Subordination in the wide sense for Lévy processes

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Abstract

Given $(\mu_t)_{t\geq 0}$ a transient convolution semigroup over $\mathbb{R}^d$, we characterize the transient semi-groups which are subordinate in the wide sense to the latter, via some quotient relation between the two Lévy generators. This is also equivalent to an inclusion between excessive cones. Thanks to a theorem of Hirsch, this allows us in big dimensions to give a probabilistic description of some measures satisfying the classical maximum principle.

Key words: Convolution kernels, Randomized stopping times, Subordinator, Excursions.

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1 Introduction

The celebrated Blumenthal-Getoor-McKean Theorem (see Chap V in [4], and also [9] for a modern approach) states that two Markov processes X and Y with the same hitting distributions are related by time-change: there exists some continuous strictly increasing additive functional of X such that if T is its right-continuous inverse, then Y and $X \circ T$ are equivalent processes. Intuitively, this means that X and Y have the same sample paths, but that they live on them with different "speeds". In the transient case, the equality of hitting distributions is equivalent to $\mathcal{S}(X) = \mathcal{S}(Y)$, where $\mathcal{S}$ stands for the cone of excessive functions.

A theorem of Shih [20] states that if the hitting distributions of X *dominate* those of Y, then Y is killed on the sample paths of X at some random time τ. Modulo time-change, τ there is the first jump of a non-homogeneous Poisson process, whose intensity is $dM/M-$, where M is a multiplicative functional of X.

In the transient case, the situation $\mathcal{S}(X) \subset \mathcal{S}(Y)$ seems to have escaped investigation, and one can naturally conjecture that this yields some relation $Y = X \circ \sigma$, where σ is a *discontinuous* time-change satisfying some additivity property $\sigma(t+s) = \sigma(t) + \sigma(s) \circ \theta_{\sigma(s)}$ where θ is the shift associated with X. This would mean that Y lives only on "portions" of the sample paths of X.

Our purpose in this paper is to study this conjecture when X and Y are Lévy processes. Under a mild assumption, we show that the condition $\mathcal{S}(X) \subset \mathcal{S}(Y)$ is equivalent to a realization of Y as $X \circ \sigma$, where σ is a subordinator whose law depends on X, such that σ_t is a stopping-time for X for each t. Besides, (X, σ) satisfies some regenerative property in the sense of Maisonneuve [15]. We call this relation a *subordination in the wide sense*, or *loose* subordination, according to a terminology of Huff [14].

However σ is not necessarily the inverse of an additive functional of X: we construct (Y, σ) on a product probability space instead of the canonical space of X (such a construction seems more delicate). In particular, the shift appearing in the additivity property of σ is not associated with X.

Actually, a very analogous result was proved recently by Bertoin [3], when X and Y are themselves subordinators. Bertoin's purposes were slightly different from ours and dealt rather with the closed ranges $M_X = \overline{\text{Im}X}$, and $M_Y = \overline{\text{Im}Y}$, viewed as regenerative sets on the half-line. His main result is that M_Y is regeneratively embedded in M_X if and only if ϕ_Y/ϕ_X is a completely monotone function.

But this complete monotonicity condition is in fact a particular case of a "quotient relation" between general Hunt kernels, introduced by Deny [7] to describe some kernels satisfying the classical maximum principle, and later developped by Hirsch [13]. We observe that in the convolution framework, this quotient relation is equivalent to an inclusion between the sets of excessive functions (and other excessive cones).

Then, we repeat essentially the same arguments as Bertoin. For the direct part, we first construct a "skeleton" of Y embedded in X, via Rost's randomized stopping times and then go to the limit, which is a bit delicate here: there is no immediate compacity as for the closed sets of $\mathbb{R}^+$, and then we must deal with the discontinuities of X. Besides, as in previous papers on embeddings (see Monroe [17], Falkner [8] and the references therein), we needed to enlarge the natural filtration of X.

For the converse part, we appeal to rather classical considerations on the excursions of X away from the homogeneous random set $\{\sigma(t) : t \geq 0\}$. As a principal corollary, we use a theorem of Hirsch to describe in dimension greater than 3 some measures satisfying the classical maximum principle in terms of these excursions.

It is very likely that the conjecture stated above is true, at least when the carrier process is a *diffusion*. The arguments used here probably hold in a more general Markovian framework, except in the limit procedure where the continuity in probability for X and σ seem crucial, as soon as X is discontinuous.

The paper is organized as follows: Section 2 presents the framework, technicalities about randomized stopping times and analytic considerations about the quotient relation and excessive cones. In section 3 we state the main result and some applications. Section 4 contains the proofs.

2 Preliminaries

2.1 Transient convolution semi-groups

We work over $\mathbb{R}^d$ equipped with $|\,.\,|$ the Euclidean norm and ∂ an isolated point that serves as a cemetery. Any measurable function on $\mathbb{R}^d$ is extended to $\mathbb{R}^d \cup \{\partial\}$ by setting $f(\partial) = 0$. We use the standard notations:

$$\left\{\begin{array}{lll} \mathcal{E} & \text{is the set of} & \text{measurable functions.} \\ \mathcal{K} & \ldots & \text{continuous functions with compact support.} \\ \mathcal{C}_b & \ldots & \text{bounded continuous functions} \\ \mathcal{M} & \ldots & \text{Radon measures.} \\ \mathcal{M}_0 & \ldots & \text{Radon measures tending to 0 at infinity.} \end{array}\right.$$

every element being considered over $\mathbb{R}^d$. The last sentence means that if $m \in \mathcal{M}_0$, then for all $\phi \in \mathcal{K}$, $\lim_{|x| \to +\infty} m * \phi(x) = 0$. For any set $\mathcal{A}$, $\mathcal{A}^+$ denotes the positive elements of $\mathcal{A}$.

Consider $(\mu_t)_{t \geq 0}$ a transient convolution semi-group of probability measures, that is:

$$\left\{\begin{array}{ll} \mu_t(1) = 1 & \forall\, t \geq 0 \\ \mu_t * \mu_s = \mu_{t+s} & \forall\, t,\, s \geq 0 \\ \lim\limits_{t \to 0} \mu_t = \delta_0 & \text{weakly} \end{array}\right.$$

and the potential-measure $u = \int_0^{+\infty} \mu_t \, dt$ is in $\mathcal{M}^+$. The inverse for convolution of $(-u)$ is the distribution:

$$T = \lim_{t \to 0} \frac{\mu_t - \delta_0}{t} \;\text{ weakly.}$$

T is called the *Lévy generator* of $(\mu_t)_{t \geq 0}$ and has a Fourier transform ψ given by the Lévy-Khintchine formula:

$$\psi(\lambda) = i < a, \lambda > - \frac{1}{2}\, Q(\lambda) + \int_{\mathbb{R}^d} (e^{i<\lambda, x>} - 1 - i < \lambda, x > \mathbf{1}_{|x|<1})\, \pi(dx),$$

where $a \in \mathbb{R}^d$, Q is a non negative definite quadratic form on $\mathbb{R}^d$ and π a measure on $\mathbb{R}^d - \{0\}$ such that $\int (1 \wedge |x|^2)\, \pi(dx) < +\infty$.

The transience property for $(\mu_t)_{t \geq 0}$ has the following characterization upon ψ (see Chap 13. in Berg/Forst [1]):

$$\Re(\frac{1}{\psi}) \text{ is locally integrable.}$$

Let X be a standard realization of $(\mu_t)_{t \geq 0}$ over some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. X is a càdlàg process with stationary, independent increments (a Lévy process), started at $X_0 = 0$ and such that for all $0 \leq t < \infty$,

$$\mathbb{P}\,[X_t \in \,.\,] = \mu_t(\,.\,).$$

We extend X to $[0,+\infty]$ by setting $X_\infty = \partial$. We denote by $(\mathcal{F}_t)_{t\geq 0}$ the natural filtration of X: $\mathcal{F}_t = \sigma(X_s,\ s \leq t) \vee \mathcal{N}$ where $\mathcal{N}$ are the sets of $\mathbb{P}$-measure 0. Notice that $(\mathcal{F}_t)_{t\geq 0}$ is necessarily right-continuous, (see Prop I.4 in Bertoin [2]).

The transience assumption ensures that $\lim_{t\to+\infty} |X_t| = +\infty$ (see Th I.19 in [2], and note that the hypothesis (H) therein is not relevant for this property). In particular X extended to $[0,+\infty]$ is still càdlàg with state space $\mathbb{R}^d \cup \{\partial\}$. By the Feller property, X is also quasi-left-continuous on $[0,+\infty]$: if $\{T_n\}$ is an increasing sequence of $\mathcal{F}_t$-stopping times such that $\lim_{n\to+\infty} T_n = T$ (T may take infinite values), then $\lim_{n\to+\infty} X_{T_n} = X_T$.

Let $\mathcal{S}(\mu_t)$ be the cone of excessive functions with respect to $(\mu_t)_{t\geq 0}$: those $f \in \mathcal{E}^+$ such that for any $x \in \mathbb{R}^d$,

$$E\left[f(x+X_t)\right] \uparrow f(x) \quad \text{as } t \downarrow 0.$$

The transience hypothesis assures that this cone $\mathcal{S}$ is rich enough. Thanks to the Markov property, it contains for example the set $\mathcal{P}$ of potential-functions: $x \mapsto \check{u} * f\,(x)$ with $f \in \mathcal{E}^+$. One can even show that $\mathcal{B}(\mathbb{R}^d) = \sigma(\mathcal{S})$ (see for instance Getoor [11] in a much more general context).

The dual cone to $\mathcal{S}(\mu_t)$ is $\mathrm{Exc}(\mu_t)$, the cone of excessive measures with respect to $(\mu_t)_{t\geq 0}$: those $m \in \mathcal{M}^+$ such that for any $g \in \mathcal{K}^+$,

$$\int E\left[g(x+X_t)\right] dm(x) \ \uparrow\ \int g(x)\, dm(x) \quad \text{as } t \downarrow 0.$$

Notice (see Proposition 16.3 in [1]), that this is equivalent to:

$$\check{\mu}_t * m \leq m \quad \forall\, t \geq 0.$$

$\mathrm{Exc}(\mu_t)$ is also rich enough under the transience hypothesis. It contains for example the set $\mathrm{Pot}(\mu_t)$ of potential-measures: $u * m$ such that m and $u * m$ are in $\mathcal{M}^+$.

