Lecture Notes

Boltzmann equations and singular cross sections

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Comments:
- Missing: More facts on local solutions in non homogeneous case;
- Missing: Improved integrability on the gain term;
- Missing: Summary of the results in the cutoff case;
- Missing: Summary of velocity averaging results....
Abstract

As suggested by the title, this set of notes is concerned about Boltzmann equation for singular cross sections, also called non cut off sections. It is assumed that the reader is well acquainted with basic facts about Boltzmann equation. However, we shall first recall standard results and notations. Furthermore, I have devoted some Chapters to recalling known results in the cutoff case, that will be found at the end of these Lectures Notes.

A large part of these notes is concerned with the homogeneous theory. But we have devoted the last chapters to the full Boltzmann non homogeneous equation.

I have also decided to include results on Landau equation, which is a limit case of Boltzmann equation. The final goal will be to show that we are then able to justify the Landau approximation, that is the rigorous mathematical justification of the transition from Boltzmann to Landau equation, in the non homogeneous framework. It was in fact my motivation when I started working on Boltzmann equation without cutoff.

Since there is now some hope to build classical solutions to non homogeneous Boltzmann equation, at least close to Maxwellians, a special Chapter will be concerned with the linearized operator.

All in all, these Lectures Notes have increased in size, but I hope that in the whole, they will be self contained. However, a number of important topics has not been even mentioned. In that case, I think that the review by Villani will complement these points.

Acknowledgments- This set of notes is an expanded version of lectures given in Shanghai in December 2006. I have tried to really make available a self-contained set of notes, and I hope that it will be profitable for the students.

I take this opportunity to thank Tong Yang, of City University of Hong Kong (CUHK) for his invitation to give these lectures, as well as for my stay in August 2006 in CUHK, where I had the pleasure to meet for the first time Seiji Ukai, Chao-Jiang Xu and Yoshinori Morimoto, and for our present collaboration. Thanks are due also (at least through their works) to Laurent Desvillettes, Clément Mouhot and Cédric Villani.

Finally, my research has benefited from Kamel Hamdache, who was my PhD advisor, and from Luc Tartar (who was the Master Thesis advisor of Kamel Hamdache).

This is the fourth version. I’ll appreciate any suggestion to correct and improve this document.
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Chapter 1

Introduction

Basic and standard references on the physics around Boltzmann equations are the books of Cercignani [38], Chapman and Cowling [42], Balescu [21], Lifschitz [80].

Though we are mostly interested in the non singular case, that is collisional kernels not satisfying Grad’s cutoff assumption, some classical references concerned with the cutoff case at the mathematical level are:

- Arlotti and Bellomo [18];
- Cercignani, Illner and Pulvirenti [41];
- The works of DiPerna and Lions [56, 57, 59, 81, 82, 83, 86].

One may also consult the Lecture Notes of Ukai and Yang [114], presenting recent results on Boltzmann equation in the cutoff case.

For the singular case considered herein, we refer to the short but recent and mathematically oriented review by Villani [118]. Though written in 2003, it does not cover fully all the material of the present Lecture Notes. However, it also contains a huge number of references on works related to Boltzmann equation, for which we refer the reader.

At this point, we wish to mention that up to the 2000’s, the main works on singular kernels were due to Pao [98] and Ukai [113]. In this last reference, it is shown existence of solutions of non homogeneous Boltzmann equation with singular kernels, in Gevrey class.

Boltzmann equation, in its usual form, reads as follows

\[ \partial_t f(t, x, v) + v \cdot \nabla_x f(t, x, v) = Q(f, f)(t, x, v). \]

Above \( f = f(t, x, v) \) is the unknown function and typically represents a (probability) density of finding particles at time \( t \geq 0 \), located around position \( x \in \mathbb{R}^n \), with velocity close to \( v \in \mathbb{R}^n \).
If the right hand side of (1.1) was set to zero, then this equation would describe the free behavior of particles. Since this is typically not the case for the considered particles, at least because of collisions, with some physical assumptions, the right hand side, called Boltzmann collision operator, is an operator acting only w.r.t. velocity variable $v$, and reads, forgetting other variables

$$Q(f,f)(v) = \int_{\mathbb{R}^n} dv_s \int_{S^{n-1}} d\sigma B(v - v_s, \sigma)(f'f'_s - ff_s).$$

Here, $S^{n-1}$ denotes the unit sphere of $\mathbb{R}^n$, $n \geq 2$. $B(v - v_s, \sigma)$ is a given positive function, depending only on the type of interactions between particles; it is called the collisional cross section, and actually only depends on $|v - v_s|$ and on $\frac{v - v_s}{|v - v_s|} \cdot \sigma$, so we shall write

$$B(v - v_s, \sigma) = B(|v - v_s|, \frac{v - v_s}{|v - v_s|} \cdot \sigma) \text{ and } \cos \theta = \frac{v - v_s}{|v - v_s|} \cdot \sigma.$$ 

$f, f_s, f'_s$ and $f'$ are usual and but shorthand notations for $f = f(v)$, $f_s = f(v_s)$, $f' = f(v')$ and $f'_s = f(v'_s)$. Finally, for a given couple of velocities $(v, v_s)$ (pre or post velocities), the couple $(v', v'_s)$ is given by

$$(1.3) \quad v' = \frac{v + v_s}{2} + \frac{|v - v_s|}{2} \sigma, \quad v'_s = \frac{v + v_s}{2} - \frac{|v - v_s|}{2} \sigma,$$

so that we have the following conservation rules

$$(1.4) \quad v' + v'_s = v + v_s \text{ and } |v'|^2 + |v'_s|^2 = |v|^2 + |v_s|^2.$$ 

One can keep in mind that the important cases are $n = 2$ or $n = 3$, but most of the calculations below do not depend on the dimensions.

The mathematical theory of Boltzmann equation is mainly separated into two parts: one can look to the homogeneous equation, that is for solutions independent of position variable $x$, or to the non-homogeneous case, that is the full Boltzmann equation above.

But there is also another mathematical separation, depending on whether or not, for a fixed $z \in \mathbb{R}^n - \{0\}$, the function $\sigma \mapsto B(|z|, \sigma)$ is integrable or not on the unit sphere $S^{n-1}$.

At the noticeable exception of hard sphere collisions, this map is never integrable, typically because of a high singularity for $\theta$ close to zero, corresponding to so called grazing collisions.

This high singularity implies additional huge difficulties in the mathematical treatment of Boltzmann equation, and this explains why Harold Grad has introduced a cutoff assumption for such collisional cross sections. Thus up to the 2000’s, the main assumption on usual works around Boltzmann equation was Grad’s cutoff assumption, namely that $\sigma \mapsto B(|z|, \sigma)$ was integrable on the
unit sphere $S^{n-1}$, or any similar kind of integrability. Another name is non singular collisional cross sections.

To be more precise, we are here referring to the non homogeneous full Boltzmann equation. In the homogeneous case, let us refer to the works [17, 47, 48, 49, 68, 117] in particular.

We just pointed out that almost all cross sections were never integrable on $S^{n-1}$. A classical example is given by inverse power laws interactions, namely for an interaction potential similar to $\phi(r) = \frac{1}{r^s}$, $s > 2$. Here $r$ denotes the intermolecular distance. In this case, though the cross section can only be computed implicitly, a good approximation is given by

$$B(|v - v_s|, \cos \theta) = |v - v_s|^s b(\cos \theta), \quad \sin^{n-2} \theta b(\cos \theta) \sim \theta^{-1} - \nu$$

where $\gamma = \frac{s - (2n - 1)}{s - 1}$, so that $\gamma = \frac{s - 5}{s - 1}$ if $n = 3$, and $\nu = \frac{2}{s - 1}$ if $n = 3$.

In particular, it is never integrable on $S^2$.

While we are introducing standard vocabulary, at least when $n = 3$, cases $\gamma > 0$, $\gamma = 0$ and $\gamma < 0$ are resp. called hard, maxwellian and soft cases.

For readers familiar with Boltzmann equation, let just recall that there is also mathematical problems related to the integrability w.r.t. $v - v_s$, and thus with the sign of $\gamma$.

Therefore, when we are talking about singular (or non cutoff) cross sections, we are just mentioning the fact that our main assumption is that the cross section will never be integrable over the unit sphere, and typically are given by the above example (1.5).

Strongly related to this framework is another well known equation: Landau equation, taking into account all grazing collisions, but this is another story, though some works around this latter equation can give (and in fact have given) better insights into Boltzmann equation. We should mention that there is still a lot of open questions around this last equation. We shall come back to Landau equation in latter chapters.

Turning to Boltzmann equation (1.1), as usual in the general theory of Partial Differential Equations, most existence results are based on a priori estimates.

The natural ones for Boltzmann equation (and up to now, the only known ones at the exception of one dimensional models) are based on symmetrization properties of the collision operator: using in particular the change of variables $(v, v_s, \sigma) \mapsto (v', v'_s, k)$, with $k = \frac{v - v_s}{|v - v_s|}$ which has unit Jacobian and is involutive, one finds that for suitable test functions $\phi = \phi(v)$

$$\int_{\mathbb{R}^n} dv Q(f, f)(v)\phi(v) = -\frac{1}{4} \int_{\mathbb{R}^n} dv \int_{\mathbb{R}^n} dv_s \int_{S^{n-1}} d\sigma (f'f'_s - ff_s) \{\phi' + \phi'_s - \phi - \phi_s\}. $$
Also by integrating over variable $x$ in the non homogeneous case, one finds the following conservation laws

(1.7) \[ \frac{d}{dt} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(t, x, v) \phi(v) dv dx = 0 \text{ if } \phi(v) = 1 \text{ or } v_i, 1 \leq i \leq n, \text{ or } |v|^2, \]

which express the global conservation of mass, momentum and energy.

Let us mention that for general initial data, and more precisely in the context of renormalised solutions introduced by Di Perna and Lions [57, 81], it is still unknown if the corresponding solutions do satisfy momentum and energy local conservation laws. These local conservation laws read here,

\[ \partial_t \rho + \nabla_x (\rho u) = 0, \]

\[ \partial_t (\rho u) + \nabla_x (\int_{\mathbb{R}^n} f v \otimes vdv) = 0 \]

and

\[ \partial_t (\rho |u|^2 + N \bar{\rho}T) + \nabla_x (\int_{\mathbb{R}^n} f |v|^2 dv) = 0. \]

One last important inequality, namely Boltzmann's H-theorem, follows by choosing $\phi(v) = \log f(v)$. Upon introducing the H-functional

(1.8) \[ H(f) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f \log f(t, x, v) dv dx, \]

and the entropy dissipation functional

(1.9) \[ D(f) = \frac{1}{4} \int_{\mathbb{R}^n} dv \int_{\mathbb{R}^n} dv_s \int_{S^{n-1}} d\sigma B(v - v_s, \sigma)(f'f'_s - ff_s) \log \frac{f'f'_s}{ff_s} \geq 0, \]

one finds also

(1.10) \[ \frac{d}{dt} H(f) + \int_{\mathbb{R}^n} D(f) dx = 0. \]

The representation of the collision operator above, given by (1.2), along with (1.3), is referred as the $\sigma$-representation. Another well known representation is given by the so-called $\omega$-representation

(1.11) \[ Q(f, f)(v) = \int_{\mathbb{R}^n} dv_s \int_{S^{n-1}} d\omega [f'_s f'_s - ff_s] 2^{n-2} \sin^{n-2}(\frac{\theta}{2}) B(|v - v_s, \cos \theta), \]

where now we have the rules

(1.12) \[ v' = v + (v_s - v) \cdot \omega \omega \text{ and } v'_s = v_s - (v_s - v) \cdot \omega \omega. \]
Finally, a third and last representation, due initially to Carleman is also useful: letting $E_{v,v'-v}$ be the hyperplane going through $v$ and orthogonal to $v - v'$, using the identity $v - v_*= 2v - v' - v'_*$, one has

\begin{equation}
Q(f,f)(v) = \int_{\mathbb{R}^n} dv' \int_{E_{v,v'-v'}} \frac{1}{|v - v'|^{n-1}} B(2v - v' - v'_*, \frac{v' - v'_*}{|v' - v'_*|}) [f(v'_*)f(v') - f(v)f(v' + v'_* - v)].
\end{equation}

We shall mainly use $\sigma$-representation, but at some places, the two other representations might be useful. For more explanation on these representation, see [118].
Chapter 2

Existence in the homogeneous case: short summary

We shall just recall the essential facts, referring to the bibliography for much more details. In particular, as for existence, we mention the works of Arkeryd [17], Goudon [68], Villani [117]. A recent synthesis is due to Desvillettes [50].

Further recent results, such as propagation of moments, unicity of solutions..., can be found in the works of Desvillettes, Mouhot... [53, 52].

Concerning regularization properties of solutions, we shall get back on this question in the following Chapters, especially starting from Chapter 6.

First of all, weak solutions can be defined by using the symmetrisation properties of the collision operators (cf. Chapter 1), in particular the duality formula (1.6) therein. This leads to the usual notion of weak solutions, and eventually also using the entropy dissipation estimates, leading then to the notion of H-solutions. That is, we are only using basic estimates (1.7) and eventually those implied by (1.9) from Chapter 1.

One has then the following summary, taken from Desvillettes [50]:

- If $\gamma \in ]0,1[$ (hard potentials), then a weak solution exists.
- If $\gamma = 0$ (maxwellian molecules) or $\gamma \in ]-2,0[$ (soft potentials), then a weak solution exists.
- If $\gamma \in ]-3, -2[$ (very soft potentials), then an H-solution exists.

Uniqueness issues have been answered (positively) in Maxwellian case, $\gamma = 0$. The situation is still unclear in the other cases, but on this point we mention recent works by Desvillettes and Mouhot [53, 52].

An important problem is get estimates on moments with respect to velocity variable. As regards
propagation of moments, it is known that if \( \gamma \in ]0,1[ \) (hard potentials), then (polynomial) moments are propagated, remain bounded, and in fact there is immediate creation of arbitrary higher moments, for positive times.

For other values of \( \gamma \), it is known that moments are at least propagated, and bounded at least in the Maxwellian case.

As for smoothness issues, since this is one of our main topic, we shall get back soon on this question.

We state the following summary taken from Villani [118], about moments estimations (in fact there is a better result stated therein, for which we refer for more details).

**Theorem 2.0.1** Assume \( B(v - v_*, \cos \theta) = |v - v_*|^{\gamma} (\cos \theta) \), with \(-3 \leq \gamma \leq 1\) and the momentum control estimate \( \int_0^\pi b(\cos \theta)(1 - \cos \theta) \sin^{n-2} \theta d\theta < +\infty \). Let \( f_0 \in L^2_2(\mathbb{R}^n_v) \geq 0 \) be an initial datum with finite mass and entropy, and \( f = f(t,v) \) a weak solution of Boltzmann equation, with decreasing kinetic energy \( \int f(t,v) |v|^2 \, dv \). Then this kinetic energy is automatically constant in time. Moreover, if we set

\[
M_s(t) = \int f(t,v) |v|^s \, dv,
\]
then

1. If \( \gamma = 0 \), then for all \( s > 2 \),

\[
\forall t > 0, M_s(t) < +\infty \iff M_S(0) < +\infty.
\]

If \( M_s(0) < +\infty \) then \( \sup_{t \geq 0} M_s(t) < +\infty \).

2. If \( \gamma > 0 \), then for all \( s > 2 \),

\[
\forall t_0 > 0, \sup_{t \geq t_0} M_s(t) < +\infty.
\]

3. If \( \gamma < 0 \), then for any \( s > 2 \),

\[
\forall t < +\infty, M_s(t) < +\infty \iff M_s(0) < +\infty.
\]

\( \square \)

The above result applies to any weak solution, where we are using the duality formula from Chapter (1), but for \( \gamma \geq -2 \). For smaller values of \( \gamma \), this duality formula is no more appropriate, and we need to shift to the notion of H-solutions. This notion was introduced by Villani [117], and is a way to encompass the kinetic singularity, i.e. bad exponent \( \gamma \).

Let us explain these difficulties more precisely.
For simplification, we assume \( n = 3 \), and take our collision cross section as

\[
B(v - v_*, \sigma) = \| v - v_* \|^\gamma b(\cos \theta),
\]

where \( b(\cos \theta) \) behaves as above, for inverse power potentials, and thus satisfies \( \int_0^\pi \theta^2 b(\cos \theta) d\theta < +\infty \) (recall we are in dimension 3).

First of all, recall the various different duality formulations of the collision operator (displayed in Chapter 1), when tested against a suitable test function \( \phi = \phi(v) \).

For instance, one possible choice is given by

\[
\int_{\mathbb{R}^n} dvQ(f, f)(v)\phi(v) = \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} dv dv_* B(v - v_*, \sigma)f f_* \{ \phi' + \phi_*' - \phi - \phi_* \}.
\]

We note that, using Taylor’s formula, that

\[
\phi' - \phi = (v' - v) \nabla \phi(v) + (v' - v) \otimes (v' - v) : \int_0^1 (1 - t) D^2 \phi(v + t(v' - v)) dt
\]

and

\[
\phi_*' - \phi_* = (v_*' - v_*) \nabla \phi(v_*) + (v_*' - v_*) \otimes (v_*' - v_*) : \int_0^1 (1 - t) D^2 \phi(v_* + t(v_*' - v_*)) dt.
\]

Since \( v_*' - v_* = -(v' - v) \), it follows that

\[
\phi' + \phi_*' - \phi - \phi_* = O(|v - v_*|^2 \theta \wedge 1).
\]

This estimation is sufficient for small singularities \( 0 < \nu < 1 \) and for \( \gamma \geq -2 \).

The problem is: what can we do for higher singularities, \( 1 \leq \nu < 2 \)?

After many preliminary works (precise references are given by Villani [117]), here is the answer. This time, without further difficulties, we consider a more general framework.

Without loss of generality, we assume that \( B(v - v_*, \sigma) \) is supported in the set \( (0 \leq \theta \leq \pi/2) \), i.e. \( (v - v_*, \sigma) \geq 0 \). If not, we can reduce to this case replacing \( B \) by its symmetrized version

\[
\overline{B}(v - v_*, \sigma) = [B(v - v_*, \sigma) + B(v - v_*, -\sigma)]1_{(v - v_*, \sigma) > 0}.
\]

We introduce the momentum transfer, associated with the collision cross section, as

\[
M(|v - v_*|) \equiv \int_{S^{n-1}} B(v - v_*, \sigma)(1 - k \cdot \sigma) d\sigma = |S^{n-2}| \int_0^{\pi/2} B(|v - v_*|, \cos \theta)(1 - \cos \theta) \sin^{n-2} \theta d\theta.
\]

Above \( k = \frac{v - v_*}{|v - v_*|} \) and \( \cos \theta = k \cdot \sigma \).

Since \( 1 - \cos \theta = 2 \sin^2(\theta/2) \) vanishes up to order 2 for \( \theta \) close to 0, this quantity is finite.
We introduce a regularity assumption on $B$ w.r.t. the relative velocity variable $z = v - v_*$ as follows.

Let us define, for $z \neq 0$,

$$B'(z, \sigma) = \sup_{1 < \lambda \leq \sqrt{2}} \frac{|B(\lambda z, \sigma) - B(z, \sigma)|}{(\lambda - 1)|z|},$$

and

$$M'(|v - v_*|) = \int_{S^{n-1}} B'(v - v_*, \sigma)(1 - k \cdot \sigma) d\sigma.$$

We shall assume

**Assumption I.**

$$M(|z|), \ |z|M(|z|) \in L^1_{loc}(\mathbb{R}^n).$$

This first condition allows of course for smooth (say, Lipschitz) functions, but also singular power laws $|v - v_*|^\gamma$ with $\gamma > -n$, since

$$|z| \sup_{1 < \lambda \leq \sqrt{2}} \frac{|\lambda z|^\gamma - |z|^\gamma}{(\lambda - 1)|z|} = |z|^\gamma \sup_{1 < \lambda \leq \sqrt{2}} \frac{\lambda^\gamma - 1}{\lambda - 1} = C_\gamma |z|^\gamma$$

is locally integrable if $\gamma > -n$.

For the borderline case $\gamma = -n$ (corresponding to Coulomb potential), we refer to another Chapter, especially Chapter 11.

We need also an assumption on the behavior at infinity

**Assumption II. (Behavior at infinity)** For $0 \leq \alpha \leq 2$, let

$$M_\alpha(|z|) = \int_{S^{n-1}} B(z, \sigma) (1 - k \cdot \sigma)^{\alpha} d\sigma, \quad k = \frac{z}{|z|}.$$

We require that for some $\alpha \in [0, 2]$, as $|z| \to \infty$,

$$M_\alpha(|z|) = o(|z|^{2-\alpha}), \quad |z|M'_\alpha(|z|) = o(|z|^2).$$

For instance, in the model case

$$B(v - v_*, \sigma) = |v - v_*|^\gamma b(\cos \theta), \text{ with } \sin^{n-2} \theta b(\cos \theta) \sim K \theta^{-1-\nu},$$

$\nu > 0$, $K > 0$, Assumption II allows $\gamma + \nu < 2$, which is always satisfied in the case of inverse $s$-power forces in dimension $n = 3$, since

$$\gamma + \nu = \frac{s - 3}{s - 1} < 1.$$

Finally, we introduce an assumption to take into account that we want to deal with really singular
kernels

Assumption III. (Angular singularity condition)

\begin{equation}
B(z, \sigma) \geq \Phi_0(|z|)b_0(k \cdot \sigma), \quad k = \frac{z}{|z|},
\end{equation}

where $\Phi_0$ is a continuous function, $\Phi_0(|z|) > 0$ if $|z| \neq 0$, and

\begin{equation}
\int_{S^{n-1}} b_0(k \cdot \sigma) d\sigma = +\infty.
\end{equation}

To keep clear ideas, in the model case

\[ B(v - v_*, \sigma) = |v - v_*|^\gamma b(\cos \theta), \quad \sin^{n-2} \theta b(\cos \theta) \sim K \theta^{-1-\nu}, \]

$\nu > 0$, $K > 0$, Assumptions I, II and III allow the following range of parameters:

\begin{equation}
\gamma \geq -n, \quad 0 \leq \nu < 2, \quad \gamma + \nu < 2.
\end{equation}

Now, we can get back to the bilinear Boltzmann operator $Q(f, f)$, defined by duality as follows. Letting $\varphi(v)$ be a smooth test-function in the velocity variable, then we define

\[ \int_{\mathbb{R}^n} Q(f, f) \varphi(v) dv = \int_{\mathbb{R}^{2n}} dv dv_* f f_* \left[ \int_{S^{n-1}} B(v - v_*, \sigma)(\varphi' - \varphi) d\sigma \right]. \]

For given $v_*$, we introduce the linear operator

\begin{equation}
T : \varphi \mapsto \int_{S^{n-1}} B(v - v_*, \sigma)(\varphi' - \varphi) d\sigma.
\end{equation}

Note that we do not use the full duality formula from Chapter 1.

We shall get back to much more precise functional properties of this operator in Chapter 7, but for the moment, under Assumptions I, II and III, the following Propositions are sufficient for our immediate purpose.

**Proposition 2.0.1 (W^{2,\infty} \rightarrow L^\infty bound for $T$)** For all $\varphi \in W^{2,\infty}(\mathbb{R}^n_0)$,

\[ |T\varphi(v)| \leq \frac{1}{2}\|\varphi\|_{W^{2,\infty}}|v - v_*| \left( 1 + \frac{|v - v_*|}{2} \right) M(|v - v_*|). \]

Moreover, for all $\alpha \in [0, 2]$ and $\varphi \in W^{2,\infty}(\mathbb{R}^n_0)$,

\[ |T\varphi(v)| \leq 2\|\varphi\|_{W^{2,\infty}}(1 + |v - v_*|)^\alpha M_\alpha(|v - v_*|), \]

where $M_\alpha$ is defined by formula (2.4).
This is enough to give

**Proposition 2.0.2** Let $B$ satisfy the hard potential case (or even with $\gamma \geq -2$), and let $f$ satisfy the usual a priori entropic bounds. Then, for all $R > 0$, $Q(f, f)$ defined by duality as above belongs to $L^\infty([0, T]; W^{-2,1}(B_R(v))))$, where $B_R(v)$ denotes $\{v \in \mathbb{R}^n, |v| \leq R\}$.

Proofs of Propositions 2.0.1 and 2.0.2 will be given in Chapter 11.

The above framework gives, with some standard works (in particular using an approximation by non singular cross sections), existence of weak solutions, in the case $\gamma > -2$.

The problem occurs then for lower values of $\gamma$, that is $\gamma > -3$ (note by the way that the same problem does not occur in the non homogeneous and renormalized framework!).

The trick, introduced by C. Villani, consists in changing the definition of solutions, shifting to so-called H-solutions, and using the entropy dissipation functional. We shall see that in a very loose sense, we use the fact that a better regularity is available on $f$. However, we shall avoid using the computations in his paper by proceeding somehow differently.

First of all, recall from Chapter 1 (in the homogeneous framework) and more precisely from (1.10), that the main assumption is that (for some unimportant constants $C$ here)

$$\int_0^T D(f)dt \leq C.$$ 

Using the classical inequality $(\log x - \log y)(x - y) \geq 4(\sqrt{x} - \sqrt{y})^2$, we find therefore the following a priori estimate

$$\int_0^T \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{S^{n-1}} dvdv_s d\sigma B(v - v_s, \sigma)(\sqrt{f'f'_s} - \sqrt{ff_s})^2 \leq C.$$ 

Our task is to give a sense to a weak form of the collision operator, allowing for values of $\gamma$, $\gamma > -3$.

At this step, two ways are possible, but both start from the simple formula

$$f'f'_s - ff_s = (\sqrt{f'f'_s} - \sqrt{ff_s})(\sqrt{f'f'_s} + \sqrt{ff_s}) = (\sqrt{f'f'_s} - \sqrt{ff_s})(\sqrt{f'f'_s} - \sqrt{ff_s} + 2\sqrt{ff_s})$$

and thus

$$f'f'_s - ff_s = (\sqrt{f'f'_s} - \sqrt{ff_s})^2 + 2(\sqrt{f'f'_s} - \sqrt{ff_s})\sqrt{ff_s}.$$ 

Therefore, we can write

$$\int Q(f, f)\phi(v)dv = -\frac{1}{2} \int (\sqrt{f'f'_s} - \sqrt{ff_s})^2(\phi' - \phi) - \int (\sqrt{f'f'_s} - \sqrt{ff_s})\sqrt{ff_s}(\phi' - \phi)$$

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and also

\[ \int Q(f, f)\phi(v)dv = -\frac{1}{4} \int (\sqrt{f'f} - \sqrt{ff^*})^2 (\phi' + \phi^* - \phi - \phi^*) - \frac{1}{2} \int (\sqrt{f'f} - \sqrt{ff^*}) \sqrt{ff^*} (\phi' + \phi^* - \phi - \phi^*). \]

By symmetry, we deduce

\[ \int Q(f, f)\phi(v)dv = -\int (\sqrt{f'f} - \sqrt{ff^*}) \sqrt{ff^*} (\phi' - \phi) \]

and

\[ \int Q(f, f)\phi(v)dv = -\frac{1}{2} \int (\sqrt{f'f} - \sqrt{ff^*}) \sqrt{ff^*} (\phi' + \phi^* - \phi - \phi^*). \]

The second formula is well defined for weak solutions: use the entropy dissipation estimate displayed above, combined with Cauchy Schwartz inequality, and note that the square leads to a good cancellation.
Chapter 3

The Cancellation Lemma

This Chapter is relatively short but we have decided to display in a separate form an important result, the cancellation Lemma, that we shall use in the following Chapters. Roughly speaking, it says that one part Boltzmann operator makes perfectly good sense, without any further regularity assumptions on functions, though at first sight, one would ask at least $W^1$ or $W^2$ type functions.

3.1 Assumptions on the collisional kernel

Some definitions were already introduced in Chapter 2, but we recall them for convenience of the reader.

For a given collisional kernel, the momentum transfer is defined by

\[(3.1) \quad M(|v - v_*|) \equiv \int_{S^{n-1}} B(v - v_*, \sigma)(1 - k \cdot \sigma) d\sigma \]

\[\quad = |S^{n-2}| \int_0^{\frac{\pi}{2}} B(|v - v_*|, \cos \theta)(1 - \cos \theta) \sin^{n-2} \theta d\theta.\]

We also define, for $z \neq 0$,

\[(3.2) \quad B'(z, \sigma) = \sup_{1 < \lambda \leq \sqrt{2}} \frac{|B(\lambda z, \sigma) - B(z, \sigma)|}{(\lambda - 1)|z|}.\]

This quantity measures smoothness of $B$ with respect to the relative velocity variable. In the same way as (3.1), we set

\[(3.3) \quad M'(|v - v_*|) = \int_{S^{n-1}} B'(v - v_*, \sigma)(1 - k \cdot \sigma) d\sigma.\]

Our first main assumption on $B$ is given by
**Assumption I.** *(At most borderline singularity)* Assume that

\[ B(z, \sigma) = \frac{\beta_0(k \cdot \sigma)}{|z|^n} + B_1(z, \sigma), \quad k = \frac{z}{|z|}, \]

for some nonnegative measurable functions \( \beta_0 \) and \( B_1 \), and define

\[ \mu_0 = \int_{S_{n-1}} \beta_0(k \cdot \sigma)(1 - k \cdot \sigma) \, d\sigma, \]

\[ M_1(|z|) = \int_{S_{n-1}} B_1(z, \sigma)(1 - k \cdot \sigma) \, d\sigma, \]

\[ M'_1(|z|) = \int_{S_{n-1}} B'_1(z, \sigma)(1 - k \cdot \sigma) \, d\sigma, \]

where

\[ B'_1(z, \sigma) = \sup_{1 < \lambda \leq \sqrt{2}} \frac{|B_1(\lambda z, \sigma) - B_1(z, \sigma)|}{(|\lambda - 1)|z|}. \]

We require that

\[ \mu_0 < +\infty, \quad \text{and} \quad M_1(|z|), \ |z|M'_1(|z|) \in L^1_{\text{loc}}(\mathbb{R}^n). \]

As a consequence,

\[ M(|z|) = M_1(|z|) + \frac{\mu_0}{|z|^n}, \]

\[ M'(|z|) \leq M'_1(|z|) + \frac{\mu_0}{|z|^n} \left( \frac{2^{n/2} - 1}{\sqrt{2} - 1} \right). \]

Note in particular that \(|z|M(|z|)\) is always integrable; this will be used in the sequel.

For cross-sections that may grow to infinity, large velocities will be controlled by

**Assumption II.** *(Behavior at infinity)* For \( 0 \leq \alpha \leq 2 \), let

\[ M^\alpha(|z|) = \int_{S_{n-1}} B(z, \sigma) \left( 1 - k \cdot \sigma \right)^{\alpha} \, d\sigma, \quad k = \frac{z}{|z|}. \]

We require that for some \( \alpha \in [0, 2] \), as \( |z| \to \infty \),

\[ M^\alpha(|z|) = o(|z|^{2-\alpha}), \quad |z|M'(|z|) = o(|z|^2). \]
In the model case

\[ B(v - v_*, \sigma) = |v - v_*|^\gamma b(\cos \theta), \quad \text{with} \quad \sin^{n-2} \theta b(\cos \theta) \sim K \theta^{1-\nu}, \]

\(\nu > 0, \ K > 0,\) Assumption II allows \(\gamma + \nu < 2.\)

Again note that this assumption is always fulfilled in the physical cases of inverse \(s\)-power forces in dimension \(n = 3,\) since \(\gamma + \nu = \frac{s-3}{s-1} < 1.\)

Since we wish to deal with collisional kernels which are really singular, we make

**Assumption III.** (Angular singularity condition)

\[
B(z, \sigma) \geq \Phi_0(|z|) b_0(k \cdot \sigma), \quad k = \frac{z}{|z|},
\]

where \(\Phi_0\) is a continuous function, \(\Phi_0(|z|) > 0\) if \(|z| \neq 0,\) and

\[
\int_{S^{n-1}} b_0(k \cdot \sigma) d\sigma = +\infty.
\]

**Summary:** in the model case

\[ B(v - v_*, \sigma) = |v - v_*|^\gamma b(\cos \theta), \quad \sin^{n-2} \theta b(\cos \theta) \sim K \theta^{1-\nu}, \]

\(\nu > 0, \ K > 0,\) our Assumptions I, II and III together allow the following range of parameters:

\[
\gamma \geq -n, \quad 0 \leq \nu < 2, \quad \gamma + \nu < 2.
\]

### 3.2 Cancellation Effects

**Proposition 3.2.1 (Cancellation Lemma)** Let \(B\) be a nonnegative measurable kernel. Then, for a.a. \(v \in \mathbb{R}^n,\)

\[
Sf \equiv \int_{\mathbb{R}^n \times S^{n-1}} dv_* d\sigma B(v - v_*, \sigma)(f'_* - f_*) = f *_v S,
\]

where

\[
S(|z|) = |S^{n-2}| \int_0^{\frac{\pi}{2}} d\theta \sin^{n-2} \theta \left[ \frac{1}{\cos^n(\theta/2)} B \left( \frac{|z|}{\cos(\theta/2)}, \cos \theta \right) - B(|z|, \cos \theta) \right].
\]

In particular, if \(B\) satisfies Assumption I, then

\[ S(|z|) = \lambda \delta_0 + S_1(|z|), \]
where $\delta_0$ is the Dirac mass at the origin,

$$\lambda = -|S^{n-2}||S^{n-1}| \int_0^{\frac{\pi}{4}} \beta_0(\cos \theta) \log \cos(\theta/2) \sin^{n-2} \theta \, d\theta,$$

and $S_1$ is a locally integrable function,

$$|S_1(|z|)| \leq \frac{2^{n+4}}{\cos^2(\pi/8)} \left[ nM_1(|z|) + |z|M_1'(|z|) \right].$$

Remark 3.2.1 1. If $B$ is homogeneous of degree $-n$ in the relative velocity variable, then $S$ has to be defined as a principal value operator,

$$Sf = \lim_{\varepsilon \to 0} \int dv_s \, d\sigma \, B(v - v_s, \sigma) 1_{|v - v_s| \geq \varepsilon} 1_{\theta \geq \varepsilon} (f'_s - f_s).$$

Thus the convolution kernel $S$ will in general be a measure. In such situations, the result of Proposition 3.2.1 should be considered as the definition of the left-hand side. At the level of the Boltzmann equation, this definition is justified by the fact that all expressions given above coincide as long as $f$ is smooth (in which case $Q(f, f)$ is easily given a sense via Taylor expansions of $f'f' - ff'$ for $v' \simeq v, \sigma' \simeq \sigma$).

2. Using the inequality $-\log u \leq (1/u) - 1$, we find

$$0 \leq \lambda \leq \frac{|S^{n-1}|}{4 \cos^2(\pi/8)} \mu_0.$$

3. In the case where the kinetic cross-section is a power law, $B(z, \sigma) = |z|^\gamma b(\cos \theta), \gamma \geq -n$, then the integrand in (3.14) is always nonnegative.

4. In fact, instead of Assumption I, we could impose that $S$ defined by (3.14) is a locally bounded measure. This kind of assumption will be useful to discuss the Landau approximation, in another Chapter.

Proof of Proposition 3.2.1:

Since $S$ is defined as a principal value operator, we shall do the computation assuming $B$ to be integrable. The conclusion will then follow by any limit procedure. Following Villani, we perform the change of variables $v_s \to v'_s$, for each $v, \sigma$ fixed. This change of variables is well-defined on $(\cos \theta > 0)$, with Jacobian given by

$$\left| \frac{dv'_s}{dv_s} \right| = \frac{1}{2^n} (1 + k \cdot \sigma) = \frac{(k' \cdot \sigma)^2}{2^{n-1}},$$

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where
\[ k' = \frac{v - v'_*}{|v - v'_*|}, \quad k' \cdot \sigma = \cos \frac{\theta}{2} > \frac{1}{\sqrt{2}}. \]

We introduce the application \( \psi_\sigma : v'_* \mapsto v_*, \) defined on \( (k' \cdot \sigma) > 1/\sqrt{2}. \)

We note that \( |v_* - \psi_\sigma(v')| = |v' - v_*|/(k' \cdot \sigma) \), or, equivalently
\[ |v_* - \psi_\sigma(v)| = \frac{|v - v_*|}{k \cdot \sigma}. \]

We apply this change of variable to the part \( \int B f'_* \) in the left-hand side of (3.13):
\[
\int_{S^{n-1} \times \mathbb{R}^n} d\sigma dv_* B(v - v_*, \sigma) f'_* = \int_{k' \cdot \sigma \geq 1/\sqrt{2}} d\sigma dv'_* \left| \frac{dv_*}{dv'_*} \right| B(v - \psi_\sigma(v'_*), \sigma) f'_* \\
= \int_{k' \cdot \sigma \geq 1/\sqrt{2}} d\sigma dv'_* \frac{2^{n-1}}{(k' \cdot \sigma)^2} B(v - \psi_\sigma(v'_*), \sigma) f'_*.
\]

Note that
\[
B(v - \psi_\sigma(v'_*), \sigma) = B(|v - \psi_\sigma(v'_*)|, k \cdot \sigma) = B(|v - \psi_\sigma(v'_*)|, 2(k' \cdot \sigma)^2 - 1).
\]

Changing the name \( v'_* \) for \( v_* \), we find that (3.13) holds with
\[
S(|v - v_*|) = \int_{k-\sigma \geq \frac{1}{\sqrt{2}}} d\sigma \frac{2^{n-1}}{(k \cdot \sigma)^2} B\left( |v_* - \psi_\sigma(v)|, 2(k \cdot \sigma)^2 - 1 \right) - \int_{k-\sigma \geq 0} d\sigma B(|v - v_*|, k \cdot \sigma).
\]

The first part is then
\[
\int_{k-\sigma \geq \frac{1}{\sqrt{2}}} d\sigma \frac{2^{n-1}}{(k \cdot \sigma)^2} B\left( \frac{|v - v_*|}{k \cdot \sigma}, 2(k \cdot \sigma)^2 - 1 \right)
\]
\[
= |S^{n-2}| \int_0^{\pi/2} d\theta \sin^{n-2} \theta \frac{2^{n-1}}{\cos^2 \theta} B\left( \frac{|v - v_*|}{\cos \theta}, \cos(2\theta) \right)
\]
\[
= |S^{n-2}| \int_0^{\pi/2} d(2\theta) \frac{\sin^{n-2}(2\theta)}{\cos^n \theta} B\left( \frac{|v - v_*|}{\cos(\theta/2)}, \cos(2\theta) \right)
\]
\[
= |S^{n-2}| \int_0^{\pi/2} d\theta \frac{\sin^{n-2} \theta}{\cos^n(\theta/2)} B\left( \frac{|v - v_*|}{\cos(\theta/2)}, \cos \theta \right).
\]

This proves our first claim.

With the notations of (3.6) of Assumption I, let us now estimate \( S \).

We consider first the contribution \( S_1 \) of \( B_1 \), i.e. that part of \( B \) which is not borderline in (3.4). Clearly,
\[
|S_1(|v - v_*|)| \leq |S^{n-2}| \int_0^{\pi/2} d\theta \sin^{n-2} \theta \frac{1}{\cos^n(\theta/2)} B_1\left( \frac{|v - v_*|}{\cos(\theta/2)}, \cos \theta \right) - B_1(|v - v_*|, \cos \theta) \]

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\[ + |S^{n-2}| \int_0^\pi d\theta \sin^{n-2} \theta \left[ \frac{1}{\cos^{n}(\theta/2)} - 1 \right] B_1(|v - v_s|, \cos \theta). \]

\[ \leq |S^{n-2}| 2^{n/2} \int_0^\pi d\theta \sin^{n-2} \theta (1 - \cos(\theta/2)) |v - v_s| B'_1(|v - v_s|, \cos \theta) \]

\[ + |S^{n-2}| 2^{n/2} \int_0^\pi d\theta \sin^{n-2} \theta (1 - \cos(\theta/2)) B_1(|v - v_s|, \cos \theta). \]

Since \(1 - \cos^n(\theta/2) \leq n(1 - \cos(\theta/2)) = 2n \sin^2(\theta/4) \leq n(1 - \cos \theta)/(4 \cos^2(\pi/8))\), there is a cancellation of order 2, and we conclude by recalling the definitions of \(M_1, M'_1\).

Now, let us estimate only the borderline contribution, i.e. assume that \(B(z, \sigma) = \beta_0(k \cdot \sigma)|z|^{-n}\). Let \(B_\varepsilon(z, \sigma) = \beta_0(k \cdot \sigma)|z|^{-n}1_{|z| \geq \varepsilon}\). To this cross-section is associated the convolution kernel

\[ S_\varepsilon(|z|) = \frac{|S^{n-2}|}{|z|^n} \int_0^\pi d\theta \sin^{n-2} \theta \left[ 1_{|z| \geq \varepsilon} \cos(\theta/2) - 1_{|z| \geq \varepsilon} \right] \beta_0(\cos \theta) \]

\[ = \frac{|S^{n-2}|}{|z|^n} \int_0^\pi d\theta \sin^{n-2} \theta \beta_0(\cos \theta) 1_{\cos(\theta/2) \leq |z| \leq \varepsilon}. \]

Let

\[ I(\delta) = \int_\delta^\pi d\theta \sin^{n-2} \theta \beta_0(\cos \theta), \]

we have

\[ S_\varepsilon(|z|) = \frac{|S^{n-2}|}{|z|^n} I \left( 2 \cos^{-1} \left( \frac{|z|}{\varepsilon} \right) \right) 1_{|z| \leq \varepsilon} = \frac{1}{\varepsilon^n} J \left( \frac{|z|}{\varepsilon} \right), \]

where \(J(z) = |S^{n-2}|I(2 \cos^{-1}(|z|))|z|^{-n}1_{|z| \leq 1}\). The fact that the integral of \(S_\varepsilon\) is constant, and that \(S_\varepsilon\) is supported in a ball of radius \(\varepsilon\) easily imply our claim, with \(\lambda = \int_{\mathbb{R}^n} J(z) \, dz\), i.e.

\[ \lambda = |S^{n-2}| |S^{n-1}| \int_0^1 \frac{dr}{r} \int_0^{2 \cos^{-1} r} \beta_0(\cos \theta) \sin^{n-2} \theta \, d\theta \]

\[ = |S^{n-2}| |S^{n-1}| \int_0^{\pi/2} d\theta \sin^{n-2} \theta \beta_0(\cos \theta) \int_0^{\cos(\theta/2)} \frac{dr}{r} \]

\[ = -|S^{n-2}| |S^{n-1}| \int_0^{\pi/2} \beta_0(\cos \theta) \log(\cos(\theta/2)) \sin^{n-2} \theta \, d\theta. \]
Chapter 4

Pseudo-differential formulations

In this Chapter, we shall display a pseudo-differential formulation of Boltzmann collision operator. Though we won’t use too much in these notes the results stated in this Chapter, I think that it has an obvious advantage: it shows very clearly that Boltzmann’s operator behaves as a fractional Laplacian, with the good sign. However, some of the computations in this Chapter will be used for displaying other types of results.

Precise references related to this chapter are:
- Alexandre [3, 6] as regards initial results, which were done in dimension 3;
- other chapters, where we shall show much more precise functional estimates, since the usual pseudo differential approach fails to give correct estimates, for real singular kernels.

4.1 General Framework

Let us recall the general definitions of Boltzmann operator, both in $\sigma$-representation and in $\omega$-representation.

Firstly, in $\sigma$-representation,

\begin{equation}
Q(f, f)(v) = \int_{\mathbb{R}^n} dv_s \int_{S^{n-1}} d\sigma B(v - v_s, \sigma) \left\{ f'_s f'_s - f f_s \right\},
\end{equation}

where

\begin{equation}
v' = \frac{v + v_s}{2} + \frac{|v - v_s|}{2} \sigma \quad \text{and} \quad v'_s = \frac{v + v_s}{2} - \frac{|v - v_s|}{2} \sigma.
\end{equation}

We shall assume that

\[ B(v - v_s, \sigma) = \Phi(|v - v_s|) b(k, \sigma) \]
where
\[ k = \frac{v - v_s}{|v - v_s|} \quad \text{and} \quad \cos \theta = k \cdot \sigma \]

We assume that
\[ \frac{1}{\theta^{1+\nu}} \lesssim \sin^{n-2} \theta b(\cos \theta) \lesssim \frac{1}{\theta^{1+\nu}}, 0 < \nu < 2. \]

and that \( \Phi \) is sufficiently smooth (but not too much!). Here \( \theta \in (-\pi, \pi) \).

Next, Boltzmann operator in \( \omega \)-representation is given by
\[
Q(f, f)(v) = \int_{\mathbb{R}^n} dv_s \int_{S^{n-1}} d\omega \tilde{B}(v - v_s, \omega) \left\{ f'_s f'_s - f_s f_s \right\},
\]
where, this time
\[
v' = v - (v - v_s, \omega) \omega \quad \text{and} \quad v'_s = v_s + (v - v_s, \omega) \omega.
\]

In this case, still with the notation \( k = \frac{v - v_s}{|v - v_s|} \), we introduce the angle \( \alpha \) by
\[ \cos \alpha = k \cdot \omega. \]
This time \( \alpha \in (-\pi/2, \pi/2) \) and grazing collisions correspond to \( \cos \alpha = 0 \), while in \( \sigma \)-representation, they correspond to \( \cos \theta = 1 \).

Assumptions in \( \omega \) representation are then translated into
\[
\tilde{B}(v - v_s, \omega) = \Phi(|v - v_s|) \tilde{b}(\cos \alpha)
\]
\[ \frac{1}{|\cos \alpha|^{1+\nu}} \lesssim \tilde{b}(\cos \alpha) \lesssim \frac{1}{|\cos \alpha|^{1+\nu}}, 0 < \nu < 2. \]

In fact, and just to simplify the exposition, we shall precisely assume equality, that is (for some constant \( \tilde{b} \))

(4.3) \[ \tilde{b}(\cos \alpha) = \tilde{b} \frac{1}{|\cos \alpha|^{1+\nu}}, 0 < \nu < 2, \]

and that for some \( \gamma \)

(4.4) \[ \Phi(|v - v_s|) = \bar{\Phi}|v - v_s|^{\gamma}. \]

In view of the underlying Physics, we shall also explicitly assume that

(4.5) \[ \gamma = \gamma(s) = \frac{s - 5}{s - 1}, \nu = \nu(s) = \frac{2}{s - 1}, \text{ for some } s > 2. \]

However, what is really important is the fact that \( 0 < \nu < 2 \), and that \(-1 < \gamma + \nu < 1 \).
4.2 Carleman representation

First of all, we can consider the bilinear form associated with $Q$, where $f'_*$ is replaced by $g'_*$ and $f_*$ by $g_*$ for another function $g$. We denote then by $Q(g, f)$ the corresponding result.

Now, we are going to use the Carleman representation of $Q(g, f)$, mentioned already in Chapter 1. For further details, we refer to [65, 115, 122].

For this purpose, let $E_{v, v'}$ be the hyperplane going through $v$ and orthogonal to $v - v'$. We introduce the parameter $q = \frac{|v - v'|}{|v - v_*|}$ so that $v' = v + q\omega$ and $v_* = v'_* + q\omega$.

Noticing that $\omega$ and $E_{v, v'}$ are orthogonal, it follows that $dv_* = dv'_* dq$, where $dv'_*$ denotes the Lebesgue measure on $E_{v, v'}$. We have also $dv'_* = q^{n-1} dq d\omega$, so that it follows that

$$Q(g, f)(v) = \int_{R^n} \frac{1}{|v - v'|^{n-1}} \int_{E_{v, v'}} dv'_* \tilde{b}(\frac{|v - v'|}{|v' - v'_*|}) \Phi(v' - v'_*) \{ g'_* f' - g_* f \}.$$  

where here $v_* = (v' - v) + v'_*$.

Then performing simple change of variables, we get

Lemma 4.2.1

$$Q(g, f)(v) = \int_{R^n} d\alpha \tilde{b}(\frac{|\alpha|}{|\alpha|}) \Phi(|\alpha|) \{ g(v + \alpha) f(v - h) - f(v) g(\alpha + v - h) \},$$  

where $E_{0,h}$ denotes the hyperplane going through $0$ and orthogonal to $h$.

4.3 Pseudo-differential decomposition

The above Lemma already shows which kind of pseudo-differential operators could arise in this context. But, there is a more explicit form, if we cut correctly the above operator.

First note, starting with the $\omega$-representation and with our assumptions (4.3), (4.4) and (4.5), that

$$Q(f, g)(v) \equiv \int_{R^n} \int_{S^n-1} \left\{ f(v') g(v'_*) - f(v) g(v_*) \right\} \ | v - v_* |^{\gamma+1+\nu} \frac{dv_* d\omega}{|v' - v|^{1+\nu}}.$$  

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From $v_* - v = (v_* - v') + (v' - v)$ and $(v_* - v').(v' - v) = 0$, one may write

\[(4.7)\]

\[Q(f, g)(v) = Q_1(f, g)(v) + Q_2(f, g)(v),\]

where

\[(4.8)\]

\[Q_1(f, g)(v) \equiv \int_{\mathbb{R}^n} \int_{S^{n-1}} \left\{ f(v') g(v'_*) - f(v) g(v_*) \right\} |v_* - v'|^{\gamma+1+\nu} \frac{dv_* d\omega}{|v' - v|^{1+\nu}},\]

and

\[(4.9)\]

\[Q_2(f, g)(v) \equiv \int_{\mathbb{R}^n} \int_{S^{n-1}} \left\{ f(v') g(v'_*) - f(v) g(v_*) \right\} \times \left\{ |v_* - v'|^2 + |v' - v|^2 \right\}^{\frac{\gamma+1+\nu}{2}} - |v_* - v'|^{\gamma+1+\nu} \right\} \frac{dv_* d\omega}{|v' - v|^{1+\nu}}.\]

Now, using the arguments involved in Lemma 4.2.1 and an obvious splitting, it follows that

\[(4.10)\]

\[Q_1(f, g) = Q_{1,1}(f, g) + Q_{1,2}(f, g),\]

where

\[(4.11)\]

\[Q_{1,1}(f, g)(v) = \int_{\mathbb{R}^n} \frac{2dh}{h|^{n+\nu}} \int_{E_{0,h}} d\alpha \left\{ f(v - h) - f(v) \right\} g(\alpha + v) |\alpha|^{\gamma+1+\nu},\]

and

\[(4.12)\]

\[Q_{1,2}(f, g)(v) = \int_{\mathbb{R}^n} \frac{2dh}{h|^{n+\nu}} \int_{E_{0,h}} d\alpha \left\{ g(\alpha + v) - g(\alpha + v - h) \right\} |\alpha|^{\gamma+1+\nu} f(v).\]

For convenience, we set down the following

**Definition 4.3.1** For $u \in S^{n-1}$, $v \in \mathbb{R}^n$, let

\[\beta_g(u, v) \equiv \int_{E_{0,u}} d\alpha g(\alpha + v) |\alpha|^{\gamma+1+\nu}.\]

Then clearly

\[Q_{1,1}(f, g)(v) = \int_{\mathbb{R}^n} \frac{2dh}{h|^{n+\nu}} \left\{ f(v - h) - f(v) \right\} \beta_g \left( \frac{h}{|h|}, v \right),\]

\[Q_{1,2}(f, g)(v) = \int_{\mathbb{R}^n} \frac{2dh}{h|^{n+\nu}} \left\{ \beta_g \left( \frac{h}{|h|}, v \right) - \beta_g \left( \frac{h}{|h|}, v - h \right) \right\} f(v),\]

\[Q_1(f, g) = Q_{1,1}(f, g) + Q_{1,2}(f, g).\]
Now an "explicit" form for $Q_{1,1}$ is stated by

Lemma 4.3.1 Let $Q_{1,1}$ be given by Definition 4.3.1. Then, there exists a fixed constant $C_s'$ depending only on $s$ (or $\nu$), such that (with the notations of pdo theory)

$$Q_{1,1}(f, g)(v) = -C_s' \int_{\mathbb{R}^n} d\alpha g(\alpha + v) | \alpha |^{\gamma + 1 + \nu} | S(\alpha).D |^\nu (f)(v),$$

where $S(\alpha)$ denotes the projection operator over the hyperplane $E_{0,\alpha}$. The value of $C_s'$ may be found below.

Proof- According to Definition 4.3.1, $\beta_g(-u, v) = \beta_g(u, v)$, so that making the change of variables $h \to -h$, we find that

$$Q_{1,1}(f, g)(v) = \int_{\mathbb{R}^n} \frac{dh}{|h|^{n+\nu}} \left\{ f(v - h) + f(v + h) - 2f(v) \right\} \beta_g \left( \frac{h}{|h|}, v \right).$$

Shifting to polar coordinates, $h = r\omega$, with $r = |h|$, $\omega = \frac{h}{|h|}$, we get also

$$Q_{1,1}(f, g)(v) = \int_0^{+\infty} dr \int_{S^{n-1}} d\omega \frac{1}{r^{1+\nu}} \left\{ f(v + r\omega) + f(v - r\omega) - 2f(v) \right\} \beta_g(\omega, v).$$

Next, letting $\hat{f}(\xi)$ for the Fourier transform of $f$, (4.14) writes as

$$Q_{1,1}(f, g)(v) = \int_{\mathbb{R}^n} d\xi \hat{f}(\xi)e^{i\xi.v} \int_{S^{n-1}} d\omega \beta_g(\omega, v) \{ \int_0^{+\infty} dr \frac{r^{1+\nu}}{r^{1+\nu}} (e^{ir\xi.\omega} + e^{-ir\xi.\omega} - 2) \}. $$

From classical homogeneity arguments, the curly brackets term above may be written as

$$\int_0^{+\infty} dr \frac{r^{1+\nu}}{r^{1+\nu}} (e^{ir\xi.\omega} + e^{-ir\xi.\omega} - 2) \equiv -C_s | \xi, \omega |^{\nu},$$

$$C_s \equiv \int_0^{+\infty} dr \frac{r^{1+\nu}}{r^{1+\nu}} (2 - e^{ir} - e^{-ir}).$$

Getting back to (4.15), we obtain

$$Q_{1,1}(f, g)(v) = -C_s \int_{\mathbb{R}^n} d\xi \hat{f}(\xi)e^{i\xi.v} \int_{S^{n-1}} d\omega \beta_g(\omega, v) | \xi, \omega |^{\nu}. $$

Next, from Definition 4.3.1, one has

$$\beta_g(\omega, v) = \int_{\mathbb{R}^n} d\alpha \delta_{\alpha,\omega=0} g(\alpha + v) | \alpha |^{\gamma + 1 + \nu},$$
according to which (4.18) transforms into

\[ Q_{1,1}(f, g)(v) = -C_s \int_{\mathbb{R}^n} d\xi \hat{f}(\xi) e^{i\xi \cdot v} \int_{S^{n-1}} d\alpha \int_{\mathbb{R}^n} d\delta_{\alpha, \omega = 0} g(\alpha + v) \mid \alpha \mid^{\gamma + 1 + \nu} \mid \xi \cdot \omega \mid^{\nu}, \]

that is also

\[ (4.20) \quad Q_{1,1}(f, g)(v) = -\int_{\mathbb{R}^n} d\xi \int_{\mathbb{R}^n} d\alpha \hat{f}(\xi) e^{i\xi \cdot v} g(\alpha + v) \mid \alpha \mid^{\gamma + 1 + \nu} \{ C_s \int_{S^{n-1}} d\omega \mid \xi \cdot \omega \mid^{\nu} \}, \]

and again, we define the constant \( C'_s \) as

\[ (4.21) \quad \{ C_s \int_{S^{n-1}} d\omega \mid \xi \cdot \omega \mid^{\nu} \} = C'_s \mid S(\alpha) \cdot \xi \mid^{\nu}. \]

Turning to (4.20), we find

\[ (4.22) \quad Q_{1,1}(f, g)(v) = -C'_s \int_{\mathbb{R}^n} d\xi \int_{\mathbb{R}^n} d\alpha \mid \alpha \mid^{\gamma + 1 + \nu} g(\alpha + v) \mid S(\alpha) \cdot \xi \mid^{\nu} \hat{f}(\xi) e^{i\xi \cdot v} \]

which is nothing else than the expression given in the statement of the Lemma.

The above arguments also apply to the analysis of \( Q_{1,2} \), but let us note that in fact this term can also be analyzed with the help of the so-called cancellation Lemma, see in particular Chapter 3.

**Lemma 4.3.2** Let \( Q_{1,2} \) be the operator in Definition 4.3.1. Then, with the same constant \( C'_s \) as in Lemma 3.1, one has

\[ Q_{1,2}(f, g)(v) = C'_s \cdot f(v) \cdot \int_{\mathbb{R}^n} d\alpha \mid \alpha \mid^{\gamma + 1 + \nu} \mid S(\alpha) \cdot D \mid^{\nu} (g)(\alpha + v). \]

**Proof** One has

\[ Q_{1,2}(f, g)(v) = f(v) \int_0^{+\infty} dr \int_{S^{n-1}} d\omega \frac{2}{r^{1+\nu}} \{ \beta_g(\omega, v) - \beta_g(\omega, v - r\omega) \}, \]

and introducing \( \hat{\beta}_g(\omega, \xi) \) for the Fourier transform of \( \beta_g(\omega, v) \) with respect to the variable \( v \), we get

\[ Q_{1,2}(f, g)(v) = f(v) \int_0^{+\infty} dr \int_{S^{n-1}} d\omega \int_{\mathbb{R}^n} d\xi \frac{2}{r^{1+\nu}} \hat{\beta}_g(\omega, \xi) e^{i\xi \cdot v} \{ 1 - e^{-ir\xi \cdot \omega} \}. \]

Shifting \( \omega \) into \(-\omega\), one finds also

\[ (4.23) \quad Q_{1,2}(f, g)(v) = f(v) \int_{S^{n-1}} d\omega \int_{\mathbb{R}^n} d\xi \hat{\beta}_g(\omega, \xi) e^{i\xi \cdot v} \{ \int_0^{+\infty} dr \frac{2}{r^{1+\nu}} \{ 2 - e^{-ir\xi \cdot \omega} - e^{ir\xi \cdot \omega} \} \}. \]

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The curly brackets term above is nothing else than $C_s \mid \xi, \omega \mid ^\nu$ ((4.16) and (4.17)), and thus

\[
Q_{1,2}(f, g)(v) = +C_s \cdot f(v) \cdot \int_{\mathbb{R}^n} d\xi \int_{\mathbb{R}^n} d\omega \int_{\mathbb{R}^n} d\delta \beta(g, \omega, \xi) e^{i\xi \cdot v} \mid \xi, \omega \mid ^\nu. 
\]

From this follows

\[
Q_{1,2}(f, g)(v) = +C_s \cdot f(v) \cdot \int_{\mathbb{R}^n} d\xi \int_{\mathbb{R}^n} d\omega \int_{\mathbb{R}^n} d\delta g(\alpha + k) \mid \alpha \mid ^{\gamma + 1 + \nu} \delta_{\alpha, \omega = 0} e^{-i\xi \cdot k e^{i\xi \cdot v}} \mid \xi, \omega \mid ^\nu 
\]

\[
= f(v) \cdot \int_{\mathbb{R}^n} d\xi \int_{\mathbb{R}^n} d\delta \int_{\mathbb{R}^n} d\alpha \mid \alpha \mid ^{\gamma + 1 + \nu} g(\alpha + k) e^{-i\xi \cdot k e^{i\xi \cdot v}} \{ C_s \int_{S^{n-1}, \omega, \alpha = 0} d\omega \mid \xi, \omega \mid ^\nu \}, 
\]

which, in view of (4.21), yields

\[
Q_{1,2}(f, g)(v) = C'_{s, f(v)} \int_{\mathbb{R}^n} d\xi \int_{\mathbb{R}^n} d\omega \mid \alpha \mid ^{\gamma + 1 + \nu} e^{i\xi \cdot v} \mid S(\alpha), \xi \mid ^\nu \int_{\mathbb{R}^n} d\delta g(\alpha + k) e^{-i\xi \cdot k} d\delta, 
\]

and finally, we obtain

\[
Q_{1,2}(f, g)(v) = +C'_{s, f(v)} \int_{\mathbb{R}^n} d\alpha \int_{\mathbb{R}^n} d\xi \mid \alpha \mid ^{\gamma + 1 + \nu} g(\xi) e^{i\xi \cdot (\alpha + v)} \mid S(\alpha), \xi \mid ^\nu, 
\]

There remains to study $Q_2$. For this purpose, set

\[
B^c(|v - v_s|, (\frac{v - v_s}{|v - v_s|}, \omega)) \equiv \frac{1}{|v' - v|^{1 + \nu}} (\mid v_s - v' \mid ^2 + |v' - v|^2) \gamma^{\frac{\gamma + 1 + \nu}{2}} - |v_s - v' |^{\gamma + 1 + \nu}. 
\]

Note that $B^c$ depends on these variables, as $\{ | v_s - v' | ^2 + |v' - v|^2 \gamma^{\frac{\gamma + 1 + \nu}{2}} = |v - v_s |^{\gamma + 1 + \nu}$ and

\[
| v_s - v' | ^2 = | v_s - v | ^2 \{ 1 - (\frac{v - v_s}{|v - v_s|}, \omega) \} ^2. 
\]

Of course $B^c \geq 0$.

Next, consider the function $j : t \in \mathbb{R}^+ \to \{ | \alpha | ^2 + t \} ^{\frac{\gamma + 1 + \nu}{2}}$, for all $\alpha \in \mathbb{R}^n - \{ 0 \}$ fixed. Then

\[
| \partial_t j | \leq C(\mid \alpha \mid ^2 + t) \gamma^{\frac{\gamma + 1 + \nu}{2}} \leq C \frac{1}{\mid \alpha \mid ^{2 - (\gamma + 1 + \nu)}}, 
\]

and since $\frac{\gamma + 1 + \nu}{2} = \frac{s - 2}{s - 1} \cdot 0 < \frac{\gamma + 1 + \nu}{2} < 1$, we deduce that

\[
| B^c(., .) \mid \leq B^c_{\max}(., .) \equiv C \frac{1}{|v_s - v'|^{2 - (\gamma + 1 + \nu)} |v' - v'|^{\nu - 1}}. 
\]
Finally, as 

\[ |v' - v| = |v_* - v| \left( \frac{v - v_*}{|v - v_*|}, \omega \right) \quad \text{and} \quad |v_* - v'| = |v_* - v| \left( 1 - \left( \frac{v - v_*}{|v - v_*|}, \omega \right)^2 \right)^{\frac{1}{2}}, \]

one gets also 

\[ B_{\max}^c(v, \ldots) = C \left| v_* - v \right|^{1 - \left( \frac{v - v_*}{|v - v_*|}, \omega \right)^2} \left( \frac{v - v_*}{|v - v_*|}, \omega \right)^{\nu - 1}. \]

Recalling that \( 0 < 1 - \left( \frac{\gamma + 1 + \nu}{2} \right) < 1, -1 < \nu - 1 < 1 \) and \(-3 < \gamma < 1\), \( B_{\max}^c \) and therefore \( B^c \) are integrable over \( S^{n-1} \), for fixed \( v_* - v \). To sum up, one has

**Theorem 4.3.1** Let \( Q \) be the following Boltzmann non linear operator

\[ Q(f, g)(v) = \int_{\mathbb{R}^n} \int_{S^{n-1}} dv_* d\omega \left\{ f(v') g(v'_*) - f(v) g(v_*) \right\} B(|v - v_*|, (\frac{v - v_*}{|v - v_*|}, \omega)), \]

with \( B \), \( \gamma \) and \( \nu \) given by (for \( s > 2 \))

\[ B(|v - v_*|, (\frac{v - v_*}{|v - v_*|}, \omega)) \equiv |v - v_*|^{1 - \left( \frac{v - v_*}{|v - v_*|}, \omega \right)^2} \left( \frac{v - v_*}{|v - v_*|}, \omega \right)^{\nu - 1}. \]

Then, one has the decomposition

\[ Q(f, g)(v) = -C_s' \int_{\mathbb{R}^n} d\alpha g(\alpha + v) |\alpha|^{\nu + 1 + \nu} \left\{ S(\alpha).D |f'(v) + C_s f(v) \right\} + \int_{\mathbb{R}^n} d\alpha |\alpha|^{\nu + 1 + \nu} \left\{ S(\alpha).D |g(\alpha + v) \right\} + \int_{\mathbb{R}^n} \int_{S^{n-1}} dv_* d\omega \left\{ f g'_* - f g_* \right\} B^c(|v - v_*|, (\frac{v - v_*}{|v - v_*|}, \omega)). \]

Above \( C_s' \) is a fixed constant depending only on \( s \), and given through (4.17) and (4.21). \( S(\alpha) \) denotes the orthogonal projection over \( E_{0, \alpha} \), the hyperplane through 0 and orthogonal to \( \alpha \). Finally

\[ B^c(|v - v_*|, (\frac{v - v_*}{|v - v_*|}, \omega)) \equiv \frac{1}{\left| v' - v \right|^{1 + \nu} \left( |v_* - v'|^2 + |v - v|^2 \right)^{\frac{\gamma + 1 + \nu}{2}} - |v_* - v'|^{\gamma + 1 + \nu}}. \]

One has \( B^c \geq 0 \), \( B^c \in L^1_{\text{loc}}(\mathbb{R}^n \times S^{n-1}) \), and more precisely

\[ B^c(v, \ldots) \leq C \left| v_* - v \right|^{\frac{1}{\left( 1 - \left( \frac{v - v_*}{|v - v_*|}, \omega \right)^2 \right)^{1 - \frac{\gamma + 1 + \nu}{2}}}} \left( \frac{v - v_*}{|v - v_*|}, \omega \right)^{\nu - 1}. \]

\[ \square \]
Let us also note the following extensions of Theorem 4.3.1, when assuming \( B \) satisfies

\[
B \chi \left( \left| v - v' \right|, \frac{v - v'}{v - v'} \right) \equiv \left| v - v' \right| \gamma_1 + 1 + 1 + \nu \chi \left( \left| v - v' \right| \right),
\]

where \( \chi \) is a smooth given function. In this case, the decomposition stated above holds true, provided we change the weight \( \left| \alpha \right|^{\gamma_1 + 1 + \nu} \) into \( \left| \alpha \right|^{\gamma_1 + 1 + \nu} \chi \left( \left| \alpha \right| \right) \) and \( B \) into \( B \chi \left( \left| v - v' \right| \left\{ 1 - \left( \left( \frac{v - v'}{v - v'} \right) \omega \right)^2 \right\}^{\frac{1}{2}} \right) \).

Allowing for such general \( \chi \) does not influence on the non angular cutoff hypothesis made on \( B \).
Chapter 5

Coercivity Results

5.1 Introduction

In Chapter 4, we have seen that Boltzmann collision operator behaves as a negative fractional Laplacian type operator. Moreover, in Chapter 3, we have seen that one part of this operator behaves exactly as a convolution type operator. It is thus interesting to know if it induces coercivity type results, up to a non important rest.

Since, the natural setting of Boltzmann equations is based on entropic bounds, the guess is that the entropy dissipation estimate should imply such results. This is the aim of this chapter to show that this is indeed the case.

Precise references for this Chapter are:
- Works of Alexandre [2, 7];
- Work of Lions [85];
- Work of Villani [116];

We shall assume $n \geq 2$ and typically the following lower bound on collision cross-sections, since we are interested in coercive type estimations

\begin{equation}
B(v - v^*, \sigma) \geq \Phi(|v - v^*|) b(k \cdot \sigma).
\end{equation}

Above the kinetic cross-section $\Phi(|z|): \mathbb{R}^n \to \mathbb{R}^+$ is continuous and strictly positive for $z \neq 0$, and

\begin{equation}
\sin^{n-2} \theta b(\cos \theta) \sim \frac{K}{\theta^{1+\nu}} \quad \text{as} \quad \theta \to 0, \quad \nu > 0.
\end{equation}
Recall the expression of the entropy dissipation functional
\[ D(f) = -\int_{\mathbb{R}^n} Q(f, f) \log f \, dv \]
(5.3)
\[ = \frac{1}{4} \int_{\mathbb{R}^n} \int_{S^{n-1}} B(v - v_*, \sigma)(f' f_*' - f f_*) \log \frac{f' f_*'}{f f_*} \, d\sigma \, dv \, dv_* \]
and the fact that there is a natural a priori bound on this quantity.

The present chapter extends all previous known results about induced regularity from this dissipation bound, but let us mention that there is still some works to be done, at the level of weighted estimates and at the level of the linearized dissipation functional. We mention some preliminary works due to Mouhot [95].

In order to state the main result, let us introduce the generalized linear (w.r.t. \(f\)) entropy dissipation functional
\[ D(g, f) = -\int_{\mathbb{R}^n} Q(g, f) \log f. \]
(5.4)

For a given cross-section \(B\) we introduce again the quantities (already introduced in Chapter 3)
\[ M(|z|) = \int_{S^{n-1}} B(z, \sigma)(1 - k \cdot \sigma) \, d\sigma \]
(5.5)
and
\[ M'(|z|) = \int_{S^{n-1}} B'(z, \sigma)(1 - k \cdot \sigma) \, d\sigma, \]
(5.6)
where
\[ B'(z, \sigma) = \sup_{1 < \lambda \leq \sqrt{2}} \frac{|B(\lambda z, \sigma) - B(z, \sigma)|}{(\lambda - 1)|z|}. \]
These quantities will be useful in order to get estimations from above, as can be guessed from Chapter 3.

