_{L^1} &\leq Ct^{-\frac{\alpha}{2}} + Ct^{-\frac{\delta}{2}}\|h_t\|_{L^1} \int_{\{|y|\geq 1\}} \frac{1}{|y|^{n+\delta}} dy + Ct^{-\frac{\alpha}{2}}\|h_t\|_{L^1} \int_{\{|y|\geq 1\}} \frac{1}{|y|^{n+\alpha}} dy \\ &\leq C\Big(t^{\frac{-\alpha}{2}} + t^{\frac{-\delta}{2}}\Big). \end{aligned}$$

■

Proof of the Lemma 3.1. We recall here that for $x \in \mathbb{R}^n$, $\varphi(x) = \phi(x/R)$, $R > 0$, where ϕ is a non-negative smooth function such that $\phi(z) = 1$ if $|z| \leq 1/2$ and $\phi(z) = 0$ if $|z| \geq 1$, $z \in \mathbb{R}^n$.

If $0 < \delta < \alpha < 1$, we have $[\mathcal{L}, \varphi]A_R(s,x) = \text{v.p.} \int_{\mathbb{R}^n} \big(\varphi(x) - \varphi(x-y)\big)A_R(s, x-y)\pi(y)dy$ and we proceed as follows.

We begin with the case $p = +\infty$ and we write:

$$|[\mathcal{L}, \varphi]A_R(s,x)| \leq C \left\{ \int_{\mathbb{R}^n} \frac{|\varphi(x) - \varphi(y)|}{|x-y|^{n+\alpha}} |A_R(s,y)|dy + \int_{\mathbb{R}^n} \frac{|\varphi(x) - \varphi(y)|}{|x-y|^{n+\delta}} |A_R(s,y)|dy \right\}. \tag{53}$$

Again, it is enough to study one of these two integrals since the other can be treated in a totally similar way. We write:

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{|\varphi(x) - \varphi(y)|}{|x-y|^{n+\alpha}} |A_R(s,y)|dy &= \int_{\{|x-y|>R\}} \frac{|\varphi(x) - \varphi(y)|}{|x-y|^{n+\alpha}} |A_R(s,y)|dy + \int_{\{|x-y|\leq R\}} \frac{|\varphi(x) - \varphi(y)|}{|x-y|^{n+\alpha}} |A_R(s,y)|dy \\ &\leq 2\|\varphi\|_{L^\infty} \int_{\{|x-y|>R\}} \frac{|A_R(s,y)|}{|x-y|^{n+\alpha}} dy + \int_{\{|x-y|\leq R\}} \frac{\|\nabla\varphi\|_{L^\infty}|x-y|}{|x-y|^{n+\alpha}} |A_R(s,y)|dy \\ &\leq 2\|\varphi\|_{L^\infty}\|A_R(s,\cdot)\|_{L^\infty} \int_{\{|x-y|>R\}} \frac{1}{|x-y|^{n+\alpha}} dy + CR^{-1} \int_{\{|x-y|\leq R\}} \frac{|A_R(s,y)|}{|x-y|^{n+\alpha-1}} dy \\ &\leq 2C\|\varphi\|_{L^\infty}\|A_R(s,\cdot)\|_{L^\infty} R^{-\alpha} + C\|A_R(s,\cdot)\|_{L^\infty} R^{-\alpha} \leq CR^{-\alpha}\|A_{0,R}\|_{L^\infty}. \end{aligned}$$

Then, with the δ-part in inequality (53) we have

$$\|[\mathcal{L}, \varphi]A_R(s,\cdot)\|_{L^\infty} \leq C(R^{-\alpha} + R^{-\delta})\|A_{0,R}\|_{L^\infty}.$$

The case $p = 1$ is very similar. Using inequality (53) we have

$$\int_{\mathbb{R}^n} |[\mathcal{L}, \varphi]A_R(s,x)|dx \leq C \left\{ \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\varphi(x) - \varphi(y)|}{|x-y|^{n+\alpha}} |A_R(s,y)|dydx + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\varphi(x) - \varphi(y)|}{|x-y|^{n+\delta}} |A_R(s,y)|dydx \right\}.$$

We only estimate one of the previous integrals.

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|\varphi(x)-\varphi(y)|}{|x-y|^{n+\alpha}}|A_R(s,y)|dydx &\leq C\|\varphi\|_{L^\infty}\int_{\mathbb{R}^n}\int_{\{|x-y|>R\}}\frac{|A_R(s,y)|}{|x-y|^{n+\alpha}}dydx\\ &\quad +R^{-1}\int_{\mathbb{R}^n}\int_{\{|x-y|\leq R\}}\frac{|A_R(s,y)|}{|x-y|^{n+\alpha-1}}dydx\\ &\leq C\|\varphi\|_{L^\infty}\|A_R(s,\cdot)\|_{L^1}R^{-\alpha}+C\|A_R(s,\cdot)\|_{L^1}R^{-\alpha}\leq CR^{-\alpha}\|A_{0,R}\|_{L^1}.\end{aligned}$$

With the other integral, we obtain

$$\|[\mathcal{L},\varphi]A_R(s,\cdot)\|_{L^1}\leq C(R^{-\alpha}+R^{-\delta})\|A_{0,R}\|_{L^1}.$$

Finally, the case $1<p<+\infty$ is obtained by interpolation. See [11] or [21] for more details about interpolation.

