Polynomial Preserving Jump-Diffusions on the Unit Interval

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February 8, 2016

Definitions

Consider the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and choose the state space $E := [0, 1]$. Denote by

- $\mathrm{Pol}_n(E)$ the set of all polynomials of degree at most n on E,
- $\mathrm{Pol}(E)$ the set of all polynomials on E.

Polynomial Preserving Operator

Consider a linear operator $\mathcal{G}$ acting on $\mathrm{Pol}(E)$ of the form

$$\mathcal{G}f\,(x) := \frac{a(x)}{2}f''(x)+b(x)f'(x)+\int_{\mathbb{R}} f(x+\xi)-f(x)-f'(x)\xi\,\nu(x,\mathrm{d}\xi),$$

where a and b are functions on E and $\nu(x,\cdot)$ is a Levy measure supported on $E-x$, for all $x \in E$.

Definition

The operator $\mathcal{G}$ is called *polynomial preserving* if and only if

$$\mathcal{G}p \in \mathrm{Pol}_n(E) \qquad \forall p \in \mathrm{Pol}_n(E),$$

for all $n \in \mathbb{N}$.

Martingale Problem for $(\mathcal{G}, E)$

Let $X := (X_t)_{t\geq 0}$ be an adapted RCLL process and ρ be a probability measure supported on E. Then the law of X is called a *solution to the martingale problem for* $(\mathcal{G}, E, \rho)$ if

$$\mathbb{P}(X_0 \in \cdot) = \rho, \qquad \mathbb{P}(X_t \in E) = 1 \quad \forall t \geq 0,$$

and the process $(N_t^p)_{t\geq 0}$, where

$$N_t^p := p(X_t) - p(X_0) - \int_0^t \mathcal{G}p(X_{s-})\mathrm{d}s$$

is a martingale $\forall p \in \mathrm{Pol}(E)$.

Polynomial Preserving Jump-Diffusions

Definition

An adapted RCLL process $X := (X_t)_{t \geq 0}$ is called *polynomial preserving* if its law is a solution to the martingale problem for $(\mathcal{G}, E, \rho)$ for some polynomial preserving operator $\mathcal{G}$ and some probability measure ρ supported on E.

Remark

Since E is compact, one can show that the law of the process X is the unique solution to the martingale problem for $(\mathcal{G}, E, \rho)$.

Question 1

Recall that

$$\mathcal{G}f\,(x) := \frac{a(x)}{2}f''(x) + b(x)f'(x) + \int_{\mathbb{R}} f(x+\xi) - f(x) - f'(x)\xi \;\nu(x, \mathrm{d}\xi).$$

Question: How to choose a, b, and ν such that $\mathcal{G}$ is polynomial preserving?

- Cuchiero, Keller-Ressel, Teichmann, 2012
  1. $b \in \mathrm{Pol}_1(E)$,
  2. $a + \int_{\mathbb{R}} \xi^2 \nu(\cdot, \mathrm{d}\xi) \in \mathrm{Pol}_2(E)$,
  3. $\int \xi^n \nu(\cdot, \mathrm{d}\xi) \in \mathrm{Pol}_n(E)$ for all $n \geq 3$.

Question 2

Recall that

$$\mathcal{G}f\,(x) := \frac{a(x)}{2} f''(x) + b(x) f'(x) + \int_{\mathbb{R}} f(x+\xi) - f(x) - f'(x)\xi \; \nu(x, \mathrm{d}\xi).$$

Question: How to choose a, b, and ν such that the martingale problem for $(\mathcal{G}, E, \rho)$ has a solution for every initial distribution ρ?

- Positive maximum principle:

$$f \in \mathrm{Pol}(E),\; x_0 \in E, \quad \text{and} \quad \sup_{x \in E} f(x) = f(x_0) \quad \Rightarrow \quad \mathcal{G}f(x_0) \leq 0.$$

- Ethier, Kurtz 2005; Filipović, Larsson 2014.

Suppose that $\nu = 0$: the diffusion case

In this case

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- $b \in \mathrm{Pol}_1(E)$ and $a \in \mathrm{Pol}_2(E)$.
- $a \geq 0$ on E.

