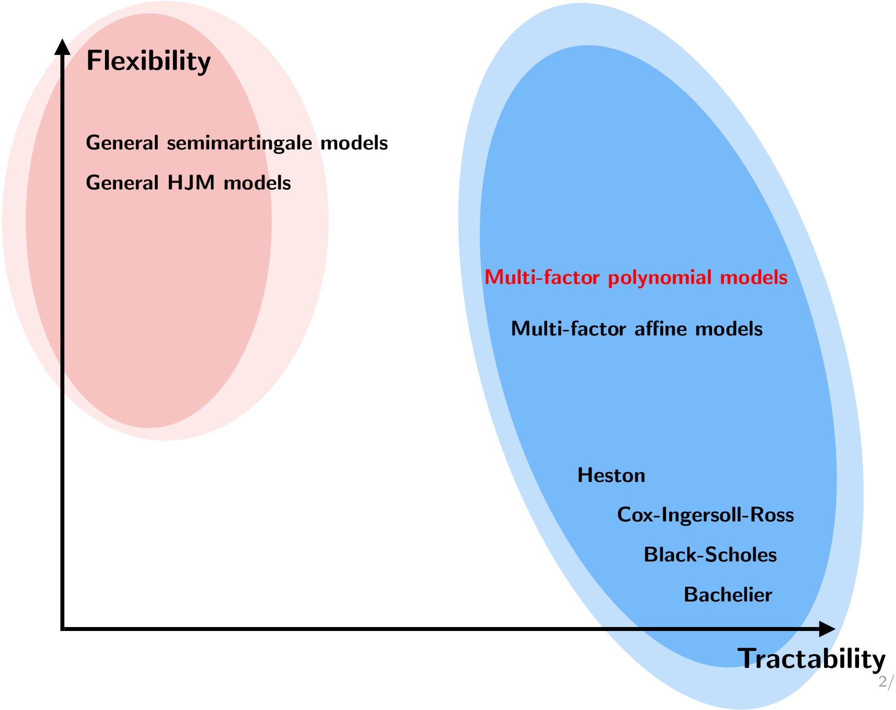
Polynomial Models in Finance

Martin Larsson

Department of Mathematics, ETH Zurich

Winter School: Frontiers in Stochastic Modelling for Finance
Padova, 8 February 2016

D MATH ETH zürich


Flexibility
General semimartingale models
General HJM models
Multi-factor polynomial models
Multi-factor affine models
Heston
Cox-Ingersoll-Ross
Black-Scholes
Bachelier
Tractability

- We want tractable stochastic models that are flexible enough to describe reality up to a satisfactory degree of accuracy.
- Polynomial preserving processes is one such class of models
- The analysis comes in two main parts:
  (1) **Theoretical study of polynomial preserving processes:** This leads to a rich set of mathematical questions involving probability as well as geometry and algebra (semi-algebraic geometry, sums of squares, the Nullstellensatz, etc.)
  (2) **Financial modeling:** Construct models that exploit the tractable structure of polynomial preserving processes.
- The two main references for this mini-course are:
  - **[FL16]:** Polynomial preserving diffusions and applications in finance (with D. Filipović), forthcoming in Fin. Stochastics.
  - **[FLT16]:** Linear-rational term structure models (with D. Filipović and A. Trolle), forthcoming in Journal of Finance.
- ... but some material is drawn from other places or is not yet available in the literature.

Polynomial preserving processes

- Definition and general characterization
- Basic properties
- Existence and uniqueness
- Examples

Applications in finance

- Overview
- State price density models
- Polynomial term structure models

Conclusions and outlook

Polynomial preserving processes

- **Definition and general characterization**
- Basic properties
- Existence and uniqueness
- Examples

Polynomial preserving processes

- State space $E \subseteq \mathbb{R}^d$
- $X = (X_t)_{t\geq 0}$ an E-valued semimartingale with extended generator

$$\mathscr{G} f(x) = b(x)^\top \nabla f(x) + \frac{1}{2} \operatorname{Tr}\left(a(x)\nabla^2 f(x)\right) + \int_{\mathbb{R}^d} \Big(f(x+\xi) - f(x) - \xi^\top \nabla f(x)\Big)\nu(x, d\xi)$$

Meaning: $f(X_t) - f(X_0) - \int_0^t \mathscr{G} f(X_s)ds \;=\;$ local martingale $\qquad (*)$

- Domain: $\operatorname{dom}(\mathscr{G}) = \left\{f \in C^2(\mathbb{R}^d)\colon\ (*) \text{ holds}\right\}$

Example. If X satisfies an SDE of the form

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t$$

then $b \equiv \mu$, $a \equiv \sigma\sigma^\top$, $\nu \equiv 0$, and $(*)$ is just Itô's formula.

Polynomial preserving processes

Remark. Existence of $\mathscr{G}$ implies that X has absolutely continuous characteristics whose densities are deterministic functions of the current state.

$\Longrightarrow$ X should "morally" be a Markov process.

Warning: X is not always a Markov process!

Assumption (A): For all $n \geq 1$, $\mathbb{E}[\|X_0\|^{2n}] < \infty$ and there exists $K_n < \infty$ such that

$$\int_{\mathbb{R}^d} \|\xi\|^{2n} \nu(x, d\xi) \leq K_n(1 + \|x\|^{2n}), \qquad x \in E.$$

Moreover, $\mathscr{G}$ is well-defined on E: $f\big|_E = 0$ implies $\mathscr{G} f\big|_E = 0$.

Definition of polynomial preserving processes

- Multi-indices, monomials and their degree:

$$\boldsymbol{k} = (k_1, \ldots, k_d) \in \mathbb{N}_0^d, \qquad x^{\boldsymbol{k}} = x_1^{k_1} x_2^{k_2} \cdots x_d^{k_d}, \qquad |\boldsymbol{k}| = \sum_i k_i$$

- Spaces of polynomials:

$$\mathrm{Pol}_n(E) = \left\{ p|_E : p \text{ is polynomial on } \mathbb{R}^d \text{ of degree} \leq n \right\}$$

- Assumption (A) implies $(*)$ holds for all $p \in \mathrm{Pol}_n(E)$: $p \in \mathrm{dom}(\mathscr{G})$

Definition. We call $\mathscr{G}$ **polynomial preserving (PP)** if

$$\mathscr{G}\mathrm{Pol}_n(E) \subseteq \mathrm{Pol}_n(E) \quad \text{for all } n \geq 1.$$

In this case X is called a polynomial preserving process.

Characterization of (PP) generators

Lemma. The extended generator

$$\mathscr{G}f(x) = b(x)^\top \nabla f(x) + \frac{1}{2}\operatorname{Tr}\left(a(x)\nabla^2 f(x)\right) + \int_{\mathbb{R}^d} \Big(f(x+\xi) - f(x) - \xi^\top \nabla f(x)\Big)\nu(x, d\xi)$$

is (PP) if and only if for all i, j,

$$b_i(x) \in \mathrm{Pol}_1(E) \qquad \text{(drift)}$$

$$a_{ij}(x) + \int_{\mathbb{R}^d} \xi_i \xi_j \nu(x, d\xi) \in \mathrm{Pol}_2(E) \qquad \text{(modified diffusion)}$$

$$\int_{\mathbb{R}^d} \xi^{\boldsymbol{k}} \nu(x, d\xi) \in \mathrm{Pol}_{|\boldsymbol{k}|}(E), \quad \forall |\boldsymbol{k}| \geq 3 \qquad \text{(jumps)}$$

Proof: Evaluate $\mathscr{G}p$ for polynomials p, collect and match terms. □

First examples of (PP) processes

The lemma immediately yields several examples of (PP) processes:

Example. The following processes are (PP):

- Ornstein-Uhlenbeck processes: $dX_t = \kappa(\theta - X_t)dt + \sigma dW_t$
- Geometric Brownian motion: $dX_t = \mu X_t dt + \sigma X_t dW_t$
- Square-root diffusions: $dX_t = \kappa(\theta - X_t)dt + \sigma\sqrt{X_t}dW_t$
- Jacobi diffusions: $dX_t = \kappa(\theta - X_t)dt + \sigma\sqrt{X_t(1 - X_t)}dW_t$
- Dunkl processes: $E = \mathbb{R}$ with extended generator

$$\mathscr{G}f(x) = f''(x) + \frac{\lambda}{2x}\int_{\mathbb{R}}\Big(f(x+\xi) - f(x) - \xi f'(x)\Big)\delta_{-2x}(d\xi)$$

- Any affine semimartingale satisfying Assumption (A)

... but we want a larger class of examples, and more information about their properties. Specifically:

Main questions

- If a (PP) process X is given a priori, what can be said in general about its properties?
- What about existence and uniqueness of (PP) processes on various state spaces E of interest? More specifically, we would like convenient parameterizations.

