Population dynamics and Microsimulation

Lesson II

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Work partially funded by the Chair "Risques financiers",
the Chair "Marchés en mutation, and the ANR project "Lolita"

Padoue, Fevrier 2016

Plan

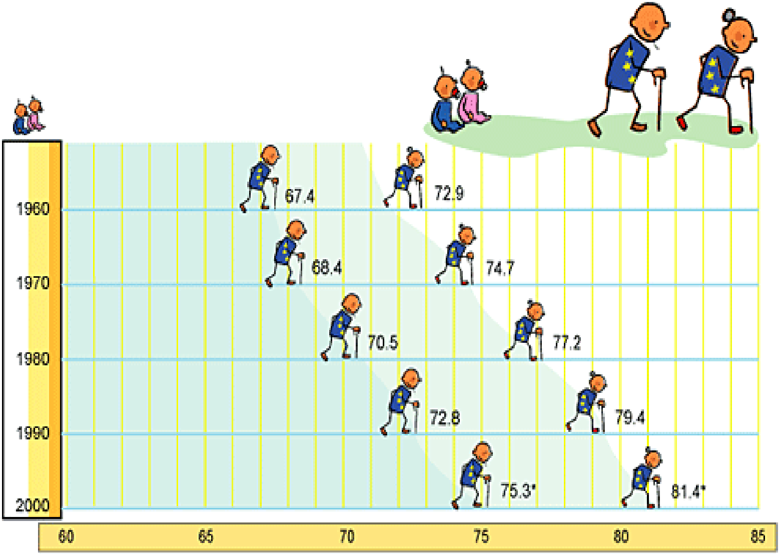


Figure: Longevity Risk

Demographic rates at individual Level

Demographic rates: an individual of traits $x_t \in \mathcal{X} \subset \mathbb{R}^d$ and age $a_t \in [0, \bar{a}]$ at time t, (born at time 0)

- Dies at rate $d\,(x_t, a_t, t, Y)$
- Gives birth at rate $b\,(x_t, a_t, t, Y)$
  and the new individual has traits $x' \sim K^b(x_t, a_t, dx')$
- Evolves during life at rate $e\,(x_t, a_t, t, Y)$
  from traits x_t to $x' \sim K^e(x_t, a_t, dx')$

Environmental factors

- Demographic rates depend on characteristics, age, time and on the stochastic environment Y
- Conditionally on the environment Y, the events for a given individual are jumps of a counting process

Thinning equations for spatial birth processes

The thinning construction can be used to define a wide variety of processes as solution to stochastic equations.

Intensity for spatial Birth Process

- naissance +"mutation"individual = rate becomes $b(x')k_b(x',x)m(dx')$
- aggregated rate of birth mutation=

$$\beta(\xi,x)=\sum_{x'\in\xi} b(x')k_b(x',x)$$

- Equation $Z(dt,dx)=\int_{\mathbb{R}_+} \mathbf{1}_{\theta\leq\beta(Z_{t-},x)}Q_b(dt,dx,d\theta)$

Birth with age

- First define the new kernel with the age
- Applied the previous relation to process $d\langle Z_t,f\rangle=\langle I^b_{Z_{t-}}f(.,0),Q_b\rangle(dt)+\langle Z_t,\partial_a f\rangle dt$
- Existence result similar to the linear case

Death process

Fondamental asymmetry

- since the newborn is from outside,
- then the death remove an individual in the population

How to select an individual by its characteristics

- the counting measure on E is not a "Radon" σ-finite measure on E
- Necessity to give a measurable and adapted process to select individual in a given population

Envelop process of population path without accumulation

- The age desagregated population $\tilde{\xi}_s(dx) = \xi_s(dx, \mathbb{R}_+)$
- The non decreasing envelop $\bar{\xi}_t = \bigcup_{s\in[0,t]} \tilde{\xi}_s$ process
- $\bar{\xi}_t$ has only finite number of jumps on $[0, t]$, denoted by (S_k)
- (S_k) are also times of jumps for the path ξ_t
- The sequence $(X_k, A_k(.))_{k\geq N_0+1}$, $X_k = \bar{\xi}_{S_{k-N_0}} \setminus \bar{\xi}_{S^-_{k-N_0}}$, and $A_k(t) = t - S_k$

Spatial death process

- A Poisson point measure $Q_d(ds, di, d\theta)$ on $\mathbb{R}_+ \times \mathbb{N}^* \times \mathbb{R}_+$
- with intensity measure $q_d(ds, di, d\theta) = ds\, n(di)\, d\theta$
- $I^d(Z_{t-}, i, \theta) = \mathbf{1}_{X_i\in Z_{t-}} \mathbf{1}_{\theta\leq d(X_i)}$
- Using the previous numbering, we see that $Z(dt, dx) = -\int_{i\in\mathbb{N}^*} \int_{\theta\in\mathbb{R}_+} I^d(Z_{t-}, i, \theta)\delta_{X^i}(dx) Q_d(dt, di, d\theta)$,
- Same transformation with age

Theorem Bezborodov (2014), Garcia 1999, ...