The main result of this Chapter is given by

**Theorem 5.1.1** Assume that \(B\) satisfies (5.1), and that \(b\) satisfies the singularity assumption (5.2). Furthermore, suppose that
\[ M(|z|) + |z|M'(|z|) \leq C_0(1 + |z|)^2. \]
Then, for all \(R > 0\) there exists a constant \(C_{g,R}\), depending only on \(b\), \(\|g\|_{L^1}\), \(\|g\|_{L^{\log L}}\), \(R\), and on \(\Phi\), such that
\[ \|\sqrt{f}\|_{H^{n/2}(|v| < R)} \leq C_{g,R} \left[ D(g, f) + \|g\|_{L^2_1} \|f\|_{L^2_1} \right]. \]
(5.7)
Corollary 5.1.1 Assume that \( B(z, \sigma) \geq B_0(z, \sigma) \), where \( B_0(z, \sigma) \) satisfies the same assumptions as \( B \) in Theorem 5.1.1. Then,

\[
\| \sqrt{f} \|_{H^{v/2, |v| < R}}^2 \leq C_{f, R} \left[ D(f) + \| f \|_L^2 \right].
\]

Actually, at least if we take power like kinetic kernel, then, the above weights are not optimal. This point will become clear from Chapter 7.

The proof of the main result is as follows.

We start from the following decomposition of \( D(g, f) \) (let us mention that there are also other possibilities)

\[
D(g, f) = -\int_{\mathbb{R}^{2n} \times S^{n-1}} B(g_s f' - g_s f) \log f \, dv \, dv_s \, d\sigma \\
= \int_{\mathbb{R}^{2n} \times S^{n-1}} B g_s f \log \frac{f}{f'} \, dv \, dv_s \, d\sigma \\
= \int_{\mathbb{R}^{2n} \times S^{n-1}} B g_s \left( f \log \frac{f}{f'} - f + f' \right) \, dv \, dv_s \, d\sigma + \int_{\mathbb{R}^{2n} \times S^{n-1}} B g_s (f - f') \, dv \, dv_s \, d\sigma,
\]

and thus

\[
D(g, f) = A + B,
\]

with obvious notations.

Thus \( D(g, f) \) has been decomposed into two terms; the first one \( A \) is positive and will control the smoothness which we are looking for. As for the second one \( B \), making the change of variables \( v \rightarrow v_s \) (or remaking a similar proof), we see that it fails under the scope of the cancellation Lemma, from Chapter 3. Thus, we find more precisely that

\[
B = \int_{\mathbb{R}^n} dv_s g_s S \ast f(v_s).
\]

Using Hardy-Littlewood-Sobolev inequality, one obtains therefore the control of \( B \) from (5.9) as follows

Corollary 5.1.2 Assume that \( M \) and \( M' \) are defined by (5.5) and (5.6) respectively, and that \( M(|v - v_s|) + M'(|v - v_s|) \leq C(|v - v_s|^\gamma + |v - v_s|^2) \).

1. If \( 0 \leq \gamma \leq 2 \), then

\[
\left| \int B g_s (f' - f) \, dv \, dv_s \, d\sigma \right| \leq C \| g \|_L^2 \| f \|_L^2.
\]
2. If $-n < \gamma < 0$, then

$$
| \int B g_*(f' - f) \, dv \, ds | \leq C \|g\|_{L^2_-} \|f\|_{L^2_+},
$$

where $p_1^{-1} + p_2^{-1} = 2 + \gamma/n$

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From Corollary 5.1.2, using the inequality $x \log x - x + y \geq (\sqrt{x} - \sqrt{y})^2$, it follows from (5.9) that

$$
\begin{align*}
\{ D(g, f) + C_1 \|g\|_{L^1_2} \|f\|_{L^1_2} \geq & \int B g_*(\sqrt{T} - \sqrt{T})^2 \, dv \, ds \, ds \\
\geq & \int \Phi(|v - v_*|) b(k \cdot \sigma) g_*(\sqrt{T} - \sqrt{T})^2 \, dv \, ds \, ds \equiv A_+.
\end{align*}
$$

The hard work is to bound from below the term $A_+$ in this inequality in terms of a suitable Sobolev norm. Though it should be clear, at least in view of the preceding Chapters, the task is not so easy, except in the special case where a good lower bound on $g$ is available. Under this assumptions, first results were proven by Alexandre and Villani. But we do not want to impose such a restriction, which is by the way not always proven.

The point is that there is also another special case which is workable: the Maxwelian case, that is when $\Phi \equiv 1$. If it was so, then using Bobylev ideas, one can use Fourier transform at this point. Therefore, we need first to reduce to the Maxwelian case.

In the following, we let $F(v) = \sqrt{f(v)}$.

For a given $R > 0$, $\chi_R(v)$ denotes a smoothed version of the characteristic function of the ball $B_R = \{|v| < R\}$, that is a smooth function such that $0 \leq \chi_R \leq 1$, $\chi_R \equiv 1$ on $B_R$, and supp $(\chi_R) \subset B_{R+1})$. More generally, $\chi_A$ denotes the characteristic function of a set $A$.

### 5.2 Truncation and Reduction to the Maxwelian case

To get a lower bound on the term $A_+$ from inequality (5.11) which still involves function $\Phi$, into a similar term which does not involve function $\Phi$, so to get 1 instead, we shall prove the following result

**Lemma 5.2.1** We have

1. Assume that $\Phi$ is continuous and bounded from below on $B_R$, for any $R > 0$. Then

$$
\int B g_*(\sqrt{T} - \sqrt{T})^2 \, dv \, ds \, ds + C_2 \|f\|_{L^1_2} \|g\|_{L^1_2}
\geq \min_{|z| \leq 2\sqrt{2}R} (1, \Phi(|z|)) \int_{\mathbb{R}^{2n} \times S^{n-1}} b(k \cdot \sigma) (g \chi_B)_*((F \chi_B)_f' - F \chi_B)_f^2 \, dv \, ds \, ds
$$

where the constant $C_2$ depends only on $n$ and $B$.  

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2. Assume that $\Phi$ as above, except that it may vanish as $|z| \to 0$. For any $R > 0$, $r_0 > 0$ a (small) positive constant, let:

- $v_j \in \mathbb{R}^n$ such that $|v_j| < R$;
- $A_j = \{v \in \mathbb{R}^n; |v - v_j| \leq r_0/4\}$;
- $B_j = \{v \in \mathbb{R}^n; |v| \leq R, |v - v_j| \geq r_0\}.$

Then,

$$\int B g_*(\sqrt{F} - \sqrt{\bar{F}})^2 \, dv \, dv_* \, d\sigma + C_2 r_0^{-2} \|f\|_{L^\infty} \|g\|_{L^\infty}$$

$$\geq \frac{r_0}{2} \min_{\frac{r_0}{2} \leq |z| \leq 2\sqrt{2}R} (1, \Phi(|z|)) \int_{\mathbb{R}^{2n} \times S^{n-1}} b(k \cdot \sigma) g_* \chi_{A_j} (F_{\chi_{A_j}} - F_{\chi_{A_j}}')^2 \, dv \, dv_* \, d\sigma.$$

In the first case, this result is enough, in view of the following sections to conclude. In the second case, we need to collect a collection of such $v_j$; this will be done in the last section, where we finish the proof.

**Proof:**

For any subset $A, B$

$$g_*(F' - F)^2 \geq g_* \chi_{B_*(F' - F)^2 \chi_A^2}$$

Furthermore, from

$$(F'_{\chi_A} - F_{\chi_A})^2 = (F'(\chi_A' - \chi_A) + (F' - F)\chi_A)^2 \leq 2F'^2(\chi_A' - \chi_A)^2 + 2(F' - F)^2 \chi_A^2,$$

it follows that

$$2 \Phi(|v - v_*|) b(k \cdot \sigma) g_*(F' - F)^2 \geq \min(1, \Phi(|v - v_*|)) b(k \cdot \sigma) g_*(F' - F)^2 \chi_A^2$$

$$\geq \frac{1}{2} \min(1, \Phi(|v - v_*|)) b(k \cdot \sigma) g_* \chi_{B_*(F'_{\chi_A} - F_{\chi_A})^2 - b(k \cdot \sigma) g_* \chi_{B_*(F' - F)^2 (\chi_A' - \chi_A)^2}.$$  

Note that

$$(\chi_A' - \chi_A)^2 \leq \|\nabla \chi_A\|_{L^\infty}^2 |v - v'|^2 = \|\nabla \chi_A\|_{L^\infty}^2 |v - v_*|^2 \sin^2 \frac{\theta}{2}.$$  

If $A$ is a ball of radius $\text{diam}(A)$, we can as well assume that $\|\nabla \chi_A\|_{L^\infty}^2 < C \max(1, (\text{diam}(A)))^{-2}$. If we denote any constant depending on $A$ in this way by $C_A$, we get

$$\left\{ \begin{array}{l}
g_* \chi_{B_*} b(k \cdot \sigma) f'(\chi_A' - \chi_A)^2 \, dv \, dv_* \, d\sigma \leq C_A \int g_* |v - v_*|^2 b(\cos \theta) \sin^2 \frac{\theta}{2} f' \, dv \, dv_* \, d\sigma \\
\leq C_A \int_{|\theta| \leq \frac{\pi}{2}} g_* |v' - v_*|^2 b(\cos \theta) \cos^{-4} \frac{\theta}{2} \sin^2 \frac{\theta}{2} f'^{2N} \, dv' \, dv_* \, d\sigma \leq C_A \|f\|_{L^\infty} \|g\|_{L^\infty}.
\end{array} \right.$$  

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For the first term in (5.14), we have
\[ |v' - v_*| = \cos \frac{\theta}{2} |v - v_*|, \]
and that \( |\theta| \leq \frac{\pi}{2} \). Therefore,
(5.16) \[ |v' - v_*| \leq |v - v_*| \leq \sqrt{2} |v' - v_*|. \]

First case of the lemma:
We just take the sets \( B = A = \{|v| \leq R\} \). Then from estimate (5.16), we see that when \( |v_*| \leq R \), \( |v - v_*|^2 < 8R^2 \) as soon as \( |v| \leq R \) or \( |v'| \leq R \). Integrating (5.14), we get
(5.17) \[
2 \int_{\mathbb{R}^{2n} \times S^{n-1}} \min(1, \Phi(|v - v_*|)) b(k \cdot \sigma) g_*(F' - F)^2 \, dv \, dv_* d\sigma + C_A \|f\|_{L^1} \|g\|_{L^2} \\
\geq \min_{|z| \leq 2\sqrt{2}R} (1, \Phi(|z|)) \int_{\mathbb{R}^{2n} \times S^{n-1}} b(k \cdot \sigma) g_* \chi_{B_*}(F'_A - F_{\chi_A})^2
\]

Second case of the lemma:
The problem is on \( |v - v_*| < r_0 \), since \( \Phi(|v - v_*|) \) may vanish on it.
Firstly, we choose a (small) constant \( r_0 \), and a \( v_j \in \chi_{B_{r_0}} \). Thus we have two sets \( A_j \) and \( B_j \) as defined in the statement of the Lemma. \( A_j \) and \( B_j \) are separated by a distance of \( 3r_0/4 \), and the support of smoothed versions of their characteristic functions of \( A_j \) and \( B_j \) may be assumed to be separated by a distance of \( r_0/2 \).
From estimate (5.16), we note that
\[
\begin{align*}
v_* &\in B_j, v \in A_j \quad \implies \quad |v_* - v| > r_0/2, \\
v_* &\in B_j, v' \in A_j \quad \implies \quad |v_* - v| > r_0/2.
\end{align*}
\]
Thus inequality (5.17) now holds with the sets \( A \) and \( B \) replaced by \( A_j \) and \( B_j \) respectively, and with \( \min_{|z| \leq 2\sqrt{2}R} (1, \Phi(|z|)) \) replaced by \( \min_{r_0/2 \leq |z| \leq 2\sqrt{2}R} (1, \Phi(|z|)) \), \( C_A \) depending on the radius \( r_0 \).

5.3 Maxwellian case

According to Lemma 5.2.1, we see that lower bounds therein do not involve function \( \Phi \) anymore, i.e. inside the integrals.
5.3.1 Fourier transform

The following lemma now shifts (with obvious change of notations) to Fourier space, denoting

\[ \hat{f}(\xi) = \mathcal{F}f(\xi) = \int_{\mathbb{R}^n} e^{-iv \cdot \xi} f(v) \, dv \]

the Fourier transform of a function \( f \in L^1(\mathbb{R}^n) \).

**Proposition 5.3.1** For any functions \( g \in L^1(\mathbb{R}^n), F \in L^2(\mathbb{R}^n) \):

\[
\begin{align*}
\int_{\mathbb{R}^n} \int_{S^{n-1}} b(k \cdot \sigma) g_* (F' - F)^2 \, dv \, d\sigma &= \frac{1}{2\pi^n} \int_{\mathbb{R}^n} \int_{S^{n-1}} b \left( \frac{\xi_{\perp}}{\|\xi\|} \cdot \sigma \right) \left[ \hat{g}(0) |\hat{F}(\xi)|^2 + \hat{g}(0) |\hat{F}(\xi^+)|^2 - \hat{g}(\xi^-) \hat{F}(\xi^+) \hat{\bar{F}}(\xi) - \hat{g}(\xi^-) \hat{\bar{F}}(\xi^+) \hat{F}(\xi) \right] \, d\xi \, d\sigma ,
\end{align*}
\]

where

\[
\xi^+ = \frac{\xi + |\xi| \sigma}{2}, \quad \xi^- = \frac{\xi - |\xi| \sigma}{2}.
\]

**Proof:**

Without loss of generality, we assume that \( b \) is integrable. Then, expanding the quadratic term in (5.18) gives three terms,

\[
F'^2 - 2FF' + F^2 .
\]

- **Middle term of (5.20)**

By usual change of variables, and Parseval’s identity,

\[
\int b(k \cdot \sigma) g_* F' F \, dv \, d\sigma = \int Q^+(g, F) F \, dv = \frac{1}{(2\pi)^n} \int \mathcal{F} \left[ Q^+(g, F) \right] \hat{F} \, d\xi .
\]

By Bobylev’s identity:

\[
\mathcal{F} \left[ Q^+(g, F) \right] = \int_{S^{n-1}} b \left( \frac{\xi_{\perp}}{\|\xi\|} \cdot \sigma \right) \hat{g}(\xi^-) \hat{F}(\xi^+) \, d\sigma ,
\]

we deduce that

\[
\int b(k \cdot \sigma) g_* F' F \, dv \, d\sigma = \frac{1}{(2\pi)^n} \int b \left( \frac{\xi_{\perp}}{\|\xi\|} \cdot \sigma \right) \hat{g}(\xi^-) \hat{F}(\xi^+) \hat{\bar{F}}(\xi) \, d\xi \, d\sigma .
\]
which is also equal to its own complex conjugate. This shows how to compute the cross-products in (5.18).

- **Last term** of (5.20)

Noticing that \( \int_{S^{n-1}} b(k \cdot \sigma) d\sigma \) does not depend on the unit vector \( k \), we get

\[
\int b(k \cdot \sigma) g_{s} F^{2} \, dv \, dv_{s} \, d\sigma = \int b(k \cdot \sigma) d\sigma \int g_{s} \, dv_{s} \int F^{2} \, dv = \frac{1}{(2\pi)^{n}} \int b \left( \frac{\xi}{|\xi|} \cdot \sigma \right) \hat{g}(0)|\hat{F}|^{2}(\xi) d\sigma d\xi,
\]

where we have applied the usual Plancherel identity.

- **First term** of (5.20)

Making the change of variables \( (v, v_{s}) \rightarrow (v - v_{s}, v_{s}) \), and then \( v \rightarrow v' \) as in Section 5, we get

\[
\int \int g(v_{s}) b \left( \frac{v}{|v|} \cdot \sigma \right) \left| \tau_{-v_{s}} F \left( \frac{v + |v|\sigma}{2} \right) \right|^{2} \, dv \, d\sigma \, dv_{s} = \int \int g(v_{s}) b(\psi(v', \sigma)) \frac{2^{n-1}}{(v'/|v'| \cdot \sigma)^{2}} \left| \tau_{-v_{s}} F(v') \right|^{2} \, dv' \, d\sigma \, dv_{s},
\]

where

\[
\psi(v', \sigma) = 2 \left( \frac{v'}{|v'|} \cdot \sigma \right)^{2} - 1,
\]

and \( \tau_{-v_{s}} F = F(v_{s} + \cdot) \). Since \( |F(\tau_{h} F)| = |F(F)| \), and since \( \int_{S^{n-1}} b(k \cdot \sigma) \, d\sigma \) does not depend on \( k \), we obtain

\[
\frac{1}{(2\pi)^{n}} \int g(v_{s}) \left( \int b(\psi(\xi, \sigma)) \frac{2^{n-1}}{(\xi/|\xi| \cdot \sigma)^{2}} |\hat{F}(\xi)|^{2} \, d\xi \, d\sigma \right) \, dv_{s}.
\]

Then, note that the inner integral does not depend on \( v_{s} \), so that, reversing the change of variables from Chapter 3, the last expression becomes, ending the proof

\[
\frac{1}{(2\pi)^{n}} \hat{g}(0) \int b \left( \frac{\xi}{|\xi|} \cdot \sigma \right) \left| \hat{F} \left( \frac{\xi + |\xi|\sigma}{2} \right) \right|^{2} \, d\xi \, d\sigma.
\]

All in all, we can conclude that, using Young’s inequality and the fact that

\[
|\hat{F}(\xi)|^{2} + |\hat{F}(\xi^{+})|^{2} \geq |\hat{F}(\xi)|^{2},
\]

**Corollary 5.3.1** For all \( f \in L^{1}(\mathbb{R}^{n}), f \geq 0 \), and \( g \in L^{2}(\mathbb{R}^{n}) \),

\[
\int_{\mathbb{R}^{2n}} \int_{S^{n-1}} b(k \cdot \sigma) g_{s} (F' - F)^{2} \, dv \, dv_{s} \, d\sigma \geq \frac{1}{2(2\pi)^{n}} \int_{\mathbb{R}^{n}} |\hat{F}(\xi)|^{2} \int_{S^{n-1}} b \left( \frac{\xi}{|\xi|} \cdot \sigma \right) \left( \hat{g}(0) - |\hat{g}(\xi^{-})| \right) \, d\sigma \, d\xi.
\]

\( \square \)
5.3.2 Decrease in Fourier space

Now, in order to finish the proof of Theorem 5.1.1, we need to bound from below the last term in the preceding Corollary, in terms of a suitable norm of \( f \), up to a multiplicative constant depending only on \( g \). The precise result is given by

**Proposition 5.3.2** Suppose that \( b \) satisfies assumption 5.2. Then, there exists a positive constant \( C_g \), depending only on \( \| g \|_{L^1}, \| g \|_{L^{log} L} \) and \( b \), such that for \( |\xi| \geq 1 \),

\[
\int_{S^{n-1}} b \left( \frac{\xi}{|\xi|} \cdot \sigma \right) (\hat{g}(0) - |\hat{g}(\xi^-)|) d\sigma \geq C_g |\xi|^\nu.
\]

This proposition is itself a consequence of the two lemmas below.

**Lemma 5.3.1** There exists a positive constant \( C'_g \), depending only on \( n, \| g \|_{L^{log} L}, \| g \|_{L^1} \), such that for all \( \xi \in \mathbb{R}^n \),

\[
\hat{g}(0) - |\hat{g}(\xi)| \geq C'_g (|\xi|^2 \wedge 1).
\]

**Lemma 5.3.2** If \( b \) satisfies assumption 5.2, then there exists a constant \( K(\nu) \), such that for all \( \xi \in \mathbb{R}^n \), \( |\xi| \geq 1 \),

\[
\int_{S^{n-1}} b \left( \frac{\xi}{|\xi|} \cdot \sigma \right) (|\xi^-|^2 \wedge 1) d\sigma \geq K(\nu)|\xi|\nu.
\]

**Proof of lemma 5.3.1:**

There exists \( \theta \in \mathbb{R} \) such that

\[
\hat{g}(0) - |\hat{g}(\xi)| = \int_{\mathbb{R}^n} g(v) (1 - \cos(v \cdot \xi + \theta)) dv
\]

\[
= 2 \int_{\mathbb{R}^n} g(v) \sin^2 \left( \frac{v \cdot \xi + \theta}{2} \right) dv
\]

\[
\geq 2 \sin^2 \varepsilon \int_{\{ |v| \leq r, \forall p \in \mathbb{Z}^n : |v \cdot \xi + \theta - 2p \pi| \geq 2 \varepsilon \}} g(v) dv
\]

\[
\geq 2 \sin^2 \varepsilon \left\{ \left| \frac{\| g \|_{L^1(\mathbb{R}^n)}}{r} \right| - \int_{\{ |v| \leq r, \exists p \in \mathbb{Z}^n : |v \cdot \xi + \theta - 2p \pi| \leq 2 \varepsilon \}} g(v) dv \right\}
\]

\[
\geq 2 \sin^2 \varepsilon \left\{ \left| \frac{\| g \|_{L^1(\mathbb{R}^n)}}{r} \right| - \frac{\sup_{|A| \leq \frac{4\varepsilon}{|\xi|} (2r)^{n-1} (1 + \frac{r}{|\xi|})^n} \int_A g(v) dv}{\frac{4\varepsilon}{|\xi|} (2r)^{n-1} (1 + \frac{r}{|\xi|})^n} \right\}.
\]

(5.25)
If $|\xi| \geq 1$, then the Lemma is proven with
\[
C_g' = 2 \sin^2 \varepsilon \left\{ \frac{\|g\|_{L^1(\mathbb{R}^n)}}{r} - \frac{\|g\|_{L^1(\mathbb{R}^n)}}{r} \sup_{|A| \leq 4 \varepsilon (2r)^{n-1}+\frac{4\pi}{3}(2r)^n} \int_A g(v) \, dv \right\},
\]
$\varepsilon > 0$ and $r > 0$ being chosen in such a way that this quantity is positive.

If $|\xi| \leq 1$, we put $\delta = \frac{\varepsilon}{|\xi|}$ in (5.25), and set
\[
C_g' = 2 \delta^2 \inf_{|\xi| \leq 1} \left| \frac{\sin^2 (\delta |\xi|)}{\delta^2 |\xi|^2} \right|
\times \left\{ \frac{\|g\|_{L^1(\mathbb{R}^n)}}{r} - \frac{\|g\|_{L^1(\mathbb{R}^n)}}{r} \sup_{|A| \leq 4 \delta (2r)^{n-1}+\frac{4\pi}{3}(2r)^n} \int_A g(v) \, dv \right\},
\]
$\delta > 0$ and $r > 0$ being chosen in such a way that this quantity is positive.

**Proof of lemma 5.3.2:**
Since $|\xi|^{-2} = \frac{|\xi|^2}{2} \left( 1 - \frac{\xi}{|\xi|} \cdot \sigma \right)$, shifting to $n$-dimensional spherical coordinates, we find for some $\theta_0 > 0$,
\[
\int_{S^{n-1}} b(\frac{\xi}{|\xi|} \cdot \sigma) (|\xi|^{-2} \wedge 1) \, d\sigma = \frac{K_{n-1}}{2} \int_0^{\theta_0} \sin^{n-2} \theta \, b(\cos \theta) \left( \frac{|\xi|^2}{2} (1 - \cos \theta) \wedge 1 \right) d\theta
\geq \frac{K_{n-1}}{4} \int_0^{\theta_0} \left( \frac{|\xi|^2}{2} \theta^2 \wedge 1 \right) \frac{d\theta}{\theta^{1+\nu}}.
\]
By the change of variables $\theta \rightarrow |\xi| \theta$, this integral is also
\[
|\xi|^\nu \int_0^{\theta_0} \left( \frac{\theta^2}{2} \wedge 1 \right) \frac{d\theta}{\theta^{1+\nu}},
\]
so that when $|\xi| \geq 1$, lemma 5.3.2 holds with
\[
K(\nu) = \frac{K_{n-1}}{2} \int_0^{\theta_0} \left( \frac{\theta^2}{2} - 1 \right) \frac{d\theta}{\theta^{1+\nu}}.
\]

**Remark 5.3.1** *In the limit case $\nu = 0$, we also find the lower bound*
\[
\int_{\sqrt{2}}^{\frac{\pi}{2}} |\xi| \frac{d\theta}{\theta} \sim \log |\xi| \text{ as } |\xi| \to \infty.
\]
*More generally,*
\[
\int_{S^{n-1}} b(k \cdot \sigma) (|\xi|^{-2} \wedge 1) \, d\sigma \geq \frac{|S^{n-1}|}{4\pi^2} \int_0^{\frac{\pi}{2}} \sin^{n-2} \theta \, b(\cos \theta) (|\xi|^2 \theta^2 \wedge 1) \, d\theta
\geq \frac{|S^{n-1}|}{4\pi^2} \int_0^{\frac{\pi}{2}} \sin^{n-2} \theta \, b(\cos \theta) \, d\theta.
\]
*This computation enables to treat arbitrary singularities.*

\[\square\]
5.4 End of the proof of the main result

By corollary (Corollary 5.1.2), we have seen that one has (5.11).

**Case** $\inf_{|z| < R} \Phi(|z|) > 0$

Then the truncation Lemma (that is, lemma 5.2.1) gives

\[
\int B_g (\sqrt{f'} - \sqrt{f})^2 dv d\sigma + C_2 \|f\|_{L^1}^2 \|g\|_{L^2}^2 
\geq \min_{|z| \leq 2\sqrt{2}R} (1, \Phi(|z|)) \int_{\mathbb{R}^{2n} \times S^{n-1}} b(k \cdot \sigma) g \ast \chi_{B_R} (F' \chi_{B_R} - F \chi_{B_R})^2 dv d\sigma.
\]  

(5.26)

From Corollary 5.3.1, it follows that

\[
\int_{\mathbb{R}^{2n} \times S^{n-1}} b(k \cdot \sigma) g \ast \chi_{B_R} (F' \chi_{B_R} - F \chi_{B_R})^2 dv d\sigma
\geq \int_{\mathbb{R}^n} d\xi \left| \widetilde{F} \chi_{B_R} (\xi) \right|^2 \left\{ \int_{S^{n-1}} d\sigma b \left( \frac{\xi}{|\xi|} \cdot \sigma \right) (\hat{g} \ast \chi_{B_R} (0) - |\hat{g} \ast \chi_{B_R} (\xi^-)|) \right\}
\]

where $\xi^- = (\xi - |\xi|\sigma)/2$.

By Proposition 5.3.2, there is a positive constant $c_g$, which depends only on $\|g\|_{L^1}$, $\|g\|_{L \log L}$, $n$ and $b$, such that

\[
\int_{S^{n-1}} d\sigma b \left( \frac{\xi}{|\xi|} \cdot \sigma \right) (\hat{g}(0) - |\hat{g}(\xi^-)|) \geq c_g |\xi|^{\nu}.
\]  

(5.27)

Therefore

\[
\|F \chi_{B_R}\|_{H^{\nu/2}}^2 \leq 2(c_g \chi_{B_R} \min_{|z| \leq 2\sqrt{2}R} (1, \Phi(|z|)))^{-1} \left( D(g, f) + (C_1 + C_2) \|g\|_{L^1} \|f\|_{L^2} \right),
\]  

(5.28)

proving the theorem, in this case.

**Case** $\Phi(|z|) \sim 0$ when $|z| \sim 0$

In this case, we need to take care of small relative velocities.

Let $v_j$ be an arbitrary point in $\{|v| \leq R\}$, and let $r_0$ be a small positive number. Then define

\[
A_j = \{ v \mid |v - v_j| < r_0/4 \} \quad \text{and} \quad B_j = \{ v \mid |v - v_j| > r_0, |v| \leq R \}
\]

and let $\chi_{B_j}$ and $\chi_{A_j}$ be the corresponding smoothed characteristic functions, in such a way that the supports of $\chi_{B_j}$ and $\chi_{A_j}$ are separated by a distance $r_0/2$. Then (5.26) becomes

\[
\left[ \min_{r_0/2 \leq |z| \leq 2\sqrt{2}R} (1, \Phi(|z|)) \right] \int_{\mathbb{R}^{2n} \times S^{n-1}} \tilde{b}(k \cdot \sigma) (g \chi_{B_j}) \ast (F' \chi_{B_R} - F \chi_{B_R})^2 dv d\sigma.
\]
while (5.28) transforms into

\[(5.29) \quad \|F_{\chi_{A_j}}\|_{H^{\nu/2}}^2 \leq 2 \left( c_{g,\chi_{B_j}} \min_{r_0/2 \leq |z| \leq 2\sqrt{2}R} (1, \Phi(|z|)) \right)^{-1} \left( D(g, f) + (C_1 + C_2)\|g\|_{L^2_1}\|f\|_{L^1_2} \right). \]

By choosing \( r_0 \) properly, the right-hand side can be made independent of \( j \). The point is that (5.27) only depends on the mass, entropy and first moment of \( g \), and on \( b \). For any function \( g \) with bounded entropy, \( r_0 \) can be chosen so that for any \( |v_0| < R \),

\[ \int_{|v-v_0| < r_0} g(v) \, dv < \frac{1}{2} \int_{|v| < R} g(v) \, dv. \]

Hence, \( c_{g,\chi_{B_j}} \) depends on the same quantities as \( c_g \) above. The choice of \( r_0 \) also changes \( C_2 \), which is of the order \( r_0^{-2} \).

All that remains to do is take a set of points \( v_j \in \{|v| \leq R\}, 1 \leq j \leq J \), such that

\[ B_R \subset \bigcup_{j=1}^J A_j. \]

Then,

\[
\|F\|_{H^{\nu/2}(|v| < R)}^2 \leq J \sum_{j=1}^J \|F_{\chi_{A_j}}\|_{H^{\nu/2}}^2 \\
\leq J^2 \left( c_g \min_{r_0/2 \leq |z| \leq 2\sqrt{2}R} (1, \Phi(|z|)) \right)^{-1} \left( D(g, f) + (C_1 + \tilde{C}_2 r_0^{-2})\|g\|_{L^2_1}\|f\|_{L^1_2} \right),
\]

which concludes the proof.
Chapter 6

Regularity- Homogeneous and maxwellian case

6.1 Introduction

References for this Section are:

- the initial works on smoothness by Desvillettes [47, 48, 49];
- the recent work of Desvillettes and Wennberg [55];
- our recent work Alexandre and Elsafadi [5] on which we have based the present chapter;

We now consider Boltzmann homogeneous equation

\[ \partial_t f(t,v) = Q(f,f)(t,v) \quad t \geq 0, \; v \in \mathbb{R}^n, \]

where \( f \) is a positive function depending only (homogeneous framework) upon the two variables \( t \geq 0 \) (time) and \( v \in \mathbb{R}^n \) (velocity) with \( f(0,v) = f_0(v) \), where \( n \geq 2 \).

The initial datum \( f_0 \neq 0 \) is supposed to satisfy the usual "entropic" hypothesis, that is

\[ f_0 \geq 0, \int_{\mathbb{R}^n} f_0(v)\{1+|v|^2 + \log(1 + f_0(v))\}dv < +\infty. \]

We shall assume here that the collision cross section \( B(v-v_*,\sigma) > 0 \) depends only on the deviation angle \( \theta \) (Maxwellian molecules):

\[ B(v-v_*,\sigma) = b(\cos \theta), \; \cos \theta = \langle \frac{v-v_*}{|v-v_*|}, \sigma \rangle, \; 0 \leq \theta \leq \frac{\pi}{2}. \]

The above range of values of \( \theta \) means more precisely that we assume that \( b(\cos \theta) \) is supported on the set \( \theta \in (0, \frac{\pi}{2}) \).
We are also interested in the case of a collision cross section \( B \) associated to an intermolecular potential of the form \( \frac{1}{r^s} \), \( s > 2 \), and we will therefore assume that \( b(\cos \theta) \) satisfies

\[
\int_0^{\frac{\pi}{2}} \sin^n \theta \ b(\cos \theta) d\theta < +\infty \quad \text{and} \quad \sin^{n-2} \theta \ b(\cos \theta) \sim \frac{\kappa}{\theta^{1+\nu}} \quad \text{when} \quad \theta \to 0,
\]

where \( \kappa > 0 \) and \( 0 < \nu < 2 \) are fixed.

We assume that a weak solution to Boltzmann (6.1) has already been constructed and that it satisfies the usual entropic estimate, for a fixed \( T > 0 \) (eventually \( T = +\infty \))

\[
\sup_{t \in [0, T]} \int_{\mathbb{R}^n} f(t, v)(1 + |v|^2 + \log(1 + f(t, v))) dv < \infty.
\]

This estimate follows from the well known formal conservation laws:
- mass conversation
  \[
  \int_{\mathbb{R}^n} f(t, v) dv = \int_{\mathbb{R}^n} f_0(v) dv;
  \]
- momentum conservation
  \[
  \int_{\mathbb{R}^n} v f(t, v) dv = \int_{\mathbb{R}^n} v f_0(v) dv;
  \]
- energy conservation
  \[
  \int_{\mathbb{R}^n} \frac{|v|^2}{2} f(t, v) dv = \int_{\mathbb{R}^n} \frac{|v|^2}{2} f_0(v) dv;
  \]
- entropy estimate
  \[
  \partial_t \int_{\mathbb{R}^n} f \log f dv + \frac{1}{4} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{S^{n-1}} b(\frac{v - v_*}{|v - v_*|}) (f' f_*' - f f_*) \log \frac{f' f_*'}{f f_*} dv dv_* d\sigma = 0.
  \]

We shall also assume that this weak solution satisfies the mass conservation.

Such weak solutions also satisfy the entropy dissipation rate estimate, that is

\[
\int_0^T \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{S^{n-1}} b(\frac{v - v_*}{|v - v_*|}) (f' f_*' - f f_*) \log \frac{f' f_*'}{f f_*} d\sigma dv dv_* dt < +\infty,
\]

but this last estimate will not be used here.

In this chapter, we are interested in regularization properties of such solutions.

That is, is this weak solution more regular than the initial datum satisfying assumption (6.2), and if so, can we have estimates on this regularity?

Here, for Maxwell cross sections, we show that any weak solution satisfying (6.5) is in \( C^\infty \). Furthermore, by known results on moments propagation, this \( C^\infty \) regularity cannot be improved in the Maxwellian case to regularity for instance in \( S \).

In a previous chapter, we have already shown some regularity estimate in a suitable Sobolev space using only the entropy dissipation estimate (6.6) and the entropic estimate (6.5), but not the full
Boltzmann equation. This result seems not to be sufficient for our purpose, because of the bad control in time of the final estimate, but anyway, we do not use assumption (6.6) here.

Here, we concentrate on the non cutoff Maxwellian case, for any velocity space dimension. The general case of non Maxwellian molecules is the subject of another chapter, with the same method as used here. There will be of course some major modifications, the main one being the use of cutoff functions in velocity space.

Our arguments use Littlewood-Paley theory. This is why a special chapter is devoted to basic facts about this topic. In fact the main point is to work on small annuli instead of the full frequency space and otherwise we will only use elementary analytical arguments. In particular, we do not use any functional interpolation inequalities. But, we do use some previous results and methods.

The exact result we want to prove here is given by

**Theorem 6.1.1** Let be given an initial datum \( f_0 \) satisfying (6.2) and a Maxwellian collision cross section \( B \) as in (6.3), with \( b \) satisfying (6.4). Let \( f \) be any weak non negative solution of Boltzmann homogeneous equation (6.1), satisfying (6.5) and the mass conservation. Then, for any \( t > 0 \), for all \( s \in \mathbb{R}^+ \), \( f(t,.) \) belongs to the Sobolev space \( H^s(\mathbb{R}^n) \).

6.2 Proof of Theorem 6.1.1

Hereafter, letter \( C \) denotes any positive constant not depending on the integer \( k \) that will appear in the proof of our result below. If we need to mention its dependence with respect to any parameter \( \beta \), then we shall use the notation \( C(\beta) \) or \( C_\beta \).

For the background on Littlewood-Paley theory and Sobolev spaces, and associated notations used herein, we refer to the final chapter.

Applying Fourier transform on Boltzmann equation (6.1), we obtain the equation (due to Bobylev)

\[
\partial_t \hat{f} = Q(f,f) = \int_{S^{n-1}} [\hat{f}(\xi^+) \hat{f}(\xi^-) - \hat{f}(\xi) \hat{f}(0)] b\left(\frac{|\xi|}{|\xi^+|^\frac{1}{2}}, \sigma\right) d\sigma,
\]

where

\[
\xi^+ = \frac{\xi}{2} + \frac{|\xi|\sigma}{2} \quad \text{and} \quad \xi^- = \frac{\xi}{2} - \frac{|\xi|\sigma}{2}.
\]

Note that this equality holds true in view of our assumptions (6.5) on \( f \).

Next, let \( k \) be an integer such that \( k \geq 3 \). The reason for choosing this minimal value of \( k \) will become clear below.
Then, it follows from (6.7) that

\[
\begin{align*}
\partial_t ||p_k f||_{L^2}^2 &= \int_{\mathbb{R}^n} Q(f, p_k f) p_k f \, dv \\
+2Re \int_{\mathbb{R}^n} \int_{S^{n-1}} \overline{p_k f} \hat{f}(\xi)^{-1} \hat{f}(\xi^-)(\xi) \frac{\psi(\xi/2\sigma)}{\sigma} d\sigma d\xi.
\end{align*}
\]

(6.8)

Using the change of variables \((v, v_\ast, \sigma) \mapsto (v', v'_\ast, -\sigma)\), the first term on the right hand side of (6.8) becomes

\[
\int_{\mathbb{R}^n} Q(f, p_k f) p_k f \, dv = \int_{\mathbb{R}^n \times \mathbb{R}^n \times S^{n-1}} f_\ast p_k f [(p_k f)' - p_k f] b\left(\frac{v - v_\ast}{|v - v_\ast|} \right) dv_\ast d\sigma dv
\]

Using the simple equality

\[
p_k f [(p_k f)' - p_k f] = \frac{1}{2} [(p_k f)^2 - p_k f^2] - \frac{1}{2} [(p_k f)' - p_k f]^2,
\]

(6.8) finally becomes

\[
\begin{align*}
\partial_t ||p_k f||_{L^2}^2 + \int_{\mathbb{R}^n \times \mathbb{R}^n \times S^{n-1}} f_\ast [(p_k f)' - p_k f]^2 b\left(\frac{v - v_\ast}{|v - v_\ast|} \right) dv_\ast d\sigma dv
= 2Re \int_{\mathbb{R}^n} \int_{S^{n-1}} \overline{p_k f} \hat{f}(\xi)^{-1} \hat{f}(\xi^-)(\xi) \frac{\psi(\xi/2\sigma)}{\sigma} d\sigma d\xi \\
+ \int_{\mathbb{R}^n \times \mathbb{R}^n \times S^{n-1}} f_\ast [(p_k f)^2 - p_k f^2] b\left(\frac{v - v_\ast}{|v - v_\ast|} \right) dv_\ast d\sigma dv.
\end{align*}
\]

(6.9)

For the second term on the l.h.s. of (6.9), we use the results from Chapter 5, and get

\[
\int \int \int f_\ast [(p_k f)^2 - p_k f^2] b\left(\frac{v - v_\ast}{|v - v_\ast|} \right) dv_\ast d\sigma dv \geq C \int_{\mathbb{R}^n} |p_k f|^2|\xi|^{\nu}\, d\xi \geq C2^{2\nu}||p_k f||_{L^2}^2.
\]

Concerning the second term on the r.h.s. of (6.9), we use this time Chapter 3 (exchanging \(v\) into \(v_\ast\)), to get

\[
\int \int \int f_\ast [(p_k f)^2 - p_k f^2] b\left(\frac{v - v_\ast}{|v - v_\ast|} \right) dv_\ast d\sigma dv \leq C\theta ||p_k f||_{L^2}^2
\]

Finally, there remains to estimate the first term on the r.h.s. of (6.9).

This integral is nothing else than

\[
2Re \int_{\mathbb{R}^n} \int_{S^{n-1}} \overline{p_k f} \hat{f}(\xi)^{-1} \hat{f}(\xi^-) A_\xi b\left(\frac{\xi}{|\xi|} \right) d\sigma d\xi,
\]

(6.10)

where

\[
A_\xi = \psi\left(\frac{\xi}{2\pi}\right) - \psi\left(\frac{\xi^+}{2\pi}\right).
\]

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Thus (6.10) is bounded from above by

\[
2 \hat{f}_0(0) \int_{\mathbb{R}^n} \int_{S^{n-1}} |\hat{p}_k f(\xi)| |\hat{f}(\xi^+)| |A^k_\xi| b\left(\frac{\xi}{|\xi|}\right) |\sigma| d\sigma d\xi.
\]

Let us notice that for the values of the variable \(\xi\), i.e. \(2^{k-1} \leq |\xi| \leq 2^{k+1}\), appearing in this integral, it follows that \(2^{k-2} \leq |\xi^+| \leq 2^{k+1}\), see Appendix A if necessary. Thus

\[
\hat{f}(\xi^+) = \hat{p}_{k-2} f(\xi^+) + \hat{p}_{k-1} f(\xi^+) + \hat{p}_k f(\xi^+) + \hat{p}_{k+1} f(\xi^+) = \hat{p}_k f(\xi^+),
\]

for another Littlewood Paley decomposition \(\tilde{p}_k\).

On the other hand, we can also easily estimate \(|A^k_\xi|\) as follows

\[
|A^k_\xi| \leq |\psi\left(\frac{\xi}{2^k}\right) - \psi\left(\frac{\xi^+}{2^k}\right)| = |\psi\left(\frac{\xi}{2^k}\right) - \psi\left(\frac{\xi^+}{2^k}\right)|
\]

\[
\leq C \frac{\xi^+ - |\xi^+|}{2^k} \frac{1}{|\xi|}
\]

\[
\leq C \frac{|\xi|^2}{|\xi|^2 - |\xi^+|^2}.
\]

Since \(|\xi^-| = |\xi| \sin \frac{\theta}{2}\), we have

\[
|A^k_\xi| \leq C \sin^2 \frac{\theta}{2}
\]

Thus, we are left to estimate from above

\[
H_k \equiv 2 \hat{f}_0(0) \int_{2^{k-1} \leq |\xi| \leq 2^{k+1}} \int_{S^{n-1}} |\hat{p}_k f(\xi)| |\hat{p}_k f(\xi^+)| |A^k_\xi| b\left(\frac{\xi}{|\xi|}\right) |\sigma| d\sigma d\xi,
\]

that we bound as

\[
H_k \leq M_k + N_k,
\]

where

\[
M_k \equiv \hat{f}_0(0) \int_{2^{k-1} \leq |\xi| \leq 2^{k+1}} \int_{S^{n-1}} |\hat{p}_k f(\xi)| |A^k_\xi| b\left(\frac{\xi}{|\xi|}\right) |\sigma| d\sigma d\xi
\]

and

\[
N_k \equiv \hat{f}_0(0) \int_{2^{k-1} \leq |\xi| \leq 2^{k+1}} \int_{S^{n-1}} |\hat{p}_k f(\xi^+)| |A^k_\xi| b\left(\frac{\xi}{|\xi|}\right) |\sigma| d\sigma d\xi.
\]

Clearly, in view of estimate (6.12), one has

\[
M_k \leq C \hat{f}_0(0) \|p_k(f)\|_{L^2}^2.
\]

As for (6.15), using (6.12) and the change of variable \(\xi^+ \rightarrow \xi\) (see for more precisions Chapter 5), we get

\[
N_k \leq C \hat{f}_0(0) \|\tilde{p}_k(f)\|_{L^2}^2.
\]
All in all, we have obtained

\[ H_k \leq C \hat{f}_0(0) \{ ||p_k(f)||^2_{L^2} + ||\tilde{p}_k(f)||^2_{L^2} \}. \]

Gluing all the above estimates, we find that

\[ \partial_t ||p_k f||^2_{L^2} + C(f_0) \int_{\mathbb{R}^n} |\tilde{p}_k f|^2 |\xi|^2 d\xi \leq C \hat{f}_0(0) \{ ||p_k f||^2_{L^2} + ||\tilde{p}_k(f)||^2_{L^2} \}, \]

and using the fact that \( 2^{k-1} \leq |\xi| \leq 2^{k+1} \) in the integral of the second term in the left hand side, we obtain (for another constant still denoted by \( C(f_0) \)), absorbing an unimportant term on the right hand side, for \( k \) big enough

(6.16) \[ \partial_t ||p_k f||^2_{L^2} + C(f_0) 2^{k\nu} ||p_k f||^2_{L^2} \leq C ||\tilde{p}_k(f)||^2_{L^2}. \]

From Bernstein’s inequality (also called Polya Plancherel Nikolski inequality), we first note that one has

\[ ||p_k f||^2_{L^2} \leq C(\tilde{\psi}) 2^n ||p_k f||^2_{L^2}. \]

Since \( ||p_k f||^2_{L^2} \leq C(\tilde{\psi}) ||f||^2_{L^1} = C(\tilde{\psi}) ||f_0||^2_{L^1}, \) clearly for all \( t \geq 0, \)

\[ ||p_k f||^2_{L^2} \lesssim 2^{nk}. \]

Then, using (6.16), one obtains that for \( t > t_0 > 0, \)

\[ ||p_k f||^2_{L^2} \lesssim \frac{2^{nk}}{2^{tk}}, \]

and by iterating also on the time interval, for all \( s \geq 0, \) for all \( t > 0, \)

\[ ||p_k f||^2_{L^2} \lesssim \frac{2^{nk}}{2^{sk}}, \]

which ends the proof.

### 6.3 Some basic facts from Littlewood-Paley theory

With the notations of Littlewood-Paley theory, we have

i) If \( \xi \in \text{supp } \psi_k \) then \( \psi_{k-1}(\xi) + \psi_k(\xi) + \psi_{k+1}(\xi) = 1 \) for all \( k \geq 1, \) and also for \( k = 0 \) if we agree that \( \psi_{-1} \equiv 0. \)

ii) If \( \xi \in \text{supp } \psi_k \) then \( 2^{k-2} \leq |\xi^+| \leq 2^{k+1}. \)

**Proof:**

i) This point is well known, and follows easily by noticing that if \( \xi \in \text{supp } \psi_k, \) then \( \xi \notin \text{supp } \psi_j, \) for either \( j \leq k - 2 \) or \( j \geq k + 2. \)

ii) Let us assume that \( \xi \in \text{supp } \psi_k \) i.e \( 2^{k-1} \leq |\xi| \leq 2^{k+1}. \)

Notice that the modulus of \( \xi^+ \) writes \( |\xi^+| = \frac{|\xi|}{\sqrt{2}}(\sqrt{1 + \cos \theta}) \) with

\[ \cos \theta = \frac{\xi}{|\xi|}, \sigma >, 0 \leq \cos \theta \leq 1. \]

Since \( \frac{1}{\sqrt{2}} \leq \frac{\sqrt{1 + \cos \theta}}{\sqrt{2}} \leq 1, \) it follows that \( 2^{k-1} \leq |\xi^+| \leq 2^{k+1}, \) which ends the proof. \(\square\)
Chapter 7

Functional estimates for the collision operator

7.1 Introduction

References for this Chapter are:
- our work [4];
- all references on Harmonic analysis, and also last Chapter 21, where for convenience of the reader, we have given the most important facts concerning Littlewood-Paley decompositions.

In this chapter, we give more precise functional properties on Boltzmann collision operator $Q$,

$$Q(f, f)(v) = \int_{\mathbb{R}^n} dv_* \int_{S^{n-1}} d\sigma B(v - v_*, \sigma)(f'f'_* - ff_*).$$

Let us recall the typical case given by inverse $s$-power repulsive forces in dimension $n = 3$,

$$B(v - v_*, \sigma) = |v - v_*|^{\gamma_s} b(\vec{k} \cdot \sigma) \text{ with } \gamma_s = \frac{s - 5}{s - 1},$$

$$\sin \theta b(\cos \theta) \sim K \theta^{1 - \nu_s} \text{ as } \theta \to 0 \text{ with } \nu_s = \frac{2}{s - 1} > 0, \quad K > 0.$$ 

Above, $s > 2$, so that $0 < \nu_s < 2$ and $-3 < \gamma_s < 1$. Note also that since $\gamma_s = 1 - 2\nu_s$, it follows that $\gamma_s + \nu_s = 1 - \nu_s < 1$, if $\nu_s < 1$.

The cases $\gamma_s > 0$, $\gamma_s = 0$ and $\gamma_s < 0$ correspond to so-called hard, maxwellian and soft potentials respectively.
One main purpose is to give as far as possible precise functional properties of $Q(f)$ with respect to functional properties of $f$ itself, in case of cross-sections having this type of singularities.

Along with operator $Q(f)$, we shall of course associate the following bilinear operator

$$(7.4) \quad Q(g, f)(v) = \int_{\mathbb{R}^n} dv_s \int_{S^{n-1}} d\sigma B(v - v_s, \sigma)(g'_s f' - g_s f).$$

In order to fix ideas, in all the paper, we shall make the following assumptions on $B$.

- **Assumption 1:**

$$(7.5) \quad B(v - v_s, \sigma) = \Phi(|v - v_s|) b(\cos \theta), \quad \sin^{n-2} \theta b(\cos \theta) \sim K \theta^{-1-\nu},$$

$\nu > 0, K > 0$, where the kinetic part $\Phi$ at least satisfies

$$(7.6) \quad \Phi(|z|) \leq C |z|^\gamma \quad \text{for} \quad |z| \leq C1 \quad \text{and} \quad \Phi(|z|) \leq C |z|^\beta \quad \text{for} \quad |z| \geq C1,$$

with the following range of parameters

$$(7.7) \quad \gamma > -n, \quad \beta \in \mathbb{R}, \quad 0 < \nu < 2.$$

- **Assumption 2:**

$$(7.8) \quad 0 < \nu < 1$$

and

$$(7.9) \quad \beta + \nu < 1.$$

**Remarks 7.1.1**

1. The $s$-power interaction, (7.2) and (7.3), enters this framework, by taking $\gamma = \beta = \gamma_s$ and assuming $\nu_s < 1$.

2. Assumption (7.8) is not a serious restriction, but enables us to use Taylor’s formula only at order 1. The second part of our work is devoted to the case $1 \leq \nu < 2$, where we shall use Taylor’s formula at order 2.

3. As for assumption (7.9), I do not know if it’s linked with the method used below, but there is some evidence in view of many works on singular integral operators that it should be imposed somehow. However, in view of the proofs below, this assumption can certainly be relaxed, at the expense of adding suitable weights on functions $f$ and $g$, entering operator (7.4).
• Notations: Herein, letter $C$ stands for any constant taking possible different values at different places. We also use weighted Lebesgue spaces $L^p_k$, $p \geq 1$, $k \in \mathbb{R}$; this is the set of functions which are $L^p$ integrable with respect to the measure $<v>^k dv$, where $<v> = (1 + |v|^2)^{1/2}$, with the usual meaning when $p = +\infty$.

Three of our main results are given by the following Theorems.

The first result uses a very particular scale of Besov spaces and is relatively the most easiest one to prove

**Theorem 7.1.1** Let Assumptions 1 and 2 hold true. Then

1. If $\beta + \nu \leq 0$ and $\gamma + \nu \geq 0$, one has

$$\|Q(f,g)\|_{B^{s,-\nu}_{1,\infty}} \leq C\|g\|_{L^1} \|f\|_{L^1}.$$ 

2. If $\beta + \nu \geq 0$ and $\gamma + \nu \geq 0$, one has, for all $\beta' \geq \beta + \nu$,

$$\|Q(f,g)\|_{B^{s,-\nu}_{1,\infty}} \leq C\|g\|_{L^1_{\beta'}} \|f\|_{L^1_{\beta'}}.$$ 

\[\Box\]

**Remarks 7.1.2**

1. In Theorem 7.1.1, other cases can also be dealt with, by using for instance Hardy-Sobolev inequality (i.e. Riesz potentials) or related arguments.

2. Using the fact that $Q$ is translation invariant, one can also deduce estimates for $\|Q(g,f)\|_{B^{s,-\nu}_{1,\infty}}$ but assuming $s \geq 0$.

\[\Box\]

Our second result uses the traditional scale of Sobolev spaces and more generally of (weighted) Besov $B^{s}_{p,q}$, but covers the specific Maxwellian case, that is the case when $\Phi$ is constant valued. This is a special case of the above assumptions, when $\gamma = \beta = 0$.

**Theorem 7.1.2** (Maxwellian case) Assume that $\Phi = 1$. Then $Q(g,f) = Q_{\text{max}}(g,f)$ satisfies the following estimates, for all $s \in \mathbb{R}$, all $1 \leq p, q < +\infty$

$$\|Q_{\text{max}}(g,f)\|_{B^{s,-\nu}_{p,q}} \leq C\|g\|_{W^{s,1}_{q}} \|f\|_{B^{s}_{p,q,\nu}}.$$ 

\[\Box\]

**Remarks 7.1.3**

1. The additional subscript $\nu$ refers to the fact that we are taking $L^p_\nu$ norm, instead of the usual $L^p$ norm, in the definition of Besov spaces.
2. The above estimate on \( Q_{\text{max}}(g, f) \) involves non weighted Besov spaces. It is likely optimal with respect to the weights on \( f \) and \( g \). If we add weights on \( f \) and \( g \), then one has the corresponding weighted Besov norm for \( Q_{\text{max}}(g, f) \). This point follows from the proofs.

With this special Maxwellian case, our last main result considers smoothed versions of kernels close to \(|v|^{\gamma}\), where we simplify the framework by taking \( \beta = \gamma \).

**Theorem 7.1.3** Assume that \( \Phi(v) = cw^{\gamma} \) for some \( \gamma < 1 \) and some positive constant \( c \). Then, for all \( s \in \mathbb{R}, 1 \leq p, q \leq +\infty \), one has

\[
\|Q(g,f)\|_{B_{s,-\nu}^{p,q}} \leq C\{\|g\|_{W^{s,1}} + \|g\|_{W^{s,1}} \gamma} \} \{\|f\|_{B_{s,p,q,+\gamma}^{s}} + \|f\|_{W^{s,p}} + \|f\|_{W^{s,p}}\}.
\]

**Remarks 7.1.4**

1. Constraint \( \gamma < 1 \) comes from assumption (7.9).

2. For the difficult singular case \( \Phi(|v|) = c|v|^\gamma \), we refer to Section 5 for more details.

3. Again, weighted Besov norms on \( Q(g,f) \) are also available.

In order to prove these results, the analysis of the bilinear Boltzmann operator (7.4) will be based on a detailed study of the following linear operator, for a given \( v_*, \)

\[
(7.10) \quad T_{v_*}^\Phi : f \mapsto \int_{S^{n-1}} B(v - v_*, \sigma)(f' - f) \, d\sigma,
\]

which has an important influence on Boltzmann equation’s theory, as it enters the weak (dual) formulations of Boltzmann operator (7.4).

In the cutoff case, the mathematical study of this operator was initiated by Lions [81], simplified by Wennberg [123] and substantially improved by Mouhot and Villani [97]. In the non cut-off case, we have stated in a Chapter 2 that \( T_{v_*}^\Phi \) is essentially bounded as an operator from \( W^{2,\infty} \) to \( L^{\infty} \) (proof will be given in Chapter-ren-titre).

Here, we wish to go much more further by lowering the exponent 2 of regularity required to give a sense to \( T_{v_*}^\Phi \). That this exponent was not the right one was already apparent if one takes into account the works around coercivity estimates see Chapter 5, and it was already clear for some linear cases. In particular, we shall show that the right exponent is given by \( \nu \) as shown by Theorems 7.1.1 and 7.1.3.

Another important motivation for getting much more informations on \( T_{v_*}^\Phi \) came from actual study around the construction of small solutions and regularity questions in the non maxwellian case.

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Remarks 7.1.5 Note that $T^\Phi_{v_*} = \Phi(v - v_*)T^\Phi_{v_*}$. One may wonder if it is not enough to study only $T^\Phi_{v_*}$, and then deduce functional properties on $T^\Phi_{v_*}$, since this one only differs by a multiplication, but we want to see exactly if we can have other cancellation properties by including $\Phi$ from the very beginning. Also, and this is maybe the most important point, let us again recall that in general, the kinetic part of the cross-section cannot be separated from the angular one.

It is by now standard that the analysis of $T^\Phi_{v_*}$ can be further simplified if one notices that $T^\Phi_{v_*} = \tau_{-v_*} \circ T^\Phi \circ \tau_{v_*}$, where $\tau_{v_*}$ denotes the usual translation operator along vector $v_*$ and $T^\Phi$ denotes the operator

$$T^\Phi f(v) \equiv \int_{S_n} B(v, \sigma) \{ f(v^+) - f(v) \} d\sigma, \quad v^+ \equiv \frac{v + |v| \sigma}{2}.$$  

It is therefore enough, at least in translation invariant functional spaces, to study this last operator. For this purpose, we have chosen to base our approach on Littlewood Paley decomposition.

Let $k \in \mathbb{N}$ be fixed. In view of Littlewood Paley decompositions, we are led to consider the following operator, defined, for smooth functions $f = f(v), v \in \mathbb{R}^n$, by

$$U^{\Phi,k} f(v) \equiv \int_{S_{n-1}} \{ (p_k f)^+ - (p_k f) \} b\left( \frac{v}{|v|} \cdot \sigma \right) \Phi(|v|) d\sigma$$

and thus, formally

$$T^\Phi f(v) = \sum_{k=0}^{+\infty} U^{\Phi,k} f(v).$$

Setting $p_k f = \phi_k * f = g$, we note first that we can write $g = \tilde{\phi}_k * g$, where $\tilde{\phi}_k(\cdot) = 2^{nk} \tilde{\phi}(2^k \cdot), \tilde{\phi} \in \mathcal{S}$ and $\tilde{\phi}$ compactly supported.

If we define the operator, at least for any smooth function $g$,

$$T^{\Phi,k} g(v) = \int_{S_{n-1}} \{ (\tilde{\phi}_k * g)^+(v) - (\tilde{\phi}_k * g)(v) \} b\left( \frac{v}{|v|} \cdot \sigma \right) \Phi(|v|) d\sigma$$

$$= \int_{\mathbb{R}^n} g(y) \int_{S_{n-1}} \{ \tilde{\phi}_k(v^+ - y) - \tilde{\phi}_k(v - y) \} b\left( \frac{v}{|v|} \cdot \sigma \right) \Phi(|v|) d\sigma dy,$$

then the Schwartz kernel of $T^{\Phi,k}$ is given by

$$K^{\Phi,k}(v, y) \equiv \int_{S_{n-1}} \{ \tilde{\phi}_k(v^+ - y) - \tilde{\phi}_k(v - y) \} b\left( \frac{v}{|v|} \cdot \sigma \right) \Phi(|v|) d\sigma.$$
The above operators are related by the (formal) rules

\[ U^{\Phi,k}f(v) = T^{\Phi,k}(p_k f)(v) \]

and

\( T^\Phi f(v) = \sum_{k=0}^{+\infty} T^{\Phi,k}(p_k f)(v). \)

We can therefore reduce our work to a detailed study of the operator \( T^{\Phi,k} \), starting with \( L^p \) type spaces, for all \( k \in \mathbb{N}_0 \).

**Proposition 7.1.1** Under the above notations and Assumptions 1 and 2, one has the following estimates:

1. If \( \beta + \nu \leq 0 \) and \( \gamma + \nu \geq 0 \), then
   \[
   \int_{\mathbb{R}^n} |K^{\Phi,k}(v,y)| \, dy \leq C 2^{k\nu} \quad \text{and} \quad \int_{\mathbb{R}^n} |K^{\Phi,k}(v,y)| \, dv \leq C 2^{k\nu}.
   \]

2. If \( \beta + \nu \geq 0 \) and \( \gamma + \nu \geq 0 \), then
   \[
   \int_{\mathbb{R}^n} |K^{\Phi,k}(v,y)| \, dy \leq C 2^{k\nu} \left\{ 1 + |v|^{|\beta+\nu|} \right\} \quad \text{and} \quad \int_{\mathbb{R}^n} |K^{\Phi,k}(v,y)| \, dv \leq C 2^{k\nu} \left\{ 1 + |y|^{|\beta+\nu|} \right\}.
   \]

3. If \( \beta + \nu \leq 0 \) and \( \gamma + \nu \leq 0 \), then
   \[
   \int_{\mathbb{R}^n} |K^{\Phi,k}(v,y)| \, dy \leq C 2^{k\nu} \left\{ 1 + |v|^{|\gamma+\nu|} \right\} \quad \text{and} \quad \int_{\mathbb{R}^n} |K^{\Phi,k}(v,y)| \, dv \leq C \sup \{ 2^{k\nu}, 2^{(-\gamma)k} \}.
   \]

4. If \( \beta + \nu \geq 0 \) and \( \gamma + \nu \leq 0 \), then
   \[
   \int_{\mathbb{R}^n} |K^{\Phi,k}(v,y)| \, dy \leq C 2^{k\nu} \left\{ 1 + |v|^{|\gamma+\nu|} \right\} \quad \text{and} \quad \int_{\mathbb{R}^n} |K^{\Phi,k}(v,y)| \, dv \leq C \sup \{ 2^{k\nu}, 2^{(-\gamma)k} \} \left\{ 1 + |y|^{|\beta+\nu|} \right\}.
   \]

**Remarks 7.1.6**

1. The proof of this result is elementary. The main cases are the two first ones.

2. Constants \( C \) appearing in the above result, depend linearly on \( \sup_{|v| \leq C_1} \frac{|\Phi(v)|}{|v|^{\lambda}} + \sup_{|v| \geq C_1} \frac{|\Phi(v)|}{|v|^{\lambda}} \).

3. At least in the generalized maxwellian case, \( \beta = \gamma = 0 \), weighted counterparts with respect to any positive power of \( <v> \) are also true. This point follows from the corresponding proofs.

\( \square \)
It is not hard then to deduce some functional properties on $T^\Phi$ by using interpolation estimates. Here is one simple example

**Corollary 7.1.1** Under case 1) of Proposition 7.1.1, the linear operator $T^\Phi$ defined by (7.11) satisfies, for all $1 \leq p \leq +\infty$

$$T^\Phi : B_{p,1}^\omega(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n),$$

with the modification $B_{p,1}^\omega(\mathbb{R}^n)$ to $b_{\infty,1}^\omega(\mathbb{R}^n)$ in case $p = +\infty$.

Proposition 7.1.1 along with Corollary 7.1.1 will give our first result Theorem 7.1.1.

For the Maxwellian operator $Q_{\text{max}}(g,f)$, that is the case when $\Phi = 1$, corresponding to a sub-case of $\gamma = \beta = 0$, a special property, on the Fourier side, will give Theorem 7.1.2, in combination with Proposition 7.1.1.

In order to get Theorem 7.1.3, we need to study commutators linked with $T^{\Phi,k}$. We have been unable to study it directly, essentially because known studies around the gain term in the cutoff case are not sufficient.

The idea is then to get back to $T^\Phi$ and write

\begin{equation}
T^\Phi \phi(v) = T^\Phi_\Delta \phi(v) + T^1(\Phi\phi)(v),
\end{equation}

where

\begin{equation}
T^\Phi_\Delta \phi(v) = \int_{S^{n-1}} \{\Phi(|v|) - \Phi(|v^+|)\} \phi(v^+) b(v^+ / |v| \cdot \sigma) d\sigma,
\end{equation}

and $T^1$ corresponds to the maxwellian case $\Phi = 1$.

It follows immediately that we have the decomposition

\begin{equation}
< Q(g,f); \phi > = < Q_\Delta(g,f); \phi > + < Q^\Delta(g,f); \phi > .
\end{equation}

Above, $Q_\Delta(g,f)$ is defined by duality as

\begin{equation}
< Q_\Delta(g,f); \phi > = \int_{\mathbb{R}^n} dv_s g(v_s) \int_{\mathbb{R}^n} dv f(v) \tau_{v_s} \circ T^\Phi_\Delta \circ \tau_{v_s} \phi,
\end{equation}

while $Q^\Delta(g,f)$ is defined by duality as

\begin{equation}
< Q^\Delta(g,f); \phi > = \int_{\mathbb{R}^n} dv_s g(v_s) \int_{\mathbb{R}^n} dv f(v) \tau_{v_s} \circ T^1 \circ (\Phi \tau_{v_s} \phi).
\end{equation}

Of course, in the Maxwellian case, $Q_\Delta(g,f) = 0$ and $Q^\Delta(g,f) = Q_{\text{max}}(g,f)$. 

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We note that operator $Q_\Delta$ defined by (7.16) is similar to the usual gain term, in the cutoff case, for instance as studied by Mouhot and Villani [97]. This point, combined with the maxwellian case given by Theorem 7.1.2, and a commutator estimate, is enough to prove Theorem 7.1.3.

We wish now to show that, eventually, under some decay assumptions on $\Phi$, the kernel $K^{\Phi,k}$ actually satisfies estimates similar to a standard Calderon-Zygmund kernel. Though the result seems plausible, we have been unable to prove it.

However, quite surprisingly, it holds true for the singular part of this kernel, defined as

\[(7.18) \quad K^{\Phi,k}_{\text{sing}}(v,y) \equiv \int_{S_{n-1,\sin \sigma} \leq \frac{\sqrt{2}}{2}} \left\{ \tilde{\phi}_k(v^+ - y) - \tilde{\phi}_k(v - y) \right\} b \left( \frac{v}{|v|} \cdot \sigma \right) \Phi(|v|) d\sigma.\]

On the other hand, the fact that we have been able to deal with this singular part can be understood if we have in mind grazing limit, that is the fact that Boltzmann operator is close to Landau operator for small deflection angles.

**Theorem 7.1.4**

1. First estimate: Assume that $(1 + |v|)^{n+\nu} \Phi \in L^\infty$. Then

\[|K^{\Phi,k}_{\text{sing}}(v,y)| \leq C 2^{k\nu} \frac{1}{|v - y|^n}.\]

2. Second estimate: Assume that $|v\Phi(|v|)$ is bounded (for all $|v|$) and that $|v|^{n+1+\nu} \Phi(|v|)$ is bounded for large $|v|$. Furthermore assume that $|v|\nabla \Phi$ is bounded for all $|v|$ and that $|v|^{n+1+\nu} \nabla \Phi$ is bounded for large $|v|$. Then

\[|\nabla_v K^{\Phi,k}_{\text{sing}}(v,y)| + |\nabla_y K^{\Phi,k}_{\text{sing}}(v,y)| \leq C 2^{k\nu} \frac{1}{|v - y|^{n+1}}.\]

\[\square\]

In Theorem 7.1.4, we do not impose Assumptions 1 and 2 on $\Phi$. Of course, assumptions on $\nu$ are in force.

We have been unable to prove similar estimates for the non singular part of the kernel. This point is linked with the fact that studies on gain term operator $Q^+$ in the cutoff case are not sufficient to conclude.

The above results, and similar ones for further derivatives, can be used to conclude for commutators results. Since the non singular part is lacking, we shall only explain the idea at the end of the paper.
7.2 $L^1$ norm of the kernel with respect to variable $y$

This section is devoted to the proof of the following Proposition, which will give one part of Proposition 7.1.1,

**Proposition 7.2.1** One has, for all functions $\Phi$

$$\|K_{\Phi,k}(v,\cdot)\|_{L^1} \leq C 2^{k\nu} |v|^{\nu} \Phi(|v|),$$

and under the more precise Assumptions 1 and 2 on $\Phi$, one has

$$\|K_{\Phi,k}(v,\cdot)\|_{L^1} \leq C 2^{k\nu} \{ |v|^{\gamma + \nu} \mathbb{I}_{|v| \leq C_1} + |v|^{\beta + \nu} \mathbb{I}_{|v| \geq C_1} \}.$$

□

**Proof of Proposition 7.2.1.**

Recalling that

$$K_{\Phi,k}(v,y) \equiv \int_{S^{n-1}} \{ \tilde{\phi}_k(v^+ - y) - \tilde{\phi}_k(v - y) \} b(\frac{v}{|v|} \cdot \sigma) \Phi(|v|) d\sigma,$$

it follows that

$$\|K_{\Phi,k}(v,\cdot)\|_{L^1} \leq \int_{S^{n-1}} \| \tilde{\phi}_k(v^+ - \cdot) - \tilde{\phi}_k(v - \cdot) \|_{L^1} b(\frac{v}{|v|} \cdot \sigma) \Phi(|v|) d\sigma.$$

A standard computation shows that

$$\| \tilde{\phi}_k(v^+ - \cdot) - \tilde{\phi}_k(v - \cdot) \|_{L^1} =$$

$$= \int_{\mathbb{R}^n} |\tilde{\phi}_k(v^+ - v + z) - \tilde{\phi}_k(z)| \, dz = \int_{\mathbb{R}^n} |\tilde{\phi}(2^k h + z) - \tilde{\phi}(z)| \, dz,$$

where $h = v^+ - v$, which from Taylor’s formula at order one, is less than $|2^k h| C(\tilde{\phi})$. Clearly, it is also less that $2\|\tilde{\phi}\| \leq C(\tilde{\phi})$.