We finally lift from Corollary 16.8 in Berg/Forst [1] that those potential-measures uniquely determines their charges: if ρ_1 and ρ_2 are Radon measures such that $\rho_1 * u$ and $\rho_2 * u$ are identical Radon measures, then:

$$\rho_1 = \rho_2.$$

2.2 Randomized stopping times

This subsection is mainly technical but crucial for our purposes in part 4. It is adapted from Meyer [16] and Fitzsimmons [10] to whom we refer for a general discussion. We are still dealing with $(\mu_t)_{t\geq 0}$ a transient convolution semi-group, and X its realisation over $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. Let $(\Omega', \mathcal{F}', \mathbb{P}')$ be an auxiliary probability space and define: $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*) = (\Omega \times \Omega', \mathcal{F} \times \mathcal{F}', \mathbb{P} \otimes \mathbb{P}')$. We assume that Ω and Ω' are Polish spaces and $\mathcal{F}$, $\mathcal{F}'$ their completed Borelian σ-fields respectively. Every process is considered implicitly over $[0,+\infty]$.

A *randomized stopping time* (RST) with respect to X with $(\Omega', \mathcal{F}', \mathbb{P}')$ as randomization space is an $\mathcal{F}^*$-measurable mapping $T^* : \Omega^* \to [0, +\infty]$ such that:

$$\{(\omega, \omega') \in \Omega^* \ : \ T^*(\omega, \omega') \le t\} \ \in \ \mathcal{F}_t \times \mathcal{F}' \qquad \forall t \ge 0.$$

If $(\Omega', \mathcal{F}', \mathbb{P}') \ = \ ([0,1], \mathcal{B}_{[0,1]}, \lambda)$ the unit interval equipped with Lebesgue measure, then we speak of a *canonical* RST with respect to X. From now on we omit the precision "with respect to X" for simplicity.

Consider now an RST T^* and define the measure M on $\Omega \times [0, +\infty]$ by $M(Z) = E^*\,[Z_{T^*}]$ for any raw process Z. Clearly $M = f(\mathbb{P}^*)$ where $f : (\omega, \omega') \mapsto (\omega, T^*(\omega, \omega'))$ and so the first marginal of M is $\mathbb{P}$. A disintegration theorem (see Sect III 71-73 in Dellacherie/Meyer [6]) gives:

$$dM\,(s, \omega) = dA_s(\omega)\, d\mathbb{P}(\omega)$$

where $A_s : [0, +\infty] \to [0,1]$ is increasing, $A_0 = 0$ and $A_\infty = 1$ $\mathbb{P}$ p.s. Besides,

$$A_t(\omega) = \mathbb{P}'\,[T^*(\omega, .) \le t],$$

and so A_t is $\mathcal{F}_t$-adapted.

We take a càdlàg version of A_t and define:

$$\widetilde{T}(\omega, z) = \inf\,\{t \ : \ A_t(\omega) > z\}, \ \ 0 \le z \le 1.$$

It is a canonical RST and by a standard change of variable, we get, for any process Z:

$$E^*\,[Z_{T^*}] = E\,[\textstyle\int_{[0,+\infty]} Z_t\, dA_t] = \int_0^1 E\,[Z_{\widetilde{T}(\,.\,,z)}]\, dz = \widetilde{E}\,[Z_{\widetilde{T}}].$$

In particular, taking $Z_t = \int_t^{+\infty} f(X_s)\, ds$ and using the strong Markov property for X yield:

$$E^*\,[Uf(X_{T^*})] = \widetilde{E}\,[Uf(X_{\widetilde{T}})].$$

Hence, by the uniqueness of charges with $\rho_1 = \mathbb{P}^*\,[X_{T^*} \in \ .\,]$ and $\rho_2 = \widetilde{\mathbb{P}}\,[X_{\widetilde{T}} \in \ .\,]$, we have:

$$\mathbb{P}^*\,[X_{T^*} \in \ .\,] = \widetilde{\mathbb{P}}\,[X_{\widetilde{T}} \in \ .\,].$$

We will refer to $\widetilde{T}$ as the *canonical rearrangement* of T^*.

Finally, Theorem 5 in [16] yields the following result concerning weak convergence of canonical RST's (due in its original form to Baxter/Chacon):

Proposition 1 *Let $\{\widetilde{T}_n\}$ be a sequence of canonical RST's. Then there exists a subsequence $\{\phi_n\}$ and a canonical RST $\widetilde{T}$ such that*

$$\widetilde{E}\,[Z_{\widetilde{T}_{\phi_n}}] \longrightarrow \widetilde{E}\,[Z_{\widetilde{T}}]$$

for any bounded, càdlàg, quasi-left-continuous process Z.

Notice that in the preceding proposition, the convergence is not so weak since Z need *not* be $\mathcal{F}_t$-adapted. However we shall not make use of this property.

2.3 The quotient relation for Lévy generators

The following analytic result is due to Hirsch ([13], p 124):

Proposition 2 *Let $K \in \mathcal{M}_0^+$. The following assertions are equivalent:*

(i) K satisfies the classical maximum principle: for all $\phi \geq 0$ in $\mathcal{K}$,

$$(\forall\, x \in Supp\,\phi,\ K * \phi(x) \leq 1) \Longrightarrow (K * \phi \leq 1).$$

(ii) There exists two Lévy generators T and S such that $T \neq 0$ and:

$$T * K = S.$$

Our purpose in this paper is to give a probabilistic interpretation of this analytic result, in the transient case. This motivates the following definition:

Definition 1 *Let T and S be two Lévy generators. S is said to be a* quotient *of T when the following relation holds:*

$$T * K = S$$

with $K \in \mathcal{M}_0^+$.

In Hirsch's theorem, T and S may correspond to Lévy processes killed at a certain exponential rate, (given by $(- < T, 1 >)$ and $(- < S, 1 >)$ respectively). But T may also be chosen such that $< T, 1 > = 0$ (see Theorem 2 in [13]), i.e T has infinite lifetime. To avoid technical difficulties, we will focus on this case only. We will also suppose that S itself has infinite lifetime (which is not necessarily the case in Hirsch's theorem).

We now give an equivalent property to the quotient relation. Recall that T and S are transient as soon as $d \geq 3$.

Proposition 3 *Consider $K \in \mathcal{M}_0^+$, T and S two transient Lévy generators. There is an equivalence:*

$$T * K = S \Longleftrightarrow u = v * K$$

where u and v are the potential-measures associated respectively with T and S.

Proof Set $\epsilon_\lambda = (\lambda\delta - T)^{*-1}$ and $\eta_\mu = (\mu\delta - S)^{*-1}$, two bounded positive measures for $\lambda, \mu > 0$. Suppose first $T * K = S$. We have:

$$(\lambda\delta - T) * K = \lambda K - \mu\delta + (\mu\delta - S)$$

and

$$(\eta_\mu * K) + (\mu\eta_\mu * \epsilon_\lambda) = (\lambda\epsilon_\lambda * \eta_\mu * K) + \epsilon_\lambda.$$

Let λ tend to 0. As K tends to 0 at infinity, the first term in the second member vanishes and we can write:

$$(\eta_\mu * K) + (\mu\eta_\mu * u) = u.$$

As $\mu \downarrow 0$, $(\mu\eta_\mu * u) \downarrow \xi$ weakly. Since $\mu\eta_\mu \to 0$ weakly and $\mu\eta_\mu * u \leq u$, we can apply Theorem 16.10 in [1] and get $\xi = 0$. This entails:

$$u = v * K.$$

Conversely suppose $u = v * K$. By the resolvent equation we have:

$$(\delta - \lambda\epsilon_\lambda) * u = \epsilon_\lambda \text{ and } (\delta - \mu\eta_\mu) * v = \eta_\mu$$

which yields:

$$\lambda(\delta - \lambda\epsilon_\lambda) * (\eta_\mu * K) = \lambda\epsilon_\lambda * (\delta - \mu\eta_\mu).$$

As K tends to 0 at infinity, we get when $\lambda \to +\infty$:

$$(-T) * \eta_\mu * K = \delta - \mu\eta_\mu.$$

Multiplying by μ and letting μ tend to $+\infty$ entail:

$$T * K = S$$

□

Notice that in the preceding proposition, S need not be transient as soon as it is not 0: if T is transient and if a quotient holds, then $v = \lim_{\mu\to 0} \eta_\mu$ is necessarily in $\mathcal{M}^+$. But K *must* tend to 0 at infinity.

We now express the quotient relation in terms of excessive cones:

Proposition 4 *Let $(\mu_t)_{t\geq 0}$ and $(\nu_t)_{t\geq 0}$ be two transient convolution semigroups, u and v their potential-measures. The following assertions are equivalent:*

(i) $\exists\, K \in \mathcal{M}^+$ such that: $u = v * K$.

(ii) $\mathcal{S}(\mu_t) \subset \mathcal{S}(\nu_t)$

(iii) Exc*$(\mu_t) \subset$* Exc *(ν_t)*

Proof: (i) $\Rightarrow$ (ii): Let $f \in \mathcal{S}(\mu_t)$. It is well-known (see for example II.2.19 in [4]) that there exists a sequence $g_n \geq 0$ of bounded functions in $\mathcal{E}$ such that $\check{u} * g_n$ is bounded and:

$$\check{u} * g_n \uparrow f$$

as $n \to \infty$. But then:

$$\check{u} * g_n = (\check{v} * \check{K}) * g_n = \check{v} * (\check{K} * g_n) \in \mathcal{S}(\nu_t).$$

And since $\{\check{u} * g_n\}$ is an increasing sequence, one checks easily that $f \in \mathcal{S}(\nu_t)$.

(ii) $\Rightarrow$ (iii): Let $m \in \text{Exc}(\mu_t)$. It is clear that for any $\phi \in \mathcal{K}^+$, $(m * \phi) \in \mathcal{S}(\mu_t)$. So $(m * \phi) \in \mathcal{S}(\nu_t)$ and:

$$\check{\nu}_t * m * \phi \leq m * \phi \text{ pointwise}$$

This entails:

$$\check{\nu}_t * m \leq m \text{ setwise}$$

and thus,

$$m \in \mathrm{Exc}(\nu_t).$$

(iii) $\Rightarrow$ (i): $u \in \mathrm{Exc}(\nu_t)$ and so there is a Riesz decomposition (see theorem 16.7 in [1]):

$$u = v * K + \eta$$

with $K \in \mathcal{M}^+$ and $\eta = \lim_{t\to+\infty} \nu_t * u$ weakly. But $\nu_t * u \leq u$ and $\lim_{t\to+\infty} \nu_t = 0$ weakly by transience. Hence we can apply theorem 16.10 in [1], which entails:

$$\eta = 0.$$

□

Notice that in the preceding proposition, we could have added two equivalent assertions: $\mathcal{P}(\mu_t) \subset \mathcal{P}(\nu_t)$ and $\mathrm{Pot}(\mu_t) \subset \mathrm{Pot}(\nu_t)$.