- If $\xi_0^1 \subset \xi_0^2$,
- $\beta_1(x, \eta^1) \leq \beta_2(x, \eta^2) \quad \eta^1 \subset \eta^2$
- $d_1(x, \eta^1) \geq d_2(x, \eta^2) \quad \eta^2 \subset \eta^1, x \in \eta^1$

The comparison theorem

There exists a cadlag process (η_t) *such that* $\eta_t \subset \xi_t^2$ *having the same law that* (ξ_t^1)

Sketch of the proof without age, and swap

$$\eta(dt, B) = \int_{B\times\mathbb{R}^+} \mathbf{1}_{[0, b_1(x, \eta_{s-})]}(\theta) dQ^b(dt, dx, d\theta)$$

$$- \int_{\mathbb{N}\times\mathbb{R}^+} \mathbf{1}_{\{x_i^2 \in \eta_{s-} \cap B\}} \mathbf{1}_{[0, d_1(x_i^2, \eta_{s-})]}(\theta) dQ^d(dt, di, d\theta)$$

Applications of comparison theorem

In progess

Study classical properties of population processes

- Agregation by traits and convexity
- Localisation and explosion
- Monotonic convergence

Stochastic order on the space of configuration

- Starting from the result of Preston (1975) on the stochastic order for the Point random field
- Property of the stochastic order on the distributions of the population processes Z_t in terms of demographic characteristics

The Poisson measures driving the equation

- Q_b, Q_d, Q_e
- $I^b(Z_{t-}, t, x, \theta)$, $I^d(Z_{t-}, t, i, x, \theta)$, $I^e(Z_{t-}, t, i, x, \theta)$

The BSD Population equation

$$d\langle Z_t, f\rangle = \langle I^b_{Z_{t-},t} f(.,0), Q_b\rangle(dt) - \langle I^d_{Z_{t-},t} f(X_., A_.(t)), Q_d\rangle(dt) \\ + \langle I^e_{Z_{t-},t}[f(., A_.(t)) - f(X_., A_.(t))], Q_e\rangle(dt) + \langle Z_t, \partial_a f\rangle dt. \quad (1)$$

Hypotheses, $E = \mathbb{R}^d$, $m(dx) = l(dx)$

- $\int b(x, \eta)dx \leq c_1|\eta| + c_2$
- $\sup_x \sup_{\{|\eta| \leq m\}} d(x, \eta) < \infty$

Then, existence and strong uniqueness

Plan

Cohort effect

An example of numerical experiment to explain an observed phenomenon

Cohort effect

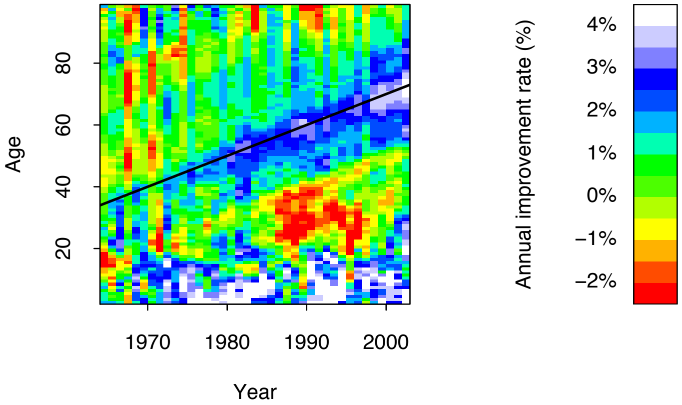
Birth cohort for the period $[t_1, t_2]$: group of individuals born between t_1 and t_2.

- Individuals of the same birth cohort share similar demographic characteristics ("cohort effect")
- Age, Period, Cohort analysis put a lot of problems in practice, in different domains, medecine, sociology,...due to the lag in data,..insurance...
- Huge literature on APC problems

Golden cohort

Golden cohort: generations born between 1925 and 1940
Cairns et al. (2009) $r_{a,t} = (q_{a,t-1} - q_{a,t})/q_{a,t}$

The Golden cohort has experienced more rapid improvements than earlier and later generations.

