Saddlepoint approximation method for pricing CDOs

Jingping Yang[*]
Department of Financial Mathematics
Peking University, Beijing 100871
P. R. China
T. R. Hurd[†] Xuping Zhang[‡]
Dept. of Mathematics and Statistics
McMaster University, Hamilton ON L8S 4K1
Canada

September 17, 2006

Abstract

A critical issue in the credit risk industry is the accurate, efficient and robust pricing of collateralized debt obligations (CDO) in a variety of mathematical models. These and many similar basket default products are very complex, due to characteristics of the large number of individual firms upon which they depend. Despite this complexity and because of their versatility, such products have become popular in the market. A central difficulty which arises in most models of CDOs is the efficient computation of conditional default loss distributions. Since exact computation is feasible only in highly symmetric situations, it is necessary to have a variety of acceptable approximation schemes. The present paper explores one general method, the saddlepoint approximation, and shows that it offers an improvement when compared with simpler methods.

Key words: large deviations; credit risk; basket credit derivative; collateralized debt obligation; tranche function

[*]Research supported by MITACS (Mathematics of Information Technology and Complex Systems) and NSFC10471008, e-mail: yangjp@math.pku.edu.cn

[†]Research supported by the Natural Sciences and Engineering Research Council of Canada and MITACS, Canada, e-mail: hurdt@mcmaster.ca

[‡]e-mail: zhangx@math.mcmaster.ca

1 Introduction

A collateralized debt obligation (CDO), or more generally any asset–backed security (ABS), is a structured product based on an underlying portfolio of default risky reference credits, such as corporate bonds, mortgages or loans. In essence, the portfolio is sliced into separate securities called "tranches" ordered by seniority, each of which receives its fair share of the revenue stream generated by the reference credits. CDOs have become very popular in the market and it is now important to price them accurately and efficiently, a problem which has been attracting more and more attention from both practitioners and academic researchers. The problem is intrinsically difficult regardless of the modeling approach adopted. The number of credits or names in a typical CDO is moderately large: for example the highly traded CDX index products are based on a portfolio of 125 credits. The credit structure of the collateral pool is also complicated, comprising firms from different sectors and with different credit ratings and in many cases the products are structured with credit having different notional amounts. Finally, since the different CDO tranches span the entire range of default probabilities, practical schemes must be very robust with respect to underlying parameters. Computational schemes which work in certain symmetric cases become infeasible in the general nonhomogeneous setting. The goal of this paper is to investigate whether the method of saddlepoint approximations, also called large deviation theory in probability, performs flexibly, robustly and accurately enough to be a reliable and general method for pricing CDOs.

The cash flow from a CDO is determined by the cumulative loss over time by default of the underlying reference credits. Understanding this involves modeling each credit's default probability, its loss given default, and the correlations among these quantities. There are a number of competing models which address this problem. [6] focused on modeling the correlation of default times by intensity-based models; [3], [11] have continued along these lines. The normal copula approach, pioneered by [15] and developed by [13], [1] and [10], is a simpler approach to modeling multifirm default, and forms the basis for most practical CDO computations because in important special cases it leads to tractable calibration and evaluation. The basic ingredient of models such as these which controls default correlations is the presence of one or more conditioning factors or common risk factors thought of as "macro–environmental variables". Conditional on knowing the values of these latent factors, firms' default times and loss amounts are assumed to be independent random variables. Therefore, conditional on the risk factors, the cumulative loss random variable is the sum of a large number of independent, but not identical Bernoulli random variables.

Many approximation schemes focus on this conditional independence structure. The normal proxy methods approximate the conditional loss by a normal random variable with the same mean and variance. More generally, Edgeworth expansions match the first n moments, and generate an asymptotic expansion for loss probabilities [9]. Other authors investigate Poisson approximations and compound Poisson approximations [12].

One promising approach to estimating conditional loss probabilities is the saddlepoint approximation method pioneered by [8] and [14]. In probability, the saddlepoint method is more generally known as large deviation theory [5], [9]. The papers [4] and [7] have successfully applied this theory to the general problem of estimating large portfolio losses. A recent work [2] has extended the saddlepoint approach to both risk measures and CDO pricing. This last paper finds that applied to typical CDOs, the saddlepoint method is a "clear winner" when compared to any moment matching method.

A key drawback of saddlepoint approximation methods is the lack of known general theoretical error estimates. Because of this gap in theory, it is important to have a good experimental knowledge of how the saddlepoint method applies to practical problems. The present paper focusses on how the saddlepoint approximation method performs when applied to the "tranche function", an important distribution function which governs the cashflows of CDOs.