It follows finally that

$$\| \tilde{\phi}_k(v^+ - \cdot) - \tilde{\phi}_k(v - \cdot) \|_{L^1} \leq C(\tilde{\phi}) \min \left\{ 1, 2^k |h| \right\} \text{ with } h = v^+ - v,$$

and thus, recalling that $|v^+ - v| = |v| \sin \frac{\theta}{2}$, we have

$$\| \tilde{\phi}_k(v^+ - \cdot) - \tilde{\phi}_k(v - \cdot) \|_{L^1} \leq C(\tilde{\phi}) \min \left\{ 2^k |v| \sin \frac{\theta}{2}, 1 \right\}.$$

All in all, we have obtained

$$\|K_{\Phi,k}(v,\cdot)\|_{L^1} \leq c_0(\tilde{\phi}) \int_{S^{n-1}} \min \left\{ 2^k |v| \sin \frac{\theta}{2}, 1 \right\} b(\cdot) \Phi(v) d\sigma.$$
Next, a rough estimate shows that
\[
\int_{S^{n-1}} \min\{2^k |v| \sin \frac{\theta}{2}, 1\} b(\frac{v}{|v|} \cdot \sigma) \Phi(\sigma) d\sigma \simeq C \Phi(|v|) \int_0^{\frac{\pi}{2}} \frac{1}{\theta^{1+\nu}} \min\{2^k |v| \sin \frac{\theta}{2}, 1\} d\theta
\]
\[
\simeq \Phi(|v|) \left\{ C 2^k |v| \int_0^{\frac{\pi}{2}} \mathbb{I}_{|v| \sin \frac{\theta}{2} \leq \frac{1}{\theta^{1+\nu}}} \frac{1}{\theta^{1+\nu}} d\theta + C \int_0^{\frac{\pi}{2}} \mathbb{I}_{|v| \sin \frac{\theta}{2} \geq 1} \frac{1}{\theta^{1+\nu}} d\theta \right\}
\]
\[
\simeq \Phi(|v|) \left\{ C 2^k |v| \int_0^{\frac{\pi}{2}} \theta^{-\nu} d\theta + C \int_0^{\frac{\pi}{2}} \theta^{1-\nu} d\theta \right\}
\]
\[
\simeq \Phi(|v|) \left\{ C 2^k |v| \left( \frac{1}{2^k |v|} \right)^{-\nu+1} - C + C \left( \frac{1}{2^k |v|} \right)^{-\nu} \right\} \leq C 2^{k\nu} |v|^{\nu} \Phi(|v|),
\]
ending the proof of Proposition 7.2.1.

\[\square\]

**Remarks 7.2.1**

1. A much more careful estimate shows that one has also the following bound on the $L^1$ norm of $K_{\Phi,k}$ with respect to $y$ variables
\[
\|K_{\Phi,k}(v,)\|_{L^1} \leq C(\tilde{\phi}) 2^{k\nu} \Phi(|v|) |v|^{\nu} \min\left\{ 2^{k(1-\nu)} |v|^{1-\nu}, 1 \right\}.
\]

2. The kernel $K_{\Phi,k}$ is actually a function of $v$ and of $v - y$, since letting $\tilde{\psi}_k = \tilde{q}_k$ it follows that, using Fourier type arguments
\[
K_{\Phi,k}(v,y) = \int_{\mathbb{R}^n} \int_{S^{n-1}} \tilde{\psi}_k(\xi) e^{i\xi \cdot (v-y)} \left\{ e^{-i(v \cdot \xi)} - 1 \right\} b(\frac{\xi}{|\xi|} \cdot \sigma) \Phi(|v|),
\]
where $\xi^- = \frac{1}{2}(\xi - |\xi| \sigma)$.

\[\square\]

### 7.3 $L^1$ norm of the kernel with respect to variable $v$

Let $\delta > 0$ be fixed, that will be chosen carefully below.

We start from the splitting
\[
(7.19) \quad K_{\Phi,k}(v,y) \equiv \int_{S^{n-1}} \{ \tilde{\phi}_k(v^+ - y) - \tilde{\phi}_k(v - y) \} b(\frac{v}{|v|} \cdot \sigma) \Phi(\sigma) d\sigma
\]
\[
= K_{\Phi,k}^{\delta,1}(v,y) + K_{\Phi,k}^{\delta,2}(v,y),
\]
where, by definition, we have settled
\[
(7.20) \quad K_{\Phi,k}^{\delta,1}(v,y) \equiv \int_{S^{n-1}} \{ \tilde{\phi}_k(v^+ - y) - \tilde{\phi}_k(v - y) \} b(\frac{v}{|v|} \cdot \sigma) \Phi(\sigma) \mathbb{I}_{|v| \sin \frac{\theta}{2} \geq \delta} d\sigma,
\]
We shall refer to the first kernel \((7.20)\) as the non singular part of the kernel and to the second kernel \((7.21)\) as the singular one.

### 7.3.1 The non singular kernel \(K_{\delta,1}^{\Phi,k}\), see formula \((7.20)\)

This sub-section is devoted to the proof of the following behavior on the term \(K_{\delta,1}^{\Phi,k}(v,y)\)

**Proposition 7.3.1** For the kernel \(K_{\delta,1}^{\Phi,k}(v,y)\) given by \((7.20)\), one has

\[
\int_{\mathbb{R}^n} |K_{\delta,1}^{\Phi,k}(v,y)| \, dv \leq \frac{C}{\delta^\nu} + I_{\beta+\nu \geq 0} \left\{ \frac{C}{\delta^\nu} \frac{1}{2(\beta+\nu)k} + C |y|^{\beta+\nu} \right\} \]

\[
\int_{\mathbb{R}^n} |K_{\delta,1}^{\Phi,k}(v,y)| \, dv \leq \frac{C}{\delta^\nu} + I_{\beta+\nu \geq 0} \left\{ \frac{C}{\delta^\nu} \frac{1}{2(\beta+\nu)k} + C |y|^{\beta+\nu} \right\} \]

and thus one has

\[
\int_{\mathbb{R}^n} |K_{\delta,1}^{\Phi,k}(v,y)| \, dv \leq I + II,
\]

with

\[
I = \int_{\mathbb{R}^n} dv \int_{S^{n-1}} |\tilde{\phi}_k(v^+ - y)| b\Phi(|v|)I_{|v|\sin \frac{\pi}{2} \geq \delta} \, d\sigma.
\]

and

\[
II = \int_{\mathbb{R}^n} dv \int_{S^{n-1}} |\tilde{\phi}_k| b\Phi(|v|)I_{|v|\sin \frac{\pi}{2} \geq \delta} \, d\sigma.
\]

The proof of Proposition 7.20 is reduced to the following two results

**Lemma 7.3.1** For \(II\) defined in \((7.24)\), we have

\[
II \leq \|C_{\delta} \| \left\{ I_{\gamma+\nu \geq 0} \frac{C}{\delta^\nu} + I_{\gamma+\nu \leq 0} C \delta^\gamma \right\} \\
+ I_{\beta+\nu \leq 0} C \frac{1}{\delta^\nu} \left\{ \max\{1, C\delta\} \right\}^{\beta+\nu} + I_{\beta+\nu > 0} \left\{ \frac{C}{\delta^\nu} \frac{1}{2(\beta+\nu)k} + C \frac{1}{\delta^\nu} |y|^{\beta+\nu} \right\}.
\]

\[
\square
\]
Lemma 7.3.2 For $I$ defined by (7.23), we have
\[
I \leq \frac{C}{\delta^\nu} + \mathbb{I}_{\beta + \nu \leq 0} \frac{C}{\delta^\nu} + \mathbb{I}_{\beta + \nu > 0} \left\{ \frac{C}{\delta^\nu} \frac{1}{2(\beta + \nu)k} + C \left| y \right|^{\beta + \nu} \right\}.
\]

\[\square\]

Proof of Lemma 7.3.1: the term $II$

We note that
\[
II \leq \int_{\mathbb{R}^n} dv \left| \tilde{\phi}_k(v - y) \right| \Phi(\left| v \right|) \int_0^{\frac{\pi}{2}} \theta^{-1-\nu} \theta \geq \frac{\delta}{\left| v \right|} d\theta.
\]

If $\left| v \right| \leq \delta \frac{\nu}{\sqrt{2}}$, then the inner characteristic function takes the value 0. We can therefore assume that $\left| v \right| \geq \delta \frac{2}{\sqrt{2}}$, abbreviated as $\left| v \right| \geq C\delta$.

It follows that, using previous spherical integral computations,
\[
II \leq C \int_{\mathbb{R}^n} dv \left| \tilde{\phi}_k(v - y) \right| \Phi(\left| v \right|) I_{\left| v \right| \geq C\delta} \left| v \right|^{\nu} \frac{\sin \theta}{\theta^{\nu}} dv.
\]

We bound from above this last integral as
\[
(7.25) \quad II \leq II_1 + II_2,
\]

where
\[
(7.26) \quad II_1 = C \int_{\mathbb{R}^n} dv \left| \tilde{\phi}_k(v - y) \right| \Phi(\left| v \right|) I_{\left| v \right| \geq C\delta} \left| v \right|^{\nu} \frac{\sin \theta}{\theta^{\nu}} I_{\left| v \right| \leq 1} dv
\]

and
\[
(7.27) \quad II_2 = C \int_{\mathbb{R}^n} dv \left| \tilde{\phi}_k(v - y) \right| \Phi(\left| v \right|) I_{\left| v \right| \geq C\delta} \left| v \right|^{\nu} \frac{\sin \theta}{\theta^{\nu}} I_{\left| v \right| \geq 1} dv.
\]

For $II_1$, one has
\[
II_1 \leq C \frac{1}{\delta^\nu} \int_{\mathbb{R}^n} \left| \tilde{\phi}_k(v - y) \right| \left| v \right|^{\gamma + \nu} I_{\left| v \right| \geq C\delta} I_{\left| v \right| \leq 1} dv.
\]

If $C \delta \geq 1$, this is zero, so there remains to consider the case $C \delta \leq 1$, for which one has
\[
II_1 \leq C \frac{1}{\delta^\nu} \int_{\mathbb{R}^n} \left| \tilde{\phi}_k(v - y) \right| \left| v \right|^{\gamma + \nu} I_{\left| v \right| \leq 1} dv.
\]

Then, we find that
\[
II_1 \leq C \frac{1}{\delta^\nu} \text{ if } \gamma + \nu \geq 0 \quad \text{and} \quad II_1 \leq C \delta^\gamma \text{ if } \gamma + \nu \leq 0.
\]

Let us turn to $II_2$ given by (7.27), for which one has
\[
II_2 = C \frac{1}{\delta^\nu} \int_{\mathbb{R}^n} \left| \tilde{\phi}_k(v - y) \right| \left| v \right|^{\beta + \nu} I_{\left| v \right| \geq \max\{1, C\delta\}} dv.
\]
If $\beta + \nu \leq 0$, then it follows that
\[
II_2 \leq C \frac{1}{\delta^\nu} \left\{ \max\{1, C\delta\}\right\}^{\beta + \nu},
\]
while if $\beta + \nu \geq 0$, we bound from above $II_2$ by
\[
II_2 \leq II_{2,1} + II_{2,2},
\]
where
\[
II_{2,1} = C \frac{1}{\delta^\nu} \int_{\mathbb{R}^n} |\tilde{\phi}_k(v - y) || v - y |^{\beta + \nu} \mathbb{I}_{|v| \geq \max\{1, C\delta\}} dv
\]
and
\[
II_{2,2} = C \frac{1}{\delta^\nu} |y|^{\beta + \nu} \int_{\mathbb{R}^n} |\tilde{\phi}_k(v - y) | \mathbb{I}_{|v| \geq \max\{1, C\delta\}} dv.
\]
It follows that
\[
II_{2,1} \leq C \frac{1}{\delta^\nu} \frac{1}{2^{(\beta + \nu)k}} \text{ and } II_{2,2} \leq C \frac{1}{\delta^\nu} |y|^{\beta + \nu},
\]
so that
\[
II_2 \leq C \frac{1}{\delta^\nu} \frac{1}{2^{(\beta + \nu)k}} + C \frac{1}{\delta^\nu} |y|^{\beta + \nu}.
\]
This ends up the proof of Lemma 7.3.1.

Proof of Lemma 7.3.2: the term $I$

We shall use the following result, which is one part of the Cancellation Lemma from Chapter 3.

**Lemma 7.3.3** For almost all $v_* \in \mathbb{R}^n$, one has
\[
\int_{\mathbb{R}^n} \int_{S^{n-1}} B(v - v_*, \sigma) f' d\sigma dv = f \ast S(v_*) = \int_{\mathbb{R}^n} f(z) S(z - v_*) dz
\]
where
\[
S(z) \equiv |S^{-2}| \int_0^{\frac{\pi}{2}} \sin^{n-2} \theta \left\{ \frac{1}{\cos^n(\frac{\theta}{2})} B\left( \frac{|z|}{\cos(\frac{\theta}{2})}, \cos \frac{\theta}{2}\right) \right\} d\theta.
\]

It follows that
\[
I \leq \int_{\mathbb{R}^n} |\tilde{\phi}_k(z - y)| C \int_0^{\frac{\pi}{2}} \frac{1}{\cos^n(\frac{\theta}{2})} \Phi\left( \frac{|z|}{\cos(\frac{\theta}{2})}, \frac{|z|}{\cos(\frac{\theta}{2})}, \frac{|y|}{\cos(\frac{\theta}{2})}, \sin \frac{\theta}{2} \geq \frac{\theta}{2} \right) \sin \frac{\theta}{2} \sin \frac{\theta}{2} \sin \frac{\theta}{2} \theta^{-1-\nu} d\theta.
\]

Again, splitting this integral according to whether or not $\frac{|z|}{\cos(\frac{\theta}{2})}$ is less or not than 1, we can bound $I$ by
\[
(7.28) \quad I \leq I_1 + I_2,
\]

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where
\[ I_1 = C \int_{\mathbb{R}^n} dz \, |\tilde{\phi}_k(z-y)| \int_0^{\frac{\pi}{2}} \frac{1}{\cos^n\left(\frac{\theta}{2}\right)} |z|^{\gamma} \frac{|z|}{\cos \frac{\theta}{2}} \sin \frac{\theta}{2} \geq \delta \cos \frac{\theta}{2} \theta^{-1-\nu} \, d\theta \]
and
\[ I_2 = C \int_{\mathbb{R}^n} dz \, |\tilde{\phi}_k(z-y)| \int_0^{\frac{\pi}{2}} \frac{1}{\cos^n\left(\frac{\theta}{2}\right)} |z|^{\beta} \frac{|z|}{\cos \frac{\theta}{2}} \sin \frac{\theta}{2} \geq \delta \cos \frac{\theta}{2} \theta^{-1-\nu} \, d\theta. \]

For \( I_1 \), a rough estimate shows that
\[ I_1 \leq C \int_{\mathbb{R}^n} dz \, |\tilde{\phi}_k(z-y)| \| z \|_{1 \leq 1} \int_0^{\frac{\pi}{2}} \frac{|z|}{\cos \frac{\theta}{2}} \sin \frac{\theta}{2} \geq C \cos \frac{\theta}{2} \theta^{-1-\nu} \, d\theta \leq C \int_{\mathbb{R}^n} dz \, |\tilde{\phi}_k(z-y)| \| z \|_{1 \leq 1} \frac{|z|^{\nu}}{\delta^{\nu}} \leq \frac{C}{\delta^{\nu}}, \]
while for \( I_2 \), we get
\[ I_2 \leq \frac{C}{\delta^{\nu}} \int_{\mathbb{R}^n} dz \, |\tilde{\phi}_k(z-y)| \| z \|^{\beta+\nu} \| z \|_{C^* \geq \delta^{\nu}}. \]
If \( \beta + \nu \leq 0 \), then a crude estimate gives that
\[ I_2 \leq \frac{C}{\delta^{\nu}}, \]
while if \( \beta + \nu \geq 0 \), we bound \( I_2 \) by
\[ I_2 \leq \frac{C}{\delta^{\nu}} \int_{\mathbb{R}^n} dz \, |\tilde{\phi}_k(z-y)| \| z \|^{\beta+\nu} \| z \|_{C^* \geq \delta^{\nu}} + C \frac{|y|^{\beta+\nu}}{\delta^{\nu}} \int_{\mathbb{R}^n} dz \, |\tilde{\phi}_k(z-y)| \| z \|_{C^* \geq \delta^{\nu}}dz, \]
so that
\[ I_2 \leq \frac{C}{\delta^{\nu}} \frac{1}{2(\beta+\nu)k} + C \frac{|y|^{\beta+\nu}}{\delta^{\nu}}. \]
Collecting all the above results, we obtain Lemma 7.3.2.

\[ \square \]
7.3.2 The singular term $K_{S,2}^{\Phi,k}(v,y)$, see formula (7.21)

Proposition 7.3.2 For the kernel $K_{S,2}^{\Phi,k}(v,y)$ defined by (7.21), one has, under assumption (7.9)

$$
\int_{\mathbb{R}^n} |K_{S,2}^{\Phi,k}(v,y)| \, dv \\
\leq C \left\{ \min\{C\delta, 1\} \right\}^{\gamma} + I_{1 \leq C\delta} \left\{ I_{\beta+1 \geq 0} C\delta^{\beta} + I_{\beta+1 \leq 0} \frac{C}{\delta} \right\} + I_{C\delta \leq 1} \left\{ I_{\gamma+\nu \geq 0} \frac{C}{\delta^\nu} + I_{\gamma+\nu \leq 0} C\delta^{\gamma} \right\}
$$

where $r > 1$ can be chosen arbitrary.

For the proof and the reader, let us recall that the term $K_{S,2}^{\Phi,k}(v,y)$ is given by (7.21)

$$
K_{S,2}^{\Phi,k}(v,y) \equiv \int_{S_{v-1}} \left( \tilde{\phi}_k(v^+ - y) - \tilde{\phi}_k(v - y) \right) b(\cos \theta) \Phi(|v|) \left| v \right|^{\gamma} \, d\sigma.
$$

Using a standard technique in harmonic analysis, we write

$$
\tilde{\phi}_k(v^+ - y) - \tilde{\phi}_k(v - y) = \tilde{\phi}_k((v - y) - (v + v^+)) - \tilde{\phi}_k(v - y)
$$

and with $X = v - y$ and $Y = v - v^+$, it follows that it is less than

$$
C \frac{|Y|}{|X|^{n+1}} \text{ if } |X| > 2 |Y|.
$$

Therefore, one has

$$
|\tilde{\phi}_k(v^+ - y) - \tilde{\phi}_k(v - y)| \leq C \frac{|v - v^+|}{|v - y|^{n+1}} \text{ if } |v - y| > 2 |v - v^+|.
$$
Note in particular, that if $|v - y| > 2\delta$, then $|v - y| > 2|v - v^+|$. We are led therefore to cut $K^{\Phi,k}_{\delta,2}(v, y)$ into two pieces and more precisely

\[\int_{\mathbb{R}^n} dv \ | K^{\Phi,k}_{\delta,2}(v, y) | dv \leq A + B,\]

where

\[A \equiv \int_{\mathbb{R}^n} dv \int_{S^{n-1}} |\tilde{\phi}_k(v^+ - y) - \tilde{\phi}_k(v - y)| b\Phi(|v|) \|v| \sin \frac{\delta}{2} \||v-y| > 2\delta d\sigma\]

and

\[B \equiv \int_{\mathbb{R}^n} dv \int_{S^{n-1}} |\tilde{\phi}_k(v^+ - y) - \tilde{\phi}_k(v - y)| b\Phi(|v|) \|v| \sin \frac{\delta}{2} \|v-y| \leq 2\delta d\sigma.\]

The proof of Proposition 7.3.2 will follow from the next two results

**Lemma 7.3.4** For $A$ given by (7.32), one has, under assumption (7.9)

\[
A \leq C\{ \min\{C\delta, 1\}\}^\gamma + \|I_{\beta+1 \geq 0}C\delta^\beta + \|I_{\beta+1 \leq 0}C\delta^\beta \}
+ \|C\delta \leq 1\left\{ I_{\gamma+\nu \geq 0} \frac{C}{\delta^\nu} + \|I_{\gamma+\nu \leq 0}C\delta^\gamma \right\}
+ \|I_{\beta+\nu \leq 0}C\left\{ \max\{C\delta, 1\}\right\}^{\beta+\nu} + \|I_{\beta+\nu \geq 0} \left\{ C\delta^\beta + C\|y|^{\beta+\nu} \frac{\delta^\nu}{\delta^\nu} \right\}.
\]

\[\square\]

For the other term, one has

**Lemma 7.3.5** For $B$ defined by (7.33), one has

\[
B \leq \|\gamma+1 \geq 0C2^{(n+1)k}\left\{ 1 + (2^k \delta)^{\frac{n}{2}} \right\} \{ \min\{1, C\delta\}\}^{\gamma+1}d^n + \|\gamma+1 \leq 0C2^{(n+1)k}\left\{ 1 + (2^k \delta)^{\frac{n}{2}} \right\} \{ \min\{1, C\delta\}\}^{n+\gamma+1}
+ \|\gamma \geq 1\left\{ I_{\gamma+\nu \geq 0}C2^{(n+1)k}\left\{ 1 + (2^k \delta)^{\frac{n}{2}} \right\} \delta^{\beta+1}d^n + \|I_{\gamma+\nu \leq 0}C2^{(n+1)k}\left\{ 1 + (2^k \delta)^{\frac{n}{2}} \right\} \delta^n \right\}
+ \|\gamma \leq 1\left\{ I_{\gamma+\nu \geq 0}C2^{(n+1)k}\delta^{\beta+1}\left\{ 1 + (2^k \delta)^{\frac{n}{2}} \right\} \delta^{\beta+1}d^n + \|I_{\gamma+\nu \leq 0}C2^{(n+1)k}\delta^{\beta-\nu+1}\left\{ 1 + (2^k \delta)^{\frac{n}{2}} \right\} \delta^{\gamma+n+1} \right\}
+ \|\gamma \geq 1\left\{ I_{\gamma+\nu \geq 0}C2^{(n+1)k}\delta^{\beta+1}\left\{ 1 + (2^k \delta)^{\frac{n}{2}} \right\} \delta^{\beta+1}d^n + \|I_{\gamma+\nu \leq 0}C2^{(n+1)k}\delta^{\beta+1}\left\{ 1 + (2^k \delta)^{\frac{n}{2}} \right\} \delta^{\beta+1}d^n \right\}
+ \|\gamma \leq 1\left\{ I_{\gamma+\nu \geq 0}C2^{(n+1)k}\delta^{\beta+1}\left\{ 1 + (2^k \delta)^{\frac{n}{2}} \right\} \delta^{\beta+1}d^n + C\|y|^{\beta+\nu}2^{(n+1)k}\delta^{\beta-\nu+1}\left\{ 1 + (2^k \delta)^{\frac{n}{2}} \right\} \delta^n \right\}.
\]

\[\square\]

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Proof of Lemma 7.3.4: the term $A$

We note that

$$A \leq C \int_{\mathbb{R}^n} dv \int_{S^{n-1}} \frac{|v - v^+|}{|v - y|^{n+1}} b \Phi(|v|) \|v\|_{|v| \sin \frac{\theta}{2} \leq \delta \|v - y\|_{n+1} > 2\delta} d\sigma$$

$$\leq C \int_{\mathbb{R}^n} dv \int_{S^{n-1}} \frac{|v - v^+|}{|v - y|^{n+1}} b \Phi(|v|) \|v\|_{|v| \sin \frac{\theta}{2} \|v - y\|_{n+1} \leq \delta \|v - y\|_{n+1} > 2\delta} d\sigma$$

(7.34)

$$\leq A_1 + A_2,$$

where we cut the last integral according to whether or not $\frac{\delta}{|v|} \geq \sqrt{\frac{n+\gamma}{2n+1}}$ (for $A_1$, respectively for $A_2$). If this is the case, i.e. for $A_1$, then we abbreviate this as $|v| \leq C\delta$.

It follows that

(7.35)

$$A_1 \leq C \int_{\mathbb{R}^n} dv \frac{1}{|v - y|^{n+1}} |v| \Phi(|v|) \|v - y\|_{|v| \|v - y\|_{n+1} > 2\delta \|v\|_{n+1} \leq C\delta} d\sigma,$$

and thus

(7.36)

$$A_1 \leq A_{1,1} + A_{1,2},$$

where

(7.37)

$$A_{1,1} = C \int_{\mathbb{R}^n} \frac{1}{|v - y|^{n+1}} |v| \Phi(|v|) \|v - y\|_{|v - y\|_{n+1} > 2\delta \|v\|_{n+1} \leq C\delta} d\sigma$$

and

(7.38)

$$A_{1,2} = C \int_{\mathbb{R}^n} \frac{1}{|v - y|^{n+1}} |v| \Phi(|v|) \|v - y\|_{|v - y\|_{n+1} > 2\delta \|v\|_{n+1} \geq 1} d\sigma.$$

For $A_{1,1}$, one has

$$A_{1,1} \leq C \int_{\mathbb{R}^n} \frac{1}{|v - y|^{n+1}} |v| \Phi(|v|) \|v - y\|_{|v - y\|_{n+1} > 2\delta \|v\|_{n+1} \leq \min(C\delta,1)} d\sigma$$

and thus

$$A_{1,1} \leq C \int_{\mathbb{R}^n} \frac{1}{|v - y|^{n+1}} |v|^{\gamma+1} \|v - y\|_{|v - y\|_{n+1} > 2\delta \|v\|_{n+1} \leq \min(C\delta,1)} d\sigma.$$

If $n + \gamma > -1$, that is $n + \gamma + 1 > 0$, it follows that

$$A_{1,1} \leq C \frac{1}{\delta^{n+1}} \{ \min\{C\delta,1\} \}^{n+\gamma+1} \leq C \{ \min\{C\delta,1\} \}^\gamma.$$

Next, for $A_{1,2}$, one has

$$A_{1,2} \leq C \int_{\mathbb{R}^n} \frac{1}{|v - y|^{n+1}} |v|^{\gamma+1} \|v - y\|_{|v - y\|_{n+1} \geq 1} d\sigma.$$
If $C\delta \leq 1$, then the inner term is zero, so we only have to see the behavior when $1 \leq C\delta$, thus
\[
A_{1,2} \leq C \int_{\mathbb{R}^n} \frac{1}{|v-y|^{n+1}} |v|^{\beta+1} \mathbb{I}_{|v|>2\delta} \mathbb{I}_{|v|\leq C\delta} dv.
\]
We get in this way
\[
A_{1,2} \leq C\delta^\beta \text{ if } \beta + 1 \geq 0 \text{ and } A_{1,2} \leq \frac{C}{\delta} \text{ if } \beta + 1 \leq 0.
\]
Therefore, we have obtained

**Lemma 7.3.6** For $A_1$ upper bounded by (7.35), one has
\[
A_1 \leq C \left\{ \min\{C\delta, 1\} \right\}^\gamma + \mathbb{I}_{1 \leq C\delta} \left( \mathbb{I}_{\beta+1 \geq 0} C\delta^\beta + \mathbb{I}_{\beta+1 \leq 0} \frac{C}{\delta} \right).
\]
\[\square\]

Let us now consider $A_2$, for which we have
\[
A_2 \leq C \int_{\mathbb{R}^n} dv \int_{S^{n-1}} \frac{|v|}{|v-y|^{n+1}} \Phi(|v|) b \sin \frac{\theta}{2} |v| \sin \frac{\theta}{2} \mathbb{I}_{|v|>2\delta} \mathbb{I}_{|v|\geq C\delta} d\sigma.
\]
Computing the spherical integral, it follows that
\[
A_2 \leq C \int_{\mathbb{R}^n} \frac{|v|}{|v-y|^{n+1}} \Phi(|v|) \mathbb{I}_{|v|>2\delta} \mathbb{I}_{|v|\geq C\delta} \delta^{-\nu+1} \mathbb{I}_{|v|<\nu} dv
\]
\[
\leq C\delta^{-\nu+1} \int_{\mathbb{R}^n} \frac{|v|}{|v-y|^{n+1}} \Phi(|v|) \mathbb{I}_{|v|>2\delta} \mathbb{I}_{|v|\geq C\delta} dv
\]
\[
\leq A_{2,1} + A_{2,2},
\]
where
\[
A_{2,1} = C\delta^{-\nu+1} \int_{\mathbb{R}^n} \frac{|v|^\nu}{|v-y|^{n+1}} \Phi(|v|) \mathbb{I}_{|v|>2\delta} \mathbb{I}_{|v|\geq C\delta} dv
\]
and
\[
A_{2,2} = C\delta^{-\nu+1} \int_{\mathbb{R}^n} \frac{|v|^\nu}{|v-y|^{n+1}} \Phi(|v|) \mathbb{I}_{|v|\geq C\delta} dv.
\]
For $A_{2,1}$, one has
\[
A_{2,1} \leq C\delta^{-\nu+1} \int_{\mathbb{R}^n} \frac{|v|^\gamma+\nu}{|v-y|^{n+1}} \mathbb{I}_{|v|>2\delta} \mathbb{I}_{|v|\leq C\delta} dv.
\]
If $C\delta \geq 1$, the inner term is zero, so we can assume $C\delta \leq 1$. In that case, one has
\[
A_{2,1} \leq C\delta^{-\nu+1} \int_{\mathbb{R}^n} \frac{|v|^\gamma+\nu}{|v-y|^{n+1}} \mathbb{I}_{|v|>2\delta} \mathbb{I}_{C\delta \leq |v|\leq 1} dv.
\]
Then, we get
\[
A_{2,1} \leq \frac{C}{\delta^\nu} \text{ if } \gamma + \nu \geq 0 \text{ and } A_{2,1} \leq C\delta^{-\nu+1} \delta^{\gamma+\nu} \frac{1}{\delta} \leq C\delta^\gamma \text{ if } \gamma + \nu \leq 0.
\]
For $A_{2,2}$, one has

$$A_{2,2} \leq C \delta^{-\nu+1} \int_{\mathbb{R}^n} \frac{|v|^{|\beta+\nu|}}{|v-y|^{n+1}} I_{|v-y|>2\delta} I_{|v| \geq \max\{C\delta, 1\}} dv,$$

that is

$$A_{2,2} = C \delta^{-\nu+1} \int_{\mathbb{R}^n} \frac{|v|^{|\beta+\nu|}}{|v-y|^{n+1}} I_{|v-y|>2\delta} I_{|v| \geq \max\{C\delta, 1\}} dv.$$

If $\beta + \nu \leq 0$, then

$$A_{2,2} \leq C \delta^{-\nu+1} \frac{1}{\delta} \max\{C\delta, 1\} \frac{\beta + \nu}{\delta} \leq C \frac{\max\{C\delta, 1\} \beta + \nu}{\delta},$$

while if $\beta + \nu \geq 0$, we split this integral as

$$A_{2,2} \leq C \delta^{-\nu+1} \int_{\mathbb{R}^n} \frac{|v-y|^{|\beta+\nu|}}{|v-y|^{n+1}} I_{|v-y|>2\delta} I_{|v| \geq \max\{C\delta, 1\}} dv$$

$$+ C \delta^{-\nu+1} |y|^{|\beta+\nu|} \int_{\mathbb{R}^n} \frac{1}{|v-y|^{n+1}} I_{|v-y|>2\delta} I_{|v| \geq \max\{C\delta, 1\}} dv$$

(note that it is exactly here that we use assumption (7.9))

$$\leq C \delta^{-\nu+1} \delta^{\beta+\nu-1} + C \delta^{-\nu+1} |y|^{|\beta+\nu|} \frac{1}{\delta},$$

and therefore, we have found that if $\beta + \nu \geq 0$, then

$$A_{2,2} \leq C \delta^\beta + C \frac{|y|^{|\beta+\nu|}}{\delta^\nu}.$$

Thus we have obtained

**Lemma 7.3.7** For $A_2$ upper bounded by (7.39), one has, under assumption (7.9)

$$\begin{cases} A_2 \leq \mathbb{I}_{C \delta \leq 1} \{ \mathbb{I}_{\beta+\nu \geq 0} \frac{C}{\delta^\nu} + \mathbb{I}_{\beta+\nu \leq 0} C \delta^\gamma \} \\
+ \mathbb{I}_{\beta+\nu \leq 0} C \frac{\max\{C\delta, 1\} \beta + \nu}{\delta^\nu} + \mathbb{I}_{\beta+\nu \geq 0} \left\{ C \delta^\beta + C \frac{|y|^{|\beta+\nu|}}{\delta^\nu} \right\}. \end{cases}$$

\[\square\]

Collecting Lemma 7.3.6 and Lemma 7.3.7 finally gives the statement of Lemma 7.3.4.
Proof of Lemma 7.3.5: the term $B$

Next, let us deal with the term $B$ from (7.33) given by

$$B = \int_{\mathbb{R}^n} dv \int_{S^{n-1}} \left\{ \tilde{\phi}_k(v^+ - y) - \tilde{\phi}_k(v - y) \right\} \Phi(|v|) b(\cos \theta) I_{\frac{\delta}{2} \leq |v| \leq \frac{\delta}{2} |v - y| \leq 2d} \, d\sigma.$$ 

Let us note that

$$\tilde{\phi}_k(\cdot) = 2^{nk} \tilde{\phi}(2^k \cdot)$$

and thus $\nabla \tilde{\phi}_k(\cdot) = 2^{(n+1)k} \nabla \tilde{\phi}(2^k \cdot)$.

Applying the maximal lemma recalled in the Appendix, it follows that

$$2^{(n+1)k} \sup_{z \in \mathbb{R}^n} \frac{\nabla \tilde{\phi}(2^k x - 2^k z)}{1 + |2^k z|^2} \leq C \sup_{z \in \mathbb{R}^n} \frac{\tilde{\phi}(2^k x - z)}{1 + |z|^2} \leq 2^{(n+1)k} C_2 \left\{ (M |\tilde{\phi}^\tau|)(2^k x) \right\}^{\frac{1}{2}},$$

where $M$ denotes the usual maximal function.

Now, the difference term inside $B$ can be written as

$$\begin{aligned}
&\left\{ \tilde{\phi}_k(v^+ - y) - \tilde{\phi}_k(v - y) \right\} \\
&= (v^+ - v) \cdot \int_0^1 \nabla \tilde{\phi}_k((v - y) + t(v^+ - v)) \, dt \\
&= (v^+ - v) 2^{(n+1)k} \int_0^1 \nabla \tilde{\phi}(2^k (v - y) + 2^k t(v^+ - v)) \, dt \\
&= (v^+ - v) 2^{(n+1)k} \int_0^1 \nabla \tilde{\phi}(2^k (v - y) + 2^k t(v^+ - v)) \frac{1 + |2^k t(v^+ - v)|^2}{1 + |2^k t(v^+ - v)|^2} \, dt,
\end{aligned}$$

and since $|v^+ - v| \leq \delta$ in the inner term of $B$, it follows that

$$|\tilde{\phi}_k(v^+ - y) - \tilde{\phi}_k(v - y)| \leq |v - v^+| 2^{(n+1)k} \left\{ 1 + (2^k \delta)^2 \right\} \left\{ (M |\tilde{\phi}^\tau|)(2^k (v - y)) \right\}^{\frac{1}{2}}.$$ 

Now we split $B$ as $B_1 + B_2$, according to whether or not $\frac{\delta}{|v|} \geq \frac{\sqrt{2}}{2}$, abbreviated as $|v| \leq C \delta$.

One has

$$B_1 \leq C \int_{\mathbb{R}^n} dv \int_{S^{n-1}} |\tilde{\phi}_k(v^+ - y) - \tilde{\phi}_k(v - y)| b \Phi I_{\frac{\delta}{2} \leq |v| \leq C \delta} \frac{d\Phi}{|v|} \leq C \int_{\mathbb{R}^n} dv \int_{S^{n-1}} |v| \Phi \sin \frac{\theta}{2} 2^{(n+1)k} \left\{ 1 + (2^k \delta)^2 \right\} \left\{ (M |\tilde{\phi}^\tau|)(2^k (v - y)) \right\}^{\frac{1}{2}} b \Phi I_{|v| \leq C \delta} \leq C \int_{\mathbb{R}^n} dv \int_{S^{n-1}} |v|^2 \Phi I_{\frac{\delta}{2} \leq |v| \leq C \delta} \leq C \int_{\mathbb{R}^n} dv \int_{S^{n-1}} |v|^2 \Phi I_{|v| \leq \frac{\delta}{2}} \, d\sigma$$

$$\leq C 2^{(n+1)k} \left\{ 1 + (2^k \delta)^2 \right\} \int_{\mathbb{R}^n} |v|^2 \Phi I_{\frac{\delta}{2} \leq |v| \leq C \delta} + C 2^{(n+1)k} \left\{ 1 + (2^k \delta)^2 \right\} \int_{\mathbb{R}^n} |v|^2 \Phi I_{|v| \leq \frac{\delta}{2}}$$

(7.40)
\[(7.41)\] 

with obvious notations.

Next, for \(B_{1,1}\), one has

\[
B_{1,1} \leq C2^{(n+1)k}\left\{ 1 + (2^k \delta)^\frac{n}{2} \right\} \int_{\mathbb{R}^n} |v|^{\gamma+1} \left\{ (M |\tilde{\phi}^r|)(2^k(v - y)) \right\} \frac{1}{r} I_{|v| \leq \min\{1, C\delta\}} I_{|v - y| \leq 2\delta} dv.
\]

If \(\gamma + 1 \geq 0\), then it follows that

\[
B_{1,1} \leq C2^{(n+1)k}\left\{ 1 + (2^k \delta)^\frac{n}{2} \right\} \left\{ \min\{1, C\delta\}\right\}^{\gamma+1} \int_{\mathbb{R}^n} |v|^{\gamma+1} I_{|v| \leq \min\{1, C\delta\}} dv
\]

which gives, using the \(L^\infty\) bound on the maximal function,

\[
B_{1,1} \leq C2^{(n+1)k}\left\{ 1 + (2^k \delta)^\frac{n}{2} \right\} \left\{ \min\{1, C\delta\}\right\}^{\gamma+1} \delta^n.
\]

If \(\gamma + 1 \leq 0\) and \(n + \gamma + 1 > 0\), the same argument yields

\[
\begin{cases}
B_{1,1} \leq C2^{(n+1)k}\left\{ 1 + (2^k \delta)^\frac{n}{2} \right\} \int_{\mathbb{R}^n} |v|^{\gamma+1} I_{|v| \leq \min\{1, C\delta\}} dv \\
< C2^{(n+1)k}\left\{ 1 + (2^k \delta)^\frac{n}{2} \right\} \left\{ \min\{1, C\delta\}\right\}^{\gamma+1} n + \gamma + 1.
\end{cases}
\]

For \(B_{1,2}\), one has

\[
B_{1,2} = C2^{(n+1)k}\left\{ 1 + (2^k \delta)^\frac{n}{2} \right\} \int_{\mathbb{R}^n} |v|^{\beta+1} \left\{ (M |\tilde{\phi}^r|)(2^k(v - y)) \right\} \frac{1}{r} I_{|v| \leq C\delta} I_{|v - y| \leq 2\delta} I_{|v| \geq 1} dv.
\]

If \(C\delta \leq 1\), then the inner term is zero, so that we may as well assume that \(C\delta \geq 1\), in which case one has

\[
B_{1,2} = C2^{(n+1)k}\left\{ 1 + (2^k \delta)^\frac{n}{2} \right\} \int_{\mathbb{R}^n} |v|^{\beta+1} \left\{ (M |\tilde{\phi}^r|)(2^k(v - y)) \right\} \frac{1}{r} I_{|v| \leq C\delta} I_{|v - y| \leq 2\delta} dv,
\]

and finally

\[
B_{1,2} \leq \mathbb{I}_{\beta+1 \geq 0} C2^{(n+1)k}\left\{ 1 + (2^k \delta)^\frac{n}{2} \right\} \delta^{\beta+1} \delta^n + \mathbb{I}_{\beta+1 \leq 0} C2^{(n+1)k}\left\{ 1 + (2^k \delta)^\frac{n}{2} \right\} \delta^n.
\]

Collecting the above computations, we have obtained

**Lemma 7.3.8** For \(B_1\) given by (7.40), one has

\[
\begin{cases}
B_1 \leq \mathbb{I}_{\gamma+1 \geq 0} C2^{(n+1)k}\left\{ 1 + (2^k \delta)^\frac{n}{2} \right\} \left\{ \min\{1, C\delta\}\right\}^{\gamma+1} \delta^n \\
+ \mathbb{I}_{\gamma+1 \leq 0} C2^{(n+1)k}\left\{ 1 + (2^k \delta)^\frac{n}{2} \right\} \left\{ \min\{1, C\delta\}\right\}^{n+\gamma+1} \\
+ \mathbb{I}_{C\delta \geq 1} \left\{ \mathbb{I}_{\beta+1 \geq 0} C2^{(n+1)k}\left\{ 1 + (2^k \delta)^\frac{n}{2} \right\} \delta^{\beta+1} \delta^n + \mathbb{I}_{\beta+1 \leq 0} C2^{(n+1)k}\left\{ 1 + (2^k \delta)^\frac{n}{2} \right\} \delta^n \right\}.
\end{cases}
\]

\[\Box\]
Next, looking to $B_2$ (see before formula (7.40)), we have

\[(7.42)\]

\[
B_2 \leq \int_{\mathbb{R}^n} dv \int_{\mathbb{R}^n} \frac{1}{2^{(n+1)k}} \left\{ (M | \tilde{\phi} |^\nu) (2^k (v - y)) \right\}^{\frac{1}{2}} \Phi b_{2^n |\nu-y| \leq 2 \delta} d\sigma
\]

\[
\leq C \int_{\mathbb{R}^n} dv \int_{\mathbb{R}^n} \frac{1}{2^{(n+1)k}} \left\{ (M | \tilde{\phi} |^\nu) (2^k (v - y)) \right\}^{\frac{1}{2}} \Phi b_{2^n |\nu-y| \leq 2 \delta} d\sigma
\]

\[
\leq \int_{\mathbb{R}^n} dv \cdot \frac{1}{2^{(n+1)k}} \left\{ (M | \tilde{\phi} |^\nu) (2^k (v - y)) \right\}^{\frac{1}{2}} \Phi b_{2^n |\nu-y| \leq 2 \delta} d\sigma
\]

where

\[
B_{2,1} = C 2^{(n+1)k} \delta^{-\nu+1} \left\{ 1 + (2^k \delta)^\frac{n}{2} \right\} \int_{\mathbb{R}^n} dv \cdot \frac{1}{2^{(n+1)k}} \left\{ (M | \tilde{\phi} |^\nu) (2^k (v - y)) \right\}^{\frac{1}{2}} \Phi b_{2^n |\nu-y| \leq 2 \delta} d\sigma
\]

and

\[
B_{2,2} = C \int_{\mathbb{R}^n} dv \cdot \frac{1}{2^{(n+1)k}} \left\{ (M | \tilde{\phi} |^\nu) (2^k (v - y)) \right\}^{\frac{1}{2}} \Phi b_{2^n |\nu-y| \leq 2 \delta} d\sigma
\]

Let us deal with $B_{2,1}$.

If $C \delta \geq 1$, the inner term is zero, so that assuming $C \delta \leq 1$, one has

\[
B_{2,1} = C2^{(n+1)k} \delta^{-\nu+1} \left\{ 1 + (2^k \delta)^\frac{n}{2} \right\} \int_{\mathbb{R}^n} dv \cdot \frac{1}{2^{(n+1)k}} \left\{ (M | \tilde{\phi} |^\nu) (2^k (v - y)) \right\}^{\frac{1}{2}} \Phi b_{2^n |\nu-y| \leq 2 \delta} d\sigma.
\]

Then bounding in $L^\infty$ the maximal function, it follows that

\[
B_{2,1} \leq \Phi b_{\gamma+\nu \geq 0} C2^{(n+1)k} \delta^{-\nu+1} \left\{ 1 + (2^k \delta)^\frac{n}{2} \right\} \delta^n + \Phi b_{\gamma+\nu \leq 0} C2^{(n+1)k} \delta^{-\nu+1} \left\{ 1 + (2^k \delta)^\frac{n}{2} \right\} \delta^n.
\]

For $B_{2,2}$, one has

\[
B_{2,2} \leq C \int_{\mathbb{R}^n} dv \cdot \frac{1}{2^{(n+1)k}} \left\{ (M | \tilde{\phi} |^\nu) (2^k (v - y)) \right\}^{\frac{1}{2}} \Phi b_{2^n |\nu-y| \leq 2 \delta} d\sigma.
\]

If $\beta + \nu \leq 0$, then

\[
B_{2,2} \leq C2^{(n+1)k} \delta^{-\nu+1} \left\{ 1 + (2^k \delta)^\frac{n}{2} \right\} \delta^n.
\]

If $\beta + \nu \geq 0$, we bound $B_{2,2}$ from above by splitting into two terms as

\[
B_{2,2} \leq B_{2,2,1} + B_{2,2,2}.
\]
where
\[ B_{2,1} = C \int_{\mathbb{R}^n} |v - y|^{\beta + \nu} 2^{(n+1)k} \delta^{-\nu + 1} \left\{ 1 + (2^k \delta)^\frac{n}{2} \right\} \left\{ (M \mid \tilde{\phi} \mid^\gamma)(2^k (v - y)) \right\} \frac{1}{\delta} \mathbb{I}_{|v - y| \leq 2\delta \beta} dv \]
and
\[ B_{2,2} = C \int_{\mathbb{R}^n} |y|^{\beta + \nu} 2^{(n+1)k} \delta^{-\nu + 1} \left\{ 1 + (2^k \delta)^\frac{n}{2} \right\} \left\{ (M \mid \tilde{\phi} \mid^\gamma)(2^k (v - y)) \right\} \frac{1}{\delta} \mathbb{I}_{|v - y| \leq 2\delta \beta} dv, \]
from which one gets
\[ B_{2,1} \leq C 2^{(n+1)k} \delta^{-\nu + 1} \left\{ 1 + (2^k \delta)^\frac{n}{2} \right\} \delta^{\beta + \nu + n}, \]
and
\[ B_{2,2} \leq C |y|^{\beta + \nu} 2^{(n+1)k} \delta^{-\nu + 1} \left\{ 1 + (2^k \delta)^\frac{n}{2} \right\} \delta^n. \]

We collect the above computations to get

**Lemma 7.3.9** For \( B_2 \) upper bounded by (7.42), one has

\[
B_2 \leq \begin{cases} 
\mathbb{I}_{\gamma + \nu \geq 0} C 2^{(n+1)k} \delta^{-\nu + 1} \left\{ 1 + (2^k \delta)^\frac{n}{2} \right\} \delta^n + \mathbb{I}_{\gamma + \nu \leq 0} C 2^{(n+1)k} \delta^{-\nu + 1} \left\{ 1 + (2^k \delta)^\frac{n}{2} \right\} \delta^{\gamma + \nu} \\
+ \mathbb{I}_{\beta + \nu \leq 0} C 2^{(n+1)k} \delta^{-\nu + 1} \left\{ 1 + (2^k \delta)^\frac{n}{2} \right\} \max\{1, C \delta\} \delta^{\beta + \nu + n} \\
+ \mathbb{I}_{\beta + \nu \geq 0} \left\{ 2^{(n+1)k} \delta^{-\nu + 1} \left\{ 1 + (2^k \delta)^\frac{n}{2} \right\} \delta^{\beta + \nu + n} + C |y|^{\beta + \nu} 2^{(n+1)k} \delta^{-\nu + 1} \left\{ 1 + (2^k \delta)^\frac{n}{2} \right\} \delta^n \right\} 
\end{cases}
\]

Finally, gluing Lemma 7.3.8 and Lemma 7.3.9 gives the statement of Lemma 7.3.5.

It remains to add Lemma 7.3.4 and Lemma 7.3.5 in order to get the statement of Proposition 7.3.2.

### 7.4 Conclusion: obtaining Proposition 7.1.1

It is only a matter to choose carefully the parameter \( \delta \) in all the cases mentioned in the statement of Proposition 7.1.1.

#### 7.4.1 The case \( \beta + \nu \leq 0 \) and \( \gamma + \nu \geq 0 \)

This is the most simple case. Note first that from Proposition 7.2.1, it follows readily that
\[
\int_{\mathbb{R}^n} |K^{\Phi,k}(v, y)| dy \leq C 2^{k\nu}.
\]

Then, we set \( \delta = C' \frac{1}{2\pi} \), where we choose \( C' \) such that \( C \delta \leq 1 \), where \( C \) is the constant entering Propositions 7.3.1 and 7.3.2. With this choice, one can check, using in particular \( \gamma + 1 \geq \gamma + \nu \), that
\[
\int_{\mathbb{R}^n} |K^{\Phi,k}(v, y)| dv \leq C 2^{k\nu}.
\]
7.4.2 The case $\beta + \nu \geq 0$ and $\gamma + \nu \geq 0$

Proposition 7.2.1 gives firstly that

$$\int_{\mathbb{R}^n} |K^{\Phi,k}(v, y)| \, dy \leq C 2^{k\nu} \{1 + |v|^{|\beta + \nu|}\}.$$  

Then again the same choice $\delta = C' \frac{1}{2} \nu$ gives, from Propositions 7.3.1 and 7.3.2

$$\int_{\mathbb{R}^n} |K^{\Phi,k}(v, y)| \, dv \leq C 2^{k\nu} \{1 + |y|^{|\beta + \nu|}\}.$$  

Remarks 7.4.1 We have tried another choice (only one!) to improve the previous bounds; for instance, the choice $\delta = C' \frac{1}{2} \nu \|y - v^*\|$ (say for large $\|y - v^*\|$) works also in Proposition 7.3.1 and gives that the non singular part has an $L^1$ norm w.r.t. $v$ variable bounded by $C 2^{k\nu}$. But then this choice of $\delta$ fails to give a similar estimate for the singular part, that is for Proposition 7.3.2. This is of course not a surprising fact!

7.4.3 The case $\beta + \nu \leq 0$ and $\gamma + \nu \leq 0$

Proposition 7.2.1 gives again that

$$\int_{\mathbb{R}^n} |K^{\Phi,k}(v, y)| \, dy \leq C 2^{k\nu} \{1 + |v|^{|\gamma + \nu|}\},$$

while the same choice of $\delta$ as above gives this time

$$\int_{\mathbb{R}^n} |K^{\Phi,k}(v, y)| \, dv \leq C \sup \{2^{k\nu}, 2^{-\gamma)k}\}.$$  

7.4.4 The case $\beta + \nu \geq 0$ and $\gamma + \nu \leq 0$

For this final case, we find again from Proposition 7.2.1 that

$$\int_{\mathbb{R}^n} |K^{\Phi,k}(v, y)| \, dy \leq C 2^{k\nu} \{1 + |v|^{|\gamma + \nu|}\}$$

and with the same choice of $\delta$, we find

$$\int_{\mathbb{R}^n} |K^{\Phi,k}(v, y)| \, dv \leq C \sup \{2^{k\nu}, 2^{-\gamma)k}\}\{1 + |y|^{|\beta + \nu|}\}.$$
7.5 Proof of Corollary 7.1.1, Theorem 7.1.1, Theorem 7.1.2 and Theorem 7.1.3

7.5.1 Proof of Corollary 7.1.1

Let $f$ be a smooth function (of variable $v$).

For the first case considered by Proposition 7.1.1, in view of formula (7.11), one has

$$\|T \Phi f\|_{L^1(\mathbb{R}^n)} \leq C \sum_{k=0}^{+\infty} 2^{k\nu} \|p_k f\|_{L^1} \leq C \|f\|_{B_{1,1}^\nu(\mathbb{R}^n)},$$

which shows that

$$T \Phi : B_{1,1}^\nu(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$$

continuously.

Next, we have

$$\|T \Phi f\|_{L^\infty(\mathbb{R}^n)} \leq \sum_{k=0}^{+\infty} 2^{k\nu} \|p_k f\|_{L^\infty(\mathbb{R}^n)} \leq C \|f\|_{B_{\infty,1}^\nu(\mathbb{R}^n)},$$

which shows that

$$T \Phi : B_{\infty,1}^\nu(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n).$$

It is then only a matter of interpolation of Besov spaces (the complex method). It is also possible to avoid using interpolation of Besov spaces, simply by noticing, firstly that, since $T \Phi, k$ is bounded on $L^1$ and on $L^\infty$, with similar bounds, it follows from interpolation that $T \Phi, k$ is also bounded on any $L^p$ with the same bound, that is $C 2^{k\nu}$. Then, we have, for smooth functions $f$

$$\|T \Phi f\|_{L^p} \leq C \sum_{k=0}^{+\infty} 2^{k\nu} \|p_k f\|_{L^p} \leq C \|f\|_{B_{p,1}^\nu},$$

which is the result, up to the modification for $p = +\infty$ to the space $b_{\infty,1}^\nu$.

7.5.2 Proof of Theorem 7.1.2

We are more precisely concerned with the Maxwellian case, that is $\Phi \equiv 1$ (or constant).

Let again $\phi$ be a test function. Then

$$< p_k Q(g, f); \phi > = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} dv_s dv_g(v_s)f(v) \tau_{-v_s} \circ T^{1,k} \circ \tau_{v_s} \tilde{p}_k \phi,$$

for another Littlewood Paley decomposition $\tilde{p}_k$.
Next, a computation on Fourier side shows that we can replace \( f \) by \( \hat{\eta}_k \) for still another Littlewood Paley decomposition \( \hat{\eta}_k \).

This fact is deduced from two points: we are working, on Fourier side, on an annulus of radius \( \sim 2^k \) for variable \( \xi \), and the corresponding variable \( \xi^+ \) is bounded below (we should work for \( k \) large enough in order to avoid overlapping 0).

It follows that

\[
< p_k Q(g, f) ; \phi > = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} dv_* dvg(v_*) \hat{\eta}_k(v) \tau_{v_*} \circ T^{1,k} \circ \tau_{v_*} \hat{\eta}_k \phi.
\]

Next, we note that, since we are in the Maxwellian case, corresponding to \( \gamma = \beta = 0 \), we can apply functional estimates on \( T^{1,k} \). Preceding arguments, and in particular Proposition 7.1.1, show that \( T^{1,k} : L^p \to L^{p-\nu} \) with a bound in \( 2^k \nu \) (corresponding weighted estimates are also true; this follows from the proof of Proposition 7.1.1). Taking into account the translation in \( v_* \), it follows that

\[
| < p_k Q(g, f) ; \phi > | \leq C 2^{k\nu} \| g \|_{L^1} \| \hat{\eta}_k \|_{L^p} \| p_k \phi \|_{L^{p'}},
\]

and thus we get

\[
\| p_k Q(g, f) \|_{L^p} \leq C 2^{k\nu} \| g \|_{L^1} \| \hat{\eta}_k \|_{L^p},
\]

which gives Theorem 7.1.2 in case \( s = 0 \). The other cases of \( s \) use Galilean invariance of the collision operator and interpolation.

### 7.5.3 Proof of Theorem 7.1.3

Again, we first consider the case \( s = 0 \).

Having in mind decomposition (7.15) of \( Q(g, f) \), we first deal with first term of decomposition (7.15) and more explicitly with \( Q_\Delta(g, f) \). This is similar to the gain term in the cutoff case.

Since \( |\Phi(|v|) - \Phi(|v^+|)| \leq C |v^+ - v| \leq |v| \sin(\theta/2) \), it follows that \( \sin^{n-2} \theta b(\frac{v}{|v|} \cdot \sigma) \cdot |\Phi(|v|) - \Phi(|v^+|)| \leq C |v| \theta^{-\nu} \), which is integrable w.r.t. variable \( \theta \).

A better estimation can be obtained if we really use the facts that \( \Phi = w^7 \) and that \( |v^+| \) is bounded below and above by a constant times \( |v| \), getting \( |\Phi(|v|) - \Phi(|v^+|)| \leq C |v|^{7-1} |v^+ - v| \leq |v|^7 \sin(\frac{\sigma}{2}) \).

Thus, for all \( 1 \leq p \leq +\infty \),

\[
\| Q_\Delta(g, f) \|_{L^p} \leq C \| g \|_{L^1} \| f \|_{L^p},
\]

and thus

\[
\| Q_\Delta(g, f) \|_{B^{-\nu}_{p,q}} \leq C \| g \|_{L^1} \| f \|_{L^p}.
\]
It remains to analyze the most difficult term $Q_\Delta(g,f)$, given by (7.17). Applying $p_k$ on it, and testing against a test function $\phi$, we find

$$< p_k Q_\Delta(g,f); \phi > = \int_{\mathbb{R}^n} dv_\ast g(v_\ast) \int_{\mathbb{R}^n} dv f(v) \tau_{-v_\ast} \circ T^1 \circ (\Phi \tau_{v_\ast} p_k \phi),$$

that we can write as follows, using one commutator,

$$< p_k Q_\Delta(g,f); \phi > = < A_k; \phi > + < B_k; \phi >.$$

Above

(7.43) $$< A_k; \phi > \equiv \int_{\mathbb{R}^n} dv_\ast g(v_\ast) \int_{\mathbb{R}^n} dv f(v) \tau_{-v_\ast} \circ T^{1,k} (\Phi \tau_{v_\ast} p_k \phi)$$

and

(7.44) $$< B_k; \phi > \equiv \int_{\mathbb{R}^n} dv_\ast g(v_\ast) \int_{\mathbb{R}^n} dv f(v) \tau_{-v_\ast} \circ T^{1,k} (\Phi \tau_{v_\ast} \tilde{p}_k (\tau_{v_\ast} \phi)).$$

For the term $< A_k; \phi >$, let us note firstly that

$$< A_k; \phi > = \int_{\mathbb{R}^n} dv_\ast g(v_\ast) \int_{\mathbb{R}^n} dv f(v) \tau_{-v_\ast} \circ T^{1,k} (\Phi \tau_{v_\ast} \phi),$$

where we have now introduced operator $T^{1,k}$. The important point is to notice that, on Fourier side, the spectrum of $T^{1,k}(.)$ is supported on an annulus $\sim 2^k$, similarly to the usual Maxwellian case. It follows that we can also replace $f$ par $\tilde{hf}$ for another Littlewood-Paley decomposition.

Therefore, we get

$$| < A_k; \phi > | \leq C \int_{\mathbb{R}^n} dv_\ast |g(v_\ast)| \| \tilde{h} f \|_{L^p} \| \tau_{-v_\ast} \circ T^{1,k} (\Phi \tau_{v_\ast} p_k \phi) \|_{L^p_{-\gamma}},$$

$$\leq C 2^{k\nu} \int_{\mathbb{R}^n} dv_\ast |g(v_\ast)| \leq \nu \nu' \| \tilde{h} f \|_{L^p} \| (\Phi \tau_{v_\ast} \phi) \|_{L^p'},$$

and thus we find

$$| < A_k; \phi > | \leq C 2^{k\nu} \| g \|_{L^1_{-\nu+\gamma}} \| \tilde{h} f \|_{L^p} \| \phi \|_{L^p_{\gamma'}}.$$

Finally, we have obtained

$$\| A_k \|_{L^p_{-\gamma+}} \leq C 2^{k\nu} \| g \|_{L^1_{-\nu+\gamma}} \| \tilde{h} f \|_{L^p},$$

and similarly, with a better weight

$$\| A_k \|_{L^p} \leq C 2^{k\nu} \| g \|_{L^1_{-\nu+\gamma}} \| \tilde{h} f \|_{L^p_{\nu+\gamma}}.$$
Turning to the term $< B_k; \phi >$, one has

$$| < B_k; \phi > | \leq C \int_{\mathbb{R}^n} dv_s |g(v_s)||f||_{L^p} \| T^1 \circ ([\Phi, \tilde{p}_k]\tau_{v_s}\phi) \|_{L^{p'}}.$$ 

Then, we note that

$$\| T^1 \circ ([\Phi, \tilde{p}_k]\tau_{v_s}\phi) \|_{L^{p'}} \leq C \sum_j \| T^1j [\Phi, \tilde{p}_k]\tau_{v_s}\phi) \|_{L^{p'}} \leq C \sum_j 2^{j\nu} \| p_j [\Phi, \tilde{p}_k]\tau_{v_s}\phi) \|_{L^{p'}}$$

We use the fact that, for $0 \leq s \leq 1$, the operator $[\Phi, \tilde{p}_k]$ maps $L^q$ into $W^{s,q}$, with norm $(\frac{1}{2\nu})^{1-s}$.

It follows that, using Hölder inequality, one has

$$\| T^1 \circ ([\Phi, \tilde{p}_k]\tau_{v_s}\phi) \|_{L^{p'}} \leq C \frac{1}{2^{k(1-\nu-\varepsilon)}} \| \phi \|_{L^{p'}};$$

for all $\varepsilon > 0$ such that $\nu + \varepsilon < 1$, and thus, for such values of $\varepsilon$,

$$\| B_k \|_{L^p} \leq C \frac{1}{2^{k(1-\nu-\varepsilon)}} \| g \|_{L^1} \| f \|_{L^p}.$$ 

Gluing these two estimates on $A_k$ and $B_k$, we have obtained

$$\| Q_A(g,f) \|_{B_{p,q}^{-\nu}} \leq C \| g \|_{L^1_{\nu+\gamma^*}} \{ \| f \|_{L^p} + \| f \|_{B_{p,q,v}^0} \}.$$

This gives the result in case $s = 0$, other values of $s$ being deduced using Galilean invariance of the collision operator and interpolation.

**Remarks 7.5.1** We consider here very briefly the singular case $\Phi(|v|) = c|v|^\gamma$, with $s = 0$, but this case remains to be studied much more deeply.

For $Q_A(g,f)$, we proceed differently than for the smooth case.

If $\Phi = c|v|^1$, it follows that $|\Phi(|v|) - \Phi(|v^+|)| \leq C|v^+ - v| \leq |v|\sin(\frac{\theta}{2})$, and thus $\sin^n \theta b(\frac{\theta}{|v|} \cdot \sigma) \cdot |\Phi(|v|) - \Phi(|v^+|)| \leq C|v|^\theta \nu$, which is again integrable on the unit sphere.

In the more general case $\Phi(|v|) = C|v|^\gamma$, $\gamma > 0$, one has similar estimations.

For instance, we can write, for $|v| \neq 0$, $\Phi(|v|) - \Phi(|v^+|) = \int_{|v^+|}^{|v|} \Phi'(s)ds$. We can assume $\gamma \neq 1$, since this case has been dealt just above.

Then, if $\gamma > 1$, one has since $\Phi$ is convex, and $|v^+| \leq |v|$, that $|\Phi(|v|) - \Phi(|v^+|)| \leq C|v^+ - v| |v|^{\gamma-1}$. If $\gamma < 1$, a similar estimate also holds true, but with $|v^+|^{\gamma-1}$ in lieu of $|v|^{\gamma-1}$. But since $|v^+|$ is bounded below and above by a constant times $|v|$, the same estimate also holds true. The same argument also works if $\gamma \leq 0$.

In any case, we have $|\Phi(|v|) - \Phi(|v^+|)| \leq C|v|^{\gamma} \sin(\frac{\theta}{2})$. Thus, for all $1 \leq p \leq +\infty$,

$$\| Q_A(g,f) \|_{L^p} \leq C \| g \|_{L^1} \| f \|_{L^p}.$$
and thus
\[ \|Q_\Delta(g, f)\|_{B_{p,q}^\nu} \leq C\|g\|_{L_1^q} \|f\|_{L_p^\nu}. \]

The remaining term \( Q_\Delta(g, f) \) involves the commutator \([\Phi, \tilde{p}_k]\). This is at this place that the situation is worse, essentially when \( \gamma \leq 0 \). If \( \gamma > 0 \), we still can get some negative power of \( 2^k \), more precisely \( 2^{-k\gamma} \) instead of \( 2^{-k} \), and thus if \( \gamma > \nu \), it is still possible to get functional estimates.

Since it is not clear how to consider other values of \( \gamma \), we skip the details.

\[ \square \]

### 7.6 Calderon-Zygmund type estimates for the singular part

The way we are going to prove our estimates will be based on some equivalent formulations on the full kernel.

The first one is given in terms of Fourier transform by

\[ K^{\Phi,k}(v, y) = \int_{S^{n-1}} \hat{\phi}_k(\xi) e^{i\xi \cdot (v-y)} \left\{ e^{-i\xi \cdot v} - 1 \right\} b\left( \frac{\xi}{|\xi|} \cdot \sigma \right) \Phi(|v|), \]

where we have used the change of variables in \( \sigma \), exchanging \( \frac{v}{|v|} \) and \( \frac{\xi}{|\xi|} \).

Let us also note that
\[ K^{\Phi,k}(v, y) = K^{\Phi,k}(v, v - y), \]

where
\[ K^{\Phi,k}(U, V) = \int_{S^{n-1}} \hat{\phi}_k(\xi) e^{i\xi \cdot V} \left\{ e^{-i\xi \cdot U} - 1 \right\} b\left( \frac{\xi}{|\xi|} \cdot \sigma \right) \Phi(|U|). \]

Without Fourier transform, it is given by
\[ \tilde{K}^{\Phi,k}(U, V) = \int_{S^{n-1}} \left\{ \tilde{\phi}_k(-U^+ + V) - \tilde{\phi}_k(V) \right\} b\left( \frac{U}{|U|} \cdot \sigma \right) \Phi(|U|), \]

which writes also as (since \(|U^-|^2 = \frac{|U|^2}{2} (1 - \frac{U}{|U|} \cdot \sigma)\))
\[ \tilde{K}^{\Phi,k}(U, V) = \int_{S^{n-1}} \left\{ \tilde{\phi}_k\left( \sqrt{|U^-|^2 + |V|^2 - 2U^-V} \right) - \tilde{\phi}_k\left( \sqrt{|V|^2} \right) \right\} b\left( \frac{U}{|U|} \cdot \sigma \right) \Phi(|U|) \]
\[ = \int_{S^{n-1}} \left\{ \tilde{\phi}_k\left( \sqrt{\frac{|U|^2}{2} \left( 1 - \frac{U}{|U|} \cdot \sigma \right) + |V|^2 - 2U^-V} \right) - \tilde{\phi}_k\left( \sqrt{|V|^2} \right) \right\} b\left( \frac{U}{|U|} \cdot \sigma \right) \Phi(|U|) \]
\[ = \int_{S^{n-1}} d\sigma \left\{ \tilde{\phi}_k\left( \sqrt{\frac{|U|^2}{2} \left( 1 - \frac{V}{|V|} \cdot \sigma \right) + |V|^2 - 2U^-V} \right) - \tilde{\phi}_k\left( \sqrt{|V|^2} \right) \right\} b\left( \frac{V}{|V|} \cdot \sigma \right) \Phi(|U|). \]

To get the last formula, we have used the change of variables in \( \sigma \) exchanging \( \frac{U}{|U|} \) and \( \frac{V}{|V|} \).
7.6.1 Getting the first estimate in Theorem 7.1.4

In view of the above computations, we need to analyze the following singular kernel, and more precisely
to get punctual estimates, compared to the averaged estimations obtained in the previous sections, of
\[(7.48) \tilde{K}^{\Phi,k}_{\text{sing}}(U,V) \equiv \int_{S^{n-1}, \sin \frac{\theta}{2} \leq \frac{1}{2k}} \{\Phi_k(-U^- + V) - \tilde{\phi}_k(V)\} b\left(\frac{U}{|U|}, \sigma\right) d\sigma \Phi(|U|).\]

We shall prove the following Lemma, giving the first estimate, but it is important to note that the
proof below contains more estimates, which will be useful in the sequel.