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- $b \in \mathrm{Pol}_1(E)$ and $a \in \mathrm{Pol}_2(E)$.
- $a \geq 0$ on E.
- $b(0) \geq 0$, $b(1) \leq 0$, and $a(0) = a(1) = 0$.

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- $a \geq 0$ on E.
- $b(0) \geq 0$, $b(1) \leq 0$, and $a(0) = a(1) = 0$.

Hence,

$$a(x) = \sigma^2 x(1-x) \quad \text{and} \quad b(x) = -\beta(x-\theta),$$

for some $\theta \in [0,1]$, and $\beta, \sigma \geq 0$.
The solution of the martingale problem associated to this $\mathcal{G}$ is called *Jacobi process*.

The structure of ν: simple polynomial jump sizes

Assume now that $(\nu(x,\cdot))_{x\in E}$ has simple polynomial jump sizes, i.e. for all $A \in \mathcal{B}(\mathbb{R})$ we have

$$\nu(x,A) = \int_A \nu(x,\mathrm{d}\xi) = \lambda(x) \int_{\mathrm{supp}(\mu)} \mathbb{1}_A\big(\gamma(x,y)\big)\mu(\mathrm{d}y),$$

where

- The measure μ is a σ-finite measure on some space $(B,\mathcal{B})$.
- The *jump size* $\gamma(x,\cdot)$ is polynomial in x on E, namely

$$\gamma(x,\cdot) = \sum_{k=0}^{N} a_k(\cdot)x^k \qquad \text{for all } x \in E,$$

  for square integrable random variables $(a_k)_{k=0}^N$ on $(B,\mathcal{B},\mu)$.
- The *jump intensity* $\lambda : E \to \mathbb{R}_+$ is a measurable function.

The operator $\mathcal{G}$

In this setting the operator $\mathcal{G}$ can be written in the following form

$$\mathcal{G}f(x) = \frac{a(x)}{2} f''(x) + b(x)f'(x) + \lambda(x) \int_{\mathrm{supp}(\mu)} f\left(x + \gamma(x,y)\right) - f(x) - f'(x)\gamma(x,y) \; \mu(\mathrm{d}y),$$

where $\gamma(x,\cdot) = \sum_{k=0}^{N} a_k(\cdot)x^k$.

Characterisation

Recall the operator

$$\mathcal{G}f(x) = \frac{a(x)}{2} f''(x) + b(x)f'(x) + \lambda(x) \int_{\mathrm{supp}(\mu)} f(x+\gamma(x,y)) - f(x) - f'(x)\gamma(x,y)\mu(\mathrm{d}y).$$

Theorem

The operator $\mathcal{G}$ is polynomial preserving and there exists a solution to the martingale problem for $(\mathcal{G}, E, \rho)$ for each initial distribution ρ on E, iff

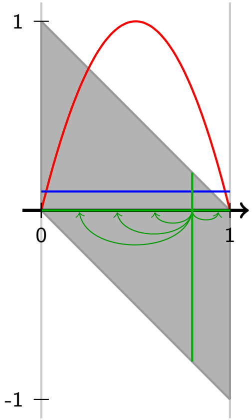
- *The measure μ and the jump size γ can be chosen such that*

$$\mathrm{supp}(\mu) \subseteq [0,1]^2, \qquad \gamma(x,y) = y_1(-x) + y_2(1-x),$$

  and y_1, y_2 are μ-square integrable.
- *$b \in \mathrm{Pol}_1(E)$, $b(0)$ is positive enough, and $b(1)$ is negative enough.*
- *One of the following four cases holds true.*

Case 1

- $\lambda \equiv$ const.
- y_1 and y_2 are μ-integrable.
- $a(x) = Ax(1-x)$ for some $A \geq 0$.



Case 2: "No jump point" $x^* \in \partial E$, wlog $x^* = 0$.