Typically, an asset backed security (ABS) such as a CDO is constructed by "tranching" the profit X derived from a portfolio of underlying assets, where X is taken to be a random variable with support on the bounded set $[0, X_{max}]$. For a set of $K+1$ "attachment points" $a_0 = 0 < a_1 < \cdots < a_K = 1$ one decomposes the payoff as follows

$$X = X_{max} \sum_{k=1}^{K} \left[(X/X_{max} - a_{k-1})^+ - (X/X_{max} - a_k)^+ \right], \qquad (x)^+ := \max(x, 0),$$

where each term of the decomposition is called a "tranche". Valuing each tranche by taking discounted expectations leads to an expression involving $F(a_{k-1}) - F(a_k) + (a_k - a_{k-1})$ where

$$F(a) = E[(a - X/X_{max})^+].$$

The tranche construction has obvious financial merits. Mathematically, it turns out that all cashflows for ABS pricing can be formulated in terms of the *tranche function* $F(a)$ (thought of as a function of time), for which the saddlepoint approximation method is admirably suited. There is even a further elegant feature related to the saddlepoint method: Since the tranche function $F(a)$ is formally the integral of the cumulative distribution function of X/X_{max}, the smoothing effect of integration actually improves the performance of the saddlepoint method for the tranche function, over and above what is possible for the probability distribution function itself.

The paper is organized as follows. Section 2 puts in place the probabilistic assumptions underlying the loss process. Section 3 describes the cashflows for a typical synthetic CDO, and derives their basic mathematical properties. In section 4 we specify two standard copula models of multifirm default which are ideal test cases for CDO computations. Section 5 presents the theory of the saddlepoint approximation, and derives certain results which are specific to computing loss distributions.

It is important to note that the specific implementation of the saddlepoint we give is different from the method of [2]. Numerical results for both the tranche function and the components of a CDO are given in Section 6 which compare exact and approximate methods for four model realizations. Section 7 summarizes the main conclusions of the paper.

2 Default and portfolio loss distributions

We begin with probabilistic assumptions for a general framework for the default of the reference credits underlying a CDO, and derive basic formulas for the distribution of portfolio loss up to any time $t \in [0, T]$. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a filtered probability space that contains all of the random elements. Throughout this paper we will interpret P as the risk–neutral or pricing probability measure. The basic entities are:

- M reference credits with notional amounts of $N_j, j = 1, 2, \ldots, M$;
- the default time τ_j of the jth credit, an $\mathcal{F}_t$ stopping time;
- the fractional recovery R_j after default of the jth credit;
- the loss $l_j = (1 - R_j)N_j/N$ caused by default of the jth credit as a fraction of the total notional $N = \sum_j N_j$;
- the cumulative portfolio loss $L(t) = \sum_j l_j I(\tau_j \le t)$ up to time t as a fraction of the total notional.

The following assumptions will hold throughout this paper.

Assumptions:

1. The discount factor is $v(t) = e^{-rt}$ for a constant interest rate $r \ge 0$.
2. The fractional recovery values R_j and hence l_j are deterministic constants;
3. There is a sub σ–algebra $\mathcal{H} \subset \mathcal{F}$ generated by a d–dimensional random variable Y, the "condition", such that the default times τ_j are mutually conditionally independent under $\mathcal{H}$. The marginal distribution of Y is denoted by P_Y and has probability density function $\rho_Y(y), y \in \mathbb{R}^d$.

Remark: Assumption 2 is for simplicity. The computational methods we develop extend easily to models with stochastic recovery. In that more general setting, Assumption 3 should then state that $\{R_j, \tau_j\}$ are mutually conditionally independent under $\mathcal{H}$.

The most important consequence of these assumptions is that conditioned on $\mathcal{H}$, the fractional loss $L(t)$ is a sum of independent (but not identical) Bernoulli random variables: this is the object of mathematical interest. For the next few sections, we fix a value for the time t and conditioning random variable Y, and denote $\hat{L} := L(t)|_Y$. Then $\hat{L} \sim \sum_j l_j X_j$ where $X_j \sim \mathrm{Bern}(p_j), p_j = \mathrm{Prob}(\tau_j \leq t | Y = y)$.

