The Fourier Transform Method – Technical Document

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CONTENTS:

- **Introduction**
- **Overview of the FTM**
- **The Underlying Models**
- **The FTM Applied to ABS/MBS Analysis**
- **The FTM Applied to CDO Analysis**
- **Further Uses of the FTM**
- **Conclusion**
- **Appendix 1: Computer Implementation of the Inverse Fourier Transform**
- **Appendix 2: Computation of Portfolio Fourier Transforms with Gaussian Quadratures**
- **Appendix 3: Calibration of the Factor Loadings**
- **Appendix 4: Moody's Expected Loss Table**
- **Appendix 5: References**

INTRODUCTION

Moody's ratings of Asset-Backed Securities (ABS), Residential Mortgage-Backed Securities (RMBS), Commercial Mortgage-Backed Securities (CMBS) or Collateralised Debt Obligations (CDO) are ultimately based on the expected loss concept. An accurate determination of the default or loss distribution for a portfolio of securitised assets is essential since it allows the computation of the expected loss on the notes according to their terms and conditions and therefore allows for the quantitative derivation of the notes' ratings.[1]

At Moody's, we often use two well-known methods to compute the assets' default or loss probability distribution: the Binomial Expansion Technique[2] (BET) for CDO or, to a lesser extent, CMBS portfolios and the Lognormal Method[3] (LNM) for granular ABS portfolios or even RMBS portfolios. Like all modelling methods, the BET and the LNM have their limits. Neither method is perfectly adapted to analyse **asset heterogeneity in terms of rating/credit risk, size or maturity** or to analyse portfolios with an **intermediate number of assets**. Ideally, the BET should be used to approximate the default distribution of a portfolio with a limited number of homogeneous assets while the LNM should be used for granular portfolios with a large number of assets.

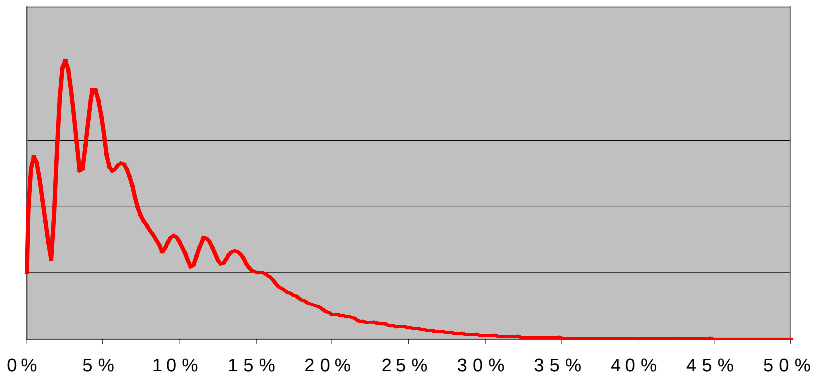
Moody's improved – and keeps improving – its existing methodologies. For instance, we developed the Multi-Binomial Technique[4] in order to extend the BET to heterogeneous portfolios. We also developed rating approaches based on Monte Carlo simulations to deal with complex pools of assets, for instance like in some CMBS transactions[5] or ith-to-default Basket Credit-Linked Notes.[6] The **Fourier Transform Method (FTM)** is one of such improvements, that presents interesting advantages compared to Monte Carlo simulations: speed, accuracy, and adaptability.

A general introduction to the FTM is provided in **Moody's special report: "*The Fourier Transform Method – Overview*"**[7], while this **technical document** aims at detailing some **technical, theoretical and practical aspects of the FTM**.

[1] Moody's ratings accounts for a wide range of factors, obviously quantitative ones but also most importantly qualitative ones. For further information, please refer to "The Combined Use of Qualitative Analysis and Statistical Models in the Rating of Securitisations", Moody's Special Report, 7 July 2001.
[2] Cf. The Binomial Expansion Method Applied to CBO/CLO Analysis, Moody's Special Report, 13 December 1996.
[3] Cf. The Lognormal Method Applied to ABS Analysis, Moody's Special Report, 27 July 2000.
[4] Cf. The Double Binomial Method and its application to a special case of CBO structures, Moody's Special Report, 20 March 1998.
[5] Cf. Moody's Approach to Rating European CMBS, Moody's Special Report, 14 June 2001.
[6] Cf. Moody's Approach to Rating ith-to-Default Basket Credit-Linked Notes, Moody's Special Report, 17 April 2002.
[7] Dated 15 January 2003.

OVERVIEW OF THE FOURIER TRANSFORM METHOD

The FTM aims at determining the default or the loss distribution of a portfolio of assets over a certain time horizon. The quickest and most accurate way to determine such distributions would be derived from an explicit analytical formula. For instance, such analytical formulas exist for the BET and the LNM. However, in more general cases, getting such explicit formulas proves impossible. The alternative to estimate the distribution lies in finding numerical algorithms or techniques that are both highly accurate and quick to run. The FTM is **a numerical technique that offers both accuracy and rapidity**. The chart below depicts **an example of a default distribution determined with the FTM** for a small heterogeneous portfolio.

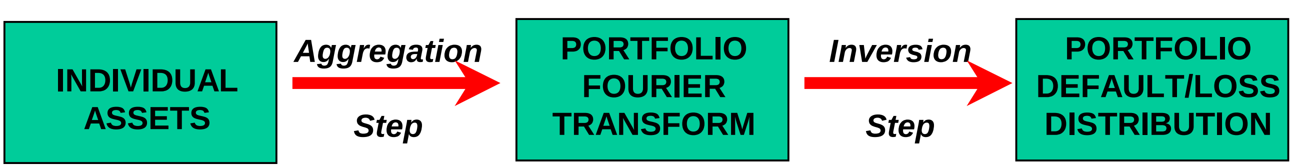


With the FTM, the **computation of the distribution** is done numerically in **two steps**:

1. ***The Aggregation Step***: the assets' risk behaviour and default correlation between assets are modelled with a factor model: the credit risk of each asset result from a dependence to common market or economic factors (systemic risks) and from individual risks (idiosyncratic risks). Under such modelling assumptions, it is possible to **numerically compute the Fourier transform** $\hat{f}(t)$ **of the portfolio's aggregate default/loss distribution** $f(x)$ **–** hereafter often simply referred to as the ***portfolio Fourier transform*.**

2. ***The Inversion Step***: the portfolio's default/loss distribution $f(x)$ is obtained by **numerically computing the inverse Fourier transform of the portfolio Fourier transform** $\hat{f}(t)$. This computation is almost immediate thanks to the use of Fast Fourier Transform algorithms (such algorithms are implemented under common spreadsheet softwares like Microsoft Excel).

The Two Steps of the FTM



The factor models realising the aggregation step of the FTM are presented in the following section, ***The Underlying Models***. Technical details regarding the implementation of the inversion step of the FTM may be found in ***Appendix 1: Computer Implementation of the Inverse Fourier Transform***. The two sections ***The FTM Applied to ABS/MBS Analysis*** and ***The FTM Applied to CDO Analysis*** deal with the tailoring of the FTM to the specific respective characteristics of ABS/MBS or CDOs: the major difference lies in the way the assets in the portfolio are accounted for, individually in CDO deals or regrouped in homogeneous subportfolios in ABS/MBS deals. The section ***Further Uses of the FTM*** essentially explains how to compute portfolio loss distributions with the FTM.

The Fourier Transform Theory in a Nutshell

A default distribution is a mathematical function $f(x)$, which indicates how likely defaults of a given level are: the probability that defaults –over a certain time horizon- will be comprised between x and $x+\Delta x$ will be $f(x).\Delta x$ for a small default bucket Δx. Such default distribution is a function of a space variable x, which "measures" the amplitude of default.

Most generally, to each – good – function $f(x)$ of a space variable x can be associated its Fourier transform $\hat{f}(t)$. The ***Fourier transform*** *can be defined by the following formula:*

$$\hat{f}(t)=\int_{-\infty}^{+\infty} f(x).\exp(-itx).dx\text{, where } i=\sqrt{-1} \text{ is the imaginary unit.}$$

The Fourier transform $\hat{f}(t)$ is not a function of the space variable x, but a function of a frequency variable t: in other words, by applying the Fourier transform, we translate ourselves from the "real space" –where things can be measured with functions of the space variable x – to another "dual space" – a frequency space where things are measured as a function of the frequency variable t. We will refer to the "real space" as the ***space domain*** *and to the "dual space" as the* ***Fourier domain (or Fourier space)****.*

Conversely, it is also possible to associate to any function $g(t)$ of the Fourier domain its inverse Fourier transform $\breve{g}(x)$ which is a function of the space domain. The ***inverse Fourier transform*** *can be defined by the following formula:*

$$\breve{g}(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} g(t).\exp(itx).dt\,.$$

*The inverse Fourier transform name comes from the fundamental "**Inversion Formula**" $\breve{\hat{f}}=f$: by applying the above defined Inverse Fourier transform to the Fourier transform $\hat{f}$ of a function f of the space domain, you'll find the original function – f – again.*

The Fourier transform of a probability distribution – also called the ***characteristic function*** *in probability theory – is another representation of the probability distribution. There is a one to one relationship between each probability distribution and its characteristic function (only valid under circumstances, which for all practical purposes will always be true…). The relationship is intimate: for instance, the moments of the probability distribution can be inferred from the derivatives of the characteristic function at the origin. A key theoretical bridge between the probability theory and the Fourier transform theory is the following: if a random variable X (like a portfolio default/loss rate) has the probability distribution $f_X(x)$ then its Fourier transform will also be defined by: $\hat{f}_X(t)=\mathrm{E}\left[e^{-itX}\right]$. It is often referred indifferently to the Fourier transform of a random variable X or to the Fourier transform of its associated probability distribution $f_X(x)$. Another very powerful property of Fourier transforms is the following: for two* ***independent*** *random variables X and Y, the Fourier transform of their sum is: $\hat{f}_{X+Y}(t)=\hat{f}_X(t).\hat{f}_Y(t)$ In other words,* ***the Fourier transform of the sum of independent random variables is the product of the transforms.***

Now, why use the Fourier transform for computing the default or loss distribution of a portfolio? In most cases, it is impossible to derive tractable formulas for default distributions in the space domain. However, under certain sets of modelling assumptions, the formulas simplify if we translate ourselves in the Fourier domain. As a matter of fact, it proves possible to find a tractable formula for the Fourier transform of the portfolio's default distribution. In order to get back to our good old "real" space, there will "only" need to apply the inverse Fourier transform. However, this is the other good news: computing an Inverse Fourier transform basically costs nothing in terms of computation time. This computation is achieved almost immediately thanks to ***Fast Fourier Transform (FFT)*** *algorithms. These algorithms were discovered some 50 years ago and are considered to be one of the most important discoveries in numerical mathematics during the last century. The combination between their speed and the growing computation capacity revolutionised some industries (electronics, radio, telecommunications, medical systems, etc.).*

THE UNDERLYING MODELS

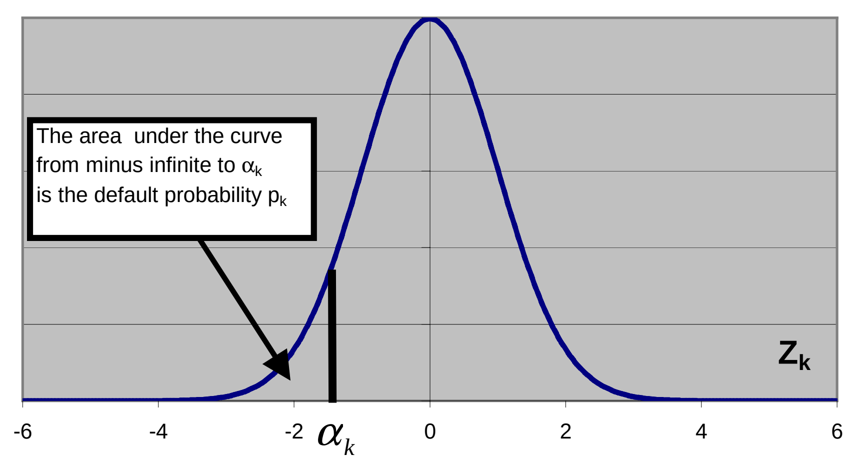
As described previously, the first step of the FTM consists in computing the Fourier transform of the portfolio's default distribution. In order to do so, we need to model the credit risk behaviour of the assets and how the default correlation is created between them. We will first introduce a very simple factor model,[8] the ***Single Factor Model***. We will then mention other possible models, for a ***Portfolio of Uncorrelated Assets***, or ***Multifactor Models*** that account for the simultaneous effects of several common factors. We will then discuss ***How to Calibrate the Factor Loadings***, which are the default correlation parameters used in the factor modelling framework, and also ***How to Deal With Amortising Assets***. Finally, we will present an important particular case, ***The Single Factor Model for a Large Portfolio of Homogeneous Assets***, where it is possible to derive a closed form formula for the default distribution, the *Normal Inverse* default distribution.

- **The Single Factor Model**

Let us consider a portfolio of N assets (bonds, loans, debentures…). The k[th] asset has an initial outstanding amount of S_k and is assumed to have a default probability of p_k over the time horizon for which we want to determine the portfolio's default distribution. For the sake of simplicity, we will assume that each asset corresponds to a different debtor and that all assets have a bullet amortisation. The first assumption may be achieved by aggregating the assets relating to the same debtor. We'll see later how to deal with the second assumption in order to account for amortising assets.

For each asset $k = 1 \ to\ N$, let us define Z_k as its **normalised credit risk indicator** at the end of the time horizon. The lower Z_k will be, the higher the credit risk of the k[th] debtor. Debtor k will default during the time horizon if Z_k falls below a certain **default threshold** α_k. α_k will be determined by the following equation: $\Pr(Z_k < \alpha_k) = p_k$. Since Z_k was assumed to have a standard normal probability distribution (with a mean of 0 and a variance of 1), we have: $\Pr(Z_k < \alpha_k) = \Phi(\alpha_k) = p_k, \ where\ \Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}}.\exp\left(-u^2/2\right).du$ is the standard normal cumulative distribution function (NORMSDIST(z) function in Microsoft Excel). Therefore $\alpha_k = \Phi^{-1}(p_k)$ (NORMSINV(p) function in Microsoft Excel). The chart below illustrates how the default threshold α_k fits to the normal probability distribution of the **normalised credit risk indicator** Z_k.

Default Threshold α_k



8 The presented factor model is derived from an article by Christopher C. Finger: "Conditional Approaches for CreditMetrics Portfolio Distributions" (1999). Another founding article for the portfolio factor models is certainly: "The Loan Loss Distribution", O. Vasicek, KMV Corporation, 1987.

The next step for this single factor model consists in modelling the default correlation –or rather the credit risk dependence– between the different individual assets. The starting point is quite general: the credit risk for each individual asset can be split between a systemic risk and an idiosyncratic risk, i.e. a risk that can only be attributed to a particular debtor. The systemic risk represents a common exposure to common factors such as the changes of a market index, interest rates or oil prices, the economic growth, the evolution of the real estate market, etc.

In a first step, we'll only consider one single factor Z (hence the name of the Single Factor Model). In this case, the factor Z is often referred to as the **state of the economy** (in reference to the macroeconomic conditions that influence the credit-worthiness of both companies and individuals).

Z will be assumed to have a standard normal probability distribution. Similarly, the idiosyncratic risks ε_k will be assumed to have a standard normal probability distribution. Consistently with the definitions of the systemic risk and the idiosyncratic risks, the $\varepsilon_k's$ and Z will be assumed to be independent random variables.