- For μ-almost every $y \in [0,1]^2$ and $x \in E$:

$$\gamma(x,y) = -(y_1 + y_2)x.$$

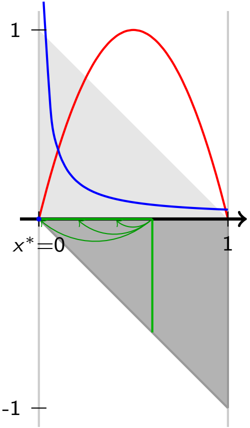
- For all $x \in E$:

$$\lambda(x) = \frac{q_1(x)}{x} \mathbb{1}_{\{x \neq 0\}}$$

  for some nonnegative $q_1 \in \mathrm{Pol}_1(E)$.

- If $q_1(1) \neq 0$, y_1 and y_2 are μ-integrable.

- $a(x) = Ax(1-x)$ for some $A \geq 0$.



Case 3: “No jump point” $x^* \in \mathrm{int}(E)$.

- For μ-almost every $y \in [0,1]^2$ and $x \in E$:

$$\gamma(x,y) = -(y_1 + y_2)(x - x^*).$$

- For all $x \in E$:

$$\lambda(x) = \frac{q_2(x)}{(x - x^*)^2} \mathbb{1}_{\{x \neq x^*\}}$$

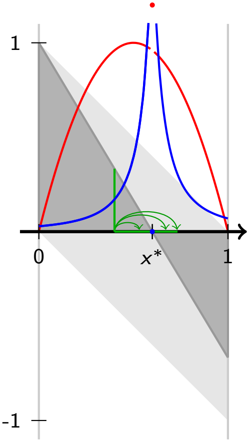
for some nonnegative $q_2 \in \mathrm{Pol}_2(E)$.

- If $q_2(0) \neq 0$ or $q_2(1) \neq 0$, y_1, y_2 are μ-integrable.

- For some $A \geq 0$:

$$a(x) = Ax(1-x) + C\mathbb{1}_{\{x=x^*\}} \quad \forall x \in E$$

for some $C > 0$ uniquely determined by λ and μ.



Case 4: No “no jump points”.

- For some $\alpha \in \mathbb{C} \setminus \mathbb{R}$:

$$\int \gamma^n(\alpha, y)\mu(dy) = 0 \quad \forall n \geq 3.$$

- For all $x \in E$:

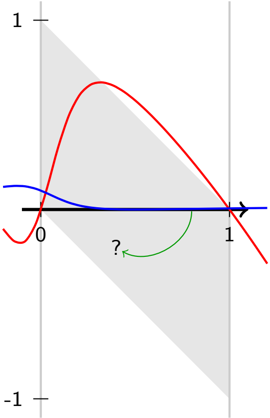
$$\lambda(x) = \frac{q_2(x)}{(x-\alpha)(x-\overline{\alpha})}$$

  for some nonnegative $q_2 \in \mathrm{Pol}_2(E)$.

- If $q_2(0) \neq 0$, y_2 is μ-integrable and
  if $q_2(1) \neq 0$, y_1 is μ-integrable.

- For some $q_2^a \in \mathrm{Pol}_2(E)$:

$$a(x) = q_2^a(x) - \lambda(x) \int \gamma^2(x, y)\mu(dy) \quad \forall x \in E.$$



Does Case 4 really exist?

Until now no probability measure μ on $[0,1]^2$ has been found, such that for all $n \geq 3$

$$\int \gamma^n(\alpha, y)\mu(dy) = \int \left(y_1(-\alpha) + y_2(1-\alpha)\right)^n \mu(dy) = 0$$

for some $\alpha \in \mathbb{C} \setminus \mathbb{R}$.
Can this condition be satisfied?

- This condition cannot be satisfied if α (or its conjugate) is not contained in the circle of radius $1/\sqrt{3}$ centered in $(1/2, -1/(2\sqrt{3}))$.
- This condition cannot be satisfied if μ is the Lebesgue measure on $[0, 1]^2$.
- ...

