American Eagle Options

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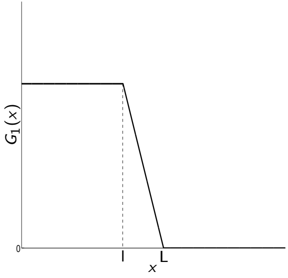
Outline of the Presentation

- Motivation to design American eagle options
- literature review for pricing American capped options
- American eagle options with balance wings
  - Structure of optimal stopping region and continuation region
  - Property of value function: continuity, part-smooth-fit
  - Property of free-boundary: monotonicity, continuity and etc.
  - EEP representation of American eagle options
  - The Solution of free-boundary is unique
- American 'disable' eagle options
  - The lower cap is inside the continuation region
- Numerical result for free-boundary, value function and Greeks.

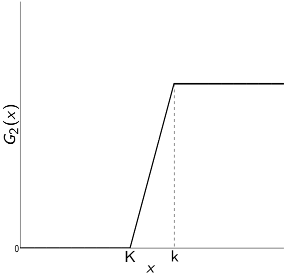
Motivation for Eagle Options

From Chapter 11 in the book *Options, Futures and Other Derivatives*, it creates

- a bear spread by buying a European put option on the stock with strike price L and selling another European put option on the same stock with a lower strike price l.
- a bull spread by purchasing a European call option with the strike price K and selling a European call option with higher strike price k



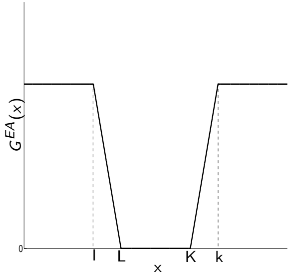
Bear Spread Payoff



Bull Spread Payoff

Motivation for Eagle Options

After we combining the bull spread and bear spread, we can get the payoff of American eagle options.



Payoff of Eagle options

Real Eagle

The payoff of eagle options is defined as

$$G^{EA}(x) = (x - K)^{+} - (x - k)^{+} + (L - x)^{+} - (l - x)^{+}. \tag{1}$$

Motivation for Eagle Options

We can simplified the payoff in (1) into

$$G^{EA}(x) = (k \wedge x - K)^+ \vee (L - l \vee x)^+, \tag{2}$$

for $l < L \leq K < k$. When $k - K = L - l$, we call it eagle options with balance wings. Otherwise, we call it 'disable' eagle options. The value function for American eagle options is defined as

$$V^{EA}(t, x) = \sup_{\tau \in [0, T-t]} E_{t,x}[e^{-r\tau} G^{EA}(X_{t+\tau})], \tag{3}$$

where τ is the stopping time over $[0, T - t]$, and stock price X satisfies geometric Brownian motion

$$dX_t = (r - \delta) X_t dt + \sigma X_t dW_t. \tag{4}$$

The infinitesimal generator of X is

$$\mathbb{L}_X = (r - \delta) x \frac{\partial}{\partial x} + \frac{\sigma^2}{2} x^2 \frac{\partial^2}{\partial x^2}. \tag{5}$$

Advantages for Eagle Options

Comparing with American strangle options in (Qiu 2014), the payoff is

$$G^{EA}(x) = (x - K)^+ \vee (L - x)^+. \quad (6)$$

The advantages for eagle options are:

- suitable for the underlying asset with high volatility
- maximum loss controlled by caps and become the attractive instruments by the options issuer
- has lower premium than the strangle option and the buyer could flexibly set the suitable cap on their preference

So the eagle options is the refinement of strangle options and can be called as American capped strangle options.

Previous Research on Capped Options

- The research of capped options started by (Boyle and Turnbull 1989) for European capped call options for forward contract, collar loans, index notes and index currency option notes.
- In 1992, Flesaker designed and valuated the capped index options, but it was not American style.
- (Detemple and Broadie, 1995) proved that the free-boundary of American capped call options was the maximum between the cap and the free-boundary of American call options. Finally, they gave the analytical solution for the value function.
- Gappeev and Lerche gave a short illustrate on perpetual American capped strangle options in 2011. From example 4.3 in their paper, the upper free-boundary of American capped strangle options was the maximum between upper cap and the upper free-boundary of American strangle options; the lower free-boundary of American capped strangle options was the minimum between the lower cap and the lower free-boundary of American strangle options.