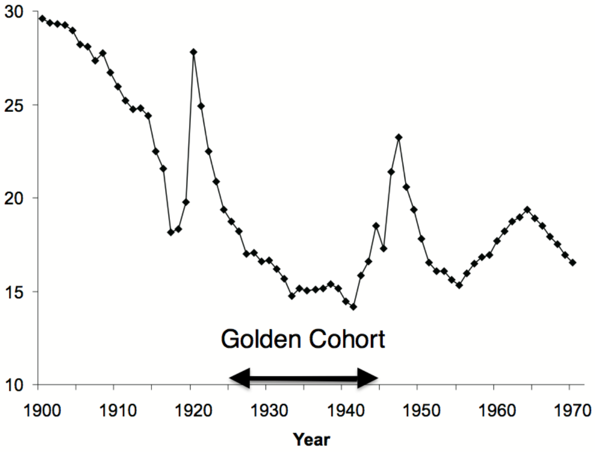


Some possible explanations:

- Impact of World War II on previous generations,
- Changes on smoking prevalence: tobacco consumption in next generations,
- Impact of diet in early life,
- Post World War II welfare state,
- Patterns of birth rates

"One possible consequence of rapidly changing birth rates is that the average child is likely to be different in periods where birth rates are very different. For instance, if trends in fertility vary by socio-economic class, the class mix of a population will change."

The Cohort Effect: Insights And Explanations, 2004, R. C. Willets



Data source: www.mortality.org

Figure 6. Crude birth rate per 1,000 population, England and Wales, 1900 to 1970

Simple toy model

The different rates

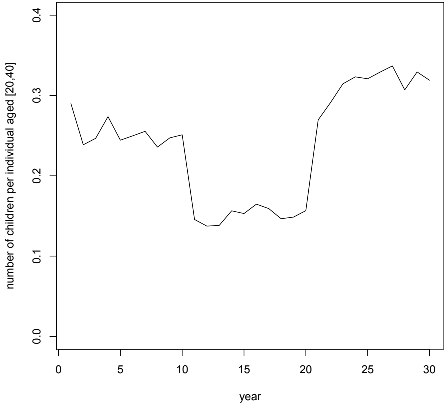
- Reference death rate $\bar{d}(a) = A\exp(Ba)$
- Parameters $A \sim 0.0004$ and $B \sim 0.073$ estimated on French national data for year 1925 to capture a proper order of magnitude
- "Upper class": time independent death rate $d^1(a) = \bar{d}(a)$ and birth rate $b^1(a) = c\mathbf{1}_{[20,40]}(a)$ (c=0.1)
- "Lower class": time independent death rate $d^2(a) = 2\bar{d}(a)$ but birth rate $b^2(a, t) = 4c\mathbf{1}_{[20,40]}(a)\mathbf{1}_{[0,t_1]\cup[t_3,\infty)}(t) + 2c\mathbf{1}_{[20,40]}(a)\mathbf{1}_{[t_2,t_3]}(t)$

Comment Constant death rates but reduction in overall fertility between times t_1 (=10) and t_2 (=20).

- Aim: Test the cohort effect by computing standard demographic indicators on the population

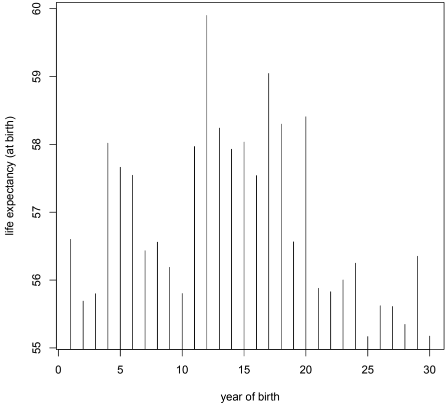
Aggregate fertility

- One trajectory with 20000 individuals (randomly) splitted between groups. Estimation of aggregate fertility



Life expectancy by year of birth

- "Cohort effect" for aggregate life expectancy



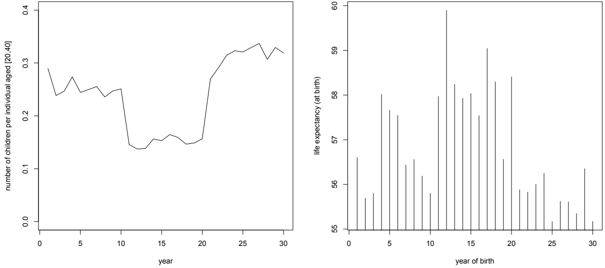


Figure: Observed fertility (left) and estimated life expectancy by year of birth (right)