3 Results and applications

3.1 Subordination in the wide sense for Lévy processes

We now state the central definition of this work:

Definition 2 *Let $(\mu_t)_{t\geq 0}$ and $(\nu_t)_{t\geq 0}$ be two convolution semigroups on $\mathbb{R}^d$. $(\nu_t)_{t\geq 0}$ is said to be a* subordination in the wide sense *of $(\mu_t)_{t\geq 0}$ if, on some probability space endowed with a filtration $(\mathcal{F}_t)_{t\geq 0}$, there exist two Lévy processes X and Y, with semi-groups $(\mu_t)_{t\geq 0}$ and $(\nu_t)_{t\geq 0}$ respectively, and a subordinator σ, such that:*

· *X is a Lévy process with respect to $\mathcal{F}_t$.*

· *σ_s is an $\mathcal{F}_t$-stopping time for each $s \geq 0$.*

· *$Y = X \circ \sigma$.*

· *(X, σ) has the following regeneration property: for all $t \geq 0$, the two-parameter process:*

$$(X_{\sigma_t+u} - X_{\sigma_t}, \sigma_{t+v} - \sigma_t)_{u,v\geq 0}$$

is independent of $\mathcal{G}_t$ and has the same law as $(X_u, \sigma_v)_{u,v\geq 0}$ where $(\mathcal{G}_t)_{t\geq 0}$ is the filtration defined by:

$$\mathcal{G}_t = \sigma\{\sigma_s, s \leq t\} \vee \sigma\{X_{u\wedge\sigma_t}, u \geq 0\}.$$

This definition mimics that of *regenerative embedding* for Markov sets introduced by Bertoin [3], and it is not difficult to see that it recovers Bochner's independent subordination [5]. Besides, notice that since, on the canonical spaces, the applications $(X, \sigma) \longrightarrow (X \circ \sigma, \sigma)$ and $(X, \sigma) \longrightarrow (X \circ \sigma, X)$ are measurable, the regeneration property entails the following properties:

· (Y, σ) is a Lévy process.

· for all $t \geq 0$, $(X_{\sigma_t+u} - X_{\sigma_t}, Y_{t+v} - Y_t)_{u,v\geq 0}$ is independent of $\mathcal{G}_t$ and has the same law as $(X_u, Y_v)_{u,v\geq 0}$.

Similarly, we will say that S (resp. v) is a subordination in the wide sense of T (resp. u) if we deal with Lévy generators (resp. potential-measures). The connection with Hirsch's theorem is the matter of our main result:

Theorem 1 *Let $(\mu_t)_{t\geq 0}$ and $(\nu_t)_{t\geq 0}$ be two transient convolution semigroups with infinite lifetime. The following statements are equivalent:*

(i) $(\nu_t)_{t\geq 0}$ is a subordination in the wide sense of $(\mu_t)_{t\geq 0}$

(ii) There exists $K \in \mathcal{M}^+$ such that:

$$u = v * K.$$

where u (resp. v) is the potential-measure of $(\mu_t)_{t\geq 0}$ (resp. $(\nu_t)_{t\geq 0}$).

Notice that by Proposition 3, if $u \in \mathcal{M}_0^+$ (which holds as soon as $d \geq 2$ by the Renewal Theorem, see [18]), those assertions are also equivalent to:

(iii) There exists $K \in \mathcal{M}_0^+$ such that:

$$S = T * K.$$

where T (resp. S) is the Lévy generator of $(\mu_t)_{t\geq 0}$ (resp. $(\nu_t)_{t\geq 0}$). Observe also that it suffices that $(\mu_t)_{t\geq 0}$ be transient.

3.2 Applications

We first turn our attention to convolution semi-groups on the half-axis $[0, +\infty)$, which are necessarily transient except in the degenerate case ($\mu_t = \delta_0\ \forall t$), and whose realizations are themselves subordinators. We still consider strict probability measures (i.e non killed subordinators). Then μ_t has a Laplace transform given by:

$$\int_0^\infty e^{-\lambda s}\, \mu_t(ds) = \exp - t\phi_\mu(\lambda)$$

where ϕ_μ is a Bernstein function - called the *Laplace exponent* of $(\mu_t)_{t\geq 0}$ - such that $\phi_\mu(0) = 0$. ϕ_μ is related to the potential-measure u by:

$$\int_0^\infty e^{-\lambda s}\, u(ds) = \frac{1}{\phi_\mu(\lambda)}.$$

On the half-axis, infinity means only "$+\infty$" and so a quotient relation is equivalent to $u = v * K$ with K any Radon measure on $\mathbb{R}^+$ (since for all $K \in \mathcal{M}(\mathbb{R}^+)$, $\phi \in \mathcal{K}(\mathbb{R}^+)$, $\int \phi(x+y)K(dy) \to 0$ as $x \to +\infty$). Hence by Bernstein's theorem, a quotient relation is also equivalent to the following condition on the Laplace exponents:

$$\frac{\phi_\nu}{\phi_\mu} \text{ is a completely monotone function.}$$

Besides, if we set M_X and M_Y for the closed ranges on the half-axis of X and Y respectively, and if we define $M = (M_Y, M_X)$ on Ω, the set of couples of closed sets of $\mathbb{R}^+$, it is not difficult to see that the subordination in the wide sense of Y to X is then equivalent to the following property on M:

· $M_Y \subset M_X$

· for all $t > 0$, f measurable,

$$\mathcal{P}(f(M \circ \tau_{d_t^Y}) \mid \mathcal{H}_t) = \mathcal{P}(f(M)),$$

where $\mathcal{P}$ is the law on Ω induced by that of (Y, X), $d_t^Y = \inf\{[0,t) \cap M_Y\}$, $M \circ \tau_t = \{(r,s) \;/\; (r+t, s+t) \in M\}$ and $\mathcal{H}_t = \sigma\{d_t^Y\,;\, M_X \cap [0, d_t^Y]\,;\, M_Y \cap [0, d_t^Y]\}$.

Thus, the subordination in the wide sense of Y to X coincides with the *regenerative embedding* of M_Y in M_X as defined by Bertoin [3], and we have shown:

Corollary 1 *M_Y is regeneratively embedded in M_X if and only if $\dfrac{\phi_\nu}{\phi_\mu}$ is completely monotone.*

which is Theorem 1 in [3].

Next we give an amusing consequence of the analytic considerations in Section 2 which is independent of the main theorem, and does not seem to appear in the literature (note that the converse part is trivial):

Corollary 2 *Let X and Y be two transient Lévy processes such that: $\mathcal{S}(X) = \mathcal{S}(Y)$. Then there exists $d > 0$ such that $(X_{dt})_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ are equivalent processes.*

Proof: Set u and v for the potential-measures associated with X and Y respectively. Proposition 4 entail:

$$u = v * K \quad \text{and} \quad v = u * K'$$

where K and K' are in $\mathcal{M}^+$. Hence $u = u * K * K'$ and by the uniqueness of charges,

$$K * K' = \delta.$$

This reduces the supports of K and K' to a point: there exist $a \in \mathbb{R}^d$, $d > 0$ such that:

$$K = d\,\delta_a \quad \text{and} \quad K' = \frac{1}{d}\,\delta_{-a}.$$

But then K tends to 0 at infinity and we can apply Proposition 3:

$$S = T * d\delta_a$$

By the Lévy-Khintchine Formula, a must be 0 and we get:

$$\psi_Y = d\psi_X,$$

where ψ_X and ψ_Y are the Lévy functions respectively. This completes the proof. □

Such a result might belong to the "folklore" around the Blumenthal-Getoor-McKean Theorem, since it is intuitively obvious that any time-change transforming a Lévy process into another is a drift. However the proof here is self-contained. Besides, notice that by Proposition 4 the same result holds with $\mathrm{Exc}(\mu_t) = \mathrm{Exc}(\nu_t)$, and so our corollary is also connected with Getoor/Glover's theory [12].

Finally, we give in big dimensions a probabilistic description of some measures satisfying the classical maximum principle, that might be of interest. The proof is straightforward via Proposition 1, Theorem 1, the uniqueness of charges and the shape of K given at the end of this paper. Recall that $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ is the set of càdlàg functions $\mathbb{R}^+ \rightarrow \mathbb{R}^d$.

Corollary 3 *Suppose $d \geq 3$ and let $K \in \mathcal{M}_0^+$ satisfy the classical maximum principle. Suppose that S has infinite lifetime in its quotient relation. Then there exist $a, \geq 0$ and n an excursion measure over $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ such that:*

$$K(dx) = a\delta(dx) + \int n(de) \int_0^\zeta \mathbf{1}_{\{e(s) \in dx\}}\, ds$$

where ζ stands for the lifetime of the excursions.

This may be correlated with another "big" theorem, that of Hunt characterizing as potentials the kernels satisfying the *complete* maximum principle and tending to 0 at infinity.

4 Proofs

4.1 Proof of the direct part of Theorem 1

The proof will be given in several steps. We first fix some notations, introducing the following objects:

$\cdot$ $\Omega_0 = \mathbb{D}_0$, the functions in $\mathbb{D}$ null at 0. It is endowed with the Skorohod topology and the Borelian corresponding σ-field $\mathcal{F}_0$. $\mathbb{P}_\mu$ (resp. $\mathbb{P}_\nu$) denotes the law corresponding to μ_t (resp. ν_t). The shift is defined by $\Theta_t\, \omega_0 = \omega_0(t + .) - \omega_0(t)$. We will use the canonical notation: $X_t(\omega_0) = \omega_0(t)$ and write $\mathbb{P}$ for $\mathbb{P}_\mu$ when there is no ambiguity.

$\cdot$ $\Omega = \Omega_0 \times \mathbf{T}$, where $\mathbf{T}$ is the space of càdlàg increasing functions from $\mathbb{R}^+$ to $\mathbb{R}^+$ equipped with the following metric that makes it into a Polish space:

$$d(\tau_1, \tau_2) = \sum_{k=1}^{\infty} 2^{-k} \min\left(\int_0^k |\tau_1(s) - \tau_2(s)| ds,\ 1\right).$$

Equipped with the product topology, Ω is also a Polish space. We denote by $\mathcal{F}$ the Borelian σ-field.