The Optimal Stopping Region for American Eagle Options with Balance Wings

We define the stopping region and continuation region for the optimal stopping problem in (3) is

$$C^{EA} = \{(t,x) \in [0,T) \times (0,\infty) | V^{EA}(t,x) > G^{EA}(x)\}, \quad (7)$$
$$\bar{D}^{EA} = \{(t,x) \in [0,T] \times (0,\infty) | V^{EA}(t,x) = G^{EA}(x)\}. \quad (8)$$

Since $x \mapsto G^{EA}(x)$ is a continuous function, applying the Corollary 2.9 in (Peskir and Shiryaev 2006), the optimal stopping time for problem (3) is

$$\tau_{\bar{D}} = \inf\{0 \leq s \leq T - t | X_{t+s} \in \bar{D}^{EA}\}. \quad (9)$$

Since $\{(t,x) \in [0,T) \times (0,\infty) | L \leq x \leq K\}$ is inside the continuation region C^{EA}, we can separate the exercised region $\bar{D}^{EA}$ into

$$\bar{D}_1^{EA} = \{(t,x) \in [0,T] \times (0,\infty) | V^{EA}(t,x) = (L - x \vee l)^+\}, \quad (10)$$
$$\bar{D}_2^{EA} = \{(t,x) \in [0,T] \times (0,\infty) | V^{EA}(t,x) = (x \wedge k - K)^+\}. \quad (11)$$

The Free-Boundary of American Eagle Options

Theorem

The continuation region and exercise region are nonempty:

- $\{(t,x) \in (0,T) \times (0,L) | x \leq l \vee b_1^{ST}(t)\} \in \bar{D}_1^{EA}$ *and* $\{(t,x) \in (0,T) \times (0,L) | L \geq x > l \vee b^P(t)\} \in C^{EA}$,
- $\{(t,x) \in (0,T) \times (K,\infty) | x \geq k \wedge b_2^{ST}(t)\} \in \bar{D}_2^{EA}$ *and* $\{(t,x) \in (0,T) \times (K,\infty) | K \leq x < k \wedge b^C(t)\} \in C^{EA}$.

Function b^C and b^P are the free-boundary for American call struck at K and put options struck at L. b_1^{ST} and b_2^{ST} are the lower free-boundary and the higher free-boundary for American strangle options struck at L and K, respectively.

Since exercise region $\bar{D}_1^{EA}$ and $\bar{D}_2^{EA}$ are nonempty, and satisfies the down connectedness and up connectedness, respectively. We can define the lower and upper free-boundary as

$$b_1^{EA}(t) = \sup\{x \in (0,\infty) | (t,x) \in \bar{D}_1^{EA}\}, \quad (12)$$

$$b_2^{EA}(t) = \inf\{x \in (0,\infty) | (t,x) \in \bar{D}_2^{EA}\}. \quad (13)$$

And

$$l \vee b_1^{ST}(t) \leq b_1^{EA}(t) \leq l \vee b^P(t) \text{ for } t \in [0,T), \quad (14)$$

$$k \wedge b^C(t) \leq b_2^{EA}(t) \leq k \wedge b_2^{ST}(t) \text{ for } t \in [0,T). \quad (15)$$

The Property of American Eagle Options

Theorem

The value function for American eagle options $(t, x) \mapsto V^{EA}(t, x)$ *defined in (3) is continuous on* $[0, T] \times (0, \infty)$.

Theorem

The lower free-boundary $t \mapsto b_1^{EA}(t)$ *is increasing function and the upper free-boundary* $t \mapsto b_2^{EA}(t)$ *is decreasing function for* $t \in [0, T]$.

Theorem

As approaching to the maturity T*, the lower free-boundary converges to* $b_1^{EA}(T-) = \max(l, \min(L, \frac{r}{\delta}L))$ *and the upper free-boundary converges to* $b_2^{EA}(T-) = \min(k, \max(K, \frac{r}{\delta}K))$.