**Lemma 7.6.1** Assume that \((1 + |U|)^{n+\nu} \Phi \in L^\infty\). Then one has
\[|\tilde{K}^{\Phi,k}_{\text{sing}}(U,V)| \leq C 2^{k\nu} \frac{1}{|V|^n}.\]

Proof of Lemma 7.6.1:

We note that if \(|V| \geq 2 \frac{\sqrt{2} - 1}{2k} |U|\), then \(|V| \geq 2 \sin \frac{\theta}{2} |U| \geq 2|U^-|\). In that case, it follows that
\[|\tilde{\phi}_k(-U^- + V) - \tilde{\phi}_k(V)| \leq C \frac{|U^-|}{|V|^{n+1}} \leq C \sin \frac{\theta}{2} \frac{|U|}{|V|^{n+1}}.\]

We therefore cut \(\tilde{K}^{\Phi,k}_{\text{sing}}\) into two parts as
\[(7.49) \tilde{K}^{\Phi,k}_{\text{sing}} = \tilde{K}^{\Phi,k}_{\text{sing}}|_{|V|\geq 2 \frac{\sqrt{2} - 1}{2k} |U|} + \tilde{K}^{\Phi,k}_{\text{sing}}|_{|V|\leq 2 \frac{\sqrt{2} - 1}{2k} |U|}.\]

Using the above estimates, one has
\[(7.50) |\tilde{K}^{\Phi,k}_{\text{sing}}|_{|V|\geq 2 \frac{\sqrt{2} - 1}{2k} |U|} \leq C \frac{1}{|V|^{n+1}} 2^{k\nu} \frac{2^{k\nu}}{|U|} |\Phi(|U|)|_{|V|\geq 2 \frac{\sqrt{2} - 1}{2k} |U|}.\]

Since we want to decrease the power \(n + 1\) on \(V\), we simply use the fact that we are on the above
characteristic set to find that
\[(7.51) |\tilde{K}^{\Phi,k}_{\text{sing}}|_{|V|\geq 2 \frac{\sqrt{2} - 1}{2k} |U|} \leq C 2^{k\nu} \frac{2^{k\nu}}{|U|} |\Phi(|U|)|_{|V|\geq 2 \frac{\sqrt{2} - 1}{2k} |U|}.\]

For the second term in \((7.49)\), an upper bound is given as follows
\[(7.52) |\tilde{K}^{\Phi,k}_{\text{sing}}|_{|V|\leq 2 \frac{\sqrt{2} - 1}{2k} |U|} \leq C 2^{k\nu} 2^{k\nu} \frac{1}{|U|} \int_{0}^{\frac{\pi}{2}} \min\{\sin \frac{\theta}{2}, \frac{1}{2k |U|}\} \theta^{\nu-1} \equiv A.\]

We separate this last estimation according to whether or not \(|U| \leq C\), so that the minimum is \(\sin \frac{\theta}{2}\),
that is
\[(7.53) A \leq A_{|U|\leq C} + A_{|U|\geq C}.\]
For the first term, we find

\begin{equation}
|A|_{U \leq C} \leq C 2^{nk} 2^{k} |U| \Phi(|U|)|U| \leq C |U| \leq 2 \frac{\sqrt{2}}{\sqrt{k}} |U|.
\end{equation}

In particular, using again the characteristic set, we find

\begin{equation}
|A|_{U \leq C} \leq C 2^{k'} \frac{1}{|V|^n} |U|^{n+1} \Phi(|U|) |U| \leq C |U| \leq 2 \frac{\sqrt{2}}{\sqrt{k}} |U|.
\end{equation}

Finally, for the second term, we find

\begin{equation}
|A|_{U \geq C} \leq C 2^{nk} 2^{k} |U| \Phi(|U|)|U| \leq C |U| \leq 2 \frac{\sqrt{2}}{\sqrt{k}} |U|,
\end{equation}

which again using the characteristic set gives

\begin{equation}
|A|_{U \geq C} \leq C \frac{1}{|V|^n} |U|^{n+1} \Phi(|U|) |U| \leq C |U| \leq 2 \frac{\sqrt{2}}{\sqrt{k}} |U|.
\end{equation}

It remains to collect (7.51), (7.55) and (7.57) to get the final estimate.

Let us note that, by using Taylor formula correctly, formula (7.50) can be improved into

\begin{equation}
|K_{\text{sing}}|_{|V| \geq 2 \frac{\sqrt{2}}{\sqrt{k}} |U|} \leq C 2^{(n+1)k} \frac{1}{1 + |2^k V|^n + 1} 2^{k'} |U| \Phi(|U|) |U| \leq 2 \frac{\sqrt{2}}{\sqrt{k}} |U|.
\end{equation}

On the other hand, using the characteristic functions, it follows also that

\begin{equation}
|A|_{U \leq C} \leq C (1 + |U|)^{n+1} 2^{nk} 2^{k} |U| \Phi(|U|) |U| \leq C |U| \leq 2 \frac{\sqrt{2}}{\sqrt{k}} |U|
\end{equation}

and

\begin{equation}
|A|_{U \geq C} \leq C (1 + |U|)^{n+1} 2^{nk} 2^{k} |U| \Phi(|U|) |U| \leq C |U| \leq 2 \frac{\sqrt{2}}{\sqrt{k}} |U|.
\end{equation}

Remarks 7.6.1

1. It is important to notice that the proof requires a better behavior with respect to the weight than what is needed in Theorem 7.1.1, for the \( L^1 \) norm w.r.t. variable \( v \) or \( y \).

2. The assumption on \( \Phi \) is only needed for large \( |v| \). For smaller values of \( |v| \), the assumption \( \Phi \in L^\infty \) is enough.
7.6.2 The second estimate

In order to check the Calderon-Zygmund conditions and more precisely the fact that in some cases, we have an associated standard kernel, we are going to estimate derivatives of $\tilde{K}^{\Phi,k}_{\text{sing}}(U,V)$ which are of course related to derivatives of $K^{\Phi,k}_{\text{sing}}(v,y)$.

The precise result we want to prove is given by the following Lemma, ending the proof of Theorem 7.1.4

**Lemma 7.6.2** Assume that $|U|\Phi(|U|)$ is bounded (for all $|U|$) and that $|U|^{n+1+\nu}\Phi(|U|)$ is bounded for large $|U|$. Furthermore assume that $|U|\nabla \Phi$ is bounded for all $|U|$ and that $|U|^{n+1+\nu}\nabla \Phi$ is bounded for large $|U|$. Then

$$|\nabla V \tilde{K}^{\Phi,k}_{\text{sing}}| \leq C2^{k\nu} \frac{1}{|V|^{n+1}} \text{ and } |\nabla U \tilde{K}^{\Phi,k}_{\text{sing}}| \leq C2^{k(\nu-1)} \frac{1}{|V|^{n+1}}.$$ 

Proof of Lemma 7.6.2: we start by estimating the simplest one, namely $\nabla V \tilde{K}^{\Phi,k}_{\text{sing}}(U,V)$, using formula (7.46).

In this case, the proof follows the lines of Lemma 7.6.1, taking into account the fact that there is a supplementary power $2^k$, everywhere.

Firstly, (7.50) gives

$$|\nabla V \tilde{K}^{\Phi,k}_{\text{sing}}|_{|V|\geq 2^{\frac{3n+1}{2}}|U|} \leq C \frac{1}{|V|^{n+1}} 2^{k\nu} |U|\Phi(|U|)_{|V|\geq 2^{\frac{3n+1}{2}}|U|}. $$

Then, formula (7.54) gives

$$|\nabla V \tilde{K}^{\Phi,k}_{\text{sing}}|_{|V|\leq 2^{\frac{3n+1}{2}}|U|} \leq C 2^{(n+1)k\nu} |U|\Phi(|U|)_{|V|\leq 2^{\frac{3n+1}{2}}|U|}. $$

Finally, formula (7.56) gives the last piece

$$|\nabla V \tilde{K}^{\Phi,k}_{\text{sing}}|_{|V|\geq 2^{\frac{3n+1}{2}}|U|} \leq C 2^{(n+1)k\nu} |1 + |U|\nu|\Phi(|U|)_{|V|\leq 2^{\frac{3n+1}{2}}|U|}. $$

In view of the above formula, we get the first estimate on the gradient with respect to variable $V$.

Turning to the derivative w.r.t. variable $U$, it is enough to study $\tilde{K}^{1,k}_{\text{sing}}$, corresponding to the case $\Phi = 1$.

Using the last formula, one has

$$\nabla U \tilde{K}^{1,k}_{\text{sing}}(U,V) = \int_{S^{n-1}, \sin \frac{\theta}{2} \leq C^n \frac{1}{2^n}} b(V \cdot \sigma) \frac{1}{2 \sqrt{\frac{E^\nu}{2} \left(1 - \frac{V}{|V|} \cdot \sigma\right) + \frac{1}{|V|^2} - 2U.V^-} \times$$

$$\times 2^{(n+1)k} (\partial \tilde{\Phi})(2^k \sqrt{\frac{|U|^2}{2} \left(1 - \frac{V}{|V|} \cdot \sigma\right) + |V|^2 - 2U.V^-}) \{U(1 - \frac{V}{|V|} \cdot \sigma) - 2V^-\}. $$

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we find, using the above change of variables back, the important formula
\[ |U(1 - \frac{V}{|V|}.\sigma) - 2V^-|^2 = |U|^2 (1 - \frac{V}{|V|}.\sigma)^2 - 4(1 - \frac{V}{|V|}.\sigma)U.V^- + 4 |V^-|^2 \]
\[ = 2(1 - \frac{V}{|V|}.\sigma)\{ \frac{|U|^2}{2} (1 - \frac{V}{|V|}.\sigma) - 2U.V^- + |V|^2 \}, \]

Noticing that
\[
|\nabla U\tilde{K}^{1,k}_{\text{sing}}(U, V)| \leq C2^{(n+1)k} \int_{S^{n-1}, \sin \frac{\theta}{2} \leq C^{\frac{1}{2k}}} |\nabla \tilde{\phi}(2^{k}(-U^- + V))| \sin \frac{\theta}{2} d\theta,
\]
we find, using the above change of variables back, the important formula
\[ |\nabla U\tilde{K}^{1,k}_{\text{sing}}(U, V)| \leq C2^{(n+1)k} \int_{S^{n-1}, \sin \frac{\theta}{2} \leq C^{\frac{1}{2k}}} |\nabla \tilde{\phi}(2^{k}(-U^- + V)) - \nabla \tilde{\phi}(2^{k}V)| \sin \frac{\theta}{2} d\theta + C2^{n+2\nu} |\nabla \tilde{\phi}(2^{k}V)| \]
with clear notations. For $E_2$, one has
\[ E_2 \leq C2^{(n+1)k} \frac{1}{|V|^n+1}. \]

As for $E_1$, this is similar to the gradient w.r.t. variable $V$, up to a supplementary term $\sin \frac{\theta}{2}$ which will modify the formula by changing $\nu$ into $\nu - 1$. We find
\[ |E_1|_{|V| \geq 2 \sqrt[4]{\frac{1}{2k}} |U|} \leq C \frac{1}{|V|^n+1} 2^{(n+1)k} |U|_{|V| \geq 2 \sqrt[4]{\frac{1}{2k}} |U|}, \]
\[ |E_1|_{|V| \leq 2 \sqrt[4]{\frac{1}{2k}} |U|} \leq C2^{(n+1)k} 2^{(n+1)k} |U|_{|V| \leq 2 \sqrt[4]{\frac{1}{2k}} |U|} \leq C, \]
and
\[ |E_1|_{|V| \leq 2 \sqrt[4]{\frac{1}{2k}} |U|} \leq C2^{(n+1)k} 2^{(n+1)k} |1 + |U|^{(n+1)k}|_{|V| \leq 2 \sqrt[4]{\frac{1}{2k}} |U|} \leq C, \]
which is enough to get the bounds on $\nabla U\tilde{K}^{1,k}_{\text{sing}}$. It remains to write
\[
\nabla U\tilde{K}^{\Phi,k}_{\text{sing}}(U, V) = \nabla U\tilde{K}^{1,k}_{\text{sing}}(U, V),\Phi(|U|) + \tilde{K}^{1,k}_{\text{sing}}(U, V).\nabla \Phi(|U|)
\]
Let us remark that, in fact, a much more precise estimate than (7.67) follows from Taylor formula as follows

\[
|E_1|_{V|\geq 2^{\frac{2}{\nu}}|U|} \leq C_2^{k(n+1)} \frac{1}{1 + |2^k V|^{n+1}} 2^{k(n-1)}|U||_{V|\geq 2^{\frac{2}{\nu}}|U|}.
\]

Furthermore, by using the precise characteristic sets, formula (7.68) and (7.69) yields also

\[
|E_1|_{V|\leq 2^{\frac{2}{\nu}}|U|\leq C} \leq C_2^{2(n+1)k} 2^{k(n-1)}|U||_{V|\leq 2^{\frac{2}{\nu}}|U|} (1 + |U|)^{n+1} (1 + |2^k V|)^{n+1} |U|\leq C
\]

and

\[
|E_1|_{V|\leq 2^{\frac{2}{\nu}}|U|\geq C} \leq C_2^{2(n+1)k} 2^{k(n-1)} |1 + |U|^{n+1} (1 + |2^k V|)^{n+1} |V|\leq 2^{\frac{2}{\nu}}|U| |U|\geq C.
\]

Of course, the term \(E_2\) can be estimated as

\[
E_2 \leq C_2^{n+1} 2^{k\nu} \frac{1}{1 + |2^k V|^{n+1}}.
\]

### 7.6.3 Commutators estimates and one step iteration

Recall that our interest is in the following operator

\[
U^\Phi_k f(v) = T^\Phi_k p_k f(v) = \int_{\mathbb{R}^n} \tilde{K}^\Phi_k (v, y) p_k f(y) dy.
\]

Using the notation \(g(y) = p_k f(y)\), it follows that

\[
U^\Phi_k f(v) = \int_{\mathbb{R}^n} \tilde{K}^\Phi_k (v, y - z) g(y) dy = \int_{\mathbb{R}^n} dz \tilde{K}^\Phi_k (v, z) g(v - z) dv.
\]

with clear notations.

If we apply the operator \(p_j\) (with respect to the variable \(v\)), we get

\[
p_j U^\Phi_k f(v) = \int_{\mathbb{R}^n} dz \ p_j \left( \tilde{K}^\Phi_k (v, z) (\tau_z g)(v) \right) = A_{sing,j,k} f(v) + B_{sing,j,k} f(v),
\]

where \(A_{sing,j,k}\) and \(B_{sing,j,k}\) are the two operators defined by

\[
A_{sing,j,k} f(v) = \int_{\mathbb{R}^n} dz \tilde{K}^\Phi_k (v, z) p_j (\tau_z g)(v) dz
\]

and

\[
B_{sing,j,k} f(v) = \int_{\mathbb{R}^n} dz \left[ p_j, \tilde{K}^\Phi_k (\tau_z g)(v) \right] (\tau_z g)(v).
\]

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Here, the brackets $[,]$ refer to the commutator between the convolution operator $p_j$ and the multiplication operator just mentioned.

By definition, one has

$$B_{\text{sing}, j, k} f(v) = \int_{\mathbb{R}^n} \left\{ p_j \tilde{K}_{\text{sing}, z}^{\Phi, k}(v) (\tau - z g)(v) - \tilde{K}_{\text{sing}, z}^{\Phi, k} (v) p_j (\tau - z g)(v) \right\} dz$$

and thus we get (recalling $g = p_k f$)

$$(7.74) \quad B_{\text{sing}, j, k} f(v) = \int_{\mathbb{R}^n} \left\{ \int_{\mathbb{R}^n} \left\{ \tilde{K}_{\text{sing}}^{\Phi, k}(y, z) g(y - z) \phi_j(y - v) - \tilde{K}_{\text{sing}}^{\Phi, k}(v, z) g(y - z) \phi_j(y - v) \right\} dz \right\} g(y) dy.$$  

Let us set

$$(7.75) \quad A_{\text{sing}}^{\Phi, k}(v, y) = \int_{\mathbb{R}^n} \left\{ \tilde{K}_{\text{sing}}^{\Phi, k}(y + z, z) - \tilde{K}_{\text{sing}}^{\Phi, k}(v, z) \right\} \phi_j(y + z - v) dz.$$  

Let us first show a way to deduce functional properties on $B_{\text{sing}, j, k}$ by analyzing its kernel, which is of course given by $A_{\text{sing}}^{\Phi, k}$. We want in particular to apply again Schur’s test, and so we shall prove the following result

**Lemma 7.6.3** Assume that:

1. $|U|\Phi(|U|)$ is bounded for all $|U|$ and that $(1 + |U|)^n + \nu \Phi(|U|)$ is bounded for large $|U|$;

2. $|U|\nabla \Phi(|U|)$ is bounded for all $|U|$ and that $(1 + |U|)^{n+1 + \nu \Phi(|U|)}$ is bounded for large $|U|$;

Then the operator $B_{\text{sing}, j, k}$ given by (7.74) is bounded in any $L^p$, with an operator norm less than $C 2^k \nu$.

Proof of Lemma 7.6.3: It is only a matter to collect the final estimations in the proofs of Lemma 7.6.1 and 7.6.2 to see that the $L^1$ norm of $\sup_U \nabla U \tilde{K}_{\text{sing}}^{\Phi, k} w.r.t$ variable $V$ can be estimated from above by $2^k \nu$. Then using Taylor’s estimation in formula (7.75), we can estimate both $L^1$ norm w.r.t. variable $v$ or variable $y$ by the constant appearing in the statement of the Lemma, using the particular scaling of $\phi_j$. This ends the proof, by using Schur’s Lemma.

\[\square\]
7.6.4 A simple application

Let us recall that, for a smooth function, one has (duality)

$$<Q(g,f);\phi> = \int_{\mathbb{R}^n} dv \int_{\mathbb{R}^n} dv g(v_s)f(v)(T_{\nu_s}^\phi)(v) = \int_{\mathbb{R}^n} dv \int_{\mathbb{R}^n} dv g(v_s)f(v)(\tau_{-v_s}T_{\nu_s}^\phi)(v).$$

It follows that, for any $k \in \mathbb{N}_0$

$$<p_k Q^k_{sing}(g,f);\phi> = \int_{\mathbb{R}^n} dv f(v)T^{\Phi,k}_{sing,g}p_k \phi(v),$$

where the operator $T^{\Phi,k}_{sing,g}$ is defined by

$$T^{\Phi,k}_{sing,g}u(v) \equiv \int_{\mathbb{R}^n} \left\{ \int_{\mathbb{R}^n} dv g(v_s)K^{\Phi,k}_{sing}(v-v_s, y-v_s) \right\} u(y)dy,$$

so that its kernel is given by the bracket term, that we denote by $K^{\Phi,k}_{sing,g}(v,y)$.

The point is that, if $g$ belongs to $L^1$, then this kernel has exactly the same properties as $K^{\Phi,k}_{sing}$, in particular in the unweighted cases. Modifying this simple idea will prove our result.

We just explain the idea in the case $\Phi$ supported on $|v| \leq C1$ and $\gamma > 0$.

Note then that necessarily $\gamma + \nu \geq 0$. Furthermore, recall that we have assumed the hypothesis of Lemma 7.6.3, here of course for small $|U|$.

Starting from $(\tilde{p_k}p_k = p_k)$

$$<p_k Q^k_{sing}(g,f);\phi> = \int_{\mathbb{R}^n} \tilde{p}_k f(v) T^{\Phi,k}_{sing,g} p_k \phi(v) dv + \int_{\mathbb{R}^n} f(v)[\tilde{T}^{\Phi,k}_{sing,g},\tilde{p}_k]p_k \phi(v) dv,$$

we note that the commutator appearing in the second term is similar to the operator $B_{sing,k,k}$ and thus, applying Holder’s inequality, we get

$$(7.76)\hspace{1cm} |<p_k Q^k_{sing}(g,f);\phi>| \leq C2^{k\nu} \|g\|_{L^1} \|\tilde{p}_k f\|_{L^p} \|\phi\|_{L^{p'}} + C\frac{\sqrt{k\nu}}{2^{k}} \|g\|_{L^1} \|f\|_{L^p} \|\phi\|_{L^{p'}},$$

since we can apply Lemma 7.6.3.

It follows that

$$\|p_k Q^k_{sing}(g,f)\|_{L^p} \leq C2^{k\nu} \|g\|_{L^1} \|\tilde{p}_k f\|_{L^p} + C2^{k(\nu-1)} \|g\|_{L^1} \|f\|_{L^p},$$

so that raising to the power $q$ and multiplying by $2^{kq}$, it follows that

$$\sum_k 2^{kq} \|p_k Q^k_{sing}(g,f)\|^q_{L^p} \leq C \|g\|^q_{L^1} \sum_k C2^{k(s+\nu)q} \|\tilde{p}_k f\|^q_{L^p} + C \sum_k 2^{kq(\nu-1+s)} \|g\|^q_{L^1} \|f\|^q_{L^p}.$$ 

Clearly, the second series will be convergent for all values of $s$ such that $s < 1 - \nu$, and one can shows that it holds true for all values of $s$.

We note that we have obtained in this way the part of the norm in $B^s_{p,q}$ of the singular part. Since the associated kernel has good properties, it can be also used to deal with the other values of $\gamma$ and $\beta$.

But the open question is quite surprisingly concerned about the usual cutoff gain collision operator.
7.6.5 Higher Derivatives of the kernel

We want to deal with higher order derivatives with respect to $U$ of $\tilde{K}^{\Phi,k}(U,V)$ on inspection of formula (7.47). This point can be helpful for higher order commutators.

The point is to get other expressions for it.

During the proof of Lemma 7.6.2, we have seen that

$$\left\{ \frac{|U|^2}{2} (1 - \frac{V}{|V|} \sigma) - 2U.V^- + |V|^2 \right\} = |\sin \frac{\theta}{2} U - \frac{1}{\sin \frac{\theta}{2}} V^-| .$$

Here the angle $\theta$ is with respect to $V$.

Note that if we perform back the change of variables exchanging $\frac{U}{|U|}$ and $\frac{V}{|V|}$, then this term transforms as $|-U^- + V|$, something we have already used during the proof of Lemma 7.6.2.

In view of these facts, we see that we have firstly

$$\tilde{K}^{\Phi,k}(U,V) = \int_{\mathbb{S}^{n-1}} \left\{ \tilde{\phi}_k(\sin \frac{\theta}{2} U - \frac{1}{\sin \frac{\theta}{2}} V^-) - \tilde{\phi}_k(V) \right\} b(\frac{V}{|V|} \sigma) d\sigma.$$

where $\theta$ refers to the variable $V$.

We can forget $\Phi$ for the moment and tackle with the derivatives of $\tilde{K}^{1,k}(U,V)$.

For the first one, one has

$$\nabla_U \tilde{K}^{1,k}(U,V) = 2^{(n+1)k} \int_{\mathbb{S}^{n-1}} \sin \frac{\theta}{2} \nabla \tilde{\phi}(2^k(\sin \frac{\theta}{2} U - \frac{1}{\sin \frac{\theta}{2}} V^-)) b(\frac{V}{|V|} \sigma) d\sigma,$$

and again performing the change of variables in $\sigma$, exchanging $\frac{U}{|U|}$ and $\frac{V}{|V|}$, we find

$$\nabla_U \tilde{K}^{1,k}(U,V) = 2^{(n+1)k} \int_{\mathbb{S}^{n-1}} \sin \frac{\theta}{2} \nabla \tilde{\phi}(2^k(- U^- + V)) b(\frac{U}{|U|} \sigma) d\sigma.$$

The point is that we can now iterate very easily for the other derivatives.

In particular, we get easily the formula, for any multi-index $l$

$$\nabla_U^{l} \tilde{K}^{1,k}(U,V) = 2^{(n+|l|)k} \int_{\mathbb{S}^{n-1}} \sin |l| \frac{\theta}{2} \nabla^{l} \tilde{\phi}(2^k(- U^- + V)) b(\frac{U}{|U|} \sigma) d\sigma,$$

where again $\theta$ refers to the variable $U$.

Note the important fact that we have used radial functions.
Chapter 8

Functional estimates for the collision operator. High singularities

8.1 Introduction

In this Chapter, we consider higher singularities $1 \leq \nu < 2$. Recall that the case of small singularities was considered in Chapter 7.

The main purpose of this paper is to continue Chapter 7, keeping same notations, giving as far as possible precise functional properties of $Q(f)$ with respect to the functional properties of $f$ itself, in the case of cross-sections having higher singularities, more precisely assuming $1 \leq \nu < 2$, with the notations introduced just below.

Recall the following Boltzmann bilinear operator

$$Q(g, f) = \int_{\mathbb{R}^n} dv_s \int_{S^{n-1}} d\sigma B(v - v_s, \sigma)(g'_s f' - g_s f).$$ (8.1)

Similarly to Chapter 7, we make the following assumptions on the kernel $B$.

**Assumption (H)**

$$B(v - v_s, \sigma) = \Phi(|v - v_s|) b(\cos \theta), \quad \sin^{n-2} \theta b(\cos \theta) \sim K \theta^{-1-\nu},$$ (8.2)

$\nu > 0, K > 0$, where the kinetic part $\Phi$ at least satisfies

$$\Phi(|z|) \leq C|z|^\gamma$$ for $|z| \lesssim 1$ and $$\Phi(|z|) \leq C|z|^\beta$$ for $|z| \gtrsim 1$. (8.3)

with the following range of parameters

$$\gamma > -n, \beta \in \mathbb{R}, 1 \leq \nu < 2.$$ (8.4)
**Notation:** Letter $C$ stands for any constant taking possible different values at different places. We shall use the notation $< \cdot > = \sqrt{1 + | \cdot |^2}$. Weighted Lebesgue space $L^p_k(p \geq 1, k \in \mathbb{R})$ are defined classically, with the norm

$$
\| f \|_{L^p_k} = \left( \int | f(v) |^p < v >^{pk} dv \right)^{1/p} = \omega(v)
$$

and the convention

$$
\| f \|_{L^\infty_k} = \sup_{\nu \in \mathbb{R}} \| f(v) \|^p < v >^k.
$$

Then, we have, using notations from 7

*Theorem 8.1.1* Let Assumption (H) holds true.

1. If $\beta + \nu \leq 0$ and $\gamma + \nu \geq 1$, one has

$$
\| Q(g,f) \|_{b^{-\nu}_{1,\infty}} \leq C(\beta, \nu) \| g \|_{L^1} \| f \|_{L^1}.
$$

2. If $\beta + \nu \geq 0$ and $\gamma + \nu \geq 1$, one has for $\beta' \geq \beta + \nu$,

$$
\| Q(g,f) \|_{b_{1,\infty}^{-\nu}} \leq C(\beta, \nu) \| g \|_{L^1_{\beta'}} \| f \|_{L^1_{\beta'}}.
$$

*Theorem 8.1.2* (Maxwellian case) Assume that $\Phi(|v|) = 1$. Then $Q(g,f) = Q_{\max}(g,f)$ satisfies the following estimates, for all $s \in \mathbb{R}$, all $1 \leq p, q < \infty$:

$$
\| Q_{\max}(g,f) \|_{B^{s-\nu}_{p,q}} \lesssim \| g \|_{W^{s,1}_p} \| f \|_{B^{s-\nu}_{p,q}}.
$$

*Theorem 8.1.3* Assume that $\Phi(v) = c \omega^\gamma$ for some $\gamma < \nu$ and some positive constant $c$. Then, for all $s \in \mathbb{R}$, all $1 \leq p, q < \infty$, one has

$$
\| Q(g,f) \|_{B^{s-\nu}_{p,q}} \lesssim (\| g \|_{W^{s,1}_p} + \| g \|_{W^{s,1}_p} + \| g \|_{W^{s,1}_p + (\gamma - 1)^+} + \| g \|_{W^{s,1}_p + (\gamma - 1)^+})

\times (\| f \|_{W^{s,p}_p} + \| f \|_{W^{s,p}_{\nu + \gamma - 1}} + \| f \|_{W^{s,p}_{\nu + \gamma - 1}} + \| f \|_{B^{s-1}_{p,q,\nu + (\gamma - 1)^+}} + \| f \|_{B^{s-1}_{p,q,\nu + (\gamma - 1)^+}}).
$$

As in Chapter 7, the analysis of the bilinear Boltzmann operator (8.1) will be based on a detailed study of the following linear operator, for a given $v^*$,

$$
(8.5) \quad T^{\Phi}_{v^*} : f \mapsto \int_{S^{n-1}} B(v - v^*, \sigma)(f' - f) d\sigma,
$$

analysis being further simplified since

$$
(8.6) \quad T^{\Phi}_{v^*} = T_{-v^*} \circ T^{\Phi} \circ T_{v^*},
$$

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where $\tau_{v\ast}$ denotes the usual translation operator along vector $v\ast$ and $T^\Phi$ denotes the operator

\begin{equation}
T^\Phi f(v) \equiv \int_{S^{n-1}} B(v, \sigma) \{(f(v^+) - f(v))\} d\sigma, \quad v^\pm \equiv \frac{v \pm |v|\sigma}{2}.
\end{equation}

Again, as in Chapter 7, let $k \in \mathbb{N}$ be fixed. In view of Littlewood-Paley decompositions, we are led to consider the following operator, defined, for smooth functions $f = f(v), v \in \mathbb{R}^n$, by

\begin{equation}
U^{\Phi,k} f(v) \equiv \int_{S^{n-1}} B(v, \sigma) \{(p_k f(v^+) - p_k f(v))\} d\sigma, \quad v^\pm \equiv \frac{v \pm |v|\sigma}{2},
\end{equation}

and thus, formally,

\begin{equation}
T^\Phi f(v) = \sum_{k=0}^{\infty} U^{\Phi,k} f(v).
\end{equation}

Setting $p_k f = \varphi \ast f = g$, we note first that we can write $g = \tilde{\varphi} \ast g$, where $\tilde{\varphi}(\cdot) = 2^{nk} \tilde{\varphi}(2^k \cdot)$, $\varphi \in \mathcal{S}$ and $\tilde{\varphi}$ compactly supported. Then for any smooth function $g$,

\begin{equation}
T^{\Phi,k} g(v) = \int_{S^{n-1}} \{ (\tilde{\varphi}_k \ast g)(v^+) - (\tilde{\varphi}_k \ast g)(v) \} b\left(\frac{v}{|v|} \cdot \sigma\right) \Phi(|v|) d\sigma
\end{equation}

and the Schwartz kernel of $T^{\Phi,k}$ is given by

\begin{equation}
K^{\Phi,k}(v, y) = \int_{S^{n-1}} \{ \tilde{\varphi}_k(v^+ - y) - \tilde{\varphi}_k(v - y) \} b\left(\frac{v}{|v|} \cdot \sigma\right) \Phi(|v|) d\sigma.
\end{equation}

The operators above are related by the rules

\begin{equation}
U^{\Phi,k} f(v) = T^{\Phi,k}(p_k f)(v),
\end{equation}

\begin{equation}
T^\Phi f(v) = \sum_{k=0}^{\infty} T^{\Phi,k}(p_k f)(v).
\end{equation}

Our work is then reduced to a detailed study of the operator $T^{\Phi,k}$, starting with $L^p$ type spaces, for all $k \in \mathbb{N}$.

**Proposition 8.1.1** Under the above notations and Assumption (H), one has the following estimates:

1. If $\beta + \nu \leq 0$ and $\gamma + \nu \geq 1$, then

\begin{equation}
\int_{\mathbb{R}^n} |K^{\Phi,k}(v, y)| dy \leq C 2^{k\nu},
\end{equation}

\begin{equation}
\int_{\mathbb{R}^n} |K^{\Phi,k}(v, y)| dv \leq C 2^{k\nu}.
\end{equation}
2. If $\beta + \nu \geq 0$ and $\gamma + \nu \geq 1$, then
\[
\int_{\mathbb{R}^n} |K^{\Phi,k}(v,y)| dy \leq C 2^{k\nu} \{1 + |v|^{\beta + \nu}\},
\]
\[
\int_{\mathbb{R}^n} |K^{\Phi,k}(v,y)| dv \leq C 2^{k\nu} \{1 + |y|^{\beta + \nu}\}.
\]

3. If $\beta + \nu \leq 0$ and $\gamma + \nu \leq 1$, then
\[
\int_{\mathbb{R}^n} |K^{\Phi,k}(v,y)| dy \leq C 2^{k\nu} \{1 + |v|^\gamma + \nu - 1\},
\]
\[
\int_{\mathbb{R}^n} |K^{\Phi,k}(v,y)| dv \leq C \sup \{2^{k\nu}, 2^{(-\gamma)k}\} \{1 + |y|^{\beta + \nu}\}.
\]

4. If $\beta + \nu \geq 0$ and $\gamma + \nu \leq 1$, then
\[
\int_{\mathbb{R}^n} |K^{\Phi,k}(v,y)| dy \leq C 2^{k\nu} \{|v|^\gamma + \nu - 1 + |v|^{\beta + \nu}\},
\]
\[
\int_{\mathbb{R}^n} |K^{\Phi,k}(v,y)| dv \leq C \sup \{2^{k\nu}, 2^{(-\gamma)k}\} \{1 + |y|^{\beta + \nu}\}.
\]

As an immediate application, let us mention Corollary 8.1.1 Under case of (1) of Proposition 8.1.1, the linear operator $T^\Phi$ defined by (8.7) satisfies, for all $1 \leq p \leq \infty$
\[
T^\Phi : B^\nu_{p,1}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)
\]
with the modification $B^\nu_{p,1}$ to $b_{\infty,1}$ in case $p = \infty$.

Proposition 8.1.1 and Corollary 8.1.1 will give our first result, Theorem 8.1.1.

For the Maxwellian operator $Q_{\text{max}}(g,f)$, which is the case when $\Phi = 1$, corresponding to a sub-case of $\gamma = \beta = 0$, a special property will give Theorem 8.1.2 in combination with Proposition 8.1.1.

In order to get Theorem 8.1.3, we need to study commutators linked with $T^{\Phi,k}$. We adopt the same idea introduced in Chapter 7, to divide the $T^{\Phi,k}$ into two parts. That is
\[
T^\Phi \varphi(v) = T^\Phi_{\Delta} \varphi(v) + T^i(\Phi \varphi)(v)
\]
where
\[
T^\Phi_{\Delta} \varphi(v) = \int_{S_{n-1}} \{\Phi(v^+) - \Phi(v)\} \varphi(v^+) b\left(\frac{v}{|v|} \cdot \sigma\right) d\sigma
\]
and $T^1$ corresponds to the maxwellian case $\Phi = 1$. Due to the higher singularity, we shall use Taylor’s formula at order 2 and momentum transfer, as mentioned in [11], to cancel this high singularity. That is, we introduce a further splitting

\begin{equation}
T^\Phi_\Delta \varphi(v) = T^\Phi_\Delta,1 \varphi(v) + T^\Phi_\Delta,2 \varphi(v)
\end{equation}

where

\begin{equation}
T^\Phi_\Delta,1 \varphi(v) = - \int_{S^{n-1}} \nabla \Phi(v) \cdot (v^+ - v)(\varphi(v^+) - \varphi(v))b(\frac{v}{|v|} \cdot \sigma)d\sigma
\end{equation}

\begin{align}
T^\Phi_\Delta,2 \varphi(v) &= - \int_{S^{n-1}} \int_0^1 (1 - s)D^2 \Phi(v + s(v^+ - v)) \cdot (v^+ - v, v^+ - v)\varphi(v^+)b(\frac{v}{|v|} \cdot \sigma)dsd\sigma \\
&\quad - \int_{S^{n-1}} \nabla \Phi(v) \cdot (v^+ - v)\varphi(v)b(\frac{v}{|v|} \cdot \sigma)d\sigma \\
&= - \int_{S^{n-1}} \int_0^1 (1 - s)D^2 \Phi(v + s(v^+ - v)) \cdot (v^+ - v, v^+ - v)\varphi(v^+)b(\frac{v}{|v|} \cdot \sigma)dsd\sigma \\
&\quad - C\nabla \Phi(v) \cdot v\varphi(v)
\end{align}

It follows immediately that we have the decomposition

\begin{equation}
\langle Q(g,f); \varphi \rangle = \langle Q_\Delta,1(g,f); \varphi \rangle + \langle Q_\Delta,2(g,f); \varphi \rangle + \langle Q_\Delta(g,f); \varphi \rangle.
\end{equation}

Above, $Q_\Delta,1(g,f)$ and $Q_\Delta,2(g,f)$ are defined by duality as

\begin{equation}
\langle Q_\Delta,1(g,f); \varphi \rangle = \int_{\mathbb{R}^n} dv_\ast g(v_\ast) \int_{\mathbb{R}^n} df(v) \tau_{-v_{\ast}} \circ T^\Phi_\Delta,1 \circ \tau_{v_{\ast}} \varphi,
\end{equation}

and

\begin{equation}
\langle Q_\Delta,2(g,f); \varphi \rangle = \int_{\mathbb{R}^n} dv_\ast g(v_\ast) \int_{\mathbb{R}^n} df(v) \tau_{-v_{\ast}} \circ T^\Phi_\Delta,2 \circ \tau_{v_{\ast}} \varphi,
\end{equation}

while $Q_\Delta(g,f)$ is defined by duality as

\begin{equation}
\langle Q_\Delta(g,f); \varphi \rangle = \int_{\mathbb{R}^n} dv_\ast g(v_\ast) \int_{\mathbb{R}^n} df(v) \tau_{-v_{\ast}} \circ T^1 \circ (\Phi_{T_{\ast}v_{\ast}} \varphi).
\end{equation}

We note that the operator $Q_\Delta,2$ defined by (8.19) is similar to the usual gain term in the cutoff case which has been studied recently in [97]. While the more interesting point is that operator $T^\Phi_{\Delta,1} \circ p_k$ looks like operator $U^\Phi,k$, but with small singularity $\nu - 1$. This point, combined with the maxwellian case given by Theorem 8.1.2, using estimates for small singularity obtained in Chapter 7 and a commutator estimate, is enough to prove Theorem 8.1.3.
8.2 $L^1$ norm of the kernel with respect to variable $y$

This section is devoted to the proof of the following proposition

**Proposition 8.2.1** One has, for all functions $\Phi$

$$
\|K^{\Phi,k}(v,\cdot)\|_{L^1} \leq C^{2k\nu} \Phi(|v|)(|v|^\nu + 2^{-k}|v|^{\nu-1})
$$

and under the more precise Assumption (H) on $\Phi$, one has

$$
\|K^{\Phi,k}(v,\cdot)\|_{L^1} \leq C^{2k\nu}\{(|v|^\gamma + 2^{-k}|v|^\gamma - 1)|v|_1 \leq 1 + (|v|^{\beta + \nu} + 2^{-k}|v|^{\beta + \nu - 1})|2|_{\nu,1}\}.
$$

**Proof.** Recalling that

$$K^{\Phi,k}(v,y) = \int_{\mathbb{R}^n} \{\tilde{\varphi}_k(v+ - y) - \tilde{\varphi}_k(v - y)\} b(\frac{v}{|v|} \cdot \sigma) \Phi(|v|) d\sigma,$$

it follows that

$$
\|K^{\Phi,k}(v,\cdot)\|_{L^1} \leq \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \{\tilde{\varphi}_k(v+ - y) - \tilde{\varphi}_k(v - y)\} b(\frac{v}{|v|} \cdot \sigma) \Phi(|v|) \|_{2k|\nu|\sin \frac{\pi}{2} \leq 1} d\sigma \right| dy
$$

$$+ \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \{\tilde{\varphi}_k(v+ - y) - \tilde{\varphi}_k(v - y)\} b(\frac{v}{|v|} \cdot \sigma) \Phi(|v|) \|_{2k|\nu|\sin \frac{\pi}{2} \geq 1} d\sigma \right| dy 1_{2k|\nu|\sin \frac{\pi}{2} \geq 1} = A + B.
$$

Using Taylor’s expansion

$$
\tilde{\varphi}_k(v+ - y) - \tilde{\varphi}_k(v - y) = \nabla \tilde{\varphi}_k(v - y) \cdot (v+ - v) + |v+ - v|^2
$$

(8.21)

$$
\times \int_0^1 (1 - s) D^2 \tilde{\varphi}_k(v - y + s(v+ - v)) \cdot \left(\frac{v+ - v}{|v+ - v|}, \frac{v+ - v}{|v+ - v|}\right) ds,
$$

we deduce that

$$A \leq \frac{1}{2} \int_{\mathbb{R}^n} \left| \nabla \tilde{\varphi}_k(v - y) \cdot \frac{v}{|v|} \int_{\mathbb{R}^n} \Phi(|v|) \|v\| \|\frac{v}{|v|} \cdot \sigma\|_{2k|\nu|\sin \frac{\pi}{2} \leq 1} d\sigma \right| dy
$$

$$+ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_0^1 |v+ - v|^2 (1 - s) D^2 \tilde{\varphi}_k(v - y + s(v+ - v)) \|b(\frac{v}{|v|} \cdot \sigma) \Phi(|v|) \|_{2k|\nu|\sin \frac{\pi}{2} \leq 1} d\sigma dy
$$

$$\leq C(\tilde{\varphi}) (2^k|\nu|^{-1})^{2-\nu} 2^k|v|(1 + 2^k|v|)
$$

$$\leq C(\tilde{\varphi}) 2^{k\nu} \Phi(|v|) (|v|^\nu + 2^{-k}|v|^{\nu-1}).$$

Since

$$\|\tilde{\varphi}_k(v^+ - \cdot) - \tilde{\varphi}_k(v - \cdot)\|_{L^1} \leq C(\tilde{\varphi}),$$

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we conclude that
\[ B \leq C(\tilde{\varphi}) \Phi(|v|)(2^k|v|)^\nu. \]

Combining the estimates for $A$ with those for $B$, the proof is ended.

**Remark 8.2.1** We can get similar $L^1$ but weighted norm estimates if we replace $K^{\Phi,k}(v,\cdot)$ by $K^{\Phi,k}(v,\cdot) < v - \cdot >^\lambda$ where $0 \leq \lambda < \nu$. In fact, recalling that $\tilde{\varphi}_k(z) = 2^{nk}\tilde{\varphi}(2^kz)$, $< z >^\lambda$ turns to $< z/2^k >^\lambda$ when we change variables from $2^kz$ to $z$. Noting that $< v - y + s(v^+ - v) >^\lambda \leq < v - y >^\lambda < v - >^\lambda$, by the same computation as what we have done above, we also divide the kernel into two parts: term $A^*$ and term $B^*$. Then we have
\[ A^* \leq C(\tilde{\varphi},\lambda)2^{k\nu}\Phi(|v|)(|v|^\nu + 2^{-k}|v|^{\nu-1}). \]
As for the term $B^*$, it is slightly different from $B$. Now we have
\[ \|\tilde{\varphi}_k(v^+ - \cdot) - \tilde{\varphi}_k(v - \cdot)\|_{L^1} \leq C(\tilde{\varphi},\lambda)(1 + |v|^{\lambda}), \]
and then, we get
\[ B^* \leq C(\tilde{\varphi},\lambda)\int_{\mathbb{R}} \Phi(|v|)(1 + |v|^\lambda|\sin \frac{\theta}{2})^\lambda|\theta|^{-1 - \nu}2^k|v|\sin \frac{\theta}{2} \geq 2 \|2^k|v|\sin \frac{\theta}{2} \geq 1 \right) d\theta \]
\[ \leq C(\tilde{\varphi},\lambda)\Phi(|v|)(2^k|v|^\nu. \]
At last, we arrive at
\[ (8.22) \quad \|K^{\Phi,k}(v,\cdot) < v - >^\lambda \|_{L^1} \leq C(\tilde{\varphi},\lambda)2^{k\nu}\Phi(|v|)(|v|^\nu + 2^{-k}|v|^{\nu-1}). \]

### 8.3 $L^1$ norm of the kernel with respect to $v$ variables

Let $\delta > 0$ be fixed which will be chosen carefully below.

We start from the splitting
\[ K^{\Phi,k}(v,y) = \int_{S^{n-1}} \{\tilde{\varphi}_k(v^+ - y) - \tilde{\varphi}_k(v - y)\} b\left(\frac{v}{|v|} \cdot \sigma\right) \Phi(|v|) d\sigma \]
\[ = K^{\Phi,k}_{\delta,1}(v,y) + K^{\Phi,k}_{\delta,2}(v,y), \]
where we set
\[ (8.24) \quad K^{\Phi,k}_{\delta,1}(v,y) = \int_{S^{n-1}} \{\tilde{\varphi}_k(v^+ - y) - \tilde{\varphi}_k(v - y)\} b\left(\frac{v}{|v|} \cdot \sigma\right) \Phi(|v|) \left|\sin \frac{\theta}{2}\right| \geq \delta d\sigma, \]
\[ (8.25) \quad K^{\Phi,k}_{\delta,2}(v,y) = \int_{S^{n-1}} \{\tilde{\varphi}_k(v^+ - y) - \tilde{\varphi}_k(v - y)\} b\left(\frac{v}{|v|} \cdot \sigma\right) \Phi(|v|) \left|\sin \frac{\theta}{2}\right| \leq \delta d\sigma. \]

We shall refer to the first kernel (8.24) as the non singular part of the kernel and to the second kernel (8.25) as the singular one.
8.3.1 The non singular kernel $K_{\delta,1}^{\Phi,k}$-see formula (8.24)

This subsection is devoted to estimating the integration of the term $K_{\delta,1}^{\Phi,k}$ with respect to variable $v$. Since the whole computation is similar to Chapter 7, we omit the details and state the final result.

**Proposition 8.3.1** For the kernel $K_{\delta,1}^{\Phi,k}$ given by (8.24), one has

\[
\int_{\mathbb{R}^n} |K_{\delta,1}^{\Phi,k}(v,y)|dv \leq \frac{C}{\delta^\nu} + \|_{\beta+\nu \geq 0} \left\{ \frac{C}{\delta^\nu} \right\} + C |y|^{(\beta+\nu)}/\delta^\nu
\]

\[+ \sum_{\gamma \geq 1} \left\{ \int_{\gamma \geq 0} C \max\{1, C\}^{\beta+\nu} \right\}
\]

8.3.2 The singular kernel $K_{\delta,2}^{\Phi,k}$-see formula (8.25)

**Proposition 8.3.2** For the kernel $K_{\delta,2}^{\Phi,k}$ given by (8.25), one has

\[
\int_{\mathbb{R}^n} |K_{\delta,2}^{\Phi,k}(v,y)|dv \leq \sum_{\nu + \gamma \geq 1} C \left\{ \max\{C \delta, 1\}^{\gamma+n} + C (\tau)^{2(\gamma+\nu)}/\delta^{\nu+\gamma} \right\}
\]

where \( r > 1 \) can be chosen arbitrary and \( \tau \) is some positive constant such that \( \tau \geq \max\{2, |\beta + \nu|\} \).

For the proof, recalling first Taylor’s formula (8.21) and the definition of $K_{\delta,2}^{\Phi,k}$ (8.25), we obtain

\[
\int_{\mathbb{R}^n} |K_{\delta,2}^{\Phi,k}(v,y)|dv \leq E + F,
\]
where
\begin{align}
(8.26) &= \int_{\mathbb{R}^n} \left| \nabla \tilde{\varphi}_k(v - y) \cdot \frac{v}{|v|} \right|_{S^{n-1}} F(\Phi(|v|)|v|b(v, \frac{v}{|v|} \cdot \sigma)) |1 - \frac{v}{|v|} \cdot \sigma| |v|_{|v| \sin \frac{\pi}{2} \leq \delta} \, dv,
\end{align}
\begin{align}
F &= \int_{\mathbb{R}^n} \int_{S^{n-1}} \int_0^1 |v^+ - v|^2 (1 - s)|D^2 \tilde{\varphi}_k(v - y + s(v^+ - v))|b(v, \frac{v}{|v|} \cdot \sigma) \Phi(|v|) |v|_{|v| \sin \frac{\pi}{2} \leq \delta} \, ds \, d\sigma \, dv.
\end{align}
(8.27)

The proof of Proposition 8.3.2 will then be deduced from the next two results.

**Lemma 8.3.1** For \( E \) given by (8.26), one has
\begin{align}
E &\leq C 2^{(n+1)k} \{ \min \{ C \delta, 1 \} \}^{1+\gamma+n} + C \Pi_{\delta \geq 1} \{ I_{1+\beta \geq 0} 2^k (C \delta)^{1+\beta} + I_{-n \leq 1+\beta < 0} 2^{(n+1)k} (C \delta)^{1+\beta+n} \\
&+ I_{1, \beta < -2^k} \} + C \Pi_{\delta \leq 1} \{ I_{\nu + \gamma - 1 \geq 0} 2^k \delta^{-\nu+2} + I_{\nu + \gamma - 1 \leq 0} 2^k \delta^{\gamma+1} \} \\
&+ C 2^k \delta^{-\nu+2} (1 + |y|^\nu) + I_{\nu + 1 \leq 0} C 2^k \delta^{-\nu+2} \{ \max \{ C \delta, 1 \} \}^{\nu+\beta-1}
\end{align}

**Lemma 8.3.2** For \( F \) given by (8.27), one has
\begin{align}
F &\leq C (\tau)^{2^{(k-2)} \delta^{-n+\nu}} \{ \min \{ C \delta, 1 \} \}^{\gamma+n+2} + C 2^{(n+2)k} \{ 1 + (2^k)^{n/\nu} \} \{ \min \{ C \delta, 1 \} \}^{\gamma+n+2} \\
&+ C 2^{(n+2)k} \{ 1 + (2^k)^{n/\nu} \} \{ \Pi_{\delta \geq 2} 2^k (C \tau)^{2(2-k) \delta^{-2+\nu}} + \Pi_{\delta \leq 0} C (C \tau)^{2(2-k) \delta^{-\nu+2}} \\
&+ C 2^k (n+2) \{ 1 + (2^k)^{n/\nu} \} \{ \Pi_{\tau \geq 0} 2^k (2-k) \delta^{-\nu+2} + \Pi_{\tau \leq C \delta} \} \{ \min \{ C \delta, 1 \} \}^{\beta+\nu} \\
&+ C 2^k (n+2) \{ 1 + (2^k)^{n/\nu} \} \delta^{-\nu+2} \{ \max \{ C \delta, 1 \} \}^{\beta+\nu} \\
&+ C 2^k (n+2) \{ 1 + (2^k)^{n/\nu} \} \delta^{-\nu+n+2} \{ \max \{ C \delta, 1 \} \}^{\beta+\nu}.
\end{align}

**Proof of Lemma 8.3.1: the term \( E \)** We note that
\begin{align}
E &\leq C \int_{\mathbb{R}^n} |\nabla \tilde{\varphi}_k(v - y)| |v| \Phi(|v|) \int_0^{\frac{\pi}{2}} \theta^{-\nu+1} |v|_{|v| \sin \frac{\pi}{2} \leq \delta} \, d\theta \, dv \\
&\leq E_1 + E_2
\end{align}
where we cut the last integral according to whether or not \( \delta/|v| \geq \sqrt{2}/2 \) (for \( E_1 \), respectively for \( E_2 \)). If this is the case, i.e., for \( E_1 \), we abbreviate this as \( |v| \leq C \delta \).

It follows that
\begin{align}
(8.28) \quad E_1 &\leq C \int_{\mathbb{R}^n} |\nabla \tilde{\varphi}_k(v - y)| |v| \Phi(|v|)|v|_{|v| \leq C \delta} \, dv
\end{align}

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and thus

$$E_1 \leq E_{1,1} + E_{1,2}, \quad (8.29)$$

where

$$E_{1,1} = C \int_{\mathbb{R}^n} \left| \nabla \tilde{\phi}_k(v - y) \right| \left| v \right| \Phi(\left| v \right|) \mathbb{I}_{\left| v \right| \leq C\delta \mathbb{I}_{|v| \leq 1} dv, \quad (8.30)$$

$$E_{1,2} = C \int_{\mathbb{R}^n} \left| \nabla \tilde{\phi}_k(v - y) \right| \left| v \right| \Phi(\left| v \right|) \mathbb{I}_{\left| v \right| \leq C\delta \mathbb{I}_{|v| \geq 1} dv. \quad (8.31)$$

For $E_{1,1}$, one has

$$E_{1,1} \leq C \int_{\mathbb{R}^n} \left| \nabla \tilde{\phi}_k(v - y) \right| \left| v \right|^{1+\gamma} \mathbb{I}_{\left| v \right| \leq \min \{C\delta, 1\}} dv \quad (8.32)$$

while, for $E_{1,2}$, one has

$$E_{1,2} \leq C \int_{\mathbb{R}^n} \left| \nabla \tilde{\phi}_k(v - y) \right| \left| v \right|^{1+\beta} \mathbb{I}_{\left| v \right| \leq C\delta dv. \quad (8.33)$$

and thus

$$E_{1,2} \leq \begin{cases} 
C \mathbb{2}^{(n+1)k} \{\min \{C\delta, 1\}\}^{1+\gamma+n}, & 1 + \beta \geq 0 \\
C \mathbb{2}^{(n+1)k} \{\min \{C\delta, 1\}\}^{1+\beta+n}, & -n \leq 1 + \beta < 0 \\
C \mathbb{2}^{k} \mathbb{I}_{C\delta \geq 1}, & 1 + \beta < -n 
\end{cases} \quad (8.34)$$

Therefore, we have obtained the following result

**Lemma 8.3.3** For $E_1$ upper bounded by $(8.28)$, one has

$$E_1 \leq C \mathbb{2}^{(n+1)k} \{\min \{C\delta, 1\}\}^{1+\gamma+n} + C \mathbb{II}_{C\delta \geq 1} \{\mathbb{II}_{1+\beta \geq 0} \mathbb{2}^{k} (C\delta)^{1+\beta} + \mathbb{II}_{-n \leq 1+\beta < 0} \mathbb{2}^{(n+1)k} (C\delta)^{1+\beta+n} + \mathbb{II}_{1+\beta < -n} \mathbb{2}^{k}\}. \quad (8.35)$$

Now considering $E_2$, we have

$$E_2 \leq C \int_{\mathbb{R}^n} \left| \nabla \tilde{\phi}_k(v - y) \right| \left| v \right| \Phi(\left| v \right|) \mathbb{I}_{\left| v \right| \geq C\delta} \left( \frac{\delta}{|v|} \right)^{-\nu-2} dv, \quad (8.36)$$

and thus

$$E_2 \leq E_{2,1} + E_{2,2}.$$
where

\begin{align}
E_{2,1} & = C \delta^{-\nu+2} \int_{\mathbb{R}^n} |\nabla \tilde{\varphi}_k(v-y)||v|^\nu \Phi(|v|) |I_v|_{|v| \geq \delta |v| \leq 1} dv, \\
E_{2,2} & = C \delta^{-\nu+2} \int_{\mathbb{R}^n} |\nabla \tilde{\varphi}_k(v-y)||v|^\nu \Phi(|v|) |I_v|_{|v| \geq \delta |v| \geq 1} dv.
\end{align}

For \( E_{2,1} \), one has

\begin{equation}
E_{2,1} \leq C \delta^{-\nu+2} \int_{\mathbb{R}^n} |\nabla \tilde{\varphi}_k(v-y)||v|^\nu \gamma - 1 \leq 0
\end{equation}

and therefore

\begin{equation}
E_{2,1} \leq \begin{cases} C 2^k \delta^{-\nu+2} |I_{C \delta \leq 1}|, & \nu + \gamma - 1 \geq 0 \\
C 2^k \delta \gamma + 1 |I_{C \delta \leq 1}|, & \nu + \gamma - 1 \leq 0
\end{cases}
\end{equation}

Next, for \( E_{2,2} \), one has

\begin{equation}
E_{2,2} \leq C \delta^{-\nu+2} \int_{\mathbb{R}^n} |\nabla \tilde{\varphi}_k(v-y)||v|^\nu \beta - 1 \leq 0, \text{ we get}
\end{equation}

\begin{equation}
E_{2,2} \leq C 2^k \delta^{-\nu+2} \{ \max \{ C \delta, 1 \} \}^{\nu + \beta - 1}.
\end{equation}

Thus we obtained

**Lemma 8.3.4** For \( E_1 \) upper bounded by (8.35), one has

\begin{equation}
E_2 \leq C 1_{C \delta \leq 1} \{ \| v + \gamma - 1 \| 2^k \delta \nu + 2 + \| v + \gamma - 1 \| 2^k \delta \gamma + 1 \} \\
+ \| v + \beta - 1 \| 2^k \delta \nu + 2 (1 + |v|^{\nu + \beta}) + \| v + \beta - 1 \| 2^k \delta \nu + 2 \{ \max \{ C \delta, 1 \} \}^{\nu + \beta - 1}.
\end{equation}

Collecting Lemma 8.3.3 and Lemma 8.3.4 finally gives the statement of Lemma 8.3.1.

**Proof of Lemma 8.3.2: the term \( F \)** We note that

\begin{equation}
F \leq 2^{(n+2)} \int_{\mathbb{R}^n} \int_{S_{n-1}} \int_0^1 |v^\nu|^2 |D^2 \tilde{\varphi}(2^k (v-y + \sigma (v^\nu - v))) b(v \cdot \sigma) \Phi(|v|) |I_v|_{|v| \sin \frac{\pi}{2} \leq \delta} ds d\sigma dv \\
\leq G + H,
\end{equation}

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where we cut the last integral according to whether or not \( |v - y| \geq 2\delta \) (for \( G \), respectively for \( H \)).

For term \( G \), noting that \( |v^-| \leq \delta \) and \( |v - y| \geq 2\delta \), we find that
\[
|v - y + s(v^+ - v)| \geq |v - y| - s|v^+ - v|
\]
\[
\geq |v - y| - \delta
\]
\[
\geq \frac{|v - y|}{2}.
\]

Then we can deduce that
\[
G \leq 2^{k(n+2)} \int_{\mathbb{R}^n} \int_{S^{n-1}} \int_{0}^{1} |D^2 \tilde{\varphi}(2^k(v - y + s(v^+ - v)))|v - y + s(v^+ - v)|^{n+\tau} 
\]
\[
\times \frac{1}{|v - y + s(v^+ - v)|^{n+\tau}} |v^-|^2 b\left(\frac{v}{|v|} \right) \cdot \sigma \Phi(|v|) I_{|v| \sin \frac{\pi}{2} \leq \delta |v - y| \geq 2\delta} ds d\sigma dv
\]
\[
(8.44) \leq C(\tilde{\varphi}, \tau) 2^{k(2-\tau)} \int_{\mathbb{R}^n} \int_{S^{n-1}} \frac{|v^-|^2}{|v - y|^{n+\tau}} b\left(\frac{v}{|v|} \right) \cdot \sigma \Phi(|v|) I_{|v| \sin \frac{\pi}{2} \leq \delta |v - y| \geq 2\delta} ds d\sigma dv
\]
\[
(8.45) \leq G_1 + G_2,
\]
where we choose the positive constant \( \tau \) such that \( \tau \geq \max\{2, |\beta + \nu|\} \) and cut the last integral according to whether or not \( \delta/|v| \geq \sqrt{2}/2 \) (for \( G_1 \), respectively for \( G_2 \)). If this is the case, i.e., for \( G_1 \), we abbreviate this as \( |v| \leq C\delta \).

For \( G_1 \), one has
\[
G_1 \leq C(\tau) 2^{k(2-\tau)} \int_{\mathbb{R}^n} \frac{|v|^2}{|v - y|^{n+\tau}} \Phi(|v|) I_{|v| \leq C\delta \geq 2\delta} dv
\]
\[
\leq G_{1,1} + G_{1,2},
\]
where
\[
(8.46) G_{1,1} = C(\tau) 2^{k(2-\tau)} \int_{\mathbb{R}^n} \frac{|v|^{\gamma + 2}}{|v - y|^{n+\tau}} I_{|v| \leq C\delta} I_{|v - y| \geq 2\delta} dv,
\]
\[
(8.47) G_{1,2} = C(\tau) 2^{k(2-\tau)} \int_{\mathbb{R}^n} |v|^{\beta + 2} \frac{1}{|v - y|^{n+\tau}} I_{|v| \leq C\delta} I_{|v - y| \geq 2\delta} dv,
\]
It is easy to estimate the term \( G_{1,1} \), i.e.
\[
G_{1,1} \leq C(\tau) 2^{k(2-\tau)} \int_{\mathbb{R}^n} \frac{|v|^{\gamma + 2}}{|v - y|^{n+\tau}} I_{|v| \leq \min\{C\delta, 1\}} I_{|v - y| \geq 2\delta} dv
\]
\[
\leq C(\tau) 2^{k(2-\tau)} \delta^{-n+\tau} \{\min\{C\delta, 1\}\}^{\gamma + n + 2}.
\]
\[
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\]
As for the term $G_{1,2}$, we have

\[
G_{1,2} \leq C(\tau)2^{k(2-\tau)} \int_{\mathbb{R}^n} \frac{|v|^{\delta + 2}}{|v-y|^{n+\tau}} I_{1 \leq |v| \leq C\delta I_{|v-y| \geq 2\delta}} dv
\]

(8.49)

Next, we treat the term $G_2$. First, we have

\[
G_2 \leq C(\tau)2^{k(2-\tau)} \int_{\mathbb{R}^n} \frac{|v|^{\gamma + \nu}}{|v-y|^{n+\tau}} I_{|v| \geq C\delta I_{|v-y| \geq 2\delta}} dv
\]

(8.50)

where

\[
G_{2,1} = C(\tau)2^{k(2-\tau)}(\delta - \nu + 2) \int_{\mathbb{R}^n} \frac{|v|^{\gamma + \nu}}{|v-y|^{n+\tau}} I_{|v| \geq C\delta I_{|v-y| \geq 2\delta}} dv
\]

(8.51)

By similar computations, we have

\[
G_{2,1} \leq \left\{ \begin{array}{ll}
C(\tau)2^{k(2-\tau)}\beta - \nu + 2 I_{\delta \leq 1}, & \gamma + \nu \geq 0 \\
C(\tau)2^{k(2-\tau)}\beta - \nu + 2 I_{\delta \leq 1}, & \gamma + \nu \leq 0
\end{array} \right.
\]

(8.52)

and if $\beta + \nu \geq 0$,

\[
G_{2,2} \leq C(\tau)2^{k(2-\tau)}\delta - \nu + 2 \int_{\mathbb{R}^n} \frac{1}{|v-y|^{n+\tau}} (|v-y|^\nu + |y|^\beta + |y|^\nu + \beta) I_{|v| \geq \max\{C\delta, 1\} I_{|v-y| \geq 2\delta}} dv
\]

\[
\leq C(\tau)2^{k(2-\tau)}\delta - \nu + 2 \left\{ \delta - \nu + 2 \right\} + |y|^\nu + \beta \delta - \tau
\]

(8.53)

While for the case $\beta + \nu \leq 0$,

\[
G_{2,2} \leq C(\tau)2^{k(2-\tau)}(\delta - \nu + 2 - \tau) \max\{C\delta, 1\}^{\beta + \nu}
\]

(8.54)

All in all, we have the following result for the term $G$

**Lemma 8.3.5** For $G$ upper bounded by (8.44), one has

\[
G \leq C(\tau)2^{k(2-\tau)}\delta - (n+\tau)\left\{ \min\{C\delta, 1\} \right\}^{\gamma + n + 2} + I_{\beta + 2 \geq 0} C(\tau)2^{k(2-\tau)}\delta^{\beta + 2} - \tau
\]

(8.55)

\[
+ I_{\beta + 2 \leq 0} C(\tau)2^{k(2-\tau)}\delta - \tau + I_{C\delta \leq 1} \left\{ \int_{\gamma + \nu \geq \gamma + \nu \leq 0} C(\tau)2^{k(2-\tau)}\delta - \nu + 2 - \tau + I_{\gamma + \nu \leq \gamma + \nu \leq 0} C(\tau)2^{k(2-\tau)}\delta - \gamma + 2 - \tau \right\}
\]

(8.56)

\[
+ I_{\beta + \nu \geq 0} C(\tau)2^{k(2-\tau)}\delta + 2 - \tau + C(\tau)2^{k(2-\tau)}|y|^{\nu + \beta} \delta - \nu + 2 - \tau
\]

(8.57)

\[
+ I_{\beta + \nu \leq 0} C(\tau)2^{k(2-\tau)}\delta - \nu + 2 - \tau \max\{C\delta, 1\}^{\beta + \nu}
\]

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Now we begin to estimate the last term $H$. One has

$$H \leq 2^{k(n+2)} \int_{\mathbb{R}^n} \int_{S^{n-1}} \int_0^1 \frac{|D^2 \tilde{F}(2^k(v - y) + 2^k s(v + v) - v))|}{1 + |2^k s(v + v) - v)|^{n/r}} \times \{1 + |2^k s(v + v)|^{n/r}] |v|^{-2b(V \cdot \sigma) \Phi(|v|) \|v\|_{v|\sin \frac{\theta}{2} \leq y |v-y| \leq 2\delta}} ds d\sigma dv$$

$$\leq C(\tilde{F}) 2^{k(n+2)} \{1 + (2^k \delta)^{n/r}\} \int_{\mathbb{R}^n} \int_{S^{n-1}} \{((M|\tilde{F}|^r)(2^k(v - y)))^{1/r} |v|^{-2} \times b(V \cdot \sigma) \Phi(|v|) \|v\|_{v|\sin \frac{\theta}{2} \leq y |v-y| \leq 2\delta}} ds d\sigma dv$$

(8.55) \quad \leq C(\tilde{F}) 2^{k(n+2)} \{1 + (2^k \delta)^{n/r}\} \int_{\mathbb{R}^n} \int_{S^{n-1}} |v|^{-2b(V \cdot \sigma) \Phi(|v|) \|v\|_{v|\sin \frac{\theta}{2} \leq y |v-y| \leq 2\delta}} ds d\sigma dv$$

(8.56) \quad \leq H_1 + H_2,$$

where we use the $L^\infty$ bound on maximal function and cut the last integral according to whether or not $\delta/|v| \geq \sqrt{2}/2$ (for $H_1$, respectively for $H_2$). If this is the case, i.e., for $H_1$, we abbreviate this as $|v| \leq C\delta$.

For $H_1$, one has

$$H_1 \leq C 2^{k(n+2)} \{1 + (2^k \delta)^{n/r}\} \int_{\mathbb{R}^n} \|v\|^{2 \Phi(|v|) \|v\|_{v|\sin \frac{\theta}{2} \leq y |v-y| \leq 2\delta}} ds d\sigma dv$$

where

(8.57) \quad H_{1,1} = C 2^{k(n+2)} \{1 + (2^k \delta)^{n/r}\} \int_{\mathbb{R}^n} \|v\|^{\gamma + 2 \|v\|_{v|\sin \frac{\theta}{2} \leq y |v-y| \leq 2\delta \|v\|_{v|\sin \frac{\theta}{2} \leq y |v-y| \leq 2\delta}} ds d\sigma dv,$$

(8.58) \quad H_{1,2} = C 2^{k(n+2)} \{1 + (2^k \delta)^{n/r}\} \int_{\mathbb{R}^n} \|v\|^{\beta + 2 \|v\|_{v|\sin \frac{\theta}{2} \leq y |v-y| \leq 2\delta \|v\|_{v|\sin \frac{\theta}{2} \leq y |v-y| \leq 2\delta}} ds d\sigma dv.$$

Then, we conclude that

(8.59) \quad H_{1,1} \leq C 2^{k(n+2)} \{1 + (2^k \delta)^{n/r}\} \{\min \{C\delta, 1\}\}^{\gamma + n+2}$$

and

$$H_{1,2} \leq C 2^{k(n+2)} \{1 + (2^k \delta)^{n/r}\} \int_{\mathbb{R}^n} \|v\|^{\beta + 2 \|v\|_{v|\sin \frac{\theta}{2} \leq y |v-y| \leq 2\delta \|v\|_{v|\sin \frac{\theta}{2} \leq y |v-y| \leq 2\delta}} ds d\sigma dv$$

(8.60) \quad \leq \left\{ \begin{array}{ll}
C 2^{k(n+2)} \{1 + (2^k \delta)^{n/r}\} \delta^{\beta + n+2} C\delta \geq 1, & \beta + 2 \geq -n \\
C 2^{k(n+2)} \{1 + (2^k \delta)^{n/r}\} \delta^{n} C\delta \geq 1, & \beta + 2 \leq -n 
\end{array} \right.$$

As for $H_2$, we have

$$H_2 \leq C 2^{k(n+2)} \delta^{\nu + 2} \{1 + (2^k \delta)^{n/r}\} \int_{\mathbb{R}^n} |v|^{-v} \Phi(|v|) \|v\|_{v|\sin \frac{\theta}{2} \leq y |v-y| \leq 2\delta \|v\|_{v|\sin \frac{\theta}{2} \leq y |v-y| \leq 2\delta}} ds d\sigma dv$$

$$\leq H_{2,2} + H_{2,2},$$

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Then, we conclude that

\[ H_{2,1} = C 2^{k(n+2)} \delta^{-\nu+2} \{1 + (2^k \delta)^{n/r}\} \int_{\mathbb{R}^n} |v|^{\gamma+\nu} 1_{|v| \geq 2\delta} 1_{|v-y| \leq 2\delta} |v| |v| \leq 1 dv, \]  

\[ H_{2,2} = C 2^{k(n+2)} \delta^{-\nu+2} \{1 + (2^k \delta)^{n/r}\} \int_{\mathbb{R}^n} |v|^{\beta+\nu} 1_{|v| \geq 2\delta} 1_{|v-y| \leq 2\delta} |v| \geq 1 dv. \]  

and

\[ H_{2,1} \leq C 2^{k(n+2)} \{1 + (2^k \delta)^{n/r}\} \delta^{-\nu+2} \int_{\mathbb{R}^n} |v|^{\gamma+\nu} 1_{|v| \geq 2\delta} 1_{|v| \leq 1} \]  

\[ \leq \begin{cases}  
C 2^{k(n+2)} \{1 + (2^k \delta)^{n/r}\} \delta^{-\nu+2} \beta \leq +n+2 \|C \delta \|_1, \quad \gamma + \nu \geq 0 \\
C 2^{k(n+2)} \{1 + (2^k \delta)^{n/r}\} \delta^{-\nu+n+2} \|C \delta \|_1, \quad \gamma + \nu \leq 0 
\end{cases} \]

(8.63)

and

\[ H_{2,2} \leq C 2^{k(n+2)} \{1 + (2^k \delta)^{n/r}\} \delta^{-\nu+2} \int_{\mathbb{R}^n} |v|^{\beta+\nu} 1_{|v| \geq \max(C \delta, 1)} 1_{|v-y| \leq 2\delta} dv \]  

\[ \leq \begin{cases}  
C 2^{k(n+2)} \{1 + (2^k \delta)^{n/r}\} \delta^{-\nu+2} \beta \leq +n+2 \{\max(C \delta, 1)\}^{\beta+\nu}, \quad \beta + \nu \geq 0 \\
C 2^{k(n+2)} \{1 + (2^k \delta)^{n/r}\} \delta^{-\nu+n+2} \{\max(C \delta, 1)\}^{\beta+\nu}, \quad \beta + \nu \leq 0 
\end{cases} \]

(8.64)

All in all,

**Lemma 8.3.6** For $H$ upper bounded by (8.55), one has

\[ H \leq C 2^{k(n+2)} \{1 + (2^k \delta)^{n/r}\} \{\min(C \delta, 1)\}^{\gamma+n+2} + \|C \delta \|_1 C 2^{k(n+2)} \{1 + (2^k \delta)^{n/r}\} \{\max(C \delta, 1)\}^{\beta+\nu} + \|C \delta \|_1 C 2^{k(n+2)} \{1 + (2^k \delta)^{n/r}\} \{\max(C \delta, 1)\}^{\beta+\nu}. \]

Finally, collecting Lemma 8.3.5 and Lemma 8.3.6 gives the statement of Lemma 8.3.2. It remains to add Lemma 8.3.1 and Lemma 8.3.2 in order to get the statement of Proposition 8.3.2.