- Death rates by specific group remain the same
- But reduction in fertility for "lower class" during 10-20 modifies the generations composition
  ⇒ "upper class" is more represented among those born between 10 and 20

Plan

The Poisson measures driving the equation

- $Q_b,\ Q_d,\ Q_e$
- $I^b(Z_{t-}, t, x, \theta)$, $I^d(Z_{t-}, t, i, x, \theta)$, $I^e(Z_{t-}, t, i, x, \theta)$

The BSD Population equation

$$\begin{aligned} d\langle Z_t, f\rangle &= \langle I^b_{Z_{t-},t} f(.,0), Q_b\rangle(dt) - \langle I^d_{Z_{t-},t} f(X_., A_.(t)), Q_d\rangle(dt) \\ &+ \langle I^e_{Z_{t-},t}[f(., A_.(t)) - f(X_., A_.(t))], Q_e\rangle(dt) + \langle Z_t, \partial_a f\rangle dt. \end{aligned} \tag{2}$$

Hypotheses, $E = \mathbb{R}^d$, $m(dx) = l(dx)$

- $\int b(x,\eta)dx \leq c_1|\eta| + c_2$
- $\sup_x \sup_{\{|\eta| \leq m\}} d(x,\eta) < \infty$

Then, existence and strong uniqueness

Martingale problem

(No evolution for simplicity of notations)

Conditionally on the complete path of Y, in conditionally independency framework

$$M_t(f) = < Z_t, f_t > - < Z_0, f_0 > - \int_0^t ds \int Z_s(dx, da)\Big[(\frac{\partial f_s}{\partial a} + \frac{\partial f_s}{\partial s})(x, a)$$

$$+ b(x, a, s, Y) \int_{\mathcal{X}} f_s(x', 0) K^b(x, a, dx') - d(x, a, s, Y) f_s(x, a)\Big] \tag{3}$$

is a square integrable martingale with quadratic variation

$$< M(f) >_t = \int_0^t ds \int_{\mathcal{X} \times [0, \bar{a}]} Z_s(dx, da) \times$$

$$\left[b(x, a, s, Y) \int_{\mathcal{X}} f_s^2(x', 0) K^b(x, a, dx') + d(x, a, s, Y) f_s^2(x, a)\right] \qquad (5)(4)$$

Macroscopic approximation

Assumptions

- Renormalization: pop. described by the measure $\bar{Z}^n_t(dx,da)=\frac{1}{n}\sum_{i=1}^{N^n_t}\delta_{(x^i_t,a^i_t)}$ (each individual has weight $1/n$)
- Weak convergence of the initial population as $n\to+\infty$: $\bar{Z}^n_0(dx,da)\Rightarrow g_0(x,a)\gamma(dx)da$ (initial size is of order n)
- By homogeneity, the quadratic variation of $\bar{M}^n(f)$ is of order $\frac{1}{n}$ and so goes to 0
- Cv in distribution on the canonical space of cadlag measure valued process of the process $(\bar{Z}^n_t)$)

Conditionnaly to Y, limit PDE

- Law of Large Numbers (Heuristic): the noise vanishes as the size n of the initial population goes to infinity
  $\Rightarrow$ deterministic behavior in time for large populations

Limit PDE

- Limit process as the size $\rightarrow +\infty$
- Weak convergence of $(Z_t^n(dx, da))_{t\geq 0}$ to the solution $\big(g_t(x,a)\gamma(dx,da)\big)_{t\geq 0}$ of conditional (wrt Y) deterministic PDE

Link between two description in a given environment:

- microscopic: stochastic behavior of each individual
- macroscopic: deterministic evolution of the whole population

Deterministic equations in demography

- Malthus (1798), Verhulst (1838): pop. structured by traits $g(x,t)$: density of individuals of trait x at time t

$$\frac{\partial g}{\partial t}(x,t) = \int_{\mathcal{X}} g(y,t)b(y)k^b(y,x)dy - d(x)g(x,t),$$
$$g(x,0) = g_0(x).$$

- McKendrick (1926), VonFoerster (1959): structured by age $g(a,t)$: density of individuals of age a at time t

$$\frac{\partial g}{\partial t}(a,t) + \underbrace{\frac{\partial g}{\partial a}(a,t)}_{\text{transport}} = -d(a)g(a,t), \quad \underbrace{g(0,t) = \int_0^{+\infty} b(a)g(a,t)da}_{renewal}$$
$$g(a,0) = g_0(a).$$