Theorem

The free-boundary $b_1^{EA}(t)$ *and* $b_2^{EA}(t)$ *are continuous function on* $[0, T)$.

Part-Smooth-fit

Theorem

As $b_1^{EA}(t) > l$ *or* $b_2^{EA}(t) < k$, *the value function satisfies the smooth-fit property,*

$$\frac{\partial V^{EA}(t,x)}{\partial x}\Big|_{x=b_1^{EA}(t)} = -1, \quad (16)$$

$$\frac{\partial V^{EA}(t,x)}{\partial x}\Big|_{x=b_2^{EA}(t)} = 1. \quad (17)$$

Theorem

As $b_1^{EA}(t) = l$ *or* $b_2^{EA}(t) = k$, *the value function dissatisfies the smooth-fit property, but*

$$\frac{\partial^- V^{EA}(t,x)}{\partial x}\Big|_{x=b_1^{EA}(t)} = 0 \quad \text{and} \quad -1 \leq \frac{\partial^+ V^{EA}(t,x)}{\partial x}\Big|_{x=b_1^{EA}(t)} \leq 0, \quad (18)$$

$$\frac{\partial^+ V^{EA}(t,x)}{\partial x}\Big|_{x=b_2^{EA}(t)} = 0 \quad \text{and} \quad 0 \leq \frac{\partial^- V^{EA}(t,x)}{\partial x}\Big|_{x=b_2^{EA}(t)} \leq 1. \quad (19)$$

Free-Boundary Problem for American Eagle Options

From the Theorem proved above, we can change the optimal stopping problem (3) into a free-boundary problem:

$$V_t^{EA} + \mathbb{L}_X V^{EA} = rV^{EA} \quad \text{in} \quad C^{EA}, \tag{20}$$

$$V^{EA}(t,x) = G^{EA}(x) = (L - x \vee l) \quad \text{for} \quad x = b_1^{EA}(t), \tag{21}$$

$$V^{EA}(t,x) = G^{EA}(x) = (x \wedge k - K) \quad \text{for} \quad x = b_2^{EA}(t), \tag{22}$$

$$V_x^{EA}(t,x) = -1 \quad \text{for} \quad x = b_1^{EA}(t) > l, \tag{23}$$

$$\frac{\partial^- V^{EA}(t,x)}{\partial x} = 0 \text{ and } -1 \leq \frac{\partial^+ V^{EA}(t,x)}{\partial x} \leq 0 \quad \text{for} \quad x = b_1^{EA}(t) = l \tag{24}$$

$$V_x^{EA}(t,x) = 1 \quad \text{for} \quad x = b_2^{EA}(t) < k, \tag{25}$$

$$\frac{\partial^+ V^{EA}(t,x)}{\partial x} = 0 \text{ and } 0 \leq \frac{\partial^- V^{EA}(t,x)}{\partial x} \leq 1 \quad \text{for} \quad x = b_2^{EA}(t) = k \tag{26}$$

$$V^{EA}(t,x) > G^{EA}(x) \quad \text{in} \quad C^{EA}, \tag{27}$$

$$V^{EA}(t,x) = G^{EA}(x) = L - x \vee l \quad \text{in} \quad D_1^{EA}, \tag{28}$$

$$V^{EA}(t,x) = G^{EA}(x) = x \wedge k - K \quad \text{in} \quad D_2^{EA}. \tag{29}$$

The EEP Representation of American Eagle Options

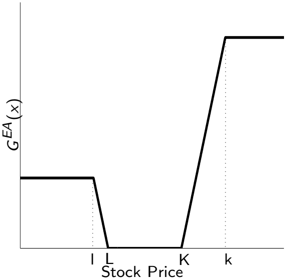
We can apply the extension of time space formula in Remark 3.2 from (Peskir 2005) on $e^{-rs}V^{EA}(t+s, X_{t+s})$, and take the $\mathrm{P}_{t,x}$ expectation from both sides. By the optional sampling theorem, the martingale term will be vanished. Finally, using the equation (20), (28), (29) and taking $s = T - t$, we get the EEP representation of American eagle options