$\cdot$ $\Omega^* = \Omega_0 \times \overline{I}$, where $\overline{I} = \prod_{k=1}^{+\infty} I_k$, $I_k = [0, 1]$ for each k. Ω^* is endowed with the product σ-field $\mathcal{F}^* = \mathcal{F}_0 \times \mathcal{B}^{\mathbb{N}}$, where $\mathcal{B}$ is the Borelian σ-field on $[0, 1]$. The probability measure on Ω^* is $\mathbb{P}^* = \mathbb{P}_\mu \otimes \lambda^{\otimes \mathbb{N}}$ where λ is the Lebesgue measure on $[0,1]$. Note that under $\mathbb{P}^*$, for any RST T with respect to X, $X \circ \Theta_T$ is independent of $\mathcal{F}_T$ and has the same law as X.

· Similarly, $\widetilde{\Omega} = \Omega_0 \times I$, where $I = [0,1]$, equipped with $\widetilde{\mathcal{F}} = (\mathcal{F}_0 \times \mathcal{B})$, and $\widetilde{\mathbb{P}} = \mathbb{P}_\mu \otimes \lambda$.

In the following, all σ-fields will be assumed to be completed with respect to the probability measures under consideration.

We now appeal to Rost's theorem on Skorohod stopping problem for transient right processes. $x \mapsto U(x,A) = \check{u} * \mathbf{1}_A(x) \in \mathcal{S}(\mu_t)$ for all Borelian A, and so we have, for all $t \geq 0$,

$$\nu_t\, U(A) \leq \delta_0\, U(A)$$

for all A Borelian. Since $\delta_0\, U = U(0,\,.\,)$ is σ-finite, the theorem of Rost ([19], Th.4) entails that for all t, $\nu_t(\,.\,)$ can be seen as the distribution of X stopped at a certain random time: there is a family $(\widetilde{T}_t)_{t\geq 0}$ of canonical RST's such that:

$$\nu_t(\,.\,) = \widetilde{\mathbb{P}}\,[X_{\widetilde{T}_t} \in \,.\,] \quad \forall t \geq 0.$$

Next we build an increasing random walk with step 2^{-n} that serves as "skeleton" for the embedding on X. We just make clear the argument of Bertoin [3], but we need this since later we must deal carefully with the randomization space. Fix n and let R_n be Rost's canonical RST corresponding to $\nu_{2^{-n}}$: $\forall\, f \in \mathcal{E}$,

$$\int_0^1 E\,[f(X_{R_n(\,.\,,z)})]\, dz \;=\; \nu_{2^{-n}}(\,f\,),$$

Consider over Ω^* the following i.i.d random variables: for $\omega^* = (\omega_0, z_1 \ldots z_p \ldots)$,

$$\begin{cases} R_n^1(\omega^*) = & R_n(\omega_0, z_1) \\ R_n^2(\omega^*) = & R_n(\Theta_{R_n^1}\omega_0, z_2) \\ \vdots & \vdots \\ R_n^p(\omega^*) = & R_n(\Theta_{R_n^{p-1}} \circ \ldots \circ \Theta_{R_n^1}\omega_0, z_p) \\ \vdots & \vdots \end{cases}$$

For any $n \geq 1$, the step-process T_n^* defined by:

$$T_n^*(t,\omega^*) = \sum_{i=1}^{k} R_n^i(\omega^*) \qquad \text{if } k\,2^{-n} \leq t < (k+1)\,2^{-n}$$

is, restricted to the dyadics, an increasing random walk such that $T_n^*(t,.)$ is an RST with respect to X for each t. Besides, by homogeneity and independence of the increments of X,

$$\mathbb{P}^*\left[X(T^*_n(1),\,.\,)\in\;.\,\right]\;=\;\nu_1(\,.\,).$$

The following remark are immediate (the second one thinking of $T^*_n(s,.)$ as a natural stopping time for X with an independent parameter):

Remark 1 *Under $\mathbb{P}^*$, for any $n\geq 1$, t, $s>0$ n-dyadic and $m\geq n$, $(T^*_m(\,t+s\,,.)-T^*_m(\,t\,,.))$ is independent of $\sigma(T^*_m(u,.),\ u\leq t)$ and has the same law as $T^*_m(t,.)$.*

Remark 2 *Under $\mathbb{P}^*$, for any n, for any t, $s>0$, $(X_{t+s}-X_t)$ is independent of $\sigma(X_u,\ u\leq t)\vee\sigma(T^*_n(v,.)\leq u,\ u\leq t)$.*

Introducing the following shift on the randomization space: $\theta(z)=\theta(z_1,z_2\ldots)=(z_2,\ldots)$, it is easy to see that for each $n\geq 1$, t n-dyadic, $m\geq n$:

$$(X_{T^*_m(t+2^{-m})}-X_{T^*_m(t)})=X_{R_m}\circ(\Theta_{T^*_m(t)},\theta^{\circ p_m})$$

where p_m is the number of m-dyadics before t. Hence, by the strong Markov property for X (as a Lévy process) and an immediate induction, we get the following reinforcement of Remark 1:

Remark 3 *Under $\mathbb{P}^*$, for any $n\geq 1$, t, s n-dyadic and $m\geq n$, $(X_{T^*_m(t+s)}-X_{T^*_m(t)})$ is independent of $\sigma\{X_{T^*_m(u)},\ u\leq t\}$ and has the same law as $X_{T^*_m(s)}$.*

Observe that in the preceding remark, t need not be dyadic, as soon as s is. Using the same shift-operators, it is a bit tedious but not difficult to observe that the following holds (with the same notations as above):

$$T^*_m(t+i.2^{-m},\omega_0,z)-T^*_m(t,\omega_0,z)=T^*_m(i.2^{-m},\Theta_{T^*_m(t)}(\omega_0),\theta^{\circ p_m}(z))$$

This entails:

$$(X_{T^*_m(t)+.}-X_{T^*_m(t)},T^*_m(t+.)-T^*_m(t))=(X(.),T^*_m(.))\circ(\Theta_{T^*_m(t)},\theta^{\circ p_m})$$

and we can write a last remark which is a reinforcement of Remark 3:

Remark 4 *Under $\mathbb{P}^*$, for any $n,p\geq 1$, $t\geq 0$, $s_1\ldots s_p$ n-dyadic and $m\geq n$, $(X_{T^*_m(t)+.}-X_{T^*_m(t)},T^*_m(t+s_1)-T^*_m(t)\ldots T^*_m(t+s_p)-T^*_m(t))$ is independent of $\sigma\{T^*_m(u),\ u\leq t\}\vee\sigma\{X_{u\wedge T^*_m(t)}\}$ and has the same law as $(X_.,T^*_m(s_1)\ldots T^*_m(s_p))$.*

We now introduce the applications

$$f_n:\begin{cases}\Omega^*\longrightarrow\Omega\\ \omega^*\longmapsto(X(\omega),T^*_n(\omega^*))\end{cases}$$

and the measures $\mathbb{Q}_n$ that they induce on Ω:

$$\mathbb{Q}_n(A)=\mathbb{P}^*[f_n^{-1}(A)]\text{ for each }A\in\mathcal{F}.$$

The next proposition shows tightness. Things here are rather easy since we decided to work on the product space $\Omega_0\times\mathbf{T}$. However we probably need more information on Rost's randomized stopping times to prove a tightness criterion on Ω_0 only, for a family such as $\{X\circ T^*_n\}$.

Proposition 5 *There exists a subsequence $\{\phi_n\}$ such that $\{\mathbb{Q}_{\phi_n}\}$ is tight.*

Proof It is clearly enough to prove that the projections of $\{\mathbb{Q}_{\phi_n}\}$ on Ω^0 and $\mathbf{T}$ are tight. The projection on Ω^0 is constant: the law $\mathbb{P}_\mu$. Besides, one can easily check that for any positive increasing sequence $\{\lambda_n\}$, the set $\{\tau \in \mathbf{T}\mathbf{:}\ \tau(n) \leq \lambda_n\ \forall n \in \mathbb{N}\}$ is relatively compact in $\mathbf{T}$. Thus, we just have to prove that for some subsequence $\{\phi_n\}$,

$$\forall \epsilon > 0,\ \ \exists \{\lambda_p\}\ \text{ such that } \mathbb{P}^*[\exists\, p\ \ /\ \ T^*_{\phi_n}(p) > \lambda_p] < \epsilon \text{ uniformly in } n.$$

Now for each n the Markov chain $\{T^*_{\phi_n}(p)\}_{p\geq 0}$ is a random walk and so it suffices to show that for some subsequence $\{\phi_n\}$:

$$\forall \epsilon > 0,\ \ \exists \lambda\ \text{ such that } \mathbb{P}^*[T^*_{\phi_n}(1) > \lambda] < \epsilon \text{ uniformly in } n.$$

Let $\{\widetilde{T}_n\}$ be the canonical rearrangements of $\{T^*_n(1)\}$: they are RST's with respect to X with randomisation space $[0,1]$, and such that:

$$\mathbb{P}^*[T^*_n > \lambda] = \widetilde{\mathbb{P}}\,[\widetilde{T}_n > \lambda].$$

By Baxter/Chacon's theorem, there exists a canonical RST $\widetilde{T}$ with respect to X and a subsequence $\{\phi_n\}$ such that:

$$\widetilde{E}\,[Z_{\widetilde{T}_{\phi_n}}] \longrightarrow \widetilde{E}\,[Z_{\widetilde{T}}]$$

for any bounded, càdlàg, quasi-left-continuous process Z on $[0,+\infty]$. Suppose $\widetilde{T}$ is finite $\widetilde{\mathbb{P}}$ a.e, then:

$$\forall \epsilon > 0,\ \ \exists \lambda\ \text{ such that } \widetilde{\mathbb{P}}\,[\widetilde{T} > \lambda] < \frac{\epsilon}{2}.$$

Consider the following continuous bounded deterministic process:

$$Z_t = \begin{cases} 0 & \text{if } t \leq \lambda \\ (t-\lambda) & \text{if } \lambda \leq t \leq \lambda + 1 \\ 1 & \text{if } t \geq \lambda + 1. \end{cases}$$