The major assumption of the Single Factor Model lies in how the systemic risk and the idiosyncratic risk combine together to make up the individual risk. They are assumed to add to each other: $Z_k = w_k Z + \theta_k \varepsilon_k$, where w_k is the level of **correlation**[9] of the kth asset to the common factor Z and θ_k the weight of the idiosyncratic risk for the kth asset. The $w_k's$ reflect the contribution of the systemic factor Z to the individual risks Z_k: therefore the $w_k's$ are often referred to as the **factor loadings**. Since Z_k, Z and ε_k were assumed to have normal standard probability distributions, and since Z and ε_k are independent, taking the variance of Z_k, it can be shown that $\theta_k = \sqrt{1-w_k^2}$. Therefore, for each asset, we will have:

$$Z_k = w_k Z + \sqrt{1-w_k^2}\,\varepsilon_k .$$

For a given state of the economy $Z = z$, the kth debtor will default if $Z_k < \alpha_k$, i.e. if $\varepsilon_k < (\alpha_k - w_k z)\big/\sqrt{1-w_k^2}$. Using the standard normality of ε_k, **the default probability of the kth debtor (conditionally to $Z = z$)** will be:

$$\boxed{p_k(z) = \Phi\left(\frac{\alpha_k - w_k z}{\sqrt{1-w_k^2}}\right)}.$$

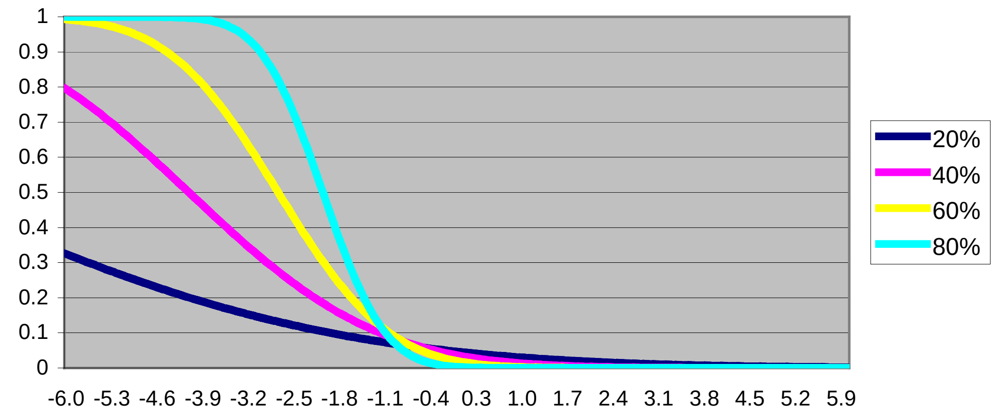
Unsurprisingly, we remark that the expected value of the r.v. $p_k(Z)$ is the default probability p_k itself:

$$\mathrm{E}[p_k(Z)] = \int_{-\infty}^{+\infty} p_k(z).\phi(z).dz = p_k \text{, where } \phi(z) = \frac{1}{\sqrt{2\pi}}\exp\left(-z^2/2\right).$$

The chart below depicts the graph of $p_k(z)$ as a function of z for different values of the factor loading w_k. Some comments about this chart: the default probability is higher during bad periods ($z < 0$) than during good periods for the economy ($z > 0$). Besides, during bad periods for the economy, the debtors who are the most sensitive to the state of the economy (i.e. those with the highest factor loading w) are the most likely to default.

[9] With these assumptions: $Corr(Z_k, Z) = w_k$ and $Corr(Z_k, Z_l) = w_k.w_l$. w_k is sometimes referred to as the *asset correlation.*

p(z) for Different Values of w
(p=5% - α=-1.64)



Let us also remark that the normality of Z was not used in establishing the above expression for $p_k(Z)$: we simply used that Z had a mean of 0, a variance of 1 and was independent from the ε_k's. Therefore, this expression also holds for a factor Z which is not normally distributed.[10] In that case however, one must be careful in determining the default thresholds α_k's: the relationship $\alpha_k = \Phi^{-1}(p_k)$ is not valid anymore (it resulted from the normality of Z_k, which resulted itself from the normality of Z). Instead the default threshold α_k will satisfy:

$$\mathrm{E}[p_k(Z)] = \int_{-\infty}^{+\infty} p_k(z).\phi(z).dz = \int_{-\infty}^{+\infty} \Phi\left(\frac{\alpha_k - w_k z}{\sqrt{1-w_k^2}}\right).\phi(z).dz = p_k,$$

where $\phi(z)$ is the probability distribution for the (non normal) factor Z. The default threshold will be determined by solving for α_k in this last equation. Note that most of what follows also holds when the factor Z is not normally distributed.

Now, let us determine **the portfolio Fourier transform** under this simple Single Factor Model. The default rate of the portfolio over the time horizon is:

$$\textit{Portfolio Default } \mathrm{Rate\ (PDR)} = \frac{S_1 X_1 + S_2 X_2 + \ldots + S_N X_N}{S_1 + S_2 + \ldots + S_N} = \sum_{k=1}^{N} s_k X_k,$$

where X_k is the default indicator for the k^th asset ($X_k = 1$, if the k^th asset defaults over the time horizon, 0 otherwise) and $s_k = S_k \Big/ \sum_{k=1}^{N} S_k$ is the weight of the k^th asset in the total portfolio (in %). Let us stress that the PDR denominator would be different for amortising assets since the defaultable amounts decrease over time.

For a given state of the economy $Z = z$, the **conditional Fourier transform of the portfolio default distribution** ($f_{PDR}(x), 0 < x < 100\%$) therefore is:[11]

$$\hat{f}_{PDR/Z=z}(t) = \mathrm{E}\left[e^{-it.\mathrm{PDR}} \middle| Z = z\right] = \mathrm{E}\left[e^{-it(s_1 X_1 + s_2 X_2 + \ldots + s_N X_N)} \middle| Z = z\right].$$

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[10] If the factor Z represents a variation in an equity index, it may be wiser to assume a lognormal or a Student distribution.

[11] This is nothing but the application of the key theoretical formula mentionned in the text box *"The Fourier Transform Theory in a Nutshell".*

Using the conditional independence given Z of the default indicators X_k and the fact that the Fourier transform of the sum of independent random variables is the product of their Fourier transforms:

$$\hat{f}_{PDR/Z=z}(t) = \mathrm{E}\left[e^{-its_1 X_1} \middle| Z = z\right] \times \mathrm{E}\left[e^{-its_2 X_2} \middle| Z = z\right] \times \ldots \times \mathrm{E}\left[e^{-its_N X_N} \middle| Z = z\right].$$

For the k[th] asset: $X_k = 1$ with the probability $\Pr(X_k = 1 | Z = z) = p_k(z)$ and $X_k = 0$ with the probability $\Pr(X_k = 0 | Z = z) = (1 - p_k(z))$. Therefore:

$$\begin{aligned}\mathrm{E}\left[e^{-its_k X_k} \middle| Z = z\right] &= \Pr(X_k = 0 | Z = z).e^{-its_k .0} + \Pr(X_k = 1 | Z = z).e^{-its_k .1} \\ &= (1 - p_k(z)) + p_k(z).e^{-its_k} = 1 + p_k(z).\left(e^{-its_k} - 1\right)\end{aligned}$$

Finally, the **conditional** Fourier transform of the portfolio's default distribution will be given by:

$$\hat{f}_{PDR|Z=z}(t) = \prod_{k=1}^{N} \left[1 + p_k(z).\left(e^{-its_k} - 1\right)\right]$$

The **unconditional** Fourier transform of the portfolio's default distribution – in short the portfolio Fourier transform – will be obtained by considering all the states of the economy Z and its related density function $\phi(z)$:

$$\hat{f}_{PDR}(t) = \mathrm{E}\left[\hat{f}_{PDR|Z}(t)\right] = \int_{-\infty}^{+\infty} \hat{f}_{PDR|Z=z}(t).\phi(z).dz \,,$$

$$\boxed{\hat{f}_{PDR}(t) = \int_{-\infty}^{+\infty} \prod_{k=1}^{N} \left[1 + p_k(z).\left(e^{-its_k} - 1\right)\right] \phi(z) dz} \qquad \textbf{(I)},$$

where $p_k(z) = \Phi\left(\frac{\alpha_k - w_k z}{\sqrt{1 - w_k^2}}\right)$ and $\phi(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-z^2/2\right)$.

Although the formula **(I)** may seem complex, **the portfolio Fourier transform can be computed numerically for any given value of *t*.**[12] The difficulty lies in the way the integral $\int_{-\infty}^{+\infty} \hat{f}_{PDR|Z=z}(t).\phi(z).dz$ is computed. Gaussian quadrature formulas prove quite useful in performing this computation efficiently (see Appendix 2).

- **Portfolio of Uncorrelated Assets**

In the case of uncorrelated assets: $p_k(z) = p_k$, and **(I)** simplifies to:

$$\boxed{\hat{f}_{PDR}(t) = \prod_{k=1}^{N} \left[1 + p_k.\left(e^{-its_k} - 1\right)\right]}$$

- **Multifactor Models**

A major assumption of the previous Single Factor model lies in the way the systemic risk is modelled. It was assumed that there was a single common factor Z. However, in certain circumstances, the default likelihood

[12] See the example of Appendix 1 for more insights on the practical computer implementation.

of the debtors may be assumed to depend on several factors Z^A, Z^B, Z^C … A typical example would be a CDO portfolio: companies in the same industry are typically exposed to the same industry risk factor – for instance $Z^{Mining}, Z^{Tourism}, Z^{Agriculture}$... At the same time, all companies across all industries may also be exposed to a global systemic risk factor Z^{Global}.

For instance, if we simply consider **two independent[13] factors** Z^A and Z^B, the individual credit risk indicators could be written: $Z_k = w_k^A Z^A + w_k^B Z^B + \sqrt{1-\left(w_k^A\right)^2-\left(w_k^B\right)^2}\,\varepsilon_k$, $k = 1 \dots N$. It would lead to the following portfolio Fourier transform:

$$\boxed{\hat{f}_{PDR}(t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\prod_{k=1}^{N}\left[1+p_k\left(z^A, z^B\right)\left(e^{-its_k}-1\right)\right]\phi\left(z^A\right)\phi\left(z^B\right)dz^A.dz^B} \quad \textbf{(II)},$$

where $p_k(z^A, z^B) = \Phi\left(\dfrac{\alpha_k - w_k^A z^A - w_k^B z^B}{\sqrt{1-\left(w_k^A\right)^2-\left(w_k^B\right)^2}}\right)$.

Again, even if this formula seems complex, it can be computed numerically for any given value of t. The computation of the double integral would however be more time consuming.

- **How to Calibrate the Factor Loadings (the w_k's)?**

Input calibration is a critical step of any credit risk modelling method. The factor models use the factor loadings w_k's as default correlation inputs. The calibration of the factor loadings is facilitated through their translations in the various "languages" commonly used to describe and quantify default correlation. Indeed factor loadings have a direct interpretation in terms of:

(1) standard deviation-over-mean ratio for a given portfolio,
(2) diversity score for a given portfolio,
(3) pairwise default correlation between two assets,
(4) asset (return) correlation between two assets,
(5) joint downgrade (and upgrade) probability for two rated assets.

Standard deviation-over-mean ratios are commonly used for ABS and MBS transactions. Diversity scores, pairwise default correlations, asset correlations (and to a lesser extent joint downgrade probabilities) are mostly used for CDO transactions. Parameters (1) to (5) are usually calibrated through estimates based on historical data[14] or by making assumptions on their value.

The existence of different possible interpretations of (1) to (5) not only makes it easier to calibrate the factor loadings; it also makes possible to check the consistency of a correlation assumption across the board. For instance, one may have a very good reason to believe that the pairwise default correlation between every pair of assets in a portfolio should be of a certain level. Such an assumption implies a certain level for the factor loadings; the factor loadings themselves imply certain levels of std dev. over mean, diversity score or asset correlation. If such levels do not make sense, the initial assumption on the pairwise default correlation should certainly be reviewed… **Appendix 3 details the relationships between the factor loadings and parameters (1) to (5) in the context of the Single Factor Model. It also gives some hints on the typical values for the factor loading for different asset classes.**

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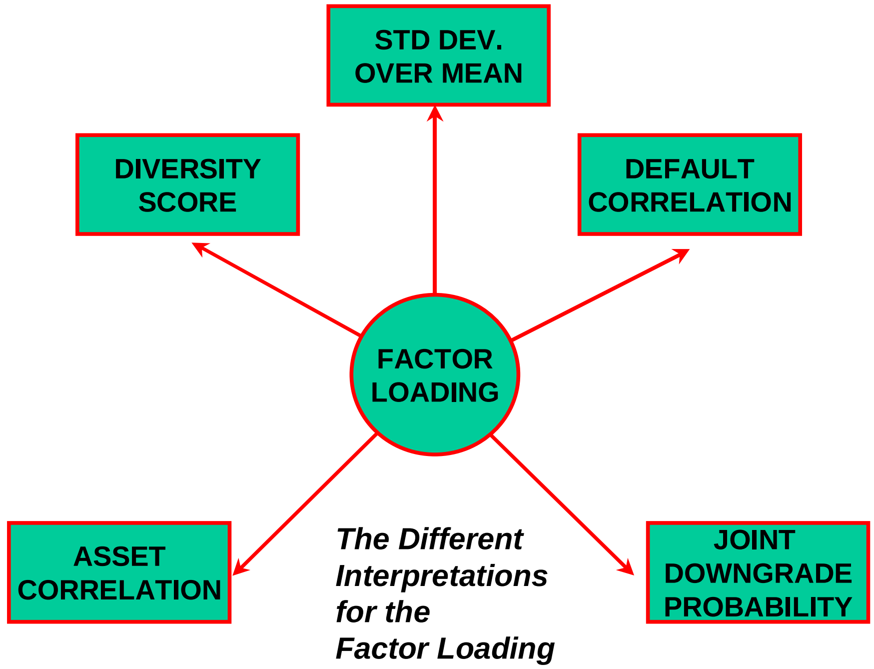
[13] The two factors could be considered as correlated between them: think of the GDP growth and the variation in an equity index. However, for two normally distributed standardised factors Z^A and Z^B, it is possible to come back to the case where they would be independent by "decorrelating" them: if we define $Z^* = \left(Z^B - \rho.Z^A\right)/\sqrt{1-\rho^2}$, Z^A and Z^* appear to be two uncorrelated normally distributed standardised factors. For more than two factors, it would still be possible to come back to the independent case thanks to a Cholevski decomposition.

[14] It is important to remember that estimates based on historical data relate to the past: they may not be fully applicable to the future.



- **How to Deal With Amortising Assets?**

Equation **(I)** was derived under the assumption that the assets have a bullet amortisation profile during the time horizon considered for the default distribution. In many transactions, especially in ABS or MBS transactions, the assets are amortising and this assumption is not directly applicable.

This issue may be addressed with two different approaches, a simple one and a more refined one.

Simple Approach
The simple approach consists in approximating the default distribution of the portfolio by the default distribution of a portfolio of **bullet** assets with equivalent initial outstanding principal. The maturity of each bullet asset will be assumed to be the weighted average life of the corresponding real asset while its default **probability** will be assumed to be the **expected default rate**[15] (EDR) of the real asset. This method is not necessary the most inaccurate: most often in ABS or MBS transactions, historical data – specifically static vintage default curves – directly permit to estimate the assets' default rate in the past and to make assumptions for future EDRs.

Refined Approach
The second approach is more complex to implement. An expression of the Portfolio Default Rate (PDR) with assets amortising over time would be:

$$PDR = \sum_{k=1}^{N} s_k(T_k).1(T_k < m_k) = \sum_{k=1}^{N} s_k(T_k).X_k \text{ , where:}$$

- For the k[th] asset, T_k is a random variable that designates the time of default ($T_k = +\infty$ if the asset never defaults)
- m_k is the minimum between the considered horizon for the PDR and the maturity of the k[th] asset
- $s_k(\tau) = S_k(\tau)/S_k$, where $S_k(\tau)$ represents the scheduled outstanding amount of the k[th] asset at time τ in the future
- $1(A) = 1$ if A is true and $1(A) = 0$ if A is false. By definition, the default indicator for the k[th] asset over the

[15] Expected default rate (EDR) means the expected defaulted amount during the time horizon divided by the initial outstanding principal. The expected default rate is the same as the default probability for a bullet asset: it is lower than the default probability for an amortising asset since the defaultable amount decreases over time. It is worth noticing that we have the following first order relationship for each asset: $EDR \approx p.WAL / Maturity$.

considered horizon is $X_k = 1(T_k < m_k)$.

The portfolio Fourier Transform then becomes:

$$\hat{f}_{PDR}(t) = E\left[e^{-itPDR}\right] = E\left[\prod_{k=1}^{N} e^{-its_k(T_k)X_k}\right]$$

Since the assets (and the default times T_k 's) are independent conditionally to Z=z:

$$\hat{f}_{PDR/Z=z}(t) = \prod_{k=1}^{N} E\left[e^{-its_k(T_k)X_k} \middle| Z = z\right] = \prod_{k=1}^{N} E\left[e^{-its_k(T_k).1(T_k<m_k)} \middle| Z = z\right]$$

Let $U_k(\tau, z)$ be the probability density function of T_k conditionally to Z=z and let $u_k(\tau, z)$ be the probability density function of T_k conditionally to Z=z and to $T_k < m_k$ (i.e conditionally to a default of the k[th] asset over the horizon). The link between $U_k(\tau, z)$ and $u_k(\tau, z)$ is simply $U_k(\tau, z) = u_k(\tau, z) \times p_k(z)$, where $p_k(z)$ is the conditional default probability of the k[th] asset over the considered horizon introduced in precedent sections.