Large population limit

PDE for the population density $g(x,a,t)$: approximation (a.s) for large populations with

- stochastic environment Y
- evolution during life

$$\left(\frac{\partial g}{\partial t}+\frac{\partial g}{\partial a}\right)(x,a,t) = -d(x,a,t,Y)g(x,a,t)$$

$$-e(x,a,t,Y)g(x,a,t)+\int_{\mathcal{X}} e(x^{'},a,t,Y)k^{e}(x^{'},a,x)g(x^{'},a,t)\gamma(dx^{'})$$

$$g(x,0,t) = \int_{\mathcal{X}\times[0,\bar{a}]} b(x^{'},a,t,Y)k^{b}(x^{'},a,x)g(x^{'},a,t)\gamma(dx^{'})da$$

$$g(x,a,0) = g_0(x,a)$$

- Take advantage of the impact of pure environment noise

Plan

Heterogeneity

Longevity patterns and longevity improvements are very different for different countries, and different geographic area.

Factor affecting mortality

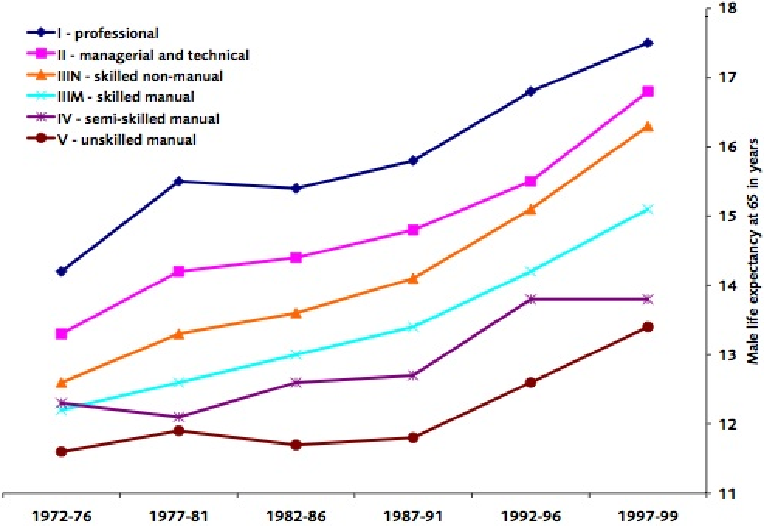
- socio-economic level (occupation, income, education, wealth...)
- gender
- marital status
- living environment (pollution, nutritional standards, hygienic...)
- Take them into account in a stochastic mortality model

Conditional calibration

- On national mortality data and on specific data (with information on individual characteristics)
- In France, specific data=Permanent demographic sample=992711 persons, died only from 1967, born in October only from 1866

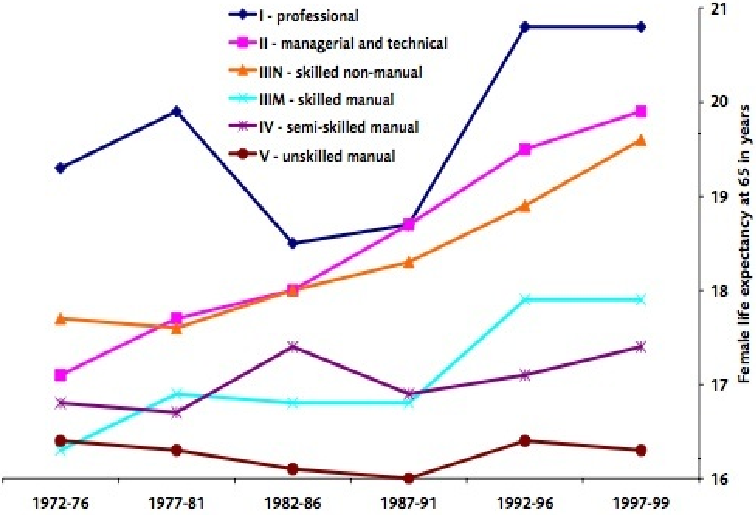
Male Life Expectancy from age 65

Figure 10. Male life expectancy from age 65 by socio-economic group



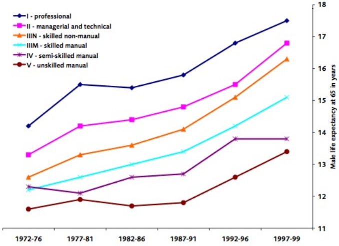
Female Life Expectancy from age 65

Figure 11. Female life expectancy from age 65 by socio-economic group



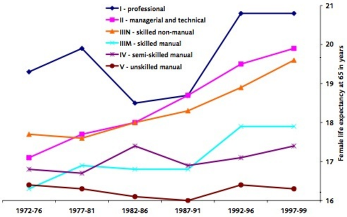
Social Heterogenity of Life Expectancy

Figure 10. Male life expectancy from age 65 by socio-economic group



Source: ONS Longitudinal Survey.