We see that for n sufficiently large,

$$\widetilde{\mathbb{P}}\,[\widetilde{T}_{\phi_n} > \lambda + 1] < \epsilon.$$

Thus, the proof will be complete if $\widetilde{T}$ is finite $\widetilde{\mathbb{P}}$ a.s. Consider $f : \mathbb{R}^d \to \mathbb{R}^+$ bounded continuous tending to 0 at infinity and define over $\Omega_0 \times [0,+\infty]$ the process Z by:

$$Z_t = f(X_t).$$

Obviously, Z is bounded, càdlàg, quasi-left-continuous on $[0,+\infty]$, and we get:

$$\widetilde{E}\,[Z_{\widetilde{T}_{\phi_n}}] \longrightarrow \widetilde{E}\,[Z_{\widetilde{T}}].$$

as $n \to +\infty$. But recall from the construction in discrete time that:

$$\begin{aligned} \widetilde{\mathbb{P}}\,[X_{\widetilde{T}_{\phi_n}} \in\ .\ ] &= \mathbb{P}^*\,[X_{T^*_{\phi_n}(1)} \in\ .\ ] \\ &= \nu_1(\ .\ ) \end{aligned}$$

In particular,

$$\widetilde{E}\,[Z_{\widetilde{T}_{\phi_n}}] = \nu_1(\,f\,) \quad \text{for all } n,$$

which entails:

$$\widetilde{E}\,[Z_{\widetilde{T}}] = \nu_1(\,f\,)$$

This being true for any $f : \mathbb{R}^d \to \mathbb{R}$ bounded continuous tending to 0 at infinity, this entails that under $\widetilde{\mathbb{P}}$, the law of $X_{\widetilde{T}}$ is given by ν_1. Hence we can write:

$$\widetilde{\mathbb{P}}\,[\widetilde{T} = +\infty] = \widetilde{\mathbb{P}}\,[X_{\widetilde{T}} = \partial] = \nu_1(\,\partial\,) = 0$$

and the proof is complete. $\square$

We now consider $\mathbb{Q}$ a cluster point for $\mathbb{Q}_m$ and write $\mathbb{Q}_m \longrightarrow \mathbb{Q}$ for simplicity. Define on $(\Omega, \mathcal{F}, \mathbb{Q})$, for $\omega = (\omega_0, \tau)$:

$$\begin{cases} X(\omega) = \omega_0 \\ \sigma(\omega) = \tau \end{cases}$$

The following lemma is important for the sequel. It is lifted from Monroe ([17] p. 52) but we give a proof for the sake of completeness. Notice that it holds in full generality on $\mathbf{T}$: there is no dependence on the limit measure $\mathbb{Q}$.

Lemma 1 *There exist dense sets of* $\mathbb{R}^+$, $\mathcal{U}$ *and* $\mathcal{V}$ *such that for each* $(u, v) \in \mathcal{U} \times \mathcal{V}$, $\{\tau(u) < v\}$ *is a* $\mathbb{Q}$*-continuity set.*

Proof For the topology we chose over $\mathbf{T}$, the boundary of $\{\tau(u_0) < v\}$ is the set:

$$\{\tau(u_0) \geq v\} \cap \{\tau(u_{0_-}) \leq v\},$$

that is:

$$\{\sup\{u : \tau(u) < v\} \leq u_0 \leq \inf\{u : \tau(u) > v\}\}$$

Now for a fixed u_i the sets depending on v: $\{\sup\{u : \tau(u) < v\} < u_i < \inf\{u : \tau(u) > v\}\}$ (those increasing càdlàg functions having a plateau at value v around u_i), are all disjoint. Thus the following set is countable:

$$\mathcal{V}_1 = \bigcup\nolimits_{u_i \in \mathcal{U}_1} (v : \mathbb{Q}[\{\sup\{u : \tau(u) < v\} < u_i < \inf\{u : \tau(u) > v\}\}] > 0)$$

where $\mathcal{U}_1$ is any dense countable subset of $\mathbb{R}^+$. Thus if $v \notin \mathcal{V}_1$, then for every $u_0 > 0$,

$$\mathbb{Q}[\{\sup\{u : \tau(u) < v\} < u_0 < \inf\{u : \tau(u) > v\}\}] = 0.$$

In particular, if $v \notin \mathcal{V}_1$ and $\{\tau(u_0) < v\}$ is not a $\mathbb{Q}$-continuity set for $\mathbb{Q}$, then

$$\mathbb{Q}[\{\sup\{u : \tau(u) < v\} = u_0 \leq \inf\{u : \tau(u) > v\}\}] > 0,$$

or

$$\mathbb{Q}[\{\sup\{u : \tau(u) < v\} \leq u_0 = \inf\{u : \tau(u) > v\}\}] > 0.$$

But the sets above are disjoint for a fixed v and so for each $v \notin \mathcal{V}_1$ the set

$$\mathcal{U}_v = \{u_0 \ : \ \{\tau(u_0) < v\} \text{ is a } \mathbb{Q}\text{-continuity set}\}$$

is the complement of a countable set. So we take for $\mathcal{V}$ any countable dense set disjoint from $\mathcal{V}_1$ and $\mathcal{U} = \bigcap_{v \in \mathcal{V}} \mathcal{U}_v$.

□

We now study the limit processes obtained from $\mathbb{Q}$:

Proposition 6 *Under $\mathbb{Q}$, σ is a generic subordinator.*

Proof σ is obviously càdlàg increasing, and we show that its increments are independent and stationary.

Let $n,\, p \geq 1$, and $h,\, w,\; v_1 < \ldots < v_p < v_p + h \leq u$, be n-dyadic. Then for $m \geq n$, under $\mathbb{Q}_m$,

$$\int_u^{u+h} [\sigma(t+w) - \sigma(t)]\, dt \;\text{ is independent of }\; \int_{v_1}^{v_1+h} \sigma(t)\, dt, \ldots \int_{v_p}^{v_p+h} \sigma(t)\, dt$$

and has the same law as $\int_0^h [\sigma(t+w) - \sigma(t)]\, dt$. (Indeed, notice that under $\mathbb{Q}_m$, if $h = k2^{-m}$,

$$\begin{cases} \int_u^{u+h} [\sigma(t+w) - \sigma(t)]\, dt = 2^{-m} \left[\sum_{j=0}^{k-1} \sigma(u + w + j2^{-m}) - \sigma(u + j2^{-m})\right] \\ \int_{v_i}^{v_i+h} \sigma(t)\, dt = 2^{-m} \left[\sum_{j=0}^{k-1} \sigma(v_i + j2^{-m})\right] \end{cases}$$

for all i, and use Remark 1). Now $\sigma \mapsto \int_u^{u+h} [\sigma(t+w) - \sigma(t)]\, dt$ is continuous on $\mathbf{T}$ and by weak convergence, we get the same property under $\mathbb{Q}$.

Dividing by h and letting h tend to 0, we get, by right-continuity: for each $w, v_1 < \ldots < v_p < u$ dyadic, under $\mathbb{Q}$,

$$(\sigma(u+w) - \sigma(u)) \;\text{ is independent of }\; \sigma(v_1) \ldots \sigma(v_p)$$

and has the same law as $\sigma(w)$.

Then, again by right-continuity, we can remove the "dyadic" restriction.

□

Consider the following filtration over Ω : $\mathcal{F}_t = \sigma(\{X_s, s \leq t\} \vee \{\sigma_u \leq s, s \leq t\})$. We make it into a complete and right-continuous one, still denoted by $\mathcal{F}_t$, and write $\mathcal{F} = \mathcal{F}_\infty$. Obviously, X is $\mathcal{F}_t$-adapted and σ_u is an $\mathcal{F}_t$-stopping time for each u.

Proposition 7 *X is a Lévy process with respect to $\mathcal{F}_t$, with semi-group $(\mu_t)_{t\geq 0}$.*

Proof $\mathbb{Q}_n \to \mathbb{Q}$ in law over $\Omega_0 \times \mathbf{T}$ and the projection of $\mathbb{Q}_n$ on Ω_0 is $\mathbb{P}_\mu$ for each n. So we check easily that the projection of $\mathbb{Q}$ on Ω_0 is $\mathbb{P}_\mu$: X is a generic Lévy process with semi-group (μ_t). Besides X is $\mathcal{F}_t$-adapted and it is enough to prove that for each $t,\; s > 0$, $X_{t+s} - X_t$ is independent of $\mathcal{F}_t$.

As in Proposition 6, we just need to show that for each $n,\, p \geq 1,\; f, f_1 \ldots f_n$ bounded continuous, $h > 0$, $0 < t_1 < \ldots < t_n < t_n + h \leq t$, $0 < v_1 < \ldots < v_p < t$, $r_1, \ldots, r_p$,

$$\mathbb{Q}\left[f\left(\int_t^{t+h} (X_{u+s} - X_u)\, du\right) \prod_{k=1}^n f_k\left(\int_{t_k}^{t_k+h} X_u\, du\right) \bigcap_{k=1}^p \{\sigma_{r_k} \leq v_k\}\right]$$

$$= \mathbb{Q}\left[f\left(\int_t^{t+h} (X_{u+s} - X_u)\, du\right)\right] \mathbb{Q}\left[\prod_{k=1}^n f_k\left(\int_{t_k}^{t_k+h} X_u\, du\right) \bigcap_{k=1}^p \{\sigma_{r_k} \leq v_k\}\right]$$

Notice that:

$$\{\sigma(r_k) \leq v_k\} = \bigcap_{i=1}^{\infty} \bigcup_{j=1}^{\infty} \{\sigma(r_k^j) < v_k^i\}$$

where $r_k^j \downarrow r_k$ and $v_k^i \downarrow v_k$. Thus by Lemma 1 and a monotone convergence argument, we only need to prove:

$$\mathbb{Q}\,[f(\textstyle\int_t^{t+h}(X_{u+s} - X_u)\,du)\prod_{k=1}^n f_k(\int_{t_k}^{t_k+h} X_u\,du)\bigcap_{k=1}^p\{\sigma_{r_k} < v_k\}]$$

$$= \mathbb{Q}\,[f(\textstyle\int_t^{t+h}(X_{u+s} - X_u)\,du)]\;\mathbb{Q}\,[\prod_{k=1}^n f_k(\int_{t_k}^{t_k+h} X_u\,du)\bigcap_{k=1}^p\{\sigma_{r_k} < v_k\}]$$

where $\{\sigma(r_k) < v_k\}$ are continuity sets for $\mathbb{Q}$. But then,

$$\mathbb{Q}\,[f(\textstyle\int_t^{t+h}(X_{u+s} - X_u)\,du)\prod_{k=1}^n f_k(\int_{t_k}^{t_k+h} X_u\,du)\bigcap_{k=1}^p\{\sigma_{r_k} < v_k\}]$$

$$= \lim_{m\to\infty} \mathbb{Q}_m[f(\textstyle\int_t^{t+h}(X_{u+s} - X_u)\,du)\prod_{k=1}^n f_k(\int_{t_k}^{t_k+h} X_u\,du)\bigcap_{k=1}^p\{\sigma_{r_k} < v_k\}]$$

$$= \lim_{m\to\infty} \mathbb{P}^*[f(\textstyle\int_t^{t+h}(X_{u+s} - X_u)\,du)\prod_{k=1}^n f_k(\int_{t_k}^{t_k+h} X_u\,du)\bigcap_{k=1}^p\{T_m^*(r_k) < v_k\}]$$

$$= \lim_{m\to\infty} \mathbb{P}^*[f(\textstyle\int_t^{t+h}(X_{u+s} - X_u)\,du)]\mathbb{P}^*[\prod_{k=1}^n f_k(\int_{t_k}^{t_k+h} X_u\,du)\bigcap_{k=1}^p\{T_m^*(r_k) < v_k\}]$$

$$= \mathbb{Q}\,[f(\textstyle\int_t^{t+h}(X_{u+s} - X_u)\,du)]\;\mathbb{Q}\,[\prod_{k=1}^n f_k(\int_{t_k}^{t_k+h} X_u\,du)\bigcap_{k=1}^p\{\sigma_{r_k} < v_k\}]$$