If $1<\delta<\alpha<2$, we have now $[\mathcal{L},\varphi]A_R(s,x)=\text{v.p.}\int_{\mathbb{R}^n}\big(\varphi(x)-\varphi(x-y)-\nabla\varphi(x)\cdot y\mathbb{1}_{|y|\leq 1}\big)A_R(s,x-y)\pi(y)dy$. With the notations of (50) and the controls of equations (51) and (52) we obtain

$$\begin{aligned}[\mathcal{L},\varphi]A_R(s,x) &= \text{v.p.}\int_{\mathbb{R}^n}\big(\varphi(x)-\varphi(x-y)-\nabla\varphi(x)\cdot y\mathbb{1}_{|y|\leq 1}\big)A_R(s,x-y)\pi(y)dy\\ &= \text{v.p.}\int_{\mathbb{R}^n}\big(\varphi(x)-\varphi(x-y)-\nabla\varphi(x)\cdot y\mathbb{1}_{|y|\leq 1}\big)A_R(s,x-y)\widetilde{\pi}(y)dy\\ &+ \text{v.p.}\int_{\mathbb{R}^n}\big(\varphi(x)-\varphi(x-y)-\nabla\varphi(x)\cdot y\mathbb{1}_{|y|\leq 1}\big)A_R(s,x-y)\underline{\pi}(y)dy.\end{aligned}$$

We start with $p=+\infty$. Using the decomposition of π in (50) and applying the maximum principle on the function A_R we have

$$\begin{aligned}\big|[\mathcal{L},\varphi]A_R(s,x)\big| &\leq \left|\text{v.p.}\int_{\mathbb{R}^n}\big(\varphi(x)-\varphi(x-y)-\nabla\varphi(x)\cdot y\mathbb{1}_{|y|\leq 1}\big)A_R(s,x-y)\widetilde{\pi}(y)dy\right|\\ &\quad +\left|\text{v.p.}\int_{\mathbb{R}^n}\big(\varphi(x)-\varphi(x-y)-\nabla\varphi(x)\cdot y\mathbb{1}_{|y|\leq 1}\big)A_R(s,x-y)\underline{\pi}(y)dy\right| \qquad (54)\\ &\leq \|A_{0,R}\|_{L^\infty}\left(\int_{\mathbb{R}^n}\big|\varphi(x)-\varphi(x-y)-\nabla\varphi(x)\cdot y\mathbb{1}_{|y|\leq 1}\big|\widetilde{\pi}(y)dy+\left|\text{v.p.}\int_{\mathbb{R}^n}\big(\varphi(x)-\varphi(x-y)\big)\underline{\pi}(y)dy\right|\right).\end{aligned}$$

We recall now that $\varphi(x)=\phi(x/R)$ and since ϕ is a smooth function by homogeneity we have for the first integral above that

$$\int_{\mathbb{R}^n}\big|\varphi(x)-\varphi(x-y)-\nabla\varphi(x)\cdot y\mathbb{1}_{|y|\leq 1}\big|\widetilde{\pi}(y)dy\leq CR^{-\alpha}.$$

For the second integral, using the definition of $\underline{\pi}$ we write

$$\begin{aligned}\left|\text{v.p.}\int_{\mathbb{R}^n}\big(\varphi(x)-\varphi(x-y)\big)\underline{\pi}(y)dy\right| &\leq \overline{c}_2\int_{\{|y|\geq 1\}}\frac{\big|\varphi(x)-\varphi(x-y)\big|}{|y|^{n+\alpha}}dy+\overline{c}_2\int_{\{|y|\geq 1\}}\frac{\big|\varphi(x)-\varphi(x-y)\big|}{|y|^{n+\delta}}dy \qquad (55)\\ &\leq \overline{c}_2\|\nabla\varphi\|_{L^\infty}\left(\int_{\{|y|\geq 1\}}\frac{1}{|y|^{n+\alpha-1}}dy+\int_{\{|y|\geq 1\}}\frac{1}{|y|^{n+\delta-1}}dy\right)\\ &\leq CR^{-1},\end{aligned}$$

so we obtain $\|[\mathcal{L},\varphi]A_R(s,\cdot)\|_{L^\infty}\leq C(R^{-\alpha}+R^{-1})\|A_{0,R}\|_{L^\infty}$.