We introduce the following functions associated to the random variable $\hat{L}$:

1. The *probability distribution function* (PDF) $\rho(x)$ (it is a sum of delta functions supported on the interval $[0, 1]$);
2. The *cumulative distribution function* (CDF) $F^{(0)}(x) = E[I(\hat{L} \leq x)]$;
3. The *higher conditional moment functions* $F^{(m)}(x) = E[((x - \hat{L})^+)^m], m = 1, 2, \dots$;
4. The *cumulant generating function of* $\hat{L}$ (CGF) $\Psi(u) = \log(E[e^{-u\hat{L}}])$.

Remarks and further definitions:

1. In later sections, we will make explicit the dependence on t, Y by writing

$$
\begin{aligned}
F^{(m)}(x|t, y) &= E[((x - L(t))^+)^m | Y = y], \\
F^{(m)}(x|t) &= E[((x - L(t))^+)^m] = \int_{\mathbb{R}^d} F^{(m)}(x|t, y) \rho_Y(y) dy. \qquad (1)
\end{aligned}
$$

2. It will be seen in later sections that the values of basic contingent claims connected with CDO tranches are all expressible in terms of the function $F^{(1)}$. We will therefore sometimes call it the "tranche function". The tranche function also arises in the study of portfolio risk measures, specifically conditional value–at–risk.

3. We shall measure the accuracy of an approximation $\tilde{F}(x|t, y)$ of $F(x|t, y)$ for $(x, t, y) \in [0, 1] \times [0, T] \times \mathbb{R}^d$ using the following norm depending on x, T, ρ_Y (T will be the maturity date of the CDO):

$$
\|\tilde{F} - F\|_{x,T,\rho_Y} := \sup_{t \in [0,T]} \int_{\mathbb{R}^d} |\tilde{F}(x|t, y) - F(x|t, y)| \rho_Y(y) dy. \qquad (2)
$$

The explicit form of the CGF of $\hat{L}$ is $\Psi(u) = \sum_j \log[1 - p_j + p_j e^{-u l_j}]$. The Fourier integral

$$
\rho(x) = \frac{1}{2\pi} \int_{-\infty + i\alpha}^{\infty + i\alpha} e^{ikx} e^{\Psi(ik)} dk \qquad (3)
$$

for the PDF exists as a distribution (generalized function) for any $\alpha \in \mathbb{R}$. The value of this integral is independent of $\alpha \in \mathbb{R}$ since the *moment generating function* $e^{\Psi(u)}$ is entire analytic in u. After noting

$$
\frac{dF^{(m)}(x)}{dx} = m F^{(m-1)}(x), \quad m = 1, 2, \dots
$$

one can prove that the CDF and higher conditional moment functions are given by related Fourier transforms

$$F^{(m)}(x) = \frac{m!}{2\pi} \int_{-\infty+i\alpha}^{\infty+i\alpha} (ik)^{-m-1} e^{ikx} e^{\Psi(ik)} dk, \quad m = 0, 1, 2, \dots. \tag{4}$$

Now one needs to account for the presence of the single pole at $k = 0$: One can check that (4) is consistent with the boundary condition $F^{(m)}(0) = 0$ for any contour with $\alpha < 0$. For use in the next sections, we also define

$$G^{(m)}(x) = \frac{m!}{2\pi} \int_{-\infty+i\alpha}^{\infty+i\alpha} (ik)^{-m-1} e^{ikx} e^{\Psi(ik)} dk, \quad m = 0, 1, 2, \dots. \tag{5}$$

for any contour with $\alpha > 0$. An easy application of the Cauchy Integral Formula shows that

$$F^{(1)}(x) = G^{(1)}(x) - E[\hat{L}] + x \tag{6}$$

with similar formulas relating $F^{(m)}$ and $G^{(m)}$ for $m = 2, 3, \dots$.

3 The structure of CDOs

CDOs can be schematically decomposed into two types of basic contingent claims whose cash flows depend on the default losses of the underlying portfolio where the reference credits are corporate bonds of similar maturity issued by a number of firms. These cash flows are analogous to insurance and premium payments paid to cover default events.

The writer (the *insurer*) of one unit of a *basic default leg with attachment level* a pays the holder (the *buyer of insurance*) all default losses up to a certain level $0 < a \leq 1$ over $[0, T]$, the term of the contract. This means that at the default time $\tau_j \leq T$ of any firm, the parties exchange cash of the amount

$$(a - L(\tau_j -))^+ - (a - L(\tau_j))^+ ,$$

where $L(\tau_j -)$ denotes the left limit of the cádlág process L_t.