In lines 1 and 4 we used that $g : X \mapsto f(\int_t^{t+h}(X_{u+s} - X_u)\,du)$ and $g_k : X \mapsto f_k(\int_{t_k}^{t_k+h} X_u\,du)$ are continuous for the Skorohod topology, and so for any $\mathbb{Q}$-continuity set A, $\mathbb{Q}_m[g.\mathbf{1}_A] \longrightarrow \mathbb{Q}\,[g.\mathbf{1}_A]$ and $\mathbb{Q}_m[g.g_1 \ldots g_n.\mathbf{1}_A] \longrightarrow \mathbb{Q}\,[g.g_1 \ldots g_n.\mathbf{1}_A]$. In line 3, we used Remark 2.

□

We now deal with the subordinate process:

Lemma 2 *For all $t \geq 0$, the law of X_{σ_t} is given by ν_t.*

Proof Fix t dyadic. Since σ is a subordinator, it is a.s continuous at t, and in particular, for all $h > 0$, $f \in \mathcal{C}_b$, the functional:

$$(\omega_0, \tau) \longmapsto \frac{1}{h}\int_0^h f(\omega_0(\tau_t + u))\,du$$

is $\mathbb{Q}$-continuous for the product topology over $\Omega_0 \times \mathbf{T}$. Hence we can write, by weak convergence:

$$\begin{aligned}\mathbb{Q}\,[\frac{1}{h}\int_0^h f(X_{u+\sigma_t})\,du] &= \lim_{m\to\infty} \mathbb{Q}_m[\frac{1}{h}\int_0^h f(X_{u+\sigma_t})\,du] \\ &= \lim_{m\to\infty} \mathbb{P}^*[\frac{1}{h}\int_0^h f(X_{u+T_m^*(t)})\,du]\end{aligned}$$

But now, by the strong Markov property for X, for m sufficiently large,

$$\begin{aligned}\mathbb{P}^*[\frac{1}{h}\int_0^h f(X_{u+T^*_m(t)})\,du] &= \mathbb{P}^*[\frac{1}{h}\int_0^h f(X_{u+T^*_m(t)} - X_{T^*_m(t)} + X_{T^*_m(t)})\,du] \\ &= \frac{1}{h}\int_0^h (\int_{\mathbb{R}^d} \mathbb{P}^*[\,f(x + X_{T^*_m(t)})]\,\mu_u(dx))\,du \\ &= \frac{1}{h}\int_0^h \mu_u * \nu_t(f)\,du\end{aligned}$$

Letting h tend to 0 yields:

$$\mathbb{Q}\,[f(X_{\sigma_t})] = \nu_t(f)$$

for all $f \in \mathcal{C}_b$, t dyadic. This completes the proof by right-continuity. □

For technical reasons, we must now focus on the case when:

$$(\nu_t)_{t\geq 0} \text{ is not the law of a compound Poisson process.}(H)$$

Thanks to Lemma 2, it turns out to be the same for σ:

Lemma 3 *Under (H), $(\sigma_t)_{t\geq 0}$ is not a compound Poisson process.*

Proof Denote by $(\eta_t)_{t\geq 0}$ the semi-group associated with σ and fix $t \geq 0$. We have:

$$\eta_t(\{0\}) = \mathbb{Q}\,[\sigma_t = 0] \leq \mathbb{Q}\,[X_{\sigma_t} = 0] = \nu_t(\{0\})$$

Proposition I.15 in [2] entails that under (H), the potential-measure associated with $(\nu_t)_{t\geq 0}$ is diffuse. In particular, $\nu_t(\{0\}) = 0$ for almost all t and the same for $\eta_t(\{0\})$. But since σ is a subordinator, we have, for all t, $s > 0$:

$$\eta_{t+s}(\{0\}) = \eta_t(\{0\})\eta_s(\{0\}),$$

and hence $\eta_t(\{0\}) = 0$ for *all* t. This entails that σ is not a compound Poisson process. □

Combined with Lemma 2, the next proposition shows that under (H), Y is a realisation of $(\nu_t)_{t\geq 0}$:

Proposition 8 *Suppose (H) holds. Then Y is a Lévy process under $\mathbb{Q}$.*

Proof By right-continuity, it is enough to show that for each n, $p \geq 1$, s, $t, > 0$, $0 < t_1 < \ldots < t_p = t$ n-dyadic, f, $f_1 \ldots f_q \in \mathcal{C}_b$,

$$\mathbb{Q}\,[f(Y_{t+s} - Y_t)\prod_{k=1}^p f_k(Y_{t_k})] \;=\; \mathbb{Q}\,[f(Y_s)]\;\mathbb{Q}\,[\prod_{k=1}^p f_k(Y_{t_k})].$$

But under (H), σ is not a compound Poisson process and hence, its sample paths are $\mathbb{Q}$ a.s *strictly* increasing. Thus, setting:

$$\mathcal{D}(X) = \{t \;\;/\;\; \Delta X_t \neq 0\},$$

a countable set, (H) entails that $\sigma^{-1}(\mathcal{D}(X))$ is $\mathbb{Q}$ a.s countable. In particular, for any $h > 0$, the functionals:

$$(\omega_0, \tau) \longmapsto \frac{1}{h} \int_0^h f(Y_{t+s+u} - Y_{t+u})\, du$$

are $\mathbb{Q}$-continuous for the product topology over $(\Omega_0, \mathbf{T})$. Denoting by $I_h(f, Y_t, Y_{t+s})$ these functionals, and choosing $h < \min\{(t_{k+1} - t_k)\}$ we get, by the continuous mapping theorem:

$$\begin{aligned}
&\mathbb{Q}\,[I_h(f, Y_t, Y_{t+s}) \textstyle\prod_{k=1}^p I_h(f_k, 0, Y_{t_k})] \\
&\qquad = \lim_{m\to\infty} \mathbb{Q}_m[I_h(f, Y_t - Y_{t+s}) \textstyle\prod_{k=1}^p I_h(f_k, 0, Y_{t_k})] \\
&\qquad = \lim_{m\to\infty} \mathbb{P}^*[I_h(f, X_{T^*_m(t)}, X_{T^*_m(t+s)}) \textstyle\prod_{k=1}^p I_h(f_k, 0, X_{T^*_m(t_k)})] \\
&\qquad = \lim_{m\to\infty} \mathbb{P}^*[I_h(f, 0, X_{T^*_m(s)})]\ \mathbb{P}^*[\textstyle\prod_{k=1}^p I_h(f_k, 0, X_{T^*_m(t_k)})] \\
&\qquad = \mathbb{Q}\,[I_h(f, 0, Y_s)]\ \mathbb{Q}\,[\textstyle\prod_{k=1}^p I_h(f_k, 0, Y_{t_k})]
\end{aligned}$$

where the third line comes from Remark 3. By right-continuity, letting h tend to 0 yields:

$$\mathbb{Q}\,[f(Y_{t+s} - Y_t) \textstyle\prod_{k=1}^p f_k(Y_{t_k})] \;=\; \mathbb{Q}\,[f(Y_s)]\ \mathbb{Q}\,[\prod_{k=1}^p f_k(Y_{t_k})].$$

and completes the proof.

□

The next proposition allows us now, under (H), to finish the proof of the direct part of Theorem 1:

Proposition 9 *Suppose (H) holds. (X, σ) has the regeneration property of Definition 2.*

Proof By right-continuity, it is plainly enough to show that for each l, n, p, q, $r \geq 1$, $s_1 \dots s_l$, $v < t$ n-dyadic, $u_1 \dots u_p$, $t_1 < \dots < t_q < t$, $v_1 < \dots < v_r$, $f_1 \dots f_l$, $F_1 \dots F_p$, $g_1 \dots g_q$, $G_1 \dots G_r \in \mathcal{C}_b$,

$$\begin{aligned}
&\mathbb{Q}\,[\textstyle\prod_{k=1}^l f_k(\sigma_{t+s_k} - \sigma_t) \prod_{k=1}^p F_k(X_{\sigma_t+u_k} - X_{\sigma_t}) \dots \\
&\qquad\qquad\qquad \dots \textstyle\prod_{k=1}^q g_k(\sigma_{t_k}) \prod_{k=1}^r G_k(X_{v_k\wedge\sigma_v})] \\
&= \mathbb{Q}\,[\textstyle\prod_{k=1}^l f_k(\sigma_{t+s_k} - \sigma_t) \prod_{k=1}^p F_k(X_{\sigma_t+u_k} - X_{\sigma_t})] \dots \\
&\qquad\qquad\qquad \dots \mathbb{Q}\,[\textstyle\prod_{k=1}^q g_k(\sigma_{t_k}) \prod_{k=1}^r G_k(X_{v_k\wedge\sigma_v})]
\end{aligned}$$

where $\{\sigma_t \geq v\}$ is a $\mathbb{Q}$-continuity set. But, for $h < \min\{(t-s), (t-t_q)\}$, the following functionals:

$$\begin{cases}
I_h(f, X_{\sigma_t+u}, X_{\sigma_t}) : (\omega_0, \tau) \longmapsto \dfrac{1}{h}\ \int_0^h f(X_{\sigma_{t+s}+u} - X_{\sigma_{t+s}})\, ds \\
J_h(f, \sigma_{t+v}, \sigma_t) : (\omega_0, \tau) \longmapsto \dfrac{1}{h}\ \int_0^h f(\sigma_{t+s+v} - \sigma_{t+s})\, ds.
\end{cases}$$

are $\mathbb{Q}$-continuous under (H). Hence we can pass to the limit as above (this time via Remark 4) and get:

$$\begin{aligned}
&\mathbb{Q}\,[\textstyle\prod_{k=1}^{l} J_h(f_k,\sigma_{t+s_k},\sigma_t)\prod_{k=1}^{p} I_h(F_k,X_{\sigma_t+u_k},X_{\sigma_t})\ldots\\
&\qquad\qquad\qquad\ldots\textstyle\prod_{k=1}^{q} J_h(g_k,\sigma_{t_k},0)\prod_{k=1}^{r} I_h(G_k,X_{v_k\wedge\sigma_v},0)]\\
=\ &\mathbb{Q}\,[\textstyle\prod_{k=1}^{l} J_h(f_k,\sigma_{t+s_k},\sigma_t)\prod_{k=1}^{p} I_h(F_k,X_{\sigma_t+u_k},X_{\sigma_t})]\ldots\\
&\qquad\qquad\qquad\ldots\mathbb{Q}\,[\textstyle\prod_{k=1}^{q} J_h(g_k,\sigma_{t_k},0)\prod_{k=1}^{r} I_h(G_k,X_{v_k\wedge\sigma_v},0)]
\end{aligned}$$