Closely related literature:

Wong (1964); Mazet (1997); Zhou (2003); Forman and Sørensen (2008); Cuchiero, Keller-Ressel, Teichmann (2012); Filipović, Gourier, Mancini (2013); Bakry, Orevkov, Zani (2014); Larsson, Pulido (2015); Larsson, Krühner (2016); etc.

Polynomial preserving processes

- Definition and general characterization
- **Basic properties**
- Existence and uniqueness
- Examples

Basic properties: Conditional moments

Given: (PP) process X, extended generator $\mathscr{G}$, satisfies Assumption (A).

Lemma. For any polynomial p on $\mathbb{R}^d$,

$$M_t^p = p(X_t) - p(X_0) - \int_0^t \mathscr{G} p(X_s) ds$$

is a (true) martingale.

Proof: Assumption (A) implies $p \in \mathrm{dom}(\mathscr{G})$, so M^p is a local martingale.

Assumption (A) and BDG imply $\sup_{t \leq T} |M_t^p|$ integrable, for any T. See for instance Lemma 2.17 in Cuchiero et al. (2012).

Hence M_t^p is a martingale since $\sup_{t \leq T} |M_t^p|$ integrable. □

Basic properties: Conditional moments

- Fix $n \in \mathbb{N}$ and set $N = \dim \mathrm{Pol}_n(E) < \infty$
- By definition of (PP), $\mathscr{G}$ restricts to an operator $\mathscr{G}|_{\mathrm{Pol}_n(E)}$ on the finite-dimensional vector space $\mathrm{Pol}_n(E)$
- Find a basis $h_1(x), \dots, h_N(x)$ of $\mathrm{Pol}_n(E)$ and denote
$$H(x) = (h_1(x), \dots, h_N(x))^\top$$
- Coordinate representation $\vec{p} \in \mathbb{R}^N$ of $p \in \mathrm{Pol}_n(E)$:
$$p(x) = H(x)^\top \vec{p}.$$
- Matrix representation $G \in \mathbb{R}^{N \times N}$ of $\mathscr{G}|_{\mathrm{Pol}_n(E)}$:
$$\mathscr{G} p(x) = H(x)^\top G \vec{p}.$$

Basic properties: Conditional moments

Theorem. For any $p \in \mathrm{Pol}_n(E)$ with coordinate vector $\vec{p} \in \mathbb{R}^N$,

$$\mathbb{E}[p(X_T) \mid \mathscr{F}_t] = H(X_t)^\top e^{(T-t)G} \vec{p}$$

is an explicit polynomial in X_t of degree $\leq n$, for all $t \leq T$.

Basic properties: Conditional moments

Theorem. For any $p \in \mathrm{Pol}_n(E)$ with coordinate vector $\vec{p} \in \mathbb{R}^N$,

$$\mathbb{E}[p(X_T) \mid \mathscr{F}_t] = H(X_t)^\top e^{(T-t)G} \vec{p}$$

is an explicit polynomial in X_t of degree $\leq n$, for all $t \leq T$.

Proof. By definition $\mathscr{G} H(x) = G^\top H(x)$. Thus for N-dim local mg M,

$$H(X_u) = H(X_t) + \int_t^u G^\top H(X_s) ds + M_u - M_t, \qquad u \geq t.$$

Lemma implies M is true martingale. Thus with $F(u) = \mathbb{E}[H(X_u) \mid \mathscr{F}_t]$,

$$F(u) = H(X_t) + \int_t^u G^\top F(s) ds.$$

Hence $\mathbb{E}[H(X_T) \mid \mathscr{F}_t] = F(T) = e^{(T-t)G^\top} H(X_t)$. □

Basic properties: Conditional moments

Theorem. For any $p \in \mathrm{Pol}_n(E)$ with coordinate vector $\vec{p} \in \mathbb{R}^N$,

$$\mathbb{E}[p(X_T) \mid \mathscr{F}_t] = H(X_t)^\top e^{(T-t)G}\vec{p}$$

is an explicit polynomial in X_t of degree $\leq n$, for all $t \leq T$.

Punchline:

- Conditional expectations of polynomials are explicit.
- Computing them only requires calculating a matrix exponential ...
- ... which should be contrasted with solving a PIDE.

Example: The scalar diffusion case

Generic scalar (PP) diffusion: $E \subseteq \mathbb{R}$,

$$dX_t = (b + \beta X_t)dt + \sqrt{a + \alpha X_t + A X_t^2}\, dW_t$$

Standard basis $\{1, x, x^2, \ldots, x^n\}$ of Pol_n:

$$p(x) = \sum_{k=0}^{n} p_k x^k \qquad \longleftrightarrow \qquad \vec{p} = (p_0, \ldots, p_n)^\top$$

Then: Matrix representation $G \in \mathbb{R}^{(n+1)\times(n+1)}$ of $\mathscr{G}$ is

$$G = \begin{pmatrix} 0 & b & 2\frac{a}{2} & 0 & \cdots & 0 \\ 0 & \beta & 2\left(b + \frac{\alpha}{2}\right) & 3 \cdot 2\frac{a}{2} & 0 & \vdots \\ 0 & 0 & 2\left(\beta + \frac{A}{2}\right) & 3\left(b + 2\frac{\alpha}{2}\right) & \ddots & 0 \\ 0 & 0 & 0 & 3\left(\beta + 2\frac{A}{2}\right) & \ddots & n(n-1)\frac{a}{2} \\ \vdots & & & 0 & \ddots & n\left(b + (n-1)\frac{\alpha}{2}\right) \\ 0 & & \cdots & & 0 & n\left(\beta + (n-1)\frac{A}{2}\right) \end{pmatrix}$$

Example: Scalar Lévy case

Suppose

$$a(x) \equiv b(x) \equiv 0 \qquad \text{and} \qquad \nu(x, d\xi) = \mu(d\xi)$$

for some measure $\eta(d\xi)$ on $\mathbb{R} \setminus \{0\}$ such that

$$\int \xi^k \mu(d\xi) < \infty, \qquad k \geq 2.$$

Then: X is a Lévy process and G is given by

$$G = \begin{pmatrix}
0 & 0 & \int \xi^2 \mu(d\xi) & \int \xi^3 \mu(d\xi) & \int \xi^4 \mu(d\xi) & \cdots & \binom{n}{0} \int \xi^n \mu(d\xi) \\
0 & 0 & 0 & 3 \int \xi^2 \mu(d\xi) & 4 \int \xi^3 \mu(d\xi) & & \vdots \\
0 & 0 & 0 & 0 & 6 \int \xi^2 \mu(d\xi) & \ddots & \\
 & & & \ddots & 0 & \ddots & \binom{n}{n-3} \int \xi^3 \mu(d\xi) \\
\vdots & & & & & \ddots & \binom{n}{n-2} \int \xi^2 \mu(d\xi) \\
 & & & & \vdots & \ddots & 0 \\
0 & \cdots & & \cdots & 0 & 0 & 0
\end{pmatrix}$$

Basic properties: New (PP) processes from old

- If $X = (X^1, \dots, X^d)$ is (PP) then

$$(X_t, \int_0^t X_s^1 ds)$$

is (PP) on the state space $E \times \mathbb{R}$.

- More generally, let $p, q \in \mathrm{Pol}_n(E)$. Define

$$\begin{aligned} \overline{X}_t &= H(X_t) \\ Y_t &= \int_0^t p(X_s) ds + \int_0^t \sqrt{q(X_s)} dW_s \end{aligned}$$

with $W \perp\!\!\!\perp X$ a Brownian motion. Then:

$$(\overline{X}, Y) \text{ is (PP) on } H(E) \times \mathbb{R} \subseteq \mathbb{R}^{N+1}.$$

- More general results hold, where Y also can have jumps.

Basic properties: New (PP) processes from old

- The proof of these statements relies on the following lemma:

Lemma. Let $k \in \mathbb{N}$. Then

$$p \in \mathrm{Pol}_{kn}(\mathbb{R}^d) \quad \Longleftrightarrow \quad p(x) = f(H(x)) \text{ for some } f \in \mathrm{Pol}_k(\mathbb{R}^N)$$

Polynomial preserving processes

- Definition and general characterization
- Basic properties
- **Existence and uniqueness**
- Examples

Existence of (PP) diffusions

- So far we have taken a (PP) process X as given a priori.
- **Question:** Which pairs $(E, \mathscr{G})$ of candidate state space and generator admit a corresponding (PP) process X?