The term of the above product may be re-written:

$$E\left[e^{-its_k(T_k).1(T_k<m_k)} \middle| Z = z\right] = \int_0^{+\infty} e^{-its_k(\tau).1(\tau<m_k)}.U_k(\tau, z).d\tau =$$

$$= \int_0^{m_k} e^{-its_k(\tau)}.U_k(\tau, z).d\tau + \int_{m_k}^{+\infty} 1 \times U_k(\tau, z).d\tau = \int_0^{m_k} e^{-its_k(\tau)}.U_k(\tau, z).d\tau + \Pr\left[T_k > m_k \middle| Z = z\right]$$

More simply:

$$E\left[e^{-its_k(T_k).1(T_k<m_k)} \middle| Z = z\right] = p_k(z).\int_0^{m_k} e^{-its_k(\tau)}.u_k(\tau, z).d\tau + (1 - p_k(z)) = 1 + p_k(z).\left(\int_0^{m_k} e^{-its_k(\tau)}.u_k(\tau, z).d\tau - 1\right)$$

Hence the expression for the conditional portfolio Fourier Transform:

$$\hat{f}_{PDR/Z=z}(t) = \prod_{k=1}^{N} E\left[e^{-its_k(T_k).1(T_k<m_k)} \middle| Z = z\right] = \prod_{k=1}^{N} \left[1 + p_k(z).\left(\int_0^{m_k} e^{-its_k(\tau)}.u_k(\tau, z).d\tau - 1\right)\right]$$

And finally, the portfolio Fourier Transform will be:

$$\boxed{\hat{f}_{PDR}(t) = \int_{-\infty}^{+\infty} \prod_{k=1}^{N} \left[1 + p_k(z).\left(\int_0^{m_k} e^{-its_k(\tau)}.u_k(\tau, z).d\tau - 1\right)\right].\phi(z).dz}$$

Let us say a few words on the **time distribution for default** for the k[th] asset, $u_k(\tau, z)$. For instance, in its discretised version, this time distribution corresponds to a probability of 5% that a defaulted asset defaults during the first period, of 10% during the second period, etc… The sum of all these probabilities over all the considered periods is of 100%.

In ABS or MBS transactions, these timing assumptions will most likely be inferred from the study of the vintage default curves over time. In CDO deals including amortising assets, these assumptions will rather be inferred from the rating of the underlying obligors and from the timing of the default generally associated to a given rating category.

As a matter of fact, the time distribution for default $u_k(\tau, z)$ may depend on the state of the economy Z=z. Further assumptions about how this time distribution may change with Z=z may be made. For z<0 (depressed economy), defaults should generally be frontloaded (i.e. defaults tend to occur more in the short term) while they should generally be backloaded for z>0 (growing economy).

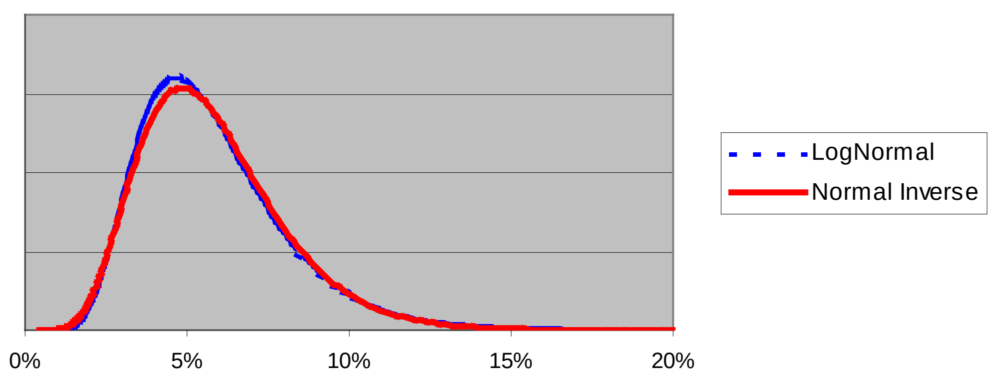
Once these timing assumptions are made, it is possible to compute the above expression for the portfolio Fourier Transform and therefore to invert it in order to find the portfolio default rate distribution.

- **The Single Factor Model for a Large Portfolio of Homogeneous Assets**

In the case of a **large** portfolio of **homogeneous** assets, it is worth mentioning that the Single Factor Model gives an **explicit** formula for the default distribution, the so-called **Normal Inverse distribution**. As suggested by the following chart, the Normal Inverse distribution closely compares to the Lognormal distribution Moody's often uses for these kinds of portfolios.

Normal Inverse Vs Lognormal Distributions

Mean (p) 5.0%/ Std Dev/Mean 40%/ For NI, w adjusted so to have Std Dev/Mean=40%



The explicit formula for the Normal Inverse distribution is directly derived from the Law of the Large Numbers (LLN). According to the LLN, the average of N random variables sharing the same probability distribution converges towards their common mean as N increases. For a given state of the economy $Z = z$, this means that the portfolio default rate $PDR = \sum_{k=1}^{N} s_k X_k = \sum_{k=1}^{N} \frac{1}{N} X_k$ will converge towards its (conditional) expected value when N increases:

$$\mathrm{E}[PDR | Z = z] = p(z) = \Phi\left(\frac{\alpha - w.z}{\sqrt{1-w^2}}\right)$$

(the homogeneity of all the assets means that they all have the same relative size $s_k = 1/N$, the same factor loading $w_k = w$ and the same default threshold $\alpha_k = \alpha$).

The LLN therefore implies that the portfolio default rate (PDR) will be equal to $p(z)$ for a large portfolio (N large) and a given state of the economy $Z = z$. In other words: $PDR = p(Z)$. As a consequence, the cumulative default distribution will be:

$$F_{PDR}(x) = \Pr[PDR < x] = \Pr[p(Z) < x] = \Pr\left[\Phi\left(\frac{\alpha - w.Z}{\sqrt{1-w^2}}\right) < x\right] = \Pr\left[Z > \frac{\alpha - \sqrt{1-w^2}.\Phi^{-1}(x)}{w}\right]$$

which gives the **Normal Inverse distribution**, using the symmetry of the normal distribution for Z :

$$F_{PDR}(x) = \Pr[PDR < x] = \Phi\left(\frac{\sqrt{1-w^2}\Phi^{-1}(x) - \alpha}{w}\right), 0\% < x < 100\%, \alpha = \Phi^{-1}(p).$$

THE FTM APPLIED TO ABS/MBS ANALYSIS

ABS and MBS transactions are characterised by granular portfolios of homogeneous assets. At Moody's, we often use the LNM to model the default distribution of the assets. However the LNM may not always be well adapted to **non granular portfolios** (i.e. with a limited number of assets) nor to **heterogeneous portfolios** (i.e. that present heterogeneities in terms of credit risk or size).

Here are a few examples of non granular or heterogeneous portfolios Moody's has encountered in rating ABS and MBS:

- ✓ Residential mortgage portfolios with an intermediate number of loans (in practice less than 500 loans originated by a small originator);
- ✓ Commercial mortgages + residential mortgages (heterogeneities in terms of size and credit risk and potentially non granularity of the commercial mortgages subportfolios);
- ✓ Real estate leases + car leases + trucks leases + equipment leases (heterogeneities in terms of size and credit risk and potential non granularity for the real estate subportfolio);
- ✓ Residential Mortgages + securities such as investment grade ABS/MBS or corporate bonds (heterogeneities in terms of size and credit risk and non granularity of the securities subportfolio).

The analysis of the risk profile of a given portfolio presenting limited granularity or heterogeneities can be successfully addressed by the FTM using the Single Factor Model. The heterogeneous portfolio must be split in homogeneous subportfolios sharing the same size (s_{SubPF} bucket) and the same default probability (p_{SubPF} bucket). All the N_{SubPF} assets in subportfolio $SubPF$ are assumed to share the same factor loading w_{SubPF}. The table below illustrates this **bucketing process** for a € 500m portfolio comprising 2000 residential mortgages, 200 commercial mortgages and 10 large commercial mortgages.

Example of Buckets for a € 500m Portfolio Comprising 3 Different Subportfolios

Breakdown of the number of loans

	Default Probability	
Individual Asset Size	**5.0%**	**10.0%**
€ 100,000	**2,000**	**-**
€ 1,000,000	**-**	**200**
€ 10,000,000	**-**	**10**

Values for the Factor Loadings (w)

	Default Probability	
Individual Asset Size	**5.0%**	**10.0%**
€ 100,000	**18.93%**	**-**
€ 1,000,000	**-**	**27.63%**
€ 10,000,000	**-**	**27.63%**

The portfolio Fourier transform in equation **(I)** may be re-written as:

$$\hat{f}_{PDR}(t) = \int_{-\infty}^{+\infty} \prod_{SubPF} \left[1 + p_{SubPF}(z).\left(e^{-its_{SubPF}} - 1\right)\right]^{N_{SubPF}} .\phi(z).dz \quad \textbf{(III)},$$

where $p_{SubPF}(z) = \Phi\left(\frac{\alpha_{SubPF} - w_{SubPF}.z}{\sqrt{1 - w_{SubPF}^2}}\right)$ and $\alpha_{SubPF} = \Phi^{-1}(p_{SubPF})$.

Let us come back to the above **example of a mixed mortgage portfolio**. The three factor loadings (the w_{SubPF}'s) were calibrated so have a ratio of std dev. over mean of 40% for the 2000 residential mortgages and of 50% for the commercial mortgages (the 200 intermediate ones and the 10 big ones).[16] The use of equation **(III)** and the inversion techniques described in Appendix 1 permit to numerically compute the following default distribution for the aggregate portfolio:

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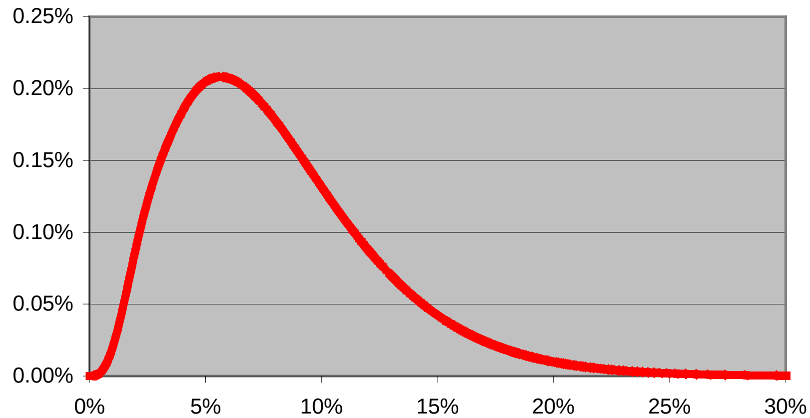
[16] The calibration of the three factor loadings was made assuming three large subportfolios of assets similar to those included in the three subportfolios. More specifically, the three factor loadings have been calibrated so to have a corresponding σ_∞ –such as defined in appendix 3– equal to respectively 40%, 50% and 50% of the respective average default rate of the three subportfolios.

Default Distribution



The tables below provide key figures associated with this default distribution as well as a rough **Aaa** / **Aa2** / **A2** / **Baa2** / equity tranching assuming a 60% recovery rate for all three subportfolios and that all losses are concentrated at year 5.[17] Observe that no **Aa2** tranche can be carved out with this portfolio and the assumed structure. The tranching was done by maximising the size of the **Aaa** tranche and each subsequent subordinated tranches: in fact, it would be possible to structure a small **Aa2** tranche mostly by carving it out of the **Aaa** tranche (and slightly out of the **A2** tranche).

Key Figures about the Default Distribution

95% percentile	**16.2%**
99% percentile	**21.1%**
99.9% percentile	**27.4%**
99.99% percentile	**33.0%**
WADP	**8.0%**
Std Dev. Over Mean	**54.0%**
Equivalent Diversity Score	**39.4**

Rough Tranching

Tranche	**%**
Aaa	**89.7%**
Aa2	**0.0%**
A2	**1.3%**
Baa2	**1.6%**
NR	**7.5%**

Going further and more realistically, assuming a time distribution for defaults and a given waterfall for the cash flows in the deal, it would be possible to determine the expected losses of each tranche and therefore their "quantitative" rating. The use of the default distribution would be exactly the same as described in the BET and LNM Moody's special reports.[18]

[17] The tranching assumes a pure sequential loss allocation with no benefit for excess spread or any other enhancement mechanism.

[18] For a detailed example on how to use a default distribution and a time distribution for defaults in a cash flow model, see Moody's special report "*The Lognormal Method Applied to ABS Analysis*", and particularly the section *An Example of a Lognormal Method Application.*

THE FTM APPLIED TO CDO ANALYSIS

CDO (or CMBS)[19] transactions are characterised by limited portfolios of assets. At Moody's, we most often use the BET to model the default distribution of CDO portfolios. Although the BET is a very flexible method that can be extended in many ways to many complex situations, it is not always applicable. In particular, the BET may not be easily adapted to CDO portfolios with:

- significant heterogeneities in terms of credit risk or size,[20] or
- very few industries,[21] or
- very few assets, typically less than 15,[22] or
- an intermediate number of assets, typically between 250 and 500.[23]

In these cases, the FTM may prove very helpful. As previously mentionned, corporates in the same industry are assumed to be exposed to the same **industry risk factor**.[24] At the same time, corporates across all industries may also be exposed to a **global systemic risk factor**. All factors are assumed independent.[25]

Assuming the industries are A,B,…,X , the portfolio Fourier transform can be written:

$$\hat{f}_{PDR}(t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\dots\int_{-\infty}^{+\infty}\prod_{k=1}^{N}\left[1+p_k\left(z^A,\dots,z^X,z^{Global}\right).\left(e^{-its_k}-1\right)\right].\phi\left(z^A\right)\dots\phi\left(z^X\right).\phi\left(z^{Global}\right).dz^A\dots dz^X.dz^{Global}$$ **(IV)**,

where $p_k(z^A,\dots,z^X,z^{Global}) = \Phi\left(\dfrac{\alpha_k - w_k^{I_k}.z^{I_k} - w_k^{Global}.z^{Global}}{\sqrt{1-\left(w_k^{I_k}\right)^2-\left(w_k^{Global}\right)^2}}\right)$, I_k being the industry of the k[th] debtor.

If no global systemic risk is assumed, all the w_k^{Global}'s are nil and a close look at **(IV)** would lead to the following simplified expression:

$$\hat{f}_{PDR}(t) = \prod_{Industry\ \mathrm{I}}\left(\int_{-\infty}^{+\infty}\prod_{\substack{\text{Debtor in}\\ \text{Industry I}}}\left[1+p_{Debtor}(z).\left(e^{-its_{Debtor}}-1\right)\right]\phi(z)dz\right)$$ **(V)**.

Computation of $\hat{f}_{PDR}(t)$ in equation **(V)** is almost as fast as computation of $\hat{f}_{PDR}(t)$ in equation **(I)** where there is only one factor. Unfortunately, computation of $\hat{f}_{PDR}(t)$ in equation **(IV)** is more time consuming.

Consider the following example:

[19] This section may also be applied to CMBS portfolios where CDO-like techniques are used.

[20] The main assumption of the BET consists in assuming that the aggregate default distribution of the N correlated assets in the portfolio is approximated by the default distribution of D (Diversity Score) uncorrelated assets with similar financial characteristics, including default probability and size. Therefore the BET will not address ideally significant heterogeneities in the portfolio.

[21] Few industries or few assets in the portfolio results in a low portfolio diversity score. With low diversity scores, the outputs of the BET are very sensitive to the value of the diversity score itself.

[22] Cf. Moody's special report "Moody's Approach to Rating-ith-to-Default Basket Credit-Linked Notes," 17 April 2002.

[23] When the number of assets increases and thus when coming closer to the ABS/MBS territory, the computation of the Diversity Score becomes more subtle and certainly less intuitive.

[24] The industry classification can also be done geographically: for instance Agriculture in Europe and Agriculture in Northern America could be considered as two independent industries.

[25] The rationale for the industry classification is precisely to regroup companies into groups considered to behave almost independently from a credit risk perspective.