The writer of one unit of a *basic premium leg with attachment level* a (the insured) pays the holder (the insurer) on prespecified payment dates $t_k < t_K = T$, $k = 0, 1, \dots, K-1$ an amount jointly proportional to the year fraction $t_{k+1} - t_k$ and the residual portfolio value below a. In practice there is also a reduction called the accrual term to account for defaults between payment dates which we ignore in this paper. Thus the cash exchanged on date t_k is

$$(t_{k+1} - t_k)(a - L(t_k))^+.$$

To avoid unnecessary complication, in this paper we treat only the limiting case when payments are made continuously in time.

Proposition 3.1. *Under the assumptions and definitions stated in section 2*

1. *The fair price of the basic default leg with attachment level* a *is*

$$W(a) = a - e^{-rT}F^{(1)}(a|T) - \int_0^T re^{-rt}F^{(1)}(a|t)dt; \tag{7}$$

2. *The fair price of the basic premium leg with attachment level* a *and continuous payments is*

$$V(a) = \int_0^T e^{-rt}F^{(1)}(a|t)dt. \tag{8}$$

The following approximation result is an easy consequence:

Corollary 3.2. *Under the assumptions and definitions stated in section 2, if* $\tilde{F}^{(1)}$ *is an approximation of the tranche function* $F^{(1)}$*, then the accuracy of the corresponding tranche leg approximations* $\tilde{V}(a), \tilde{W}(a)$ *is given by*

$$|\tilde{V}(a) - V(a)| \leq r^{-1}(1 - e^{-rT})\|\tilde{F}^{(1)} - F^{(1)}\|_{a,T,\rho_Y} \tag{9}$$
$$|\tilde{W}(a) - W(a)| \leq \|\tilde{F}^{(1)} - F^{(1)}\|_{a,T,\rho_Y} \tag{10}$$

where the norm is defined by (2).

Remarks and definitions:

1. This proposition and corollary underscore the need for accurate and efficient computation of the tranche functions $F^{(1)}(x|t, y), F^{(1)}(x|t)$ over the full range of parameters t, y, x, for portfolios with a large number M of possibly non-homogeneous reference credits. This, the main challenge for any numerical algorithm, will be efficiently solved by the saddlepoint method. The t and y integrals implicit in (7) and (8) can be computed by a discrete approximation of the form

$$\int_{[0,T]\times\mathbb{R}^d} f(t, y)\rho_Y(y)dtdy \sim \sum_{i,j} w_i v_j f(t_i, y_j) \tag{11}$$

   over a grid of points (t_i, y_j) with suitable integration weights w_i, v_j.

2. A *CDO tranche* is defined by upper and lower attachment levels $0 \leq a < b \leq 1$, usually expressed as percentages. The so-called *base tranche* are those with $a = 0$. In the following, we take tranches defined by the attachment levels 3%, 7%, 10%, 15%, 30% as used for the CDX index products. The lowest tranche $[0, .03]$ is called the *equity tranche* or *toxic waste*; the middle tranches are called *mezzanine tranches*; the highest tranche $[0.3, 1]$ is called the *super-senior tranche*. The *default* $[a, b]$*–tranche leg* pays the amount

$$(b - L(\tau_j-))^+ - (b - L(\tau_j))^+ - (a - L(\tau_j-))^+ - (a - L(\tau_j))^+$$

at each default time $\tau_j \le T$ and its fair price is therefore $W(b) - W(a)$. The *premium* $[a, b]$*–tranche leg* pays periodic amounts

$$(t_{k+1} - t_k)\left((b - L(t_k))^+ - (a - L(t_k))^+\right)$$

and hence the fair price for the continuous time premium leg is $V(b) - V(a)$.

3. The $[a, b]$*–tranche spread* is that multiplier X of the premium tranche leg which solves the balance equation

$$X\,((V(b) - V(a)) = W(b) - W(a). \tag{12}$$

Proof of Proposition 3.1:

1. Over the term of the basic default leg, the payments have present value

$$\sum_j I(\tau_j \le T)e^{-r\tau_j}\left[(a - L(\tau_j -))^+ - (a - L(\tau))^+\right] = -\int_0^T e^{-rt}\ d(a - L(t))^+.$$

Since e^{-rt} is differentiable and $(a - L(t))^+$ is right-continuous, integration by parts gives the value

$$a - e^{-rT}(a - L(T))^+ - \int_0^T (a - L(t))^+ re^{-rt} dt. \tag{13}$$

Formula (7) is given by taking expectations and using the definition (1).