Setup (I): Consider operator $\mathscr{G}$ of diffusion type:

$$\mathscr{G}f(x) = b(x)^\top \nabla f(x) + \frac{1}{2} \operatorname{Tr}\left(a(x)\nabla^2 f(x)\right)$$

with (see Lemma characterizing (PP) generators):

$$b_i \in \mathrm{Pol}_1, \qquad a_{ij} \in \mathrm{Pol}_2$$

Existence of (PP) diffusions

Setup (II): Consider basic closed semialgebraic state space:

$$E = \{x \in \mathbb{R}^d \ : \ p(x) \geq 0 \text{ for all } p \in \mathscr{P}\}$$

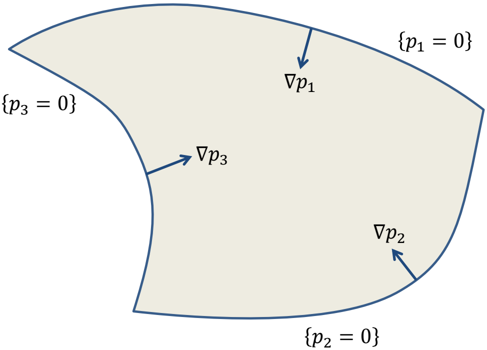
with $\mathscr{P}$ a finite collection of polynomials on $\mathbb{R}^d$.

Existence of (PP) diffusions

Setup (II): Consider basic closed semialgebraic state space:

$$E = \left\{x \in \mathbb{R}^d \ : \ p(x) \geq 0 \text{ for all } p \in \mathscr{P}\right\}$$

with $\mathscr{P}$ a finite collection of polynomials on $\mathbb{R}^d$.



Existence of (PP) diffusions

Setup (II): Consider basic closed semialgebraic state space:

$$E = \{x \in \mathbb{R}^d \ : \ p(x) \geq 0 \text{ for all } p \in \mathscr{P}\}$$

with $\mathscr{P}$ a finite collection of polynomials on $\mathbb{R}^d$.

Examples:

$\mathbb{R}^d_+$: $\quad \mathscr{P} = \{p_i(x) = x_i, i = 1, \ldots, d\}$

$[0,1]^d$: $\quad \mathscr{P} = \{p_i(x) = x_i, \ p_{d+i}(x) = 1 - x_i, \ i = 1, \ldots, d\}$

unit ball : $\quad \mathscr{P} = \{p(x) = 1 - \|x\|^2\}$

$\mathbb{S}^m_+$: $\quad \mathscr{P} = \{p_I(x) = \det x_{II}, \ I \subset \{1, \ldots, m\}\}$,

(In the last example, $\mathbb{S}^m_+ \subset \mathbb{S}^m \cong \mathbb{R}^d$, $d = m(m+1)/2$.)

Existence of (PP) diffusions

Goal: Look for E-valued (weak) solutions to SDE of the form

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \qquad X_0 = x_0, \qquad (*)$$

for some $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ with $\sigma\sigma^\top \equiv a$ on E.

Existence of (PP) diffusions

Goal: Look for E-valued (weak) solutions to SDE of the form

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \qquad X_0 = x_0, \qquad (*)$$

for some $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ with $\sigma\sigma^\top \equiv a$ on E.

Theorem (necessary conditions). Assume $(*)$ admits an E-valued solution for any $x_0 \in E$. Then for all $p \in \mathscr{P}$,

$$a\,\nabla p = 0 \text{ and } \mathscr{G}\,p \geq 0 \text{ on } E \cap \{p = 0\}.$$

Existence of (PP) diffusions

Goal: Look for E-valued (weak) solutions to SDE of the form

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \qquad X_0 = x_0, \qquad (*)$$

for some $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ with $\sigma\sigma^\top \equiv a$ on E.

Theorem (necessary conditions). Assume $(*)$ admits an E-valued solution for any $x_0 \in E$. Then for all $p \in \mathscr{P}$,

$$a\nabla p = 0 \text{ and } \mathscr{G}p \geq 0 \text{ on } E \cap \{p = 0\}.$$

Proof: X is E-valued implies $p(X) \geq 0$, $\forall\, p \in \mathscr{P}$. On the other hand,

$$p(X_t) = p(x_0) + \int_0^t \mathscr{G}p(X_s)ds + \int_0^t \nabla p(X_s)^\top \sigma(X_s)dW_s$$

$$\langle p(X)\rangle_t = \int_0^t \|\sigma(X_s)^\top \nabla p(X_s)\|^2 ds. \qquad \square$$

Existence of (PP) diffusions

Goal: Look for E-valued (weak) solutions to SDE of the form

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \qquad X_0 = x_0, \qquad (*)$$

for some $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ with $\sigma\sigma^\top \equiv a$ on E.

Theorem (existence). Assume

- $a(x) \in \mathbb{S}^d_+$ for all $x \in E$,
- $a\nabla p = 0$ on $\{p = 0\}$ and $\mathscr{G}p > 0$ on $E \cap \{p = 0\}$, $\forall\, p \in \mathscr{P}$,
- each $p \in \mathscr{P}$ is irreducible and changes sign on $\mathbb{R}^d$.

Then $\exists\, \sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ with $\sigma\sigma^\top \equiv a$ on E such that $(*)$ has an E-valued solution for every $x_0 \in E$. Furthermore, one has

$$\int_0^t \mathbf{1}_{\{p(X_s)=0\}} ds \equiv 0 \qquad \forall\, p \in \mathscr{P}.$$

Existence of (PP) diffusions

Proof: Consider the metric projection $\pi : \mathbb{S}^d \to \mathbb{S}^d_+$, and define

$$\widehat{a}(x) = \pi(a(x)), \qquad \widehat{\sigma}(x) = \widehat{a}(x)^{1/2}.$$

Then (see Ikeda/Watanabe, 1981) there exists $\mathbb{R}^d$-valued solution to

$$dX_t = b(X_t)dt + \widehat{\sigma}(X_t)dW_t.$$

To do: For all $p \in \mathscr{P}$, show $p(X) \geq 0$ and $\int_0^t \mathbf{1}_{\{p(X_s)=0\}} ds \equiv 0$.

Lemma (See [FL16], Lemma A.1). Let Y be a continuous semimartingale

$$Y_t = Y_0 + \int_0^t \mu_s ds + M_t, \qquad Y_0 \geq 0, \qquad \mu \text{ continuous.}$$

If $\mu_t > 0$ on $\{Y_t = 0\}$ and $L^0(Y) = 0$, then $Y \geq 0$ and $\int_0^t \mathbf{1}_{\{Y_s=0\}} ds \equiv 0$.

Take $Y = p(X)$, $p \in \mathscr{P}$. After stopping, $\mu_t = \mathscr{G} p(X_t) > 0$ on $\{p(X_t) = 0\}$.

To do: Show $L^0(p(X)) = 0$.

Existence of (PP) diffusions

Proof (cont'd): Occupation density formula (see [RY99], Corollary VI.1.6):

$$\int_0^\infty \frac{1}{y} L_t^y(p(X))dy = \int_0^t \mathbf{1}_{\{p(X_s)>0\}} \frac{\nabla p(X_s)^\top \widehat{a}(X_s) \nabla p(X_s)}{p(X_s)} ds$$

Want $\dfrac{\nabla p^\top \widehat{a} \nabla p}{p}$ locally bounded. Let's show this for $\dfrac{\nabla p^\top a \nabla p}{p}$ instead!

Lemma from real algebra on real principal ideals (See [BCR98], Theorem 5.4.1): Assume $p \in \mathrm{Pol}(\mathbb{R}^d)$ is irreducible. The following are equivalent:

(i) p changes sign on $\mathbb{R}^d$

(ii) Any $q \in \mathrm{Pol}(\mathbb{R}^d)$ with $q = 0$ on $\{p = 0\}$ satisfies $q = pr$ for some $r \in \mathrm{Pol}(\mathbb{R}^d)$.

By assumption $a \nabla p = 0$ on $\{p = 0\}$. Hence

$$a \nabla p = pF, \qquad F = (f_1, \ldots, f_d)^\top \text{ polynomial.}$$

Thus $\dfrac{\nabla p^\top a \nabla p}{p} = \nabla p^\top F =$ polynomial. □

Existence of (PP) diffusions

Remarks.