**Remark 8.3.1** From the whole proof in the section 3, we shall pay much attention to the question why the weight $1 + |y|^{\beta+\nu}$ occurs on the rightside of the estimation. The main reason is that we have to estimate the integrals such as (8.41),(8.51) and (8.62). So if we replace the kernel $K^\Phi,k(v,y)$ by $K^\Phi,k(v,y) < v >^{-\kappa}$ where $\kappa$ is some reasonable positive constant, then the weight for $y$ will disappear.
8.4 Conclusion: obtaining Proposition 8.1.1

It is only matter of choosing carefully the parameter $\delta$ in all the cases mentioned in the statement of Proposition 8.1.1.

8.4.1 The case $\beta + \nu \leq 0$ and $\gamma + \nu \geq 1$

This is the simplest case. Note first that from Proposition 8.2.1, it readily follows that

$$\int_{\mathbb{R}^n} |K^{\Phi,k}(v,y)|dy \leq C 2^{k\nu}.$$ 

Then, we set $\delta = C'/2^k$, where we choose $C'$ such that $C\delta \leq 1$, where $C$ is the constant entering Proposition 8.3.1 and 8.3.2. With this choice, one can check that

$$\int_{\mathbb{R}^n} |K^{\Phi,k}(v,y)|dy \leq C 2^{k\nu}.$$ 

8.4.2 The case $\beta + \nu \geq 0$ and $\gamma + \nu \geq 1$

Proposition 8.2.1 tells us that

$$\int_{\mathbb{R}^n} |K^{\Phi,k}(v,y)|dy \leq C 2^{k\nu} \{1 + |v|^{\beta+\nu}\}.$$ 

Then, again with the same choice $\delta = C'/2^k$, from Proposition 8.3.1 and 8.3.2,

$$\int_{\mathbb{R}^n} |K^{\Phi,k}(v,y)|dy \leq C 2^{k\nu} \{1 + |y|^{\beta+\nu}\}.$$ 

8.4.3 The case $\beta + \nu \leq 0$ and $\gamma + \nu \leq 1$

Proposition 8.2.1 gives again that

$$\int_{\mathbb{R}^n} |K^{\Phi,k}(v,y)|dy \leq C 2^{k\nu} \{1 + |v|^{\gamma+\nu-1}\}.$$ 

While the same choice $\delta = C'/2^k$, from Proposition 8.3.1 and 8.3.2,

$$\int_{\mathbb{R}^n} |K^{\Phi,k}(v,y)|dy \leq C \sup\{2^{k\nu}, 2^{-(\gamma)}k\}.$$
8.4.4 The case $\beta + \nu \geq 0$ and $\gamma + \nu \leq 1$

For the last case, Proposition 8.2.1 gives again that

$$\int_{\mathbb{R}^n} |K^{\Phi,k}(v,y)| dy \leq C 2^{k\nu} \{ |v|^{\gamma + \nu - 1} + |v|^{\beta + \nu} \}.$$

While the same choice $\delta = C'/2^k$, from Proposition 8.3.1 and 8.3.2,

$$\int_{\mathbb{R}^n} |K^{\Phi,k}(v,y)| dv \leq C \sup \{ 2^{k\nu}, 2^{(-\gamma)k} \} \{ 1 + |y|^{\beta + \nu} \}.$$

8.5 Proof of Corollary 8.1.1, Theorem 8.1.1, Theorem 8.1.2 and Theorem 8.1.3

Since we have obtained Proposition 8.1.1, the proofs of Corollary 8.1.1, Theorem 8.1.1 and Theorem 8.1.2 are similar to the corresponding results proven in Chapter 7. We omit the details and only prove the Theorem 8.1.3. Before we start the work, we state the following proposition which will be very useful to the proof of the theorem.

**Proposition 8.5.1** Assume $\Phi = 1$ and let $\lambda$ such that $|\lambda| < \nu$. Then one has

$$T^{1,k} : L^p_\lambda \to L^p_{\lambda - \nu}$$

with a bound $C(\lambda) 2^{k\nu}$.

**Proof of Proposition 8.5.1**: Actually, by Schur’s Lemma, we only want to prove that

\begin{align}
(8.65) & \quad \int_{\mathbb{R}^n} |K^{\Phi,k}(v,y) < v >^{\lambda - \nu} < y >^{-\lambda} | dy \leq C(\lambda) 2^{k\nu}, \\
(8.66) & \quad \int_{\mathbb{R}^n} |K^{\Phi,k}(v,y) < v >^{\lambda - \nu} < y >^{-\lambda} | dv \leq C(\lambda) 2^{k\nu}.
\end{align}

Using $< v >^{\lambda} \cdot < y >^{-\lambda} \leq < v - y >^{|\lambda|}$ and (8.22), we arrive at formula (8.65). So the remainder is to prove

$$\int_{\mathbb{R}^n} |K^{\Phi,k}(v,y) < v >^{-\nu} < y >^{|\lambda|} | dv \leq C(\lambda) 2^{k\nu}.$$

Recalling that

$$K^{\Phi,k}(v,y) = \int_{\mathbb{S}^{n-1}} \{ \tilde{\varphi}_k(v^+ - y) - \tilde{\varphi}_k(v - y) \} b(v) \cdot \Phi(|v|) d\sigma$$
Proof of Theorem 8.1.3: We first consider the case constant the proof done in Chapter 7 works. Then we get that the non singular part is also bounded with a constant $C$.

Finally recalling Remark 8.3.1, we can conclude that the singular part is bounded with a constant $\phi(\lambda)2^{k\nu}$.

For the non singular part, we can use the arguments mentioned in Remark 8.2.1 to check that the proof done in Chapter 7 works. Then we get that the non singular part is also bounded with a constant $C(\lambda)2^{k\nu}$.

This ends the proof of Proposition 8.5.1.

Proof of Theorem 8.1.3: We first consider the case $s = 0$.

Having in mind the decomposition (8.17) of $Q(g, f)$, we first deal with the term $Q_\triangle(g, f)$ which is similar to the main term in the cutoff case.

Since $\Phi = \omega^7$ and $|v^+|$ is bounded below and above by a constant times $|v|$, we have

$$\langle v^- > |\lambda| \leq C(1 + \frac{C'}{2^k}) |\lambda| \leq C.$$

Finally recalling Remark 8.3.1, we can conclude that the singular part is bounded with a constant $C(\lambda)2^{k\nu}$.

For the non singular part, we can use the arguments mentioned in Remark 8.2.1 to check that the proof done in Chapter 7 works. Then we get that the non singular part is also bounded with a constant $C(\lambda)2^{k\nu}$.

This ends the proof of Proposition 8.5.1.
and thus

\[(8.71) \quad \|Q_{\Delta,2}(g,f)\|_{B^{-\gamma}_{p,q}} \leq \|g\|_{L^1_p} \|f\|_{L^\gamma_p},\]

As for the operator \(Q_{\Delta,1}(g,f)\), we first go back to the operator \(T_{\Delta,1}^{p,k}\). Actually,

\[(8.72) \quad T_{\Delta,1}^{p,k}(p_k\varphi)(v) = - \int_{S_{\nu-1}} \nabla \Phi(v) \cdot (v^+ - v) \{(p_k\varphi)(v^+) - (p_k\varphi)(v)\}\{\cdot\} b\left(\frac{v}{|v|} \cdot \sigma\right) d\sigma.
\]

Comparing it with the definition of \(U^{\Phi.k}(v)\) (see (8.8)), we can regard the operator \(T_{\Delta,1}^{p,k}(p_k\varphi)(v)\) as \(T^{\Phi,k}(p_k\varphi)\) (see (8.9)) but with small singularity \(\nu - 1\). Then Proposition 1.11 from Chapter 7 tells us that

\[(8.73) \quad T_{\Delta,1}^{p,k} \circ p_k : L^p_p \to L^p_{\nu+(\nu-1)},\]

with a bound in \(2^{k(\nu-1)}\). Then

\[(8.74) \quad |\langle p_kQ_{\Delta,1}(g,f); \varphi\rangle| \leq 2^{k(\nu-1)} \|g\|_{L^1_{\nu+(\nu-1)}} \|f\|_{L^\gamma_{\nu+(\nu-1)}} \|p_k\varphi\|_{L^{\gamma'}} ,\]

and thus we get

\[(8.75) \quad \|Q_{\Delta,1}(g,f)\|_{B^{-\gamma}_{p,q}} \leq \|g\|_{L^1_{\nu+(\nu-1)}} \|f\|_{L^\gamma_{\nu+(\nu-1)}} .\]

It remains to analyze the most difficult term \(Q_{\Delta}(g,f)\), given by (8.20). Applying \(p_k\) on it, and then testing against a test function \(\varphi\), we find

\[(8.76) \quad \langle p_kQ_{\Delta}(g,f); \varphi\rangle = \int_{\mathbb{R}^n} dv x g(v_a) \int_{\mathbb{R}^n} df(v)\tau_{v_a} \circ T^1 \circ (\Phi\tau_{v_a} p_k\varphi) = \langle A_k; \varphi \rangle + (B_k; \varphi) .\]

where

\[(8.77) \quad \langle A_k; \varphi \rangle = \int_{\mathbb{R}^n} dv x g(v_a) \int_{\mathbb{R}^n} df(v)\tau_{v_a} \circ T^1 \circ \tilde{p}_k(\Phi\tau_{v_a} \varphi)\]

\[(8.78) \quad \langle B_k; \varphi \rangle = \int_{\mathbb{R}^n} dv x g(v_a) \int_{\mathbb{R}^n} df(v)\tau_{v_a} \circ T^1 \circ [\Phi, \tilde{p}_k]\tau_{v_a} \varphi .\]

For the term \(\langle A_k; \varphi \rangle\), let us first note that

\[(8.79) \quad \langle A_k; \varphi \rangle = \int_{\mathbb{R}^n} dv x g(v_a) \int_{\mathbb{R}^n} df(v)\tau_{v_a} \circ T^{1,k} \Phi\tau_{v_a} \varphi ,\]

where we have now introduced the operator \(T^{1,k}_1\). The important point is to notice that, on the Fourier side, the spectrum of \(T^{1,k}(\cdot)\) is supported on an annulus \(\sim 2^k\), similarly to the usual Maxwellian case. It follows that we can also replace \(f\) par \(\tilde{p}_k f\) for another Littlewood-Paley decomposition.
Therefore, we get that for the case $\gamma \leq 0$,

$$
|\langle A_k; \varphi \rangle| \leq 2^{k\nu} \|g\|_{L^{1+\gamma}} \|\hat{p}_k f\|_{L^p} \|\varphi\|_{L^{p'}},
$$

(8.79)

while for the case $\gamma \geq 0$, we have

$$
|\langle A_k; \varphi \rangle| \leq \int_{\mathbb{R}^n} dv \ast |g(v)| \|\hat{p}_k f\|_{L^{p'}} \|\tau_{-v} \circ T^{1,k}(\Phi_{\tau_v} \varphi)\|_{L^{p'}_{(\nu+\gamma)}} \leq 2^{k\nu} \|g\|_{L^{1+\gamma}} \|\hat{p}_k f\|_{L^p} \|\varphi\|_{L^{p'}}.
$$

(8.80)

So anyway, we conclude

$$
|\langle A_k; \varphi \rangle| \leq 2^{k\nu} \|g\|_{L^{1+\gamma}} \|\hat{p}_k f\|_{L^p} \|\varphi\|_{L^{p'}}.
$$

(8.81)

Finally, we have obtained

$$
\|A_k\|_{L^p} \leq 2^{k\nu} \|g\|_{L^{1+\gamma}} \|\hat{p}_k f\|_{L^p} \|\varphi\|_{L^{p'}}.
$$

(8.82)

Turning to the term $\langle B_k; \varphi \rangle$, we know from the fact that

$$
\Phi(x) - \Phi(y) = \nabla \Phi(y) \cdot (x - y) + \int_0^1 (1 - s) D^2 \Phi(y + s(x - y)) \cdot (x - y, x - y) ds
$$

then

$$
[\Phi, \hat{p}_k] \psi(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y) \cdot \xi} \{\Phi(x) - \Phi(y)\} \tilde{\varphi}_k(\xi) \psi(y) d\xi dy = P_1 \psi(x) + P_2 \psi(x).
$$

(8.84)

where

$$
P_1 \psi(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y) \cdot \xi} \{\nabla \Phi(y) \cdot (x - y)\} \tilde{\varphi}_k(\xi) \psi(y) d\xi dy,
$$

$$
P_2 \psi(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_0^1 (1 - s) e^{i(x-y) \cdot \xi} \{D^2 \Phi(y + s(x - y)) \cdot (x - y, x - y)\} \tilde{\varphi}_k(\xi) \psi(y) ds d\xi dy.
$$

(8.85)

Integral by parts, we get

$$
P_1 \psi(x) = -i \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y) \cdot \xi} \{\nabla \Phi(y) \cdot \nabla \xi \tilde{\varphi}_k\} \psi(y) d\xi dy,
$$

$$
P_2 \psi(x) = - \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_0^1 (1 - s) e^{i(x-y) \cdot \xi} \{D^2 \Phi(y + s(x - y)) : D^2 \xi \tilde{\varphi}_k\} \psi(y) ds d\xi dy.
$$

(8.86)

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From the theory of pseudo-differential operator, we can regard the operator \( P_1 \) as \( 2^{-k} \tilde{p}_k \circ \partial_i \Phi \), while the operator \( P_2 \) maps \( L^q \) into \( W^{s,q} \) with bound \( (1/2^k)^{2-s} \) for \( 0 \leq s \leq 2 \).

So we divide the \( B_k \) into parts: \( B_{k,1} \) and \( B_{k,2} \) which correspond to the operator \( P_1 \) and \( P_2 \). As for \( B_{k,1} \), we can get the similar estimates as for \( A_k \) due to the fact we state above. Then

\[
|\langle B_{k,1}; \varphi \rangle| \leq 2^{k\nu} 2^{-k} \|g\|_{L^1_{\nu+\gamma+1}} \|\tilde{p}_k f\|_{L^p_{\nu+\gamma+1}} \|\varphi\|_{L^p}. \tag{8.89}
\]

As for \( B_{k,2} \), we have

\[
|\langle B_{k,2}; \varphi \rangle| \leq \int_{\mathbb{R}^n} dv_s |g(v_s)| < v_s >^\nu \|f\|_{L^p} \|T^1 \circ P_2(\tau_{v_s} \varphi)\|_{L^p}. \tag{8.90}
\]

Then, we note that

\[
\|T^1 \circ P_2(\tau_{v_s} \varphi)\|_{L^p} \leq \sum_j \|T^{1,j} \circ P_2(\tau_{v_s} \varphi)\|_{L^p}. \tag{8.91}
\]

We use the fact we state above, then for all \( \epsilon \) such that \( \nu + \epsilon < 2 \),

\[
\|T^1 \circ P_2(\tau_{v_s} \varphi)\|_{L^p} \leq \frac{1}{2^{(2-\nu-\epsilon)}} \|\varphi\|_{L^p}. \tag{8.92}
\]

Then

\[
\|B_{k,1}\|_{L^p} \leq 2^{k\nu} 2^{-k} \|g\|_{L^1_{\nu+\gamma+1}} \|\tilde{p}_k f\|_{L^p_{\nu+\gamma+1}}, \tag{8.93}
\]

\[
\|B_{k,2}\|_{L^p} \leq \frac{1}{2^{(2-\nu-\epsilon)}} \|g\|_{L^p_1} \|f\|_{L^p}. \tag{8.94}
\]

Gluing these two estimates on \( A_k \) and \( B_k \), we obtain

\[
\|Q(g, f)\|_{B^{-1}_{p,q}} \leq (\|g\|_{L^1_{\nu+\gamma+1}} + \|g\|_{L^1_{\nu}} + \|g\|_{L^1_{\nu+\gamma+1}}) (\|f\|_{L^p} + \|f\|_{B^{0}_{p,q,\nu+\gamma+1}} + \|f\|_{B^{-1}_{p,q,\nu+\gamma+1}}). \tag{8.95}
\]

So we arrive at the result

\[
\|Q(g, f)\|_{B^{-1}_{p,q}} \leq (\|g\|_{L^1_{\nu}} + \|g\|_{L^1_{\nu}} + \|g\|_{L^1_{\nu+\gamma+1}} + \|g\|_{L^1_{\nu+\gamma+1}}) \times (\|f\|_{L^p} + \|f\|_{L^p_{\nu+\gamma-1}} + \|f\|_{L^p} + \|f\|_{B^{-1}_{p,q,\nu+\gamma+1}} + \|f\|_{B^{0}_{p,q,\nu+\gamma+1}}). \tag{8.96}
\]

This gives the result in the case \( s = 0 \), other values of \( s \) being deduced using Galilean invariance of the collision operator and interpolation. The proof of Theorem 8.1.3 is now complete.
Chapter 9

Regularity issues: homogeneous equations and non maxwellian molecules

9.1 Introduction

This Chapter is devoted to regularization properties of weak solutions to Boltzmann homogeneous equation, by using technics from harmonic analysis. It is the natural continuation of Chapter 6. Recall that in that Chapter, we considered a very special case of collision cross sections, namely non cutoff maxwellian molecules. Here, we wish to consider a larger class of collision sections, namely those corresponding to so called hard potentials. More precisely, we shall consider smoothed versions of this case, see assumptions below for precise definitions.

Since we have already given precise references on the framework considered herein in Chapter 6, we shall be rather concise in this Introduction, but we again refer to [38, 47, 48, 53, 52].

Let us just recall that Boltzmann homogeneous equation reads as

(9.1) \[ \partial_t f(t, v) = Q(f, f)(t, v) \quad t \geq 0, \ v \in \mathbb{R}^n, \]

where \( f \) is a positive function depending only (homogeneous framework) upon the two variables \( t \geq 0 \) (time) and \( v \in \mathbb{R}^n \) (velocity) with \( f(0, v) = f_0(v) \), where \( n \geq 2 \).

The initial datum \( f_0 \neq 0 \) is supposed to satisfy the usual ”entropic” hypothesis, that is

(9.2) \[ f_0 \geq 0, \int_{\mathbb{R}^n} f_0(v) \{1 + |v|^2 + \log(1 + f_0(v))\} dv < +\infty. \]
Boltzmann quadratic operator \(Q\) appearing on the r.h.s. of (9.1) depends on \(v\) as follows

\[
Q(f, f) = \int_{\mathbb{R}^n} \int_{S^{n-1}} dv_* d\sigma B(v - v_*, \sigma) (f' f_* - f f_*) d\sigma dv_*
\]

where \(v_* \in \mathbb{R}^n, \sigma \in S^{n-1}\) (unit sphere of \(\mathbb{R}^n\)), \(f = f(v), f_* = f(v_*), f' = f(v')\) and \(f'_* = f(v'_*)\), and

\[
v' = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma, \quad v'_* = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma
\]

are the so called post (or pre) collisional velocities.

As in Chapter 6, we shall assume that the collision cross section \(B(v - v_*, \sigma) > 0\) is given under the following multiplicative form

\[
B(v - v_*, \sigma) = \Phi(|v - v_*|) b(\cos \theta), \quad \cos \theta = \frac{\langle v - v_* | \sigma \rangle}{|v - v_*|}, \quad 0 \leq \theta \leq \frac{\pi}{2}.
\]

and that it satisfies the following non cutoff assumption

\[
\int_{0}^{\frac{\pi}{2}} \sin^n \theta b(\cos \theta) d\theta < +\infty \text{ and } \sin^{n-2} \theta b(\cos \theta) \sim \frac{\kappa}{\theta^{1+\nu}} \text{ when } \theta \to 0,
\]

where \(\kappa > 0\) and \(0 < \nu < 1\) are fixed.

Chapter 6 was concerned with the so called maxwellian case, corresponding to \(\Phi \equiv 1\) (or constant) with the above notations.

Herein, the velocity part of the kernel, that is function \(\Phi\), shall be assumed to correspond to a smoothed version of the so called hard potentials, that is

\[
\Phi(|v|) = (1 + |v|^2)^{\gamma},
\]

with the range of parameters \(0 < \gamma \leq 1\); the real hard potentials case corresponds to the case \(\Phi(|v|) = |v|^{\gamma}\).

We assume that a weak solution to Boltzmann (9.1) has already been constructed and that it satisfies the usual entropic estimate, for a fixed \(T > 0\) (eventually \(T = +\infty\)), together with mass conservation, and decrease of energy. It is then known that such a weak solution has then all moments w.r.t. velocity, for strictly positive time. Again, we refer for precise references to Chapter 6 and to bibliography.

In this second part of our work, we are still interested in regularization properties of such solutions.

In Chapter 6, we have provided a very simple proof of \(C^\infty\) regularization property of weak solutions, for maxwellian molecules, that is when \(\Phi\) is taken to be constant, so a case which is now excluded by our assumption (9.5).
It should be mentioned that the proof performed in Chapter 6, though extremely simple, does not (at least for us) adapt for non maxwellian molecules, and this is so from the first computations.

In this non maxwellian and non cutoff case, the up to date recent results about this regularization property question are due to Desvillettes and Wennberg [55], showing $S$ (in fact through weighted Sobolev spaces) regularity. However, the point is that, actually, Desvillettes and Wennberg show that, under suitable assumptions on the cross section, a solution in $S$ does exist, with $f_0$ satisfying (9.2).

Here, exactly as in Chapter 6, we wish to show the stronger result that any weak solution is smooth. Up to an assumption of weighted $L^2$ bounds, we shall show that this is indeed the case. Thus, our result is strictly not comparable to [55]. A similar result was established in the context of Landau homogeneous equation by El Safadi [62].

As in Chapter 6, our arguments are based on Littlewood-Paley theory. However, we need here commutators estimates, to take into account the fact that $\Phi$ is now really a non constant function. In particular, we shall need some results extracted from Chapter 7.

The bad point is that we need further integrability assumption with respect to variable $v$. This point is in fact easy to understand, see the remarks at the end of this Chapter.

Thus, according to [53], we shall furthermore assume that for some $t_0 > 0$,

$$f \in L^\infty([t_0, +\infty); L^2_q(\mathbb{R}^n)), \text{ for all } q > 2.$$  \hspace{1cm} (9.6)

It is an open problem to show that any weak solution satisfying only the usual entropic bounds enjoy this $L^2$ integrability (9.6), even if we do take $t_0 = 0$ and an initial datum in the same class. This is in particular due to the lack of a good uniqueness result, and also to the fact that power of $f$, for an exponent less that 1 belongs to a worse Besov type space, see [101] for instance, and in view of the (quite optimal w.r.t. index of regularity) functional properties of Boltzmann operator, cf. Chapter 7.

The exact result we want to prove here is given by

**Theorem 9.1.1** Let be given an initial datum $f_0$ and a collision cross section $B$ such that (9.2), (9.3), (9.4) and (9.5) hold true. Let $f$ be any weak non negative solution of Boltzmann homogeneous equation (9.1). Furthermore, we assume that (9.6) holds true for some $t_0 > 0$. Then, for any $t > t_0$, for all $s, \alpha \in \mathbb{R}^+$, $f(t, \cdot)$ belongs to the weighted Besov-Sobolev space $B^{s,2,\alpha}_{2,2}(\mathbb{R}^n)$.

\[ \square \]

In particular, it follows that for $t > t_0$, $f(t)$ belongs immediately to Schwartz space $\mathcal{S}(\mathbb{R}^n)$. **Plan of the paper:** Section 2 is devoted to the proof of our main result. Then we make some final comments in Section 3.
9.2 Proof of the theorem

For any $j \in \mathbb{N}$, $k \in \mathbb{N}$, one has, using notations from Chapter 7 and Chapter 21, $< \cdot ; \cdot >$ denoting the usual duality bracket,

\begin{equation}
< \psi_j p_k Q(g, f); \psi_j p_k f > = \int_{\mathbb{R}^n} dv_s g_s \int_{\mathbb{R}^n} df \tau_{-v_s} \circ T^\Phi \circ [p_k \psi_j^2 p_k f].
\end{equation}

Above, we have introduced $g = f$ in order to show clearly on which functions we are going to perform fractional differentiation.

Furthermore, for any suitable test function, see Chapter 7 for more precisions,

\[ T^\Phi \phi(v) = \int_{S^{n-1}} \left[ \phi(v^+) - \phi(v) b \left( \frac{v}{|v|} \right) \sigma \right] \Phi(v), \]

$\tau_v$, denoting the usual translation.

In particular, when $\Phi \equiv 1$, corresponding to the Maxwellian case, let us recall that, see [11] for instance, that

\[ v \rightarrow T^1 \phi(v) \]

is adjoint to

\[ f \rightarrow Q_{Max}(\delta_{v,-0}, f). \]

We can then write

\[ < \psi_j p_k Q(g, f); \psi_j p_k f > = A + B + C, \]

where

\begin{equation}
A = \int_{\mathbb{R}^n} dv_s g_s \int_{\mathbb{R}^n} df \tau_{-v_s} \circ T^\Phi_\Delta \circ \tau_v \{ p_k \psi_j^2 p_k f \},
\end{equation}

\begin{equation}
B = \int_{\mathbb{R}^n} dv_s g_s \int_{\mathbb{R}^n} df \tau_{-v_s} \circ T^1 \circ \{ [\Phi, p_k] \tau_v \psi_j^2 p_k f \},
\end{equation}

and

\begin{equation}
C = \int_{\mathbb{R}^n} dv_s g_s \int_{\mathbb{R}^n} df \tau_{-v_s} \circ T^1 (p_k \Phi \tau_v \psi_j^2 p_k f).
\end{equation}

Above, $T^\Phi_\Delta$ is defined as in Chapter 7 by

\[ T^\Phi_\Delta \phi(v) = \int_{S^{n-1}} \phi(v^+)[\Phi(v^+) - \Phi(v)] b \left( \frac{v}{|v|} \sigma \right) d\sigma \]

Using the fact that

\[ C = \int_{\mathbb{R}^n} dv_s g_s \int_{\mathbb{R}^n} df \tau_{-v_s} \circ T^1 \circ (p_k \Phi \tau_v \psi_j^2 p_k f) \]
\[
= \int_{\mathbb{R}^n} dv^* g^* \int_{\mathbb{R}^n} dv(\tau_v, f) T^1(p_k \Phi \tau_v, \psi^2_j p_k f)
= \int_{\mathbb{R}^n} dv^* g^* Q_{\max}(\delta_{\omega=0}, \tau_v, f) p_k \Phi \tau_v, \psi^2_j p_k f
= \int_{\mathbb{R}^n} dv^* g^* \int_{\mathbb{R}^n} d\xi (Q_{\max}(\delta_{\omega=0}, \tau_v, f)) \psi_k(\xi) \{ \Phi \tau_v, \psi^2_j p_k f \}
= \int_{\mathbb{R}^n} dv^* g^* \int_{\mathbb{R}^n} d\xi \int_{\mathbb{S}^{n-1}} d\sigma \{ \Phi \tau_v, \psi^2_j p_k f \}.
\]

It follows that
(9.11) \quad \mathcal{C} = \mathcal{D} + \mathcal{E},
where
(9.12) \quad \mathcal{D} = \int_{\mathbb{R}^n} dv^* g^* \int_{\mathbb{R}^n} d\xi \int_{\mathbb{S}^{n-1}} d\sigma b(\frac{\xi}{|\xi|}) \{ \psi_k(\xi) - \psi_k(\xi^+) \} e^{-iv \cdot \xi^+} \hat{g}(\xi^+) \{ \Phi \tau_v, \psi^2_j p_k f \},
\quad \mathcal{E} = \int_{\mathbb{R}^n} dv^* g^* \int_{\mathbb{R}^n} dv p_k f \tau_{-v} \circ T^1 \circ (\Phi \tau_v, \psi^2_j p_k f),
by performing back the above computations.

We then write
(9.14) \quad \mathcal{E} = \mathcal{F} + \mathcal{G},
where
(9.15) \quad \mathcal{F} = - \int_{\mathbb{R}^n} dv^* g^* \int_{\mathbb{R}^n} dv p_k f \tau_{-v} \circ T^1_{\Delta} \tau_v, \psi^2_j p_k f,
and
(9.16) \quad \mathcal{G} = \int_{\mathbb{R}^n} dv^* g^* \int_{\mathbb{R}^n} dv p_k f \tau_{-v} \circ T^1 \circ (\Phi \tau_v, \psi^2_j p_k f),
that is also
(9.17) \quad \mathcal{G} = \mathcal{H} + \mathcal{I},
where
(9.18) \quad \mathcal{H} = \int_{\mathbb{R}^n} dv^* g^* \int_{\mathbb{R}^n} dv \int_{\mathbb{S}^{n-1}} d\sigma b(\frac{v - v^*}{|v - v^*|}) p_k f \Phi(v - v) b(\frac{v - v^*}{|v - v^*|}, \sigma) (\psi^2_j p_k f)' - (\psi^2_j p_k f)],
For this last term, one has
(9.19) \quad \mathcal{H} = \mathcal{J} + \mathcal{K},
where
(9.20) \quad \mathcal{J} = \int_{\mathbb{R}^n} dv^* g^* \int_{\mathbb{R}^n} dv \int_{\mathbb{S}^{n-1}} d\sigma b(\frac{v - v^*}{|v - v^*|}) p_k f \Phi(v - v^*) (\psi_j - \psi_j) (\psi_j p_k f)'.
and

\[ I = \int_{\mathbb{R}^n} dv^* g^* \int_{\mathbb{R}^n} dv \int_{S^{n-1}} d\sigma b \left( \frac{v - v^*}{|v - v^*|}, \sigma \right) \psi_j p_k f \Phi(v - v^*) [(\psi_j p_k f)' - (\psi_j p_k f)] . \]

By using the simple identity \( a(b - a) = -\frac{1}{2}(b - a)^2 + \frac{1}{2}(b^2 - a^2) \), it follows that

\[ I = J - K , \]

where

\[ J = \frac{1}{2} \int_{\mathbb{R}^n} dv^* g^* \int_{\mathbb{R}^n} dv \int_{S^{n-1}} d\sigma b \left( \frac{v - v^*}{|v - v^*|}, \sigma \right) \Phi(v - v^*) (\langle \psi_j p_k f \rangle' - (\psi_j p_k f)^2) , \]

and

\[ K = \frac{1}{2} \int_{\mathbb{R}^n} dv^* g^* \int_{\mathbb{R}^n} dv \int_{S^{n-1}} d\sigma b \left( \frac{v - v^*}{|v - v^*|}, \sigma \right) \Phi(v - v^*) (\langle \psi_j p_k f \rangle' - (\psi_j p_k f)^2) . \]

All in all, we have obtained, by applying operator \( \psi_j p_k \) on Boltzmann equation and integrating against \( \psi_j p_k f \), operations which are perfectly allowed even for entropic weak solutions, that is not even without assumption (9.6), the following differential equality

\[ \frac{d}{dt} \| \psi_j p_k f \|^2_{L^2} + K = A + B + D + F + H + J . \]

In the following, our task will be to found upper bounds on each term on the right hand side, while we shall look for a lower bound on \( K \).

**Upper bound on \( J \)**

Since

\[ J = \frac{1}{2} \int_{\mathbb{R}^n} dv^* g^* \int_{\mathbb{R}^n} dv \int_{S^{n-1}} d\sigma b \left( \frac{v - v^*}{|v - v^*|}, \sigma \right) \langle \psi_j p_k f \rangle^2 \Phi(v - v^*) \{ g^* - g_s \} , \]

it follows that, using the results from Chapter 3, one may write, for a suitable kernel \( S \)

\[ J = C \int_{\mathbb{R}^n} dv (\psi_j p_k f)^2 S * g(v) . \]

Since we have assumed all moments on \( f \) (and thus on \( g \)) bounded, we find

\[ |J| \lesssim 2^j \| \psi_j p_k f \|^2_{L^2} . \]

**Upper bound on \( H \)**

Firstly, we note immediately that

\[ |H| \lesssim \frac{1}{2^j} \int_{\mathbb{R}^n} dv^* g^* \int_{\mathbb{R}^n} dv \int_{S^{n-1}} d\sigma b \left( \frac{v - v^*}{|v - v^*|}, \sigma \right) g_s |p_k f| \Phi(v - v^*) |v - v^*| |\hat{b}| |\psi_j p_k f| \]
and thus similarly to [97] (see also Chapter 7), we find

\[ |\mathcal{H}| \lesssim \frac{1}{2^j} \|g\|_{L^1_{\gamma+1}} \|p_k f\|_{L^2} \|\psi_j p_k f\|_{L^2_{\gamma-1}}. \]

It follows that

\[ (9.25) \quad |\mathcal{H}| \lesssim \frac{1}{2^j(\gamma+2)} \|g\|_{L^1_{\gamma+1}} \|p_k f\|_{L^2} \|\psi_j p_k f\|_{L^2}. \]

**Upper bound on \( F \)**

Using notations from Chapter 7, one has

\[ F = - \langle Q \Delta (g, p_k f); \psi_j^2 p_k f \rangle, \]

so that

\[ |F| \lesssim \|Q \Delta (g, p_k f)\|_{L^2} \|\psi_j^2 p_k f\|_{L^2}. \]

Thus

\[ (9.26) \quad |F| \lesssim \|g\|_{L^1} \|p_k f\|_{L^2} \|\psi_j p_k f\|_{L^2}. \]

**Upper bound on \( A \)**

In the same way,

\[ |A| = |\langle Q \Delta (g, f); p_k \psi_j^2 p_k f \rangle| \lesssim \|Q \Delta (g, f)\|_{L^2} \|p_k \psi_j^2 p_k f\|_{L^2}, \]

and thus

\[ (9.27) \quad |A| \lesssim \|g\|_{L^1} \|f\|_{L^2} \|\psi_j p_k f\|_{L^2}. \]

**Upper bound on \( B \)**

Similarly, again with notations from Chapter 7

\[ |B| = |\langle B_k; \psi_j^2 p_k f \rangle|, \]

and thus

\[ (9.28) \quad |B| \lesssim \|g\|_{L^1} \|f\|_{L^2} \|\psi_j p_k f\|_{L^2}. \]

**Estimate on \( D \)**
Taking into account the fact that $|\xi^+|$ is bounded above and below by a constant times $|\xi|$ on the support of $\psi_k$, we can introduce another Littlewood-Paley partition $\tilde{p}_k$ to get

$$\mathcal{D} = \int_{\mathbb{R}^n} dv_s \int_{\mathbb{R}^n} d\xi \int_{S^{n-1}} d\sigma b(\frac{\xi}{|\xi|}) g_s \{\tau_{\nu_s} \tilde{p}_k f\}(\xi^+) A_k^\xi \{\Phi_{\tau_{\nu_s}} \psi_j^2 p_k f\}$$

where $A_k^\xi \equiv \psi_k(\xi^+) - \psi_k(\xi)$. Since $|A_k^\xi| \lesssim \sin \frac{\theta}{2}$, it follows that

$$|\mathcal{D}| \lesssim \int_{\mathbb{R}^n} dv_s \int_{\mathbb{R}^n} d\xi \int_{S^{n-1}} d\sigma b(\frac{\xi}{|\xi|}) g_s \{\tau_{\nu_s} \tilde{p}_k f\}(\xi^+) A_k^\xi \{\Phi_{\tau_{\nu_s}} \psi_j^2 p_k f\}$$

$$\lesssim \int_{\mathbb{R}^n} dv_s g_s \{\int_{\mathbb{R}^n} d\sigma |\{\Phi_{\tau_{\nu_s}} \tilde{p}_k f\}(\xi^+)|^2 b(\xi)\}^{\frac{1}{2}} \{\int_{\mathbb{R}^n} d\xi \int_{S^{n-1}} d\sigma |\Phi_{\tau_{\nu_s}} \psi_j^2 p_k f| |\tilde{b}(\xi)\}^{\frac{1}{2}},$$

where $\tilde{b}(\cdot) = \sin \frac{\theta}{2} b(\cdot)$.

Thus making the change of variables $\xi^+ \mapsto \xi$, see Chapter 3, we get

(9.29) $|\mathcal{D}| \lesssim \|g\|_{L^1} \|\tilde{p}_k f\|_{L^2} 2^{j\gamma} \|\psi_j p_k f\|_{L^2}$.

**Lower bound on $\mathcal{K}$**

From Peetre’s inequality, it follows that

$$\mathcal{K} \gtrsim \int_{\mathbb{R}^n} dv_s \int_{\mathbb{R}^n} d\xi \int_{S^{n-1}} d\sigma b(\frac{\nu - \nu_s}{|\nu - \nu_s|}) g_s < v_s >^{-\gamma} < v >^\gamma \left\{ (\psi_j p_k f)^' - (\psi_j p_k f) \right\}^2$$

$$\gtrsim \int_{\mathbb{R}^n} dv_s \int_{\mathbb{R}^n} d\xi \int_{S^{n-1}} d\sigma b(\frac{\nu - \nu_s}{|\nu - \nu_s|}) g_s < v_s >^{-\gamma} 2^{j\gamma} \tilde{\psi}_j^2(v) \left( (\psi_j p_k f)^' - (\psi_j p_k f) \right)^2$$

$$\gtrsim \int_{\mathbb{R}^n} dv_s \int_{\mathbb{R}^n} d\xi \int_{S^{n-1}} d\sigma b(\cdot) g_s < v_s >^{-\gamma} 2^{j\gamma} \left\{ (\psi_j p_k f)^' - (\psi_j p_k f) \right\}^2$$

$$\gtrsim \int_{\mathbb{R}^n} dv_s \int_{\mathbb{R}^n} d\xi \int_{S^{n-1}} d\sigma b(\cdot) g_s < v_s >^{-\gamma} \left\{ (\psi_j p_k f)^' - (\psi_j p_k f) \right\}^2 - c 2^{j\gamma} g_s < v_s >^{-\gamma} |\tilde{\psi}_j - \tilde{\psi}_j'|^2 |(\psi_j p_k f)^'|^2$$

$$\gtrsim 2^{j\gamma} \|\psi_j p_k f\|^2_{H^2} - C 2^{j(\gamma - 2)} \int_{\mathbb{R}^n} dv_s \int_{\mathbb{R}^n} d\xi \int_{S^{n-1}} d\sigma b(\cdot) g_s < v_s >^{-\gamma} |v - v_s|^2 \sin^2 \left(\frac{\theta}{2}\right) |(\psi_j p_k f)^'|^2$$

using similar computations as those from Chapter 3. Therefore, using one commutator, we find

(9.30) $\mathcal{K} \gtrsim 2^{j\gamma} 2^{k\nu} \|\psi_j p_k f\|^2_{L^2} - C 2^{j(\gamma - 2)} 2^{k(\nu - 2)} \|p_k f\|^2_{L^2}$,

for all $\alpha \geq 0$.

**Differential inequality**
Collecting all the above estimates, we have found that, setting \( U_{j,k} = \|\psi_j p_k f \|^2_{L^2} \), that one has

\[
\begin{aligned}
\partial U_{j,k} + C 2^{j\gamma} 2^{k \nu} U_{j,k} & \\ 
2^j U_{j,k} + 2^{-j(\gamma+2)} \|g\|_{L^1_{\gamma+1}} \|p_k f\|_{L^2_j} U_{j,k}^{1/2} + \|g\|_{L^1_j} \|p_k f\|_{L^2_j} U_{j,k}^{1/2} \\
+ \|g\|_{L^1_j} \|f\|_{L^2_j} U_{j,k}^{1/2} + \|g\|_{L^1_j} \|f\|_{L^2_j} U_{j,k}^{1/2} \\
+ \|g\|_{L^1_j} \|\bar{p}_k f\|_{L^2_j} 2^{j\gamma} U_{j,k}^{1/2} + 2^{j(\gamma-2)} 2^{k(\nu-2)} \|p_k f\|_{H^2}^2.
\end{aligned}
\]

(9.31)

**Iteration - First step**

By assumption, for all \( t \geq t_0 \), \( U_{j,k} \lesssim \frac{1}{2^{j\beta}} \), for all \( \beta \geq 0 \), \( \|p_k f\|_{L^2_j} \lesssim C \) and \( \|g\|_{L^1_j} \lesssim C \). It follows that we found

\[
\partial U_{j,k} + 2^{j\gamma} 2^{k \nu} U_{j,k} \lesssim 2^{j(\gamma-2)}.
\]

Thus, it follows from (9.31) that for \( t \geq t_1 > t_0 \),

\[
U_{j,k} \lesssim 2^{j(\gamma-3)} 2^{-k \nu}.
\]

Since we have also

\[
U_{j,k} \lesssim 2^{-j\alpha}
\]

it follows that, for any \( \varepsilon > 0 \) small, any \( \alpha \geq 0 \)

\[
U_{j,k} \lesssim 2^{-j\alpha} 2^{-k(\nu-\varepsilon)}.
\]

Thus, we have obtained that for any \( \varepsilon > 0 \) small, \( f \in B_{2,2,\infty,\alpha}^{\nu-\varepsilon} \) \((\alpha \text{ referring to the weight})\). These bounds were obtained by using punctual (in \( j \) and \( k \)) estimates. But, if we take into account that we have also summability, then we can relax the parameter \( \varepsilon \), and we get in fact that \( f \in B_{2,2,\infty,\alpha}^{\nu} \), for all \( t \geq t_1 > t_0 \).

**Iteration - Second step**

We now want to improve the index of regularity. For this purpose, we need to work back on the terms \( A \) and \( B \), from which we deduce the two estimates appearing on the third line of (9.31). In order to improve these two estimates, the simplest way is to use the results from Chapter 7. Then, in view of the regularity obtained in the first step, we obtain immediately that

\[
|A| \lesssim \|g\|_{W^{1,\frac{\nu}{2}}_{\alpha}} \|f\|_{B_{2,2}\nu,\alpha} 2^{-k\nu} 2^{-j\alpha} \|\psi_j p_k f\|_{L^2_j} \lesssim 2^{-k\nu} 2^{-j\alpha} \|\psi_j p_k f\|_{L^2}
\]

(9.32)

(for all big \( \alpha \)). Similarly, taking into account the results on \( T^1 \) from Chapter 7 and the fact that there is a commutator appearing in \( B \), we get

\[
|B| \lesssim 2^{-j\alpha} 2^{-k(1-\nu)} \|\psi_j p_k f\|_{L^2}.
\]

(9.33)
Replacing the two estimates on the third line of (9.31) by the estimates obtained in (9.32) and (9.33), we get this time from (9.31)
\[ \partial_t U_{j,k} + 2^{j\gamma} 2^{k\nu} U_{j,k} \lesssim 2^{j(\gamma - 2)} 2^{-k\nu} \]
and by iterating, we get
\[ U_{j,k} \lesssim 2^{-j\alpha} 2^{-k(2\nu - \varepsilon)} \]
and finally \( f \in B^{\nu}_{2,2,\alpha} \) for all \( \alpha \geq 0 \), by the same type of arguments.

In conclusion, we have passed from the regularity index \( \frac{\nu}{2} \) to the regularity index \( \nu \).

We can now bootstrap this new index of regularity, by using it to again get improved estimates on \( \mathcal{A} \) and \( \mathcal{B} \). That is, we get estimates similar to (9.32) and (9.33) but with \( \nu \) replaced by \( 2\nu \). This concludes the proof.

### 9.3 Final comments

We wish to finish on some remarks connected in particular with assumption (9.6).

1) First of all, we assumed that \( \nu \in (0, 1) \). This is only for convenience, since we have used results from Chapter 7. The range \( \nu \in [1, 2) \) is in fact available, see [12]. Thus, our main result can be also extended to this case.

2) Next, what about relaxing assumption (9.6)? Then, note that adding the assumption of boundedness on entropy dissipation rate (which is in fact part of the definition of a good entropic weak solution), we can assume that

\[ \int_0^T \| <v>^\gamma \sqrt{f(s)} \|_{H^{\frac{\nu}{2}}}^2 ds < +\infty. \]

From the books quoted in the bibliography, in particular [101], we get

\[ \int_0^T \| <v>^{\gamma} f(s) \|_{H_{p_1}} ds < +\infty, \]

where \( p_1 = \frac{n}{n - \frac{\nu}{2}} > 1 \), but \( p_1 < 2 \).

To simplify the exposition, let’s forget about integrability w.r.t. time \( t \). Then it follows from Sobolev embedding that \( <v>^{\gamma} f \in L^{p_2} \), where \( p_2 = \frac{n}{n - \nu} \). Of course \( p_2 > 1 \), but we note that \( p_2 \geq 2 \) iff \( \nu \geq \frac{n}{2} \). In particular, in dimension \( n = 2 \), this is the case iff \( \nu \geq 1 \), while in dimension 3, this is the case iff \( \nu \geq \frac{3}{2} \). In conclusion when \( \nu \) is really very close to 2, then this \( L^2 \) bound is available.

In conclusion, in dimension \( n = 2 \) or \( n = 3 \), it should be certainly possible (with some extra work) to relax assumption (9.6) and get our result.
We also note, that having in mind [53], small power of $f$ should have good regularity. These small remarks explain also the fact that Landau equation, corresponding to a version of Boltzmann equation with $\nu = 2$ is much more easiest to deal with, see for instance [62].

3) Finally, as regards the non homogeneous version of Boltzmann equation, let us note our work in progress [13], where we show regularization properties, for solutions satisfying very weak assumptions. This is has to be compared to [40], where initial assumptions are quite strong.
Chapter 10

Remarks on $L^p$ estimates for Homogeneous Boltzmann equation

The purpose of this short chapter is to show some weighted $L^p$ estimates for solutions of the homogeneous Boltzmann equation, and for singular kernels. First results in this direction are due to Desvillettes and Mouhot [52]. However, our method of proof uses some ideas already introduced in the context of renormalisation for the full (i.e. non homogeneous) Boltzmann equation. As such, it is then possible to consider more general nonlinearities. More precisely, we shall use essentially Taylor formula at order one for functions $\beta : \mathbb{R}^+ \to \mathbb{R}$, in the following form (assuming smoothness): for all $a, b$ in $\mathbb{R}^+$, $\beta'$ denoting the derivative of $\beta$

$$\beta'(a)(b-a) = [\beta(b) - \beta(a)] - R_{\beta}(a,b)$$

(10.1)

where $R_{\beta}(a,b)$ denotes the reminder in Taylor formula; so it is defined alternatively as

$$R_{\beta}(a,b) = [\beta(b) - \beta(a)] - \beta'(a)(b-a)$$

(10.2)

or also by

$$R_{\beta}(a,b) = \frac{(b-a)^2}{2} \int_{0}^{1} (1-u)\beta''(a+u(b-a))du.$$ 

(10.3)

Of particular interest is the case when $\beta$ is convex on $\mathbb{R}^+$, so that when $R_{\beta}(a,b) \geq 0$; in particular, this the case for the choice $\beta(s) = s^p$, $p \geq 1$, made by Desvillettes and Mouhot.

Let us recall homogeneous Boltzmann equation, in the context of singular kernels, see [11, 118] for more details. Let a positive function $f = f(t,v) \geq 0$, $t \in \mathbb{R}^+$ time variable, $v \in \mathbb{R}^n$ velocity variable, be solution in standard sense of homogeneous Boltzmann equation:

$$\partial_t f(t,v) = Q(f,f)(t,v),$$

(10.4)
where $Q$ is Boltzmann collisional operator. For smooth functions $g$ and $f$, it acts only w.r.t. variable $v$ and is given by

\[(10.5)\]

\[Q(g, f)(v) = \int_{\mathbb{R}^n} dv_* \int_{S^{n-1}} d\sigma B(|v - v_*|, \cos \theta)[g'_* f' - g_* f],\]

where $g'_* = g(v'_*)$, $g_* = g(v_*)$, $f' = f(v')$ and $f = f(v)$ are standard short hand notations. For a given couple $(v, v_*)$ in $\mathbb{R}^{2n}$, $v'$ and $v'_*$ are given ($\sigma$-representation) by the rules

\[(10.6)\]

\[v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma \quad \text{and} \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma.\]

Here $\theta$ is defined by

\[\cos \theta = \frac{v - v_*}{|v - v_*|} \sigma, \theta \in (0, \pi/2).\]

Function $B$ in (10.5), in general, may be assumed to satisfy the singular assumptions as in [11], [14]. However, only for simplifications, as may be checked from the proofs, it is enough to have in mind the following special case:

\[(10.7)\]

\[B(v - v_*, \cos \theta) = |v - v_*|^\gamma b(\cos \theta).\]

Above $\gamma > 0$, $b(\cos \theta)$ is bounded away from $\theta = 0$ and has a singularity for $\theta \sim 0$ as follows

\[(10.8)\]

\[\sin^{n-2} \theta b(\cos \theta) \sim \frac{c}{\theta^{1+\nu}} \quad \text{as} \quad \theta \to 0,\]

for some fixed non negative constants $c$ and $0 < \nu < 2$ with the following link

\[\gamma = \frac{s - 5}{s - 1} \quad \text{and} \quad \nu = \frac{2}{s - 1}, \quad \text{for some} \quad s > 2.\]

We shall also simply write $B(,,)$, omitting the arguments of $B$.

One has the following result, choosing $\beta(u) = u^p$, $p \geq 1$,

**Proposition 10.0.1** Under the above assumptions, choose $\beta(u) = u^p$, for some $p \geq 1$. Then, for any positive and smooth function $f$, for all $\alpha \geq 0$, with $p\alpha \geq 1$, one has

\[\int_{\mathbb{R}^n} dv Q(f, f) \beta'(f) < v >^{p\alpha} + \int_{\mathbb{R}^n} dv \int_{\mathbb{R}^n} dv_* \int_{S^{n-1}} d\sigma B(,,) g'_* R \beta(f, f') \leq C^+ \|f\|_{L^p}^p - K^- \|f\|_{L^p}^{p + \frac{2}{p}}.\]

Above, $C^+$ and $K^-$ are positive constants depending on an upper bound of $\|f\|_{L^1}$, on an upper bound of the entropy and on a lower bound of $\|f\|_{L^1}$.

This result is only but a slight improvement of Proposition 2.5 of [52]. Of course, from there, one can deduce the main Theorem 1.1 of [52], by using the improved integrability for the gain term, as shown
by Mouhot and Villani [97], following previous works of Wennberg [122], simplifying the first proof by Lions [81]. Let us note that it is still an open question to get this improved integrability directly.

The (slight) improvements w.r.t. [52] of the proof below are:

- it is possible to consider more general nonlinearities $\beta$;
- we are also able to get a term similar to the entropy dissipation rate; this is the second term on the l.h.s of the above inequality, see [11] for possible implications.

The rest of the chapter is devoted to the short proof of the above Proposition. Moreover, from time to time, $g$ denotes any smooth positive function, that we shall set to be $f$, but only in the final steps. By this way, it should be certainly possible to have estimations, but in a linear context.

### 10.1 Singular kernel and renormalisation

We first use ideas from renormalised solutions, in the context of singular kernels, see [14, 10]. Let $\phi = \phi(v)$ be a smooth function. In the applications we have in mind, it will be taken as $\phi(v) = \langle v \rangle^p$, for $p \geq 1$, $\alpha \in \mathbb{R}^+$. Let $\beta$ a smooth function, $\beta'$ its derivative, and $R_{\beta}$ its remainder.

Then, $<\cdot,\cdot>$ denoting the standard duality bracket, one has

$$<Q(g,f);\phi\beta(f)> = \int_{\mathbb{R}^n} dv \int_{\mathbb{R}^n} dv_* \int_{\mathbb{S}^{n-1}} d\sigma B(\cdot,\cdot) \{g'_*(f' - f) + f(g'_* - g_*)\} \phi\beta'(f).$$

Beware of the fact that $\beta'$ is the derivative and $f'$ is the value of $f$ at $v'$!

Splitting the term, we get

$$<Q(g,f);\phi\beta(f)> = A_1 + A_2,$$

where

$$A_1 \equiv \int_{\mathbb{R}^n} dv \int_{\mathbb{R}^n} dv_* \int_{\mathbb{S}^{n-1}} d\sigma B(\cdot,\cdot) g'_*(f' - f) \phi\beta'(f),$$

$$A_2 \equiv \int_{\mathbb{R}^n} dv \phi f \beta'(f) \int_{\mathbb{R}^n} dv_* \int_{\mathbb{S}^{n-1}} d\sigma B(\cdot,\cdot) (g'_* - g_*).$$

The treatment of $A_2$ from (10.10) is clear in view of [11], [14]. That is, using the Cancellation Lemma therein, one has

$$A_2 = \int_{\mathbb{R}^n} dv \phi f \beta'(f) S * g(v),$$

where the convolution kernel $S$ is given by the rule in the above mentioned papers. Thus, one can get upper bounds estimates on it.
For $A_1$ given by (10.9, we now use the convexity of $\beta$ to get

\[(10.11)\]

\[A_1 \leq \int_{\mathbb{R}^n} dv \int_{\mathbb{R}^n} dv_s \int_{S^{n-1}} d\sigma B(\cdot, \cdot) g'_s \phi [\beta(f') - \beta(f)] \equiv B.\]

Note that the term we have missed, denoted by $D_\beta$ can be (having in mind Boltzmann equation) put on the l.h.s. of the equation, which also gives an information if for example $\beta$ was superquadratic. Of course, we have defined $D_\beta \geq 0$ by the formula

\[(10.12)\]

\[D_\beta \equiv \int_{\mathbb{R}^n} dv \int_{\mathbb{R}^n} dv_s \int_{S^{n-1}} d\sigma B(\cdot, \cdot) g'_s \phi [\beta(f') - \beta(f)].\]

Then one can write $B$ given by (10.11) as follows

\[(10.13)\]

\[B = \int_{\mathbb{R}^n} dv \int_{\mathbb{R}^n} dv_s \int_{S^{n-1}} d\sigma B(\cdot, \cdot) g'_s \phi [\beta(f') - \beta(f)] + \int_{\mathbb{R}^n} dv \int_{\mathbb{R}^n} dv_s \int_{S^{n-1}} d\sigma B(\cdot, \cdot) g'_s \phi [\beta(f') - \beta(f)].\]

We note that the first term is 0 since this is nothing else than the integral of $Q(g, \phi \beta(f))$.

By gluing all the above estimates, we have obtained:

\[(10.13)\]

\[<Q(g, f); \phi \beta(f) > + D_\beta = \int_{\mathbb{R}^n} dv \phi [\beta(f') - \beta(f)] \cdot S * g(v) + <Q(g, \beta(f)); \phi > \equiv B_1 + B_2.\]

Note also that

\[B_2 \equiv <Q(g, \beta(f)); \phi > = \int_{\mathbb{R}^n} dv \int_{\mathbb{R}^n} dv_s \int_{S^{n-1}} d\sigma B(\cdot, \cdot) g_s \beta(f)[\phi' - \phi].\]

From [11] and also its proof, we know that, for a.a. $v \in \mathbb{R}^n$,

\[(10.14)\]

\[Sg \equiv \int_{\mathbb{R}^n \times S^{n-1}} dv_s d\sigma B(v - v_s, \sigma)(g'_s - g_s) = g *_v S,\]

where

\[(10.15)\]

\[S(|z|) = |S^{n-2}| \int_{0}^{\pi/2} d\theta \sin^{n-2} \theta \left[ \frac{1}{\cos^n(\frac{\theta}{2})} B \left( \frac{|z|}{\cos(\frac{\theta}{2})}, \cos \theta \right) - B(|z|, \cos \theta) \right].\]

Making the explicit choice of the special assumption (though it is not really necessary), we obtain $S(|z|) = c_b |z|^{\gamma}$, and thus

\[Sg(v) = c_b \int_{\mathbb{R}^n} dv_s |v - v_s|^{\gamma} g(v_s)\]

and thus

\[|Sg(v)| \leq c_b < v >^\gamma \|g\|_{L^1_v}.\]
Making now the choice $\phi = <v >^{p\alpha}$ and $\beta(s) = s^p$, for $p \geq 1$, we obtain
\[
B_1 \equiv \int \phi[f\beta'(f) - \beta(f)].S * g(v) \leq C\|g\|_{L^1_{\alpha+\gamma}}\|f\|_{L^p_{\alpha+\gamma}}^p.
\]
It remains to deal with the last term $B_2$ and for this purpose, we need to have an estimate for
\[
E = |v - v_*|^7 \int_{S_{\nu-1}} b(\cos \theta)|\phi' - \phi|,
\]
where recall that $\phi = <v >^{p\alpha}$; this fails under the scope of previous papers, and in particular [14], but we can proceed differently here; we sketch the proof when $0 < \nu < 1$.

First note that $| < v' >^{p\alpha} - < v >^{p\alpha} | \lesssim < v >^{p\alpha} + < v_* >^{p\alpha}$.

Furthermore, by using Taylor formula at order one (this is enough in the case $0 < \nu < 1$; in the case $\nu \geq 1$, we need Taylor's formula at order two, the symetrization used in [14] and similar computations as done below), one obtains
\[
| < v' >^{p\alpha} - < v >^{p\alpha} | \lesssim |v - v_*| \sin \frac{\theta}{2} [ < v >^{p\alpha-1} + < v_* >^{p\alpha-1}]
\]
Set $M_1 = |v - v_*|(< v >^{p\alpha-1} + < v_* >^{p\alpha-1})$ and $M_2 = < v >^{p\alpha} + < v_* >^{p\alpha}$ (up to unimportant constants).

We have just shown that
\[
| < v' >^{p\alpha} - < v >^{p\alpha} | \lesssim \min \{ M_1 \sin \frac{\theta}{2}, M_2 \}
\]
Computing a spherical integral, one gets
\[
E \lesssim |v - v_*|^7 \{ < v >^{p\alpha} + < v_* >^{p\alpha} \}^{\nu+1} \{ < v >^{p\alpha-1} + < v_* >^{p\alpha-1} \}^{\nu}
\]
\[
\lesssim < v >^{p\alpha+\gamma} < v_* >^{\gamma+\nu} + < v >^{\gamma+\nu} < v_* >^{p\alpha+\gamma} +
\]
\[
+ < v >^{p\alpha+\gamma} < v_* >^{p\alpha(-\nu+1)+\gamma+\nu} + < v_* >^{p\alpha\nu+\gamma} < v >^{p\alpha(-\nu+1)+\gamma+\nu}
\]
Thus we have obtained
\[
B_2 \equiv < Q(g, \beta(f)); \phi > \lesssim \|g\|_{L^1_{\alpha+\gamma}}\|f\|_{L^p_{\alpha+\gamma+\nu}}^p + \|g\|_{L^1_{\alpha+\gamma+\nu}}\|f\|_{L^p_{\alpha+\gamma+\nu}}^p.
\]
(10.16)
For the usual inverse power law potentials, one has for some $s > 2$
\[
\gamma = \frac{s - 5}{s - 1} \quad \text{and} \quad \nu = \frac{2}{s - 1}
\]
In particular $\gamma + \nu = \frac{s - 3}{s - 1}$, $-1 < \gamma + \nu < 1$. So if we choose $p$ and $\alpha$ such that $p\alpha$ is bigger than 1, then we get directly Corollary 2.2 from [52].

Next, the point is that if we consider singular kernels, but cutoff for large values of $\theta$, that is with factor $\|\sin \frac{\theta}{2} \leq \delta\$, then the same result holds, up to a factor of $\delta^{2-\nu}$ ($\delta^{1-\nu}$ also if we assume that $0 < \nu < 1$).
10.2 Non singular kernel and renormalisation

Similarly, we now use ideas of renormalisation, but this time for non singular kernels, as introduced by Di Perna and Lions [57, 81]. According to the last part of the previous section, let us fix $\delta > 0$ and denote by $B_\delta$ the kernel truncated at level $\delta$; that is multiplied by $I_{\sin \frac{\theta}{2} \geq \delta}$. We denote by $Q_\delta$ the corresponding collision operator; then it follows that, with usual notations:

$$< Q_\delta(g, f); \phi \beta'(f) > = < Q_\delta^+(g, f); \phi \beta'(f) > - < Q_\delta^-(g, f); \phi \beta'(f) > .$$

By definition, one has

$$< Q_\delta^+(g, f); \phi \beta'(f) > = \int_{\mathbb{R}^n} dv \int_{\mathbb{R}^n} dv_* \int_{S^{n-1}} d\sigma_g f_{[v]} \beta'(f') B_\delta,$$

that we split into two parts according to whether or not $|v| \leq R$ for some large $R$,

$$< Q_\delta^+(g, f); \phi \beta'(f) > = I_{R^-} + I_{R^+},$$

where

\begin{align}
I_{R^-} & \equiv \int_{\mathbb{R}^n} dv \int_{\mathbb{R}^n} dv_* \int_{S^{n-1}} d\sigma_g f_{[v]} [v] \leq R \phi(v') \beta'(f') B_\delta \\
I_{R^+} & \equiv \int_{\mathbb{R}^n} dv \int_{\mathbb{R}^n} dv_* \int_{S^{n-1}} d\sigma_g f_{[v]} [v] \geq R \phi(v') \beta'(f') B_\delta
\end{align}

We first deal with $I_{R^-}$ that we split, for some large parameter $\mu_1$ as follows

$$I_{R^-} = \int_{\mathbb{R}^n} dv \int_{\mathbb{R}^n} dv_* \int_{S^{n-1}} d\sigma_g f_{[v]} [v] \leq R \phi(v') \beta'(f') [v] \leq \mu_1 f B_\delta + \int_{\mathbb{R}^n} dv \int_{\mathbb{R}^n} dv_* \int_{S^{n-1}} d\sigma_g f_{[v]} [v] \geq R \phi(v') \beta'(f') [v] \leq \frac{1}{\mu_1} f B_\delta$$

With our choice of $\beta(u) = u^p$, it follows that

$$I_{R^-} \leq \int_{\mathbb{R}^n} dv \int_{\mathbb{R}^n} dv_* \int_{S^{n-1}} d\sigma_g f_{[v]} [v] \leq R \mu_1^{p-1} \beta(f) B_\delta + \int_{\mathbb{R}^n} dv \int_{\mathbb{R}^n} dv_* \int_{S^{n-1}} d\sigma_g f_{[v]} [v] \geq R \mu_1^{p-1} \beta(f) B_\delta$$

$$\leq \frac{1}{\delta^p} R^\frac{2}{p} \mu_1^{p-1} \|g\|_{L^1_{\delta \alpha+\gamma}} \|f\|_{L^p_{\alpha}} + \frac{1}{\delta^p} \|g\|_{L^1_{\delta \alpha+\gamma}} \|f\|_{L^p_{\alpha+\gamma}}.$$}

The factor $\frac{1}{\delta^p}$ comes from the explicit computation of the spherical integral of $B_\delta$; we have also used the Cancellation Lemma (or more precisely its proof, see [11]) to get the second term.

Next, for $I_{R^+}$, making the change of variables $"v \to v_*"$, one gets first

$$I_{R^+} = \int_{\mathbb{R}^n} dv \int_{\mathbb{R}^n} dv_* \int_{S^{n-1}} d\sigma_g f_{[v_*]} [v_*] \geq R \phi(v'_*) \beta'(f'_*) B_\delta.$$
Then, we decompose, for a parameter $\mu_2$, this last integral according to whether or not $g \leq \mu_2 f'$:

$$I_{R^+} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{S^{n-1}} d\sigma g f_* I_{|v_*| \geq R} g \leq \mu_2 f_\ast \phi(v_\ast) \beta'(f_\ast) B_\delta + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{S^{n-1}} d\sigma g f_* I_{|v_*| \geq R} f_* \leq \frac{1}{\mu_2} g \phi(v_\ast) \beta'(f_\ast) B_\delta$$

Using moreover the change of variables $\sigma \to -\sigma$ for the first integral (as in [52]), we obtain, taking now $g = f$, for a suitable kernel $\tilde{B}$

$$I_{R^+} \lesssim \mu_2 p \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{S^{n-1}} d\sigma f_* I_{|v_*| \geq R} \phi(v_\ast) \beta'(f_\ast) B_\delta + p \frac{1}{\mu_2} - \frac{1}{\delta P} \|f\|_{L^{1+\gamma}_p} \|f\|_{L^p}^p$$

$$\lesssim \mu_2 p \frac{1}{\delta^p} \|f\|_{L^{1+\gamma}_p} \|f\|_{L^p}^p + \frac{1}{\mu_2} \frac{1}{\delta^p} \|f\|_{L^{1+\gamma}_p} \|f\|_{L^p}^p.$$

It follows that

$$<Q^+ (g, f); \phi \beta'(f)> \lesssim \frac{1}{\delta^p} \frac{1}{\mu_1^{\gamma - 1}} \|g\|_{L^{1+\gamma}_p} \|f\|_{L^p}^p + \frac{1}{\delta^p} \frac{1}{\mu_2^{\gamma - 1}} \|g\|_{L^p} \|f\|_{L^p}^p$$

$$+ \mu_2 p \frac{1}{\delta^p} \frac{1}{\mu_1^{\gamma - 1}} \|f\|_{L^{1+\gamma}_p} \|f\|_{L^p}^p + \frac{1}{\mu_2} \frac{1}{\delta^p} \|f\|_{L^{1+\gamma}_p} \|f\|_{L^p}^p.$$

Then we note finally that (again as in [52]),

$$- <Q^- (f, f); \phi \beta'(f)> \lesssim -K^p.$$

To finish, we choose $\delta$ small enough, then $\mu_1$ large enough, then $\mu_2$ large enough and finally $R$ large enough to get the result.
Chapter 11

Renormalised solutions

11.1 Introduction

References for this chapter are:
- the averaging results [41, 60, 66, 86];
- the works of DiPerna and Lions [56, 57, 58, 59];
- the works of Lions [81, 82, 83, 84, 86];
- our work Alexandre [6, 10];
- our joint work with Villani [14] on which is based this chapter.

The general motivation for this Chapter is to extend or modify DiPerna Lions theory of renormalised solutions for the full non homogeneous Boltzmann, [57, 81, 82, 83], which was designed for non singular kernels (cutoff) to the case of non singular kernels.

There is also another (personal) motivation which is the justification of the so called grazing limit. We shall get back on this point in another Chapter.

We now consider the full non homogeneous Boltzmann equation

\[
\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f, f).
\]

Let us recall the notations

\[
k = \frac{v - v_*}{|v - v_*|}, \quad k \cdot \sigma = \cos \theta, \quad 0 \leq \theta \leq \pi,
\]

and again we assume that \(B(v - v_*, \sigma)\) is supported in the set \((0 \leq \theta \leq \pi/2)\), i.e. \((v - v_*, \sigma) \geq 0\).

We also fix an arbitrary time interval \([0, T]\), and we recall the following conservation laws
\[
\frac{d}{dt} \int_{\mathbb{R}^n \times \mathbb{R}^n} f(t, x, v) \begin{pmatrix} 1 \\ \frac{v}{|v|^2} \end{pmatrix} \, dx \, dv = 0;
\]

\[(11.3) \quad \frac{d}{dt} \int_{\mathbb{R}^n \times \mathbb{R}^n} f(t, x, v) \log f(t, x, v) \, dx \, dv = -\frac{1}{4} \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} dv \int_{S^{n-1}} d\sigma B(v-v_*, \sigma)(f'f_*' - ff_*) \log \frac{f'f_*'}{ff_*}.
\]

The entropy of a nonnegative function \(f(x, v)\) is defined by

\[(11.4) \quad H(f) = \int_{\mathbb{R}^n_x \times \mathbb{R}^n_v} f \log f
\]

and we set

\[
L \log L(\mathbb{R}^n_x \times \mathbb{R}^n_v) = \left\{ f \in L^1(\mathbb{R}^n_x \times \mathbb{R}^n_v) ; \int_{\mathbb{R}^n_x \times \mathbb{R}^n_v} |f| \log(1 + |f|) \, dx \, dv < +\infty \right\}.
\]

Also, given a nonnegative function \(f(v)\), we let

\[(11.5) \quad D(f) = \frac{1}{4} \int_{\mathbb{R}^n} dv \int_{\mathbb{R}^n} dv_* \int_{S^{n-1}} d\sigma B(v-v_*, \sigma)(f'f_*' - ff_*) \log \frac{f'f_*'}{ff_*}
\]

for the (nonnegative) entropy dissipation functional associated to \(f\).

Finally, one has an (local in time) estimate on

\[
\int_{\mathbb{R}^n_x \times \mathbb{R}^n_v} f(t, x, v) |x|^2 \, dx \, dv.
\]

It follows that one has the following a priori estimates

\[(11.6) \quad f \in L^\infty_t ([0, T]; L^1_{x,v}((1 + |v|^2 + |x|^2) \, dx \, dv) \cap L \log L(\mathbb{R}^n_x \times \mathbb{R}^n_v)),
\]

and Di Perna and Lions succeeded in building from these estimates a mathematical theory of weak (renormalized) solutions to equation (11.1), in the following sense, and under the main assumption that collisional cross section satisfy Grad’s cutoff assumption.