Example of a Heterogeneous and Poorly Diversified Portfolio

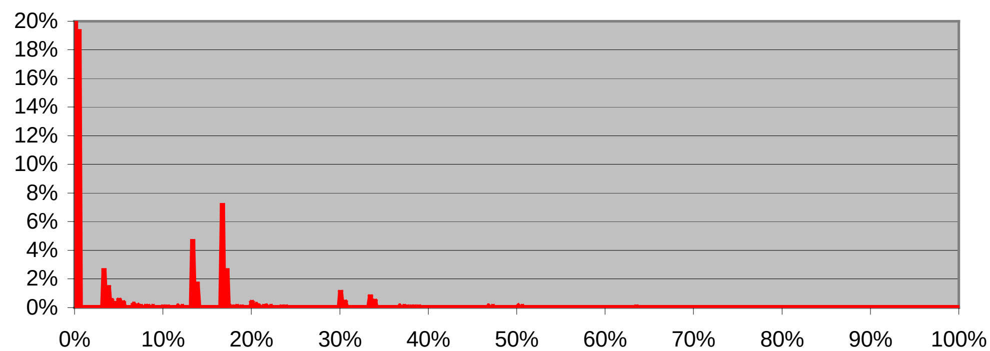
Total Outstanding Amount: € 598m
Weighted Average Rating: Ba1

Bond	*Size (€m)*	*Maturity (Y)*	*Industry*	*Rating*	*Factor Loading*
Bond 1	**200**	**5**	**A**	**Baa2**	**35%**
Bond 2	**40**	**5**	**A**	**A2**	**35%**
Bond 3	**30**	**5**	**A**	**A3**	**35%**
Bond 4	**20**	**5**	**A**	**Ba1**	**35%**
Bond 5	**25**	**5**	**A**	**Baa1**	**35%**
Bond 6	**30**	**5**	**A**	**A2**	**35%**
Bond 7	**70**	**5**	**A**	**Aa3**	**35%**
Bond 8	**3**	**5**	**A**	**B3**	**35%**
Bond 9	**100**	**5**	**B**	**Ba3**	**40%**
Bond 10	**80**	**5**	**B**	**Ba2**	**40%**

Using equation **(V)** and the inversion techniques of Appendix 1 leads to the default distribution[26] depicted in the two following charts with two different scales on the Y axis:

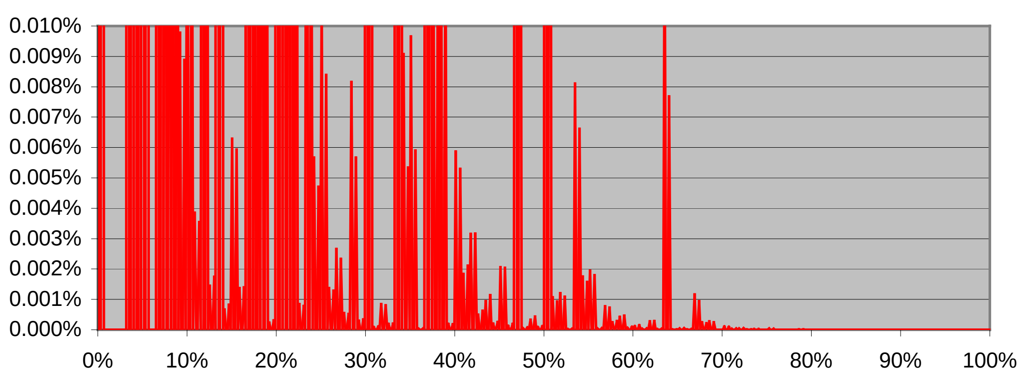
Default Distribution

(Truncated on the Y axis at 20%)



Default Distribution

(Truncated on the Y axis at 0.01%)



[26] For each bond in the portfolio, the default probability over the 5 year horizon was deduced from the bond's rating by dividing the associated expected loss –as shown in Appendix 4– by an assumed average LGD of 55%. As with the BET, please note that stress factors should be applied on these default probabilities in order to account for the fact that a rating corresponds to a range of expected loss values and not to a single value of expected loss. For the sake of simplicity, these stress factors were not applied in the given example.

These charts illustrate the peculiar nature of the default distribution of small and heterogeneous CDO portfolios:

- the default distribution is not as “smooth” as the distribution of a granular portfolio: it is characterised by spikes,
- there are ranges of default rate values that are impossible or very unlikely (in the example: 1% to 3%, 5% to 6%, 19% to 20%, 32% to 33%, 47% to 50%, etc.).

The tables below provide key figures associated with this default distribution and a rough **Aaa** / **Aa2** / **A2** / **Baa2** / **Ba2** / equity tranching assuming a 30% recovery rate and that all losses are concentrated at year 5.[27] No **Baa2** tranche can be carved out with this portfolio and the assumed structure. As already mentionned in the previous section, a more realistic way to compute the notes’ expected losses and their associated “quantitative” ratings would consit in incorportating the default distribution in a cash flow model that requires assumptions about the time distribution of defaults.

Key Figures about the Default Distribution

No Default Probability	54.6%
95% percentile	17.2%
99% percentile	33.4%
99.9% percentile	50.7%
99.99% percentile	64.0%
WADP	4.1%
Std Dev. Over Mean	196.3%
Equivalent Diversity Score	6.1

Rough Tranching

Tranche	%
Aaa	59.7%
Aa2	5.3%
A2	11.5%
Baa2	0.0%
Ba2	17.4%
NR	6.1%

[27] The tranching assumes a pure sequential loss allocation with no benefit for excess spread or any other enhancement mechanism.

FURTHER USES OF THE FTM

The FTM may also be easily adapted and used for instance to determine the **loss distribution,** the **market value distribution** and the **return distribution** of a portfolio of assets during a given time horizon. We will specifically focus on the loss distributions.

- **The Theoretical Background for Loss Distributions**

The first step of the FTM requires to determine the Fourier transform of the portfolio loss rate (that is the portfolio Fourier transform). Using the same terminology as the one introduced in the *Single Factor Model* section of the *Underlying Models* part, the loss rate of the portfolio over the time horizon is:

$$Portfolio\ Loss\ \text{Rate (PLR)} = \frac{S_1 X_1 LGD_1 + S_2 X_2 LGD_2 + \ldots + S_N X_N LGD_N}{S_1 + S_2 + \ldots + S_N} = \sum_{k=1}^{N} s_k X_k LGD_k \,,$$

where X_k and LGD_k are respectively the default indicator and the Loss Given Default (or indifferently the loss severity) for the k[th] asset. The $LGD_k{}'s$ are random variables that typically range between 0% and 100%.

For a given state of the economy $Z = z$, the conditional Fourier transform of the portfolio loss distribution ($f_{PLR}(x), 0 < x < 100\%$) is therefore under the Single Factor Model:

$$\hat{f}_{PLR/Z=z}(t) = \mathrm{E}\left[e^{-it.\mathrm{PLR}} \middle| Z = z\right] = \mathrm{E}\left[e^{-it\left(s_1 X_1 LGD_1 + s_2 X_2 LGD_2 + \ldots + s_N X_N LGD_N\right)} \middle| Z = z\right].$$

As seen previously, the default indicators X_k are conditionally independent given $Z = z$. Assuming the same conditional independence for the $LGD_k{}'s$:[28]

$$\hat{f}_{PLR/Z=z}(t) = \mathrm{E}\left[e^{-its_1 X_1 LGD_1} \middle| Z = z\right] \times \mathrm{E}\left[e^{-its_2 X_2 LGD_2} \middle| Z = z\right] \times \ldots \times \mathrm{E}\left[e^{-its_N X_N LGD_N} \middle| Z = z\right].$$

For the k[th] asset:

$$\mathrm{E}\left[e^{-its_k X_k LGD_k} \middle| Z = z\right] = \Pr\left(X_k = 0 \middle| Z = z\right).\mathrm{E}\left[e^{-its_k .0.LGD_k} \middle| Z = z\right] + \Pr\left(X_k = 1 \middle| Z = z\right).\mathrm{E}\left[e^{-its_k .1.LGD_k} \middle| Z = z\right]$$
$$= \left(1 - p_k(z)\right).1 + p_k(z).\hat{f}_{LGD_k|Z=z}(ts_k) = 1 + p_k(z).\left(\hat{f}_{LGD_k|Z=z}(ts_k) - 1\right)$$

Finally, the **conditional** portfolio Fourier transform will be given by:

$$\hat{f}_{PLR|Z=z}(t) = \prod_{k=1}^{N} \left[1 + p_k(z).\left(\hat{f}_{LGD_k|Z=z}(ts_k) - 1\right)\right],$$

and the **unconditional** portfolio Fourier transform will therefore be:

$$\hat{f}_{PLR}(t) = \mathrm{E}\left[\hat{f}_{PLR|Z}(t)\right] = \int_{-\infty}^{+\infty} \hat{f}_{PLR|Z=z}(t).\phi(z).dz \,,$$

$$\boxed{\hat{f}_{PLR}(t) = \int_{-\infty}^{+\infty} \prod_{k=1}^{N} \left[1 + p_k(z).\left(\hat{f}_{LGD_k|Z=z}(ts_k) - 1\right)\right] \phi(z) dz} \quad \textbf{(VI)},$$

[28] This assumption means that the potential correlation between LGDs would only be created *through* a common dependence to the factor Z, the very one that is the source of default correlation between the debtors. This assumption may be a bit simplistic in certain cases. Consider a portfolio of residential mortgage loans: the main factor driving defaults would rather be the growth of the economy (for instance measured by the GDP growth), while the factor driving the LGDs would be the state of the real estate market. Both factors are certainly strongly correlated but are nonetheless different.

where $p_k(z)=\Phi\left(\frac{\alpha_k - w_k.z}{\sqrt{1-w_k^2}}\right)$ and $\phi(z)=\frac{1}{\sqrt{2\pi}}\exp\left(-z^2/2\right)$.

Comparing **(I)** and **(VI)** shows that the portfolio default rate or the portfolio loss rate have very similar Fourier transforms: the loss Fourier transform is obtained by substituting e^{-its_k} by $\hat{f}_{LGD_k|Z=z}(ts_k)$ in the expression of the default Fourier transform.

There are three types of modelling assumptions for the LGDs: fixed LGDs, variable LGDs or variable LGDs correlated with the defaults in the portfolio. Each of these assumptions gives a different expression for $\hat{f}_{LGD_k|Z=z}(t)$ and leads to a different computation formula for the portfolio Fourier transform $\hat{f}_{PLR}(t)$.

- **Computation of the Portfolio Loss Distribution with Fixed LGDs**

The LGDs are assumed to be deterministic. The LGD for the k[th] asset takes a fixed value $LGD_k = \mu_k$. In that case, $\hat{f}_{LGD_k|Z=z}(t)=\mathrm{E}\left[e^{-itLGD_k}\right]=e^{-it\mu_k}$ and equation **(VI)** becomes:

$$\hat{f}_{PLR}(t)=\int_{-\infty}^{+\infty}\prod_{k=1}^{N}\left[1+p_k(z).\left(e^{-its_k\mu_k}-1\right)\right]\phi(z).dz\ .$$

- **Computation of the Portfolio Loss Distribution with Variable LGDs**

The LGDs are assumed to be random variables ranging between 0% and 100%. In addition, the LGDs are assumed to be independent from the factor Z and therefore from the portfolio default rate. Here are the formulas[29] giving $\hat{f}_{LGD_k|Z=z}(t)=\hat{f}_{LGD_k}(t)$ for some workable LGD distributions with mean μ and standard deviation σ:

- ❑ **Normal LGD:[30]** $\hat{f}_{LGD}(t)=\int_{-\infty}^{+\infty}f_{LGD}(x).e^{-itx}.dx=\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}.e^{-\frac{(x-\mu)^2}{2\sigma}}.e^{-itx}.dx=e^{-it\mu}.e^{-\sigma^2t^2/2}$

- ❑ **Triangular LGD:** $\hat{f}_{LGD}(t)=e^{-it\mu}.\left[sc\left(\frac{\sqrt{6}\sigma t}{2}\right)\right]^2$, where $sc(u)=\frac{\sin u}{u}$ for $u\neq 0$ and $sc(0)=1$

- ❑ **Gamma LGD:** $\hat{f}_{LGD}(t)=(1+i\beta t)^{-\alpha}$, where $\alpha=\frac{\mu^2}{\sigma^2}$ and $\beta=\frac{\sigma^2}{\mu}$

- ❑ **Beta LGD:** Beta distributions are a common assumption (that seems empirically justified) for LGD distributions. Unfortunately, there does not seem to be any closed form formula for the Fourier transform. It can however be approximated by the Fourier transform of the its discretised distribution :

$$\hat{f}_{LGD}(t)=\frac{1}{B(n,p).M}.\sum_{k=0}^{M-1}\left(\frac{k}{M}\right)^{n-1}.\left(1-\frac{k}{M}\right)^{p-1}.e^{-it\frac{k}{M}},$$

[29] These formulas are obtained by applying the Fourier Transform definition, as it can be seen for the Normal LGD.

[30] The Normal, Triangular or the Gamma distributions may not always be fully suitable LGD distributions: high standard deviations or low means may yield non negligible probabilities for negative LGD values or LGD values larger than 100%. However, the impact is generally limited to the left part of the loss distribution and does not affect its tail (it is a consequence of the Law of the Large Numbers –see next footnoote for more details).

where M is the number of discretisation points of the LGD distribution, $B(n,p)=\frac{\Gamma(n).\Gamma(p)}{\Gamma(n+p)}$, $n=\mu.\left[\frac{(1-\mu)\mu}{\sigma^2}-1\right]$ and $p=(1-\mu).\left[\frac{(1-\mu)\mu}{\sigma^2}-1\right]$.

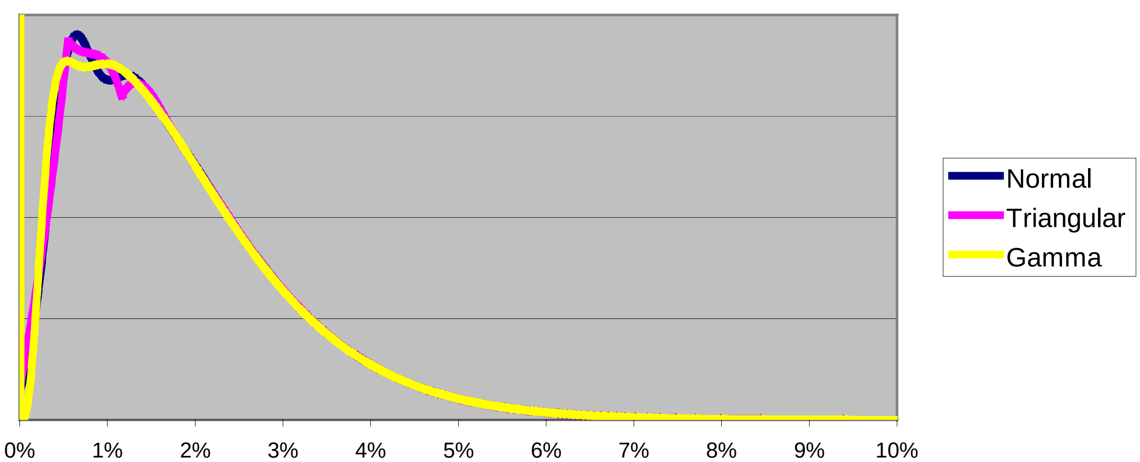
As a consequence, the portfolio Fourier transform of a portfolio with normally distributed LGDs will be:

$$\hat{f}_{PLR}(t)=\int_{-\infty}^{+\infty}\prod_{k=1}^{N}\left[1+p_k(z).\left(e^{-its_k\mu_k}.e^{-t^2\sigma_k^2s_k^2/2}-1\right)\right]\phi(z).dz\ ,$$

where μ_k and σ_k are the respective mean and standard deviation for LGD_k.

The chart below depicts different loss distributions for different types of LGD distributions. Observe that the tail of the loss distribution does not depend on the assumed LGD distribution.[31]

Portfolio Loss Distributions for Different Kinds of LGD Distributions
(100 assets / same size / p=3% / w=20% / MeanLGD=55% / StDevLGD=25%)

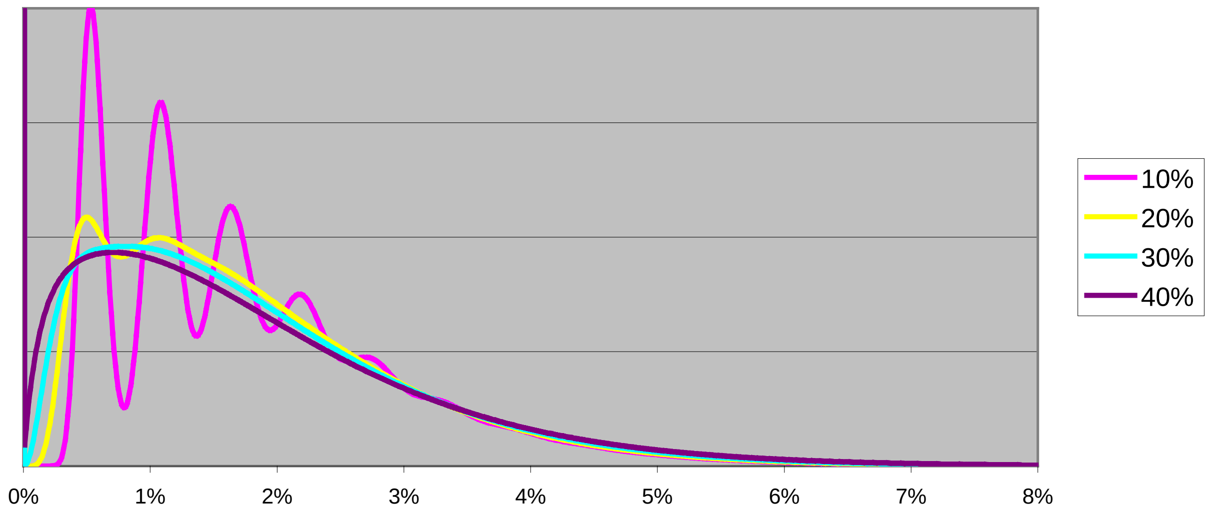


The chart below shows the evolution of the shape of the loss distribution when the standard deviation of the LGD increases while the following table provides the key numbers associated with these loss distributions, including a rough **Aaa** / **Baa2** / equity tranching.[32] It should be noticed that the expected losses in the portfolio do not change and are equal to the product of the average default probability and the LGD mean. As expected, an increase of the LGD volatility translates into increased unexpected losses.