$$\begin{aligned} V^{EA}(t,x) \quad = \quad & e^{-r(T-t)}E_{t,x}G^{EA}(X_T) - E_{t,x}\int_0^{T-t} e^{-ru}(rK - rk)I(X_{t+u} > k)du \\ & - E_{t,x}\int_0^{T-t} e^{-ru}(rK - \delta X_{t+u})I(k > X_{t+s} > b_2^{EA}(t+u))du \\ & - E_{t,x}\int_0^{T-t} e^{-ru}(-rL + rl)I(X_{t+u} < l)du \\ & - E_{t,x}\int_0^{T-t} e^{-ru}(-rL + \delta X_{t+u})I(l < X_{t+u} < b_1^{EA}(t+u))du \\ & + \frac{1}{2}E_{t,x}\int_0^{T-t} e^{-ru}V_x^{EA}(t+u, k-)I(b_2^{EA}(t+u) = k)d\ell_u^k(X) \\ & - \frac{1}{2}E_{t,x}\int_0^{T-t} e^{-ru}V_x^{EA}(t+u, l+)I(b_1^{EA}(t+u) = l)d\ell_u^l(X). \end{aligned} \tag{30}$$

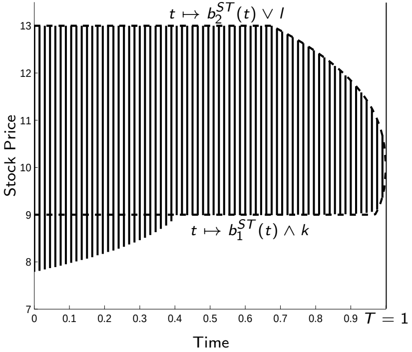
where $e^{-r(T-t)}E_{t,x}G^{EA}(X_T)$ is the value of European eagle options and the term $\ell_u^k(X)$ is the local time for X at k, and $\ell_u^k(X) = \mathrm{P} - \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_0^u I(k-\varepsilon < X_{t+r} < k+\varepsilon)\, d\langle X, X\rangle_{t+r}$.

The Definition American 'Disable' Eagle Options

For the 'disable' eagle, we have $k - K \neq L - l$. Without losing generality, the discussion is based on the assumption $k - K > L - l$. As l approaching to L, the lower cap will be inside the continuation region.



Payoff of Eagle options



Continuation Region of Eagle Options

The figure on the right hand side use the parameter: $l = 9$, $L = 10$, $K = 10$, $k = 13$, $r = \delta = 0.06$, $\sigma = 0.2$, $T = 1$. The bar region is the continuation region

The Property of American 'Disable' Eagle Options

We can analyze the 'disable' eagle options in the same way, except for the following two theorems,

Theorem

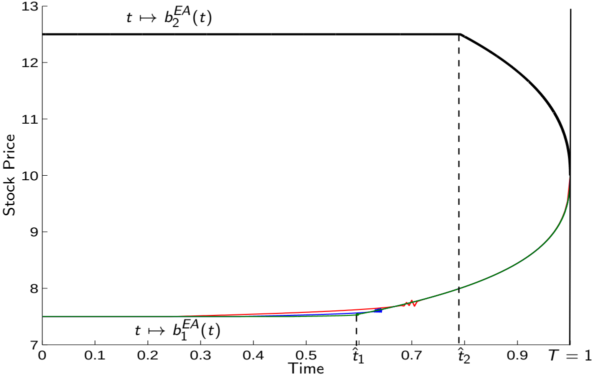
If $V^{ST}(t, l, L, K) - V^{ST}(t, l, l, k) - (L - l) > 0$, (t, l) is inside the continuation region C^{EA}. Notation $V^{ST}(t, x, L, K)$ is the value of American strangle options at (t, x) with lower strike price L and upper strike price K.