**Definition 11.1.1** A nonnegative function \(f \in C(\mathbb{R}^+; L^1(\mathbb{R}^n_x \times \mathbb{R}^n_v))\) is called a renormalized solution of the Boltzmann equation (11.1) if for all nonlinearity \(\beta \in C^1(\mathbb{R}^+, \mathbb{R}^+)\), such that \(\beta(0) = 0, \beta'(f) \leq C/(1 + f),\)

\[(11.7) \quad \frac{\partial \beta(f)}{\partial t} + v \cdot \nabla_x \beta(f) = \beta'(f)Q(f, f),
\]

in the sense of distributions.

\[\square\]
Through weak compactness estimates, Di Perna and Lions made use of
- 1) a renormalized formulation, i.e. a (distributionally) meaningful definition of $\beta'(f)Q(f,f)$ under the above a priori estimates and Grad’s angular cut-off assumption;
- 2) the averaging lemmas, first introduced in the context of kinetic equations by Golse, Lions, Perthame, Sentis and further developed by many others.

The renormalized formulation used by Di Perna and Lions made an explicit use of the cutoff assumption: one splits the collision operator into a positive (“gain”) and a negative (“loss”) part:

$$Q = Q^+ - Q^-,$$

where

$$Q^+(f,f) = \int_{\mathbb{R}^n} dv_s \int_{S^{n-1}} d\sigma B(v - v_s, \sigma) f' f'^*,$$
$$Q^-(f,f) = f(A*v f).$$

Then, one notes that under the assumptions on $\beta$ in Definition 11.1.1,

$$\beta'(f)Q^-(f,f) \leq \frac{Cf}{1 + f}(A*f)$$

lies in $L^\infty([0,T]; L^1_{loc}(\mathbb{R}^n_x \times \mathbb{R}^n_v))$. Finally, integrating equation (11.7) in all variables, one finds the additional a priori estimate

$$\beta'(f)Q^+ (f,f) \in L^1_{loc}([0,T] \times \mathbb{R}^n_x \times \mathbb{R}^n_v).$$

The above study was latter extended, simplified by Lions, through the introduction of a new result, namely regularity of the gain term. In particular, DiPerna and Lions have shown

**Theorem 11.1.1** Assume Grad’s angular cut-off, together with a suitable growth at infinity. Let $(f^m)_{m \in \mathbb{N}}$ be a sequence of renormalized solutions of the Boltzmann equation with respective initial datum $(f^m_0)$, satisfying uniform estimates of mass, energy and entropy. Assume w.l.o.g that $f^m \rightharpoonup f$ weakly in $L^p([0,T], L^1(\mathbb{R}^n_x \times \mathbb{R}^n_v))$ ($1 \leq p < +\infty$). Then

(i) $f$ is a renormalized solution of the Boltzmann equation;

(ii) let $f_0$ denote $f(0, \cdot, \cdot)$; then

$$f^m \rightarrow f \text{ strongly if and only if } f^m_0 \rightarrow f_0 \text{ strongly in } L^1(\mathbb{R}^n_x \times \mathbb{R}^n_v).$$

\[\square\]

In this chapter, our aim is to remove this cutoff assumption, and thus to consider singular cross-sections, including also the special case of Coulomb potential, for which there is a singularity with respect to the relative velocity, cases not covered by DiPerna and Lions theory.
This extension will mainly use three tools:

- smoothness estimates associated with the entropy dissipation in presence of grazing collisions; that is more precisely results from Chapter 5;
- a different procedure of renormalization;
- cancellation effects associated with the symmetries of the Boltzmann kernel. More precisely results from Chapter 3.

On the other hand, the price to pay for such an extension will be a weakening of the notion of renormalized solution, similar to the existing theory for non homogeneous Landau equation. Thus, a lot remains to be done!

**Definition 11.1.2** A nonnegative function $f$

$$f \in C(R^+; \mathcal{D}'(R^2_+ \times R^n)) \cap L^\infty(R^+; L^1((1 + |v|^2 + |x|^2), dx, dv))$$

is a renormalized solution of the Boltzmann equation with a defect measure, if for all nonlinearity $\beta \in C^2(R^+, R^+)$ satisfying

(11.10) \[ \beta(0) = 0, \quad 0 < \beta'(f) \leq \frac{C}{1 + f}, \quad \beta''(f) < 0, \]

the inequality

(11.11) \[ \frac{\partial \beta(f)}{\partial t} + v \cdot \nabla_x \beta(f) \geq \beta'(f)Q(f, f) \]

holds in distribution sense, together with the mass-conservation condition

(11.12) \[ \forall t \geq 0, \quad \int_{R^n \times R^n} f(t, x, v) \, dx \, dv = \int_{R^n \times R^n} f(0, x, v) \, dx \, dv. \]

**Example :** A typical choice for $\beta(f)$ will be $f/(1 + \delta f)$, $\delta > 0$.

We shall precise the sense of $\beta'(f)Q(f, f)$ in (11.11) latter on.

### 11.2 Main results

In this Chapter, we assume that Assumptions I, II and III from Chapter 3 hold true. Thus, in comparison with the DiPerna-Lions theory, the changes are the parameter $\nu$ of course, but also the possibility of letting $\gamma = -n$. 

\[ 131 \]
Theorem 11.2.1 (Stability and appearance of strong compactness) Make assumptions I, II, III from Chapter 3. Let $(f^n)$ be a sequence of solutions to the Boltzmann equation, in the sense of Definition 11.1.2, satisfying the natural entropic a priori estimates. Assume w.l.o.g that $f^n \rightharpoonup f$ weakly in $L^p([0,T], L^1(\mathbb{R}^n_x \times \mathbb{R}^n_v))$ ($1 \leq p < +\infty$). Then

(i) $f$ is a solution to the Boltzmann equation, in the sense of Definition 11.1.2;

(ii) moreover, automatically, $f^n \to f$ strongly. \hfill $\Box$

Corollary 11.2.1 (Existence of weak solutions) Make assumptions I, II, III from Chapter 3. Let $f_0$ be an initial datum satisfying the natural assumption

$$\int_{\mathbb{R}^n_x \times \mathbb{R}^n_v} f_0(x,v)[1 + |v|^2 + |x|^2 + \log f_0(x,v)] \, dx \, dv < +\infty.$$ 

Then, there exists a solution $f$ to the Boltzmann equation, in the sense of Definition 11.1.2, with initial datum $f(0,\cdot,\cdot) = f_0$. This solution satisfies, for all $t \geq 0$,

(11.13) $$\int_{\mathbb{R}^n_x \times \mathbb{R}^n_v} f(t,x,v) \, dx \, dv = \int_{\mathbb{R}^n_x \times \mathbb{R}^n_v} f_0(x,v) \, dx \, dv,$$

(11.14) $$\int_{\mathbb{R}^n_x \times \mathbb{R}^n_v} f(t,x,v)v \, dx \, dv = \int_{\mathbb{R}^n_x \times \mathbb{R}^n_v} f_0(x,v)v \, dx \, dv,$$

(11.15) $$\int_{\mathbb{R}^n_x \times \mathbb{R}^n_v} f(t,x,v) \frac{|v|^2}{2} \, dx \, dv \leq \int_{\mathbb{R}^n_x \times \mathbb{R}^n_v} f_0(x,v) \frac{|v|^2}{2} \, dx \, dv,$$

and the entropy inequality

(11.16) $$H(f(T,\cdot,\cdot)) + \int_0^T dt \int_{\mathbb{R}^n} dx D(f(t,x,\cdot)) \leq H(f_0).$$ \hfill $\Box$

11.3 Renormalized formulation

In this section, we consider a fixed function $\beta : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the assumptions of Definition 11.1.2. In particular, $\beta$ is 1-to-1, grows at most logarithmically, and $\beta^{-1}$ is convex. Typical choices are

$$\beta(f) = \frac{1}{\delta} \log(1 + \delta f), \quad \beta(f) = \frac{f}{1 + \delta f}, \quad \delta > 0.$$
In fact, for simplicity, we shall also assume in the sequel that $\beta$ is bounded, and leave the general case to the reader. We write, for all nonnegative numbers $f, f'$,

$$f' - f = \beta^{-1}(\beta(f')) - \beta^{-1}(\beta(f)) = \frac{1}{\beta'(f)} \left[ \beta(f') - \beta(f) \right] + \Theta(f, f')[\beta(f') - \beta(f)]^2,$$

where

$$\Theta(f, f') = \int_0^1 ds (1 - s)(\beta^{-1})''\left(\beta(f) + s[\beta(f') - \beta(f)]\right).$$

**Example:** For $\beta(f) = f/(1 + \delta f)$, one finds

$$\Theta(f, f') = \delta(1 + \delta f)^2 (1 + \delta f').$$

Using identity (11.17), we obtain

$$\beta'(f)Q(f, f) = \left( R_1 \right) + \left( R_2 \right) + \left( R_3 \right),$$

where

$$\left( R_1 \right) = \int_{\mathbb{R}^n \times S^{n-1}} dv_* \ d\sigma B(f'_* - f_*),$$

$$\left( R_2 \right) = \int_{\mathbb{R}^n \times S^{n-1}} dv_* \ d\sigma B\left[ f'_* \beta(f') - f_* \beta(f) \right],$$

$$\left( R_3 \right) = \int_{\mathbb{R}^n \times S^{n-1}} dv_* \ d\sigma B\beta'(f) f'_* \Theta(f, f')[\beta(f') - \beta(f)]^2.$$

(We have omitted the arguments of $B$ for notational convenience.)

We define $S$ as the linear operator

$$Sf = \int_{\mathbb{R}^n \times S^{n-1}} dv_* \ d\sigma B(v - v_*, \sigma)(f'_* - f_*).$$
Thus,
\[(R_1) = [f \beta'(f) - \beta(f)] Sf.\]

Note that this is exactly the operator analyzed in Chapter 3.

The structure of the renormalized formulation is much clearer when expressed in terms of the (asymmetrical) bilinear Boltzmann operator,
\[Q(g,f) = \int_{\mathbb{R}^n \times S^{n-1}} dv_\ast d\sigma B(v - v_\ast, \sigma)(g_\ast f' - g_\ast f).\]

Indeed, note that
\[(R_2) = Q(f, \beta(f)).\]

Moreover, the same manipulation as above shows that
\[
\beta'(f)Q(g,f) = [f \beta'(f) - \beta(f)] Sg + Q(g, \beta(f)) + \int_{\mathbb{R}^n \times S^{n-1}} dv_\ast d\sigma B g_\ast \beta'(f) \Theta(f, f') [\beta(f') - \beta(f)]^2.
\]

This formula should actually be taken as the definition of the renormalization of the asymmetrical Boltzmann operator without cut-off.

### 11.3.1 Symmetry-induced cancellation effects: the term \((R_1)\)

As regards the term \((R_1)\), we shall of course use the Cancellation Lemma from Chapter 3. It follows that one has the following corollary.

**Corollary 11.3.1** Let \(B\) satisfy assumptions I and II from Chapter 3, and let \(f\) satisfy an entropic bound. Then, \((R_1)\) defined by (11.19) lies in \(L^\infty([0,T];L^1(\mathbb{R}_x^n \times B_R(v)))\), for all \(R > 0\), where \(B_R(v) = \{v \in \mathbb{R}^n, |v| \leq R\}\).

\[\square\]

**Proof of Corollary 11.3.1:**

Since \(f \beta'(f) \in L^\infty\) by assumption, we only need to show that \(f \ast S \in L^1(\mathbb{R}_x^n \times B_R(v))\). But
\[
\|f \ast S\|_{L^1(\mathbb{R}_x^n \times B_R(v))} \leq \int_{|v| \leq R} f(v_\ast)|S(v - v_\ast)| \, dv \, dv_\ast \, dx
\]
\[
\leq \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^N} dv_\ast f(v_\ast) \int_{|z + v_\ast| \leq R} d|z||S(z)|
\]
\[
\leq \|f\|_{L^1(\mathbb{R}_x^n \times \mathbb{R}_x^n)} \|S\|_{TV(|z| \leq R + R')} + \frac{1}{(R')^2} \left[ \int dx \, dv_\ast f(v_\ast)|v_\ast|^2 \right] \sup_{|v_\ast| \geq R'} \int_{|z + v_\ast| \leq R} d|z||S(z)|,
\]
for all \( R' > 0 \). Here \( \| \cdot \|_{TV} \) denotes the norm in total variation.

From the Cancellation Lemma and Assumption II, from Chapter 3, we have \( |S(z)| = o(|z|^2) \) as \( |z| \to \infty \), and the above expression is finite for \( R' \) large enough.

### 11.3.2 Dual formulation of the bilinear non-cutoff Boltzmann operator: the term \((R_2)\)

Now, we tackle \((R_2)\). This term, which is nothing but the action of the bilinear Boltzmann operator on \( f \) and \( \beta(f) \), and it essentially involves fractional derivatives of these functions, as we have seen in previous Chapters. For the results of this Chapter, one could use the precise results (or methods) of Chapters 7 or 8, but here it is enough to proceed with a weaker result.

The above term is given a sense by duality as we know. Let \( \phi(v) \) be a (smooth) test-function in the velocity variable, then

\[
\int (R_2) \phi(v) \, dv = \int_{\mathbb{R}^n \times S^{n-1}} dv \, dv_\ast \, d\sigma \, B[f'_\ast \beta(f') - f_\ast \beta(f)] \phi
\]

\[
= \int_{\mathbb{R}^n} dv \, dv_\ast \, f_\ast \beta(f) \left[ \int_{S^{n-1}} B(v - v_\ast, \sigma)(\phi' - \phi) \, d\sigma \right].
\]

For given \( v_\ast \), let us introduce the linear operator

\[
(11.22) \quad T : \phi \mapsto \int_{S^{n-1}} B(v - v_\ast, \sigma)(\phi' - \phi) \, d\sigma.
\]

**Proposition 11.3.1** \((W^{2,\infty} \to L^\infty \text{ bound for } T)\) Let \( B \) satisfy Assumption I from Chapter 3. Then, for all \( \varphi \in W^{2,\infty}(\mathbb{R}^n) \),

\[
|T \varphi(v)| \leq \frac{1}{2} \| \varphi \|_{W^{2,\infty}} |v - v_\ast| \left( 1 + \frac{|v - v_\ast|}{2} \right) M(|v - v_\ast|).
\]

Moreover, for all \( \alpha \in [0, 2] \) and \( \varphi \in W^{2,\infty}(\mathbb{R}^n) \),

\[
|T \varphi(v)| \leq 2 \| \varphi \|_{W^{2,\infty}} (1 + |v - v_\ast|)^\alpha M^\alpha(|v - v_\ast|),
\]

using notations from Chapter 3.

**Corollary 11.3.2** Let \( B \) satisfy Assumptions I and II from Chapter 3, and let \( f \) satisfy an entropic bound. Then, for all \( R > 0 \), the term \((R_2)\) defined by (11.20) lies in \( L^\infty([0, T]; L^1(\mathbb{R}^n; W^{-2,1}(B_R(v)))) \), where \( B_R(v) \) still denotes \( \{ v \in \mathbb{R}^n, |v| \leq R \} \).
Proof of Corollary 11.3.2:
For a.a. \( t, x \),
\[
\| (R_2) \|_{W^{2,1}(B_R(v))} = \sup \left\{ \int_{\mathbb{R}^n} (R_2) \varphi \, dv; \quad \varphi \in W^{2,\infty}(B_R(v)), \quad \| \varphi \|_{W^{2,\infty}} \leq 1 \right\}
\]
\[
\leq \int_{\mathbb{R}^n \times B_R(v)} dv \, dv_*, \beta(f) f_* |T \varphi|
\]
\[
\leq \int_{\mathbb{R}^n \times B_R(v)} dv \, dv_* \beta(f) f_* (|v - v_*| M(|v - v_*|) \| v - v_* \| \leq R + 2 |v - v_*| \| v - v_* \| M^\alpha(|v - v_*|) \| v - v_* \| \geq R)
\]
\[
\leq C(1 + R) \| f \|_{L^1} \| \varphi \|_{L^1(\varphi \leq R)} + \varepsilon(R) \int_{\mathbb{R}^n \times B_R(v)} dv \, dv_* |v - v_*|^2 \, dv \, dv_*
\]
where we used assumption II from Chapter 3. It then suffices to integrate with respect to \( x \).

Proof of Proposition 11.3.1:
By Taylor formula,
\[
(11.23) \quad \varphi(v') - \varphi(v) = (v' - v) \cdot \nabla \varphi(v) + \frac{1}{2} |v' - v|^2 \left[ \int_0^1 ds (1 - s) D^2 \varphi(v + s(v' - v)) \cdot \left( \frac{v' - v}{|v' - v|}, \frac{v' - v}{|v' - v|} \right) \right].
\]
By symmetry,
\[
\int_{S^{n-1}} d\sigma B(v - v_*, \sigma)(v' - v) = \int_{S^{n-1}} d\sigma B(v - v_*, \sigma)(v' - v, k)k,
\]
where \( k = (v - v_*)/|v - v_*| \). But \( (v' - v, k)k = -(1/2)v - v_*|1 - \cos \theta)k = -(v - v_*) \sin^2(\theta/2) \). Since also \( |v' - v|^2 = |v - v_*|^2 \sin^2(\theta/2) \), we find precisely
\[
(11.24) \quad T \varphi = -\frac{1}{2} M(|v - v_*|)(v - v_*) \cdot \nabla \varphi(v) + \frac{1}{2} |v - v_*|^2 \left[ \int_0^1 ds (1 - s) D^2 \varphi(v + s(v' - v)) \cdot \left( \frac{v' - v}{|v' - v|}, \frac{v' - v}{|v' - v|} \right) \right].
\]
In particular,
\[
(11.25) \quad |T \varphi| \leq \| \varphi \|_{W^{2,\infty}} \left[ \frac{1}{2} M(|v - v_*|)|v - v_*| + \frac{1}{4} |v - v_*|^2 M(|v - v_*|) \right]
\]
\[
= \frac{1}{2} \left( 1 + \frac{|v - v_*|^2}{2} \right) |v - v_*| M(|v - v_*|) \| \varphi \|_{W^{2,\infty}} \leq \frac{1}{2} (1 + |v - v_*|^2) M(|v - v_*|) \| \varphi \|_{W^{2,\infty}}.
\]

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Note that, by the same estimate,

$$|T \varphi| \leq \left( \frac{1}{2} + \frac{R}{4} \right) \|\varphi\|_{W^{2,\infty}} |v - v_s| M(|v - v_s|) \quad \text{if } |v - v_s| \leq R.$$  

But from the definition of $T$ also follows

$$(11.26) \quad |T \varphi| \leq 2 M^0(|v - v_s|) \|\varphi\|_{L^\infty},$$

and since $M = M^2$, by combining (11.25) and (11.26), we obtain the a priori bound

$$|T \varphi| \leq 2(1 + |v - v_s|^2)^{\alpha/2} M^\alpha(|v - v_s|) \|\varphi\|_{W^{2,\infty}}.$$

### 11.3.3 Integrability of the nonnegative remainder: the term $(R_3)$

**Proposition 11.3.2** Let $B$ satisfy assumptions I and II from Chapter 3, and let $f$ be a solution of the Boltzmann equation, satisfying an entropic bound. Then, for all $R > 0$,

$$(11.27) \quad \left(R_3\right) \in L^1([0,T]; L^1(\mathbb{R}^n_x \times B_R(v))),$$

where $(R_3)$ is defined in (11.21).

Proof of Proposition 11.3.2:

The argument is similar to the one due to Lions. Let us integrate the equation (11.11), in the form

$$\frac{\partial \beta(f)}{\partial t} + v \cdot \nabla \beta(f) \geq (R_1) + (R_2) + (R_3),$$

in all variables, against a test-function $\varphi(v) \geq 0$, $\varphi \equiv 1$ on $B_R(v)$, $\varphi \equiv 0$ on $\mathbb{R}^n \setminus B_{2R}(v)$, $\|\varphi\|_{W^{2,\infty}} \leq CR^{-2}$. Thanks to our assumptions on $\beta$, we have $\beta(f) \leq Cf$, so that

$$\int_{\mathbb{R}^n_x \times B_{2R}(v)} \beta(f(T, x, v)) \, dx \, dv \leq C \|f_0\|_{L^1(\mathbb{R}^n_x \times \mathbb{R}^n_v)}.$$  

Also $\int v \cdot \nabla x \beta(f) \varphi(v) \, dv \, dx = 0$ (this computation is easily justified by approximation, using the entropic bounds which ensure decay of $f$ at infinity). Also the integrals of $(R_1)$ and $(R_2)$ are bounded thanks to Corollaries 11.3.1 and 11.3.2. Since $(R_3)$ is nonnegative, we are left with (11.27).

### 11.4 Strong compactness and passage to the limit

In this section, we prove Theorem 11.2.1. Since some parts of it are quite similar to existing proofs due to DiPerna and Lions, we only insist on the new features of the proof.
We proceed by approximation. There are several ways of doing this, but it will be simpler to start from the known results of DiPerna and Lions, of existence in the cutoff case, and thus to deal only with solutions of true Boltzmann equations, where the only approximation will be performed at the level of the cross-section.

**Definition 11.4.1** Let \((B_m)_{m \in \mathbb{N}} \cup \{B\}\) be a sequence of cross-sections satisfying Assumptions I and II from Chapter 3. We denote quantities attached to each \(B_m\) as in Chapter 3, by \(M_m, S_m, \) etc. We shall say that \(B_m\) approximates \(B\) (and write \(B_m \rightarrow B\)) if

\[
\begin{align*}
(i) & \quad S_m \to S, \text{ locally in weak-measure sense;} \\
(ii) & \quad T_m \to T, \text{ in weak (distributional) sense;} \\
(iii) & \quad B_m \to B \text{ a.e. on } \mathbb{R}^n \times S^{n-1}; \\
(iv) & \quad \text{As } |z| \to \infty, \, M_m^\alpha = o(|z|^{2-\alpha}) \text{ for some } \alpha \in [0, 2], \text{ and } |z| M_m'(|z|) = o(|z|^2), \text{ uniformly in } m.
\end{align*}
\]

\[\square\]

**Remark 11.4.1** If \(|z| M_m(|z|) \to |z| M(|z|)\) (locally in weak-measure sense) and

\[|z|^{2-\epsilon} B^m(z, \sigma)(1 - k \cdot \sigma)^{1-\delta} \to |z|^{2-\epsilon} B(z, \sigma)(1 - k \cdot \sigma)^{1-\delta}\]

for some \(\epsilon, \delta > 0\), then \(T_m \to T\) weakly, according to formula (11.24) (note that for any \(\epsilon, \delta > 0\), the function \(|v - v_*|^\epsilon(1 - k \cdot \sigma)^\delta|v' - v|/|v'|\) is a continuous function of both \(\sigma\) and \(v - v_*\)).

\[\square\]

**Example :** If \(B\) satisfies assumption I from Chapter 3, the most simple way to approximate \(B\) is of course to choose \(B_m(z, \sigma) = B(z, \sigma) 1_{|z| \geq 1/m, \theta \geq 1/m}\).

We also need a condition to express the fact that “on the whole”, the sequence \((B_m)\) is singular enough:

**Assumption III’ (Overall angular singularity condition).** We require that for all \(m\),

\[
B_m(z, \sigma) \geq \Phi_0(|z|) b_{0,m}(k \cdot \sigma), \quad k = \frac{z}{|z|},
\]

for some fixed continuous function \(\Phi_0(|z|)\) such that \(\Phi_0(|z|) > 0 \text{ if } z \neq 0\), and

\[
\int_{S^{n-1}} \lim_{m \to \infty} b_{0,m}(k \cdot \sigma) \, d\sigma = +\infty.
\]

Taking into account the results of DiPerna and Lions, Theorem 11.2.1 will be a byproduct of the following extended stability theorem.
Theorem 11.4.1 (Extended stability) Let $B$ satisfy Assumptions I, II, III from Chapter 3, and let $(B_m)_{m \in \mathbb{N}}$ be a sequence of cross-sections such that $B_m \longrightarrow B$, satisfying the overall singularity condition III’. Let $(f^m)_{m \in \mathbb{N}}$ be a sequence of solutions to the Boltzmann equation with respective cross-section $B_m$, in the sense of Definition 11.1.2. Assume that the sequence $(f^m)$ satisfies the natural entropic a priori bounds (with $B$ replaced by $B_m$, of course). Assume w.l.o.g that

$$f^m \longrightarrow f \text{ weakly in } L^p([0, T], L^1(\mathbb{R}^n_x \times \mathbb{R}^n_v)) \quad (1 \leq p < +\infty).$$

Then

i) $f^m$ converges strongly towards $f$;

ii) $f$ is a renormalized solution with a defect measure of the Boltzmann equation with cross-section $B$;

iii) $f$ satisfies (11.13)–(11.16).

Proof of Theorem 11.4.1:

By classical arguments, we may assume that

$$f^m \longrightarrow f \quad \text{in } w-L^p([0, T], L^1(\mathbb{R}^n_x \times \mathbb{R}^n_v)), \quad 1 \leq p < \infty.$$

We first prove that the convergence is strong, and for this purpose we follow the general strategy exposed by Lions. Once this is done, it will remain to pass to the limit in the equation as $m \rightarrow \infty$.

1) The first step is to write the renormalized formulation, that is,

$$\frac{\partial \beta(f^m)}{\partial t} + v \cdot \nabla_x \beta(f^m) = \beta'(f^m)Q(f^m, f^m) + \mu^m,$$

where, for all $R > 0$, $\mu^m$ is a nonnegative measure, with finite mass on $[0, T] \times \mathbb{R}^n_x \times B_R(v)$. From all the bounds of section 11.3,

$$\frac{\partial \beta(f^m)}{\partial t} + v \cdot \nabla_x \beta(f^m) = g^m + \sum_i \frac{\partial}{\partial v_i} g_i^m + \sum_{ij} \frac{\partial^2}{\partial v_i \partial v_j} g^m_{ij},$$

where $g^m$, $g_i^m$, $g^m_{ij}$ are locally integrable, or locally bounded measures. This entails, by the so-called averaging lemmas, that averages of the form $\int \beta(f^m)\varphi(v) \, dv$, for $\varphi \in C^\infty_0(\mathbb{R}^n)$, are strongly compact in the variables $(t, x)$. Reasoning as DiPerna and Lions, and using a priori bounds on $f^m$, one deduces from this that convolution products of the form $\beta(f^m) \ast \varphi$ are also strongly compact in $L^1([0, T] \times \mathbb{R}^n_x \times \mathbb{R}^n_v)$. 

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Replacing $\beta(f^m)$ by a concave approximation of the square root function and using the entropic a priori bounds..., this also implies that convolution products of the form $\sqrt{f^m} \ast \varphi$ are strongly compact in $L^1$. This result is a somewhat general consequence of the existence of a renormalized formulation. Another useful information, obtained by combining the averaging lemmas, the renormalized formulation and the bounds on $(f^m)$, is that $\int f^m \, dv$ is strongly compact in $(t,x)$, and thus converges strongly to $\int f \, dv$.

2) The next and most delicate step is to use a smoothness estimate in the velocity variable, as shown in Chapter 5. It will be a consequence of the entropy dissipation estimate and the overall singularity assumption III'. Let

$$Z_m(a) = |S^{n-2}| \int a^2 \, b_{0,m}(\cos \theta) \sin^{n-2} \theta \, d\theta,$$

and let $f^m_R = f^m \chi_R$, where $\chi(v)$ is a smooth cutoff function with support in $B_{3R}(v)$, identically equal to 1 on $B_{2R}(v)$. Then: for all $0 < L < +\infty$, $0 < \varepsilon < +\infty$, for a.a $(t,x)$ such that

$$H^m(t,x) \equiv \int f^m(t,x,v)(1 + |v|^2 + \log f^m(t,x,v)) \, dv \leq L < +\infty,$$

and

$$\rho^m(t,x) \equiv \int f^m(t,x,v) \, dv \geq \varepsilon > 0,$$

the following pointwise estimate in $(t,x)$ holds:

$$\int_{|\xi| \geq 1} |\mathcal{F}\sqrt{f^m_R}|^2 Z_m \left( \frac{1}{|\xi|} \right) \, d\xi \leq C(f^m,R,\Phi_0) \left[ D(f^m) + \|f^m\|_{L^1}^2 \right].$$

Here $\mathcal{F}$ denotes the usual Fourier transform with respect to the velocity variable, and $C(f^m,R,\Phi_0)$ is a constant depending only on $R$, $\Phi_0$, and on $L$, $\varepsilon$.

We refer to Chapter 5 for more details.

Thus (loosely speaking), taking into account the bounds

$$\sup_m \sup_t \int_{\mathbb{R}^n} dx \, D(f^m) < +\infty, \quad \sup_n \sup_t \int_{\mathbb{R}^n} dx \, H^m(t,x) < +\infty,$$

we see that the entropy dissipation estimate implies a smoothness estimate for $\sqrt{f^m}$ in the velocity variable, out of

(i) a set of small measure in $(t,x)$ where $f^m$ may have infinite mass, energy or entropy;

(ii) a set where the density $\rho^m = \int f^m \, dv$ may be very small.

A precise formulation is easy, in the same spirit as by Lions. Let us fix a large number $R$. By the a priori bounds on $f$, it is clear that $g^m = \sqrt{f^m} \rightharpoonup g$ for some function $g \in L^2$. To prove strong
convergence, it is enough to prove that $g^m \to g$ a.e on each $W_\varepsilon \times B_R(v)$, where

$$W_\varepsilon = \{ (t,x) : t \in [0,T], |x| < R, \int g^2 \, dv > \varepsilon \}.$$ 

Indeed, note that $g^m \to g = 0$ in $L^1(|x| < R, |v| < R, \int g^2 \, dv = 0)$ (a sequence of nonnegative functions, converging to 0 weakly in $L^1$, automatically converges to 0 strongly in $L^1$).

Now, on $W_\varepsilon$, by convexity of the square function,

$$\varepsilon < \int g^2 \, dv \leq \lim_{m \to \infty} (g^m)^2 \, dv = \lim_{m \to \infty} f^m \, dv = \int f \, dv$$

where we have used the fact that $\int f^m \, dv \to \int f \, dv$ strongly. By Egorov's theorem, for all $\delta > 0$ there is a Borel set $U_\delta$, with measure less than $\delta$, such that $\int f^m \, dv \to \int f \, dv$ uniformly on $W_\varepsilon \setminus U_\delta$.

For $m$ large enough, this implies that $\int f^m \, dv \geq \varepsilon/2$ on $W_\varepsilon \setminus U_\delta$.

Next, let

$$V^m_L = \{ (t,x) ; H^m(t,x) > L \}.$$ 

Our bounds imply that $|V^m_L| \leq CL^{-1}$, where $C$ is a constant independent on $n$. And our entropy dissipation estimate implies that there is another constant $C$ such that for all $A \geq 1$,

$$\int_{W^\varepsilon \setminus (U_\delta \cup V^m_L)} dt \, dx \int_{|\xi| \geq A} d\xi \sqrt{\frac{f^m}{2}} \leq \frac{C}{Z_\infty(\frac{1}{A})}.$$ 

Passing to the lim sup on both sides, we find

$$\lim_{m \to \infty} \int_{W^\varepsilon \setminus (U_\delta \cup V^m_L)} dt \, dx \int_{|\xi| \geq A} d\xi \sqrt{\frac{f^m}{2}} \leq \frac{C}{Z_\infty(\frac{1}{A})},$$

where

$$Z_\infty(a) = \lim_{m \to \infty} |S^{n-2}| \int_a^\pi b_{0,m}(\cos \theta) \sin^{n-2} \theta \, d\theta \geq |S^{n-2}| \int_a^\pi \lim_{m \to \infty} b_{0,m}(\cos \theta) \sin^{n-2} \theta \, d\theta.$$ 

From formula (11.29) we know that

$$Z_\infty(a) \xrightarrow{a-0} +\infty.$$ 

Thus, for each $L$, $\varepsilon$, $\delta$, we have

$$\lim_{A \to \infty} \lim_{m \to \infty} \int_{W^\varepsilon \setminus (U_\delta \cup V^m_L)} dt \, dx \int_{|\xi| \geq A} d\xi \sqrt{\frac{f^m}{2}} = 0.$$ 

On the other hand, since $|U_\delta \cup V^m_L| \leq \delta + L^{-1}$, and since $(f^m)$ is uniformly equi-integrable,

$$\lim_{L \to \infty} \sup_{m \in \mathbb{N}} \int_{U_\delta \cup V^m_L} dt \, dx \int_{\mathbb{R}^n} d\xi \sqrt{\frac{f^m}{2}} = 0.$$ 

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On the whole,

\[(11.31) \lim_{A \to +\infty} \lim_{m \to \infty} \int_{W^*} dt \, dx \int_{|\xi| \geq A} d\xi \|F_{\sqrt{f^m_R}}\|^2 = 0.\]

This is the “velocity smoothness” estimate that we needed.

3) Now, let \(\rho_\delta = \delta^{-n} \rho(\cdot/\delta)\), \(\delta > 0\), be a family of mollifiers in the velocity variable (\(\rho\) smooth, nonnegative, compactly supported, \(\int \rho = 1\)). From (11.31) follows

\[\lim_{\delta \to 0} \lim_{m \to \infty} \|\sqrt{f^m - \sqrt{f^m_R} * \rho_\delta}\|_{L^2(W_\varepsilon \times B_R(v))} = 0.\]

Clearly, in view of our truncation procedure this is the same as

\[\lim_{\delta \to 0} \lim_{m \to \infty} \|\sqrt{f^m - \sqrt{f^m_R} \ast \rho_\delta}\|_{L^2(W_\varepsilon \times B_R(v))} = 0.\]

Since, for any \(\delta > 0\), \(\sqrt{f^m} \ast \rho_\delta\) lies in a strongly compact set in \(L^2\), this entails that \((\sqrt{f^m})\) is also relatively strongly compact.

4) We conclude to the strong compactness as by Lions. From the strong compactness of \((\sqrt{f^m})\) follows the a.e. convergence of \((f^m)\). Since \((f^m)\) already converges weakly towards \(f\), this entails that

\[f^m \rightharpoonup f \quad \text{in} \quad L^1([0,T] \times R^m_x \times R^m_v).\]

5) Once the strong convergence has been established, it is easy to pass to the limit in the renormalized formulation with a defect measure. First, it is clear that

\[\frac{\partial \beta(f^m)}{\partial t} \rightharpoonup \frac{\partial \beta(f)}{\partial t}, \quad v \cdot \nabla_x \beta(f^m) \rightharpoonup v \cdot \nabla_x \beta(f)\]

in weak sense.

Then, we handle the term \((R_1)\). With similar notations as in Chapter 3, for any test-function \(\varphi\) (smooth with compact support),

\[\int (R_1)^m \varphi = \int_{\mathbb{R}^2} [f^m \beta'(f^m) - \beta(f^m)] f^m_s S^m(v - v_s) \varphi(v) \, dv \, dv_s.\]

Since \(f^m \beta'(f^m) - \beta(f^m)\) is compact in weak-* \(L^\infty\), it suffices to show that \(f^m \ast S^m\) is strongly compact in \(L^1\). This follows immediately from the bounds on \(S^m\) and \(f^m\) at infinity, the fact that \(S^m\) converges locally in weak-measure sense, and the strong convergence of \(f^m\).

Next, we consider \((R_2)\). For a sufficiently smooth test-function \(\varphi\), we have

\[\int (R_2)^m \varphi \, dv = \int_{\mathbb{R}^2} f^m_s \beta(f^m)(T^m \varphi) \, dv \, dv_s,\]

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where $\mathcal{T}^m$ is the linear operator defined in (11.22). Thus, it is immediate that

$$\int (\mathcal{R}_2)^m \varphi \, dv \, dx \longrightarrow \int (\mathcal{R}_2) \varphi \, dv \, dx,$$

since $\mathcal{T}^m \rightarrow \mathcal{T}$ in weak sense, as an operator on $L^1(\mathbb{R}_v^n \times \mathbb{R}_v^n)$, and $f^m \beta(f^m) \rightarrow f \beta(f)$ in strong $L^1$ sense, locally on $\mathbb{R}_x^n \times \mathbb{R}_v^n \times \mathbb{R}_v^n$.

Finally, $(\mathcal{R}_3)^m$ is bounded in measure sense, locally in velocity space, so that up to extraction, it converges weakly to some nonnegative limit. Let $\varphi(t, x, v)$ be a nonnegative test-function; since $f^m \rightarrow f$ and $B_m \rightarrow B$ a.e., it follows at once from Fatou’s lemma that

$$\int (\mathcal{R}_3) \varphi \leq \lim_{m \to \infty} \int (\mathcal{R}_3)^m \varphi,$$

which proves that, in weak sense, $(\mathcal{R}_3) \leq \lim (\mathcal{R}_3)^m$.

6) As for the mass and momentum conservation laws (11.13) and (11.14), they are easy consequences of the convergence of $f^m$ towards $f$ and the uniform energy bounds, while the identities (11.15) and (11.16) are implied by Fatou’s lemma again.
Chapter 12

Local Solutions between maxwellians

Precise references for this chapter are:
- Initial works in the cutoff case [76, 69];
- Our work [7] and some results mentioned previously.

Furthermore, let us note that additional results are available in [7]; in particular, the case of a perturbation by \(-\varepsilon \Delta v\) is considered therein, as well as other cases of collisional kernels. However, those results were stated in 3 dimensional case. Here, we shall consider any dimension, and for simplicity we shall only consider the case of small singularities \(0 < \nu < 1\).

For the first time in these Notes, we consider the full non homogeneous Boltzmann equation in dimension \(n\), and look for a solution \(f = f(t, x, v), t \in \mathbb{R}^+\) (or in \((0, T)\), with \(T > 0\) fixed), \((x, v) \in \mathbb{R}^{2n}\), solution in a suitable sense of

\[
\begin{align*}
\partial_t f + v \cdot \nabla_x f &= Q(f), \\
 f \mid_{t=0} &= f_0,
\end{align*}
\]

(12.1)

denoted also hereafter by problem (B).

\(f_0 = f_0(x, v)\) is a given initial datum and we shall assume in this chapter that it satisfies

\[
(12.2) \quad m^- M(0, x, v) \leq f_0(x, v) \leq m^+ M(0, x, v),
\]

where \(0 < m^- \leq m^+\) are given constants and

\[
(12.3) \quad M(t, x, v) = e^{-|v|^2} e^{-|x-tv|^2}.
\]

Let us mention here that the assumption \(m^- > 0\) is not necessary for the existence results presented below, but it simplifies the regularity questions dealt with.
The Boltzmann operator $Q$ which appears in $(B)$ is given by the $\sigma$-representation, see Chapter 1. Our aim is to show that there exists $T > 0$ so that problem $(B)$ admits weak solutions (to be defined below) in the class $L^1 \cap L^\infty((0,T) \times \mathbb{R}^{2n}_{x,v})$, for initial data satisfying (12.2), and this will hold true for any value of $m^+$.

The above comparison assumption is classical, and we recall that the function $M$ is a special solution of $(B)$ as $Q(M) = 0$ and $\partial_t M + v \nabla_x M = 0$.

Such studies already exist in the cutoff (or non singular) case, but here we deal with the non cut-off (singular) case.

As usual, we shall also make the assumption that our kernel $B$ is given by a multiplicative form, $\gamma$ denoting the exponent of the kinetic part, and $\nu$ the exponent of the angular part.

**Definition 12.0.2** Under the above assumptions, we shall say that, for $T > 0$, possibly $T = +\infty$, $f \geq 0$ is a weak solution of $(B)$ if

$$
\begin{cases}
  f \in L^\infty(0,T; L^1 \cap L^\infty(\mathbb{R}^{2n})), \\
  \int_0^b \int_{\mathbb{R}^{2n}_x} \int_{\mathbb{R}^{2n}_v} B(f' f' - f f) \ln \frac{f}{f'} d\sigma dv dt < +\infty, \text{ for any finite } b \leq T, \\
  |v|^2 f \in L^\infty(0,T; L^1(\mathbb{R}^{2n}))
\end{cases}
$$

and for all $h \in C^\infty_0([0,T] \times \mathbb{R}^{2n})$,

$$
\int_{[0,T] \times \mathbb{R}^{2n}} f \{-\partial_t h - v \nabla_x h\} dv dt = \langle Q(f); h \rangle + \int_{\mathbb{R}^{2n}} f_0 h(0,x,v) dx dv,
$$

where

$$
\langle Q(f); h \rangle = \int_{[0,T] \times \mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}_v} \int_{S^{n-1}} B f f \{h' - h\} d\sigma dv dt.
$$

The fact that $\langle Q(f); h \rangle$ for (at least) such $h$ as defined above is meaningfull has been explained in previous chapters.

The main result of this chapter is given by

**Theorem 12.0.2** Assume that $\gamma < -1$. Then, for all $m^+ > 0$, there exists $T \in \mathbb{R}^+$ and two $C^1$ functions $\delta, \beta : [0,T] \to \mathbb{R}^+$, with $\delta(0) = m^-$ and $\beta(0) = m^+$, such that problem $(B)$ admits a weak solution $f$ satisfying

$$
\delta(t) M(t,x,v) \leq f(t,x,v) \leq \beta(t) M(t,x,v).
$$

Moreover, if $\gamma \geq 1 - n$, then $T = +\infty$ is allowed.
The next theorem gives a partial regularity result on these solutions

**Theorem 12.0.3** The solutions constructed above satisfy

\[ hf \in L^2(0,T; H^{\frac{\nu}{\nu+\sigma}}(\mathbb{R}^{2n})) \]

for all \( h \in C_0^\infty([0,T]\times\mathbb{R}^{2n}) \).

Let us note that, in fact, we only need in this last result two facts: that a solution is bounded in \( L^1 \cap L^\infty \), but also that it is bounded from below by a strictly positive function which is also in \( L^1 \cap L^\infty \). Of course, this is the case in particular for the solutions constructed by the first result.

In Section II below, we shall prove the existence Theorem 12.0.2. Then, the regularity result is proven in the last Section III.

As far as I know, this is (still) the first regularity result for the non homogeneous Boltzmann equation without cutoff.

Finally, it follows from the proofs that, if \( T > 0 \) is fixed, we can construct weak solutions of (B) on the time interval \((0,T)\), if \( m^+ \) is sufficiently small. This is a kind of statement in use in the framework of non linear pde. We have choosen the opposite presentation of our results, in the hope of showing the existence of global in time solutions, for any value of \( m^+ \). However, even if we have failed in this direction, the solutions constructed herein are renormalised solutions, so that they continue to exist as such for time bigger that \( T \).

### 12.1 Proof of Theorem 12.0.2

**First Step: A cutoff problem**

Let us fix a parameter \( n' \in \mathbb{N}^* \), large enough, and we consider the following cutoff type problem

\[
\begin{aligned}
\partial_t f + v \cdot \nabla_x f &= Q_{n'}(f,f),

f \big|_{t=0} &= f_0,
\end{aligned}
\]

where \( Q_{n'} \) is the non singular Boltzmann operator corresponding to the non singular cross section

\[
B_{n'} = B_{\frac{\sin \theta}{\cos \theta \geq \frac{1}{n'}}}.
\]

More precisely, we want to find \( T > 0 \), two \( C^1 \) functions \( \delta, \beta : [0,T] \to \mathbb{R}^+ \), \( \delta(0) = m^- \), \( \beta(0) = m^+ \), which do not depend on \( n' \), and such that problem (12.5) admits a weak solution \( f(=f^{n'}) \) such that \( \delta M \leq f \leq \beta M \).
We shall use previous ideas from the cutoff case that we adapt to our singular framework. Note that we could apply (somehow directly) these results in order to solve (12.5), but constants appearing in these works do depend on \( n' \), and this is definitively useless for our second step, which consists in sending \( n' \) to \(+\infty\).

**We shall explicitly display the parameter \( n' \) on any constant iff it does depend on it.** Furthermore, we abbreviate notations below.

For any \( 0 < T \leq +\infty \), \( \delta \) and \( \beta \) given functions in \( C^0_\text{+}(0,T) \), and \( l \) a fixed and given function satisfying

\[
\delta(t)M \leq l \leq \beta(t)M,
\]

we introduce the following linear problem

\[
\begin{aligned}
\partial_t g + v.\nabla g &= \int_{\mathbb{R}^n} \int_{S^{n-1}} B_{n'}(...) l^t' s d\sigma dv_s - \int_{\mathbb{R}^n} \int_{S^{n-1}} B_{n'}(...) d\sigma dv_s B_{n'} l^t_g d\sigma dv_s,

\left| g \right|_{t=0} = f_0
\end{aligned}
\]

that we can write also as

\[
\begin{aligned}
\partial_t g + v.\nabla g &= \int_{\mathbb{R}^n} \int_{S^{n-1}} B_{n'}(...) l^t' s - \int_{\mathbb{R}^n} \int_{S^{n-1}} B_{n'}(...) l^t'_s g + g \int_{\mathbb{R}^n} \int_{S^{n-1}} B_{n'}(...) (l^t'_s - l^t_s),

\left| g \right|_{t=0} = f_0
\end{aligned}
\]

Then one has, for all time \( t \)

\[
\int_{\mathbb{R}^n} \int_{S^{n-1}} B_{n'}(...) l^t' s \leq C\beta^2(t)
\]

for some constant depending on \( n' \).

This shows that for all \( T > 0 \) fixed, problem (12.8) admits an unique solution \( g \) (in the mild and distributional sense) which is in \( L^1 \cap L^\infty((0,T) \times \mathbb{R}^{2n}_{x,v}) \) and \( \geq 0 \).

Next, we look for an upper solution of problem (12.9) in the form \( a \equiv \beta(t)M \). More precisely, we wish to find a sufficient condition on \( \beta \) for this purpose, and so we first compute

\[
\begin{aligned}
\partial_t [a - g] + v.\nabla [a - g] &= \\
&= \beta'(t)M - \int \int B_{n'} l^t' + \int \int B_{n'} l^t'_s g - g \int \int B_{n'} (l^t'_s - l^t_s) = \\
&= \beta'(t)M - \int \int B_{n'} l^t' + \int \int B_{n'} l^t'_s g + (a - g) \int \int B_{n'} (l^t'_s - l^t_s) - a \int \int B_{n'} (l^t'_s - l^t_s).
\end{aligned}
\]

Since \( l \leq a \), one has

\[
\begin{aligned}
\partial_t [a - g] + v.\nabla [a - g] &\geq \\
&\geq \beta'(t)M - \int \int B_{n'} l^t'_s a' + \int \int B_{n'} l^t'_s g + (a - g) \int \int B_{n'} (l^t'_s - l^t_s) - a \int \int B_{n'} (l^t'_s - l^t_s) \geq \\
\end{aligned}
\]
Thus

\[ \geq \beta'(t)M - \int_\mathbb{R} \int_\mathbb{R} B_n'(l'_a - l_a) - \int_\mathbb{R} \int_\mathbb{R} B_n'(l'_a - l_a). \tag{12.10} \]

We have

\[ + (a - g) \int_\mathbb{R} \int_\mathbb{R} B_n'(l'_a - l_a) - a \int_\mathbb{R} \int_\mathbb{R} B_n'(l'_a - l_a). \]

Since we want \( a \) to be an upper solution of problem (12.8), it is enough to ask for \( \beta \) to satisfy

\[ \beta'(t)M - \int_\mathbb{R} \int_\mathbb{R} B_n'(l'_a - a) - \int_\mathbb{R} \int_\mathbb{R} B_n'(l'_a - l_a) \geq 0, \tag{12.11} \]

and therefore, we are led to look for \( \beta \) such that \( (\beta \geq 0) \)

\[ \begin{cases} \beta'(t)M \geq \beta \int_\mathbb{R} \int_{S^{n-1}} B_n'(\gamma, \beta, \alpha)(l'_a - l_a) + \beta M \int_\mathbb{R} \int_{S^{n-1}} B_n'(\gamma, \beta, \alpha)(l'_a - l_a), \\ \beta(0) \geq m^+. \end{cases} \tag{12.12} \]

Note that the right hand side is nothing else than \( \beta Q_n'(l, M) \).

Firstly, one has

\[ \int_\mathbb{R} \int_{S^{n-1}} B_n'(\gamma, \beta, \alpha)(l'_a - l_a) = \int_\mathbb{R} S_n'(|v - v_0|)l_a dv_a, \tag{12.13} \]

where

\[ S_n'(|v - v_0|) = \left| S^{n-2} \int_\mathbb{R} \sin^{n-2} \theta \left[ \frac{1}{\cos^n \frac{\theta}{2}} B_n'(\frac{|v - v_0|}{\cos \frac{\theta}{2}}, \cos \theta) - B_n'(|v - v_0|, \cos \theta) \right] d\theta, \tag{12.14} \]

By known results, one has

\[ |S_n'(|v - v_0|)| \leq c_1 |v - v_0|^\gamma \tag{12.15} \]

where \( c_1 \) does not depend on \( n' \).

Let us introduce

\[ F(t, x, v) \equiv \int_\mathbb{R} |v - v_0|^\gamma e^{-\{v_0^2 + |x + tv_0|^2\}} dv_a \]

It follows that for \(-v\), one has changing \( v_0 \) into \(-v_0\)

\[ F(t, x, -v) = \int_\mathbb{R} |v - v_0|^\gamma e^{-\{v_0^2 + |x - tv_0|^2\}} dv_a \leq \int_\mathbb{R} |u|^\gamma e^{-2|x + (t+1)v_0|^2} du \leq \int_\mathbb{R} |u|^\gamma e^{-2|x + (t+1)v - (t+1)u|^2} du. \]

Thus

\[ F(t, x, v) \leq \frac{1}{(t + 1)^{\gamma+n}} \int_\mathbb{R} |u|^\gamma e^{-|x - (t+1)v - u|^2} du = \frac{1}{(t + 1)^{\gamma+n}} G(x - (t + 1)v) \]

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where
\[ G(U) = \int_{\mathbb{R}^n} |u|^\gamma e^{-|U-u|^2} du. \]
In particular, since \( \gamma \leq 0 \), it follows that
\[ F(t, x, v) \leq \frac{1}{(t+1)^{\gamma+n}} \]
and therefore
\[(12.16) \int_{\mathbb{R}^n} S_n'(|v - v_*|) l_* dv_* \leq \beta(t) c_2 \frac{1}{(1+t)^{\alpha+\gamma}}.\]
In conclusion, it is enough to choose \( \beta \geq 0 \) such that
\[(12.17) \begin{cases} \beta'(t) M \geq \int_{\mathbb{R}^n} \int_{S^{n-1}} B_n'(\ldots) l'_s(a' - a) d\sigma dv_* + \beta^2(t) M c_2 \frac{1}{(1+t)^{\alpha+\gamma}}, \\ \beta(0) \geq m^+. \end{cases}\]
and for this purpose, there remains to analyse the most difficult term \( \int_{\mathbb{R}^n} \int_{S^{n-1}} B_n'(\ldots) l'_s(a' - a) d\sigma dv_* \).

We estimate
\[ | \int_{\mathbb{R}^n} \int_{S^{n-1}} B_n'(\ldots) l'_s(a' - a) d\sigma dv_* | \leq \beta^2(t) \int_{\mathbb{R}^n} \int_{S^{n-1}} |M' - M| \leq \beta^2(t) M \int_{\mathbb{R}^n} |M_* - M'_s|. \]

Next note that
\[ |M_* - M'_s| \leq \min \{ t|v - v_*| \sin \frac{\theta}{2}, M_* + M'_s \}. \]
In fact, one has also
\[ |M_* - M'_s| \leq t^{\frac{1}{2}} |v - v_*| \sin \frac{\theta}{2} \]
and
\[ |M_* - M'_s| \leq M_*^{\frac{1}{2}} + M'_s^{\frac{1}{2}} \]
and thus
\[ |M_* - M'_s| \leq t^{\frac{1}{2}} |v - v_*| \sin \frac{\theta}{2} \{ M_*^{\frac{1}{2}} + M'_s^{\frac{1}{2}} \}. \]
We wish to take \( \frac{1}{\beta} \) close to 1, so that \( \frac{1}{\beta'} \) will be close to 0.
In fact we choose \( \beta \) such that \( \frac{1}{\beta} > \nu \), then we must have \( \gamma + \frac{1}{\beta} \leq 0 \), so that it restricts \( \gamma \) to \( \gamma < -1 \).
With these choices, we find that
\[ | \int_{\mathbb{R}^n} \int_{S^{n-1}} B_n'(\ldots) l'_s(a' - a) d\sigma dv_* | \leq \beta^2(t) M \frac{1}{(1+t)^{\alpha+\gamma}}. \]
By the above computations, it is enough to choose \( \beta \geq 0 \) such that
\[(12.18) \begin{cases} \beta(t) M \geq \beta^2(t) M c_3 \frac{1}{(1+t)^{\alpha+\gamma}}, \\ \beta(0) \geq m^+. \end{cases}\]
that is also

\[
\begin{align*}
\beta'(t) &\geq \beta^2(t) c_3 \frac{1}{(1+t)^{n+\gamma}}, \\
\beta(0) &\geq m^+.
\end{align*}
\]

Of course, one may choose \( \beta \) solution of

\[
\begin{align*}
\beta'(t) &= \beta^2(t) c_3 \frac{1}{(1+t)^{n+\gamma}}, \\
\beta(0) &= m^+.
\end{align*}
\]

which is given by

\[\beta(t) = \frac{1}{m^+ - c_4(1+t)^{-n-\gamma+1} + c_4}.\]

Let us note immediately that if \( \gamma \geq 1 - n \), then \( \beta \) is a global solution. If not, for any value of \( m^+ \), there exists still a \( T > 0 \) such that \( \beta \) is well defined on \( [0,T] \).

Once we get at this point, looking for a lower solution in the form \( \delta(t) M \) on the same time interval \( [0,T] \) is classical. Clearly, we can choose such \( \delta \) independent from \( n' \).

In conclusion, we have therefore achieved the following.

For \( \gamma < -1 \), there exists \( T > 0 \) and \( \beta, \delta \) \( C^1 \) functions from \( [0,T] \) in \( \mathbb{R}^{+\ast} \), \( \beta(0) = m^+ \), \( \delta(0) = m^- \), which do not depend on \( n' \), such that for all

\[ l \in L^1 \cap L^\infty((0,T) \times \mathbb{R}^{2n}) \] \( \delta M \leq l \leq \beta M \),

problem (12.9) has an unique solution \( g \) such that

\[ \delta M \leq g \leq \beta M. \]

If moreover, \( \gamma \geq 1 - n \), then we can take \( T = +\infty \).

**Remark 12.1.1** Is it possible to extend the range of values of \( \gamma \)?

Of course the trouble comes from the term \( \int \int B_{a'} l'_s(a' - a) \). We compute it as follows, setting \( n' = +\infty \).

First of all, up to immaterial multiplicative terms, one has, now using \( \omega \)-representation and then Carleman one

\[
\begin{align*}
\int \int B_{a'} l'_s(a' - a) &= \int \int |v - v_*|^{\gamma+1} \frac{1}{|v' - v_*|^{1+\nu}} l'_s(a' - a) = \\
&= \int \int |v - v_*|^{\gamma+\nu+1} \frac{1}{|v' - v|^{1+\nu}} l'_s(a' - a) = \\
&= \int \int |v' - v_*|^{\gamma+\nu+1} \frac{1}{|v' - v|^{1+\nu}} l'_s(a' - a) + \int \int |v - v_*|^{\gamma+\nu+1} - |v_* - v'|^{\gamma+\nu+1} \frac{1}{|v' - v|^{1+\nu}} l'_s(a' - a)
\end{align*}
\]

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(2.15) \[ = I + II. \]

Note that the bracket term inside \( II \) is positive. Furthermore,
\[
II \leq \phi \int \int \left[ |v - v_*| \gamma^{\nu+1} - |v_* - v'| \gamma^{\nu+1} \right] \frac{1}{v' - v} |v' - v| \, dv' \, dv 
\leq \beta^2(t) MC_3 \int v_* \, v \, M_*
\]
by following the computations made in Chapter 4, and finally,
\[
II \leq \beta^2(t) MC_4 \frac{1}{(1+t)^{n+\gamma}}.
\]

Next, we deal with \( I \). Using the same computations as in Chapter 4, one has firstly, using Carlemann’s representation
\[
I = \beta(t) \int R^n_h \frac{dh}{h^{v+\nu}} \int \left| \alpha \right| \gamma^{\nu+1} 1_l'(M' - M)
\]
where \( M' = M(v - h) \) and \( M = M(v) \), \( E_{0,h} \) denotes the hyperplane orthogonal to \( h \) and containing 0. If we let \( \bar{M}' = M(v + h) \), \( \bar{M}' = M(\alpha + v) \), \( M_* = M(\alpha + v - h) \), \( \bar{M}_* = M(\alpha + v + h) \), one has
\[
I = \beta(t) \int R^n_h \frac{dh}{h^{v+\nu}} \int _E \left| \alpha \right| \gamma^{\nu+1} 1_l'(M' + \bar{M}' - 2M)
\]
where \( F = \frac{1}{\beta M} \).

Since \( M'M_* = MM_* \) and \( \bar{M}'M_* = \bar{M}_* \), we obtain finally
\[
I \leq \beta^2(t) c_5 M \int R^n_h \frac{dh}{h^{v+\nu}} \int _E \left| \alpha \right| \gamma^{\nu+1} F_l'M'_* (M_* + \bar{M}_* - 2M'_*) \, d\alpha,
\]
By the same computations as in Chapter 4, on has
\[
I \sim \beta^2 M \int _{S^{n-1}} \int _{E_{0,\omega}} \, d\omega |\alpha| \gamma^{\nu+1} F(\alpha + v) \int IR^\nu_\xi e^{i\xi(\alpha+v)M(\xi)\xi \omega} |\nu
\sim \beta^2 M \int \int \left| \alpha \right| \gamma^{\nu+1} F(\alpha + v) e^{i\xi(\alpha+v)M(\xi)} |S(\alpha).\xi| |\nu
\]
and on this expression, it is clear that we have indeed bad control w.r.t. variable \( v \).

Indeed, for a fixed \( \alpha \neq 0 \), using an orthogonal basis with \( \alpha/|\alpha| \) as first vector, expression \( \xi \) and \( v \) in this basis, with \( v = (v_1, v') \) and \( \xi = (\xi_1, \xi') \) then it follows that
\[
\int \xi e^{i\xi(\alpha+v)M(\xi)} |S(\alpha).\xi| |\nu = \int \xi e^{i\xi(\alpha+\xi_1,v_1+v') \bar{M}(\xi)\xi' |\nu
\]

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and thus for any power $m$, this is less than $(|\alpha| + v_1)^{-m}$. Also we have upper bound in terms of $|v'|$ but powers depend on the values of $v$ and $n$.

More exactly, this is also less than $|v'|^{-m}$, but for power $m$ such that $m < n + \nu$.

This small computation shows that the weight $|\alpha|^{7+\nu+1}$ is not good enough for this method. But for weights with a good decreasing behavior at infinity, we can still get local solutions.

Newt, getting back, by a classical fixed point argument, we can assert that there exists $g_n$ solution of the following Boltzmann equation with cutoff

\begin{align}
\partial_t g_n + v \cdot \nabla_x g_n &= Q_n(g_n), \\
g_n|_{t=0} &= f_0
\end{align}

on the time interval $[0, T]$, such that $\delta M \leq g^n \leq \beta M$, where $T$, $\delta$ and $\beta$ are as above, not depending on $n$.

Furthermore, $g_n$ satisfies the following uniform entropic dissipation rate bound estimate

\begin{align}
\int_0^T \int_{\mathbb{R}^2} \int_{\mathbb{R}^n} \int_{S^{n-1}} B_n'(\cdot, \cdot) \{g'_n g_{n*} - g_n g_{n*}\} \ln \frac{g'_n g_{n*}}{g_n g_{n*}} d\sigma dv dx dt \leq C_T,
\end{align}

as it is clear by multiplying (12.21) by $\ln g^n$.

In fact, one can also use similar computations as in Chapter 10.

**Second Step: sending $n'$ to $+\infty$**

From (12.22) and the (uniform in $n'$) $L^\infty$ bound on $g_{n'}$, one deduces that

\begin{align}
\int_0^T \int_{\mathbb{R}^2} \int_{\mathbb{R}^n} \int_{S^{n-1}} B_n'(\cdot, \cdot) \{g'_n g_{n*} - g_n g_{n*}\}^2 d\sigma dv dx dt \leq C_T
\end{align}

This is enough to apply know results and arguments, see in particular Chapter 11. In particular, there exists $f \in L^1 \cap L^\infty$, $\delta M \leq f \leq \beta M$ such that (for a suitable sub-sequence)

$g_{n'} \rightharpoonup f$ in $L^p((0, T) \times \mathbb{R}^{2n})$

strongly ($1 \leq p < +\infty$). Writing the distributional formulation associated to (12.21) (as in Definition 12.0.2), it follows that $f$ is a weak solution in the sense of Definition 12.0.2, ending the proof.

**Remark 12.1.2** Note that $Q(f)$ as defined there satisfies

\begin{align}
Q(f) \in L^2((0, T) \times \mathbb{R}^n; H^{-\frac{\nu}{2}}(\mathbb{R}^n))
\end{align}

Indeed for all $h \in L^2((0, T) \times \mathbb{R}^n; C_c^\infty(\mathbb{R}^n))$
\[ \langle Q(f); h \rangle = | \int_0^T \int_{\mathbb{R}^n_x} \int_{S^N_{v-1}} B \{ f' f_*' - f f_* \} \{ h' - h \} | \]
\[ = | \int_0^T \int_{\mathbb{R}^{2n}_x} \int_{S^N_{v-1}} B \{ \sqrt{f' f_*'} - \sqrt{f f_*} \} \{ \{ \sqrt{f' f_*'} + \sqrt{f f_*} \} h' - h \} | \]
\[ \lesssim \left\{ \int B | \sqrt{f' f_*'} - \sqrt{f f_*} |^2 \right\}^{1/2} \left\{ \int B f f_* | h' - h |^2 \right\}^{1/2} \]
\[ \leq C_T \| h \|_{L^2((0,T) \times \mathbb{R}^n_x; H^{2}_{\nu}(\mathbb{R}^n_v))}. \]

### 12.2 Proof of Theorem 12.0.3

Since \( f \) is a weak solution in the sense of Definition 12.0.2, more precisely as it satisfies the entropic dissipation rate bound, and as it is bounded below and above by a Maxwellian, it follows that for all \( h \in C_c^\infty((0,T) \times \mathbb{R}^{2n}_x) \)

\[ hf \in L^2((0,T) \times \mathbb{R}^n_x; H^{2}_{\nu}(\mathbb{R}^n_v)) \tag{12.24} \]

Setting \( F = hf \), one has

\[ \partial_t F + v \cdot \nabla_x F = hQ(f) + f[\partial_t h + v \cdot \nabla_x h]. \]

From the preceding Section, we also know that

\[ hQ(f) \in L^2((0,T) \times \mathbb{R}^n_x; H^{2}_{\nu}(\mathbb{R}^n_v)). \tag{12.25} \]

If \( \hat{\cdot} \) denotes the Fourier transform with respect to the variables \((x,v)\) and \((\xi,\mu)\) the dual variables, then letting \( G = hQ(f), H = f[\partial_t h + v \cdot \nabla_x h], \) one has

\[ \partial_t \hat{F} - \xi \cdot \nabla_\mu \hat{F} = \hat{G} + \hat{H}. \tag{12.26} \]

By (12.25) and (12.24), we can write

\[ \hat{G} + \hat{H} = \hat{g}_1 + | \mu |^{\frac{\nu}{2}} \hat{g}_2, \tag{12.27} \]

where each \( g_i \) belongs to \( L^2 \). On each side of (12.26), we add \( | \mu |^{\nu} \hat{F} \) to get

\[ \partial_t \hat{F} - \xi \cdot \nabla_\mu \hat{F} + | \mu |^{\nu} \hat{F} = \hat{g}_1 + | \mu |^{\frac{\nu}{2}} \hat{g}_2 + | \mu |^{\nu} \hat{F}. \tag{12.28} \]

By (12.24) \( | \mu |^{\frac{\nu}{2}} \hat{F} \in L^2 \). Therefore, one may write

\[ \partial_t \hat{F} - \xi \cdot \nabla_\mu \hat{F} + | \mu |^{\nu} \hat{F} = \hat{g}_3 + | \mu |^{\frac{\nu}{2}} \hat{g}_4, \tag{12.29} \]

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where again, each $g_i \in L^2$.

At this point, one can use Perthame’s argument to get

$$
\partial_t \left| \hat{F} \right|^2 - \xi \cdot \nabla_{\mu} \left| \hat{F} \right|^2 + \left| \mu \right|^r \left| \hat{F} \right|^2 \leq \left| \hat{F} \hat{g}_3 \right| + \left| \mu \right|^r \left| \hat{\mu} \right| \left| \hat{F} \right|^2 + \left| \hat{g}_4 \right|^2,
$$

and thus

$$
\left| \hat{F}(t, \xi, \mu) \right|^2 \leq \left| \hat{F}_0(\xi, \mu + t\xi) \right|^2 + \int_0^t (| \hat{F} \hat{g}_3 | + | \hat{g}_4 |^2)(\xi, \mu + s\xi, t - s)ds.
$$

Fix $r \geq 0$ and $D \geq 0$. Then

$$
\int_0^T dt \int_{\mathbb{R}^n_\mu} \left| \mu \right|^D \left| \hat{F}(t, \xi, \mu) \right|^2 \leq \int_0^T dt \int_{|\mu| \geq D} \left| \xi \right|^r \left| \hat{F}(t, \xi, \mu) \right|^2 + \int_0^T dt \int_{|\mu| \leq D} \left| \xi \right|^r \left| \hat{F}(t, \xi, \mu) \right|^2 \leq \int_0^T dt \int_{|\mu| \geq D} \left| \xi \right|^r \left| \hat{F}_0(\xi, \mu + t\xi) \right|^2 + \int_0^T dt \int_{|\mu| \leq D} \int_0^t (| \hat{F} \hat{g}_3 | + | \hat{g}_4 |^2)(\xi, \mu + s\xi, t - s)ds
$$

$$
\equiv I + II + III.
$$

One has

$$
II = \int_0^T dt \int_{|\mu| \leq D} \left| \xi \right|^r \left| \hat{F}_0(\xi, \mu + t\xi) \right|^2 = \int_0^T dt \int_{|\mu - t\xi| \leq D} \left| \xi \right|^r \left| \hat{F}_0(\xi, \mu) \right|^2 \leq \int_0^T dt \int_{\mathbb{R}^n_\mu} \left| \mu - \frac{\left| \xi \right|^r (s) - \frac{s}{|\xi|^r} \right| \left| \hat{F}_0(\xi, \mu) \right|^2 \leq \left| \xi \right|^r D \int_{\mathbb{R}^n_\mu} \left| \hat{F}_0(\xi, \mu) \right|^2.
$$

In fact, this term is 0, since we are integrating, starting from $t = 0$.

In the same fashion, one gets

$$
III \leq \left| \xi \right|^r D \int_{\mathbb{R}^n_\mu \times (0, T)} (| \hat{F} \hat{g}_3 | + | \hat{g}_4 |^2)(s, \xi, \mu)dsd\mu.
$$

In conclusion, one obtains

$$
\int_0^T dt \int_{\mathbb{R}^n_\mu} \left| \xi \right|^r \left| \hat{F}(t, \xi, \mu) \right|^2 dtd\mu \leq
$$
\begin{equation}
(12.35) \quad \leq |\xi|^{-r} D A + |\xi|^{-1} \int_{|\mu| \geq D} |\hat{F}|^2 d\mu dt,
\end{equation}
where
\begin{equation}
A = \int_{\mathbb{R}^n} |\hat{F}_0(\xi, \mu)|^2 + \int_{\mathbb{R}^n \times (0, T)} (|\hat{F}\hat{g}_3| + |\hat{g}_4|^2(s, \xi, \mu)) d\mu dt.
\end{equation}
Of course, from here, one has
\begin{equation}
(12.36) \quad \int_0^T \int_{\mathbb{R}^n} |\xi|^{-r} |\hat{F}(t, \xi, \mu)|^2 d\mu dt \leq |\xi|^{-1} D A + \left|\frac{\xi}{D^\nu}\right| \int_0^T \int_{\mathbb{R}^n} \mu |\nu| |\hat{F}|^2 d\mu dt.
\end{equation}
Choosing \( D = |\xi|^{\frac{1}{1+\nu}} \) and then \( r = \frac{\nu}{1+\nu} \), we get
\begin{equation}
(12.37) \quad \int_0^T \int_{\mathbb{R}^n} |\xi|^{-1} |\hat{F}(t, \xi, \mu)|^2 d\mu dt \leq C,
\end{equation}
and thus we have obtained finally
\begin{equation}
(12.38) \quad hf \in L^2(0, T; H^{\frac{\nu}{2}}(\mathbb{R}^{2n}_x, c)),
\end{equation}
and this concludes the proof of the regularity result.

\textbf{Remark 12.2.1} It is clear that any improvement of the regularity must take care of functional properties of \( Q(f) \), as those shown in Chapter 7.