We treat now the case $p=1$. Using the decomposition $\pi=\widetilde{\pi}+\underline{\pi}$ and inequalities (54) and (55) we can write

$$\begin{aligned}\int_{\mathbb{R}^n}\big|[\mathcal{L},\varphi]A_R(s,x)\big|dx &\leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\big|\varphi(x)-\varphi(x-y)-\nabla\varphi(x)\cdot y\mathbb{1}_{|y|\leq 1}\big||A_R(s,x-y)|\widetilde{\pi}(y)dydx\\ &\quad +\overline{c}_2\int_{\mathbb{R}^n}\int_{\{|y|\geq 1\}}\frac{\big|\varphi(x)-\varphi(x-y)\big|\,|A_R(s,x-y)|}{|y|^{n+\alpha}}dydx\\ &\quad +\overline{c}_2\int_{\mathbb{R}^n}\int_{\{|y|\geq 1\}}\frac{\big|\varphi(x)-\varphi(x-y)\big|\,|A_R(s,x-y)|}{|y|^{n+\delta}}dydx.\end{aligned}$$

Using the same arguments for the two last integrals we obtain

$$\begin{aligned}\int_{\mathbb{R}^n}\big|[\mathcal{L},\varphi]A_R(s,x)\big|dx &\leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\big|\varphi(x)-\varphi(x-y)-\nabla\varphi(x)\cdot y\mathbb{1}_{|y|\leq 1}\big||A_R(s,x-y)|\widetilde{\pi}(y)dydx + C\|A_R(s,\cdot)\|_{L^1}\|\nabla\varphi\|_{L^\infty}\\ &\leq \|A_R(s,\cdot)\|_{L^\infty}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\big|\varphi(x)-\varphi(x-y)-\nabla\varphi(x)\cdot y\mathbb{1}_{|y|\leq 1}\big|\widetilde{\pi}(y)dydx + C\|A_R(s,\cdot)\|_{L^1}R^{-1}.\end{aligned}$$

Using the definition of $\varphi(x)=\phi(x/R)$ and the maximum principle we obtain

$$\int_{\mathbb{R}^n}\big|[\mathcal{L},\varphi]A_R(s,x)\big|dx \leq C\big(\|A_{0,R}\|_{L^\infty}R^{-\alpha+n}+\|A_{0,R}\|_{L^1}R^{-1}\big).$$

With the L^∞-L^1 inequalities, the L^p case follows by interpolation:

$$\|[\mathcal{L},\varphi]A_R\|_{L^p} \leq C\big(\|A_{0,R}\|_{L^\infty}R^{-\alpha+n}+\|A_{0,R}\|_{L^1}R^{-1}\big)^{\frac{1}{p}}\big(R^{-\alpha}+R^{-1})\|A_{0,R}\|_{L^\infty}\big)^{1-\frac{1}{p}}.$$

■

Appendix B

We will need the following results concerning Morrey-Campanato spaces:

Lemma B-1 *Let $1\leq q<+\infty$, $0<a<+\infty$, $x_0\in\mathbb{R}^n$, $0<\rho<1$ and $k\in\mathbb{N}$.*

- *We have the inequality*
$$\|f-\overline{f}_{B(x_0,\rho)}\|_{L^q(B(x_0,\rho))}\leq C\rho^{\frac{a}{q}}\|f\|_{M^{q,a}},$$
- *If $0<a<n$ we have*
$$|\overline{f}_{B(x_0,2^k\rho)}-\overline{f}_{B(x_0,\rho)}|\leq C\rho^{\frac{a-n}{q}}\|f\|_{M^{q,a}}, \tag{56}$$
- *If $n<a<n+q$ we have*
$$|\overline{f}_{B(x_0,2^k\rho)}-\overline{f}_{B(x_0,\rho)}|\leq C(2^k\rho)^{\frac{a-n}{q}}\|f\|_{M^{q,a}}. \tag{57}$$

See [25] and [1] for a proof of these facts.

We will prove here Lemma 4.1 in a slightly more general framework.

Proposition B-1 *Consider a time $s_N\in[0,T]$, a real $0<\omega<1$ and a real $0<\rho<1$. Let $x(s_N)$ be a point in $\mathbb{R}^n$. If $v(s_N,\cdot)\in M^{q,a}$ with $1\leq q<+\infty$ and $0<a<n+q$, if $\psi(s_N,\cdot)\in L^p$ with $1\leq p\leq+\infty$, then we have the inequality*

$$\begin{aligned}\int_{\mathbb{R}^n}|x-x(s_N)|^{\omega-1}|v-\overline{v}_{B_\rho}||\psi(s_N,x)|dx &\leq C\|v(s_N,\cdot)\|_{M^{q,a}}\rho^{\omega-1}\Big(\rho^{\frac{n}{p}+\frac{a}{q}}\|\psi(s_N,\cdot)\|_{L^z}+\rho^{\frac{a}{q}}\|\psi(s_N,\cdot)\|_{L^{q'}}\\ &\quad+\rho^{\frac{a-n}{q}-n+\frac{n}{\bar{p}}-\gamma}\|\psi(s_N,\cdot)\|_{L^{\bar{p}}}\Big),\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}+\frac{1}{z}=1$, $\frac{1}{q}+\frac{1}{q'}=1$, $\frac{1}{\bar{p}}+\frac{1}{\bar{q}}=1$ with $\omega-1+\frac{a}{q}<0$, $\omega-1+\frac{a-n}{q}+\frac{n}{\bar{q}}<0$.