2. Over the term of the basic premium leg, the payments have present value which is a Riemann sum approximation of the integral

$$\sum_j (t_{j+1} - t_j)e^{-rt_j}(a - L(t_j))^+ \to \int_0^T e^{-rt}(a - L(t))^+ dt.$$

Formula (8) is given by taking expectations of the right hand side and using the definition (1).

□

4 Standard copula default models

We now briefly describe two standard models of multifirm default which are ideal test cases for comparing the saddlepoint method to the exact computation.

4.1 One factor normal copula default model

In this model, the joint distribution of default times $\{\tau_j\}$ is specified by an arbitrary choice of marginal cumulative distribution function $F_j(t) := E[I(\tau_j \le t)]$ for each firm and the selection of a one factor normal copula to describe the correlation structure. The default times $\{\tau_j\}$ are defined in terms of a multidimensional normal random variable $\vec{Z} = (Z_1, \ldots, Z_M)$ by $Z_j = H_j(\tau_j),\ j = 1, 2, \ldots, M$. The marginals of Z_j are assumed to be standard normals, and $H_j = \Phi^{-1} \circ F_j$ where Φ^{-1} denotes the inverse cumulative distribution function of the standard normal random variable. This guarantees that the marginal CDF of τ_j is F_j.

The joint distribution function of default is given by

$$P[\tau_j \le t_j, j = 1, \ldots, M] = P\big[Z_j \le H_j(t_j), j = 1, \ldots, M\big].$$

The one factor normal copula arises by taking $\vec{Z}$ to have mean zero and covariance matrix

$$E[Z_i Z_j] = \begin{cases} 1 & i = j \\ a_i a_j & i \ne j \end{cases}$$

with the values a_j in $[-1, 1]$.

Assumption 3 can be verified for this model. If we let $\{Y, Y_1, \ldots, Y_M\}$ be iid standard normal random variables, then the random variables $Z_j = a_j Y + \sqrt{1 - a_j^2} Y_j, j = 1, \ldots, M$ have the required joint distribution. If $\mathcal{H}$ is defined to be $\sigma(Y)$, the sigma algebra of the single factor Y, then one can compute that the Z's are independent conditionally on Y:

$$\begin{aligned} P[\tau_j \le t_j, j = 1, \ldots, M] &= E\big[P[Z_j \le z_j, j = 1, \ldots, M | \mathcal{H}]\big] \\ &= \int_{\mathbb{R}} P\big[a_j Y + \sqrt{1 - a_j^2} Y_j \le z_j, j = 1, \ldots, M | Y = y\big] \phi(y) dy \\ &= \int_{\mathbb{R}} \prod_j P\big[Y_j \le (z_j - a_j y)/\sqrt{1 - a_j^2}\big] \phi(y) dy \\ &= \int_{\mathbb{R}} \prod_j \Phi\Big((z_j - a_j y)/\sqrt{1 - a_j^2}\Big) \phi(y) dy, \end{aligned} \tag{14}$$

Here Φ, ϕ are the CDF and PDF respectively of the standard normal random variable.

4.2 Clayton copula default model

In this model, discussed in [13], the joint distribution of default times $\{\tau_j\}$ is specified by an arbitrary choice of marginal cumulative distribution function $F_j(t) := E[I(\tau_j \le t)]$ for each firm and correlation structure given by a Clayton copula parametrized by $\theta > 0$. The M–dimensional Clayton copula $C(u_1, \ldots, u_M)$ is the joint cumulative distribution function of random variables $\{U_j\}_{j=1}^M$ constructed in the form $U_j = \psi_\theta(Z_j)$. Here $Z_j = -Y^{-1}\log Y_j$ where $\{Y, Y_1, \dots, Y_M\}$ are mutually independent random variables. Y has a Gamma $(1/\theta, 1)$ distribution with distribution function $\Gamma(y)$ and Laplace transform $\psi_\theta(z) = (z+1)^{-1/\theta}$. Each Y_j has a uniform $[0,1]$ distribution. One can then check that the marginals of U_j are uniform:

$$
\begin{aligned}
P[U_j \le u] &= \int_0^\infty P[Z_j \ge \psi_\theta^{-1}(u) | Y = y] d\Gamma(y) \\
&= \int_0^\infty P[Y_j \le \exp(-y\psi_\theta^{-1}(u))] d\Gamma(y) \\
&= \int_0^\infty \exp(-y\psi_\theta^{-1}(u)) d\Gamma(y) \\
&= \psi_\theta(\psi_\theta^{-1}(u)) = u.
\end{aligned} \tag{15}
$$