[31] According to the Law of the Large Numbers, the average of a given number of random variables sharing the same probability distribution converges towards their common mean as their number increases. As a direct consequence, the average LGD will be almost equal to its expected value for high default rates: the right tail of the loss distribution won't depend on the shape of the LGD distribution for each asset. Going further, as a consequence of the Central Limit Theorem, the average LGD distribution given n default even converges towards a normal distribution (with a standard deviation decreasing with the inverse of the square root of n or the default rate) whatever may be the shape of the individual LGD distribution (provided however that the LGDs have the same probability distribution and are independent).

[32] The tranching assumes a pure sequential loss allocation with no benefit for excess spread or any other credit enhancement mechanism and that all losses occur at year 5.

Portfolio Loss Distributions for Different Values of LGD Std Deviations
(100 assets / same size / p=3% / w=20% / Mean LGD=55% / Gamma LGD distributions)



Impact of the LGD Std Deviation on the Loss Distribution

Portfolio and LGD Characteristics: 100 assets / same size / p=3% / w=20% / Mean LGD =55% / Gamma LGD distributions

LGD Std Dev	0%	10%	20%	30%	40%
Mean Loss	1.65%	1.65%	1.65%	1.65%	1.65%
Std Dev. Loss	1.21%	1.23%	1.26%	1.32%	1.40%
Std Dev./Mean	73.61%	74.35%	76.54%	80.06%	84.70%
99% percentile	5.51%	5.42%	5.56%	5.79%	6.11%
99.9% percentile	7.16%	7.36%	7.57%	7.93%	8.44%
99.99% percentile	9.36%	9.21%	9.49%	9.99%	10.64%
Aaa	93.45%	93.36%	93.14%	92.77%	92.26%
Baa2	2.23%	2.29%	2.44%	2.69%	3.01%
NR	4.32%	4.34%	4.42%	4.55%	4.73%
Credit Enhancement Aaa	6.55%	6.64%	6.86%	7.23%	7.74%

- **Computation of the Portfolio Loss Distribution with Variable LGDs Correlated With the Defaults in the Portfolio**

As with variable LGDs, the variable correlated LGDs are assumed to be random variables ranging between 0% and 100%. However, the LGDs are assumed to be correlated with the portfolio default rate (PDR). Recent studies[33] suggest that this correlation level was approximately of 20% to 30% for US bonds during the three past decades.

Assumptions on the LGDs

A (positive) correlation between defaults in the portfolio and LGDs reflects that, on average, LGDs increase when the porfolio default rate increases. Let $\mu_k(z)$ designate the expected LGD for the k[th] asset conditionally to the state of the economy Z=z:

$$E\left[LGD_k \middle| Z = z = \mu_k(z)\right].$$

[33] See Appendix 5: References.

$\mu_k(z)$ is expected to be a decreasing function of the state of the economy Z=z: on average, LGDs tend to decrease during bad times for the economy while they rather decrease during good times. A simple and natural choice is to assume a linear relationship between $\mu_k(z)$ and z :

$$\mu_k(z) = \mu_k - \Sigma_k.z \qquad \textbf{(VII)},$$

where $\mu_k = E[LGD_k]$ and Σ_k is a positive coefficient that will further be determined in function of correlation assumptions between LGD_k and the defaults in the portfolio.[34]

Let us make a further natural assumption on LGD_k :

$$LGD_k = \mu_k(Z) + \eta_k \qquad \textbf{(VIII)},$$

where η_k and Z (or $\mu_k(Z)$) are independent random variables. We obviously have $E[\eta_k] = E[\eta_k | Z = z] = 0$.

Correlation inputs

Natural inputs to consider in the modelling process would be the correlation between LGD_k and the portfolio default rate PDR , that is $Corr(LGD_k, PDR)$. However the theoretical expressions for $\hat{f}_{LGD_k|Z=z}(t)$ quickly become complicated. It proves much easier to consider the correlation between the LGD_k's and the factor Z , that is:

$$\theta_k = -Corr(LGD_k, Z), \text{ where } -100\% \le \theta_k \le 100\% \qquad \textbf{(IX)}.$$

Indeed the factor Z drives the defaults in the portfolio. As a matter of fact, it can be verified that $Corr(PDR, Z)$ is very close to –100% for many portfolios: in practice, the θ_k's may be considered as the level of correlation between the portfolio default rate (PDR) and the corresponding LGDs (the LGD_k's) .

Expression for the Portfolio Fourier Transform

Equation **(VI)** for the portfolio Fourier Transform $\hat{f}_{PLR}(t)$ requires an explicit expression for $\hat{f}_{LGD_k|Z=z}(t)$. We will naturally assume that all the conditional distributions for $LGD_k | Z = z$ belong to the same family of distributions when z fluctuates. For instance, assuming that all conditional distributions for $LGD_k | Z = z$ are normal distributions will yield:

$$\hat{f}_{LGD_k|Z=z}(t) = \exp\left(-i.\mu_k(z).t - \sigma_k^2(z).t^2/2\right),$$

where $\sigma_k(z) = \sqrt{Var(LGD_k | Z = z)}$ is the conditional standard deviation of $LGD_k | Z = z$. The only

[34] More elaborate choices may be made for the function $\mu_k(z)$. Let $F(x) = \Pr[\mu_k(Z) < x]$ be the cumulative distribution function of the LGD average , i.e. the r.v. $\mu_k(Z)$. In that case, it can be proved that $\mu_k(z) = F^{-1}[\Phi(-z)]$.

unknown variables in the expression above (as with other expressions for $\hat{f}_{LGD_k|Z=z}(t)$ when other families of distribution are used) are $\mu_k(z)$ and $\sigma_k(z)$. They will be determined thanks to **(VII)**, **(VIII)** and **(IX)**.

From **(IX)**:

$$-\theta_k = Corr[LGD_k, Z] = \frac{Cov[LGD_k, Z]}{\sqrt{Var(LGD_k)}.\sqrt{Var(Z)}.} = \frac{Cov[\mu_k - \Sigma_k.Z - \eta_k, Z]}{\sigma_k.1} = \frac{0 - \Sigma_k - 0}{\sigma_k.1} = -\frac{\Sigma_k}{\sigma_k},$$

which means:

$$\boxed{\Sigma_k = \theta_k.\sigma_k}$$

and:

$$\boxed{\mu_k(z) = \mu_k - \theta_k.\sigma_k.z}$$

From **(VIII):**

$$Var(LGD_k) = \sigma_k^2 = Var[\mathrm{E}(LGD_k|Z)] + \mathrm{E}[Var(LGD_k|Z)] = Var[\mu_k(Z)] + \mathrm{E}[Var(\eta_k)] = \Sigma_k^2 + Var(\eta_k).$$

Therefore:

$$\sigma_k^2(z) = Var(\eta_k) = \sigma_k^2 - \Sigma_k^2 = \sigma_k^2(1-\theta_k^2),$$

and:

$$\boxed{\sigma_k(z) = \sigma_k.\sqrt{1-\theta_k^2}}$$

Coming back to the example of normal conditional LGD distributions:

$$\hat{f}_{LGD_k|Z=z}(t) = \exp[-i.(\mu_k - \theta_k.\sigma_k).t - \sigma_k^2.(1-\theta_k^2).t^2/2],$$

which permits to compute the portfolio Fourier transform with **(VI)**:

$$\hat{f}_{PLR}(t) = \int_{-\infty}^{+\infty}\prod_{k=1}^{N}[1 + p_k(z).(\exp[-i.(\mu_k - \theta_k.\sigma_k).s_k.t - \sigma_k^2.(1-\theta_k^2).t^2.s_k^2/2] - 1)].\phi(z)dz.$$

The chart below depicts different loss distributions for different levels of correlation between defaults and LGDs. As in the previous section, a table then provides key numbers associated with these loss distributions, including a rough **Aaa** / **Baa2** / equity tranching.[35]

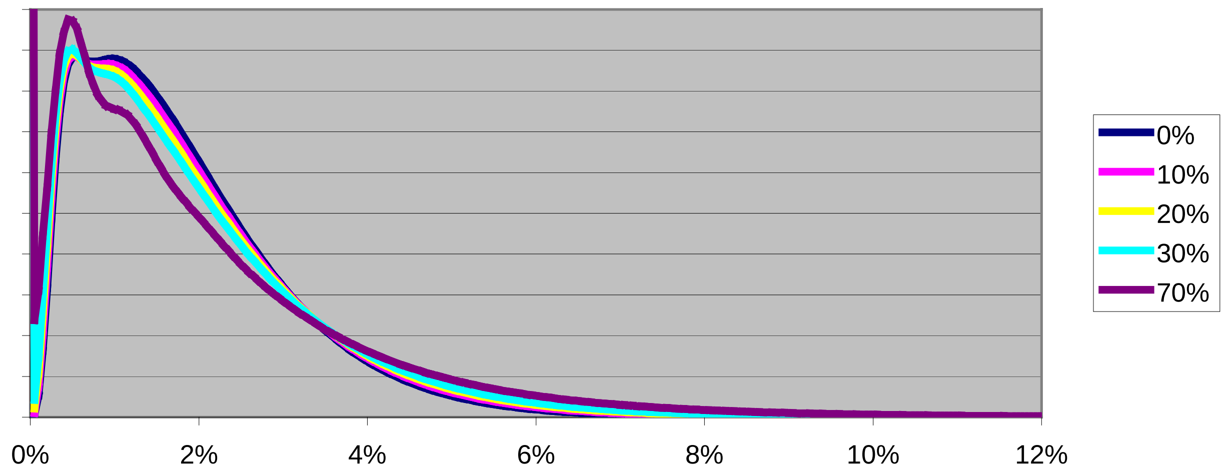
Observe that the expected losses in the portfolio increase with the correlation between defaults and LGDs: in the case where LGDs are correlated to defaults, the expected losses in the portfolio are no longer the product of the average default probability and the LGD mean (in the example, respectively 3.0% and 55% regardless of the LGD/default correlation level).

Note also (as a rule of thumb) that the additional required **Aaa** credit enhancement resulting from the correlation between LGDs and defaults corresponds more or less to the correlation percentage itself. For instance, for a correlation of 20%, the required **Aaa** CE is of 8.66%, which represents an increase of 23.3% (close to 20%) compared to the case where there is no such LGD/default correlation.

[35] The tranching assumes a pure sequential loss allocation with no benefit for excess spread or any other credit enhancement mechanism and that all losses occur at year 5.

Portfolio Loss Distributions for Different Levels of Correlation Between LGDs and Defaults in the Portfolio

(100 assets / same size / p=3% / w=20% / Mean LGD=55% / Std Dev LGD =25% / Gamma conditional LGD Distributions)



Impact of the correlation between LGDs and Defaults on the Loss Distribution

Portfolio and LGD Characteristics: 100 assets / same size / p=3% / w=20% / Mean LGD =55% / Std Dev. LGD = 25% / Gamma conditional LGD distributions

Correlation LGD/Default	0%	10%	20%	30%	70%
Mean Loss	1.65%	1.68%	1.72%	1.75%	1.89%
Std Dev Loss	1.29%	1.36%	1.44%	1.52%	1.87%
StdDev/Mean	78.1%	81.0%	83.9%	86.9%	98.9%
99% percentile	5.65%	6.04%	6.44%	6.85%	8.58%
99.9% percentile	7.72%	8.44%	9.20%	9.98%	13.18%
99.99% percentile	9.70%	10.82%	11.99%	13.20%	17.60%
Aaa	92.97%	92.19%	91.34%	90.43%	86.73%
Baa2	2.55%	3.18%	3.90%	4.66%	7.71%
NR	4.48%	4.62%	4.76%	4.91%	5.56%
CE Aaa	7.03%	7.81%	8.66%	9.57%	13.27%
% chge CE Aaa / no correlation	0.0%	11.1%	23.3%	36.2%	88.9%

CONCLUSION

The FTM is a powerful tool. It presents interesting advantages compared to Monte Carlo-based approaches. The strongest feature of the FTM is its speed: in most cases, it runs in five to thirty seconds. At the same time, it gives accurate outputs and therefore avoids the convergence issues that are encountered with Monte Carlo simulations. Its adaptability permits its use for a wide range of portfolios. It is also simple to implement with common spreadsheet softwares such as Microsoft Excel, thanks to their pre-programmed Fast Fourier Transform (FFT) routines.

The **FTM is only a tool, however**. As with any other modelling methodology, one must be aware of its **limits**:

- The FTM is a **one-period model**: it does not address the issue of the **timing of the defaults or the losses** in the analysed portfolio. Additional timing assumptions should be made in the cash flow analysis integrating the default or loss distribution generated by the FTM.
- The FTM makes **assumptions on how default correlation is created between assets**, namely through exposure to systemic risk. Besides, the FTM requires an **adequate calibration of the level of correlation** to the systemic risk for each asset.

Moreover, the FTM only deals with quantifying the aggregate risk profile of a portfolio of assets. Although an important step, it is **only one step in Moody's overall rating process** for a structured finance deal. Analysing, understanding and questioning the data is often a more critical step than the choice of the modelling methodology itself.

Moody's views the FTM as an additional quantitative methodology to analyse the risk profile of complex portfolios of assets. We intend to use it mainly once our usual methods become difficult to apply. We may also use the FTM in conjunction with other methods such as Monte Carlo simulations, as we prefer not to rely on a single model in the case of complex transactions.

We believe that – apart from being a useful tool for the analysis of structured finance transactions – the FTM can also **help credit risk professionals** to model, understand and quantify complex credit risk phenomena at a portfolio level, for instance, in the complex cases where defaults and LGDs are correlated.

APPENDIX 1: COMPUTER IMPLEMENTATION OF THE INVERSE FOURIER TRANSFORM

INTRODUCTION

The first step of the FTM – the aggregation step – makes it possible to compute the portfolio Fourier transform $\hat{f}(t)$ for any value of t. This technical appendix deals with the practical implementation of the second step of the FTM, the inversion step. Precisely, this appendix aims at answering the following question: **knowing the portfolio Fourier transform $\hat{f}(t)$, how is it possible to compute its inverse Fourier transform, i.e. the portfolio distribution $f(x)$?**

Answering this question requires a minimum understanding of some key notions or tools like the *Discrete Fourier Transform*, the *Discrete Inverse Fourier Transform*, the *Fast Fourier Transform algorithm*, the *MS Excel Fourier Analysis tool*. ***We must stress that any computer implementation of the inversion step of the FTM will prove very difficult or even impossible without this minimum understanding. This being said, readers mostly interested in the computer implementation may skip the theoretical sections below in a first step and only refer to the two grey "Inversion Recipe" textboxes.***

The inversion must be computed differently depending on the discrete or continuous nature of the portfolio default/loss distribution $f(x)$. Portfolio default distributions are generally discrete distributions: the number of possible values for the default rate is at most 2^N for a portfolio of N assets. They converge to a continuous distribution for large values of N. Portfolio loss distributions are also discrete distributions if LGDs are assumed to take discrete values between 0% and 100%, while they are (almost) continuous distributions if the LGDs are assumed to take continuous values between 0% and 100%.

Before going any further, let us notice that the kind of distributions $f(x)$ we are trying to determine are concentrated in an interval $[0;V_{\max}]$: $f(x)$ is equal to zero outside this interval. For instance, if $f(x)$ is a default distribution, $V_{\max}$ can be equal to 100% if the distribution refers to the default rate distribution (there cannot be more defaults than 100% of the portfolio principal outstanding); if $f(x)$ refers to the defaulted amount distribution, $V_{\max}$ can be equal to the total principal outstanding of the portfolio.