Proof. We begin by considering the space $\mathbb{R}^n$ as the union of a ball with dyadic coronas centered around $x(s_N)$, more precisely we set $\mathbb{R}^n=B_\rho\cup\bigcup_{k\geq 1}E_k$ where

$$B_\rho=\{x\in\mathbb{R}^n:|x-x(s_N)|\leq\rho\}\quad\text{and}\quad E_k=\{x\in\mathbb{R}^n:2^{k-1}\rho<|x-x(s_N)|\leq 2^k\rho\}\quad\text{with } 0<\rho<1, \tag{58}$$

and we write

$$\begin{aligned}\int_{\mathbb{R}^n}|x-x(s_N)|^{\omega-1}|v-\overline{v}_{B_\rho}||\psi_+(s_N,x)|dx &= \int_{B_\rho}|x-x(s_N)|^{\omega-1}|v-\overline{v}_{B_\rho}||\psi_+(s_N,x)|dx\\ &\quad+\sum_{k\geq 1}\int_{E_k}|x-x(s_N)|^{\omega-1}|v-\overline{v}_{B_\rho}||\psi_+(s_N,x)|dx.\end{aligned}$$

(i) Estimations over the ball B_ρ. Applying Hölder's inequality we obtain

$$\begin{aligned}\int_{B_\rho}|x-x(s_N)|^{\omega-1}|v-\overline{v}_{B_\rho}||\psi_+(s_N,x)|dx &\leq \||x-x(s_N)|^{\omega-1}\|_{L^p(B_\rho)}\|v-\overline{v}_{B_\rho}\|_{L^q(B_\rho)}\\ &\quad\times\|\psi_+(s_N,\cdot)\|_{L^z(B_\rho)},\end{aligned}\tag{59}$$

where $\frac{1}{p}+\frac{1}{q}+\frac{1}{z}=1$ and $p,q,z>1$. We treat each of the previous terms separately:

Observe that for $1<p<n/(1-\omega)$ we have for the first term above:

$$\||x-x(s_N)|^{\omega-1}\|_{L^p(B_\rho)}\leq C\rho^{\frac{n}{p}+\omega-1}.$$

For the second term, by hypothesis we have $v(s_N,\cdot)\in M^{q,a}$ and we just apply the first inequality of Lemma B-1 to obtain

$$\|v-\overline{v}_{B_\rho}\|_{L^q(B_\rho)}\leq C\|v(s_N,\cdot)\|_{M^{q,a}}\,\rho^{\frac{a}{q}}.$$

For the last term we simply write $\|\psi_+(s_N,\cdot)\|_{L^z(B_\rho)}\leq\|\psi(s_N,\cdot)\|_{L^z}$.

We combine all these inequalities to obtain the following estimate for (59):

$$\int_{\mathbb{R}^n}|x-x(s_N)|^{\omega-1}|v-\overline{v}_{B_\rho}||\psi_+(s_N,x)|dx\leq C\|v(s_N,\cdot)\|_{M^{q,a}}\rho^{\omega-1}\left(\rho^{\frac{n}{p}+\frac{a}{q}}\|\psi(s_N,\cdot)\|_{L^z}\right).\tag{60}$$

(ii) Estimations for the dyadic corona E_k. Let us note I_{E_k} the integral

$$I_{E_k}=\int_{E_k}|x-x(s_N)|^{\omega-1}|v-\overline{v}_{B_\rho}||\psi_+(s_N,x)|dx.$$

Since over E_k we have[3] $|x-x(s_N)|^{\omega-1}\leq C(2^k\rho)^{\omega-1}$ we write

$$I_{E_k}\leq C(2^k\rho)^{\omega-1}\left(\int_{E_k}|v-\overline{v}_{B_{2^k\rho}}||\psi_+(s_N,x)|dx+\int_{E_k}|\overline{v}_{B_\rho}-\overline{v}_{B_{2^k\rho}}||\psi_+(s_N,x)|dx\right),$$

where we have denoted $B_{2^k\rho}=B(x(s_N),2^k\rho)$, then

$$\begin{aligned}I_{E_k}&\leq C(2^k\rho)^{\omega-1}\left(\int_{B_{2^k\rho}}|v-\overline{v}_{B_{2^k\rho}}||\psi_+(s_N,x)|dx+\int_{B_{2^k\rho}}|\overline{v}_{B_\rho}-\overline{v}_{B_{2^k\rho}}||\psi_+(s_N,x)|dx\right)\\ &\leq C(2^k\rho)^{\omega-1}\left(\|v-\overline{v}_{B_{2^k\rho}}\|_{L^q(B_{2^k\rho})}\|\psi(s_N,\cdot)\|_{L^{q'}}+\int_{B_{2^k\rho}}|\overline{v}_{B_\rho}-\overline{v}_{B_{2^k\rho}}||\psi_+(s_N,x)|dx\right),\end{aligned}$$

where we used the Hölder inequality with $\frac{1}{q}+\frac{1}{q'}=1$.