- A more general existence theorem is in [FL16], Theorem 5.3:

$$E = \{x \in M \ : \ p(x) \geq 0 \text{ for all } p \in \mathscr{P}\}$$

  where

$$M = \left\{x \in \mathbb{R}^d \ : \ q(x) = 0 \text{ for all } q \in \mathscr{Q}\right\}$$

  with $\mathscr{P}$, $\mathscr{Q}$ finite collections of polynomials on $\mathbb{R}^d$. This requires further conditions involving polynomial ideals and their varieties.

> **Example:** Unit simplex $\Delta^d = \{x \in \mathbb{R}^d_+ : x_1 + \cdots + x_d = 1\}$

- Can relax $\mathscr{G}p > 0$ to $\mathscr{G}p \geq 0$ near $E \cap \{p = 0\}$.

  $\Longrightarrow$ Boundary absorption. Here we don't yet have the full picture.

- Conditions for boundary attainment: [FL16], Theorem 5.7.

Uniqueness of (PP) processes

- Let $(\mathscr{G}, E)$ be given with Assumption (A) satisfied.
- Notion of uniqueness:

X, X' two E-valued semimartingales with extended generator $\mathscr{G}$ $X_0 = X_0'$ deterministic	$\Longrightarrow$	$\text{Law}(X) = \text{Law}(X')$

"*Uniqueness in law among E-valued solutions to the local martingale problem for $\mathscr{G}$.*"

Uniqueness of (PP) processes

- Non-trivial in general: Non-Lipschitz, non–uniformly elliptic.
- Scalar diffusion case:

$$dX_t = (b + \beta X_t)dt + \sqrt{a + \alpha X_t + AX_t^2}\,dW_t$$

Yamada-Watanabe gives pathwise uniqueness, and hence:

Theorem. If $d = 1$ and $\nu \equiv 0$, then uniqueness holds.

- What about the general case?

Uniqueness of (PP) processes

- **Observation:** $\mathscr{G}$ and X_0 determine all mixed moments

$$\mathbb{E}\left[X_{t_1}^{\boldsymbol{k}_1} \cdots X_{t_m}^{\boldsymbol{k}_m}\right], \qquad 0 \le t_1 < \cdots < t_m, \quad \boldsymbol{k}_i \in \mathbb{N}_0^d.$$

Theorem. Let X be (PP) on E with extended generator $\mathscr{G}$. If

$$\text{for each } t \ge 0, \text{ there is } \varepsilon > 0 \text{ with } \mathbb{E}[e^{\varepsilon \|X_t\|}] < \infty \qquad (**)$$

then the law of X is uniquely determined by $\mathscr{G}$ and X_0.

Proof: Using MGFs, $(**)$ implies $\mathrm{Law}(X_t^i)$ determined by its moments.

By Petersen (1982), so are all FDMDs $\mathrm{Law}(X_{t_1}^{i_1}, \ldots, X_{t_m}^{i_m})$. □

Uniqueness of (PP) processes

Lemma. Assume $\nu \equiv 0$ (diffusion case) and there exists $C < \infty$ such that $\|a(x)\| \leq C(1 + \|x\|)$ for all $x \in E$. Then $(**)$ holds.

These results cover:

- Scalar (PP) diffusions,
- (PP) processes on compact sets,
- Any affine diffusions,
- ... etc.

Remark. Uniqueness does not always hold: P. Krühner has constructed a (PP) process on $\mathbb{R}$ for which uniqueness fails. This also leads to an example of a non-Markovian (PP) process.

An open problem

- The proof of the Theorem uses moment determinacy of each X_t.
- If $dX_t = X_t dW_t$ (Geometric Brownian motion) then X_t is lognormal.

  $\Longrightarrow$ Moment determinacy of X_t fails (see Heyde, 1963)

  $\Longrightarrow$ Uniqueness can't be proved in this way
- But could the mixed moments still pin down the law of X?
- **Open problem:** Find a process Y, not geometric Brownian motion, such that for all $0 \leq t_1 < \ldots < t_m$, $(k_1, \ldots, k_m) \in \mathbb{N}_0^m$,

  $$\mathbb{E}\left[Y_{t_1}^{k_1} \cdots Y_{t_m}^{k_m}\right] = \mathbb{E}\left[X_{t_1}^{k_1} \cdots X_{t_m}^{k_m}\right],$$

  where X is geometric Brownian motion.

  (Related to "weak" and "fake" Brownian motion, see Föllmer/Wu/Yor (2000), Hobson (2012), etc.)

Polynomial preserving processes

- Definition and general characterization
- Basic properties
- Existence and uniqueness
- **Examples**

Examples of (PP) diffusions

- Diffusion case only.
- Three examples: Unit cube $[0,1]^d$, unit ball $\mathscr{B}^d$, unit simplex Δ^d.
- All of them are compact, hence no issue with uniqueness.
- Compactness is also nice thanks to Weierstrass: polynomial approximation is possible.
- An affine diffusion on a compact state is necessarily deterministic. This is one reason to go beyond affine processes.
- Geometry of the state space crucially affects the possible dynamics.

The unit cube $[0,1]^d$

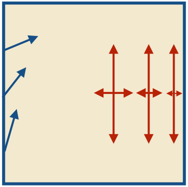
$E = [0,1]^d$

Proposition. The conditions of the existence theorem are satisfied if and only if

$$a(x) = \begin{pmatrix} \gamma_1 x_1(1-x_1) & & 0 \\ & \ddots & \\ 0 & & \gamma_d x_d(1-x_d) \end{pmatrix}, \qquad b(x) = \beta + Bx,$$

where $\gamma_i \geq 0$ and $\sum_{j\neq i} B_{ij}^- < \beta_i < -B_{ii} - \sum_{j \neq i} B_{ij}^+$.

- Interaction occurs only through the drifts.
- Volatility is componentwise of Jacobi type.



The unit simplex Δ^d

$E = \Delta^d = \{x \in \mathbb{R}^d_+ : x_1 + \cdots + x_d\}$

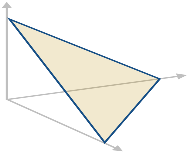
Proposition. The conditions of the (general) existence theorem are satisfied if and only if $a(x)$ and $b(x)$ are given by

$$a_{ii}(x) = \sum_{j \neq i} \alpha_{ij} x_i x_j \qquad a_{ij}(x) = -\alpha_{ij} x_i x_j \quad (i \neq j)$$

$$b(x) = \beta + Bx,$$

with $\alpha_{ij} \geq 0$, $\alpha_{ij} = \alpha_{ji}$, $B^\top \mathbf{1} + (\beta^\top \mathbf{1})\mathbf{1} = 0$ and $\beta_i + B_{ji} > 0$ for all i and $j \neq i$.

- Generalizes the multivariate Jacobi process: take $\alpha_{ij} = \sigma^2$, $i \neq j$; see Gourieroux/Jasiak (2006).



The unit ball $\mathscr{B}^d$

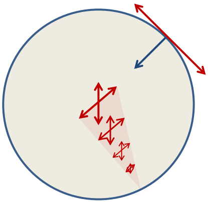
$E = \mathscr{B}^d = \{x \in \mathbb{R}^d : \|x\| \leq 1\}$. Details are in Larsson/Pulido (2015).

Example. Let $d = 2$ and consider

$$dX_t = -X_t dt + \sqrt{1 - \|X_t\|^2}\, \sigma\, dW_t + AX_t dB_t$$

with $\sigma \in \mathbb{R}^{2\times 2}$, $W = \begin{pmatrix} W^1 \\ W^2 \end{pmatrix}$, $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ skew-symmetric, B is one-dimensional Brownian motion.

- Mean-reverting drift.
- Volatility has both tangential and radially scaled components.