\section*{12.3 Final Remarks}
We wish to extend the above range of values of \( \gamma \). For this purpose, let us first consider the cutoff linear problem
\[
\partial_t f + v \cdot \nabla_x f = \int \int B_{n'}[l' f' - l f], \quad f_{t=0} = f_0.
\]
Let \( a \) a smooth function (which can be \( \beta M \)). Then, one has, setting \( F = f - a \),
\[
\partial_t F + v \cdot \nabla_x F + \partial_t a + v \cdot \nabla_x a = \int \int B_{n'}[l' F' - l F] + \int \int B_{n'}[l' a' - l a]
\]
For the moment, we just assume that \( l \geq 0 \) and that \( a \) is sufficiently smooth.
Let \( \alpha(s) = s^+ \), so that \( \alpha'(s) = \mathbb{I}_{s \geq 0} \). Then, it follows that
\[
\partial_t \alpha(F) + v \cdot \nabla_x \alpha(F) + [\partial_t a + v \cdot \nabla_x a] \alpha'(F) \leq \int \int B_{n'}[l' \alpha'(F) - \alpha(F)] + \int \int B' \alpha(F)(l' - l) + \alpha'(F) \int \int B_{n'}[l' a' - l a]
\]
\[
\leq \int \int B_n[l'_s \alpha(F)' - l_s \alpha(F)] + \alpha'(F) \int \int B_n[l'_s a' - l_s a]
\]
Now, if we assume that \( l \leq a \), with \( a = \beta M \), then it follows that \( l' \leq a' \), \( l'_s \leq a'_s \), and \( l'l'_s \leq aa_s \).
Let us look for an inequality such as
\[
\beta'(t)M \leq \int \int B_n[l'_s a' - l_s a]
\]
In view of previous computations, using positivity, it is enough to ask for
\[
\beta'(t)M \leq \int \int B_n[l'_s a' - a]
\]
Since \( l_s \leq a_s \), and thus \(-l_s \geq -a_s\), it is enough to look for \( \beta \) such that
\[
\beta'(t)M \leq \int \int B_n[l'_s a' - a_s a]
\]
Chapter 13

Landau Asymptotics

Landau equation was introduced by Landau himself, and models the behavior of a dilute plasma interacting through binary collisions, see for instance [21, 45] and the references therein. As for Boltzmann equation, the unknown is the distribution function $f(t,x,v)$ of the plasma, at time $t$, at position $x \in \mathbb{R}^3$, with velocity $v \in \mathbb{R}^3$, solution of Landau equation

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q_L(f,f).$$

$Q_L$ is the Landau collision operator, and acts only on the velocity dependence of $f$ ($f_s = f(v_s)$),

$$Q_L(f,f) = \nabla_v \cdot \left( \int_{\mathbb{R}^3} dv_s a(v-v_s) \left[ f_s \nabla f - f(\nabla f)_s \right] \right),$$

$$a_{ij}(z) = \frac{L}{|z|} \left[ \delta_{ij} - \frac{z_i z_j}{|z|^2} \right].$$

For simplicity, we are only considering a single species of particles, for instance electrons, while plasma phenomena usually involve at least two species (typically, ions and electrons). For more details, see for instance Chapter ??.

The value of the physical constant $L$ in (13.3) will be discussed later on.

At this point, let us recall that Landau collision operator $Q_L(f,f)$ is usually obtained as a formal limit of the Boltzmann collision operator,

$$Q_B(f,f) = \int_{\mathbb{R}^3} dv_s \int_{S^2} d\sigma B(v-v_s, \sigma) (f'_s f'_s - f f_s),$$

where we used notations from previous chapters, and in particular $\sigma$-representation.

Taking into account only elastic collisions, there are several different types of electrostatic interactions in plasmas: Coulomb interaction between two charged particles, Van der Waals interaction between two neutral particles, or Maxwellian interaction between one neutral and one charged particle.
In totally ionized plasmas, dominant effects are due to the interaction between charged particles; moreover the mathematical analysis of the Boltzmann equation is much simpler for Van der Waals or Maxwellian interaction, than for Coulomb interaction. Therefore we restrict to this last case.

Thus assuming the interaction between particles is governed by the Coulomb potential,

\[
\phi(r) = \frac{e^2}{4\pi\epsilon_0 r},
\]

the kernel \( B \) is given by the well-known Rutherford formula,

\[
B^C(v - v_*, \sigma) = \left( \frac{e^2}{4\pi\epsilon_0 m} \right)^2 \frac{2}{|v - v_*|^3 \sin^4(\theta/2)}.
\]

In the above formulae, \( \epsilon_0 \) is the permittivity of vacuum, \( m \) is the mass of the electron and \( e \) its charge.

It is now well known that Boltzmann collision operator is meaningless for Coulomb interactions, since in view of the last formula, \( B^C \) is extremely singular as \( \theta \to 0 \). This singularity for zero deviation angle reflects the great abundance of grazing collisions, i.e collisions in which interacting particles are hardly deviated. From the physical point of view, these collisions correspond to encounters between particles which are microscopically very far apart, and this abundance is a consequence of the long range of Coulomb interaction.

Mathematically, the Boltzmann equation can be used only if the mean transfer of momentum between two colliding particles of velocities \( v, v_* \) is well-defined. Here, the typical amount of momentum which is communicated to a particle of velocity \( v \) by collisions with particles of velocity \( v_* \) is given by

\[
\int_{S^2} B(v - v_*, \sigma)(v' - v) \, d\sigma = \frac{|S^1|}{2} \left( \int_0^\pi B(|v - v_*|, \cos \theta)(1 - \cos \theta) \sin \theta \, d\theta \right) (v - v_*).
\]

In the case of the cross-section (13.6), the integral in the r.h.s of (13.7) does not converge since

\[
\frac{\cos(\theta/2)(1 - \cos \theta)}{\sin^4(\theta/2)} \, d\theta \sim 4 \frac{d\theta}{\theta}
\]

defines a logarithmically divergent integral as \( \theta \to 0 \).

The physical meaning of this divergence is that for a Coulomb potential, grazing collisions are so frequent as to be the only ones to count, in some sense: the mechanism of momentum transfer is dominated by small-angle deviations, and a given particle is extremely sensitive to the numerous particles which are very far apart. This fact is referred as the “collective behavior” of electrons in a plasma. At this point, let us mention that it is widely admitted, though not completely clear a
priori, that these collective effects can still be described by binary collisions, because corresponding deflections are very small.

On the other hand, that the physical phenomenon of the screening tends to tame the Coulomb interaction at large distances, i.e when particles are separated by distances much larger than the so-called Debye (or screening) length.

This screening effect may be induced by the presence of two species of particles with opposite charges: typically, the presence of ions constitutes a background of positive charge which screens the interaction of electrons at large distances [21, 45].

One is therefore led to assume that particles do not interact via the “bare” Coulomb potential $1/(4\pi\varepsilon_0 r)$, but rather via the so-called Debye (or Yukawa) potential, $e^{-r/\lambda_D}/(4\pi\varepsilon_0 r)$, where $\lambda_D$ is the Debye length. For this screened interaction, note that the Boltzmann operator can be used.

However, this screened potential is not very interesting, because the corresponding cross-section is not explicit, and because the really interesting regime is when the Debye length is very large w.r.t the characteristic length $r_0$ for collisions, so that the potential is approximately Coulomb. Thus we are really interested in letting $\lambda_D$ go to infinity.

Of course, in the limit $\lambda_D/r_0 \to \infty$, the Boltzmann operator $Q_B$ diverges. But Landau has shown formally that in this limit, it is to leading order proportional to the operator $Q_L$. The proportionality factor is the so-called Coulomb logarithm, since it is, roughly speaking, proportional to $\log(\lambda_D/r_0)$. We refer to [80] for Landau’s original argument, to [21, 45] for more physical background, and to [44] for a slightly more mathematically oriented presentation, see also [117].

This approximation procedure, called herein the Landau approximation, is one of the main theoretical justifications for the Landau equation.

Here are some standard references. First of all, Cercignani [38], Degond and Lucquin [44], Desvillettes [46] showed that

$$Q_B(f, f) \simeq Q_L(f, f)$$

in a certain asymptotic procedure, for a fixed, smooth $f$ depending only on the velocity variable.

Then, concerning justification at the level of the equations, this problem was solved by Arsen’ev and Buryak [19], Goudon [68], Villani [117] in the framework of spatially homogeneous solutions, the true Debye cross-section being never considered, because it is not explicit.

In the present chapter, we shall solve the problem of the Landau approximation, in the general, spatially inhomogeneous setting, discussing also the true cross-section associated with the Debye screening.

Of course, the main tools will be the results developed in the previous Chapters, and the results of
Villani [121] concerning the theory of non homogeneous Landau equation.

However, there is one last serious physical restriction: at this moment, we are unable to take into account the effect of a mean-field interaction between particles, but only under certain assumptions which are not satisfied in the classical theory of plasmas, see explanations in next section.

13.1 Analysis of scales

Recall that the Rutherford cross-section is given by

\begin{equation}
B(v - v_*, \sigma) = \left( \frac{e^2}{4\pi \epsilon_0 m} \right)^2 |v - v_*|^{-3} b^C(\cos \theta),
\end{equation}

where the angular cross-section $b^C$ satisfies

\begin{equation}
b^C(\cos \theta) \sin \theta = \frac{2 \cos(\theta/2)}{\sin^3(\theta/2)}.
\end{equation}

Having in mind a kinetic description, we take the thermal velocity as the natural scale of velocity

\begin{equation}
v_{th} = \sqrt{\frac{k\Theta}{m}},
\end{equation}

where $k$ is Boltzmann’s constant, $m$ the mass of the electron, and $\Theta$ the mean temperature of the plasma.

For Coulomb interaction, the natural length scale is given by the Landau length $r_0$, corresponding to the distance at which particles have an interaction energy of the order of their kinetic energy:

\begin{equation}
r_0 = \frac{e^2}{4\pi \epsilon_0 k\Theta}.
\end{equation}

Thus the dimensional constant in front of (13.8) is exactly $(r_0 v_{th}^2)^2$.

The Rutherford cross-section (13.8) presents two types of singularities:

- first, the strongly singular kinetic cross-section: $|v - v_*|^{-3} \notin L^1_{loc}(\mathbb{R}^3)$.
- next, the strongly angular cross-section, $b^C(\cos \theta)$:

\begin{equation}
b^C(\cos \theta)(1 - \cos \theta) \notin L^1(\sin \theta d\theta).
\end{equation}

This last singularity is the reason why the Boltzmann operator with Rutherford cross-section is meaningless.

Since the Coulomb interaction cannot be handled, consider instead the Debye potential, see for instance [21, 45]

\begin{equation}
\phi_D(r) = \frac{e^2}{4\pi \epsilon_0 r} e^{-r/\lambda_D}.
\end{equation}
Roughly speaking, it amounts to a sheer truncation of the Rutherford cross-section, yielding a kernel proportional to

\[(13.12) \quad |v - v_*|^3 b^C \cdot (\cos \theta) \mathbb{1}_{\theta \geq \theta_D},\]

where \[45\]

\[(13.13) \quad \sin \frac{\theta_D}{2} = \frac{r_0}{2 \lambda_D}.\]

Let introduce the classical physical notation

\[(13.14) \quad \Lambda = 2 \lambda_D.\]

When \(\Lambda\) is large but finite, the cross-section for momentum transfer is finite:

\[
|S^1| \int_{\theta_D}^{\pi} \frac{2(\cos \theta/2)}{\sin^3(\theta/2)} (1 - \cos \theta) \, d\theta = 8 |S^1| \log \Lambda < +\infty.
\]

In this case, as we saw already in previous Chapters, the corresponding Boltzmann equation makes sense, written here as

\[(13.15) \quad \frac{\partial f}{\partial t} + v \cdot \nabla_x f = (r_0 v_{th}^2)^2 Q_{\Lambda}(f, f),\]

where \(Q_{\Lambda}\) is the Boltzmann collision operator with nondimensional cross-section given by (13.12), as an approximation to the true Debye cross-section.

Since this equation diverges as \(\Lambda\) goes to infinity, we now look for physical scales on which there is a meaningful limit equation, changing the scales of time, length and density. This means that we consider the new distribution function \(\tilde{f}\) in adimensional variables,

\[(13.16) \quad \tilde{f}(t, x, v) = \frac{1}{N v_{th}^3} f(Tt, Xx, v_{th} v),\]

where \(N\) is a typical density, \(T\) is a typical time and \(X\) is a typical length of the system under study.

In this rescaling we impose the velocity scale to be \(v_{th}\), and to be consistent we impose \(X = v_{th} T\). As for the density scaling, it ensures that one may assume the mass of \(f\) to be of order 1 without physical inconsistency (recall that the Boltzmann equation is established in a regime when the density is small enough that only binary collisions should be taken into account). The same holds true for the kinetic energy because of our choice of the velocity scale.

Plugging (13.16) into the Boltzmann equation (13.15), and denoting \(\tilde{f}\) by \(f\) again, we arrive at the rescaled Boltzmann equation

\[
\frac{\partial f}{\partial t} + v \cdot \nabla_x f = r_0^2 v_{th}^2 N T Q_{\Lambda}(f, f).
\]
In order to make the limit $\Lambda \to \infty$ meaningful, we consider a time scale such that (say)

$$T = \frac{1}{(\log \Lambda)^{r_0^2} v_{th} N}.$$  

With this system of units the rescaled Boltzmann equation now reads

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \frac{Q_{\Lambda}(f, f)}{\log \Lambda} - 3.$$  

and now the total angular cross-section for momentum transfer of the rescaled Boltzmann collision operator will converge to a finite limit:

Moreover, for any $\theta_0 > 0$, the contribution of deviation angles $\theta \geq \theta_0$ in this total cross-section goes to $0$ as $\Lambda \to \infty$, because of the division by $\log \Lambda$. In this sense, only grazing collisions have an influence in the limit.

It is precisely the combination of these two points, namely that

- the total angular cross-section for momentum transfer stays finite;
- only grazing collisions count in the limit,

which ensures that, as the parameter $\Lambda$ goes to infinity, the collisions term is given by the action of a Landau operator.

A generalized mathematical framework for this was introduced in Villani [117]: a family $(b_m)_{m \in \mathbb{N}}$ of angular cross-sections is said to concentrate on grazing collisions if

$$\begin{cases} 
|S^1| \int_0^\pi b_m(\cos \theta)(1 - \cos \theta) \sin \theta \, d\theta \to \infty \quad & m \to \infty \quad \mu_{\infty} \in (0, +\infty) \\
\forall \theta_0 > 0, \quad \sup_{\theta \geq \theta_0} b_m(\cos \theta) \to 0 \quad & m \to \infty.
\end{cases}$$

These conditions allow a number of generalizations of the limit at large screening length. From this mathematical point of view (contrary to what is often believed), the scalings considered in [44] and in [46] are just the same, and correspond respectively to the two model cases below:

**Case 1:** $\int_0^\pi b(\cos \theta)(1 - \cos \theta) \sin \theta \, d\theta = +\infty$. Then, define

$$b_m(\cos \theta) = \frac{b(\cos \theta) 1_{\theta \geq m^{-1}}}{|S^1| \int_0^\pi b(\cos \theta)(1 - \cos \theta) \sin \theta \, d\theta};$$
Case 2: \( \int_0^\pi b(\cos \theta)(1 - \cos \theta) \sin \theta \, d\theta < +\infty \). Then, with the notation \( \zeta_m(\theta) = b(\cos \theta) \sin \theta \), define \( b_m(\cos \theta) \) in such a way that
\[
\zeta_m(\theta) = m^3 \zeta(m\theta)
\]
(by convention \( \zeta \) vanishes for angles greater than \( \pi \)).

However, the above definition is too restrictive; it applies only when the cross-section factors into the product of a kinetic cross-section and an angular cross-section (and where the former is kept fixed). This has no real physical basis, and actually, in the “true” cross-section associated with a Debye approximation, this property does not hold. Therefore, in the next section we shall introduce a generalization of (13.19) which applies in generality to all relevant situations.

To summarize: we shall show the following.

Consider a family of solutions \( (f^m) \) to the Boltzmann equation with respective cross-section \( B_m \), where \( (B_m)_{m \in \mathbb{N}} \) concentrates on grazing collisions as \( m \to \infty \). Then \( f^m \) converges strongly towards the solution of a certain Landau-type equation.

Before turning to precise statements, let us discuss the physical relevance of our results.

For the Boltzmann equation to be relevant, it is physically required that
\[
Nr_0^3 << 1, \quad N\lambda_D^3 >> 1.
\]
Taking into account the definition of \( \Lambda \), this is satisfied if
\[
(13.20) \quad \frac{r_0}{v_{th}(\log \Lambda)} << T << \frac{r_0}{v_{th} \log \Lambda} \Lambda^3.
\]
We note that the quantity
\[
(13.21) \quad g = \frac{1}{N\lambda_D^3}
\]
is called the plasma parameter. Classical plasmas are often defined as those in which \( g \) is very small.

In the classical theory of plasmas, the Debye length depends on the mean density \( N \):
\[
(13.22) \quad \lambda_D = \sqrt{\frac{\epsilon_0 k \Theta}{Ne^2}} = \frac{1}{\sqrt{4\pi Ne^2}};
\]
and as a consequence
\[
\Lambda = \frac{2}{\sqrt{4\pi N r_0^3}};
\]
If we use the law (13.22), we find from (13.17)
\[
T = \frac{r_0}{v_{th} \log \Lambda} \left( \frac{1}{Nr_0^3} \right) = \frac{\pi r_0 \Lambda^2}{v_{th} \log \Lambda}.
\]
and the validity of our limit is ensured if $1 \ll \Lambda^2 \ll \Lambda^3$, which is the case as $\Lambda \to \infty$. We note that in plasma physics, $\Lambda$ ranges from $10^2$ to $10^{30}$, so that $\Lambda$ is actually very large, but its logarithm is not so large (this is why physicists most often do not worry about the precise value to attribute to the multiplicative constant in front of the Landau operator...)

However, our results are far from being fully satisfactory! In particular, we have assumed that the interaction between particles can be modelled by binary collisions, while a more precise description of a plasma is obtained when one also takes into account collective effects modelled by a \textit{mean-field} self-consistent force term of Poisson type.

Of course, it is easy to add such a term at the level of (13.18), and treat the Landau approximation for the model

\begin{equation}
\frac{\partial f}{\partial t} + v \cdot \nabla_x f + F \cdot \nabla_v f = \frac{Q\Lambda(f,f)}{\log \Lambda},
\end{equation}

where

\begin{equation}
F(x) = -\nabla V(x), \quad V(x) = \frac{1}{4\pi r} \star \int f \, dv.
\end{equation}

Then, the self-consistent coupling can be handled in exactly the same way as in Lions [86, 83], and our main result would apply. But this mathematical problem would not take into account physical scales!

Actually, writing physical constants explicitly, the Boltzmann equation with a mean field term should be

\begin{equation}
\frac{\partial f}{\partial t} + v \cdot \nabla_x f + 4\pi r_0 v_{th}^2 F \cdot \nabla_v f = (r_0 v_{th}^2)^2 Q\Lambda(f,f),
\end{equation}

with $F$ defined as in (13.24), or

\begin{equation}
\frac{\partial f}{\partial t} + v \cdot \nabla_x f + 4\pi r_0 v_{th}^2 F_\Lambda \cdot \nabla_v f = (r_0 v_{th}^2)^2 Q\Lambda(f,f),
\end{equation}

with $F_\Lambda$ defined as $F$ but with a Debye potential. Note that which equation should be chosen at this point is not clear at all.

Then, going back to (13.16), we obtain the following rescaled Boltzmann equation (with the notation of (13.24)) :

\begin{equation}
\frac{\partial f}{\partial t} + v \cdot \nabla_x f + 4\pi r_0 v_{th}^2 N T^2 F \cdot \nabla_v f = r_0^2 v_{th} N T Q\Lambda(f,f).
\end{equation}

Up to numerical constants, the quantity

\begin{equation}
\omega = \sqrt{4\pi v_{th}^2 r_0 N}
\end{equation}

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is called the *plasma oscillation frequency*. It is supposed to give the inverse time scale for oscillations due to the Poisson coupling.

If we now wish to consider physical scales $N, T$ on which the relevant equation is the Landau equation with a mean-field term, say
\[
\frac{\partial f}{\partial t} + v \cdot \nabla_x f + F \cdot \nabla_v f = Q_L(f, f) \quad \left( \simeq \frac{Q_{\Lambda}(f, f)}{\Lambda} \right),
\]
we have to impose
\[
T = \omega^{-1} = \frac{1}{\sqrt{v_{\text{th}}^2 r_0 N}}
\]
and at the same time (eq. (13.17))
\[
T = \frac{1}{(\log \Lambda) v_{\text{th}} r_0^2 N}.
\]
Identification of these two formulae implies that $log \Lambda = \sqrt{\frac{4\pi}{\Lambda}} (N r_0^2)^{-1/2}$, or
\[
(13.27) \quad \frac{2(\log \Lambda)^2}{\pi \Lambda^3} = g.
\]
Under this assumption, which implies that the typical length for oscillations is much smaller than the Debye length, we are able to recover the “full” Landau equation (with a mean-field term) from the Boltzmann equation.

But (13.27) is not satisfied in the classical theory of plasmas, since it is incompatible with (13.22). Instead, one should have
\[
(13.28) \quad \lambda_D = \omega^{-1} v_{\text{th}}, \quad \Lambda = \frac{8\pi}{g}.
\]
In particular, up to numerical constants, the typical length for oscillations should coincide with the Debye length, see [45] for a more precise discussion, and much more on these scale problems.

The physical content of this obstruction is the following: in the classical theory of plasmas, the Landau equation with a mean-field term,
\[
\frac{\partial f}{\partial t} + v \cdot \nabla_x f + F \cdot \nabla_v f = Q_L(f, f)
\]
is relevant on no physical scale! As the parameter $\Lambda$ goes to infinity, or equivalently as $g \to 0$, the Boltzmann equation with a mean-field term should converge to the “pure” (collisionless) Vlasov-Poisson equation, and the effect of collisions should only be felt as large-time corrections to the Vlasov-Poisson equation.

Two possible ways toward a mathematical justification of the Landau approximation when a mean-field term is present and when the Debye length satisfies (13.22), (13.28), are as follows.
1) First possibility: adopt the time scale for the Landau equation. Choose

\[ T = \frac{2\pi \Lambda}{\log \Lambda} \omega^{-1}, \]

then the rescaled Boltzmann equation is

\[ \frac{\partial f}{\partial t} + v \cdot \nabla_x f + \left( \frac{2\pi \Lambda}{\log \Lambda} \right)^2 F(\Lambda) \cdot \nabla_v f = \frac{Q_\Lambda(f, f)}{\log \Lambda}. \]

Problem: prove that on a fixed time interval, as \( \Lambda \to \infty \), solutions to this equation are close to solutions of

\[ \frac{\partial f}{\partial t} + v \cdot \nabla_x f + \left( \frac{\Lambda}{2\log \Lambda} \right)^2 F(\Lambda) \cdot \nabla_v f = Q_L(f, f). \]

It is not clear at all if such a statement has a chance to hold true. As \( \Lambda \to \infty \), the very large mean-field term is expected to induce very fast oscillations, and the strong compactification effects induced by the entropy dissipation mechanism will be lost. Passing to the limit in the collision operator when such oscillations are present seems a desperate task, and the only hope would be to prove that solutions to both equations are wildly oscillating in exactly the same way, but asymptotically close to each other in strong sense.

2) Second possibility: adopt the time scale for the Vlasov-Poisson equation. Choose

\[ T = \omega^{-1}. \]

This choice is consistent with (13.20) since

\[ 1 << \frac{\Lambda}{\log \Lambda} << \Lambda^3. \]

Then the rescaled Boltzmann equation is

\[ (13.29) \quad \frac{\partial f}{\partial t} + v \cdot \nabla_x f + F(\Lambda) \cdot \nabla_v f = \frac{1}{2\pi \Lambda} Q_\Lambda(f, f) = \frac{\log \Lambda}{2\pi \Lambda} \frac{Q_\Lambda(f, f)}{\log \Lambda}. \]

Problem: prove that, as \( \Lambda \to \infty \), on a large time interval of size \( O(\Lambda/\log \Lambda) \), solutions of this equation are close to solutions of

\[ (13.30) \quad \frac{\partial f}{\partial t} + v \cdot \nabla_x f + F \cdot \nabla_v f = \frac{\log \Lambda}{2\pi \Lambda} Q_L(f, f). \]

(compare with [44]). Note that on any fixed time interval, solutions of both systems converge to solutions of the collisionless Vlasov-Poisson equation (because \( \log \Lambda/\Lambda \to 0 \)), so that this problem can only be expressed in terms of long-time corrections. Note also that formally, \( F_\Lambda - F = O(1/\Lambda) \), so that the error due to the screening at the level of the mean field should be of order \( O(1/\log \Lambda) \) – hence negligible – on this time scale.
13.2 Main result

Without further difficulties, we consider a more general framework. We set the problem on \([0, T] \times \mathbb{R}_x \times \mathbb{R}^n_v\) : here \([0, T]\) is an arbitrary time interval fixed once for all, and both the position and the velocity take values in \(\mathbb{R}^n, n \geq 2\). Thus the equations on consideration will be of the form

\[
\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f, f),
\]

where \(Q\) is either a Boltzmann-type operator,

\[
Q_B(f, f) = \int_{\mathbb{R}^n} dv_s \int_{S^{n-1}} d\sigma B(v - v_s, \sigma)(f'_s - ff_*)
\]

or a Landau-type operator,

\[
Q_L(f, f) = \nabla_v \left( \int_{\mathbb{R}^n} dv_s a(v - v_s)[f_s \nabla f - f(\nabla f)_s] \right).
\]

In the Boltzmann case, we shall use assumptions and notations already used many times in previous Chapters.

In the Landau case, the matrix-valued function \(a(z)\) will be of the form

\[
a(z) = \Psi(|z|) \Pi(z), \quad \Pi_{ij}(z) = \delta_{ij} - \frac{z_i z_j}{|z|^2},
\]

where \(\Psi\) is a nonnegative measurable function. Of course \(\Pi(z)\) is the orthogonal projection upon \(z^\perp\).

We have already seen previously that two mathematical objects play a central role for the Boltzmann equation:

- 1) the cross-section for momentum transfer (at a given relative velocity),

\[
M(|v - v_s|) = \int_{S^{n-1}} B(v - v_s, \sigma)(1 - k \cdot \sigma) d\sigma
\]

\[
= |S^{n-2}| \int_0^\frac{\pi}{2} B(|v - v_s|, \cos \theta)(1 - \cos \theta) \sin^{n-2} \theta d\theta,
\]

- 2) the compensated integral kernel for \(Q_B(\cdot, 1)\),

\[
S(|v - v_s|) = \int_{S^{n-2}} \left[ \frac{1}{\cos^n(\theta/2)} B \left( \frac{|v - v_s|}{\cos(\theta/2)}, \cos \theta \right) - B(|v - v_s|, \cos \theta) \right] \sin^{n-2} \theta d\theta.
\]

Here again \(Q_B(g, f)\) stands for the bilinear Boltzmann operator. Recall that

\[
Sf \equiv S \ast f = Q_B(f, 1) = \int_{\mathbb{R}^n \times S^{n-1}} B(v - v_s, \sigma)(f'_s - ff_*) dv_s d\sigma \equiv Sf.
\]
Definition 13.2.1 Let \((B_m)_{m \in \mathbb{N}}\) be a sequence of admissible cross-sections for the Boltzmann equation, and \(M_m, S_m\) be defined as in formulas (13.34), (13.36). We say that \((B_m)\) concentrates on grazing collisions if

i) \(S_m(|z|), |z| M_m(|z|)\) define sequences of measures which are bounded in total variation on compact sets, uniformly in \(m\);

ii) There exists a nonnegative, radially symmetric measurable function \(M_\infty\) such that

\[
S_m^0(|z|), \quad |z| M_m^0(|z|) \xrightarrow{[n \to \infty]} 0 \quad \text{locally in weak measure sense,}
\]

where \(S_m^0, M_m^0\) are associated to the truncated cross-sections \(B_m^{\theta_0} = B_m \mathbf{1}_{\theta \geq \theta_0}\) via formulas similar to (13.34), (13.36).

Remarks 13.2.1 1. Admissible cross-sections are those such that \(M\) and \(S\) define meaningful mathematical objects.

2. Assumption (ii) is formally equivalent to the convergence of \(M_n\) towards \(M_\infty\), but our formulation is less sensitive to the behavior of \(M_n\) at the origin. This is important because of the nonintegrable singularity of the Rutherford cross-section.

3. Condition (iii) means that only grazing collisions count in the limit.

Example: The following family of cross-sections concentrates on grazing collisions:

\[
B_m(z, \sigma) = \Phi_m(|z|) b_m(\cos \theta) + \frac{\beta_m(\cos \theta)}{|z|^n},
\]

as soon as \(\Phi_m \xrightarrow{} \Phi\) in \(L^1_{loc}(\mathbb{R}^n)\),

\[
\sup_{1 < \lambda \leq \sqrt{2}} \frac{\Phi_m(\lambda |z|) - \Phi_m(|z|)}{\lambda - 1} \in L^1_{loc}(\mathbb{R}^n_+), \quad \text{uniformly in } m,
\]

and \(b_m, \beta_m\) concentrate on grazing collisions in the sense of (13.19).

This general definition enables to deal with a general class of cases. For instance, the following approximation of the Debye cross-section:

\[
B_n(|v - v_\ast|, \cos \theta) \sin \theta = \frac{|v - v_\ast| \sin \theta}{(|v - v_\ast|^2 \sin^2 \frac{\theta}{2} + \frac{1}{2})^\frac{1}{2}},
\]

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used in quantum physics, is discussed in Appendix 13.6. Another kind of cross-sections that we can now handle is the true Debye cross-section, see Appendix 13.7.

We need further technical but otherwise always satisfied conditions.

**Condition of decay at infinity :**

\[(13.39) \quad M_m(|z|) = o(1) \quad \text{as } |z| \to \infty, \text{ uniformly in } m.\]

**Overall singularity condition :** We require that

\[(13.40) \quad B_m(z, \sigma) \geq \Phi_0(|z|) b_{0,m}(k \cdot \sigma),\]

where \(\Phi_0(|z|)\) is a continuous nonnegative function, nonvanishing for \(|z| \neq 0\), and

\[(13.41) \quad |S^{n-2}| \int_0^\pi b_{0,m}(\cos \theta)(1 - \cos \theta) \sin^{n-2} \theta d\theta \to \mu > 0.\]

**Technical condition :** Let

\[(13.42) \quad \eta_m(|v - v_*|, \cos \theta) = |S^{n-2}| \sin^{n-2} \theta \frac{B_m(|v - v_*|, \cos \theta)(1 - \cos \theta)}{\sqrt{M_m(|v - v_*|)}}.\]

We require that for all bounded smooth open set \(\Omega\) with \(0 \notin \bar{\Omega}\),

\[(13.43) \quad \eta_m(|z|, \cos \theta) \to \sqrt{M_\infty(|z|)} \delta_{\theta=0}\]

in weak-measure sense on \(\Omega \times [0, \pi/2]\).

Requirement (13.43) is somewhat natural since

\[\int \eta_m(|z|, \cos \theta) d\theta = \sqrt{M_m(|z|)},\]

and \(\eta_m\) is supposed to concentrate on \((\theta = 0)\). In realistic cases, this assumption is always satisfied, because \(M_m\) is uniformly smooth away from the origin.

These three technical conditions will suffice to prove the Landau approximation. But before formulating a statement we have to precise our notion of solutions. For the Boltzmann, it is quite clear; we shall use the notion of renormalised solutions, simply called weak solutions herein, introduced previously, see Chapter 11.

We are now ready to state our main result.
Theorem 13.2.1 Let \((f^m)\) be a sequence of weak solutions to the Boltzmann equation with respective cross-section \(B_m\), on \([0,T] \times \mathbb{R}_x^n \times \mathbb{R}_v^n\). Assume that \((B_m)\) concentrates on grazing collisions, in the sense of Definition 13.2.1, and satisfies the technical conditions (13.39), (13.40), (13.43). Further assume that the sequence \((f^m)\) satisfies the physical assumptions of finite mass, energy and entropy, i.e.

\[
\sup_m \sup_{t \in [0,T]} \int_{\mathbb{R}_x^n \times \mathbb{R}_v^n} f^n(t,x,v) \left[ 1 + |v|^2 + |x|^2 + \log f^n(t,x,v) \right] \, dx \, dv < +\infty,
\]

as well as the assumption of finite entropy dissipation, i.e.

\[
\int_0^T dt \int_{\mathbb{R}_x^n} dx D_m(f^n(t,x,\cdot)) < +\infty,
\]

where \(D_m\) is the entropy dissipation functional associated with the cross-section \(B_m\), as defined in section 13.4. Assume, w.l.o.g. that as \(m \to \infty\), \(f^m \rightharpoonup f\) weakly in \(w-L^p([0,T];L^1(\mathbb{R}_x^n \times \mathbb{R}_v^n))\), for all \(p \in (1, +\infty)\).

Then, \(f\) is a weak solution of the Landau equation with cross-section

\[
\Psi(|z|) = \frac{|z|^2 M_\infty(|z|)}{4(n-1)}.
\]

Moreover, the convergence of \(f^m\) towards \(f\) is automatically strong.

The proof of this theorem is performed in sections 13.4 and 13.5. We shall be using notations from Chapter ??, as regards Boltzmann equation. However, first of all, let us recall the results of Villani, as regards Landau equation.

13.3 Renormalised solutions for Landau equation

The renormalized formulation of the Landau collision operator was studied by Lions [84] and Villani [121].

Let \(a = a_{ij}(z)\) be the matrix appearing in the definition of the Landau collision operator. We define

\[
b = \nabla \cdot a, \quad c = \nabla \cdot b,
\]

where \(\nabla \cdot\) denotes the divergence operator. More explicitly,

\[
b_j = \sum_i \partial_i a_{ij}, \quad c = \sum_{ij} \partial_{ij} a_{ij}.
\]
As the matrix $a$ is of the form $a(z) = \Psi(|z|)\Pi(z)$, with $\Pi$ a projection operator, one has in particular

\begin{align}
(13.47) & \quad b(z) = -(n - 1)\frac{z\Psi(|z|)}{|z|^2}, \\
(13.48) & \quad c(z) = -(n - 1)\nabla \cdot \left[ \frac{z\Psi(|z|)}{|z|^2} \right].
\end{align}

We also define

\[ \tilde{a} = a \ast f, \quad \tilde{b} = b \ast f, \quad \tilde{c} = c \ast f, \]

and we immediately note that the Landau operator can be rewritten as

\[ Q_L(f, f) = \nabla \cdot (\tilde{a}\nabla f - \tilde{b}f). \]

In order to define a renormalized formulation of the Landau equation, we only need the following assumptions.

**Assumption L.1** *(Integrability).* We require that $|b(z)| \in L^1_{\text{loc}}(\mathbb{R}^n)$, and $|c(z)|$ is a locally bounded measure.

**Assumption L.2.** $\Psi(|z|) = o(|z|^2)$ as $|z| \to \infty$.

Let us note that, when $n \geq 3$ and $\Psi(|z|) = |z|^{-(n-2)}$, then

\[ c(z) = -(n - 1)\nabla \cdot \left( \frac{z}{|z|^n} \right) = -(n - 1)|S^{n-1}|\delta_0. \]

**Definition 13.3.1** Let $f$ be a distribution function with finite mass and energy, satisfying the additional a priori estimate

\begin{align}
(13.49) & \quad \tilde{a}\nabla \beta(f) \nabla \beta(f) \in L^1_{\text{loc}}([0, T]; \mathbb{R}^n \times \mathbb{R}^n).
\end{align}

Then, by convention, the renormalized Landau collision operator is given by

\begin{align}
\beta'(f)Q_L(f, f) = & \quad -\tilde{c} [f \beta'(f) - \beta(f)] & (\mathcal{R}^1_L) \\
& \quad + \nabla \cdot \left[ \nabla \cdot (\tilde{a}\beta(f)) - 2\tilde{b}\beta(f) \right] & (\mathcal{R}^2_L) \\
& \quad - \frac{\beta''(f)}{\beta'(f)^2} \tilde{a}\nabla \beta(f) \nabla \beta(f) & (\mathcal{R}^3_L).
\end{align}
Remarks 13.3.1 1. Expressions (R^4_L) and (R^5_L) in (13.50) are well-defined since \( \tilde{a}, \tilde{b}, \tilde{c} \in L^1_{\text{loc}} \), while (R^4_L) is well-defined as an a.e. finite function in view of assumption (13.49). Moreover, (R^5_L) is signed since \( \beta \) is concave. For \( \delta = 1 \), we find \(-\beta''(f)/\beta'(f)^2 = 2(1 + f)\).

2. Formally,
\[
\nabla \cdot (\tilde{a}\beta(f)) - 2\tilde{b}\beta(f) = \tilde{a}\nabla \beta(f) - \tilde{b}\beta(f),
\]
so that (R^5_L) can be rewritten as
\[
Q_L(f, \beta(f)),
\]
where \( Q_L \) is now the bilinear Landau operator, defined by duality:
\[
\int_{\mathbb{R}^n} Q_L(f, \beta(f)) \, dv = \int_{\mathbb{R}^2} f, \beta(f) T_L \varphi \, dv, \]
where
\[
(13.51) \quad [T_L \varphi](v, v_*) = -2b(v - v_*) \cdot \nabla \varphi(v) + a(v - v_*) : D^2 \varphi(v).
\]
Thus there is an excellent analogy between (R^5_L) (coming from Boltzmann theory) and (R^5_L).

And since (R^4_L) has the form \(-[f \beta'(f) - \beta(f)] S f\), where \( S \) is a convolution operator, there is a complete analogy between Boltzmann case and (13.50).

3. The exact analogues of estimates from Chapter 11 hold for the Landau equation under assumptions L.1 and L.2. In particular, this includes the additional estimate (13.49) which is required in the definition of the renormalized formulation, see for instance Lions [84].

4. One could be curious about a precise a priori definition of \( \tilde{a}\nabla \beta(f) \nabla \beta(f) \), for a function \( f \) which is only assumed to have finite mass and energy. A simple (and natural) way to define it is by
\[
\lim_{\varepsilon \to 0} \tilde{a}_{\varepsilon} \nabla \beta(f) \nabla \beta(f),
\]
where \( \tilde{a}_{\varepsilon} \) is defined as \( \tilde{a} \), except that \( \Psi(|z|) \) is replaced by a cutoffed version \( \Psi_{\varepsilon}(|z|) = \Psi(|z|) \chi_{\varepsilon}(|z|) \), with \( \chi_{\varepsilon} \) smooth, identically vanishing for \( \varepsilon \leq \varepsilon \) and identically equal to 1 for \( \varepsilon \geq 2\varepsilon \) (this definition does not depend on the particular choice of \( \chi_{\varepsilon} \)). Then we can say that \( \tilde{a}_{\varepsilon} \nabla \beta(f) \nabla \beta(f) \) lies in \( L^1_{\text{loc}}([0, T] \times \mathbb{R}^n \times \mathbb{R}^n) \) if
\[
\sqrt{\Psi_{\varepsilon}(|v - v_*|)} \Pi(v - v_*) \sqrt{f, \nabla \beta(f)} \in L^2([0, T] \times \mathbb{R}^n \times \mathbb{R}^n; L^2_{\text{loc}}(\mathbb{R}^n \times \mathbb{R}^n)).
\]
This last object always makes sense as a distribution since \( \beta(f) \in L^\infty, \sqrt{f} \in L^2, \) and \( \nabla \cdot (\Pi \sqrt{\Psi_{\varepsilon}}) = \sqrt{\Psi_{\varepsilon}}(\nabla \cdot \Pi) \in L^2 \) (because \( \Psi_{\varepsilon} \) vanishes near the origin). Note that the identity \( \nabla \cdot (\Pi \sqrt{\Psi_{\varepsilon}}) = \sqrt{\Psi_{\varepsilon}}(\nabla \cdot \Psi) \) follows from \( \Pi \nabla \sqrt{\Psi_{\varepsilon}} = 0 \).
5. In the proof of our main result, we shall show that assumptions L.1 and L.2 are automatically satisfied for the limit cross-section, and that the a priori estimate (13.49) automatically holds for the limit distribution function.

13.4 Damping of oscillations via entropy dissipation

Taking into account Chapter 5, we can state

Proposition 13.4.1 (Control of velocity oscillations) Assume that $B_m$ satisfies the overall singularity condition (13.40). Then, there exists a sequence $\alpha(m) \to 0$, such that

\[
\begin{aligned}
\int_0^{\alpha(m)} b_{0,m} (\cos \theta) (1 - \cos \theta) \sin^{n-2} \theta d\theta & \longrightarrow [m \to \infty] \mu > 0, \\
\int_{\frac{\pi}{2}}^\pi b_{0,m} (\cos \theta) \sin^{n-2} \theta d\theta & \equiv \psi(m) \longrightarrow [m \to \infty] + \infty.
\end{aligned}
\]

Let $\chi_R(v)$ be a smooth cutoff function, identically 1 for $|v| \leq R$ and identically 0 for $|v| \geq R + 1$, and $f^m(v)$ be a distribution function depending on the $v$ variable. Let $f^m_R = \chi_R f^m$ be the localized distribution function, and $\mathcal{F}\sqrt{f^m_R}$ the Fourier transform of its square root. Then for all $A > 0$, and $m$ large enough,

\[
\int_{|\xi| \geq A} |\mathcal{F}\sqrt{f^m_R}(\xi)|^2 d\xi \leq C(n, \chi_R, f^m) \frac{1}{\min(\psi(m), \mu A^2)} \left[ D_m(f^m) + \int_{\mathbb{R}^n} f^m(v)(1 + |v|^2) dv \mathbb{R} \right].
\]

Moreover, the constant $C$ depends on $f^m$ only via the density $\int f^m dv$ (which must be neither too large neither too low), the energy $\int f^m |v|^2 dv$ and the entropy $\int f^m \log f^m dv$.

Combining Proposition 13.4.1 with the renormalized formulation, velocity-averaging lemmas [60, 86] and some work, one concludes to the statement in Theorem 13.2.1 that the convergence is automatically strong. The (long) proof is omitted, because it is exactly similar to the one given in Chapter 11 for sequences of solutions to the Boltzmann equation.

We end this section by displaying two basic examples for the sequences $\alpha(m), \psi(m)$ which control the oscillations in Proposition 13.4.1. The more delicate case of the true Debye screening will be examined in Appendix 13.7.

Examples:
1. Consider the case \( \sin^{n-2} \theta b_m(\cos \theta) = \zeta_m(\theta) \), and let \( \zeta_m(\theta) = m^3 \zeta(m \theta) \), where \( \zeta \) has compact support in \([0, \pi/2]\). This framework is equivalent to the one considered in Desvillettes [46]. Then if \( a \) is any point such that \( \int_0^a \zeta \) and \( \int_{\pi/2}^a \zeta \) are positive, we can let \( \alpha(m) = am^{-1} \), and it follows that

\[
\int_0^{\alpha(m)} d\theta \left(1 - \cos \theta \right) \zeta_m(\theta) = \int_0^{a} m^2 \left(1 - \cos \frac{\theta}{m} \right) \zeta(\theta) d\theta \longrightarrow_{m \to \infty} \frac{1}{2} \int_0^{a} \theta^2 \zeta(\theta) d\theta.
\]

On the other hand,

\[
\int_{am^{-1}}^{\pi} \zeta_m(\theta) d\theta = m^2 \int_{a}^{\pi} \zeta(\theta) d\theta \longrightarrow_{m \to \infty} \infty.
\]

2. For the approximate Debye potential in three dimensions of space, eq. (13.38), we write

\[
B_m(|z|, \cos \theta) \geq \left( \frac{|z|}{4 \max(|z|, 1)^4} \right) \left( \frac{1}{\log m} \frac{1}{\sin^4(\theta/2)} \right),
\]

and we choose \( \alpha(m) = m^{-1/2} \), so that on one hand

\[
\frac{1}{\log m} \int_{(\pi/2)m^{-1}}^{m^{-1/2}} \frac{1 - \cos \theta}{\sin^4(\theta/2)} \sin \theta d\theta \sim \frac{8}{\log m} \int_{(\pi/2)m^{-1}}^{m^{-1/2}} \frac{d\theta}{\theta} \longrightarrow_{m \to \infty} 4,
\]

while on the other hand

\[
\frac{1}{\log m} \int_{m^{-1/2}}^{\pi} \frac{\sin \theta d\theta}{\sin^4(\theta/2)} \sim \frac{m}{4 \log m} \longrightarrow_{m \to \infty} \infty.
\]

### 13.5 Proof of the Landau approximation

We begin with two important lemmas.

**Lemma 13.5.1** If \((B_m)\) concentrates on grazing collisions in the sense of Definition 13.2.1 in section 13.2, then

\[
S_m(|z|) \longrightarrow_{m \to \infty} \frac{1}{4} \nabla \cdot (zM_{\infty}(|z|))
\]

in weak-measure sense.

**Remarks.**

1. Formally, \( \nabla \cdot (zM_{\infty}(|z|)) = |z| M'_{\infty}(|z|) + N M_{\infty}(|z|) \), where \( M'_{\infty} \) denotes the (distributional) derivative of \( M_{\infty} \) on the real line.

2. The limit coincides with the negative of the kernel \( c \) in (13.48) if \( M_{\infty}(|z|) = (4n - 1) \Psi(|z|)|z|^2 \).
3. As a particular case, if \( B_m(|v - v_*|, \sigma) = b_m(k \cdot \sigma)/|z|^n \), with an total angular cross-section for momentum transfer \( \mu_m \rightarrow \mu_\infty \), then \( S_m = \lambda_m \delta_0 \), where

\[
\lambda_m \equiv -|S^{n-1}| |S^{n-2}| \int_0^{\pi/2} d\theta \sin^{n-2} \theta b_m(\cos \theta) \log \cos(\theta/2) d\theta \rightarrow_{m \rightarrow \infty} \frac{|S^{n-1}|}{4} \mu_\infty.
\]

**Proof of Lemma 13.5.1**

By assumption, \( S_m \) converges (up to extraction), locally in weak-measure sense, and for all \( \theta_0 \),

\[
\int_{\theta_0}^{\pi/2} d\theta \sin^{n-2} \theta \left[ \frac{1}{\cos^n(\theta/2)} B_m \left( \frac{|z|}{\cos(\theta/2)}, \cos \theta \right) - B_m(|z|, \cos \theta) \right] \rightarrow 0 \text{ weakly}.
\]

Thus we may consider only the contribution of angles \( \theta \leq \theta_0 \), where \( \theta_0 \) is small enough. If \( \varphi(z) \) is a test-function with compact support, by the change of variables \( z \rightarrow z/\cos(\theta/2) \),

\[
|S^{n-2}| R^n dz \int_{\theta_0}^{\pi/2} d\theta \sin^{n-2} \theta B_m(|z|, \cos \theta) [\varphi(z \cos(\theta/2)) - \varphi(z)] dz
\]

For small \( \theta \),

\[
\varphi(z \cos(\theta/2)) - \varphi(z) \approx -\nabla \varphi(z) \cdot z (1 - \cos(\theta/2)),
\]

and therefore (13.54) is approximately

\[
-\frac{|S^{n-2}|}{4} R^n dz \int_{\theta_0}^{\pi/2} d\theta \sin^{n-2} \theta B_m(|z|, \cos \theta) (1 - \cos \theta) z \cdot \nabla \varphi(z) \rightarrow_{m \rightarrow \infty} -\frac{1}{4} \int_{R^n} dz \left[ M_\infty(|z|) \right] \cdot \nabla \varphi(z).
\]

This proves our claim.

In the next lemma, we are interested in the behavior of the \( T_m \)'s, recalling notations from Chapter 11.

**Lemma 13.5.2** Let \( T_m \) be the operator associated with \( B_m \), where \( (B_m) \) is a sequence of cross-sections concentrating on grazing collisions. Then,

\[
T_m \rightarrow T_\infty \quad \text{in distributional sense},
\]

where

\[
T_\infty \varphi = -\frac{1}{2} M_\infty(|v - v_*|)(v - v_*) \cdot \nabla \varphi(v) + \frac{M_\infty(|v - v_*|)|v - v_*|^2}{4(n-1)} \Pi(v - v_*) : D^2 \varphi(v).
\]
In particular, note that the linear operator $T_\infty$ coincides with $T_L$ in formula (13.51) if $M_\infty(|z|) = 4(n-1)\Psi(|z|)|z|^2$.

**Proof of Lemma 13.5.2**

Let us recall from Chapter 11 that

\begin{align*}
T_m \varphi &= - \int_{S^{n-1}} d\sigma B_m(v - v_\star, \sigma)(1 - k \cdot \sigma) \frac{v - v_\star}{2} \cdot \nabla \varphi(v) \\
& \quad + \int_{S^{n-1}} d\sigma B_m(v - v_\star, \sigma)|v - v_\star|^2 \int_0^1 ds \, (1 - s) D^2 \varphi(v + s(v' - v)) \cdot \left( \frac{v' - v}{|v - v_\star|}, \frac{v' - v}{|v - v_\star|} \right) .
\end{align*}

By assumption (iii) of Definition 13.2.1, the contribution of “large” deviation angles in (13.56) is negligible and one can assume that $\theta \leq \theta_0$ where $\theta_0$ is very small. By our assumptions on the decay at infinity, eq. (13.39), one may also neglect the contribution of large velocities, and assume that $v, v_\star$ are bounded.

We change variables to separate $\theta$ from the other coordinates, writing

$$
\sigma = (\cos \theta, \sin \theta \phi), \quad \phi \in S^{n-2},
$$

where the first component is the projection onto $k$. We note that

$$
\frac{v' - v}{|v - v_\star|} = \frac{k - \sigma}{2} .
$$

In the regime when $\theta$ is small, according to the smoothness of $\varphi$, we find that $T_m \varphi$ is equivalent to

\begin{align*}
- M_m(|v - v_\star|) \frac{v - v_\star}{2} \cdot \nabla \varphi(v) &+ \int_0^\frac{\pi}{2} d\theta \sin^{n-2} \theta \, B_m(|v - v_\star|, \cos \theta)|v - v_\star|^2 \\
&\quad \frac{1}{2} \left[ \int_{S^{n-2}} d\phi \, D^2 \varphi(v) \cdot \left( \frac{k - \sigma}{2}, \frac{k - \sigma}{2} \right) \right] .
\end{align*}

Thus, we turn to estimate the behavior of the integral in square brackets.

Let $(\lambda_{ij})_{1 \leq i, j \leq n}$ be the components of the symmetric matrix $D^2 \varphi(v)$, where the first component corresponds to the axis $k$. Separating the first component from the other ones, the components of $k - \sigma$ are $(\cos \theta - 1, \sin \theta \phi)$. Therefore, in the term involving $\lambda_{11}$ there is a factor $(\cos \theta - 1)^2$, of order 4 in $\theta$, and this term disappears in the limit $m \to \infty$. By symmetry with respect to $\phi$, the terms with $\lambda_{jk}, j \neq k$, also disappear. We are only left with the $\lambda_{ii}, i = 2, ..., n$, and they all appear with the same coefficient, which is

$$
\sin^2 \theta \int_{S^{n-2}} (e \cdot \phi)^2 d\phi, \quad |e| = 1.
$$
A classical computation leads to the value
\[
|S^{n-2}| \int_0^\pi \cos^2 \alpha \sin^{n-3} \alpha \, d\alpha = \frac{|S^{n-2}|}{n-1}.
\]
On the whole, using \(\sin^2 \theta \simeq 2(1 - \cos \theta)\),
\[
\int_{S^{n-2}} d\phi \int_0^1 D^2 \varphi(v) \cdot \left( \frac{v' - v}{|v - v_*|} \frac{v' - v}{|v - v_*|} \right) \simeq \frac{|S^{n-2}|(1 - \cos \theta)}{4(n-1)} \Pi(v - v_*): D^2 \varphi(v).
\]
The conclusion follows immediately.

After these preparations, we turn to the proof of our main result.

**Proof of theorem 13.2.1**

Let \((f^m)\) be a sequence of weak solutions of the Boltzmann equation, satisfying the assumptions of Theorem 13.2.1. W.l.o.g we can assume that \(f^m \rightharpoonup f\) weakly, and our goal is to pass to the limit in the renormalized equation satisfied by \(f^m\). By the strong compactness discussed in section 13.4, it is immediate that \((\partial_t + v \nabla_x) \beta(f^m) \rightharpoonup (\partial_t + v \nabla_x) \beta(f)\). Next, combining lemmas 13.5.1 and 13.5.2 above, the estimates recalled in sections ?? and 13.3, the technical assumption (13.39) and the strong compactness again, using similar arguments as in Chapter 11, one can show that
\[
(R_1)^m \rightharpoonup (R_1)^\infty, \ (R_2)^m \rightharpoonup (R_2)^\infty \quad \text{in weak sense},
\]
with obvious notations.

It remains to show that
\[
(13.57) \quad (R_3)^\infty \leq \lim_{m \to \infty} (R_3)^m.
\]
This is considerably more difficult than the similar problem treated in Chapter 11 because we cannot rely on Fatou’s lemma any more. All the rest of this section is devoted to the proof of (13.57).

Without loss of generality, we consider only the case \(\delta = 1\), i.e. \(\beta(f) = f/(1 + f)\), and we recall that
\[
(R_3)^m = \int_{\mathbb{R}^n \times S^{n-1}} dv_* \sigma (f^m)^\prime (1 + f^m)^\prime B_m(v - v_*, \sigma) \left[ \beta(f^m)^\prime - \beta(f^m) \right]^2.
\]
The following argument is somewhat lengthy and we shall sometimes skip easy verifications. In a preliminary step, we establish a crucial a priori bound stemming out (once again) of the entropy dissipation estimate.

**Lemma 13.5.3 (Entropy dissipation bounds)**

\[
\sup_{n \in \mathbb{N}} \int_0^T dt \int_{\mathbb{R}^n \times \mathbb{R}^{2n} \times S^{n-1}} dx dv \, dv_* \, d\sigma \, B_m(v - v_*, \sigma) \left[ f^m + (f^m)^\prime \right] \left[ \beta(f^m)^\prime - \beta(f^m) \right]^2 < +\infty.
\]
Proof of Lemma 13.5.3

Since \((x - y) \log(x/y) \geq 4(\sqrt{x} - \sqrt{y})^2\), we deduce from the entropy dissipation bounds that

\[
\sup_{m \in \mathbb{N}} \int dt \, dx \, dv \, ds \, B_m \left( (f^m)'(f^m)' - \sqrt{f^m} \right)^2 < +\infty.
\]

Choose an increasing function \(P\), to be precised later, such that \(\|P\|_{\text{Lip}} < +\infty\). Then,

\[ (13.58) \quad \sup_{m \in \mathbb{N}} \int dt \, dx \, dv \, ds \, B_m \left( (f^m)'(f^m)' - \sqrt{f^m} \right)^2 \frac{P(f^m)' - P(f^m)}{(f^m)' - f^m} < +\infty. \]

Then we apply the strategy in Villani [116]: we write

\[
\sqrt{f^m} - \sqrt{f^*_m} = \frac{1}{2}(\sqrt{f^m} + \sqrt{f^*_m})(\sqrt{f^m} - \sqrt{f^*_m}) + \frac{1}{2}(\sqrt{f^m} - \sqrt{f^*_m})(\sqrt{f^m} + \sqrt{f^*_m}),
\]

plug this inside (13.58) and expand, to find

\[ (13.59) \quad \sup_{m \in \mathbb{N}} \left\{ \int B_m \left[ \sqrt{(f^m)_*'} + \sqrt{f^*_m} \right]^2 \frac{P(f^m)' - P(f^m)}{(f^m)' - f^m} \right. \]

\[
+ \int B_m \left[ (f^m)_*'' - f^*_m \right] \frac{P(f^m)' - P(f^m)}{(f^m)' - f^m} \left. \right\} < +\infty.
\]

(there is also another nonnegative term, that we throw away) The second integral is also

\[
\int B_m \left[ (f^m)_*'' - f^*_m \right] \left[ (f^m)' - f^m \right] \frac{P(f^m)' - P(f^m)}{(f^m)' - f^m} = \int B_m P(f^m)' \frac{(f^m)_*'' - f^*_m}{(f^m)_*'} \]

\[
= \int dt \, dx \, dv \, ds \, S_m(|v - v_s|) P(f^m)' f^m_*',
\]

where we have used the Cancellation lemma 13.37.

It is a priori bounded as soon as \(P \in L^\infty\), since in that case \((S_m * P)(v_s) \leq C(1 + |v_s|^2)\).

Choosing \(P(f) = \beta(f) = f/(1 + f)\), from the preceding remark and (13.59) we get

\[
\sup_{m} \int B_m \left[ \sqrt{(f^m)_*'} + \sqrt{f^*_m} \right]^2 \frac{\beta(f^m)'}{\beta(f^m)} \frac{2}{\sqrt{f^m} + \sqrt{(f^m)_*'}} < +\infty,
\]

and we conclude by noting that \((1 + f)(1 + f')/\sqrt{f + \sqrt{f}})^2 \geq 1/2).

Remark 13.5.1 This lemma implies the estimate

\[ \int_0^T dt \int_{\mathbb{R}^n \times \mathbb{R}^{2n}} dx \, dv \, ds \, \beta \nabla \beta(f) \nabla \beta(f) < +\infty. \]

To see this, write \(B_m(g' - g)^2 = B_m |v - v_s|^2 \sin^2(\theta/2)(g' - g)^2/|v' - v|^2\) and pass to the limit as \(m \to \infty\) in the same way as below. Thus the condition \(\beta \nabla \beta(f) \nabla \beta(f) \in L^1_{\text{loc}}\) is satisfied (and in fact, this function belongs to \(L^1\) globally!).

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We now come back to the main argument.

**Proof of (13.57)**

In view of our a priori bounds, we only need to show that for all smooth test-function \( \varphi(t,x,v) \) with compact support,

\[
(13.60) \quad \int (R_3)^{\infty} \varphi^2 \leq \lim_{m \to \infty} \int (R_3)^m \varphi^2.
\]

**Step 1.** By the usual pre-postcollisional change of variables \(((v,v_s,\sigma) \to (v',v'_s,k)) \) in our notations, we rewrite \( \int (R_3)^m \varphi^2 \) as

\[
\int (R_3)^m \varphi^2 = \int f_*^m (1 + f^m) B_m([v - v_s], \cos \theta) \left[ \beta(f^m)' - \beta(f^m) \right]^2 (\varphi')^2.
\]

**Step 2.** By monotonicity, we only need to prove the result when \( B_m \) is replaced by \( \chi_\varepsilon([v - v_s], \cos \theta) \), where \( \chi_\varepsilon(\cdot) \) is a smooth cutoff function, identically vanishing for \( |z| \leq \varepsilon \), \( |z| \geq \varepsilon^{-1} \).

**Step 3.** Then we introduce

\[
\delta_m([v - v_s], \theta) = \frac{|S_n - 2| \sin^{-2} \theta B_m([v - v_s], \cos \theta) |v - v_s|^2 (1 - \cos \theta)}{M_m([v - v_s]) |v - v_s|^2},
\]

whose link with the quantity (13.42) introduced in the Technical Condition is obvious. By construction, \( \int_0^{\frac{\pi}{2}} \delta_m([v - v_s], \theta) d\theta = 1 \), and by Jensen’s inequality \( \int (R_3)^m \varphi^2 \) is greater than

\[
\frac{1}{2} \int f_*^m (1 + f^m) M_m([v - v_s]) |v - v_s|^2 \left[ \int_0^{\frac{\pi}{2}} \delta_m(v - v_s, \theta) \varphi' \frac{\beta(f^m)' - \beta(f^m)}{|v' - v|} d\theta \right]^2
\]

\[
= \frac{1}{2} \int_0^T dt \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^2} dv \int_{S_n - 2} d\phi (a_m)^2,
\]

where

\[
(a_m) = \sqrt{f_*^m} \sqrt{1 + f^m} \sqrt{M_m([v - v_s]) |v - v_s|^2}
\]

\[
\int_0^{\frac{\pi}{2}} d\theta \sin^{-2} \theta \frac{|S_n - 2| B_m([v - v_s], \cos \theta) |v - v_s|^2 (1 - \cos \theta)}{\sqrt{M_m([v - v_s]) |v - v_s|^2}} \varphi' \left[ \frac{\beta(f^m)' - \beta(f^m)}{|v' - v|} \right].
\]

There is a factor \( 1/2 \) since \( \frac{1}{|v - v'_s|^2} = \frac{1}{2} \left( \frac{1 - \cos \theta}{|v - v_s|^2} \right) \), and we have once again used a spherical system of coordinates \((\theta, \phi)\) for \( \sigma \).

Due to the convexity of the square function, it would be enough, in order to conclude, to show that \((a_m) \to (a)_\infty\) in distributional sense, where

\[
(a)_\infty = \sqrt{f_\infty} \sqrt{1 + f} \sqrt{M_\infty([v - v_s]) |v - v_s|^2} \varphi \nabla \beta(f) \cdot e_\phi.
\]
Here $e_\phi$ is a unit vector orthogonal to $k = (v - v_*)/|v - v_*|$, with coordinate $\phi$ in a system of spherical coordinates around $k$. The convergence should hold in $D^0([0, T] \times \mathbb{R}_x^n \times \mathbb{R}_v^n \times \mathbb{R}_v^n \times S^{n-2})$ (as in the other weak limits considered below). Then the proof of (13.60) would follow by Jensen’s inequality and the formula

$$\forall \ell \in \mathbb{R}^n, \quad \int_{S^{n-2}} (\ell \cdot e_\phi)^2 \, d\phi = \left| \frac{S^{n-2}}{n-1} \right| |\Pi\ell|^2.$$ 

We immediately note a slight subtlety: it is not always possible to define $e_\phi$ in a smooth way on the sphere $S^{N-1}$ (think that there is no smooth field of unit tangent vectors on spheres of even dimension). However, since we wish to prove a local property (that is, $S$ on the sphere $S$), recall that by assumption $s$ is a smooth (Lipschitz) bounded approximation of the square root,

$$\tilde{s}(x) \quad (13.61) \quad \text{and the formula}$$

This will entail that $\tilde{s}(x) \leq \sqrt{x}$. Denoting by $\tilde{(a)}_m$ this modification of $(a)_m$, we shall prove that

$$\tilde{(a)}_m \longrightarrow_{m \to \infty} (a)_\infty \quad \text{in distribution sense.}$$

This will entail that

$$\int (\tilde{R}_3)^\infty \varphi^2 \leq \lim_{m \to \infty} \int (R_3)^m \varphi^2,$$

where $(\tilde{R}_3)^\infty$ is just as $(R_3)^\infty$, but with $f_*$ replaced by $s(f_*)^2$. Once (13.62) is established, if we let the function $s$ approximate the square root function in a monotone way, Beppo Levi’s monotone convergence theorem will imply the same inequality with $(R_3)^\infty$ in place of $(\tilde{R}_3)^\infty$.

**Step 4.** Next, we already know that $\sqrt{1 + f^m R} \to \sqrt{1 + f}$, strongly in $L^2_{\text{loc}}$, and thus, to prove (13.61) it is sufficient to prove the convergence in weak $L^2$ of the distribution

$$\int_0^\pi \eta_m(|v - v_*|, \theta) \left[ \varphi' \frac{\beta'(f^m)' - \beta(f^m)}{|v' - v|} \right] d\theta.$$ 

Recall that by assumption

$$\eta_m|v - v_*| \longrightarrow \delta_{\theta = 0} \otimes \sqrt{M_\infty(|v - v_*|)|v - v_*|^2}$$

in weak measure-sense (remember that we have cut out small values of $|v - v_*|$ in Step 2). But

$$\sup_m \| (b)_m \|_{L^2} \leq C \sup_m \| (a)_m \|_{L^2} < +\infty,$$

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and therefore we only have to check that

\[(b)_m \rightarrow_{m \rightarrow \infty} (b)_\infty \] in distributional sense.

**Step 5.** At this point, we want to partly symmetrize the integrand in (13.63). Therefore, we first write \((b)_m\) as

\[s(f^*_m) \int \eta_m \left( \frac{f^*_m}{2} \right) \beta(f^*_m) d\theta + s(f^*_m) \int \eta_m \left( \frac{f^*_m}{2} \right) \beta(f^*_m) d\theta.\]

If \(\varphi\) is smooth, by Taylor formula \(|\varphi' - \varphi| \leq C|v' - v|\), so that the first integral in (13.65) is bounded in \(L^2\). We want to check that it actually goes weakly to 0. Since functions of the form \(P(f^m)\), where \(P\) is smooth and bounded, converge towards \(P(f)\) in all \(L^p_{loc}\) spaces, \(1 \leq p < \infty\), this will be a consequence of the

**Lemma 13.5.4** If \(\chi(v, w)\) is a smooth function and \(F^m\) converges towards \(F\) in \(L^1\), then

\[\int_0^{\pi/2} \int_0^d \eta_m(|v - v_*|, \theta) \chi(v, v') F^m(v') \rightarrow \sqrt{M_\infty(|v - v_*|)} \chi(v, v) F(v)\]

in weak sense.

**Proof of Lemma 13.5.4**

Note that for given \(\phi, v'\) is only a function of \(v\) and \(\theta\). For each \(\theta, \phi, v_*\), the change of variables \(v \rightarrow v'\) is allowed, and its Jacobian goes to 1 as \(\theta\) goes to 0 (see [14]). Thus we pick a test function \(\zeta(v)\) and we write (for each \(\phi, v_*\)) as in [14]

\[\int dv \int_0^{\pi/2} d\theta \eta_m(|v - v_*|, \theta) \chi(v, v') F^m(v') \zeta(v) = \int_0^{\pi/2} d\theta \int dv' \left| \frac{dv}{dv'} \right| \eta_m(|\psi_\sigma(v') - v_*|, \theta) \chi(\psi_\sigma(v'), v') F^m(v') \zeta(\psi_\sigma(v')).\]

Since we work on a compact domain, \(|v' - v|\) goes to 0 as \(\theta \rightarrow 0\). We can pick another test-function \(\kappa\) in the variable \(v_*\), and perform the change of variables \(z = v_* - \psi_\sigma(v')\), Then our claim follows from the strong convergence of \(F^m\) and the (strong) convergence, pointwise in \(v'\), of

\[\int d\theta dz \eta_m(|z|, \theta) \chi(\psi_\sigma(v'), v') \zeta(\psi_\sigma(v')) \kappa(z + \psi_\sigma(v')) \frac{dv}{dv'} \]

towards

\[\int \sqrt{M_\infty(|z|)} |z|^{3/2} \chi(v', v') \zeta(v') \kappa(z + v') dz.\]
Remark 13.5.2 A similar proof shows that if $F^m$ and $G^m$ converge strongly towards $F$ and $G$, respectively, then

$$\int_0^\pi d\theta \eta_m(|v-v_*|, \theta) \chi(v, v') F^m(v') G_m(v'_\star) \longrightarrow M_\infty(|v-v_*|) \chi(v, v) F(v) G(v_*),$$

in weak sense. We shall use this later on.

We come back to the proof of (13.64). We continue to symmetrize: we now wish to prove that

$$\int d\theta \eta_m(|v-v_*|, \theta)[s(f^m)_* - s(f^m)'_\star] \left( \frac{\varphi + \varphi'}{2} \right) \frac{\beta(f^m)' - \beta(f^m)}{|v' - v|} \longrightarrow 0$$

in weak sense.

By another application of Lemma 13.5.4 (or rather the variant given as a remark after the proof), and using smooth cutoff functions in the variable $v' - v$, it is sufficient to control the contribution of small values of $|v' - v|$. But, if $s$ is chosen in such a way that $|s(x) - s(y)| \leq C|\beta(x) - \beta(y)|$, using the fact that $|v'_\star - v_*| = |v' - v|$, we bound the integral of the contribution of ($|v' - v| \leq \delta$) by

$$C\delta \int \eta_m(|v-v_*|, \theta) \left| \frac{\beta(f^m)_* - \beta(f^m)'_\star}{|v'_\star - v_*|} \right| \left| \frac{\beta(f^m) - \beta(f^m)'}{|v' - v|} \right|$$

$$\leq C\delta \int \eta_m(|v-v_*|, \theta) \left( \frac{\beta(f^m)_* - \beta(f^m)'_\star}{|v'_\star - v_*|} \right)^2 d\sigma \, dv \, dv_* \, dx \, dt.$$

Using Lemma 13.5.3 again, this expression is bounded like $O(\delta)$ and hence negligible.

As a conclusion of Step 5, we have shown that (13.64) holds if

$$(c)_m \longrightarrow_{m \to \infty} (c)_{\infty}$$

in distributional sense, where $(c)_m$ is the symmetric expression

$$\int d\theta \eta_m(|v-v_*|, \theta) \frac{s(f^m)_* + s(f^m)'_\star}{2} \left( \frac{\varphi + \varphi'}{2} \right) \frac{\beta(f^m)' - \beta(f^m)}{|v' - v|}.$$

**Step 6.** We can now use a duality argument to prove (13.66): multiplying $(c)_m$ by a test-function $\psi(v, v_*)$ (and also a test-function in $t, x, \phi$, that we omit), applying the pre-postcollisional change of variables, the formula

$$\psi(v, v_* - \psi(v', v'_\star) \approx |v - v'|(|\nabla - \nabla_*|) \psi(v, v_\star) \cdot e_\phi,$$

we only have to prove convergence in weak formulation, i.e. that

$$\int dx \, d\phi \, dv \, dv_* \, d\theta \frac{s(f^m)_* + s(f^m)'_\star}{2} \eta_m(|v-v_*|, \theta) \left( \frac{\varphi + \varphi'}{2} \right) \frac{\beta(f^m) + \beta(f^m)'}{|v' - v|} \psi' - \psi$$

converges towards

$$\int s(f^\star_\star) \varphi \beta(f)(|\nabla - \nabla_*|) \psi(v, v_\star) \cdot e_\phi.$$

This follows by a last application of Lemma 13.5.4.
13.6 Appendix 1: an approximate Yukawa cross-section

The Yukawa potential is similar to the Debye potential, except that it is usually considered in a quantum context (for instance, scattering of electrons or ions by neutrons).