Now, since $v(s_N,\cdot)\in M^{q,a}(\mathbb{R}^n)$, using Lemma B-1 we have

- if $0<\delta<\alpha<1$ and then $\frac{a-n}{q}=1-\alpha>0$, so $n<a<n+q$:

$$I_{E_k}\leq C(2^k\rho)^{\omega-1}\left((2^k\rho)^{\frac{a}{q}}\|v(s_N,\cdot)\|_{M^{q,a}}\|\psi(s_N,\cdot)\|_{L^{q'}}+(2^k\rho)^{\frac{a-n}{q}+\frac{n}{q}}\|v(s_N,\cdot)\|_{M^{q,a}}\|\psi(s_N,\cdot)\|_{L^{\bar{p}}}\right).$$

- if $1<\delta<\alpha<2$ and then $\frac{a-n}{q}=1-\alpha<0$, so $0<a<n$:

$$I_{E_k}\leq C(2^k\rho)^{\omega-1}\left((2^k\rho)^{\frac{a}{q}}\|v(s_N,\cdot)\|_{M^{q,a}}\|\psi(s_N,\cdot)\|_{L^{q'}}+\rho^{\frac{a-n}{q}}(2^k\rho)^{\frac{n}{q}}\|v(s_N,\cdot)\|_{M^{q,a}}\|\psi(s_N,\cdot)\|_{L^{\bar{p}}}\right),$$

with the condition $\omega-1+\frac{a}{q}<0$ which implies of course that $\omega-1+\frac{a-n}{q}<0$.

Since $0<\omega<1$, summing over each dyadic corona E_k, we have in both cases the inequality

$$\sum_{k\geq1}I_{E_k}\leq C\|v(s_N,\cdot)\|_{M^{q,a}}\,\rho^{\omega-1}\left(\rho^{\frac{a}{q}}\|\psi(s_N,\cdot)\|_{L^{q'}}+\rho^{\frac{a-n}{q}+\frac{n}{q}}\|\psi(s_N,\cdot)\|_{L^{\bar{p}}}\right).\tag{61}$$

[3] recall that we always have $0<\gamma<\omega<1$.

Finally, gathering together inequalities (60) and (61) we obtain the desired conclusion. ■

We now prove Lemma 4.2 with the following proposition.

Proposition B-2 *Consider a time $s_N \in [0,T]$, a real $0<\omega<1$ and a real $0<\rho<1$. Let $x(s_N)$ be a point in $\mathbb{R}^n$. If $\psi(s_N,\cdot)\in L^p$ with $1\le p\le+\infty$ and if $\mathcal{L}$ is a Lévy-type operator under the hypotheses (3) and (4), for $0<\delta<\alpha<2$ we have the inequality*

$$\int_{\mathbb{R}^n}\left|\mathcal{L}\big(|x-x(s_N)|^\omega\big)\right||\psi(s_N,x)|dx\le C\rho^{\omega-\alpha}\left(\rho^{\frac{n}{2}}\|\psi(s_N,\cdot)\|_{L^2}+\rho^{\frac{n}{q}}\|\psi(s_N,\cdot)\|_{L^{\bar p}}\right).$$

where $\frac{1}{\bar p}+\frac{1}{q}=1$ and $\omega-\delta+\frac{n}{q}<0$.

Proof. As for Proposition B-1, we consider $\mathbb{R}^n$ as the union of a ball of radius ρ with dyadic coronas centered on the point $x(s_N)$ (cf. (58)).

$$\begin{aligned}\int_{\mathbb{R}^n}\left|\mathcal{L}\big(|x-x(s_N)|^\omega\big)\right||\psi(s_N,x)|dx &= \int_{B_\rho}\left|\mathcal{L}\big(|x-x(s_N)|^\omega\big)\right||\psi(s_N,x)|dx\\ &\quad+\sum_{k\ge1}\int_{E_k}\left|\mathcal{L}\big(|x-x(s_N)|^\omega\big)\right||\psi(s_N,x)|dx.\end{aligned}$$