Note: $a(x) = (1 - \|x\|^2)\sigma\sigma^\top + Axx^\top A^\top$

The unit ball $\mathscr{B}^d$

Proposition. $\mathscr{G}$ is the extended generator of a (PP) diffusion on E if and only if

$$a(x) = (1 - \|x\|^2)\alpha + c(x),$$
$$b(x) = b + Bx,$$

for some $b \in \mathbb{R}^d$, $B \in \mathbb{R}^{d\times d}$, $\alpha \in \mathbb{S}^d_+$, and $c \in \mathscr{C}_+$ such that

$$b^\top x + x^\top Bx + \frac{1}{2}\operatorname{Tr}(c(x)) \le 0 \quad \text{for all} \quad x \in \mathscr{S}^{d-1}.$$

Here $\mathscr{S}^{d-1}$ is the unit sphere in $\mathbb{R}^d$, and

$$\mathscr{C}_+ = \left\{ c : \mathbb{R}^d \to \mathbb{S}^d \ : \ \begin{array}{l} c_{ij} \in \mathrm{Hom}_2 \text{ for all } i,j \\ c(x)x \equiv 0 \\ c(x) \in \mathbb{S}^d_+ \text{ for all } x \end{array} \right\}$$

The unit ball $\mathscr{B}^d$

$$\mathscr{C}_+ = \left\{ c : \mathbb{R}^d \to \mathbb{S}^d \ : \ \begin{array}{l} c_{ij} \in \mathrm{Hom}_2 \text{ for all } i,j \\ c(x)x \equiv 0 \\ c(x) \in \mathbb{S}^d_+ \text{ for all } x \end{array} \right\}$$

Examples of $c \in \mathscr{C}_+$:

- Take $A_1 \in \mathrm{Skew}(d)$ and set

$$c(x) = A_1 x x^\top A_1^\top$$

The unit ball $\mathscr{B}^d$

$$\mathscr{C}_+ = \left\{ c : \mathbb{R}^d \to \mathbb{S}^d \ : \ \begin{array}{l} c_{ij} \in \mathrm{Hom}_2 \text{ for all } i,j \\ c(x)x \equiv 0 \\ c(x) \in \mathbb{S}^d_+ \text{ for all } x \end{array} \right\}$$

Examples of $c \in \mathscr{C}_+$:

- Take $A_1, \ldots, A_m \in \mathsf{Skew}(d)$ and set

$$c(x) = A_1 x x^\top A_1^\top + A_2 x x^\top A_2^\top + \cdots + A_m x x^\top A_m^\top$$

The unit ball $\mathscr{B}^d$

$$\mathscr{C}_+ = \left\{ c : \mathbb{R}^d \to \mathbb{S}^d \;:\; \begin{array}{l} c_{ij} \in \mathrm{Hom}_2 \text{ for all } i, j \\ c(x)x \equiv 0 \\ c(x) \in \mathbb{S}^d_+ \text{ for all } x \end{array} \right\}$$

Examples of $c \in \mathscr{C}_+$:

- Take $A_1, \ldots, A_m \in \mathrm{Skew}(d)$ and set

$$c(x) = A_1 x x^\top A_1^\top + A_2 x x^\top A_2^\top + \cdots + A_m x x^\top A_m^\top$$

- This leads to a convenient parameterization of a large class of elements of $\mathscr{C}_+$ ...

The unit ball $\mathscr{B}^d$

$$\mathscr{C}_+ = \left\{ c : \mathbb{R}^d \to \mathbb{S}^d \quad : \quad \begin{array}{l} c_{ij} \in \mathrm{Hom}_2 \text{ for all } i, j \\ c(x)x \equiv 0 \\ c(x) \in \mathbb{S}^d_+ \text{ for all } x \end{array} \right\}$$

Examples of $c \in \mathscr{C}_+$:

- Take $A_1, \ldots, A_m \in \mathrm{Skew}(d)$ and set

$$c(x) = A_1 x x^\top A_1^\top + A_2 x x^\top A_2^\top + \cdots + A_m x x^\top A_m^\top$$

- This leads to a convenient parameterization of a large class of elements of $\mathscr{C}_+$ ...
- ... but is this exhaustive?

The unit ball $\mathscr{B}^d$

$c(x)$ with $c_{ij} = c_{ji} \in \mathrm{Hom}_2$	$\iff$	$\mathrm{BQ}(x,y) := y^\top c(x) y$ is a **biquadratic form**
$c(x)x \equiv 0$	$\iff$	$\mathrm{BQ}(x,x) \equiv 0$
$c(x)$ positive semidefinite for all x	$\iff$	$\mathrm{BQ}(x,y) \geq 0$ for all x, y
$c(x) = \sum_{p=1}^m A_p x x^\top A_p^\top$	$\iff$	$\mathrm{BQ}(x,y) = \sum_p (y^\top A_p x)^2$ $=$ sum of squares (**SOS**)

$\mathscr{C}_+ \cong$ {all nonnegative biquadratic forms with vanishing diagonal}

$\stackrel{?}{=}$ {all **SOS** biquadratic forms with vanishing diagonal}

Answer: $d \leq 4$: Yes! $d \geq 6$: No! $d = 5$: Don't know!

Other interesting state spaces

- $[0,1]^m \times \mathbb{R}^n_+$ and $[0,1]^m \times \mathbb{R}^n_+ \times \mathbb{R}^l$ are straightforward extensions of the unit cube; see [FL16].
- The unit ball analysis can be brought to bear on parabolic and hyperbolic sets, although this has not been done and will require some effort.
- A nice feature of the unit sphere is that it is compact (polynomial approximation) with no boundary (simulation easier). This has yet to be exploited in applications.
- Partial parameterization exists for $E = \mathbb{S}^m_+$: the affine case is fully understood, see Cuchiero et al. (2011).
- Partial parameterization exists for $E = \mathfrak{C}^m$ (correlation matrices), see Ahdida/Alfonsi (2013), but work remains.

Applications in finance

- **Overview**
- State price density models
- Polynomial term structure models

Overview

(PP) processes have been used in a variety of applications

- Term structure of interest rates (See [FLT15] and Glau/Grbac/Keller-Ressel, 2015)
- Stochastic volatility models (Ackerer/Filipović/Pulido, 2016)
- Variance swap rates (Filipović/Gourier/Mancini, 2016)
- Credit risk (Ackerer/Filipović, 2016)
- Stochastic portfolio theory (Cuchiero, 2016)

The crucial property of (PP) processes — closed-form expressions for conditional moments — are exploited in different ways in these papers.

Here I will focus on models for the **term structure of interest rates**.

Applications in finance

- Overview
- **State price density models**
- Polynomial term structure models

State price density models

Recipe for building arbitrage-free asset pricing models:

Let $\zeta > 0$ be a positive semimartingale on $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$. For any claim C_T maturing at some $T < \infty$, **define**

$$\text{model price at } t = \frac{1}{\zeta_t}\mathbb{E}[\,\zeta_T C_T \mid \mathscr{F}_t] \qquad (t \leq T).$$

We call ζ the **state price density**.

State price density models

Recipe for building arbitrage-free asset pricing models:

Let $\zeta > 0$ be a positive semimartingale on $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$. For any claim C_T maturing at some $T < \infty$, **define**

$$\text{model price at } t = \frac{1}{\zeta_t}\mathbb{E}[\zeta_T C_T \mid \mathscr{F}_t] \qquad (t \leq T).$$

We call ζ the **state price density**.

Remarks:

- Usually $\mathbb{P}$ is not a risk-neutral measure ...
- ... but need not be the historical measure either.
- In the applications to interest rate modeling presented here, $\mathbb{P}$ is the so-called **long forward measure**; see Hansen/Scheinkman (2009) and Qin/Linetsky (2015), etc.

State price density models

Recipe for building arbitrage-free asset pricing models:

Let $\zeta > 0$ be a positive semimartingale on $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$. For any claim C_T maturing at some $T < \infty$, **define**

$$\text{model price at } t = \frac{1}{\zeta_t}\mathbb{E}[\,\zeta_T C_T \mid \mathscr{F}_t] \qquad (t \le T).$$

We call ζ the **state price density**.

Remarks:

- Zero-coupon bond prices, $C_T = 1$:

$$P(t, T) = \frac{1}{\zeta_t}\mathbb{E}[\,\zeta_T \mid \mathscr{F}_t]$$

State price density models

Such models are arbitrage-free on any finite time horizon $[0, T^*]$:

- Asset prices $S^1, \ldots, S^m$:

$$S_t^i = \frac{1}{\zeta_t}\mathbb{E}[\,\zeta_{T^*} S_{T^*}^i \mid \mathscr{F}_t]$$

- Suppose $S^1 > 0$ and choose this as numeraire.
- Define $\mathbb{Q}^1 \sim \mathbb{P}$ with Radon-Nikodym density process

$$Z_t = \frac{\zeta_t S_t^1}{\zeta_0 S_0^1}$$

- Then S^i/S^1 is a $\mathbb{Q}^1$-martingale for all i,

$$\frac{S_t^i}{S_t^1} Z_t = \frac{\zeta_t S_t^i}{\zeta_0 S_0^1} = \mathbb{P}\text{-martingale}$$

... and hence NFLVR holds with respect to the numeraire S^1.