Letting h tend to 0 completes the proof. □

We now turn to the case when $(\nu_t)_{t\geq 0}$ is the law of a compound Poisson process. For all $t \geq 0$ we can write:

$$\nu_t = \sum_{n\geq 0} e^{-\lambda t}\frac{(\lambda t)^n \rho^{*n}}{n!}$$

where $\lambda > 0$ and ρ is a probability measure over $\mathbb{R}^d$. In particular, the condition $\mathcal{S}(\mu_t) \subset \mathcal{S}(\nu_t)$ entails:

$$\frac{\nu_t - \delta}{t} * \check{u} \ \leq\ 0 \ \text{ setwise}$$

for all $t \geq 0$, which yields, letting t tend to 0:

$$\rho * \check{u} \ \leq\ \check{u} \ \ \text{setwise.}$$

By Rost's theorem, there exists a canonical RST $\widetilde{T}$ such that:

$$\widetilde{\mathbb{P}}\,[\,X_{\widetilde{T}} \in .\,] = \rho(\,.\,).$$

Using the same arguments as above, we can construct over Ω^* a random walk $\{X_{T_n^*}\}_{n\geq 0}$, with measure ρ. Considering $(N_t^{\lambda*})_{t\geq 0}$ an independent standard Poisson process with parameter λ, we see that the image-measure of:

$$(X_{T^*_{N_t^{\lambda*}}})_{t\geq 0}$$

over $(\Omega_0, \mathbf{T})$ is that of a compound Poisson process, which jumps at exponential times with parameter λ. Its realization is:

$$(X\circ\sigma_t)_{t\geq 0}$$

where σ is a subordinator and $X\circ\sigma$ writes:

$$X\circ\sigma_t = \sum_{k=1}^{N_t^\lambda} \xi_k$$

where $(\xi_k)_{k\geq 0}$ is a family of i.i.d random variables with law ρ and $(N_t^\lambda)_{t\geq 0}$ an independent standard Poisson process with parameter λ. Thus, $X\circ\sigma$ is a Lévy process with semi-group $(\nu_t)_{t\geq 0}$, and taking over $(\Omega_0, \mathbf{T})$ the same filtration $\mathcal{F}_t$ as in the general case yields the 3 first properties of Definition 2.

To show the regeneration property, we first observe that, as in Remark 4, the increment $(X_{T_n^*+u}-X_{T_n^*}, T_{n+p}^*-T_n^*)$ is independent of $\sigma\{T_k^*, k\leq n\}\vee\sigma\{X_{u\wedge T_n^*}, u\geq 0\}$ and has the same law as (X_u, T_p^*) for all (t,p). Considering $(N_t^{\lambda^*})_{t\geq 0}$ and $(P_t^{\lambda^*})_{t\geq 0}$ two independent standard Poisson process with parameter λ and writing $n=N_t^{\lambda^*}$, $p=P_v^{\lambda^*}$ yield :

$$(X_{\sigma_t+u}-X_{\sigma_t}, \sigma_{t+v}-\sigma_t)_{u,v\geq 0}$$

is independent of $\mathcal{G}_t$ and has the same law as $(X_u, \sigma_v)_{u,v\geq 0}$ where $(\mathcal{G}_t)_{t\geq 0}$ is the filtration appearing in Definition 2.

We end this subsection in stressing that the regeneration property may not hold with the filtration $(\mathcal{F}_{\sigma_t})_{t\geq 0}$ instead of $(\mathcal{G}_t)_{t\geq 0}$ (which would simplify the form of Definition 2): it is easy to see that $\mathcal{G}_t\subset\mathcal{F}_{\sigma_t}$ for all t, but the converse inclusion is false, since for example σ is not necessarily a Lévy process with respect to $\mathcal{F}_{\sigma_t}$.

4.2 Proof of the converse part of Theorem 1

Before proceeding to the proof, we observe that in the case of independent subordination, the result is easily obtained via analytic considerations. Indeed the subordinate semi-group $(\nu_t)_{t\geq 0}$ writes in this case:

$$\nu_t=\int_0^{+\infty}\mu_s\,\eta_t(ds)$$

where $(\eta_t)_{t\geq 0}$ is the subordination semi-group over $\mathbb{R}^+$. So the Lévy generator of $(\nu_t)_{t\geq 0}$ is:

$$S=\lim_{t\to 0}\frac{\displaystyle\int_0^{+\infty}\mu_s\,\eta_t(ds)-\delta}{t}=\int_0^{+\infty}\mu_s\,(\lim_{t\to 0}\frac{\eta_t-\delta}{t})(ds)$$

By the Lévy-Khintchine formula for subordinators, we have:

$$\lim_{t\to 0}(\frac{\eta_t-\delta}{t})=-a\delta'+\int_0^{+\infty}(\delta_t-\delta)\,\pi(dt)$$

where a is the drift coefficient and π the Lévy measure of σ. Thus:

$$\begin{aligned}S &= aT+\int_0^{+\infty}(\mu_t-\delta)\,\pi(dt)\\ &= aT+\int_0^{+\infty}(T*[\int_0^t\mu_s\,ds])\,\pi(dt)\end{aligned}$$

Since $(\mu_t)_{t\geq 0}$ is transient we can apply Fubini's theorem:

$$S\;=\;T*(a\delta+\int_0^{+\infty}[\int_0^t\mu_s\,ds]\,\pi(dt))$$

Hence:

$$S\;=\;T*K$$

where K the following positive Radon measure over $\mathbb{R}^d$:

$$K(dx) \;=\; a\delta(dx) + \int_0^{+\infty} [\int_0^t \mu_s(dx)\, ds]\, \pi(dt) \;=\; a\delta(dx) + \int_0^{+\infty} \mu_t(dx)\, \overline{\pi}(t) dt.$$

where $\overline{\pi}$ is the tail of the Lévy measure. Since $\overline{\pi}$ tend to 0 at infinity and u is shift-bounded (see Prop 13.7 in [1]), it is easy to see that K is in fact an element of $\mathcal{M}_0^+$. Hence we can apply Proposition 3 and get:

$$u \;=\; v * K$$

In the non-independent case, the method is different. First we need a lemma about the graph of subordinators whose statement is rather obvious:

Lemma 4 *Let g be a Borel function: $[0, +\infty) \longrightarrow \mathbb{C}$ such that $\int_0^{+\infty} |g(t)|\, dt \; < +\infty$, and f be the graph of a strict subordinator with drift coefficient $a \geq 0$. Then:*

$$\int_0^{+\infty} g(t)\, dt \;=\; a \int_0^{+\infty} g(f_s)\, ds \;+\; \sum_{u>0} \int_{f_{u-}}^{f_u} g(s)\, ds.$$

Proof It is clearly enough to show the lemma for $g = \mathbf{1}_{[0,u]},\ u \in \mathbb{R}^+$. Recall that f is càdlàg increasing, $f_0 = 0$, $f_t \to +\infty$ as $t \to +\infty$, and that f admits a Lévy-Itô decomposition:

$$f_s = as + \sum_{u \leq s} \Delta f_u$$

where $\Delta f_u = f_u - f_{u-}$ for all $u > 0$. In particular, if $\mathcal{R}$ stands for the closed range of f on $\mathbb{R}^+$, then:

$$\mathcal{R}^c \;=\; [0, \infty) - \mathcal{R} \;=\; \bigcup_{u \in \mathcal{I}} (f_{u-}, f_u)$$

where $\mathcal{I}$ denotes the jump times of f. Hence we can write:

$$\begin{aligned} \int_0^{+\infty} g(t)\, dt \;&=\; \int_{\mathcal{R}} g(s)\, ds \;+\; \sum_{u>0} \int_{f_{u-}}^{f_u} g(s)\, ds \\ &=\; m(\mathcal{R} \cap [0, u]) + \sum_{u>0} \int_{f_{u-}}^{f_u} g(s)\, ds \end{aligned}$$

where m is the Lebesgue measure. Setting $t = \inf\{s \,/\, f_s > u\}$, we get:

$$m(\mathcal{R} \cap [0, u]) = m(\mathcal{R} \cap [0, f_t]) = at = a \int_0^{+\infty} g(f_s)\, ds$$

where the second equality comes from the Lévy-Itô decomposition. This completes the proof.

$\square$

We now finish the proof of the converse part of Theorem 1. The method is very similar to that of Song [21], who needed however stronger assumptions.