One can also check that the copula has the "arithmetic" form:

$$
\begin{aligned}
C(u_1, \dots, u_M) &= \int_0^\infty \prod_j \left(P[Z_j \ge \psi_\theta^{-1}(u_j) | Y = y] \right) d\Gamma(y) \\
&= \int_0^\infty \prod_j \left(P[Y_j \le \exp(-y\psi_\theta^{-1}(u_j))] \right) d\Gamma(y) \\
&= \int_0^\infty \exp\left(-y \sum_j \psi_\theta^{-1}(u_j)\right) d\Gamma(y) \\
&= \psi_\theta\left(\psi_\theta^{-1}(u_1) + \cdots + \psi_\theta^{-1}(u_M)\right).
\end{aligned} \tag{16}
$$

Putting these pieces together leads to default times τ_j which are defined by

$$
\tau_j = F_j^{-1}(U_j) = F_j^{-1}(\psi_\theta(-Y^{-1}\log Y_j)). \tag{17}
$$

Note in particular that Assumption 3 holds for $\mathcal{H} = \sigma(Y)$.

5 The saddlepoint method

This method is a part of the theory of large deviations which develops generalizations of the central limit theorem. In the present context, we use it to approximate Fourier integrals such as (4) which have the form

$$
\int_{-\infty+i\alpha}^{\infty+i\alpha} e^{G(ik)} dk
$$

by exercising our freedom to choose α so that the contour passes through a *critical point* of G, i.e. a value $u^* \in \mathbb{C}$ where $G'(u^*) = 0$. Taylor expansion of G about u^* then leads to an asymptotic expansion of the integral which in some circumstances yield very useful approximations. For a review of the method, see [5].

We illustrate the method for the PDF Fourier integral (3) in the symmetric case $p_j = p, l_j = l$. Now $\Psi(u) = M \log[1 - p + pe^{-ul}]$ and for any $x \in (0, lM)$ the unique critical point u^* solving $x + \Psi'(u) = 0$ is given by

$$u^* = u^*(x, M) = \frac{1}{l} \log \left[\frac{p(lM - x)}{(1 - p)x} \right]. \tag{18}$$

The Taylor expansion $ikx + \Psi(ik) = u^*x + \Psi(u^*) + \sum_{n=2}^{\infty} \frac{1}{n!} \Psi^{(n)}(u^*)(ik - u^*)^n$ is plugged into the exponent, and the contour is chosen with $\alpha = -u^*$, leading to

$$\begin{aligned} \rho(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp \left[u^*x + \Psi(u^*) + \sum_{n=2}^{\infty} \frac{1}{n!} \Psi^{(n)}(u^*)(ik)^n \right] dk & (19) \\ &\sim \frac{e^{u^*x+\Psi(u^*)}}{2\pi} \int_{-\infty}^{\infty} e^{-\Psi^{(2)}(u^*)k^2/2} \left[1 - \frac{i}{3!} \Psi^{(3)}(u^*)k^3 + \frac{1}{4!} \Psi^{(4)}(u^*)k^4 + \cdots \right] dk \end{aligned}$$

One can do the resulting Gaussian integrals, noting that the odd order terms vanish, and obtain the asymptotic expansion

$$\rho(x) \sim \frac{e^{u^*x+\Psi(u^*)}}{\sqrt{2\pi\Psi^{(2)}(u^*)}} \left[1 + \frac{\Psi^{(4)}(u^*)}{8(\Psi^{(2)}(u^*))^2} - \frac{5(\Psi^{(3)}(u^*))^2}{24(\Psi^{(2)}(u^*))^3} + \cdots \right] \tag{20}$$

This example is interesting because the left hand side is a sum of delta functions: the approximation must be understood in a weak or distributional sense. In this example, $\Psi^{(n)}(u^*) = O(M)$ for all orders of differentiation, from which one can see that the second and third terms in (20) are $O(M^{-1})$ while the first omitted term is $O(M^{-2})$. We therefore expect the asymptotic estimate for the relative error of the first order approximation (which omits the second and third terms) to be $O(M^{-1})$ while the second order approximation, which includes these two terms, should be $O(M^{-2})$. Figure 1 illustrates the first and second approximation functions of the loss distribution over a five year time horizon for baskets of $M = 32$ and $M = 128$ firms, with identical independent $Exponential(\lambda)$ distributed default times with $\lambda = 0.1$. Note that both approximations fit the tail of the distribution well, but the second approximation fits the mode better.