State price density models

Such models are arbitrage-free on any finite time horizon $[0, T^*]$:

- Suppose $\mathbb{Q} \sim \mathbb{P}$ is a (local) martingale measure associated with the usual bank account numeraire

$$B_t = e^{\int_0^t r_s ds}.$$

Then

$$\zeta_t = e^{-\int_0^t r_s ds}\, \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \mid \mathscr{F}_t\right]$$

is the "discounted density process".

State price density models

Closely related literature:

Constantinides (1992); Flesaker and Hughston (1996); Rogers (1997); Rutkowski (1997); Brody and Hughston (2005), Carr, Gabaix, Wu (2010); Nguyen and Seifried (2015) Crépey, Macrina, Nguyen, Skovmand (2015), etc.

Applications in finance

- Overview
- State price density models
- **Polynomial term structure models**

Polynomial term structure models

Let X be a (PP) process on $E \subseteq \mathbb{R}^d$ with extended generator $\mathscr{G}$. Specify the state price density by

$$\zeta_t = e^{-\alpha t} p(X_t)$$

for some positive $p \in \mathrm{Pol}(E)$ and some $\alpha \in \mathbb{R}$.

Polynomial term structure models

Let X be a (PP) process on $E \subseteq \mathbb{R}^d$ with extended generator $\mathscr{G}$. Specify the state price density by

$$\zeta_t = e^{-\alpha t} p(X_t)$$

for some positive $p \in \mathrm{Pol}(E)$ and some $\alpha \in \mathbb{R}$.

Example. X is a scalar square-root diffusion

$$dX_t = \kappa(\theta - X_t)dt + \sigma\sqrt{X_t}dW_t$$

and the state price density is given by

$$\zeta_t = e^{-\alpha t}(1 + X_t).$$

Polynomial term structure models

Let X be a (PP) process on $E \subseteq \mathbb{R}^d$ with extended generator $\mathscr{G}$. Specify the state price density by

$$\zeta_t = e^{-\alpha t} p(X_t)$$

for some positive $p \in \mathrm{Pol}(E)$ and some $\alpha \in \mathbb{R}$.

Fix the following notation:

- $n \geq \deg(p)$
- $N = \dim \mathrm{Pol}_n(E)$
- $H(x) = (h_1(x), \ldots, h_N(x))^\top$ basis for $\mathrm{Pol}_n(E)$
- G matrix representation of $\mathscr{G}$
- $\vec{p}$ coordinate representation of p

Bond prices and short rate

Explicit zero-coupon bond prices:

$$P(t, T) = e^{-\alpha(T-t)} \frac{H(X_t)^\top e^{(T-t)G} \vec{p}}{H(X_t)^\top \vec{p}}$$

Proof: $P(t, T) = \frac{1}{\zeta_t}\mathbb{E}[\, \zeta_T \mid \mathscr{F}_t] = \frac{e^{-\alpha T}\mathbb{E}[p(X_T)|\mathscr{F}_t]}{e^{-\alpha t} p(X_t)}$ □

Explicit short rate:

$$r_t = \alpha - \frac{H(X_t)^\top G \vec{p}}{H(X_t)^\top \vec{p}}$$

Proof: $r_t = -\partial_T \log P(t, T)\big|_{T=t} = \alpha - \frac{H(X_t)^\top G e^{(T-t)G}\vec{p}}{H(X_t)^\top e^{(T-t)G}\vec{p}}\Big|_{T=t}$ □

- Elucidates the role of α as a shift to the short rate

Bond prices and short rate

Example (cont'd). X is a scalar square-root diffusion

$$dX_t = \kappa(\theta - X_t)dt + \sigma\sqrt{X_t}dW_t$$

and the state price density is given by

$$\zeta_t = e^{-\alpha t}(1 + X_t).$$

Then

$$P(t,T) = e^{-\alpha(T-t)}\frac{1 + \theta + e^{-\kappa(T-t)}(X_t - \theta)}{1 + X_t}$$

$$r_t = \alpha + \frac{1 + \theta(1+\kappa) - \kappa X_t}{1 + X_t}$$

The short rate is bounded:

$$\alpha - \kappa \quad \leq \quad r_t \quad \leq \quad \alpha + 1 + \theta(\kappa + 1)$$

α as infinite-maturity yield

- The **yield** $y(t, T)$ is by definition

$$P(t, T) = e^{-(T-t)y(t,T)}$$

- Since $\mathscr{G}1 = 0$, G has at least one zero eigenvalue. Suppose it has exactly one. Suppose also that every other eigenvalue λ satisfies

$$\mathrm{Re}(\lambda) < 0.$$

- Assume $\inf_{x \in E} p(x) > 0$

Under these conditions, $\alpha = \lim_{T \to \infty} y(t, T)$.

Proof: $y(t, T) = \alpha - \frac{1}{T-t} \log \mathbb{E}[p(X_T) \mid \mathscr{F}_t] + \frac{1}{T-t} \log p(X_t)$.

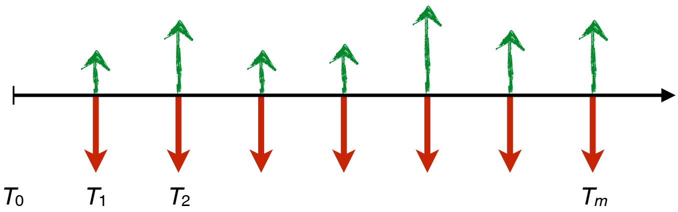
Eigenvalue assumption $\implies$ moments $\mathbb{E}[X_t^{\boldsymbol{k}}]$, $|\boldsymbol{k}| \leq n$, bounded in t. □

Interest rate swaps

- Tenor structure $T_0 < T_1 < \cdots < T_m$, $\Delta = T_i - T_{i-1}$
- Fixed annualized rate K
- Value per dollar notional of **payer swap** (pay fixed, receive floating):

$$\Pi_t^{\text{swap}} = P(t, T_0) - P(t, T_n) - \Delta K \sum_{i=1}^{m} P(t, T_i), \qquad t \leq T_0$$

- The **swap rate** $S_t^{T_0, T_m}$ is the value of K that yields $\Pi_t^{\text{swap}} = 0$.



Swaptions

- Option with expiry date T_0 written on the swap
- Payoff at time T_0:

$$C_{T_0} = \left(\Pi^{\mathrm{swap}}_{T_0}\right)^+$$

- Note that

$$\Pi^{\mathrm{swap}}_{T_0} = \sum_{i=0}^{m} c_i P(T_0, T_i) = \frac{1}{\zeta_{T_0}} \sum_{i=0}^{m} c_i q_i(X_{T_0})$$

  for some constants c_i and polynomials q_i.

- Option price at $t \leq T_0$:

$$\Pi^{\mathrm{swaption}}_t = \frac{1}{\zeta_t}\mathbb{E}[\zeta_{T_0} C_{T_0} \mid \mathscr{F}_t] = \frac{1}{\zeta_t}\mathbb{E}\left[\left(\sum_{i=0}^{m} c_i q_i(X_{T_0})\right)^+ \Bigm| \mathscr{F}_t\right]$$

- $\Longrightarrow$ Must compute $\mathbb{E}[q(X_{T_0})^+ \mid \mathscr{F}_t]$ for $q \in \mathrm{Pol}_n(E)$
- More coupon payments yield no increase in complexity!

Swaptions: Comparison with affine models

- Consider (for this slide only) an **affine interest rate model**:

$$r_t = \alpha + a^\top X_t \text{ for some } \alpha \in \mathbb{R},\ a \in \mathbb{R}^d$$

X is an **affine process** under $\mathbb{Q}$.

- Then $\overline{X}_t = (\int_0^t r_s ds, X_t)$ is again affine, and bond prices are given by

$$P(t,T) = \mathbb{E}_{\mathbb{Q}}\left[e^{u^\top \overline{X}_{T_0}} \;\middle|\; \mathscr{F}_t \right] = e^{A(T-t)+B(T-t)^\top \overline{X}_t}$$

where $u = (-1, 0, \dots, 0)^\top$ and (A, B) solves a system of quadratic ODEs called the (generalized) Riccati equations.

- Hence $\Pi_t^{\text{swaption}} = \mathbb{E}\left[\left(\sum_{i=0}^m c_i e^{A_i + B_i^\top \overline{X}_{T_0}} \right)^+ \;\middle|\; \mathscr{F}_t \right] \dots$
- ... but linear combinations of exponentials are unfriendly!
- See Filipović (2009) for more on affine term structure models.