(i) Estimations over the ball B_ρ. From the Cauchy-Schwarz inequality, we write:

$$I_{2,B_\rho}=\int_{B_\rho}\left|\mathcal{L}(|x-x(s_N)|^\omega)\right||\psi(s_N,x)|dx\le\|\psi(s_N,\cdot)\|_{L^2(B_\rho)}\|\mathcal{L}|x-x(s_N)|^\omega\|_{L^2(B_\rho)},$$

and we need now to study the term $\|\mathcal{L}|x-x(s_N)|^\omega\|_{L^2(B_\rho)}$ which is equivalent up to a change of variables to

$$\left(\int_{B(0,\rho)}|\mathcal{L}|x|^\omega|^2\,dx\right)^{\frac12}.$$

We use decomposition (50) to obtain:

$$\begin{aligned}\left(\int_{B(0,\rho)}|\mathcal{L}|x|^\omega|^2\,dx\right)^{\frac12}&\le\left(\int_{B(0,\rho)}\left|\text{v.p.}\int_{\mathbb{R}^n}[|x|^\omega-|x-y|^\omega]\widetilde{\pi}(y)dy\right|^2dx\right)^{\frac12}\\&\quad+\left(\int_{B(0,\rho)}\left|\text{v.p.}\int_{\mathbb{R}^n}[|x|^\omega-|x-y|^\omega]\underline{\pi}(y)dy\right|^2dx\right)^{\frac12}.\end{aligned}$$

We will start assuming $0<\omega<\delta<\alpha<1$. Then, using inequality (52) and by homogeneity we have

$$\begin{aligned}\left(\int_{B(0,\rho)}|\mathcal{L}|x|^\omega|^2\,dx\right)^{\frac12}&\le C\rho^{\omega-\alpha+\frac{n}{2}}\left(\int_{\{|x|\le1\}}\left(\text{v.p.}\int_{\mathbb{R}^n}\frac{\big||x|^\omega-|x-y|^\omega\big|}{|y|^{n+\alpha}}dy\right)^2dx\right)^{\frac12}\\&\quad+\overline{c}\rho^{\omega-\alpha+\frac{n}{2}}\left(\int_{\{|x|\le1\}}\left(\int_{\{|y|\ge1/\rho\}}\frac{\big||x|^\omega-|x-y|^\omega\big|}{|y|^{n+\alpha}}dy\right)^2dx\right)^{\frac12}\\&\quad+\overline{c}\rho^{\omega-\delta+\frac{n}{2}}\left(\int_{\{|x|\le1\}}\left(\int_{\{|y|\ge1/\rho\}}\frac{\big||x|^\omega-|x-y|^\omega\big|}{|y|^{n+\delta}}dy\right)^2dx\right)^{\frac12}.\end{aligned}$$

Since $0<\rho<1$ and $||x|^\omega-|x-y|^\omega|\le c|y|^\omega$, the two last integrals in the right hand side can be bounded by a uniform constant so we only need to study the first integral above that can be decomposed in the following way:

$$\begin{aligned}\left(\int_{\{|x|\le1\}}\left(\text{v.p.}\int_{\mathbb{R}^n}\frac{\big||x|^\omega-|x-y|^\omega\big|}{|y|^{n+\alpha}}dy\right)^2dx\right)^{\frac12}&\le\left(\int_{\{|x|\le1\}}\left(\text{v.p.}\int_{\{|y|\le1\}}\frac{\big||x|^\omega-|x-y|^\omega\big|}{|y|^{n+\alpha}}dy\right)^2dx\right)^{\frac12}\\&\quad+\left(\int_{\{|x|\le1\}}\left(\text{v.p.}\int_{\{|y|>1\}}\frac{\big||x|^\omega-|x-y|^\omega\big|}{|y|^{n+\alpha}}dy\right)^2dx\right)^{\frac12}.\end{aligned}$$

For the first integral in the right hand side we use the inequality $||x|^\omega - |x-y|^\omega| \leq |y||x|^{\omega-1}$, for the second integral we apply the same arguments used before (*i.e.* $||x|^\omega - |x-y|^\omega| \leq c|y|^\omega$). In any case all these quantities are bounded by constants and we obtain:

$$\|\mathcal{L}|x-x(s_N)|^\omega\|_{L^2(B_\rho)} \leq C\big(\rho^{\omega-\alpha+\frac{n}{2}} + \rho^{\omega-\delta+\frac{n}{2}}\big).$$

The case $1 < \delta < \alpha < 2$ can be treated in a very similar way performing a Taylor expansion of second order, reasoning as in the proof of Lemma 3.1 for that case (see [9], Section 3 for more details).

Finally, recalling that $0 < \rho < 1$ and since $0 < \delta < \alpha < 2$ we obtain $\rho^{\omega-\delta+\frac{n}{2}} \leq \rho^{\omega-\alpha+\frac{n}{2}}$ so we have

$$I_{2,B_\rho} \leq C\rho^{\omega-\alpha+\frac{n}{2}}\|\psi(s_N,\cdot)\|_{L^2}. \tag{62}$$

(ii) Estimations for the dyadic corona E_k. By homogeneity we have

$$\int_{E_k} |\mathcal{L}(|x-x(s_N)|^\omega)||\psi(s_N,x)|dx \leq \|\psi(s_N,\cdot)\|_{L^{\bar{p}}}(2^{k-1}\rho)^{\omega+n(1+\frac{1}{\bar{q}})} \underbrace{\sup_{1\leq|x|\leq 2}\left|\text{v.p.}\int_{\mathbb{R}^n}[|x|^\omega - |x-y|^\omega]\pi(2^{k-1}\rho y)dy\right|}_{I}. \tag{63}$$