Example 1: Extension of the Jacobi P. (Cuchiero, 2011)

Definition

The Jacobi process is the solution of the stochastic differential equation

$$dX_t = -\beta(X_t - \theta)dt + \sigma\sqrt{(X_t(1 - X_t))}dW_t, \quad X_0 = x \in [0, 1],$$

on $[0, 1]$, where $\theta \in [0, 1]$ and $\beta, \sigma > 0$.

Its (extended) infinitesimal generator is given by

$$\mathcal{G}f(x) := \frac{1}{2}\sigma^2(x(1 - x))f''(x) - \beta(x - \theta)f'(x).$$

Hence the Jacobi process is a PP process on $[0, 1]$.

Example 1: Extension of the Jacobi P. (Cuchiero, 2011)

This example can be extended by adding jumps, where the jump times correspond to those of a Poisson process with intensity λ and if a jump occurs, then the process is reflected at $\frac{1}{2}$. The (extended) infinitesimal generator is then given by (Case1)

$$\mathcal{G}f = \frac{1}{2}\sigma^2(x(1-x))f''(x) + \big(-\beta(x-\theta) + \lambda(1-2x)\big)f'(x) + \lambda\int_{[0,1]^2} f(x+\gamma(x,y)) - f(x) - f'(x)\gamma(x,y)\ \delta_{(1,1)}(dy),$$

where, $\gamma(x,y) := y_1(-x)+y_2(1-x) = 1-2x\ \delta_{(1,1)}$-almost sure.

Example 2

Consider an operator of the form

$$\mathcal{G}f(x) = \frac{a(x)}{2}f''(x)+b(x)f'(x)+\lambda(x)\int_{[0,1]^2} f(x+\gamma(x,y))-f(x)-\gamma(x,y)f'(x)\mu(\mathrm{d}y)$$
$$= (-2x)f'(x)+\frac{1}{x}\int_{[0,1]} f(x+\gamma(x,y_1))-f(x)-\gamma(x,y_1)f'(x)\gamma(x,y_1)\,\mathrm{d}y_1.$$

where

$$\gamma(x,y_1) := \sin^2((x+y_1)\pi)(-x).$$

One can show that $\mathcal{G}$ is polynomial preserving and the martingale problem for $(\mathcal{G},E,\rho)$ has a solution for every initial distribution ρ.

Example 2

Consider an operator of the form

$$\mathcal{G}f(x) = \frac{a(x)}{2}f''(x)+b(x)f'(x)+\lambda(x)\int_{[0,1]^2} f(x+\gamma(x,y))-f(x)-\gamma(x,y)f'(x)\mu(\mathrm{d}y)$$
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where

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Is this example not covered by our theory?

Example 2

Consider an operator of the form

$$\mathcal{G}f(x) = \frac{a(x)}{2}f''(x) + b(x)f'(x) + \lambda(x)\int_{[0,1]^2} f(x+\gamma(x,y)) - f(x) - \gamma(x,y)f'(x)\mu(\mathrm{d}y)$$
$$= (-2x)f'(x) + \frac{1}{x}\int_{[0,1]} f(x+\gamma(x,y_1)) - f(x) - \gamma(x,y_1)f'(x)\gamma(x,y_1)\,\mathrm{d}y_1.$$

where

$$\gamma(x,y_1) := \sin^2((x+y_1)\pi)(-x).$$

One can show that $\mathcal{G}$ is polynomial preserving and the martingale problem for $(\mathcal{G}, E, \rho)$ has a solution for every initial distribution ρ.

Is this example not covered by our theory? The answer is no.

Example 2

Indeed, the described operator coincide with

$$\widetilde{\mathcal{G}}f := (-2x)f'(x) + \frac{1}{x}\int_{[0,1]} f(x+\gamma(x,y_1)) - f(x) - f'(x)\gamma(x,y_1)\ \tilde{\mu}(\mathrm{d}y_1),$$

where $\tilde{\mu} := \sin^2(y_1\pi) * \mu$ and

$$\gamma(x, y_1) := y_1(-x).$$

We can see that $\widetilde{\mathcal{G}}$ is of the form considered until now (Case 2).