In practice, it is more interesting to **choose the lowest possible value for $V_{\max}$ in order to reduce the computation time**. For instance, we may ***a priori*** think that the default/loss rate distribution will be concentrated in the interval $[0\%;25\%]$; defaults/losses larger than 25% are theoretically possible but their probability is so small – for instance lower than 1E-14, close to the double precision computation limit for most computers – that their corresponding probabilities can be assumed to be zero. In that case, we may make an ***a priori* guess** of $V_{\max} = 25\%$ and run the computation in order to get the default rate distribution $f(x)$; this *a priori* guess for $V_{\max}$ would however **need to be validated *ex post* by checking that the probabilities obtained for the end of the interval** – let's say in the 24% to 25% area for the sake of the example[36] – **are effectively negligible.** Making such an *a priori* guess for $V_{\max}$ speeds the computation of the distribution $f(x)$. For instance, in the previous case, the computer won't lose time in computing the default/loss rate distribution between 25% and 100%, because it will have been assumed to be equal to zero.

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[36] As a matter of fact, the end of the computation interval will be larger than $V_{\max}$ (see point 6 in the first *Inversion Recipe* grey textbox: $x_{\mathrm{Re}\,sF-1} = (\mathrm{Re}\,sF - 1).\Delta x$, the last point of the computation interval, is larger than $V_{\max}$).

COMPUTING DISCRETE DISTRIBUTIONS $f(x)$

- ***Determination of the Step of the Distribution*** Δx

Let Δx designate the "Greatest Common Divisor" (GCD) of the discrete values that may be taken by the default (or loss) distribution. The distribution will take its values at the discrete points $x_n = n.\Delta x, \mathrm{n} = 0,\ldots,\mathrm{N}_{\max} - 1$, where $\mathrm{N}_{\max} = Int[V_{\max}/\Delta x] + 1$: Δx will be the **distribution's *step***.

If we refer to a default distribution, the **default rate step** Δx will be the GCD of the size percentages s_k of the assets. If we refer to a discrete loss distribution, the **loss rate step** will be the product of the default rate step and the LGD step.

For instance, if we consider the default distribution of a portfolio of three assets with respective principal outstanding amount of € 3.5 m, € 8.5 m and € 10.0 m, the "GCD" of the principal outstanding amounts will be of € 0.5 m,[37] and therefore $\Delta x = 0.5/(3.5+8.5+10.0) = 0.5/22 \approx 2.27\%$. If we add a € 5.56m asset to this portfolio, the GCD of the principal outstanding amounts is changed to € 0.02m[38] and therefore $\Delta x = 0.02/(3.5+8.5+10.0+5.56) = 0.02/27.56 \approx 0.07\%$.

For a discrete LGD distribution taking values at $0\%, 5\%, \ldots, 100\%$, the LGD step will be of 5% . As a result, the step of the loss distribution will be of $2.27\% \times 5\% \approx 0.11\%$ for the portfolio of three assets and of $0.07\% \times 5\% \approx 0.0035\%$ for the portfolio of four assets.

Computing the discrete distribution $f(x)$ consists in determining the f_n's, the "values" that $f(x)$ "takes" at the points $x_n = n.\Delta x, \mathrm{n} = 0,\ldots,\mathrm{N}_{\max} - 1$ (recall that $f(x)$ is equal to zero for values of x different from the x_n's).

- ***Theoretical Expression for the Distribution*** $f(x)$ ***and its Fourier Transform*** $\hat{f}(t)$

Strictly speaking, the last sentence in the previous section is an abuse of language. Strictly speaking, $f(x)$ was not defined as the probability distribution function. It was defined as the density probability function. However, the density function $f(x)$ for a <u>discrete</u> probability distribution is not a classical function taking values at some points but rather as a ***mathematical distribution***[39] which is a sum of *"Dirac spikes"* with energy f_n concentrated at the points $x_n = n.\Delta x, \mathrm{n} = 0,\ldots,\mathrm{N}_{\max} - 1$:

$$\boxed{f(x) = \sum_{n=0}^{N_{\max}-1} f_n.\delta_{n\Delta x}(x)} \quad \textbf{(A)},$$

where $\delta_a(x)$ is the so called ***Dirac distribution*** or *Dirac delta function* at point a:

- ❑ $\delta_a(x) = 0 \ \ for \ x \neq a$, and
- ❑ $\int_{-\infty}^{+\infty} \delta_a(x).dx = 1$.

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[37] $0.5 = GCD(3.5\times 10; 8.5\times 10; 10\times 10)/10$: all outstanding amounts are multiplied by 10 so as to make them integers.

[38] $0.02 = GCD(3.5\times 100; 8.5\times 100; 10\times 100; 5.56\times 100)/100$.

[39] For more details on the Theory of Distributions, see for instance: *"Introduction to the Theory of Distributions"*, F.G. Friedlander, Cambridge University Press, 1982 *or "The Analysis of Linear Partial Differential Operators: distributions theory and Fourier analysis",* L. Hörmander, Springer Verlag, NY, 1990.

For readers not accustomed to mathematical distributions, let us mention how equation **(A)** is connected with the classical discrete probabilities. The cumulative default distribution function implied by **(A)** is:

$$F_{PDR}(X) = \Pr(PDR \le X) = \int_{-\infty}^{X} f(x).dx = \sum_{x_n < X} f_n \int_{-\infty}^{X} \delta_{x_n}(x).dx = \sum_{x_n < X} f_n ,$$

which is equivalent to say that the discrete probability $\Pr(PDR = x_n)$ is equal to f_n.

Going further, it can be proved that the Fourier transform of the *Dirac distribution* $\delta_a(x)$ is the function e^{-iat}. Therefore after application of the Fourier transform to equation **(A)** :

$$\boxed{\hat{f}(t) = \sum_{n=0}^{N_{\max}-1} f_n.e^{-in\Delta xt} = \sum_{n=0}^{N_{\max}-1} f_n.e^{-in\frac{2\pi}{T}t}}$$ **(B)**, where $T = \dfrac{2\pi}{\Delta x}$.

Observe that $\hat{f}(t)$ is a periodic function of t, with a period T: $\hat{f}(t) = \hat{f}(t+T)$ (remember that $e^{-i2\pi} = 1$). As a matter of fact, it is not difficult to see that **discrete probability distributions** are characterised by a **periodic Fourier transform.**

At this stage, recall that $\hat{f}(t)$ can be computed for any value of t (for instance with **(VI)**). Equation **(B)** and the *Discrete Fourier Transform* theoretical framework will lead to the values of the f_n's, $n = 0 \text{ to } N_{max} - 1$.

- ***The Discrete Fourier Transform Framework***

The ***Discrete Fourier Transform*** of a given vector $[h_0, h_1, \ldots, h_{M-1}]$ of M complex numbers is defined itself as a vector of M complex numbers $[H_0, H_1, \ldots, H_{M-1}] = DFT([h_0, h_1, \ldots, h_{M-1}])$ where:

$$\boxed{H_m = DFT([h_0, h_1, \ldots, h_{M-1}])_m = \sum_{k=0}^{M-1} h_k.e^{-2\pi ikm/M}, \quad m = 0,\ldots, M-1}.$$

Similarly the ***Discrete Inverse Fourier Transform*** of the vector $[H_0, H_1, \ldots, H_{M-1}]$ is defined as the vector $[h_0, h_1, \ldots, h_{M-1}] = DIFT([H_0, H_1, \ldots, H_{M-1}])$ where:

$$\boxed{h_m = DIFT([H_0, H_1, \ldots, H_{M-1}])_m = \frac{1}{M}.\sum_{k=0}^{M-1} H_k.e^{2\pi ikm/M}, \quad m = 0,\ldots, M-1}.$$

Like their continuous Fourier transform counterparts $\hat{f}$ and $\breve{g}$, the DFT and the DIFT vectors are defined in almost the same way: the only differences are a different sign in the exponential and a normalising factor ($1/M$) for the DIFT. The **inversion formula also holds for the discrete case**. If the DFT is applied to a given vector, then the DIFT to the resulting vector, it yields the original vector; that means $DIFT = DFT^{-1}$ (hence the name of the Discrete Inverse Fourier Transform).

The major interest of the DFT and the DIFT comes from the existence of the ***Fast Fourier Transform (FFT)*** algorithms. A rapid examination of the above formulas would suggest that the DFT or the DIFT require in the order of M^2 computer operations (a number of operations – additions, multiplications…– linear in M is required to compute each of the M component of the transform vector). In fact, the FFT algorithms allow the computation of the DFT or the DIFT in the order of $M.Log(M)$ operations. The gain $M/Log(M)$ is absolutely considerable in terms of computing time. For example, for $M=10^6$, it would take approximately 2 weeks of CPU time for a microsecond cycle time computer to do what could be done in 30 seconds of CPU time with the FFT.[40] FFT's only constraint is that M must be a power of 2, i.e. $M=2^p$.

M, the number of points of the vectors of complex numbers, will be referred to as the **Fourier resolution**: the need for higher accuracy (i.e. the need to have a good "resolution" to be able to capture enough details) will indeed mean larger values for M.

FFT computations of the DFT and the DIFT are already implemented under ***MS Excel*** in its Analysis ToolPak add-in.[41] To install the Analysis ToolPak add-in in MS Excel, go to Tools/Adds-In. To compute a DFT or a DIFT of a vector of 2^p complex numbers, go to Tools/Data Analysis/Fourier Analysis. Complex numbers are in the format "x+/-yi" or "x+/-yj" in MS Excel and are treated as strings (for instance "5.32-7.213i").

There is however one important practical constraint to remember: the maximum available Fourier resolution for the DFT under MS Excel is $2^{12}=4096$. It can create **resolution issues** for portfolios with a large number of assets, or in general when Δx, the step of the distribution, is very small. Not using a high enough resolution will give strange results, most often a distribution having non-zero probabilities at the end of the computation interval. This maximum resolution constraint may be overcome, alternatively and by order of increasing efficiency, by trying to:

- reduce $V_{\max}$: generally the maximum gain in resolution is limited to a factor 2;
- increase Δx, the step of the distribution, by slightly modifying the size of the assets in the portfolio. This is the **replication principle**: the real portfolio is approximated by a replicated one. The sizes of the assets in the replicated portfolio are rounded to the closest integer multiple of the desired step. This replication mechanism works quite well for CDO portfolios;
- program and use your own FFT VB macro;
- switch to the computation algorithm of the continuous distributions by assuming an LGD distribution with mean 100% and with a very small standard deviation (for instance 0.1%).

- ***Computation of the Discrete Probabilities*** f_n's

This last step in order to achieve the inversion consists in computing $\hat{f}(t)$ for appropriate values of t. It will generate a vector of complex numbers, the Fourier vector. The application of the DIFT to this Fourier vector will give the discrete probabilities f_n's.

Let M be an integer superior or equal to $N_{\max}$, the number of points x_n between 0 and $V_{\max}$. As a matter of fact, M will be the Fourier resolution introduced in the previous section. It will be chosen to be the lowest power of 2 larger than $N_{\max}$.

Let us define $t_m = mT/M = 2\pi m/M\Delta x$ for $m=0,1,\ldots,M-1$. With equation **(B)**:

$$\hat{f}(t_m)=\sum_{n=0}^{N_{\max}-1} f_n.e^{-in\frac{2\pi}{T}.\frac{mT}{M}}=\sum_{n=0}^{M-1} f_n.e^{-i2\pi nm/M}=DFT([f_0,f_1,\ldots,f_{M-1}])_m,\ \mathrm{m}=0,\ldots,\mathrm{M}\text{-}1.$$

[40] Example taken from "Numerical Recipes in C", Chapter 12 "Fast Fourier Transform", to which readers may refer for a much more detailed presentation of the Discrete Fourier Transform.
[41] Refer to "Numerical Recipes in C" to write a Visual Basic FFT macro. Alternatively, existing VB FFT code may be downloaded from the Internet (for instance at www.fullspectrum.com/deeth/programming/fft.html).

Thus, applying the inversion formula to the **Fourier vector** $\left[\hat{f}(t_0), \hat{f}(t_1), \ldots, \hat{f}(t_{M-1})\right]$:

$$[f_0, f_1, \ldots, f_{M-1}] = DIFT\left(\left[\hat{f}(t_0), \hat{f}(t_1), \ldots, \hat{f}(t_{M-1})\right]\right).$$

Noticing that the values of $\hat{f}(t_m)$ for m = 1 + M / 2 to M - 1 are given by $\hat{f}(t_m) = \overline{\hat{f}(t_{M-m})}$,[42] where $\bar{z}$ designates the complex conjugate of z :

$$\boxed{[f_0, f_1, \ldots, f_{M-1}] = DIFT\left(\left[\hat{f}(t_0), \ldots, \hat{f}(t_{M/2}), \overline{\hat{f}(t_{M/2-1})}, \ldots, \overline{\hat{f}(t_1)}\right]\right)},$$

which concludes the determination of the f_n's.

The textbox below summarizes the main points of the inversion algorithm (it also determines the appropriate Fourier resolution M as a power of 2 in function of the other main parameters).

Inversion Recipe #1

How to Invert the Portfolio Fourier Transform $\hat{f}(t)$ and Get the (Discrete) Distribution $f(x)$?

1. *Choose the lowest possible value for $V_{\max}$, a percentage level above which $f(x)$ is assumed to be equal to zero.*
2. *Compute the step of the distribution Δx as the "Greatest Common Divisor" (GCD) of the discrete potential values that may be taken by the distribution. If $f(x)$ refers to a default distribution, Δx is the "GCD" of the relative weights of the assets (the s_k's). If $f(x)$ refers to a loss distribution, Δx is the product of the step of the default distribution (the "GCD" of the s_k's) by the step of the discrete (or discretised) LGD distribution. $\hat{f}(t)$ is a T-periodic function, with $T = 2\pi/\Delta x$.*
3. *Compute $N_{\max} = Int[V_{\max}/\Delta x]+1$, the number of points of the distribution between 0 and $V_{\max}$.*
4. *Compute $p_{\min} = Int[Ln(N_{\max}-1)/Ln(2)]+1$. ResF $= 2^{p_{\min}}$ is the Fourier resolution required for the inversion, i.e. the number of points of the Fourier Transform vector or the number of computed default (or loss) probabilities.*
5. *Compute $\hat{f}(t_m)$ where $t_m = mT/\mathrm{Re}\,sF$ for m = 0 to ResF/2.*
6. *Apply the Discrete Inverse Fourier Transform to the Fourier vector $\left[\hat{f}(t_0), \ldots, \hat{f}(t_{\mathrm{Re}\,sF/2}), \overline{\hat{f}(t_{\mathrm{Re}\,sF/2-1})}, \ldots, \overline{\hat{f}(t_1)}\right]$ to obtain the vector $[f_1, f_2, \ldots, f_{\mathrm{Re}\,sF-1}]$ ($\bar{z}$ designates the complex conjugate of z). In MS Excel, go to Tools/Data Analysis/Fourier Analysis to run the Discrete Inverse Fourier Transform. The resulting f_m's are the values of the inverse Fourier transform $f(x)$ at the points $x_m = m.\Delta x, \mathrm{m} = 0, \ldots, \mathrm{ResF-1}$ (provided that the f_m's are negligible for m close to ResF). In other words, the resulting f_m's are the probabilities associated with the potential default (or loss) levels (the x_m's).*
7. *….Serve the f_m's with a nice and colourful chart.*

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[42] Given the T-periodicity of $\hat{f}(t)$ and the fact that $f(x)$ takes real values, $\hat{f}(T-t) = \hat{f}(-t) = \int f(x).e^{ixt}.dx = \overline{\int f(x).e^{-ixt}.dx} = \overline{\hat{f}(t)}$, which leads to $\hat{f}(t_m) = \overline{\hat{f}(t_{M-m})}$. This symmetry divides the computation time by a factor 2.