Consider the space-time processes (t, X_t) and (σ_t, X_{σ_t}). By assumption they are both transient Lévy processes and we denote by $\tilde{u}$ and $\tilde{v}$ their potential-measures. Set, for $\lambda > 0$ and $\xi \in \mathbb{R}^d$,

$$f_{\lambda,\xi}(t, x) = \exp(-\lambda t + \mathrm{i}\xi.x).$$

Let a be the drift coefficient of σ. Thanks to Lemma 1, we can write:

$$\begin{aligned} < \tilde{u}, f_{\lambda,\xi} > &= E\,[\int_0^\infty \exp(-\lambda t + \mathrm{i}\xi.X_t)\, dt] \\ &= aE\,[\int_0^\infty \exp(-\lambda\sigma_s + \mathrm{i}\xi.X_{\sigma_s})\, ds] + E\,[\sum_{u>0}\int_{\sigma_{u-}}^{\sigma_u} \exp(-\lambda t + \mathrm{i}\xi.X_t)\, dt] \\ &= a < \tilde{v}, f_{\lambda,\xi} > + E\,[\sum_{u>0}\int_{\sigma_{u-}}^{\sigma_u} \exp(-\lambda t + \mathrm{i}\xi.X_t)\, dt] \end{aligned}$$

As in the preceding subsection we first consider the case where σ is not a compound Poisson process. We then have:

$$\begin{aligned} E\,[\sum_{u>0}\int_{\sigma_{u-}}^{\sigma_u} \exp(-\lambda t + \mathrm{i}\xi.X_t)\, dt] &= E\,[\sum_{u>0}\int_{\sigma_{u-}}^{\sigma_u} \exp(-\lambda t + \mathrm{i}\xi.X_{t-})\, dt] \\ = E\,[\sum_{u>0} \exp(-\lambda\sigma_{u-} + \mathrm{i}\xi.X_{\sigma_{u-}}) & \int_0^{\sigma_u - \sigma_{u-}} \exp(-\lambda t + \mathrm{i}\xi.(X_{(\sigma_{u-}+t)-} - X_{\sigma_{u-}}))\, dt] \\ = E\,[\sum_{u>0} \exp(-\lambda\sigma_{u-} + \mathrm{i}\xi.X_{(\sigma_{u-})-}) & \int_0^{\sigma_u - \sigma_{u-}} \exp(-\lambda t + \mathrm{i}\xi.(X_{\sigma_{u-}+t} - X_{(\sigma_{u-})-}))\, dt]. \end{aligned}$$

Consider now the following point process $(e_s)_{s\geq 0}$, valued in the space $\mathbb{D}$:

$$\text{if } \sigma_u \neq \sigma_{u-},\quad e_u(r) = \begin{cases} X_{\sigma_{u-}+r} - X_{(\sigma_{u-})-} & \text{if } r \leq \sigma_u - \sigma_{u-} \\ \partial & \text{otherwise.} \end{cases}$$

Define, for $x \in \mathbb{D}$, $\zeta(x) = \inf\{t > 0 \,/\, x(t) = \partial\,\}$ and set:

$$\mathbb{D}^{(a)} = \{x \in \mathbb{D} \text{ such that } \zeta(x) \geq a\}.$$

Restricted to $\mathbb{D}^{(a)}$, e is discrete for any $a > 0$, and hence, e is σ-discrete. For any B in the Borelian σ-field of $\mathbb{D}$, consider the counting measure:

$$N_t^B = \mathrm{card}\{0 < u \leq t : e_u \in B\} = F(X_u,\ u \leq \sigma_t).$$

It follows from the regeneration property of (X, σ) that for any t, $s > 0$, its increment:

$$N_{t+s}^B - N_t^B = \mathrm{card}\{t < u \leq t+s : e_u \in B\} = F(X_{\sigma_t+u} - X_{\sigma_t},\ u \leq \sigma_{t+s} - \sigma_t)$$

is independent of $\mathcal{G}_t$ and have the same law as N_t^B, where $(\mathcal{G}_t)_{t\geq 0}$ is given in Definition 2: $N_\cdot^B$ is an $\mathcal{G}_t$-adapted standard Poisson process and e is a $\mathcal{G}_t$-*Poisson* point process.

Since $u \mapsto (\sigma_{u-}, X_{(\sigma_{u-})-}) = (\sigma_{u-}, Y_{u-})$ is $\mathcal{G}_t$-predictable (it is left-continuous and $\mathcal{G}_t$-adapted), we can apply the Compensation Formula:

$$
\begin{aligned}
&E\,[\sum_{u>0} \int_{\sigma_{u-}}^{\sigma_u} \exp(-\lambda t + \mathrm{i}\xi.X_t)\,dt] \\
&\qquad = \; E\,[\int_0^\infty \exp(-\lambda\sigma_s + \mathrm{i}\xi.X_{(\sigma_s)-})\,ds]\,[\int n(de) \int_0^\zeta \exp(-\lambda t + \mathrm{i}\xi.e_t)\,dt] \\
&\qquad = \; E\,[\int_0^\infty \exp(-\lambda\sigma_s + \mathrm{i}\xi.X_{\sigma_s})\,ds]\,[\int n(de) \int_0^\zeta \exp(-\lambda t + \mathrm{i}\xi.e_t)\,dt]
\end{aligned}
$$

since σ is $\mathbb{Q}$ a.e strictly increasing, and where n is the characteristic measure of e.

When σ is a compound Poisson process, we can get an analogous formula with the following $\mathcal{G}_t$-Poisson point process:

$$
\text{if } \sigma_u \neq \sigma_{u-}, \quad e_u(r) = \begin{cases} X_{\sigma_{u-}+r} - X_{\sigma_{u-}} & \text{if } r \leq \sigma_u - \sigma_{u-} \\ \partial & \text{otherwise.} \end{cases}
$$

because here $(\sigma_{u-}, X_{\sigma_{u-}})$ is left-continuous. Hence we can write:

$$
< \tilde{u}, f_{\lambda,\xi} > = < \tilde{v}, f_{\lambda,\xi} >< \tilde{K}, f_{\lambda,\xi} >
$$

where $\tilde{K}$ is the positive measure over $\mathbb{R}^+ \times \mathbb{R}^d$ defined by:

$$
\tilde{K}(dt, dx) = a\delta(dt, dx) + \int n(de) \int_0^\zeta \mathbf{1}_{\{v \in dt, e(v) \in dx\}}\,dv.
$$

Note that for any $\lambda > 0$,

$$
< \tilde{v}, f_{\lambda,0} > = \frac{1}{\phi(\lambda)} < +\infty \quad \text{and} \quad < \tilde{K}, f_{\lambda,0} > = \frac{\phi(\lambda)}{\lambda} < +\infty.
$$

where ϕ is the Laplace exponent of σ. In particular, $\tilde{K}$ is a *Radon* measure and we can define formally $\tilde{v} * \tilde{K}$. Applying twice Fubini's theorem yields:

$$
< \tilde{u}, f_{\lambda,\xi} > = < \tilde{v} * \tilde{K}, f_{\lambda,\xi} >
$$

By density of the $f_{\lambda,\xi}$, we get:

$$
\tilde{u} \;=\; \tilde{v} * \tilde{K}
$$

Now since $u = \tilde{u}(\mathbb{R}^+, .)$ is a Radon measure over $\mathbb{R}^d$ we can integrate over $\mathbb{R}^+$ and get:

$$
u \;=\; v * K
$$

where K is a positive measure over $\mathbb{R}^d$:

$$
K(dx) = a\delta(dx) + \int n(de) \int_0^\zeta \mathbf{1}_{\{e(v) \in dx\}}\,dv.
$$

K is necessarily a Radon measure as u, and the proof is complete.

$\square$

This shape of K allows us to state more precisely the loss of information from Bochner's subordination to subordination in the wide sense. Indeed, $n[\zeta > a] = \pi[a; +\infty)$ where π is the jump-measure of σ. So n can be thought as a measure over $\mathbb{R}^+ \times \Omega_0$ whose first marginal is π, and K writes:

$$K(dx) = a\delta(dx) + \int n(dt, d\omega_0) \int_0^t \mathbf{1}_{\{\omega_0(s) \in dx\}} \, ds.$$

In the case of independent subordination, n is the product measure $\pi(dt)\mathbb{P}^t_\mu(d\omega_0)$, where $\mathbb{P}^t_\mu$ is the law of X killed at time t. This entails:

$$\begin{aligned} K &= a\delta + \int_0^{+\infty} [\int_0^t \mathbb{P}^t_\mu \left[\mathbf{1}_{\{\omega_0(s) \in .\}}\right] ds] \, \pi(dt) \\ &= a\delta + \int_0^{+\infty} [\int_0^t \mu_s \, ds] \, \pi(dt) \end{aligned}$$

which is the formula above, obtained via different means.

In the general case, n is no more a product measure, since there is a coupling between X and the random part of σ: its jumps.

We end this section with a counter-example when the carrier process is recurrent. Let $(B_t)_{t \geq 0}$ be the standard Brownian motion on $\mathbb{R}$, and $(T_t)_{t \geq 0}$ the process of its first passage times:

$$T_t = \inf\{s \;/\; B_s > t\}.$$

T is a stable-$(\frac{1}{2})$ subordinator and if we consider Y another subordinator, independent of B, then $\sigma_t = T \circ Y_t$ defines again a subordinator. Take for probability space $\mathbf{W} \times \mathbf{T}$ where $\mathbf{W}$ is the Wiener-space and $\mathbf{T}$ as in section 4, and endow it with $(\mathcal{H}_t = \mathcal{F}_t \vee \mathcal{G})_{t \geq 0}$ where $(\mathcal{F}_t)_{t \geq 0}$ is the natural brownian filtration and $\mathcal{G}$ the completed borelian σ-field of $\mathbf{T}$. Obviously B is a Lévy process with respect to $\mathcal{H}_t$ and σ_s a $\mathcal{H}_t$-stopping-time for all s. Besides, by continuity of the brownian paths, for all $t \geq 0$,

$$B_{\sigma_t} = Y_t.$$

Finally, defining our processes on the product canonical space $\Omega \times \Omega'$, where Ω is Wiener space and Ω' is $\mathbf{T}$, it is easy to see that the following holds for all (u, v):

$$(B_{\sigma_t + u} - B_{\sigma_t}, \sigma_{t+v} - \sigma_t)(\omega, \omega') \;=\; (B_u, \sigma_v)(\Theta_{\sigma_t}(\omega), \Theta_t(\omega'))$$

which entails the regeneration property (integrating first on ω, then on ω'). Hence, Y is a *subordination in the wide sense* of B.

Suppose now Y has no drift coefficient and that his Lévy measure is a probability π on $\mathbb{R}^+$ such that $\int_0^{+\infty} t\pi(dt) = +\infty$. Then the Lévy generator of Y writes:

$$S \;=\; \pi - \delta$$

But the Lévy generator of B is:

$$T \;=\; \frac{\delta''}{2},$$

and if K is a positive Radon measure such that $T * K = S$, then:

$$K'' \;=\; 2(\pi - \delta)$$

By direct integration,

$$K(x) = 2\left[x^{+}(\pi[0,x]-1) - \int_0^{x^+} t\pi(dt) + ax + b\right]$$

and it is not difficult to see that K cannot be everywhere positive.

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