Figure 11. Female life expectancy from age 65 by socio-economic group



Source: ONS Longitudinal Survey.

Basis risk I

Difference : national mortality versus that of specific group

- Insurance companies can use national reliable mortality estimates on large samples
- but the final goal is to model mortality rates specific to subpopulations with owns traits
  - population of a small country or region,
  - individuals with a specific disease,
  - insurance portfolio,
  - annuitants of sectorial pension funds.
- But also how take into account other informations
  - They know the exact ages at death and not only the year of death (time continuous data)
  - Cause of death are specified

Basis risk II

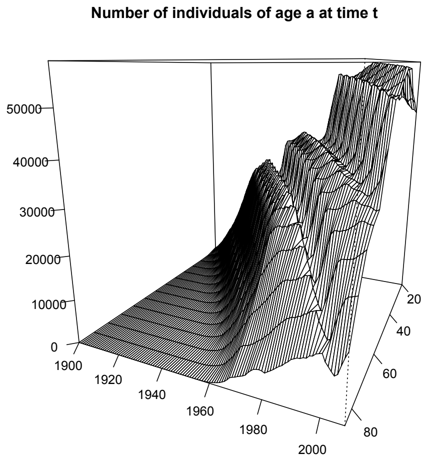
- Characteristics of the policyholders : socio economic level, living conditions ...
- selection bias

- BUT
  - limited size of their portfolios (in comparison to national populations : 700 000 individuals from 19 different insurance companies)
  - small range of the observation period

This heterogeneity is very important for longevity risk transfer based on national indices: for too important basis risk, the hedge would be too imperfect

Permanent Demographic Sample

- Number of individuals at each year by 5 years age groups in the sample



Some tests on three characteristics of interest

- Education level
  - Group 1: Diploma $\leq$ Baccalaurat (high school diploma)
  - Group 2: Diploma $>$ Baccalaurat (high school diploma)
- Socio-professional category
  - Groupe 1: Employees and workers
  - Groupe 2: Executives and higher intellectual professions, intermediaries professional categories
- Marital status
  - Group 1: Single or divorced
  - Group 2: Married or widowed

Mortality heterogeneity: education level

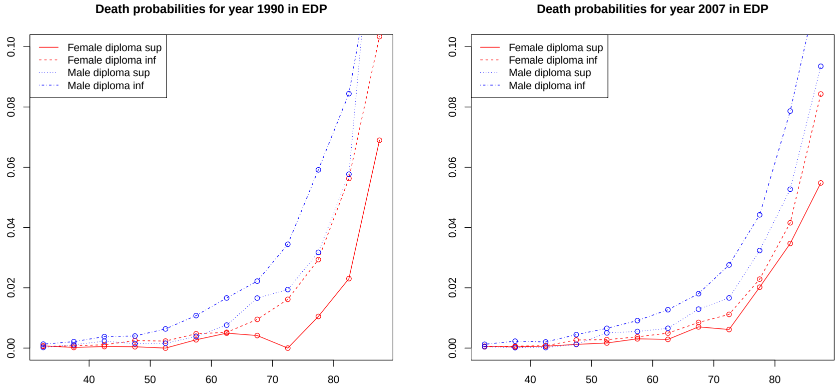


Figure: Death probabilities by education level: years 1990 and 2007

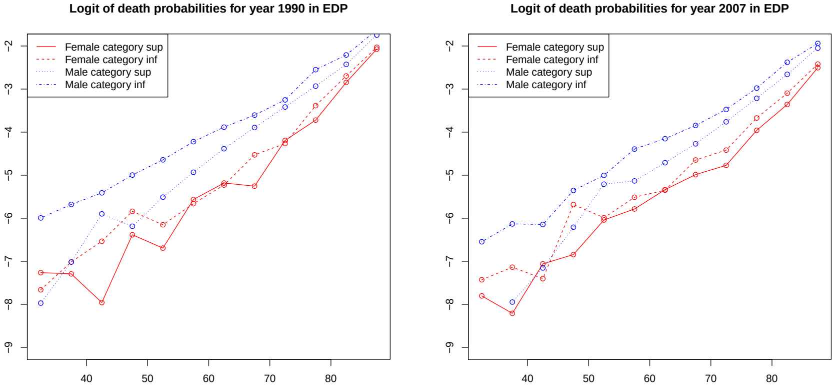


Figure: Logit of death probabilities by socio-professional category: years 1990 and 2007