Example 3

Consider the operator of Case 3 given by

$$\mathcal{G}f(x) = b(x)f'(x) + \frac{x(1-x)}{(x-1/2)^2}\int_{[0,1]^2} f(x+\gamma(x,y)) - f(x) - \gamma(x,y)f'(x)\mu(\mathrm{d}y)$$

where

$$\gamma(x,y) := -(y_1+y_2)(x-1/2) \quad \mu\text{-a.s.}$$

Since $\lambda(0) = \lambda(1) = 0$ we are free to choose $b \equiv 0$.
The solution of the associated martingale problem will then be a true martingale on $[0,1]$.

A cone of PP operators

Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be PP such that the respective martingale problems have a solution for each initial distribution.

$\Rightarrow$ $\mathcal{G} := c_1\mathcal{G}_1 + c_2\mathcal{G}_2$ is a PP operator such that the respective martingale problem has a solution for each initial distribution, for all $c_1, c_2 \geq 0$.

Combining Cases (1)-(3) we thus obtain a cone of operators with those properties.

A cone of PP operators

An element of this cone is given by

$$\mathcal{G}f(x) = \frac{1}{2}a(x)f''(x) + b(x)f'(x) + \int_{\mathbb{R}\setminus\{0\}} f(x+\xi) - f(x) - \xi f'(x)\nu(x, \mathrm{d}\xi)$$

such that

- $a(x) = Ax(1-x)$ for a.e. $x \in E$,
- $b(x) \in \mathrm{Pol}_1(E)$ enough inward pointed at the boundary, and
- $\nu(x,\cdot) = \gamma(x,\cdot)_* F(x,\cdot)$ where $\gamma(x,y) = y_1(-x) + y_2(1-x)$ and

$$F(x,\mathrm{d}y) = m(\mathrm{d}y) + \frac{1-x}{x}\mu_1^{(1)}(\mathrm{d}y) + \frac{x}{1-x}\mu_2^{(1)}(\mathrm{d}y)$$
$$+ \sum_{k=3}^{K} \frac{1}{(x-x_k)^2}\left(x^2\mu_k^{(0)}(\mathrm{d}y) + 2x(1-x)\mu_k^{(1)}(\mathrm{d}y) + (1-x)^2\mu_k^{(2)}(\mathrm{d}y)\right)$$

for (signed) measures $m, \mu_k^{(j)}$ on $(0,1]^2$ and distinct points $x_k \in (0,1)$, satisfying some technical conditions.

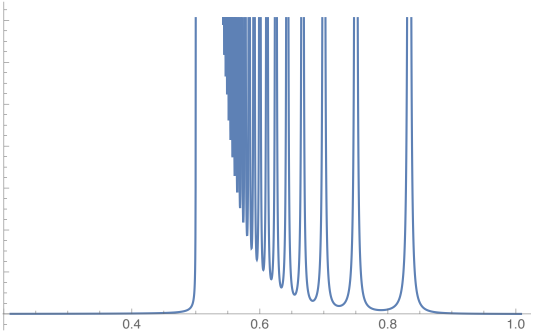
A CLOSED cone of PP operators

Let $(\mathcal{G}_n)_{n\in\mathbb{N}}$ be PP such that the respective martingale problems have a solution for each initial distribution.

- Suppose that $\mathcal{G}f(x) := \lim_{n\to\infty} \mathcal{G}_n f(x)$ is well defined for all $f \in \mathrm{Pol}(E)$ and $x \in E$,

$\Rightarrow$ $\mathcal{G}$ is a PP operator and the respective martingale problem has a solution for each initial distribution.

Example: $\mathcal{G} := \sum_{n=3}^{\infty} \mathcal{G}_n$



A graphical representation of

$$\sum_{n=3}^{\infty} \lambda_n(x)\mu_n([0,1]^2)$$

- $\mathcal{G}_n f(x) = \frac{a_n(x)}{2} f''(x) + \lambda_n(x) \int f\left(x+\gamma(x,y)\right) - f(x) - \gamma(x,y) f'(x)\, \mu_n(dy)$.
- Define $(\mu_n)_{n\geq 3}$ such that $y_1 + y_2$ is uniformly distributed on $[0,1]$ and for $x_n^* = \frac{1}{2} + \frac{1}{n}$: $\gamma(x_n^*, y) = 0$ μ_n-a.s.
- $\lambda_n(x) = n^{-2} \frac{x(1-x)}{(x-x_n^*)^2} \mathbb{1}_{\{x \neq x_n^*\}}$ and $a_n(x) = \frac{1}{3n^2} x_n^*(1-x_n^*) \mathbb{1}_{\{x = x_n^*\}}$.