The saddlepoint approximation method can be extended to the computation of the integrals (4) for $F^{(m)}(x)$ and (5) for $G^{(m)}(x)$ in the fully nonsymmetric case with p_j, l_j all different. Care must now be taken over the placement of the contour around the pole at $k = 0$. Since $E[\hat{L}]$ is explicit and $F^{(1)}(x) = G^{(1)}(x) - E[\hat{L}] + x$ (with similar formulas relating $F^{(m)}$ and $G^{(m)}$), one can compute $F^{(m)}$ either directly by (4) or indirectly from (5). In both cases, the saddlepoint method is implemented by rewriting the integrands in exponential form, for example,

$$F^{(m)}(x) = \frac{m!}{2\pi} \int_{-\infty+i\alpha}^{\infty+i\alpha} e^{ikx+\Psi(ik)-(m+1)\log ik} dk, \quad \alpha < 0, \tag{21}$$

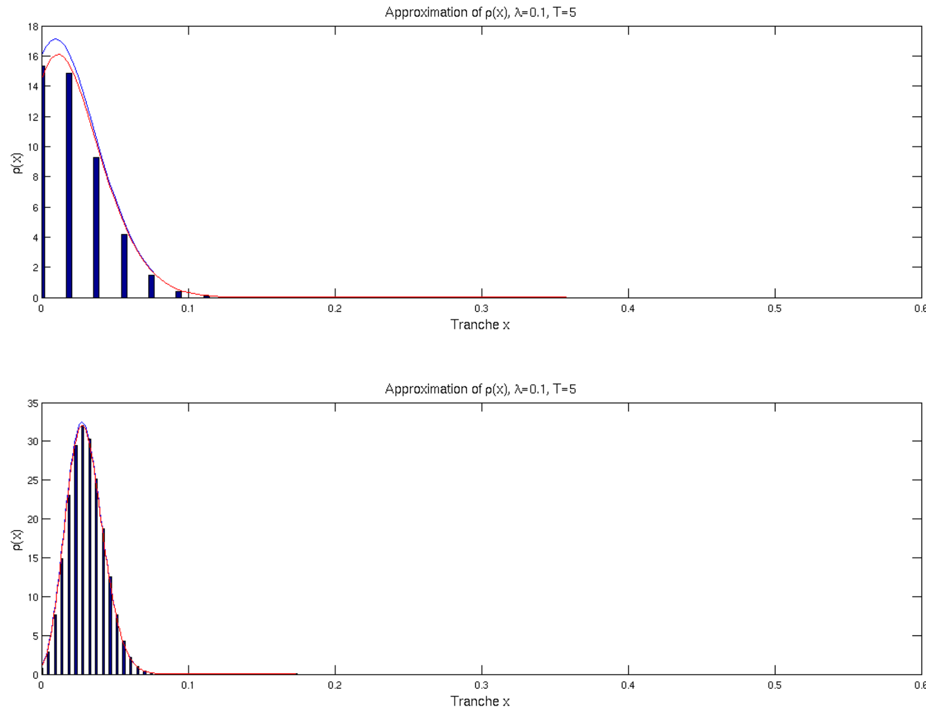

Figure 1: The first and second order saddlepoint approximation functions of the discrete binomial distribution when $M = 32, 128$, $\lambda = 0.1$ and $T = 5$. In the first graph, the second approximation is the lower curve.

and numerically solving the saddlepoint condition

$$x + \Psi'(u) - (m+1)/u = 0. \tag{22}$$

The next proposition shows that a choice of two saddlepoints solving this equation is typically available:

Proposition 5.1. *Suppose that $p_j,\ l_j > 0$ for all j. Then*

1. *There is a solution u^*, unique if it exists, of $x + \Psi'(u) = 0$ if and only if $0 < x < \sum_j l_j$. If $E[\hat{L}] > x > 0$ then $u^* > 0$ and if $E[\hat{L}] < x < \sum_j l_j$ then $u^* < 0$.*

2. *For each $m \geq 0$ there is exactly one solution u_m^- of (22) on $(-\infty, 0)$ if $x < \sum_j l_j$ and no solution on $(-\infty, 0)$ if $x \geq \sum_j l_j$. Moreover, when $x < \sum_j l_j$ the sequence $\{u_m^-\}_{m \geq 0}$ is monotonically decreasing in m.*

3. *For each $m \geq 0$ there is exactly one solution u_m^+ of (22) on $(0, \infty)$ if $x > 0$ and no solution on $(0, \infty)$ if $x \leq 0$. Moreover, when $x > 0$ the sequence $\{u_m^+\}_{m \geq 0}$ is monotonically increasing in m.*

Remarks:

1. Interestingly, [2] apply the saddlepoint method differently: In that paper, the saddlepoint u^* is used rather than one of the points $u_m^\pm$. We have not attempted to compare the relative merits of the two schemes.