AN EXAMPLE OF A DEFAULT DISTRIBUTION COMPUTATION

Consider the following portfolio:

- *50 assets (50 different debtors) – bullet amortisation after 5 years*
- *Individual asset size goes from € 1 m to € 10.8 m with a step of € 0.2m*
- *The individual default probability of the smallest asset is 1.0% and goes up to 15.7% for the biggest asset with a step of 0.3% on the default probability for each asset in between (ranked by increasing principal outstanding); the resulting weighted average default probability – WADP – is of 10.47%*
- *No default correlation exist between the assets (for the sake of simplicity)*

As mentionned in the Underlying Models section, the portfolio Fourier transform for a portfolio of uncorrelated assets is:

$$\hat{f}_{PDR}(t) = \prod_{k=1}^{N} \left[1 + p_k \left(e^{-its_k} - 1 \right) \right].$$

The following table is a copy of the first rows of the MS Excel spreadsheet:[43]

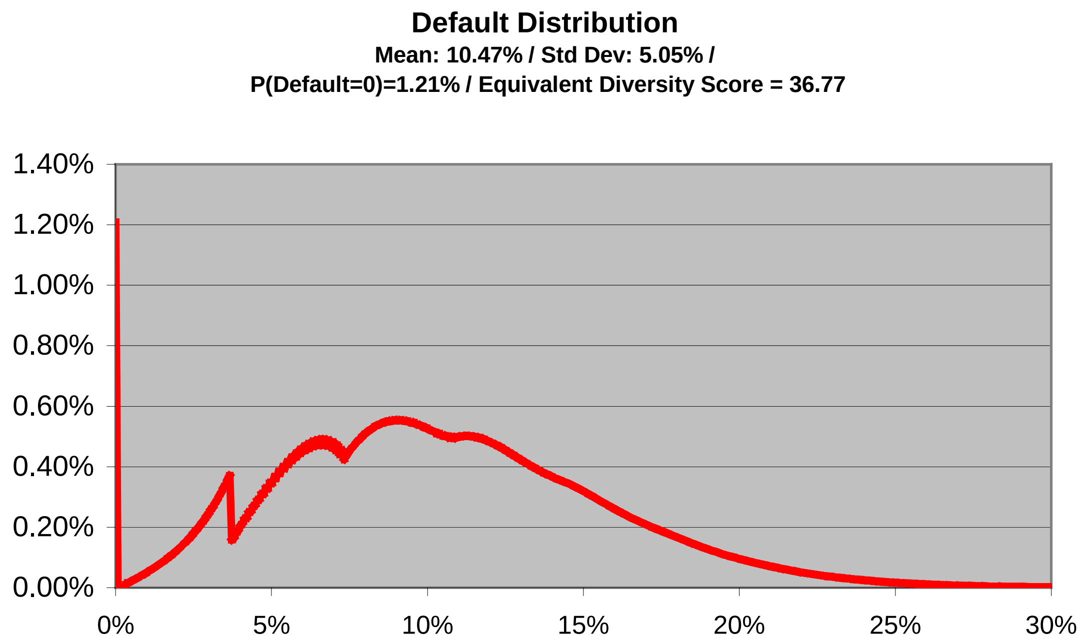
Example of computation of a portfolio default distribution for uncorrelated assets

N	**50**	***WADP***
Vmax	**66.00%**	***10.47%***
GCD Size (EURm)	**0.20**	
Total Porfolio Size (EURm)	**295.00**	
GCD Size (%) (=DeltaX)	**0.068%**	
Fourier Transform Period (T)	**9267.7**	
Nmax	**974**	Size Multiplier
pmin	**10**	100
ResF	**1024**	GCD Multiplied#
Distribution goes up to	**69.36%**	20

(Column A) # Asset	(B) Size (EURm)	(C) Size (%)	(D) Default Proba	(E) GCD Determination	(G) m (0 to ResF-1)	(H) tm	(I) Fourier Vector	(K) xm	(L) Default Distribution
1	**1.00**	0.34%	**1.0%**	100	*0*	*0.00*	*1*	*0.00%*	**1.20878314315226E-002**
2	**1.20**	0.41%	**1.3%**	120	*1*	*9.05*	*0.531045844593*	*0.07%*	**0**
3	**1.40**	0.47%	**1.6%**	140	*2*	*18.10*	*-0.175512330582*	*0.14%*	**0**
4	**1.60**	0.54%	**1.9%**	160	*3*	*27.15*	*-0.351304304111*	*0.20%*	**0**
5	**1.80**	0.61%	**2.2%**	180	*4*	*36.20*	*-0.195193147742*	*0.27%*	**0**
6	**2.00**	0.68%	**2.5%**	200	*5*	*45.25*	*-6.463086848861*	*0.34%*	**1.22099307389113E-004**
7	**2.20**	0.75%	**2.8%**	220	*6*	*54.30*	*-1.757028368970*	*0.41%*	**1.59211558875167E-004**
8	**2.40**	0.81%	**3.1%**	240	*7*	*63.35*	*-6.062814911846*	*0.47%*	**1.96550104577593E-004**
9	**2.60**	0.88%	**3.4%**	260	*8*	*72.40*	*-3.103651662501*	*0.54%*	**2.3411702059014E-004**
10	**2.80**	0.95%	**3.7%**	280	*9*	*81.45*	*-1.744314232357*	*0.61%*	**2.71914408480051E-004**
11	**3.00**	1.02%	**4.0%**	300	*10*	*90.50*	*-8.003355780704*	*0.68%*	**3.09944395680058E-004**
12	**3.20**	1.08%	**4.3%**	320	*11*	*99.56*	*-1.029167670550*	*0.75%*	**3.49817333451871E-004**

In this spreadsheet, a Visual Basic macro was used to compute the Fourier vector $\left[\hat{f}(t_0), \ldots, \hat{f}(t_{\mathrm{Res}F/2}), \overline{\hat{f}(t_{\mathrm{Res}F/2-1})}, \ldots, \overline{\hat{f}(t_1)} \right]$ in column (I) and to invert it and get the default distribution in column (L). The resulting default distribution is depicted in the chart below.

[43] In this spreadsheet, columns (A) to (E) have 50 rows corresponding to N=50 assets. Columns (G) to (L) have 1024 rows corresponding to a Fourier resolution of ResF=1024.



COMPUTING (ALMOST) CONTINUOUS DISTRIBUTIONS $f(x)$

This case refers to loss distributions where the LGD distributions take continuous values between 0% and 100%. Although for quite different reasons, the inversion recipe will surprisingly prove – almost – similar to the discrete case.

- ***Introducing the Bandwith Limited Function*** $f_{BWL}(x)$

For most continuous (or almost continuous) LGD distributions such as the ones seen in the *Further Uses of the FT*M section, $\hat{f}_{LGD}(t)$ is a continuous function converging towards zero for large values of t. A close look at expression **(VI)** permits to conclude that $\hat{f}(t) = \hat{f}_{PLR}(t)$ is a continuous function converging towards a positive value $\hat{f}_{\infty}$ for large values of t.[44]

If we define the function $\hat{f}_{BWL}(t) = \hat{f}(t) - \hat{f}_{\infty}$, $\hat{f}_{BWL}(t)$ is a continuous function converging towards zero for large values of t. Besides, the symmetry property for $\hat{f}(t)$: $\hat{f}(-t) = \overline{\hat{f}(t)}$ will translate into $\hat{f}_{BWL}(-t) = \overline{\hat{f}_{BWL}(t)}$. Therefore, in practice, $\hat{f}_{BWL}(t)$ will be assumed to be negligible for numerical purposes outside a certain interval $[-T_{\max}, T_{\max}]$. Its inverse Fourier Transform $f_{BWL}(x)$ is therefore a so called ***bandwith limited function*** (hence the index BWL), namely a function whose Fourier transform is equal to zero outside an interval centered in zero.

As will be obvious later, $f_{BWL}(x)$ is a continuous function. As the application of the inverse Fourier Transform to $\hat{f}(t) = \hat{f}_{BWL}(t) + \hat{f}_{\infty}$ yields $f(x) = f_{BWL}(x) + \hat{f}_{\infty}.\delta(x)$, it means that $f(x)$ is an almost

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44 $\hat{f}_{\infty} = \int_{-\infty}^{+\infty} \prod_{k=1}^{N} [1 - p_k(z)].\phi(z)dz$. Note that $\hat{f}_{\infty}$ is a positive real number.

continous density function. More precisely, it is not continous in zero where there is a Dirac spike with energy $\hat{f}_\infty$: $\hat{f}_\infty$ can be intrepreted as the probability of no loss associated with the scenario where no asset defaults in the portfolio.

According to the **Nyquist Sampling Theorem**, bandwith limited functions may be sampled (i.e. recorded) using a **sampling step** δx smaller than a critical value $\Delta x = \pi / T_{\max}$ **without any loss of information**. In other words, the knowledge of the sampled values $f_{BWL}(x_n)$ for $x_n = n.\delta x, \mathrm{n\ integer}$ permits to completely and exactly determine $f_{BWL}(x)$ for any value of x.

A key step in the inversion of $\hat{f}(t)$ will be to determine the critically sampled values of $f_{BWL}(x)$ (i.e. $f_{BWL}(x_n)$ for $x_n = n.\Delta x, \mathrm{n\ integer}$) in order to get an expression for $f_{BWL}(x)$. $f(x) = f_{BWL}(x) + \hat{f}_\infty.\delta(x)$ will then obviously be known once $\hat{f}_\infty$ will have been computed.

- ***Expressions for*** $\hat{f}_{BWL}(t)$ ***and*** $f_{BWL}(x)$

As a continuous function on $[-T_{\max}, T_{\max}]$, $\hat{f}_{BWL}(t)$ can be expanded into a Fourier series:

$$\boxed{\hat{f}_{BWL}(t) = \sum_{n=-\infty}^{+\infty} c_n.e^{-in\frac{\pi}{T_{\max}}t} = \sum_{n=-\infty}^{+\infty} c_n.e^{-in.\Delta x.t}, \mathrm{for} |\mathrm{t}| \le T_{\max}} \quad \textbf{(C),}$$

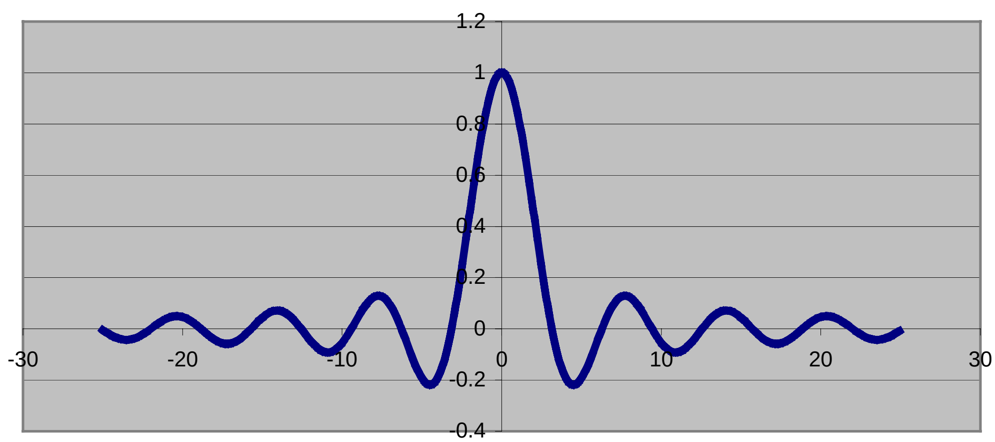
Introducing the functions $g_a(t)$ defined for a real number a by $g_a(t) = e^{-iat}$, $for\ |t| \le T_{\max}$, and $g_a(t) = 0,\ for\ |t| > T_{\max}$:

$$\boxed{\hat{f}_{BWL}(t) = \sum_{n=-\infty}^{+\infty} c_n.g_{x_n}(t),\ for\ any\ value\ of\ t} \quad \textbf{(D)}, \text{ where } x_n = n.\Delta x, \mathrm{n\ integer}.$$

Going further, it can be proved that the inverse Fourier transform of $g_a(t)$ is:

$$\breve{g}_a(x) = \frac{1}{\Delta x} sc\left(\frac{x-a}{\Delta x}\right), \text{ where } sc(u) = \frac{\sin u}{u},\ for\ u \ne 0, and\ sc(0) = 1.$$

The sc(u) Function



Notice that $\breve{g}_{x_n}(x_m) = 1/\Delta x, \text{if } m = n \text{ and } 0 \text{ if } m \neq n$.

If we therefore invert **(D)**:

$$\boxed{f_{BWL}(x) = \sum_{n=-\infty}^{+\infty} c_n . \breve{g}_{x_n}(x) = \frac{1}{\Delta x} . \sum_{n=-\infty}^{+\infty} c_n . sc\left(\frac{x - x_n}{\Delta}\right)} \textbf{ (E),}$$

and combining **(E)** with the previous remark:

$$\boxed{f_{BWL}(x_m) = \frac{c_m}{\Delta x}} \textbf{ (F)}, \text{ for } m \text{ integer.}$$

Since $f(x) = f_{BWL}(x) + \hat{f}_\infty . \delta(x)$ is equal to zero outside $[0; V_{\max}]$, we will have: $c_n = f_{BWL}(x_n).\Delta x = 0$ for $n < 0$ and $n \geq N_{\max} = Int[V_{\max}/\Delta x] + 1$, and therefore **(E)** becomes:

$$\boxed{f_{BWL}(x) = \frac{1}{\Delta x} . \sum_{n=0}^{N_{\max}-1} c_n . sc\left(\frac{x - x_n}{\Delta x}\right)} \textbf{ (G)},$$

while **(C)** becomes:

$$\boxed{\hat{f}_{BWL}(t) = \sum_{n=0}^{N_{\max}-1} c_n . e^{-in.\Delta x.t}, \text{ for } |t| \leq T_{\max}} \textbf{ (H)}.$$

The explicit expression for $f_{BWL}(x)$ in **(G)** proves that $f_{BWL}(x)$ is indeed a continuous function. This expression also leads to an explicit determination of $f(x) = f_{BWL}(x) + \hat{f}_\infty . \delta(x)$ provided we manage to compute $\hat{f}_\infty$ and $c_0, c_1, \ldots, c_{N_{\max}-1}$. Like in the discrete case, the latter computation will be possible thanks to the *Discrete Fourier Transform* theoretical framework.

- ***Using the Discrete Fourier Transform Framework***

Let M be an even integer superior or equal to $N_{\max} = Int[V_{\max}/\Delta x] + 1$. Let us define $t_m = m.\frac{2T_{\max}}{M} = m.\frac{2\pi}{M\Delta x}$, for $m = 0, \ldots, M-1$. Let us distinguish <u>two cases</u> according to the values of m.

- For $m = 0, \ldots, M/2$, $|t_m| \leq T_{\max}$ and therefore using **(H)**:

$$\hat{f}_{BWL}(t_m) = \sum_{n=0}^{N_{\max}-1} c_n . e^{-in.\Delta x.t_m} = \sum_{n=0}^{N_{\max}-1} c_n . e^{-in\Delta x.m.\frac{2\pi}{M\Delta x}} = \sum_{n=0}^{M-1} c_n . e^{-i.2\pi.nm/M} = DFT([c_0, c_1, \ldots, c_{M-1}])_m \textbf{ (I).}$$

- For $m = 1 + M/2, \ldots, M-1$, unfortunately **(H)** can not be applied since $|t_m| > T_{\max}$. However it can be applied to $2T_{\max} - t_m = t_{M-m}$:

$$\hat{f}_{BWL}(t_{M-m}) = \sum_{n=0}^{N_{\max}-1} c_n . e^{-in.\Delta x.(2T_{\max} - t_m)} = \sum_{n=0}^{M-1} c_n . e^{+i.2\pi.nm/M}$$

  and therefore, since the c_n's are real numbers (according to **(F)**):

$$\overline{\hat{f}_{BWL}(t_{M-m})} = \sum_{n=0}^{M-1} c_n . e^{-i.2\pi.nm/M} = DFT([c_0, c_1, \ldots, c_{M-1}])_m \textbf{ (J).}$$

Combining **(I)** and **(J)**:

$$\left[\hat{f}_{BWL}(t_0),\ldots,\hat{f}_{BWL}(t_{M/2}),\overline{\hat{f}_{BWL}(t_{M/2-1})},\ldots,\overline{\hat{f}_{BWL}(t_1)}\right]=DFT([c_0,c_1,\ldots,c_{M-1}]) \quad \textbf{(K)}$$

Besides, it can be proved that:

$$[1,1,1,\ldots,1]=DFT([1,0,0,\ldots,0]) \text{ (vectors with M points),}$$

so that:

$$\left[\hat{f}_\infty,\hat{f}_\infty,\hat{f}_\infty,\ldots,\hat{f}_\infty\right]=DFT\left(\left[\hat{f}_\infty,0,0,\ldots,0\right]\right) \quad \textbf{(L)}.$$

Adding **(K)** to **(L)** and remembering that $\hat{f}(t)=\hat{f}_{BWL}(t)+\hat{f}_\infty$:

$$\left[\hat{f}(t_0),\ldots,\hat{f}(t_{M/2}),\overline{\hat{f}(t_{M/2-1})},\ldots,\overline{\hat{f}(t_1)}\right]=DFT\left(\left[c_0+\hat{f}_\infty,c_1,\ldots,c_{M-1}\right]\right).$$

Applying the discrete inversion formula to the **Fourier vector** $\left[\hat{f}(t_0),\ldots,\hat{f}(t_{M/2}),\overline{\hat{f}(t_{M/2-1})},\ldots,\overline{\hat{f}(t_1)}\right]$:

$$\left[c_0+\hat{f}_\infty,c_1,\ldots,c_{M-1}\right]=DIFT\left(\left[\hat{f}(t_0),\ldots,\hat{f}(t_{M/2}),\overline{\hat{f}(t_{M/2-1})},\ldots,\overline{\hat{f}(t_1)}\right]\right)$$

In the end, we are able to determine all the $c_n's$ and $\hat{f}_\infty=\lim_{t\to+\infty}\hat{f}(t)=\hat{f}(T_{\max})$: the inversion is finished. The main points of the inversion algorithm are summarized in the textbox below.