Swaptions: How to evaluate $\mathbb{E}[q(X_T)^+]$?

- **Transform method** if $\widehat{q}(z) = \mathbb{E}[e^{zq(X_T)}]$ is available: The identity

$$s^+ = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(\mu+\mathrm{i}\lambda)s}\frac{1}{(\mu+\mathrm{i}\lambda)^2}d\lambda \qquad \text{(any } \mu > 0\text{)}$$

  implies

$$\mathbb{E}[q(X_T)^+] = \frac{1}{\pi}\int_0^{\infty} \mathrm{Re}\left(\frac{\widehat{q}(\mu+\mathrm{i}\lambda)}{(\mu+\mathrm{i}\lambda)^2}\right)d\lambda$$

- **Polynomial expansion:** Fix a weight function $w(x)$ and consider Hilbert space L^2_w with inner product $\langle f, g\rangle_w = \int f(x)g(x)w(x)dx$. Let $Q_n, n \geq 0$ be an orthonormal polynomial basis. Then

$$\int q(x)^+ f_{X_T}(x)dx = \langle q^+, \frac{f_{X_T}}{w}\rangle_w = \sum_{n\geq 0}\langle q^+, Q_n\rangle_w \langle \frac{f_{X_T}}{w}, Q_n\rangle_w$$

  (Filipović/Mayerhofer/Schneider, '13; Ackerer/Filipović/Pulido, '15)

Unspanned stochastic volatility

Empirical fact: Volatility risk cannot be hedged using bonds

- Collin-Dufresne, Goldstein (2002): Interest rate swaps can hedge only 10%–50% of variation in ATM straddles (a volatility-sensitive instrument)
- Heidari, Wu (2003): Level/curve/slope explain 99.5% of yield curve variation, but 59.5% of variation in swaption implied vol

This phenomenon is called **Unspanned Stochastic Volatility (USV)**.

- Other types of factors can be similarly unspanned
- Joslin, Priebsch, Singleton (2014): Bonds cannot be used to hedge macro-economic risks

> How to operationalize this in a polynomial term structure model?

Unspanned stochastic volatility

- Assume we are in the linear case:

$$\zeta_t = e^{-\alpha t}\left(\phi + \psi^\top X_t\right)$$

for some $\phi \in \mathbb{R}$ and $\psi \in \mathbb{R}^d$.

- This is **w.l.o.g.**: $\zeta_t = e^{-\alpha t} p(X_t)$ is linear in $\overline{X}_t = H(X_t)$, which is again (PP).
- Since X is (PP) it has affine drift. Thus, in mean-reversion form:

$$dX_t = \kappa(\theta - X_t)dt + dM_t,$$

where $\kappa \in \mathbb{R}^{d\times d}$, $\theta \in \mathbb{R}^d$, and M is a martingale.

- Bond prices are **linear-rational** in X_t,

$$P(t,T) = e^{-\alpha(T-t)} \frac{\phi + \psi^\top X_t + \psi^\top e^{-\kappa(T-t)}(X_t - \theta)}{\phi + \psi^\top X_t},$$

which does not depend on the specification of M.

Unspanned stochastic volatility

Consider an extended factor process (X, U) such that:

- (X, U) is jointly (PP)
- X has autonomous linear drift,

$$dX_t = \kappa(\theta - X_t)dt + dM_t$$

- U feeds into the characteristics of M.

Then U acts as an unspanned volatility factor:

- Does not affect $P(t, T) = e^{-\alpha(T-t)} \frac{\phi + \psi^\top X_t + \psi^\top e^{-\kappa(T-t)}(X_t - \theta)}{\phi + \psi^\top X_t}$
- But does generically affect the "volatility" $\langle P(\cdot, T) \rangle_t$

Unspanned stochastic volatility

Example. Consider a model on $\mathbb{R}_+ \times [0,1]$ of the form

$$dX_t = \kappa(\theta - X_t)dt + \sigma\sqrt{U_t X_t}dW_t$$
$$dU_t = \gamma(\eta - U_t)dt + \nu\sqrt{U_t(1-U_t)}dB_t$$

with W and B independent Brownian motions. Let

$$\zeta_t = e^{-\alpha t}(1 + X_t).$$

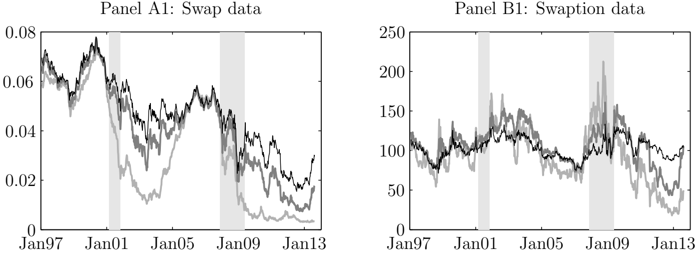
Then with $\tau = T - t$,

$$P(t,T) = e^{-\alpha\tau}\frac{1 + \theta + e^{-\kappa\tau}(X_t - \theta)}{1 + X_t}$$
$$\langle P(\cdot,T)\rangle_t = \sigma^2(1+\theta)^2 e^{-2\alpha\tau}(1 - e^{-\kappa\tau})^2\frac{X_t U_t}{(1+X_t)^4}$$

This leads to **USV**: Delta-hedging is ineffective for risks that depend on $\langle P(\cdot,T)\rangle$.

Empirics

- Panel data set of swaps and ATM swaptions
- Swap maturities: 1Y, 2Y, 3Y, 5Y, 7Y, 10Y
- Swaptions on 1Y, 2Y, 3Y, 5Y, 7Y, 10Y forward starting swaps with option expiries 3M, 1Y, 2Y, 5Y
- 866 weekly observations, Jan 29, 1997 – Aug 28, 2013



Empirics

Linear-rational square root (LRSQ) model: $E = \mathbb{R}_+^d$

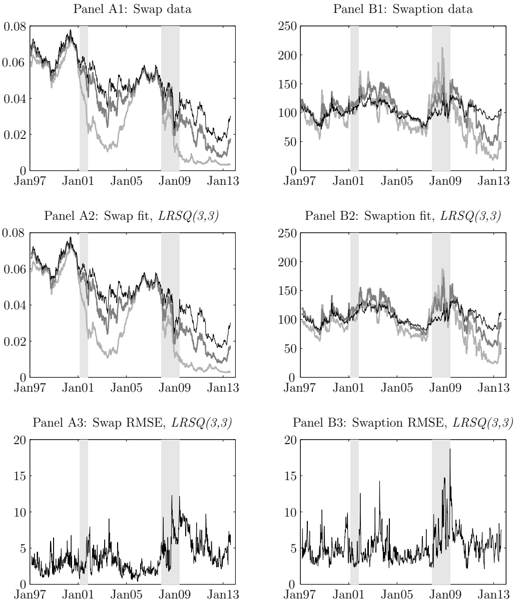
$$\mathrm{d}X_t = \kappa(\theta - X_t)\mathrm{d}t + \begin{pmatrix} \sigma_1\sqrt{X_{1t}} & & 0 \\ & \ddots & \\ 0 & & \sigma_d\sqrt{X_{dt}} \end{pmatrix} \mathrm{d}W_t$$

$$\zeta_t = e^{-\alpha t}(1 + \mathbf{1}^\top X_t)$$

LRSQ(m, n)**:**

- Constrained to have m term structure factors and n USV factors ($m \geq n$, $m + n = d$)
- Number of parameters: $m^2 + 2m + 2n$
- Estimation approach: Quasi-maximum likelihood in conjunction with the unscented Kalman Filter

Fit to data



Panel A1: Swap data
0.08
0.06
0.04
0.02
0
Jan97 Jan01 Jan05 Jan09 Jan13
Panel B1: Swaption data
250
200
150
100
50
0
Jan97 Jan01 Jan05 Jan09 Jan13
Panel A2: Swap fit, *LRSQ(3,3)*
0.08
0.06
0.04
0.02
0
Jan97 Jan01 Jan05 Jan09 Jan13
Panel B2: Swaption fit, *LRSQ(3,3)*
250
200
150
100
50
0
Jan97 Jan01 Jan05 Jan09 Jan13
Panel A3: Swap RMSE, *LRSQ(3,3)*
20
15
10
5
0
Jan97 Jan01 Jan05 Jan09 Jan13
Panel B3: Swaption RMSE, *LRSQ(3,3)*
20
15
10
5
0
Jan97 Jan01 Jan05 Jan09 Jan13