Theorem

As $b_1^{EA}(t) < l$, the value function satisfies the smooth-fit property,

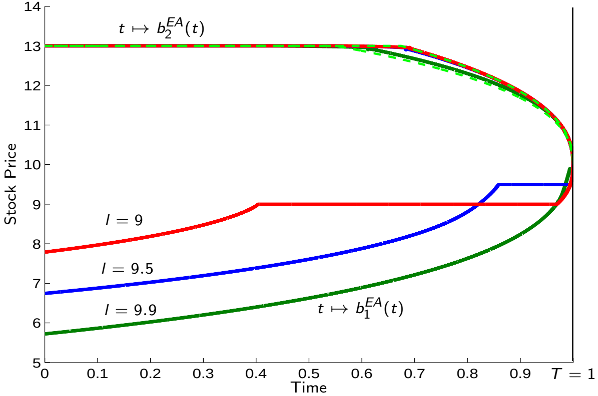
$$\left.\frac{\partial V^{EA}(t,x)}{\partial x}\right|_{x=b_1^{EA}(t)} = 0. \tag{31}$$

The Free-Boundary of American Eagle Options with Balanced Wings



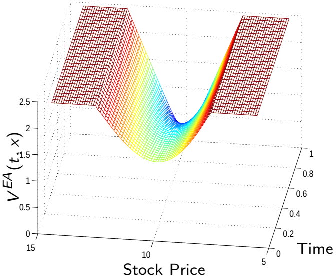
The figure shows the free-boundary of American eagle options with balanced wings. The parameter is: $l = 7.5$, $L = K = 10$, $k = 12.5$, $r = \delta = 0.06$, $\sigma = 0.2$, $T = 1$. The black line label by b_2^{EA} is the upper boundary. The lower boundary in red is approximated by n=200 discretization points in [0,T], the blue line uses n=1000 discretization points and the green line uses n=10000 discretization points.

The Free-Boundary of American 'Disable' Eagle Options

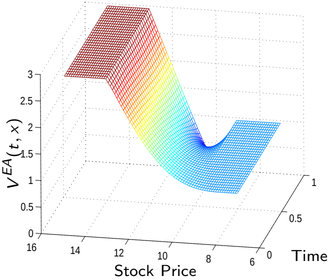


The figure shows the free-boundary of American 'disable' eagle options. The parameter is:$L = K = 10$, $k = 13$, $r = \delta = 0.06$, $\sigma = 0.2$, $T = 1$. The green solid line is the free-boundary for $l = 9.9$, the blue solid line is for $l = 9.5$ and the red line is for $l = 9$. The three upper solid lines labeled by b_2^{EA} is the upper boundary. The three lower boundaries are labeled by b_1^{EA}. The upper dash line is $b_2^{ST}(t) \wedge k$ and the lower dash line is $b_1^{C}(t) \wedge k$. All the line is approximated by n=10000 discretization points in [0,T].

The Value of American Eagle Options



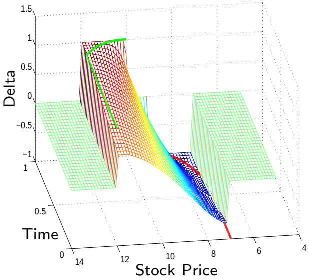
Value of Eagle Options with Balanced Wings



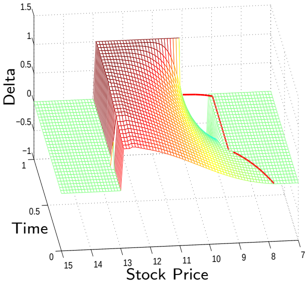
Value of 'Disable' Eagle Options

The left figure shows value of eagle options with balance wings with the parameter $K = 10$, $L = 9$, $k = 12$, $l = 7$, $r = \delta = 0.06$, $\sigma = 0.2$, $T = 1$. Right figure shows value of 'disable' eagle options with parameter $K = L = 10$, $k = 13$, $l = 9$, $r = \delta = 0.06$, $\sigma = 0.2$, $T = 1$.

The Delta of American Eagle Options



Delta of Eagle Options with Balanced Wings



Delta of 'Disable' Eagle Options

The left figure shows Delta value of eagle options with balance wings with the parameter $K = 10$, $L = 9$, $k = 12$,$l = 7$, $r = \delta = 0.06$, $\sigma = 0.2$, $T = 1$. The right figure shows Delta value of 'disable' eagle options with parameter $K = L = 10$, $k = 13$, $l = 9$, $r = \delta = 0.06$, $\sigma = 0.2$, $T = 1$. The green line is upper free-boundary and red line is the lower free-boundary.

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