Mortality heterogeneity: marital status

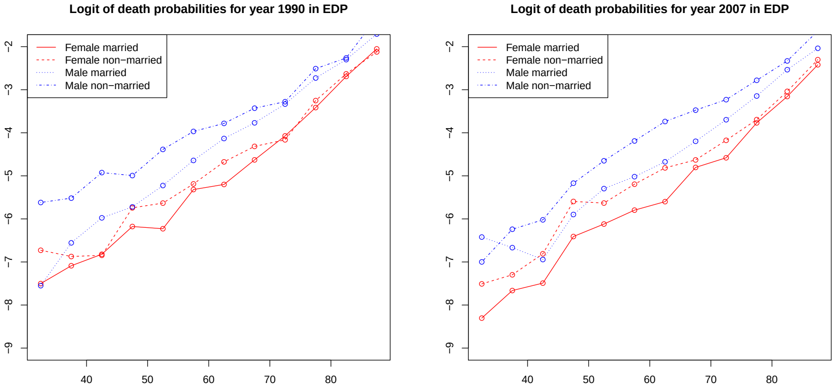
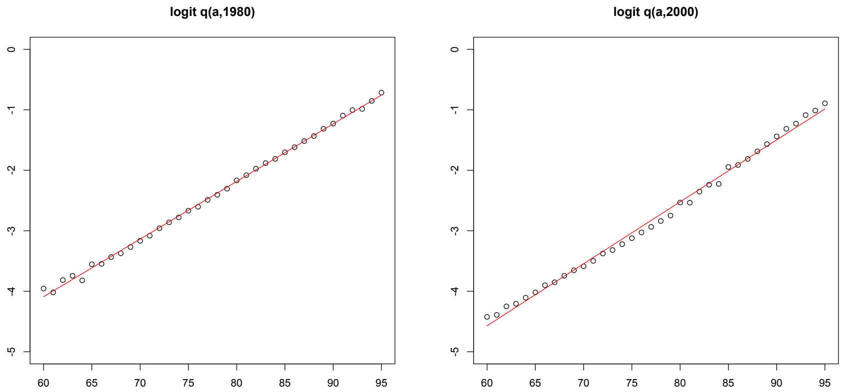


Figure: Logit of death probabilities by marital status: years 1990 and 2007

Cairns-Blake-Dowd model

- Logit of annual death probabilities for years 1980 and 2000 (French males)



Model for high ages (Cairns, Blake, Dowd, 2006):

$$\text{logit }(q(a,t)) = Y_1(t) + a.Y_2(t) + \epsilon_{a,t},$$

- $Y_1(t)$: overall reduction in mortality through time, for all ages,
- $Y_2(t)$: specific adjustment at each age,
- $\epsilon_{a,t}$ is the residual noise.

$\Rightarrow$ choice of a particular form of age dependency (Compertz=linear)

$\Rightarrow$ 2 time factors

Estimating parameters: for each year t between 1980 and 2007, we perform the linear regression over ages between 60 and 95, which gives parameters $Y_1(t)$ and $Y_2(t)$ (for men and women separately)

- Compression effect: constraint linking Y_1 and Y_2
  $\Rightarrow$ Mortality improvement transferred from old (~ 95) to younger ages (~ 60)

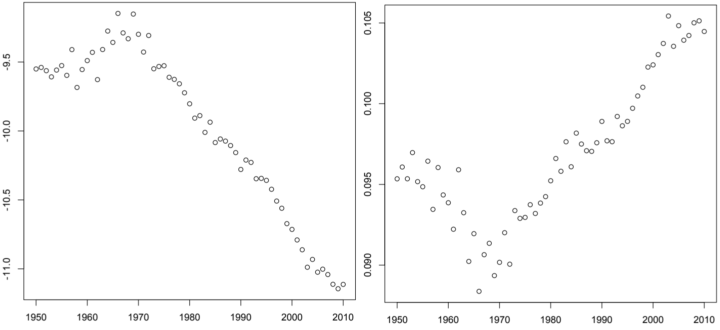


Figure: Processes Y_1 (left) and Y_2 (right) estimated for French males (ages 60-95) between 1950 and 2010