Comparison of model specifications

Specification	Swaps	Swaptions				
		All	3 mths	1 yr	2 yrs	5 yrs
LRSQ(3,1)	7.11	6.63	8.27	5.54	5.25	5.71
LRSQ(3,2)	3.83	5.77	7.87	5.12	3.98	4.19
LRSQ(3,3)	3.72	5.19	7.20	4.40	3.88	3.70
LRSQ(3,2)-LRSQ(3,1)	-3.28^{***}	-0.86^{**}	-0.40	-0.42	-1.27^{***}	-1.52^{**}
	(-8.95)	(-2.18)	(-0.74)	(-1.04)	(-3.66)	(-2.55)
LRSQ(3,3)-LRSQ(3,2)	-0.12	-0.58^{**}	-0.67^{*}	-0.72^{***}	-0.11	-0.49^{**}
	(-0.78)	(-2.52)	(-1.82)	(-2.97)	(-0.46)	(-2.06)

Figure: Average RMSE (bps)

- $LRSQ(3,1)$ and $LRSQ(3.2)$ both have reasonable fit
- ... but $LRSQ(3,3)$ is the preferred model
- Captures level-dependence in swaption implied vol at low rates
- Upper bounds on short rate:

LRSQ(3,1)	*LRSQ(3,2)*	*LRSQ(3,3)*
0.20	1.46	0.72

Conclusions and outlook

Conclusions and outlook

- Polynomial models represent an attractive tradeoff between flexibility and tractability.
- Significant progress has already been made both on the theoretical side and in applications.
- Nonetheless this is a wide open area . . .

Conclusions and outlook

... the following being but a few examples of unexplored territory:

- Statistical estimation. E.g. martingale estimating functions (see Forman/Sørensen (2008) and Kessler/Sørensen (1999)) and generalized method of moments (see Zhou (2003)).
- Filtering. Exploit the (PP) property to improve existing approximate filters, such as the extended and unscented Kalman filters.
- Improved existence/uniqueness theory. Various natural state spaces like $\mathfrak{C}^d$ are not well-understood. Uniqueness in the diffusion case should hold but is not completely settled. Same for boundary absorption.
- Other open questions, such as existence of "fake" GBM and the sum-of-squares problem for the unit ball.

Thank you!

D. Ackerer and D. Filipović.
Linear credit risk models.
Technical report, Swiss Finance Institute, 2016.

D. Ackerer, D. Filipović, and S. Pulido.
The Jacobi stochastic volatility model.
Technical report, Swiss Finance Institute, 2016.

A. Ahdida and A. Alfonsi.
A Mean-Reverting SDE on Correlation matrices.
Stoch. Proc. Appl., 123(4):1472–1520, 2013.

D. Bakry, S. Orevkov, and M. Zani.
Orthogonal polynomials and diffusion operators.
arXiv:1309.5632v2, 2014.

J. Bochnak, Coste M., and M.-F. Roy.
Real Algebraic Geometry.
Springer-Verlag Berlin Heidelberg, 1998.

D. C. Brody and L. P. Hughston.

Chaos and coherence: a new framework for interest-rate modelling.
Proceedings of the Royal Society A, 460(4010):85–110, 2004.
Brody, D. C., and L. P. Hughston, , Proceedings of the Royal Society A 460, 2041, 85-110 (2005).

P. Carr, X. Gabaix, and L. Wu.
Linearity-generating processes, unspanned stochastic volatility, and interest-rate option pricing.
Working paper, New York University, 2009.

P. Collin-Dufresne and R. Goldstein.
Do bonds span the fixed income markets? Theory and evidence for unspanned stochastic volatility.
Journal of Finance, 57:1685–1730, 2002.

G. Constantinides.
A theory of the nominal term structure of interest rates.
Review of Financial Studies, 5:531–552, 1992.

S. Crépey, A. Macrina, T. M. Nguyen, and D. Skovmand.

Rational Multi-Curve Models with Counterparty-Risk Valuation Adjustments.
arXiv:1502.07397, 2015.

C. Cuchiero.
Polynomial processes in stochastic portfolio theory.
Working paper, 2016.

C. Cuchiero.
Affine and Polynomial Processes.
PhD thesis, ETH Zurich, 2011.

C. Cuchiero, M. Keller-Ressel, and J. Teichmann.
Polynomial processes and their applications to mathematical finance.
Finance and Stochastics, 16:711–740, 2012.

D. Filipović.
Term Structure Models – A Graduate Course.
Springer, 2009.

D. Filipović, E. Gourier, and L. Mancini.

Quadratic variance swap models.
Journal of Financial Economics, 119(1):44–68, 2016.

D. Filipović and M. Larsson.
Polynomial preserving diffusions and applications in finance.
Forthcoming in Finance and Stochastics, 2016.
Available at SSRN: http://ssrn.com/abstract=2479826.

D. Filipović, M. Larsson, and A. Trolle.
Linear-rational term structure models.
Forthcoming in Journal of Finance, 2016.
Available at SSRN: http://ssrn.com/abstract=2397898.

D. Filipović, E. Mayerhofer, and P. Schneider.
Density approximations for multivariate affine jump-diffusion processes.
Journal of Econometrics, 176:93–111, 2013.

B. Flesaker and L. P. Hughston.
Positive interest.
Risk, 9(1):46–49, 1996.

J. L. Forman and M. Sørensen.
The Pearson diffusions: a class of statistically tractable diffusion processes.
Scand. J. Statist., 35(3):438–465, 2008.

H. Föllmer, C.-T. Wu, M. Yor.
On weak Brownian motions of arbitrary order.
Ann. de l'IHP (B), 36(4):447–487, 2000.

K. Glau, Z. Grbac, and M. Keller-Ressel.
Polynomial preserving processes and discrete-tenor interest rate models.
https://indico.math.cnrs.fr/event/699/contribution/31/material/slide
2015.

C. Gourieroux and J. Jasiak.
Multivariate Jacobi process with application to smooth transitions.
Journal of Econometrics, 131:475–505, 2006.

L. P. Hansen and J. A. Scheinkman.

Long-Term Risk: An Operator Approach.
Econometrica, 77(1):177–234, 2009.

M. Heidari and L. Wu.
Are interest rate derivatives spanned by the term structure of interest rates?
Journal of Fixed Income, 13:75–86, 2003.

C. C. Heyde.
On a property of the lognormal distribution.
Journal of the Royal Statistical Society, Series B, 25(2):392–393, 1963.

D. Hobson.
Fake Exponential Brownian Motion.
arXiv:1210.1391, 2012.

N. Ikeda and S. Watanabe.
Stochastic Differential Equations and Diffusion Processes.
North-Holland, 1981.

S. Joslin, M. Priebsch and K. Singleton.

Risk Premiums in Dynamic Term Structure Models with Unspanned Macro Risks.
Journal of Finance, 69(3):1197–1233, 2014.

M. Kessler and M. Sørensen.
Estimating equations based on eigenfunctions for a discretely observed diffusion process.
Bernoulli, 5(2):299–314, 1999.

M. Larsson and P. Krühner.
Affine processes with compact state space.
Working paper, 2016.

M. Larsson and S. Pulido.
Polynomial preserving diffusions on compact quadric sets.
arXiv:1511.03554, 2015.

O. Mazet.
Classification des semi-groupes de diffusion sur $\mathbb{R}$ associé à une famille de polynômes orthogonaux.
Séminaire de probabilités (Strasbourg), 31:40–53, 1997.

T. A. Nguyen and F. Seifried.
The multi-curve potential model.
Int. J. Theor. Appl. Finan., 18, 2015.

L. C. Petersen.
On the relation between the multidimensional moment problem and the one-dimensional moment problem.
Math. Scand., 51:361–366, 1982.

L. Qin and V. Linetsky.
Long Term Risk: A Martingale Approach.
arXiv:1411.3078, 2015.

D. Revuz and M. Yor.
Continuous Martingales and Brownian Motion.
Springer-Verlag, third edition, 1999.

L. C. G. Rogers.
The potential approach to the term structure of interest rates and foreign exchange rates.
Mathematical Finance, 2:157–164, 1997.

M. Rutkowski.
A note on the Flesaker-Hughston model of the term structure of interest rates.
Applied Mathematical Finance, 4:151–163, 1997.

E. Wong.
The construction of a class of stationary Markoff processes.
In *Stochastic Processes in Mathematical Physics and Engineering*, pages 264–276, 1964.

H. Zhou.
Itô conditional moment generator and the estimation of short-rate processes.
Journal of Financial Econometrics, 1:250–271, 2003.