Up to a dimensional multiplicative factor, the Yukawa potential is given by

\[ V(r) = \frac{e^{-r/\lambda}}{r}, \]

where \( \lambda \) is the screening length, essentially given by

\[ \lambda = \frac{\hbar^2}{me^2}. \]

When the relative velocity \( z \) satisfies the Born approximation, i.e

\[ |z| >> \frac{e^2}{\hbar}, \]

then the scattering cross-section can be approximated by

\[ B(|z|, \cos \theta) \sin \theta = \frac{1}{4} \left( \frac{e^2}{m} \right)^2 \frac{|z| \sin \theta}{\left( |z|^2 \sin^2 \frac{\theta}{2} + \left( \frac{\hbar}{2m\lambda} \right)^2 \right)^2}. \]

This expression does not take into account exchange terms that arise due to the Pauli exclusion principle. At the level of the Rutherford cross-section, these corrective terms can be computed explicitly, see for instance [45, eq. (3.83)], and it is clear that they are negligible for small deviation angles. We shall admit that the same holds true here.

Note that

\[ \left( \frac{\hbar}{2m\lambda} \right)^2 = \frac{e^2}{4m\lambda}. \]

Turning to adimensional units, denote this parameter by \( 1/m \), so that, up to a multiplicative factor, \( m \) coincides with the screening length. Beware that \( m \) is not the mass! Then we may rewrite the approximate cross-section as

\[ B(|z|, \cos \theta) \sin \theta = \frac{|z| \sin \theta}{\left( |z|^2 \sin^2 \frac{\theta}{2} + \left( \frac{1}{m} \right)^2 \right)^2}. \]

Now, when we consider the Landau approximation, the effect of the change of scales will be to renormalize this cross-section by a factor proportional to \( \log n \). This leads us to our final expression

(13.67)

\[ B_{m}(|z|, \cos \theta) \sin \theta = \frac{1}{\log m} \frac{|z| \sin \theta}{\left( |z|^2 \sin^2 \frac{\theta}{2} + \left( \frac{1}{m} \right)^2 \right)^2}. \]
We claim that this sequence of cross-sections concentrates on grazing collisions, in the sense of Definition 13.2.1, and satisfies the technical assumptions of section 13.2 as well. Let us sketch the proof of this claim.

First of all, $B_m$ is admissible (for fixed $m$), because it is just a nonsingular cross-section. Next, let

$$M_m(|z|) = \frac{|S^1|}{\log m} \int_0^\frac{\pi}{2} \frac{|z| \sin \theta (1 - \cos \theta) d\theta}{(|z|^2 \sin^2 \frac{\theta}{2} + \frac{1}{m})^2}.$$  

For any $\theta_0 > 0$,

$$M_{m_0}^\theta(|z|) \leq \frac{C(\theta_0)}{|z|^3 \log m},$$

and we see that $M_{m_0}^\theta \leq \lambda_m \delta_0$, with $\lambda_m \to 0$ as $m \to \infty$. Hence the contribution of large angles in (13.68) is asymptotically negligible, and we can assume that $\theta \leq \theta_0$ where $\theta_0$ is very small. Then, locally in $z$, we can replace the integrand by its equivalent, and $M_m$ by

$$= \frac{|S^1|}{|z|^3} \left[ 1 + \frac{1}{\log m} \log \left( \frac{|z|^2}{2 \theta_0^2 + \frac{1}{m}} \right) + O \left( \frac{1}{\log m} \right) \right].$$

As a consequence, $z M_m(|z|)$ converges weakly (locally in measure) towards $z M_\infty(|z|)$, with

$$M_\infty(|z|) = \frac{|S^1|}{|z|^3}. $$

Further note that, since $(2/\pi)\theta \leq \sin \theta \leq \theta$ and $1 - \cos \theta \leq \theta^2/2$ for $\theta \in [0, \pi/2]$, we also have

$$|M_m(|z|)| \leq C|z| \int_0^\frac{\pi}{2} \frac{\theta^3 d\theta}{(|z|^2 \theta^2 + \frac{1}{m})^2} \to |z| \to \infty 0,$$

uniformly in $m$, by the same estimate as above. Thus assumption 13.39 holds.

Next, we compute (see (13.36))

$$s_m(|z|, \theta) = \sin \theta \left[ \frac{1}{\cos^3(\theta/2)} B_m \left( \frac{|z|}{\cos(\theta/2)}, \cos \theta \right) - B_m(|z|, \cos \theta) \right],$$

and investigate the behavior of the $\theta$-integral of this expression, which gives the kernel $S(|z|)$. Again, one can check that the contribution of large deviation angles is negligible in the limit $m \to \infty$, and we concentrate on small values of $\theta$. For small $\theta$ we find that (13.69) is equivalent to

$$\frac{1}{\log m} \frac{4}{|z|^3 \theta} \left( 1 + \frac{1}{|z|^2 \theta^2 m/4} \right)^3.$$
By a homogeneous change of variables, 
\[ \frac{1}{\log m} \int_{|z| \leq A} dz \int_0^\theta \frac{d\theta}{|z|^3} \left( \frac{1}{1 + |z|^2(m/4)} \right)^3 = \frac{1}{\log m} \int_0^{\sqrt{\pi}} \frac{d\theta}{\theta \left( 1 + \frac{1}{|z|^2} \right)^3} \left( \int_{|z| \leq A\theta} \frac{1}{|z|^3} \right). \]

Since 
\[ \int_{|z| \leq A\theta} \frac{dz}{|z|^3} \left( \frac{1}{1 + |z|^2} \right)^3 \leq C \max(\theta^4, 1), \]
we see that \( \int_{|z| \leq A} S_m(|z|) \, dz = \int_{|z| \leq A} d\theta \, s_m(|z|, \theta) \) is bounded uniformly in \( m \) (this is the remaining part of Assumption (i) in Definition 13.2.1).

Next, for any \( \theta_0 > 0 \), 
\[ s_m(|z|, \theta) \chi_{\theta \geq \theta_0} \leq \frac{C}{m \log m} \frac{|z|}{(|z|^2 + \frac{1}{m})^3} \, \chi_{\theta \geq \theta_0}, \]
which easily leads to
\[ \int_{|z| \leq A} \frac{dz}{\theta} \int_{\theta_0}^\theta s_m(|z|, \theta) \, d\theta \leq \frac{C(\theta_0)}{m \log m} \int_{\mathbb{R}^3} \frac{|z| \, dz}{(|z|^2 + 1)^3} \xrightarrow{m \to \infty} 0. \]

Thus \( S_m^{\theta_0} \) converges towards 0 in \( L^1_{\text{loc}}(\mathbb{R}^3) \).

As for the technical conditions (13.40) and (13.43) : the first one was established in section 13.4, and the second one is a consequence of the uniform smoothness of \( B_m \) in the \( z \) variable away from the origin.

**Remarks 13.6.1**

1. We have not considered the part of the kernel \( B \) which comes from the symmetrization for deviation angles larger than \( \pi/2 \). This part has no influence on the estimates.

2. If we denote by \( a = 1/(4|z|^2) \), we have, again by a homogeneous change of variable,
\[ \frac{1}{\log m} \int_0^{\sqrt{\pi}} \frac{d\theta}{\theta \left( 1 + \frac{a}{m\theta} \right)^3} = \frac{1}{\log m} \int_0^{\sqrt{\pi}m} \frac{d\theta}{\theta \left( 1 + \frac{a}{\theta} \right)^3} \simeq \frac{Ca}{m \log m}. \]
This expression goes to 0 for each \( z \), and in fact uniformly for \( |z| \geq \varepsilon \), which shows that \( S_m \) will converge weakly to a Dirac measure at the origin. This was expected in view of lemma 13.5.2.

### 13.7 True Debye potential

The Debye potential is given by (in adimensional units)
\[ \phi_\Lambda(r) = \frac{e^{-r/\Lambda}}{r}, \quad \Lambda = \frac{2\lambda_D}{r_0}. \]
and the rescaled cross-section by \( \tilde{B}_\Lambda = \frac{1}{\log \Lambda} B_\Lambda \), where \( B_\Lambda \) is the cross-section associated with \( \phi_\Lambda \), see the standard books on kinetic theory:

\[
\theta = \pi - 2b \int_{s_0}^{+\infty} \frac{ds/s^2}{\sqrt{1 - b^2/s^2 - 4\phi_\Lambda(s)/|z|^2}}.
\]

Above \( b \) is the impact parameter, and \( s_0 \) is (the) solution of

\[
1 - b^2/s_0^2 - 4\phi_\Lambda(s_0)/|z|^2 = 0
\]

Then \( B_\Lambda \) is defined by

\[
B_\Lambda(|z|, \cos \theta) = \frac{b}{\sin \theta} \int_0^{\pi/2} \tilde{B}_\Lambda(|z|, \cos \theta)(1 - \cos \theta) \sin \theta d\theta.
\]

The point is to show the following four assertions:

1) Let

\[ M_\Lambda(|z|) = |S^1| \int_0^{\pi/2} \tilde{B}_\Lambda(|z|, \cos \theta)(1 - \cos \theta) \sin \theta d\theta \]

Then

1) \( M_\Lambda(|z|) = o(1) \) for \( |z| \to \infty \), uniformly in \( \Lambda \)

2) \( |z| |M_\Lambda(|z|)\) is locally bounded in measure, uniformly in \( \Lambda \)

3) \( zM_\Lambda(|z|) \to M_{loc.} K \frac{|z|}{|z|^2} \), for a constant \( K > 0 \) to compute.

2) Let

\[ S_\Lambda(|z|) = |S^1| \int_0^{\pi/2} \frac{1}{\cos^3(\theta/2)} \tilde{B}_\Lambda\left(\frac{|z|}{\cos(\theta/2)}, \cos \theta\right) - \tilde{B}_\Lambda(|z|, \cos \theta) \right) \sin \theta d\theta \]

Then \( S_\Lambda \) is bounded locally in measure, uniformly in \( \Lambda \).

3) Let, for \( \theta_0 > 0 \) fixed

a) \( M_{\theta_0}^\Lambda(|z|) = |S^1| \int_0^{\pi/2} \tilde{B}_\Lambda(|z|, \cos \theta)(1 - \cos \theta) \sin \theta d\theta \)

and

b) \( S_{\theta_0}^\Lambda(|z|) = |S^1| \int_{\theta_0}^{\pi/2} \frac{1}{\cos^3(\theta/2)} \tilde{B}_\Lambda\left(\frac{|z|}{\cos(\theta/2)}, \cos \theta\right) - \tilde{B}_\Lambda(|z|, \cos \theta) \right) \sin \theta d\theta \)

Then both go to 0 locally in measure when \( \Lambda \to \infty \).

4) There exists \( \Phi_0 \) continuous, \( \Phi_0(|z|) > 0 \) if \( |z| \neq 0 \) and \( b_\Lambda(\cos \theta) \) such that
a) \( \lim_{\Lambda \to \infty} \int_0^\infty b_\Lambda(\cos \theta) \sin \theta d\theta = +\infty. \)

b) \( \int_0^\infty b_\Lambda(\cos \theta)(1 - \cos \theta) \sin \theta d\theta \to_{\Lambda \to \infty} \mu > 0 \)
c) \( \forall \theta_0 > 0, \sup_{\theta \geq \theta_0} b_\Lambda(\cos \theta) \to 0. \)

We first prove point 1), and start from

\[
\theta = \pi - 2b \int_{s_0}^\infty \frac{dr}{r^2 \sqrt{1 - b^2/r^2 - 4\phi_\Lambda(r)/|z|^2}}
\]

where \( s_0 \) is the biggest root (in \( t \)) of

\[
1 - b^2/t^2 - 4\phi_\Lambda(t)/|z|^2 = 0
\]

We perform the change of variables \( x = b/r \), ie \( r = b/x \) and we get

\[
\theta = \pi - 2 \int_0^{s_0} \frac{dx}{\sqrt{1 - x^2 - 4\phi_\Lambda(b/x)/|z|^2}}
\]

If we perform the change \( y = s_0 x \), then we get also

\[
\theta = \pi - 2 \int_0^1 \frac{x_0 dy}{\sqrt{1 - x_0^2 y^2 - 4\phi_\Lambda(b/(x_0 y))/|z|^2}}
\]

where \( x_0 = b/s_0 \) satisfies

\[
1 - x_0^2 - 4\phi_\Lambda(b/x_0)/|z|^2 = 0
\]

Note that if \( \theta \to \pi \) then \( x_0 \to 0. \)

For all \( r \in ]0, +\infty[, \) letting \( \psi(r) = 1 - b^2/r^2 - \frac{4}{|z|^2} e^{-\frac{2}{r}} \), it follows that for all \( |z| \neq 0 \) and \( b \neq 0 \)

\[
\psi'(r) = 2b^2 r^{-3} + \frac{4}{|z|^2} r^{-2} e^{-\frac{2}{r}} + \frac{8}{|z|^2} \frac{1 - e^{-\frac{2}{r}}}{r}
\]

Therefore, \( \psi \) is strictly increasing , and as \( \lim_{r \to 0} \psi(r) = -\infty \) and \( \lim_{r \to +\infty} \psi(r) = 1 \), there exists an unique \( s_0 > 0 \) such that

\[
1 - \frac{b^2}{s_0^2} - \frac{4}{|z|^2} e^{-\frac{2}{s_0}} = 0
\]

Note that since \( \psi(b) < 0 \), then \( s_0 > b \) and \( x_0 = \frac{b}{s_0} < 1. \) We improve this as follows. Using the definition of \( x_0 \), it follows easily that

\[
x_0 \geq \frac{1}{1 + \frac{4}{|z|^2}}
\]

One has also

\[
1 - x_0^2 \leq \frac{4}{|z|^2} \frac{1}{b} x_0
\]

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and as $\Delta = \frac{16}{|z|^2 b^2} + 4$, one has
\[
x_0 \geq -\frac{4}{|z|^2 b^2} + \sqrt{\frac{16}{|z|^2 b^2} + 4}
\]

Now, let us introduce
\[
A \equiv b \int_{s_0}^{\infty} \frac{dr}{r^2 \sqrt{1 - \frac{r^2}{s_0^2} - \frac{4}{|z|^2 \phi_A(r)}}}
\]

Thus $\theta = \pi - 2A$ and
\[
1 - \cos \theta = 2 \sin^2 \frac{\theta}{2} = 2 \sin^2 \left( \frac{\pi}{2} - A \right) = 2 \cos^2 A
\]

If $\theta \to 0$ then $A \to \frac{\pi}{2}$, $b \to +\infty$. Next (if one goes up to $\theta = 0$)
\[
M_A(|z|) = |S^1| \left( \int_0^{\frac{\pi}{2}} B_A(|z|, \cos \theta) (1 - \cos \theta) \sin \theta d\theta \right) = \frac{|S^1||z|}{\log \Lambda} \int_0^{+\infty} (1 - \cos \theta(b)) bdb = 2 \frac{|S^1||z|}{\log \Lambda} \int_0^{+\infty} \cos^2 A bdb
\]
and also
\[
M_A(|z|) = \frac{|S^1||z|}{\log \Lambda} \int_0^{+\infty} \sin^2 \left( \frac{\pi}{2} - A \right) bdb
\]

Let us provide some estimates on $A$, and we first begin with a lower bound. From
\[
A \equiv b \int_{s_0}^{\infty} \frac{dr}{r^2 \sqrt{1 - \frac{r^2}{s_0^2} - \frac{4}{|z|^2 \phi_A(r)}}} = b \int_{s_0}^{\infty} \frac{dr}{r^2 \sqrt{\psi(r)}}
\]
since for all $r \geq s_0$, $0 \leq \psi(r) \leq 1$, one deduces that $A \geq b \int_{s_0}^{\infty} \frac{dr}{r^2}$, $(s_0 > 0)$, and thus $A \geq x_0$. ($x_0 \rightarrow 1$ when $b \rightarrow \infty$)

Next, for all $r \geq s_0$ we write
\[
\psi(r) = \psi(r) - \psi(s_0) = \int_{s_0}^{r} \psi'(t) dt
\]

Since
\[
\psi'(t) = \frac{2b^2}{t^3} + \frac{4}{|z|^2 t^2} e^{-\frac{2}{|z|^2}} + \frac{8}{|z|^2 A} \frac{1}{t^2} e^{-\frac{2}{|z|^2}} \geq \frac{2b^2}{t^3}
\]
one deduces that
\[ \psi(r) \geq \frac{b^2[r^2 - s_0^2]}{r^2s_0^2} \]

Finally, one get the upper bound \( A \leq \frac{\pi}{2} \).

One can improve the above lower bound. Indeed, from the explicit expression for \( A \), one gets

\[ A \geq b \int_{s_0}^{\infty} \frac{dr}{r^2 \sqrt{1 - \frac{b^2}{r^2}}} \]

Making the change of variables \( u = b/r \), it follows \( A \geq \arcsin(x_0) \). In conclusion

\[ \arcsin x_0 \leq A \leq \frac{\pi}{2} \]

In particular, when \( |z|^2 b \to +\infty \), we have \( A \simeq \arcsin x_0 \), uniformly in \( \Lambda \). Finally

\[ 0 \leq \sin^2\left(\frac{\pi}{2} - A\right) = \cos^2 A \leq 1 - x_0^2 \]

and

\[ 0 \leq \sin^2\left(\frac{\pi}{2} - A\right) \leq \frac{4}{|z|^2 b} e^{-\frac{2b}{|z|^2 b}} \]

This estimate is not sufficient in order to get the behaviour of \( M_{\Lambda}(|z|) \) when \( |z| \to \infty \).

We need in fact to improve the estimates for \( A \). For this purpose, since

\[ A = b \int_{s_0}^{\infty} \frac{dr}{r^2 \sqrt{1 - \frac{b^2}{r^2}}} - \frac{4}{|z|^2 b} \frac{1}{r^2} e^{-\frac{2b}{|z|^2 b}} \]

setting \( x = b/r \), it follows that

\[ A = \int_{x_0}^{1} \frac{dx}{\sqrt{1 - x^2}} - \frac{4}{|z|^2 b} x e^{-\frac{2b}{|z|^2 b}} \]

If we set \( x = x_0 u \), we get also

\[ A = x_0 \int_{0}^{1} \frac{du}{\sqrt{1 - x_0^2 u^2}} - \frac{4x_0}{|z|^2 b} \frac{1}{u} e^{-\frac{2b}{|z|^2 b}} \]

where

\[ B \equiv 1 - u^2 + \frac{4x_0}{|z|^2 b} e^{-\frac{2b}{|z|^2 b}} u^2 - \frac{4x_0}{|z|^2 b} e^{-\frac{2b}{|z|^2 b}} u \]

Since \( 0 \leq u \leq 1 \), one has \( u \geq u^2 \), \( -u \leq -u^2 \) and thus
\[ B \leq 1 - u^2 + \frac{4x_0}{|z|^2} u^2 \left( e^{-\frac{2b}{\Lambda x_0}} - e^{-\frac{2b}{\Lambda x_0} \frac{1}{b}} \right) \]

Note the term in brackets is positive. For \( t = 1/u \geq 1 \), let \( \phi(t) = e^{-\frac{2b}{\Lambda x_0} t} \). Then \( \phi'(t) = -\frac{2b}{\Lambda x_0} e^{-\frac{2b}{\Lambda x_0} t} \), and thus \( \sup_{t \geq 1} |\phi'(t)| = \frac{2b}{\Lambda x_0} e^{-\frac{2b}{\Lambda x_0}}. \) We find that

\[ B \leq 1 - u^2 + \frac{8}{|z|^2} \Lambda e^{-\frac{2b}{\Lambda x_0} u(1 - u)} \]

and thus, we have

\[ A \geq x_0 \int_0^1 \frac{1}{\sqrt{1 - u^2 + \frac{8}{|z|^2} \Lambda e^{-\frac{2b}{\Lambda x_0} u(1 - u)}}} \]

\[ \geq x_0 \int_0^1 \frac{1}{\sqrt{1-u^2}} \left[ \frac{8}{|z|^2} \Lambda e^{-\frac{2b}{\Lambda x_0}} \frac{u}{1+u} \right]^{\frac{1}{2}} \]

Using a Taylor formula for \((1 + t)^{-\frac{1}{2}}\) around 0 at order 2, we find that

\[ A \geq x_0 \frac{\pi}{2} - C \frac{x_0}{|z|^2} e^{-\frac{2b}{\Lambda x_0}} \]

Since \( \sin^2 \left( \frac{\pi}{2} - A \right) \leq \left( \frac{\pi}{2} - A \right)^2 \), we deduce

\[ \sin^2 \left( \frac{\pi}{2} - A \right) \leq \frac{\pi}{2} (1 - x_0) + \frac{C x_0}{|z|^2} e^{-\frac{2b}{\Lambda x_0}} \]

Thus

\[ \sin^2 \left( \frac{\pi}{2} - A \right) \leq C_1 (1 - x_0)^2 + \frac{C_2 x_0^2}{|z|^4} e^{-\frac{4b}{\Lambda x_0}} \]

On the other hand, one checks that

\[ (1 - x_0)^2 \leq \frac{C}{|z|^4} e^{-\frac{4b}{\Lambda x_0}} \]

Therefore

\[ \sin^2 \left( \frac{\pi}{2} - A \right) \leq C_1 \frac{1}{|z|^4} e^{-\frac{4b}{\Lambda x_0}} + C_2 \frac{1}{|z|^4} e^{-\frac{4b}{\Lambda x_0}} \]

We can get back to

\[ M_\Lambda (\|z\|) = \frac{\|S_1\| \|z\|}{\log \Lambda} \int_0^\infty \sin^2 \left( \frac{\pi}{2} - A \right) bdb = I + II \]
where

\[ I = \frac{|S_1||z|}{\log \Lambda} \int_0^1 \sin^2(\frac{\pi}{2} - A)db \quad \text{and} \quad II = \frac{|S_1||z|}{\log \Lambda} \int_1^\infty \sin^2(\frac{\pi}{2} - A)db \]

Since

\[ I \leq \frac{|S_1||z|}{\log \Lambda} \int_0^1 \frac{4}{|z|^2} e^{-\frac{4x}{\pi}} db \leq \frac{4}{|z| \log \Lambda} \left\{ 1 - e^{-\frac{2x}{\pi}} \right\} \]

It follows that \( I = o(1) \), uniformly wrt \( \Lambda \). (In fact one has \( I = o(1/\log \Lambda) \)).

Next, for \( II \), we find that

\[ II \leq II' + II'' \]

where

\[ II' = \frac{C_1}{|z|^3 \log \Lambda} \int_1^\infty \frac{1}{b} e^{-\frac{4b}{\pi}} db \quad \text{and} \quad II'' = \frac{C_2}{|z|^3 \log \Lambda^2} \int_1^\infty be^{-\frac{4b}{\pi}} \]

By setting \( x = b/\Lambda \) for \( II'' \), one finds that

\[ II'' \leq \frac{C_2'}{|z|^3 \log \Lambda} \]

Next

\[ II' = \frac{C_1}{|z|^3 \log \Lambda} \int_\frac{x}{\Lambda}^\infty \frac{1}{x} e^{-4x} dx \]

For say \( \Lambda \geq 10 \), one has

\[ II' \leq \frac{C_1'}{|z|^3 \log \Lambda} e^{-\frac{8}{\pi}} \int_1^\infty \frac{1}{x} dx + \frac{C_2''}{|z|^3 \log \Lambda} \]

\[ \leq \frac{C_1'}{|z|^3} + \frac{C_2''}{|z|^3 \log \Lambda} \]

Finally, one has obtained

\[
I = \frac{1}{|z|} o_{\Lambda}(1) \\
II = \frac{1}{|z|^3} O_{\Lambda}(\frac{1}{\log \Lambda}) + \frac{1}{|z|^3} O_{\Lambda}(1) + \frac{1}{|z|^3} O_{\Lambda}(\frac{1}{\log \Lambda})
\]

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and it follows that

\[ M(\|z\|) = \frac{1}{\|z\|} O(1) + \frac{1}{\|z\|^3} O(1) + \frac{1}{\|z\|^2} O(1) \]

This is enough to end the first two points of assertion 1).

In order to get the limit of \( z M(\|z\|) \), and thus third point of 1), we start from

\[ A = x_0 \int_0^1 \frac{1}{\sqrt{1 - u^2}} \left[ 1 - \frac{4 x_0}{|z|^2 b} e^{-\lambda} u^2 - \frac{4 x_0}{|z|^2 b} u e^{-\lambda} \frac{1}{u} \right] du \]

where \( \lambda \equiv \frac{2b}{\Lambda x_0} > 0 \).

Set \( \phi(t) = e^{-\lambda t} \) for all \( t \geq 1 \). Then, one finds easily, that for all \( 0 < u < 1 \), there exists \( 0 < u' < 1 \), \( u' > u \) such that

\[ e^{-\lambda \frac{u}{u'}} = e^{-\lambda} - \lambda e^{-\lambda} \frac{1}{u'} \]

Using this writing, we have after some simplifications

\[ A = \int_0^1 \frac{1}{\sqrt{1 - u^2}} \left[ 1 + \frac{4}{|x_0|^2 b} e^{-\lambda} \frac{1}{1 + u} + \frac{8}{|z|^2 \Lambda x_0^3} e^{-\lambda} \frac{1}{1 + u} \right] \]

Since for all \( t > 0 \), there exists \( c \in [0; t] \) such that

\[ (1 + t)^{-\frac{1}{2}} = 1 - \frac{1}{2} t + \frac{3}{8} \frac{1}{(1 + c)^2} t^2 \]

it follows

\[ A = \int_0^1 \frac{1}{\sqrt{1 - u^2}} \left[ \frac{4}{|x_0| b} e^{-\lambda} \frac{1}{1 + u} + \frac{8}{|z|^2 \Lambda x_0^3} e^{-\lambda} \frac{1}{1 + u} \right] \]

In conclusion, recalling that \( \pi - A = \frac{\theta}{2} \), we get

\[ \frac{\pi}{2} - A = A_1 + A_2 + A_3 \]

where

\[ A_1 = 2 \int_0^1 \frac{du}{\sqrt{1 - u^2} (1 + u)} \frac{1}{|z|^2 b} e^{-\lambda} \]
\[ A_2 = 4\int_0^1 \frac{e^{-\lambda \sqrt{u}}}{\sqrt{1 - u^2}(1 + u)} \frac{1}{|z|^2 \Lambda x_0^2} \, du \]
\[ A_3 = -\frac{3}{8} \int_0^1 (1 - u^2)^{-1/2}(1 + c)^{-1/2} \left[ \frac{4}{x_0} \frac{1}{|z|^2 b} e^{-\lambda} \frac{1}{1 + u} + \frac{8}{|z|^2 \Lambda x_0^2} e^{-\lambda \sqrt{u}} \right]^2 \]

In the following, we will use the asymptotic

\[ \sin^2 \left( \frac{\pi}{2} - A \right) = \left( \frac{\pi}{2} - A \right)^2 + O((\frac{\pi}{2} - A)^3) + O((\frac{\pi}{2} - A)^4) \]

From the preceding computations, to get the behaviour of \( M_\Lambda(|z|) \), it is enough to get the behaviour of

\[ M'_\Lambda(|z|) = \frac{|S_1||z|}{\log \Lambda} \int_1^\Lambda \sin^2 \left( \frac{\pi}{2} - A \right) b db. \]

Indeed

\[ M_\Lambda(|z|) = M'_\Lambda(|z|) + \frac{|S_1||z|}{\log \Lambda} \int_0^1 \sin^2 \left( \frac{\pi}{2} - A \right) b db + \frac{|S_1||z|}{\log \Lambda} \int_\Lambda^{+\infty} \sin^2 \left( \frac{\pi}{2} - A \right) b db \]

The second term on the rhs of this last equality is nothing else than \( I \) and we have seen that (say for \( \Lambda \geq 10 \)) \( I = O(\frac{1}{|z| \log \Lambda}) \). As for the last term, it is less than

\[ \left| \frac{S_1||z|}{\log \Lambda} \int_\Lambda^{+\infty} \frac{C_1}{|z|^4 b} e^{-\frac{4b}{\pi}} + C_2 \frac{1}{|z|^4 \Lambda^2} be^{-\frac{4b}{\pi}} \right| = O\left( \frac{1}{|z|^3 \log \Lambda} \right) \]

Therefore

\[ M_\Lambda(|z|) = M'_\Lambda(|z|) + O\left( \frac{1}{|z| \log \Lambda} \right) + O\left( \frac{1}{|z|^3 \log \Lambda} \right) \]

We can therefore work on

\[ M'_\Lambda(|z|) \equiv \frac{|S_1||z|}{\log \Lambda} \int_1^\Lambda \sin^2 \left( \frac{\pi}{2} - A \right) b db \]

We set from the previous computations

\[ M'_\Lambda(|z|) = A_1 + A_2 + A_3 \]

where

\[ A_1 \equiv \frac{|S_1||z|}{\log \Lambda} \int_1^\Lambda \left( \frac{\pi}{2} - A \right)^2 b db \]
\[ A_2 \equiv -\frac{|S_1||z|}{\log \Lambda} \int_1^\Lambda O((\frac{\pi}{2} - A)^3)bdb \]
\[ A_3 \equiv -\frac{|S_1||z|}{\log \Lambda} \int_1^\Lambda O((\frac{\pi}{2} - A)^4)bdb \]

From the following computations, it will be clear that \( A_2 \) and \( A_3 \) go to 0 as \( \Lambda \to +\infty \), and we work only on \( A_1 \). One has

\[
A_1 = \frac{|S_1||z|}{\log \Lambda} \int_1^\Lambda (A_1)^2bdb + \frac{|S_1||z|}{\log \Lambda} \int_1^\Lambda (A_2)^2bdb + \\
+ \frac{|S_1||z|}{\log \Lambda} \int_1^\Lambda (A_3)^2bdb + 2 \frac{|S_1||z|}{\log \Lambda} \int_1^\Lambda (A_1A_2)bdb + \\
+ 2 \frac{|S_1||z|}{\log \Lambda} \int_1^\Lambda (A_2A_3)bdb + 2 \frac{|S_1||z|}{\log \Lambda} \int_1^\Lambda (A_1A_3)bdb.
\]

One has

\[
\frac{|S_1||z|}{\log \Lambda} \int_1^\Lambda (A_2)^2bdb = \\
= \frac{16 |S_1||z|}{\log \Lambda} \int_1^\Lambda \left[ \int_0^1 \frac{e^{-\lambda \frac{u}{2}}}{\sqrt{1-u^2(1+u)}} |z|^4 \Lambda^2 x_0^4 bdb \right] \\
\leq \frac{C}{\log \Lambda |z|^3} \int_0^\Lambda \frac{1}{x_0^4} e^{\frac{-2b}{\Lambda}} bdb.
\]

Since \( x_0 \geq (1 + \frac{4}{|z|^2b})^{-1} \), and since \( b \geq 1 \), it follows that \( x_0 \geq (1 + \frac{4}{|z|^2})^{-1} \), and so

\[
\frac{|S_1||z|}{\log \Lambda} \int_1^\Lambda (A_2)^2bdb = O\left( \frac{1}{|z|^2 \log \Lambda (1 + \frac{4}{|z|^2})^4} \right)
\]

The same computations show also that

\[
\frac{|S_1||z|}{\log \Lambda} \int_1^\Lambda (A_3)^2bdb = O\left( \frac{1}{|z|^3 \log \Lambda (1 + \frac{4}{|z|^2})^4} + O\left( \frac{1}{|z|^3 \log \Lambda \Lambda^2} \right) (1 + \frac{4}{|z|^2})^8 \right)
\]

Next for the cross product terms, one has for instance

\[
2 \frac{|S_1||z|}{\log \Lambda} \int_1^\Lambda A_1A_2bdb \leq \\
\leq \frac{C |z|}{\log \Lambda} \int_0^\Lambda b \frac{1}{x_0 |z|^2 b} e^{\frac{-2b}{\Lambda}} e^{\frac{-2b}{\Lambda}} \frac{1}{|z|^2 \Lambda x_0^2}
\]

and so

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The other terms can be treated in the same fashion, and it follows that $M'_\Lambda$ has the same behaviour as

$$L = \left[ \frac{|S_1|}{\log \Lambda} \right] \int_1^\Lambda A_2^2 db$$

For this term, we have

$$L = \frac{4}{|z|^3 \log \Lambda} \int_0^1 \frac{du}{\sqrt{1-u^2} (1+u)^2} \int_1^\Lambda \frac{1}{x_0 b} e^{-\frac{2b}{x_0^2}} = \frac{4}{|z|^3 \log \Lambda} \int_1^\Lambda \frac{1}{x_0 b} e^{-\frac{2b}{x_0^2}}$$

For all $t \geq 1$, set $\phi(t) = t^2 e^{-\frac{2b}{t^2}}$. Then, there exists $0 \leq \alpha \leq 1$, $\alpha \geq x_0$ such that $\phi(\frac{1}{x_0}) = \phi(1) + \phi'(\frac{1}{x_0})(\frac{1}{x_0} - 1)$. Thus

$$\frac{1}{x_0} e^{-\frac{2b}{x_0}} = I + II$$

where

$$I = e^{-\frac{2b}{x_0}} \text{ and } II = \left\{ \frac{2}{\alpha} e^{-\frac{2b}{x_0^2}} - \frac{2b}{\alpha} e^{-\frac{2b}{x_0^2}} \right\} \left( \frac{1}{x_0} - x_0 \right)$$

and it follows that

$$L = \frac{4}{|z|^3 \log \Lambda} \int_1^\Lambda \frac{1}{b} e^{-\frac{2b}{x_0^2}} + \frac{4}{|z|^3 \log \Lambda} \int_1^\Lambda \frac{1}{b} II db$$

One has easily $\frac{1-x_0}{x_0} \leq \frac{4}{|z|^3 \log \Lambda} e^{-\frac{2b}{x_0^2}}$, and thus one finds that

$$II \leq C(1 + \frac{4}{|z|^2}) \frac{1}{b} e^{-\frac{2b}{x_0^2}} + \frac{C}{|z|^2 \Lambda} (1 + \frac{4}{|z|^2}) \frac{1}{b} e^{-\frac{2b}{x_0^2}}$$

Finally

$$\frac{C}{|z|^3 \log \Lambda} \int_1^\Lambda \frac{1}{b} II db \leq C(1 + \frac{4}{|z|^2}) \frac{1}{|z|^5 \log \Lambda} \int_1^\Lambda \frac{1}{b} e^{-\frac{2b}{x_0^2}} + \frac{C}{|z|^5 \Lambda \log \Lambda} (1 + \frac{4}{|z|^2})^2 \int_1^\Lambda \frac{1}{b} e^{-\frac{2b}{x_0^2}} \rightarrow 0 \text{ as } \Lambda \rightarrow +\infty$$

In conclusion, since $\int_1^\Lambda \frac{1}{b} e^{-\frac{2b}{x_0^2}} db \simeq \log \Lambda$, we have find

$$zM(\|z\|) \rightarrow 4 \left| S_1 \right| \left| \frac{z}{|z|^3} \right| \text{ in } M_{loc} - * \text{ as } \Lambda \rightarrow +\infty$$

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In the following we set \( K \equiv 4 \mid S_1 \mid. \)

This ends the proof of the first point \( 1) \).

Next, let us show a) of point \( 3) \). This is clear if we show that the corresponding \( b \) are finite (ie do not take large values). For this purpose, we start from

\[
\frac{\pi}{2} - A \leq \frac{\pi}{2} (1 - x_0) + \frac{4x_0}{|z|^2} e^{\frac{4x_0}{|z|^2}}
\]

which leads for a suitable constant \( C \) to

\[
b \leq \frac{C}{|z|^2} \frac{1}{\theta}
\]

Thus, if \( \theta \geq \theta_0 > 0 \), it follows that \( b \leq \frac{C}{|z|^2} \frac{1}{\theta_0} \) and so

\[
M_{\Lambda}^{\theta_0} \leq \frac{C \mid \theta \mid}{\log \Lambda} \int_{0}^{C \mid \theta \mid \theta_0} \frac{4}{|z|^2} e^{\frac{4x_0}{|z|^2}} dB \leq \frac{C}{|z|^3} \log \Lambda \theta_0
\]

This proves a) of point \( 3) \).

We now turn to point \( 4) \) and we first return to the upper bound for \( A \). We have seen that

\[
A = x_0 \int_{0}^{1} \left[ 1 - u^2 \right]^{\frac{1}{2}} \left[ 1 - \frac{4x_0}{|z|^2} e^{-\lambda u} \frac{1}{1 + u} \right] dB
\]

where \( \lambda = \frac{2b}{x_0} \). Since \( \frac{4x_0}{|z|^2} e^{-\lambda u} \frac{1}{1 + u} < 1 \), it follows that

\[
A \leq x_0 \int_{0}^{1} \left[ 1 - u^2 \right]^{\frac{1}{2}} \left[ 1 - \frac{4x_0}{|z|^2} e^{-\lambda u} \frac{1}{1 + u} \right] dB
\]

Since \( \frac{u}{1 + u} \leq \frac{1}{2} \), one has finally

\[
A \leq x_0 \int_{0}^{1} \left[ 1 - u^2 \right]^{\frac{1}{2}} \left[ 1 - (1 - x_0^2) \right]^{\frac{1}{2}}
\]

and thus it follows that \( A \leq \frac{\pi}{2} \sqrt{x_0} \). From this we get easily that

\[
\frac{1}{1 - x_0^2} \geq \frac{\pi}{4} \frac{1}{\theta}
\]

and also

\[
x_0 \geq \frac{(\pi - \theta)^2}{\pi^2}
\]

Next we turn to the expression of \( \theta \) given by

\[
\theta = \pi - 2x_0 \int_{0}^{1} \left[ 1 - x_0^2 u^2 - \frac{4x_0}{|z|^2} b e^{-\frac{2b}{x_0} \frac{1}{2}} \right] dB
\]
where $b = b(\theta)$ and $x_0 = x_0(b(\theta))$. Deriving wrt to $\theta$, we find
\[
1 = -2x_0' b' \int_0^1 \left[ 1 - x_0'^2 u^2 - \frac{4x_0}{|z|^2 b} u e^{-\frac{2b}{x_0^2}} \right] \frac{1}{u} du + \]
\[
x_0 \int_0^1 \left[ 1 - x_0'^2 u^2 - \frac{4x_0}{|z|^2 b} u e^{-\frac{2b}{x_0^2}} \right] \frac{3}{u} \left[ 1 - x_0'^2 u^2 - \frac{4x_0}{|z|^2 b} u e^{-\frac{2b}{x_0^2}} \right] \frac{1}{u} du
\]
where after some computations, we find that
\[
\left[ 1 - x_0'^2 u^2 - \frac{4x_0}{|z|^2 b} u e^{-\frac{2b}{x_0^2}} \right] \frac{1}{u} du = -2u^2 x_0 x_0' b' - \frac{4b'}{|z|^2} (x_0' b - x_0) e^{-\frac{2b}{x_0^2}} \left\{ \frac{u}{b^2} + \frac{2}{Abx_0} \right\}
\]
On the other hand, deriving wrt to $b$, the equation satisfied by $x_0$, we find
\[
-2x_0 x_0' = \frac{4}{|z|^2} (x_0' b - x_0) e^{-\frac{2b}{x_0^2}} \left\{ \frac{1}{b^2} + \frac{2}{Abx_0} \right\}
\]
In particular, note that $x_0' b - x_0 \leq 0$, and thus $x_0 \geq x_0' b$. We note this under the form
\[
\frac{4}{|z|^2} (x_0' b - x_0) = -2x_0 x_0' e^{-\frac{2b}{x_0^2}} \left\{ \frac{1}{b^2} + \frac{2}{Abx_0} \right\}^{-1}
\]
In conclusion, we have found that
\[
\left[ 1 - x_0'^2 u^2 - \frac{4x_0}{|z|^2 b} u e^{-\frac{2b}{x_0^2}} \right] \frac{1}{u} du = 2x_0 x_0' \left[ -u + e^{-\frac{2b}{x_0^2}} \left\{ \frac{1}{b^2} + \frac{2}{Abx_0} \right\}^{-1} \left\{ \frac{u}{b^2} + \frac{2}{Abx_0} \right\} \right]
\]
It follows that
\[
1 = -2x_0 x_0' \int_0^1 \left( 1 - x_0'^2 u^2 - \frac{4x_0}{|z|^2 b} e^{-\frac{2b}{x_0^2}} \right) du
\]
\[
+2x_0^2 x_0' b' \int_0^1 \left( 1 - x_0'^2 u^2 - \frac{4x_0}{|z|^2 b} e^{-\frac{2b}{x_0^2}} \right) \times
\]
\[
\times \left[ -u + e^{-\frac{2b}{x_0^2}} \left\{ \frac{1}{b^2} + \frac{2}{Abx_0} \right\}^{-1} \left\{ \frac{u}{b^2} + \frac{2}{Abx_0} \right\} \right] du
\]
One checks easily that $b' \geq 0$. The conclusion is that
\[
|b'| = -b' = \frac{1}{2x_0} \frac{1}{B - C}
\]
$(B \geq C)$, with
\[
B = \int_0^1 \left( 1 - x_0'^2 - \frac{4x_0}{|z|^2 b} e^{-\frac{2b}{x_0^2}} \right) \frac{1}{u} du
\]
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\[ C = x_0^2 \int_0^1 \left( 1 - x_0^2 - \frac{4x_0}{|z|^2} u e^{-\frac{2b}{x_0} \frac{1}{u}} \right) \frac{1}{u^3} \times \]

\[ [-u^2 + \left\{ \frac{1}{b^2} + \frac{2}{\Lambda x_0} \right\}^{-1} \left\{ \frac{u}{b^2} + \frac{2}{\Lambda b x_0} \right\} e^{\frac{2b}{x_0} \left( 1 - \frac{1}{u} \right)}] du \]

And finally, one has

\[ \tilde{B}_\Lambda(|z|, \cos \theta) = \frac{1}{\log \Lambda} \frac{b}{\sin \theta} \left| b'(\theta) \right| \left| z \right| = \frac{1}{\log \Lambda} \frac{b}{\sin \theta} \frac{1}{B - C} \left| z \right| \]

It follows that

\[ \tilde{B}_\Lambda \sin \theta \geq \frac{1}{\log \Lambda} b \frac{1}{2x_0} \frac{1}{B} \left| z \right| \]

Since \( A = x_0 B \), we get

\[ \tilde{B}_\Lambda \sin \theta \geq \frac{1}{\pi \log \Lambda} b \frac{1}{2x_0} \frac{x_0}{A} \left| z \right| \]

Since \( A \leq \frac{\pi}{2} \), we get

\[ \tilde{B}_\Lambda \sin \theta \geq \frac{1}{\pi \log \Lambda} b^2 \frac{1}{x_0} \frac{1}{x_0} \left| z \right| \]

We have seen that \( x_0' b \leq x_0 \), thus

\[ \tilde{B}_\Lambda \sin \theta \geq \frac{1}{\pi \log \Lambda} b^2 \left| z \right| \]

Next, since

\[ 1 - x_0^2 = \frac{4x_0}{|z|^2} e^{-\frac{2b}{x_0}} \]

that is

\[ |z|^2 (1 - x_0^2) b = 4x_0 e^{-\frac{2b}{x_0}} \]

it follows that

\[ b \geq \frac{4x_0}{|z|^2 (1 - x_0^2) + \frac{8}{\Lambda}} \]

Since \( x_0 \geq \frac{(\pi - \theta)^2}{\pi^2} \) and so \( x_0 \geq 1/4 \), it follows that, using also \( 1 - x_0^2 \leq \frac{4}{\pi} \theta \)

\[ b \geq \frac{1}{\max \left( 1, \frac{2|z|^2}{\pi^2} \right)} \frac{1}{\theta + \frac{x}{\Lambda}} \]

and finally

\[ \tilde{B}_\Lambda \sin \theta \geq \frac{1}{\pi \log \Lambda} \frac{1}{\max \left( 1, \frac{16|x|^4}{\pi^2} \right)} \frac{1}{4\theta^2} \frac{1}{1 \geq \frac{x}{\Lambda}} \]

We have to improve this estimate. For this, we return to

\[ \tilde{B}_\Lambda \sin \theta \geq \frac{1}{\pi \log \Lambda} b \frac{1}{x_0} \left| z \right| \]
Above, we have seen that

\[ x'_0 = \frac{4x_0}{|z|^2} \cdot \frac{\frac{4b}{|z|^2}}{4b + 2x_0e^{-\Lambda x_0} \left( \frac{1}{b^2} + \frac{2}{\Lambda bx_0} \right)^{-1}} \]

From this, we get

\[ \tilde{B}_A \sin \theta \geq \frac{1}{\pi \log \Lambda} \cdot \frac{1}{|z|^2} \cdot \frac{4b}{|z|^2} \cdot \frac{2x_0e^{-\Lambda x_0} \left( \frac{1}{b^2} + \frac{2}{\Lambda bx_0} \right)^{-1}} \]

In particular, it follows that

\[ \tilde{B}_A \sin \theta \geq C \log \Lambda \]

and using the lower bound on \( b \) found just above, we get

\[ \tilde{B}_A \sin \theta \geq C \log \Lambda \max \left( |z|^3 b^3, \frac{1}{|z|^3 \theta} \right) \]

and this ends the proof of the assertion 4).

For the remaining parts concerning \( S_A \) and \( S_{\theta_0}^A \), that is point 2) and point b) 3), this is clear testing against smooth functions, and using the above results.
Chapter 14

Linearized Boltzmann operator
Chapter 15

Boltzmann Dirac model

15.1 Introduction

We have seen that the usual Boltzmann equation describes the evolution of a phase-space density of classical particles under the assumption that they only interact by pairwise (elastic) collisions, within classical mechanics

\begin{equation}
\frac{\partial}{\partial t} f + v \cdot \nabla_x f = Q_B(f, f).
\end{equation}

Now, if we want to describe a gas of Fermi-Dirac particles satisfying Sommerfeld’s degeneracy condition (see [42]), one has to modify the collision integral in order to take into account quantum effects. Then a well known modification of equation (15.1) is given by the following equation, called Boltzmann-Dirac equation

\begin{equation}
\partial_t f + v \cdot \nabla_x f = Q_{BD}(f, f),
\end{equation}

where

\begin{equation}
Q_{BD}(g, f)(v) = \int_{v^*} \int_{\sigma} B(v-v^*, \sigma)[g^* f'(1-g^*)](1-f^*) - g^* f(1-g^*)](1-f^*)\, dv^* \, d\sigma.
\end{equation}

Again, we assume that the scattering cross section \(B(v-v^*, \sigma) > 0\) writes

\begin{equation}
B(v-v^*, \sigma) = \Phi(|v-v^*|) b(\cos \theta), \quad \cos \theta = <\frac{v-v^*}{|v-v^*|}, \sigma>, \quad 0 \leq \theta \leq \frac{\pi}{2}.
\end{equation}

As in the previous Chapters, we are interested in taking a collision section \(B\) associated to an intermolecular potential \(\frac{1}{r^s}\), under the non cutoff assumption, and more precisely,

\begin{equation}
\sin^{n-2} \theta b(\cos \theta) \sim \frac{k}{\theta^{1+\nu}}, \quad \theta \to 0, \quad \nu = \frac{2}{s-1}.
\end{equation}
Physical background and derivation of such quantum Boltzmann models can be found in Chapman and Cowling ([42], chap.17) and Uehling and Uhlenbeck [112].

Equation (15.2) provides a good approximation to the Boltzmann equation (15.1). One of the main interest of this approximation is due to the fact that suitable solution of (15.2) has a natural $L^\infty$-estimate (due to Pauli’s exclusion principle):

\[
0 \leq f(t, x, v) \leq 1, \ 0 \leq g(t, x, v) \leq 1 \quad (t, x, v) \in [0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3.
\]

Under integrability assumptions on $B$, that is within the cutoff case, global existence of solutions to (15.2) in the distributional sense have been proven by Dolbeault [61].

The non cutoff case was initiated by Alexandre [8] and one can find also in Lu [88] a recent compactness result under rather general assumptions on the cross section.

For the spatially homogeneous solutions, some results on equilibrium states and long-time behavior of solutions have been also obtained in [89], [90].

Our aim is to improve the results from [8, 88] in view of our work with Desvillettes, Villani and Wennberg, and in particular results displayed in previous Chapters.

First of all, we shall show that using the entropy dissipation linked with Boltzmann-Dirac equation, one can obtain coercivity type results, as in Chapter 5. Here, we shall use the same method of proof as in [11], to get the following estimation

\[
\|f\|_{H^2}^2 \leq C_g [D^{BD}(g, f) + \|g\|_{L^1_\infty} \|f\|_{L^1_\infty}],
\]

where

\[
D^{BD}(g, f) = - \int_{\mathbb{R}^n} Q^{BD}(g, f) \log \frac{f}{1-f} dv,
\]

is the entropy dissipation for the Boltzmann-Dirac equation.

With this result at hand, the existence of weak solutions will follow immediately, by following the arguments from Chapter 11. But note the usual weak formulation involving $\phi' - \phi$ will allow only small singularities $0 < \nu < 1$. However, by using arguments similar to the notion of H-solutions of Villani, see Chapter 1, one can allow higher singularities.

It is not at all clear what can be done, once one has existence of such weak solutions. For instance, trying to show that solutions gain some regularity is still an open problem. Because of this fact, we shall limit ourself to the coercivity estimate. Results from this Chapter are due to my former PhD student Mouhamad El Safadi.
15.2 Coercivity result

Following [11] and previous Chapters, we introduce the cross-section for momentum transfer:

$$\Lambda(|v - v_*|) = \int_\sigma B(v - v_*, \sigma)(1 - k.\sigma)d\sigma$$

We also define for $|z| \neq 0$

$$B'(z, \sigma) = \sup_{1 \leq \lambda \leq \sqrt{2}} \frac{|B(\lambda z, \sigma) - B(z, \sigma)|}{(\lambda - 1)|z|},$$

and, in the same way as (15.7)

$$\Lambda'(|v - v_*|) = \int_\sigma B'(v - v_*, \sigma)(1 - k.\sigma)d\sigma.$$ 

**Theorem 15.2.1** Assume that

$$B(v - v_*, \sigma) \geq \Phi(|v - v_*|)b(k.\sigma),$$  

where $\Phi$ is continuous and bounded below by a strictly positive in every interval $0 < r \leq |z| \leq 2\sqrt{2}$. Assume also that $b$ satisfies the singularity assumption (15.5), and that

$$\Lambda(|z|) + |z|\Lambda'(|z|) \leq C(1 + |z|)^2.$$ 

Then there exists a constant $C_g$, depending only on $b$, $\|g\|_{L^1}$, $\|g\|_{L^1_{\log L}}$ and on $\Phi$, such that

$$\|f\|_{H^2(\mathbb{R}^n)} \leq C_g[ D^{BD}(g, f) + \|g\|_{L^1} \|f\|_{L^1}],$$

where

$$D^{BD}(g, f) = - \int_{\mathbb{R}^n} Q^{BD}(g, f) \log \frac{f}{1 - f} dv.$$

**Proof:** First, from the generalized entropy dissipation functional defined just above, using the change of the variable $(v, v_*, \sigma) \to (v', v'_*, \sigma)$, one has

$$D^{BD}(g, f) = - \int_{\mathbb{R}^n} Q^{BD}(g, f) \log \frac{f}{1 - f} dv$$

$$= - \int_{\mathbb{R}^n \times S^{n-1}} Bg'_s(1 - g_s)f'(1 - f) \left[ \log \frac{f}{1 - f} \frac{1 - f'}{f'} \right] dv dv_* d\sigma$$

$$= \int_{\mathbb{R}^n \times S^{n-1}} Bg'_s(1 - g_s)f'(1 - f) \log \frac{f'(1 - f)}{f(1 - f')} dv dv_* d\sigma.$$
Splitting the last integral into two terms of the form
\[ D^{BD}(g, f) = \int_{\mathbb{R}^{2n} \times S^{n-1}} Bg'_s(1 - g_s) \left( f'(1 - f) \log \frac{f'(1 - f)}{f(1 - f')} - f'(1 - f) + f(1 - f') \right) dv dv_s d\sigma \]
\[ + \int_{\mathbb{R}^{2n} \times S^{n-1}} Bg'_s(1 - g_s)[f'(1 - f) - f(1 - f')]dv dv_s d\sigma = A_1 + A_2. \]

\( A_2 \) can be written as
\[ A_2 = \int_{\mathbb{R}^{2n} \times S^{n-1}} Bg'_s(1 - g_s)[f' - f]dv dv_s d\sigma. \]

Applying the change of variables in prime, we obtain
\[ A_2 = -\int_{\mathbb{R}^{2n} \times S^{n-1}} Bf(g_s' - g_s)dv dv_s d\sigma. \]

Using the cancellation lemma [11], see also Chapter 3, we obtain
\[ \int_{\mathbb{R}^{n} \times S^{n-1}} B(g_s' - g_s)dv d\sigma = g \ast S(v), \]
where
\[ |S(z)| \leq C(\Lambda(|z|)) + |z|\Lambda'(|z|)). \]

Taking into account (15.12), one has
\[ (15.13) \quad A_2 \leq C\|g\|_{L_1^2}\|f\|_{L_1^2}. \]

Returning to the estimation \( A_1 \), since
\[ \forall \; X \leq 1, \; Y \leq 1, \; \text{one has} \; X \log \frac{X}{Y} - X + Y \geq (X - Y)^2, \]
choosing \( X = f'(1 - f), \; Y = f(1 - f') \), we obtain
\[ A_1 \geq \int_{\mathbb{R}^{2n} \times S^{n-1}} Bg'_s(1 - g_s)[f'(1 - f) - f(1 - f')]^2 dv dv_s d\sigma, \]
or also
\[ (15.14) \quad A_1 \geq \int_{\mathbb{R}^{2n} \times S^{n-1}} Bg'_s(1 - g_s)(f' - f)^2 dv dv_s d\sigma. \]

Once again, we use the change of variable in prime for the last integral, and get
\[ (15.15) \quad A_1 \geq \int_{\mathbb{R}^{2n} \times S^{n-1}} Bg'_s(1 - g'_s)(f' - f)^2 dv dv_s d\sigma. \]
Now, adding (15.14) and (15.15), we obtain

\[ A_1 \geq \frac{1}{2} \int_{\mathbb{R}^{2n} \times S^{n-1}} B[g_*(1 - g_*) - g'_*(1 - g_*)](f' - f)^2 dv d\sigma. \]

Taking into account the inequality

\[ g_*(1 - g_*) - g'_*(1 - g_*) \geq g'_*(1 - g'_*) \]

We obtain

\[ A_1 \geq \frac{1}{2} \int_{\mathbb{R}^{2n} \times S^{n-1}} Bg'_*(1 - g'_*)(f' - f)^2 dv d\sigma. \]

Applying again the change of variables in prime, we have

\[ A_1 \geq \frac{1}{2} \int_{\mathbb{R}^{2n} \times S^{n-1}} BG'_*(f' - f)^2 dv d\sigma. \]

such that \( G_* = g_*(1 - g_*) \), we know that \( ||G_*||_{L^1} \leq ||g_*||_{L^1} \) and \( ||G_*||_{L \log L} \leq \infty \).

We can then use the same manipulations as in the work of Alexandre, Villani, Desvillettes and Wennberg [11]. By definition, the functions \( f \) and \( g \) are already bounded. The minoration of \( A_1 \) gives then

\[ A_1 \geq \min_{|\xi| \leq 2\sqrt{2}(1, \Phi(|\xi|))} \int_{\mathbb{R}^{2n} \times S^{n-1}} G_*(f' - f)^2 b(k, \sigma) dv d\sigma. \]

Finally, the following two inequalities

\[
\begin{aligned}
&\int_{\mathbb{R}^{2n} \times S^{n-1}} G_*(f' - f)^2 b(k, \sigma) dv d\sigma \\
&\geq \frac{1}{2(2\pi)^n} \int_{\mathbb{R}^n} \left| \hat{f}(\xi) \right|^2 \left\{ \int_{\sigma} b(\xi, \sigma) \hat{G}(0) - |\hat{G}(\xi^-)|d\sigma \right\} d\xi \\
&\quad \text{and} \int_{\mathbb{R}^n} \left| \hat{f}(\xi) \right|^2 \left\{ \int_{\sigma} b(\xi, \sigma) \hat{G}(0) - |\hat{G}(\xi^-)|d\sigma \right\} d\xi \geq C_g |\xi|^\gamma.
\end{aligned}
\]

suffice to deduce that

(15.16) \[ A_1 \geq C_g ||f||_{L^2(\mathbb{R}^n)}^2. \]

Here, \( C_g \) depends only on \( b, ||g||_{L^1}, ||g||_{L \log L} \) and on \( \Phi \).

Estimates (15.13) and (15.16) together ensure the theorem holds.

### 15.3 Appendix: Fourier transformation of the Boltzman-Dirac operator

In the spirit of the calculation in [11], we are interested here in computing the Fourier transformation of \( Q^{BD} \) in the case of Maxwellian molecules. In this case, the collision kernel \( B \) of (15.4) reduces to
function $b$ (i.e. $\Phi = 1$), that we assume integrable. For simplicity, we decompose this operator into two terms,

$$Q^{BD} = Q^{BD^+} - Q^{BD^-},$$

where

$$Q^{BD^+}(g, f) = \int_{\mathbb{R}^n \times S^{n-1}} g(v_s') \overline{g(v)} f(v') \overline{f}(v) b(v - v_s) \frac{1}{|v - v_s|} \sigma dv_s d\sigma$$

and

$$Q^{BD^-}(g, f) = \int_{\mathbb{R}^n \times S^{n-1}} \overline{g(v_s')} g(v_s) f(v') \overline{f}(v) b(v - v_s) \frac{1}{|v - v_s|} \sigma dv_s d\sigma,$$

where the notations $\overline{g} = 1 - g$ and $\overline{f} = 1 - f$ are used. These operators have the same structure by changing the prime, so it suffices to calculate the Fourier transform of the first one and conclude directly for the second one.

By definition, the Fourier transform of the first operator writes

$$\hat{Q}^{BD^+}(g, f) = \int_{\mathbb{R}^n \times S^{n-1}} g(v_s') \overline{g(v)} f(v') \overline{f}(v) e^{-iv' \xi} b(v - v_s) \frac{1}{|v - v_s|} \sigma dv_s d\sigma.$$

Using the classical change of variables in prime, we get

$$\hat{Q}^{BD^+}(g, f) = \int_{\mathbb{R}^n \times S^{n-1}} g(v_s) \overline{g(v_s')} f(v) \overline{f}(v') e^{-iv \xi} b(v - v_s) \frac{1}{|v - v_s|} \sigma dv_s d\sigma.$$

Introducing the inverse Fourier transform, we obtain

$$\hat{Q}^{BD^+}(g, f) = \int_{\mathbb{R}^{2n} \times S^{n-1}} g(v_s) \overline{\mathbf{\hat{g}}(v_s)} f(v) \overline{\mathbf{\hat{f}}}(v') e^{iv \xi} \xi_1 e^{iv'(\xi_2 - \xi)} b(v - v_s) \frac{1}{|v - v_s|} \sigma d\xi_1 d\xi_2 dv_s d\sigma.$$

Writing the definition of $v'$ and $v_s'$, it becomes

$$\hat{Q}^{BD^+}(g, f) = \int_{\mathbb{R}^{2n} \times S^{n-1}} d\xi_1 d\xi_2 dv_s d\sigma$$

$$g(v_s) \overline{\mathbf{\hat{g}}(v_s)} f(v) \overline{\mathbf{\hat{f}}}(\xi_2) e^{-i\frac{v + v_s}{2}(\xi - \xi_1)} e^{-i\frac{|v - v_s|}{2} \sigma} e^{i\frac{v - v_s}{2} \sigma} b(v - v_s).$$

By the isometry on $S^{n-1}$, we can exchange the unity vectors in the sense

$$\int_{\sigma} F(k, \sigma, l, \sigma) d\sigma = \int_{\sigma} F(l, \sigma, k, \sigma) d\sigma, \quad |k| = |l| = 1.$$

We introduce the following definitions, for any vector $w \in \mathbb{R}^n$

$$w^+ = \frac{w}{2} + \frac{|w|}{2} \sigma \quad \text{and} \quad w^- = \frac{w}{2} - \frac{|w|}{2} \sigma.$$

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Taking account (15.18) and applying (15.17) to the vectors \( l = \frac{\xi - \xi_2 + \xi_1}{|\xi - \xi_2 + \xi_1|} \) and \( k = \frac{v - v_*}{|v - v_*|} \), we get

\[
\hat{Q}^{BD+} (g,f) = \int_{\mathbb{R}^{2n} \times S^{n-1}} d\xi_1 d\xi_2 dv_* d\sigma \\
g(v_*) \hat{g}(\xi_1) f(v) \hat{f}(\xi_2) e^{-iv \cdot [(\xi - \xi_2 + \xi_1)^+ - \xi_1]} e^{-iv_* \cdot [(\xi - \xi_2 + \xi_1)^- - \xi_1]} b\left( \frac{v - v_*}{|v - v_*|}, \sigma \right).
\]

Thus,

\[
\hat{Q}^{BD+} (g,f) = \int_{\mathbb{R}^{2n} \times S^{n-1}} d\xi_1 d\xi_2 d\sigma \\
\hat{g} \left( (\xi - \xi_2 + \xi_1)^- - \xi_1 \right) \hat{g}(\xi_1) \hat{f} \left( (\xi - \xi_2 + \xi_1)^+ - \xi_1 \right) \hat{f}(\xi_2) b\left( \frac{\xi - \xi_2 + \xi_1}{|\xi - \xi_2 + \xi_1|}, \sigma \right).
\]

Making the following change of variable \( \xi - \xi_2 + \xi_1 = \xi_* \), we get

\[
\hat{Q}^{BD+} (g,f) = \int_{\mathbb{R}^{2n} \times S^{n-1}} d\xi_1 d\xi_2 d\sigma \\
\hat{g} \left( \xi_* - \xi_1 \right) \hat{g}(\xi_1) \hat{f} \left( \xi_*^+ - \xi_1 \right) \hat{f}(\xi_2) b\left( \xi_* \frac{\xi_*^+}{|\xi_*^+|}, \sigma \right) d\xi_1 d\xi_* d\sigma.
\]

By definition of \( \mathcal{F} \) and \( \mathcal{g} \), we get

\[
\hat{Q}^{BD+} (g,f) = \int_{\mathbb{R}^{2n} \times S^{n-1}} d\xi_1 d\xi_* d\sigma \\
\hat{g} \left( \xi_* - \xi_1 \right) \left[ \delta_{[\xi_* = 0]} - \hat{g}(\xi_1) \right] \hat{f} \left( \xi_*^+ - \xi_1 \right) \left[ \delta_{[\xi_*^+ - \xi_* = 0]} - \hat{f}(\xi_1 - \xi_*^+) \right] b\left( \xi_* \frac{\xi_*^+}{|\xi_*^+|}, \sigma \right).
\]

Taking into account the change of variables in prime, we obtain the Fourier transform of \( Q^{BD-} \):

\[
\hat{Q}^{BD-} (g,f) = \int_{\mathbb{R}^{2n} \times S^{n-1}} d\xi_1 d\xi_* d\sigma \\
\left[ \delta_{[\xi_* - \xi_1 = 0]} - \hat{g}(\xi_* - \xi_1) \right] \hat{g}(\xi_1) \left[ \delta_{[\xi_*^+ - \xi_* = 0]} - \hat{f}(\xi_*^+ - \xi_1) \right] \hat{f} (\xi + \xi_1 - \xi_*^+) b\left( \frac{\xi_*}{|\xi_*^+|}, \sigma \right).
\]
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Appendix: Harmonic Analysis

This final Chapter fixes some notations used throughout the text and recalls some standard facts from
Harmonic Analysis.

Precise references for basic facts related to Harmonic Analysis are:
- the book by Lemarié [79];
- the book by Runst and Sickel [101];
- the book by Stein [104];
- the books by Taylor [105, 106, 107];
- the books by Triebel [108, 109];
- the Lecture notes by Tao [110];
- the Lecture notes by Tartar [111].

Let us recall quickly the Littlewood-Paley decomposition.

We fix once for all in this paper a collection \( \{ \psi_k = \psi_k(\xi) \}_{k \in \mathbb{N}} \) of smooth radial functions such that
\[
\text{supp } \psi_0 \subset \{ \xi \in \mathbb{R}^n, \ |\xi| \leq 2 \}, \\
\text{supp } \psi_k \subset \left\{ \xi \in \mathbb{R}^n, 2^{k-1} \leq |\xi| \leq 2^{k+1} \right\} \text{ for all } k \geq 1,
\]
for every multi-index \( \alpha \), there exists a positive number \( c_\alpha \) such that
\[
2^{k|\alpha|} |D^\alpha \psi_k(\xi)| \leq c_\alpha \text{ for all } k = 0, 1, 2, \ldots \text{ and all } \xi \in \mathbb{R}^n
\]
and
\[
\sum_{k=0}^{+\infty} \psi_k(\xi) = 1 \text{ for all } \xi \in \mathbb{R}^n.
\]
All functions $\psi_k$, for $k \geq 1$, are constructed from a single one $\psi \geq 0$, i.e. we are given $\psi$ such that

$$\text{supp } \psi \subset \left\{ \xi \in \mathbb{R}^n, \frac{1}{2} \leq |\xi| \leq 2 \right\}$$

and $\psi > 0$ if $1/\sqrt{2} \leq |\xi| \leq \sqrt{2}$.

Defining the Littlewood-Paley projection operators $p_k$, for $k \geq 0$, by

$$\hat{p}_k f(\xi) = \psi_k(\xi) \hat{f}(\xi),$$

it follows that one has the Littlewood-Paley decomposition

$$f = \sum_{k=0}^{+\infty} p_k f$$

for all $f \in \mathcal{S}'$.

We shall denote by $\phi_k$ the Fourier inverse function of $\psi_k$, so that actually, $p_k$ is nothing else than the convolution by $\phi_k$.

Besov spaces $B^s_{p,q}$ are defined by those tempered distributions $f$ such that $\sum_{k \in \mathbb{N}} 2^{ksq} \|p_k f\|_L^q < \infty$, with usual limitations on the indices $s$, $p$ and $q$. For instance, if $p = q = 2$, then we recover the usual Sobolev spaces $H^s$. We shall also need Besov spaces with a suitable weight $\delta$, $\delta \in \mathbb{R}$; the corresponding Besov space will be denoted by $B^s_{p,q,\delta}$. This corresponds to the usual Besov space, but with the $L^p_\delta$ norm instead of the usual $L^p$ one, see more details in the references on harmonic analysis.

Finally, the (small) space $b^s_{p,q}$ refers to the completion of $\mathcal{S}$ for the $B^s_{p,q}$ norm, in the extreme cases $p = \infty$ or $q = \infty$.

In the text, from time to time, we made use of Muckenhoupt classes of weight functions. These are defined as follows.

Firstly, for a given weight function $w$, let us denote by $L^p(w)$ the standard Lebesgue space with respect to the measure $w(v)dv$. $L^{1,\infty}(w)$ denotes the corresponding Lorentz space and $M$ refers to the usual Hardy-Littlewood maximal function.

In case $p = 1$, a weight $w$ is said to satisfy the $A_1$ condition if there exists a constant $C$ such that, for every cube $Q$

$$\frac{w(Q)}{|Q|} \leq C w(v) \text{ for a.e. } v \in Q,$$

which is equivalent to

$$M w(v) \leq C w(v) \text{ for a.e. } v \in \mathbb{R}^n.$$
Finally $A_\infty = \bigcup_{p \geq 1} A_p$.

As an example to have in mind, $|v|^a \in A_1$ if $-n < a \leq 0$ and is in $A_p$ for $p > 1$ if $-n < a < n(p-1)$.

**Remarks 21.0.1**

1. There are much more results available in Harmonic Analysis.

2. Let us note that in this work, we have based our approach on the use of Littlewood-Paley partition of unity, localized in frequency, and through the use of the usual maximal functions. However, we mention the case of local weights and not using such localized partitions of unity. This point might be useful elsewhere. Also, some work remains to be done if one is interested in two weights type estimates.

Finally, we made use of the following maximal lemma

**Lemma 21.0.1 (Maximal Lemma)** If $\hat{\phi}$ is smooth, with $\text{supp} \hat{\phi}$ compact, then for all $r > 0$, it follows that

$$\sup_{z \in \mathbb{R}^n} \left| \nabla \hat{\phi}(x - z) \right| \leq C_1 \sup_{z \in \mathbb{R}^n} \left| \hat{\phi}(x - z) \right| \leq C_2 \left\{ \left( M \left| \hat{\phi} \right|^{\frac{r}{r'}}(x) \right) \right\}^{\frac{1}{r}},$$

where $M$ denotes the usual maximal function.
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