Time series Y_1 and Y_2 can be viewed as a fluctuating environment

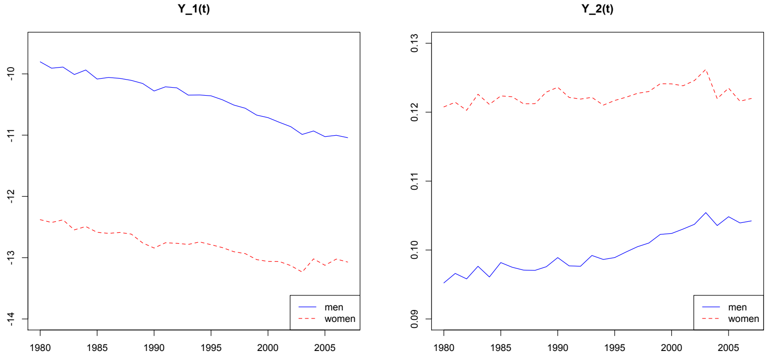
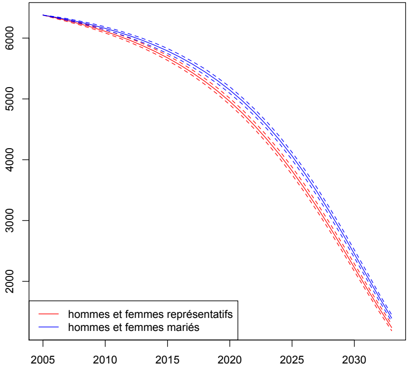


Figure: Estimated environment four factors on French data for ages 60-95 and years 1980-2007.

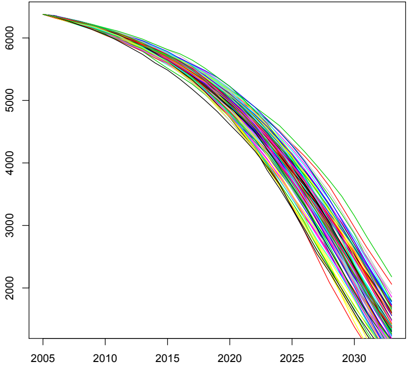
Numerical example: French sub-population

- Cohort of French (males and females) aged 61 at the beginning of year 2005 in the Permanent Demographic Sample
- Confidence intervals at 90% for the number of individuals without environment noise



Numerical example: French sub-population

- The model allows to simulate the evolution of the population, subject to various death rates dues to different environment scenarios



Numerical example: French sub-population

- Application to an insurance portfolio: initial age distribution

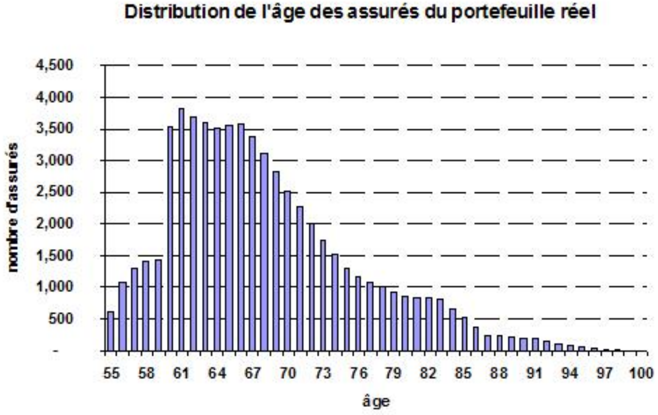
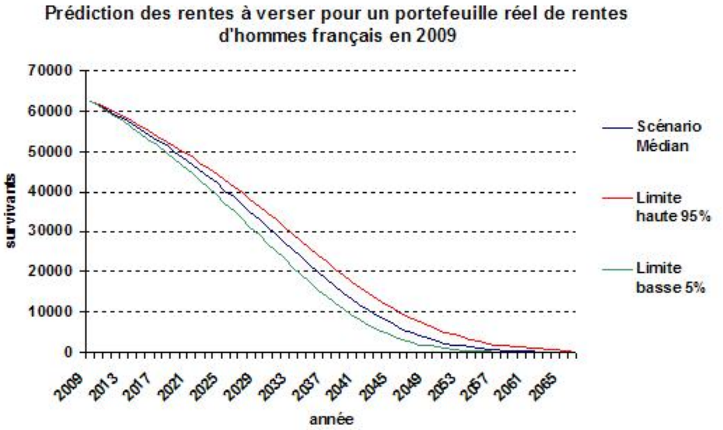


Figure: Confidence interval at 90% on the size of the insured population

- Application to an insurance portfolio: pension amount



With the indulgence of Sau, Gold of health and longevity

Thank you