Conclusion

- We defined PP processes as solution of a MP, whose operator $\mathcal{G}$ is of the form

$$\mathcal{G}f(x) = \frac{a(x)}{2}f''(x) + b(x)f'(x) + \int_{[0,1]^2} f(x+\xi) - f(x) - f'(x)\xi \; \nu(x, \mathrm{d}y)$$

and maps $\mathrm{Pol}_n(E)$ to itself.

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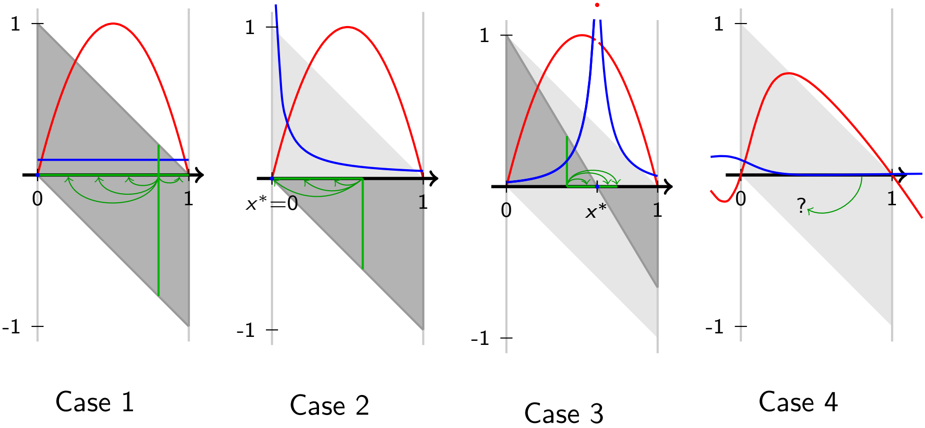
and maps $\mathrm{Pol}_n(E)$ to itself.

- We completely characterised the parameters a, b, γ, and λ s.t.

$$\mathcal{G}f(x)=\frac{a(x)}{2}f''(x)+b(x)f'(x)+\lambda(x)\int_{\mathrm{supp}(\mu)} f(x+\gamma(x,y))-f(x)-f'(x)\gamma(x,y)\,\mu(\mathrm{d}y)$$

is PP and the MP for $(\mathcal{G},E,\rho)$ has a solution for every initial distribution ρ on E, assuming γ polynomial in x.

Conclusion



Conclusion

And now?

- Find a probability measure μ on $[0,1]^2$ and an $\alpha \in \mathbb{C} \setminus \mathbb{R}$ s.t.

$$\int \big(y_1(-\alpha) + y_2(1-\alpha)\big)^n \mu(\mathrm{d}y) = 0 \qquad \forall n \geq 3;$$

  or show that they do not exist.
- What about boundary attainment?
- What about higher dimensional simplices as state space?

References

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Thank you!

The structure of ν (generalisation)

Consider N polynomial preserving operators $(\mathcal{G}_i)_{i=1}^N$ of the form

$$\mathcal{G}_i f(x) := \frac{a_i(x)}{2} f''(x) + b_i(x) f'(x) + \lambda_i(x) \int_{[0,1]^2} \Big(f\big(x+\gamma(x,y)\big) - f(x) - f'(x)\gamma(x,y) \Big)\, \mu_i(\mathrm{d}y).$$

where $\gamma(x,y) = y_1(-x) + y_2(1-x)$, such that for each i the martingale problem for $(\mathcal{G}_i, E, \rho)$ has a solution for every initial distribution ρ.