Inversion Recipe #2

How to Invert the Portfolio Fourier Transform $\hat{f}(t)$ and Get the (Continuous) Distribution $f(x)$?

1. *Choose the lowest possible value for $V_{\max}$, a percentage level above which $f(x)$ is assumed to be equal to zero.*
2. *Choose Δx, the step of the distribution. This step must be small enough so to ensure that the Fourier transform $\hat{f}(t)$ will be practically equal to a constant for $t>T_{\max}=\pi/\Delta x$.*
3. *Compute $\hat{f}_\infty=\lim_{t\to+\infty}\hat{f}(t)\approx\hat{f}(T_{\max})$, the probability that no asset in the portfolio defaults. $f(x)$ is the sum of a continuous function $f_{BWL}(x)$ corresponding to the loss distribution when there are defaulted assets and a (Dirac) spike with energy $\hat{f}_\infty$ in x=0 corresponding to the scenario when there are no defaulted assets. $f(x)$ and $f_{BWL}(x)$ coincide everywhere except in x=0.*
4. *Compute $N_{\max}=Int[V_{\max}/Q]+1$ and $p_{\min}=Int[Ln(N_{\max}-1)/Ln(2)]+1$. $ResF=2^{p_{\min}}$ is the lowest Fourier resolution required for the inversion, i.e. the number of points of the Fourier Transform vector or the number of computed (loss) probabilities.*
5. *Compute $\hat{f}(t_m)$ where $t_m=mT_{\max}/\mathrm{Re}\,sF$ for m = 0 to ResF/2.*
6. *Apply the Discrete Inverse Fourier Transform to the Fourier vector $\left[\hat{f}(t_0),\ldots,\hat{f}(t_{\mathrm{Re}\,sF/2}),\overline{\hat{f}(t_{\mathrm{Re}\,sF/2-1})},\ldots,\overline{\hat{f}(t_1)}\right]$ to obtain the vector $\left[c_0+\hat{f}_\infty,c_1,\ldots,c_{\mathrm{Re}\,sF-1}\right]$ ($\bar{z}$ designates the complex conjugate of z). In MS Excel, go to Tools/Data Analysis/Fourier Analysis to run the Discrete Inverse Fourier Transform.*
7. *$c_m/\Delta x$ is the value of $f_{BWL}(x)$ at the point $x_m=m.\Delta x,\ \mathrm{m}=0,\ldots,\mathrm{ResF-1}$ (provided to check that the $c_m's$ are negligible for m close to ResF). For other values of x, $f_{BWL}(x)=\frac{1}{\Delta x}.\sum_{n=0}^{N_{\max}-1}c_n.sc\left(\frac{x-x_n}{\Delta x}\right)$.*
8. *In practice, the sum of $\hat{f}_\infty$ and the $c_m's$ equals 100%, and $c_0+\hat{f}_\infty,c_1,\ldots,c_{\mathrm{Re}\,sF-1}$ may be used as the discrete probabilities associated with the discretised loss levels $x_m=m.\Delta x,\ \mathrm{m}=0,\ldots,\mathrm{ResF-1}$.*
9. *….Serve $\hat{f}_\infty$ and the $c_m's$ with a nice and colourful chart.*

APPENDIX 2:
COMPUTATION OF PORTFOLIO FOURIER TRANSFORMS WITH GAUSSIAN QUADRATURES[45]

As seen for instance in the Underlying Models section, the portfolio Fourier transform requires the computation of improper integrals such as:

$$\int_{-\infty}^{+\infty} g(x).\phi(x).dx\,, \text{ where } \phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-x^2/2\right).$$

This integral does not usually have any analytic expression and therefore must be computed numerically. The first approach would be to use a “classic“ formula like:

$$\int_{-\infty}^{+\infty} h(x).dx \approx \Delta.\sum_{j=0}^{N} h(x_j), \text{ where } x_j = x_0 + j\Delta,\ \ j = 0,1,\ldots,N \text{ , for } h(x) = g(x).\phi(x).$$

With this approach, N must typically be larger than 200 to achieve a sufficient accuracy. An equivalent level of accuracy may be reached with the Gaussian quadrature approach with N roughly divided by a factor of 10, therefore significantly enhancing the computation speed.

The Gaussian quadrature consists in approximating the integral:

$$\int_{-\infty}^{+\infty} g(x).W(x).dx\,,$$

where $W(x)$ is a known function – the weight function – by the following formula:

$$\int_{-\infty}^{+\infty} g(x).W(x).dx \approx \sum_{j=0}^{N} g(x_j).w_j\,,$$

where the $x_j\text{'}s,\ j = 0,\ldots,N$ are appropriately chosen abscissas and the $w_j\text{'}s$ the corresponding appropriate weights. Algorithms leading to the values of the abscissas $x_j\text{'}s$ and the weights $w_j\text{'}s$ have been determined for certain key weight functions $W(x)$, including the Hermite function $W(x) \equiv e^{-x^2}$. Let us call them the $x_j^{GH}\text{'}s$ and the $w_j^{GH}\text{'}s$ (the *GH* exponent stands for Gauss-Hermite). Now, making the substitution $x = \sqrt{2}.x'$ in $\phi(x)$, it can be proved that:

$$\boxed{\int_{-\infty}^{+\infty} g(x).\phi(x).dx \approx \sum_{j=0}^{N} g\left(x_j^{FTM}\right).w_j^{FTM}\,,\ with\ \ x_j^{FTM} = \sqrt{2}.x_j^{GH}\,,\ and\ \ w_j^{FTM} = \pi^{-1/2}.w_j^{GH}}\,.$$

An interesting theoretical result is that the $w_j\text{'}s$ and the $x_j\text{'}s$ can be determined by studying the set of orthogonal polynomials associated to $W(x)$. For the Hermite function $W(x) \equiv e^{-x^2}$, these polynomials are generated by the recurrence:

$$H_{-1}(x) = 0\,, \qquad H_0(x) = \frac{1}{\pi^{1/4}}\,, \qquad H_{j+1}(x) = x\sqrt{\frac{2}{j+1}}H_j(x) - \sqrt{\frac{j}{j+1}}H_{j-1}(x).$$

The N abscissas x_j^{GH} are the N roots of $H_N(x)$. A classical Newton algorithm[46] may be used to determine them.

The N Gauss Hermite weights are given by: $w_j^{GH} = \dfrac{2}{\left[H'_{j-1}\left(x_j^{GH}\right)\right]^2}$, with $H'_j(x) = \sqrt{2j}H_{j-1}(x)$.

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[45] This appendix draws heavily from chapter 4.5 of “Numerical Recipes in C: the Art of Scientific Computing” (see References). Chapter 4.5 also includes a computer code that permits to determine the $x_j^{GH}\text{'}s$ and the $w_j^{GH}\text{'}s$.

[46] The Newton algorithm starts at a point x_0, the initial guess for the root , then usually converges with the iteration: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$.

APPENDIX 3: CALIBRATION OF THE FACTOR LOADINGS

This appendix details the interpretation of the factor loadings in terms of:

- standard deviation of the default distribution,
- diversity score of the portfolio,
- pairwise default correlations between two assets,
- asset (return) correlations between two assets,
- joint downgrade (and upgrade) probability for two rated assets,

in the context of the Single Factor Model (however the appendix does not detail the proofs).

For two assets A and B, let us define:

$$\phi_A^2 = \mathrm{E}\left[p_A^2(Z)\right] = \int p_A^2(z).\phi(z).dz\ ,$$

$$\phi_B^2 = \mathrm{E}\left[p_B^2(Z)\right] = \int p_B^2(z).\phi(z).dz\ ,$$

$$\phi_{AB} = \mathrm{E}\left[p_A(Z).p_B(Z)\right] = \int p_A(z).p_B(z).\phi(z).dz\ ,$$

where $p_A(z) = \Phi\left(\frac{\alpha_A - w_A z}{\sqrt{1-w_A^2}}\right)$, $p_B(z) = \Phi\left(\frac{\alpha_B - w_B z}{\sqrt{1-w_B^2}}\right)$ and $\phi(z) = \frac{1}{\sqrt{2\pi}}\exp\left(-z^2/2\right)$.

❑ *Factor Loadings and Standard Deviation of the Default Rate Distribution*

- Standard deviation of the default distribution for a large portfolio of assets with similar characteristics (each asset has the same size, the same maturity, the same default probability p_A):

$$\boxed{\sigma_\infty = \sqrt{\phi_A^2 - p_A^2}}.$$

- Standard deviation of the default distribution for a portfolio of N assets with similar characteristics (each asset has the same size, the same maturity, the same default probability p_A):

$$\boxed{\sigma_N = \sqrt{\left(\phi_A^2 - p_A^2\right) + \left(p_A - \phi_A^2\right)/N} = \sqrt{\sigma_\infty^2 + \left(p_A.q_A - \sigma_\infty^2\right)/N}}$$

In these expressions, the terms depending on the value of the factor loadings are ϕ_A and σ_∞.

❑ *Factor Loadings and Diversity Score*

The Diversity Score of a portfolio of assets (non necessarily homogeneous) with a weighted average default probability of p is:

$$\boxed{D = pq/\sigma^2},$$

where $q = 1 - p$, and where σ is the standard deviation of the portfolio default rate derived from the computed default standard deviation $f_{PDR}(x)$: $\sigma^2 = \int f_{PDR}(x).(x-p)^2.dx$.

In the expression for D, the term depending on the value of the factor loadings is σ.

- ***Factor Loadings and Pairwise Default Correlation***

Pairwise Default Correlation between two assets: $\rho_{AB} = \frac{\phi_{AB} - p_A p_B}{\sqrt{p_A q_A}\sqrt{p_B q_B}}$

In this expression, the term depending on the value of the factor loadings is ϕ_{AB}.

- ***Factor Loadings and Asset Return Correlation***

- If the factor Z represents the return of a market index (typically an equity index for a given industry or a given geographical region), then the correlation between the return of A and the return of the index over the considered time horizon is given by:

$$\rho_{A,Index} = Corr(Z_A, Z) = w_A$$

- If there is no avalaible index, one may consider the equity return of one relevant asset as the reference and estimate the asset correlation using this asset as a reference. Alternatively, if one looks at three assets A, B and C simultaneously, we shall have:

$$w_A = \sqrt{\rho_{AB}.\rho_{AC} / \rho_{BC}}$$

- ***Factor Loadings and Joint Downgrade (and Upgrade) Probability***

The factor model is not only adapted to the modelling of default events, but also to the modelling of **downgrade** or **upgrade** events for rated assets. Similarly to the case of the default event, a **downgrade threshold** α_A^{DWN} can be defined for asset A:

$$\Pr\left[Z_A < \alpha_A^{DWN}\right] = p_A^{DWN} \text{ or equivalently } \alpha_A^{DWN} = \Phi^{-1}\left(p_A^{DWN}\right),$$

where p_A^{DWN} is the downgrade probability, typically derived from a rating transition matrix.

The downgrade probability conditionally to Z=z would be found to be:

$$p_A^{DWN}(z) = \Phi\left(\frac{\alpha_A^{DWN} - w_A.z}{\sqrt{1-\left(\alpha_A^{DWN}\right)^2}}\right),$$

while for the **joint downgrade probability** for assets A and B would be given by:

$$p_{A,B}^{DWN} = \int p_A^{DWN}(z).p_B^{DWN}(z).\phi(z).dz \quad \textbf{(A)}.$$

Similarly, the **joint upgrade probability** for assets A and B would have the expression:

$$p_{A,B}^{UP} = \int p_A^{UP}(z).p_B^{UP}(z).\phi(z).dz \quad \textbf{(B)},$$

where $p_A^{UP}(z) = \Phi\left(\frac{w_A.z - \alpha_A^{UP}}{\sqrt{1-\left(\alpha_A^{UP}\right)^2}}\right)$ is the conditional upgrade probability for asset A , $\alpha_A^{UP} = -\Phi^{-1}\left(p_A^{UP}\right)$ its

upgrade threshold (similar definitions obviously hold for asset B).

The knowledge of both the joint downgrade and upgrade probabilities ($p_{A,B}^{DWN}$ and $p_{A,B}^{UP}$ are typically derived from a joint rating transition matrix) permits to **solve for** w_A **and** w_B in **(A)** and **(B)**.

- ***Hints for the factor loadings***
  (<u>indicative</u> levels subject to changes according to the considered assets)

- w =10% to 25% for consumer loans, leases, mortgages
- w= 20% to 35% for middle market loans
- w = 15% to 80% for large corporate bonds or loans
- The shorter the time horizon, the lower the factor loading w.[47]
- The more concentrated the industry (sector) of a debtor, the higher its w (for the industry risk factor).[48]
- The higher the rating (or the lower the credit risk), the higher w.[49] See the following <u>indicative</u> table for large corporates :

	Factor Loading w
Aaa	80%
Aa	65%
A	55%
Baa	45%
Ba	35%
B	25%
Caa-C	15%

[47] It seems that pairwise default correlation increases in a linear way with the time horizon up to approx. 3 years.
[48] In a concentrated industry, an industry player shapes and influences the industry as a whole.
[49] The rating of a company generally reflects the level of diversification of its activities (although diversification is obviously only one parameter for the rating). As a result, the higher the rating, the closer the risk profile of a company to the systemic risk.

APPENDIX 4: MOODY'S EXPECTED LOSS TABLE

	Years									
Rating	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Aaa	0.000028%	0.000110%	0.000390%	0.000990%	0.001600%	0.002200%	0.002860%	0.003630%	0.004510%	0.005500%
Aa1	0.000314%	0.001650%	0.005500%	0.011550%	0.017050%	0.023100%	0.029700%	0.036850%	0.045100%	0.055000%
Aa2	0.000748%	0.004400%	0.014300%	0.025850%	0.037400%	0.048950%	0.061050%	0.074250%	0.090200%	0.110000%
Aa3	0.001661%	0.010450%	0.032450%	0.055500%	0.078100%	0.100650%	0.124850%	0.149600%	0.179850%	0.220000%
A1	0.003196%	0.020350%	0.064350%	0.103950%	0.143550%	0.181500%	0.223300%	0.264000%	0.315150%	0.385000%
A2	0.005979%	0.038500%	0.122100%	0.189750%	0.256850%	0.320650%	0.390500%	0.455950%	0.540100%	0.660000%
A3	0.021368%	0.082500%	0.198000%	0.297000%	0.401500%	0.500500%	0.610500%	0.715000%	0.836000%	0.990000%
Baa1	0.049500%	0.154000%	0.308000%	0.456500%	0.605000%	0.753500%	0.918500%	1.083500%	1.248500%	1.430000%
Baa2	0.093500%	0.258500%	0.456500%	0.660000%	0.869000%	1.083500%	1.325500%	1.567500%	1.782000%	1.980000%
Baa3	0.231000%	0.577500%	0.940500%	1.309000%	1.677500%	2.035000%	2.381500%	2.733500%	3.063500%	3.355000%
Ba1	0.478500%	1.111000%	1.721500%	2.310000%	2.904000%	3.437500%	3.883000%	4.339500%	4.779500%	5.170000%
Ba2	0.858000%	1.908500%	2.849000%	3.740000%	4.625500%	5.373500%	5.885000%	6.413000%	6.957500%	7.425000%
Ba3	1.545500%	3.030500%	4.328500%	5.384500%	6.523000%	7.419500%	8.041000%	8.640500%	9.190500%	9.713000%
B1	2.574000%	4.609000%	6.369000%	7.617500%	8.866000%	9.839500%	10.521500%	11.126500%	11.682000%	12.210000%
B2	3.938000%	6.418500%	8.552500%	9.971500%	11.390500%	12.457500%	13.205500%	13.832500%	14.421000%	14.960000%
B3	6.391000%	9.135500%	11.566500%	13.222000%	14.877500%	16.060000%	17.050000%	17.919000%	18.579000%	19.195000%
Caa	14.300000%	17.875000%	21.450000%	24.134000%	26.812500%	28.600000%	30.387500%	32.175000%	33.962500%	35.750000%

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