Using again the decomposition $\pi = \widetilde{\pi} + \underline{\pi}$ given in (50) and (51) page 22 we have

$$I \leq \sup_{1\leq|x|\leq 2}\left(\left|\text{v.p.}\int_{\mathbb{R}^n}[|x|^\omega - |x-y|^\omega]\widetilde{\pi}(2^{k-1}\rho y)dy\right| + \left|\text{v.p.}\int_{\mathbb{R}^n}[|x|^\omega - |x-y|^\omega]\underline{\pi}(2^{k-1}\rho y)dy\right|\right). \tag{64}$$

We will study each one of these two terms separately.

- For the first one we have:

$$\begin{aligned}\left|\text{v.p.}\int_{\mathbb{R}^n}[|x|^\omega - |x-y|^\omega]\widetilde{\pi}(2^{k-1}\rho y)dy\right| \leq & \sup_{1\leq|x|\leq 2}\left|\text{v.p.}\int_{B(0,1)}[|x|^\omega - |x-y|^\omega]\widetilde{\pi}(2^{k-1}\rho y)dy\right| \\ & + \sup_{1\leq|x|\leq 2}\int_{B(0,1)^c}\left|[|x|^\omega - |x-y|^\omega]\widetilde{\pi}(2^{k-1}\rho y)\right|dy.\end{aligned} \tag{65}$$

For the first integral above we recall that the function $\tilde{\pi}(y)$ is equivalent up to some constants to the function $|y|^{-n-\alpha}$ and we remark that the function $|x|^\omega$ is smooth in the annulus $\{x \in \mathbb{R}^n : 1 \leq |x| \leq 2\}$. Thus we can write for $0 < \alpha < 1$,

$$\begin{aligned}\sup_{1\leq|x|\leq 2}\left|\text{v.p.}\int_{B(0,1)}[|x|^\omega - |x-y|^\omega]\widetilde{\pi}(2^{k-1}\rho y)dy\right| &\leq \sup_{1\leq|x|\leq 2}\text{v.p.}\int_{B(0,1)}||x|^\omega - |x-y|^\omega|\,\widetilde{\pi}(2^{k-1}\rho y)dy \\ &\leq \sup_{1\leq|x|\leq 2}\int_{B(0,1)}\frac{|y||x|^{\omega-1}}{|2^{k-1}\rho y|^{n+\alpha}}dy \\ &\leq (2^{k-1}\rho)^{-n-\alpha}\sup_{1\leq|x|\leq 2}|x|^{\omega-1}\int_{B(0,1)}|y|^{1-n-\alpha}dy \\ &\leq C(2^{k-1}\rho)^{-n-\alpha}.\end{aligned}$$

The case $1 \leq \alpha < 2$ can be treated in a completely similar way by performing a Taylor expansion of second order (see [9], Section 3 for more details).

The last integral of (65) can be easily controlled since

$$\int_{B(0,1)^c}\left|[|x|^\omega - |x-y|^\omega]\widetilde{\pi}(2^{k-1}\rho y)\right|dy \leq C\int_{B(0,1)^c}\frac{||x|^\omega - |x-y|^\omega|}{|2^{k-1}\rho y|^{n+\alpha}}dy \leq C(2^{k-1}\rho)^{-n-\alpha}\int_{B(0,1)^c}\frac{|y|^\omega}{|y|^{n+\alpha}}dy,$$

and as we have $0 < \omega < \alpha < 2$, the previous integral is bounded and we have

$$\int_{B(0,1)^c}\left|[|x|^\omega - |x-y|^\omega]\widetilde{\pi}(2^{k-1}\rho y)\right|dy \leq C(2^{k-1}\rho)^{-n-\alpha}.$$

- We study now the second part of the formula (64). By definition of $\underline{\pi}$ we can write:

$$\sup_{1\leq|x|\leq 2}\left|\text{v.p.}\int_{\mathbb{R}^n}[|x|^\omega-|x-y|^\omega]\underline{\pi}(2^{k-1}\rho y)dy\right| \leq \sup_{1\leq|x|\leq 2}\left(\left|\text{v.p.}\int_{\mathbb{R}^n}[|x|^\omega-|x-y|^\omega]\pi(2^{k-1}\rho y)\mathbb{1}_{\{2^{k-1}\rho|y|\geq 1\}}dy\right|\right) + \sup_{1\leq|x|\leq 2}\left(\left|\text{v.p.}\int_{\mathbb{R}^n}[|x|^\omega-|x-y|^\omega]\widetilde{\pi}(2^{k-1}\rho y)\mathbb{1}_{\{2^{k-1}\rho|y|\geq 1\}}dy\right|\right),$$

since $\widetilde{\pi}(2^{k-1}\rho y)\sim|2^{k-1}\rho\ y|^{-n-\alpha}$ the second term above can be treated in the same way as explained previously and we only need to focus on the following term

$$\sup_{1\leq|x|\leq 2}\left|\text{v.p.}\int_{\mathbb{R}^n}[|x|^\omega-|x-y|^\omega]\pi(2^{k-1}\rho y)\mathbb{1}_{\{2^{k-1}\rho|y|\geq 1\}}dy\right|,$$

and we will distinguish two cases depending on the size of $2^{k-1}\rho$:

– If $0<2^{k-1}\rho<1$ we have the inequality

$$\sup_{1\leq|x|\leq 2}\left|\text{v.p.}\int_{\mathbb{R}^n}[|x|^\omega-|x-y|^\omega]\pi(2^{k-1}\rho y)\mathbb{1}_{\{2^{k-1}\rho|y|\geq 1\}}dy\right| \leq \sup_{1\leq|x|\leq 2}\int_{\mathbb{R}^n}\frac{||x|^\omega-|x-y|^\omega|}{|2^{k-1}\rho y|^{n+\delta}}dy \leq C(2^{k-1}\rho)^{-n-\delta}.$$

– If $1<2^{k-1}\rho$ we can bound this term by the quantity

$$\begin{aligned}
&\leq \sup_{1\leq|x|\leq 2}\left|\text{v.p.}\int_{\{\frac{1}{2^{k-1}\rho}\leq|y|\leq 1\}}[|x|^\omega-|x-y|^\omega]\pi(2^{k-1}\rho y)dy\right| \\
&\quad+\sup_{1\leq|x|\leq 2}\left|\text{v.p.}\int_{\mathbb{R}^n}[|x|^\omega-|x-y|^\omega]\pi(2^{k-1}\rho y)\mathbb{1}_{\{1\leq|y|\}}dy\right| \\
&\leq \sup_{1\leq|x|\leq 2}\left|\text{v.p.}\int_{\{|y|\leq 1\}}[|x|^\omega-|x-y|^\omega]\pi(2^{k-1}\rho y)dy\right|+\sup_{1\leq|x|\leq 2}\int_{\mathbb{R}^n}\frac{||x|^\omega-|x-y|^\omega|}{|2^{k-1}\rho y|^{n+\delta}}dy.
\end{aligned}$$

With the same ideas used in the previous item we obtain

$$\sup_{1\leq|x|\leq 2}\left|\text{v.p.}\int_{\mathbb{R}^n}[|x|^\omega-|x-y|^\omega]\pi(2^{k-1}\rho y)\mathbb{1}_{\{2^{k-1}\rho|y|\geq 1\}}dy\right| \leq C(2^{k-1}\rho)^{-n-\alpha}+C(2^{k-1}\rho)^{-n-\delta},$$

and, thus, in any case we have the following inequality for this term

$$\sup_{1\leq|x|\leq 2}\left|\text{v.p.}\int_{\mathbb{R}^n}[|x|^\omega-|x-y|^\omega]\pi(2^{k-1}\rho y)\mathbb{1}_{\{2^{k-1}\rho|y|\geq 1\}}dy\right| \leq C(2^{k-1}\rho)^{-n-\alpha}+C(2^{k-1}\rho)^{-n-\delta}.$$

Finally, with these two inequalities for the terms of (64) one obtains

$$\int_{E_k}|\mathcal{L}(|x-x(s_N)|^\omega)||\psi(s_N,x)|dx \leq C\|\psi(s_N,\cdot)\|_{L^{\bar{p}}}(2^{k-1}\rho)^{\omega+n(1+\frac{1}{\bar{q}})}\left((2^{k-1}\rho)^{-n-\alpha}+(2^{k-1}\rho)^{-n-\delta}\right).$$

Since $0<\gamma<\omega<\delta<\alpha<2$, summing over $k\geq 1$, we obtain

$$\sum_{k\geq 1}\int_{E_k}|\mathcal{L}(|x-x(s_N)|^\omega)||\psi(s_N,x)|dx \leq \|\psi(s_N,\cdot)\|_{L^{\bar{p}}}\left(\rho^{\omega-\alpha+\frac{n}{\bar{q}}}+\rho^{\omega-\delta+\frac{n}{\bar{q}}}\right).$$

Repeating the same argument used before (*i.e.* the fact that $0<\rho<1$ and that $\rho^{\omega-\delta+\frac{n}{\bar{q}}}\leq\rho^{\omega-\alpha+\frac{n}{\bar{q}}}$), we finally obtain

$$\sum_{k\geq 1}I_{2,E_k}\leq C\rho^{\omega-\alpha+\frac{n}{\bar{q}}}\|\psi(s_N,\cdot)\|_{L^{\bar{p}}}. \tag{66}$$

In order to finish the proof of the proposition, it is enough to gather the inequalities (62) and (66). ■

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