The structure of ν (generalisation)

Then the operator $\mathcal{G}$ given by

$$\mathcal{G}f(x) = \frac{1}{2}\sum_{i=1}^{N} a_i(x)f''(x) + \sum_{i=1}^{N} b_i(x)f'(x) + \sum_{i=1}^{N} \lambda_i(x)\int_{[0,1]^2}\Big(f\big(x+\gamma(x,y)\big)-f(x)-f'(x)\gamma(x,y)\Big)\,\mu_i(\mathrm{d}y).$$

is polynomial preserving and the martingale problem for $(\mathcal{G}, E, \rho)$ has a solution for every initial distribution ρ. Note that in this case $\nu(x,\cdot) = \sum_{i=1}^{N} \lambda_i(x)\mu_i(x,\cdot)$, where

$$\mu_i(x, A) := \int_{[0,1]^2} \mathbb{1}_A\big(\gamma(x,y)\big)\mu_i(\mathrm{d}y).$$

Example 3

We have seen that given N PP operators $(\mathcal{G}_i)_{i=1}^N$ of the form

$$\mathcal{G}_i f(x) := \frac{a_i(x)}{2} f''(x) + b_i(x) f'(x) \\ + \lambda_i(x) \int_{[0,1]^2} \Big(f\big(x+\gamma(x,y)\big) - f(x) - f'(x)\gamma(x,y) \Big) \, \mu_i(\mathrm{d}y),$$

the operator $\mathcal{G}$ given by

$$\mathcal{G} f(x) = \frac{1}{2} \sum_{i=1}^N a_i(x) f''(x) + \sum_{i=1}^N b_i(x) f'(x) \\ + \sum_{i=1}^N \lambda_i(x) \int_{[0,1]^2} \Big(f\big(x+\gamma(x,y)\big) - f(x) - f'(x)\gamma(x,y) \Big) \, \mu_i(\mathrm{d}y).$$

is PP, too.

Example 3

We have also seen that the measures $(\nu(x,\cdot))_{x\in E}$ associated to $\mathcal{G}$ are then given by

$$\nu(x,\cdot)=\sum_{i=1}^{N}\lambda_i(x)\mu_i(x,\cdot), \quad \mu_i(x,A):=\int \mathbb{1}_A(\gamma(x,y))\mu_i(\mathrm{d}y). \quad (1)$$

The natural question is then: given a collection of measures $(\tilde{\nu}(x,\cdot))_{x\in E}$ of the form described in (1) and associated to a PP operator $\widetilde{\mathcal{G}}$, there always exist PP operators $(\widetilde{\mathcal{G}}_i)_{i=1}^{N}$ with associated measures $\lambda_i(x)\mu_i(x,\cdot)$, respectively?

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The answer is no.

Example 3

Consider an operator of the form

$$\widetilde{\mathcal{G}}f(x) := b(x)f'(x) + \tilde{\lambda}_1(x)\int f(x+\gamma(x,y)) - f(x) - f'(x)\gamma(x,y)\ \delta_{(1,0)}(\mathrm{d}y) + \tilde{\lambda}_2(x)\int f(x+\gamma(x,y)) - f(x) - f'(x)\gamma(x,y)\ \delta_{(0,1/2)}(\mathrm{d}y)$$

where $b(x) = 1-2x$, $\tilde{\lambda}_1(x) = \frac{1}{x(x+1)}$, $\tilde{\lambda}_2(x) = \frac{2}{(1-x)(x+1)}$. Computing

$$q_1(x) := \tilde{\lambda}_1(x)\int \gamma^n(x,y)\ \delta_{(1,0)}(\mathrm{d}y) = -\frac{(-x)^{n-1}}{x+1}$$

$$q_2(x) := \tilde{\lambda}_2(x)\int \gamma^n(x,y)\ \delta_{(0,1/2)}(\mathrm{d}y) = \frac{((1-x)/2)^{n-1}}{x+1}$$

we see that $q_1 + q_2 \in \mathrm{Pol}_n(E)$ but neither q_1 nor q_2 is in $\mathrm{Pol}_n(E)$.