2. In the range of interest $0 < x < \sum_j l_j$, for each $m \geq 0$, we always have the choice between the positive and negative saddlepoints. When the positive saddlepoint u_m^+ is used, $F^{(m)}$ is computed directly by (20) with $\Psi(u)$ replaced by $\Psi(u) - (m+1)\log|u|$. On the other hand, when the negative saddlepoint u_m^- is used, $F^{(m)}$ is computed indirectly using (6) and its generalizations, with $G^{(m)}$ approximated by (20) with $\Psi(u)$ replaced by $\Psi(u) - (m+1)\log|u|$. We argue below that it is reasonable to pick the saddlepoint with the same sign as u^*.

3. The monotonicity in m seems to relate to the beneficial effects of integration in the saddlepoint method.

Proof of Proposition 5.1:

1. For $\Psi(u) = \sum_j \log(1 - p_j + p_j e^{-ul_j})$, we have

$$\begin{aligned}\Psi'(u) &= \sum_j \frac{-p_j l_j e^{-ul_j}}{1 - p_j + p_j e^{-ul_j}} \\ \Psi^{(2)}(u) &= \sum_j \frac{p_j l_j^2 e^{-ul_j}(1-p_j)}{(1 - p_j + p_j e^{-ul_j})^2} > 0.\end{aligned}$$

These equations show that $\Psi'(u)$ is strictly increasing on $u \in (-\infty, \infty)$ and that

$$\Psi'(0) = -E(\hat{L}), \quad \Psi'(\infty) = 0, \quad \Psi'(-\infty) = -\sum_j l_j.$$

The above properties tell us that u^* exists if and only if $0 < x < \sum_j l_j$. If $0 < x < E[\hat{L}]$, then $u^* > 0$; if $E[\hat{L}] < x < \sum_j l_j$, then $u^* < 0$.

2. Notice that for fixed x, $x + \Psi'(u) - \frac{m+1}{u}$ is strictly increasing over the interval $(-\infty, 0)$ and

$$\lim_{u \to -\infty} \left(x + \Psi'(u) - \frac{m+1}{u} \right) = x - \sum_j l_j, \quad \lim_{u \uparrow 0} \left(x + \Psi'(u) - \frac{m+1}{u} \right) = \infty.$$

Thus for $-\infty < x < \sum_j l_j$, there is exactly one solution u_m^- on $(-\infty, 0)$ of the equation $x + \Psi'(u) - \frac{m+1}{u} = 0$; For $x \geq \sum_j l_j$, there is no solution on $(-\infty, 0)$. To prove the monotonicity result of part 2, note that the mean value theorem implies that for any $m \geq 1$

$$\left(\Psi'(u_m^-) - m/u_m^-\right) - \left(\Psi'(u_{m-1}^-) - m/u_{m-1}^-\right) = (\Psi^{(2)}(\xi) + m/\xi^2)(u_m^- - u_{m-1}^-)$$

for some intermediate value ξ. Combining this equation with (22) for $m-1$ and m yields the equation

$$(\Psi^{(2)}(\xi) + m/\xi^2)(u_m^- - u_{m-1}^-) = 1/u_m^-.$$

Since $(\Psi^{(2)}(\xi) + m/\xi^2)$ is everywhere positive, $u_m^- < 0$ implies $u_m^- - u_{m-1}^- < 0$.

3. The results of part 3 are proved in a similar manner.

□

To approximate the tranche function $F^{(m)}$ on a grid of (t, y) values, the solution of (22) must be computed numerically for each (t, y) pair. This can always be done using a modified Newton-Raphson method, by starting from an initial value $u^{(0)}$ and iterating the formula:

$$u^{(k)} = u^{(k-1)} - \frac{\Psi'(u^{(k-1)}) - (m+1)/u^{(k-1)} + x}{\Psi^{(2)}(u^{(k-1)}) + (m+1)/(u^{(k-1)})^2}, \quad k = 1, 2, \ldots . \tag{23}$$

We searched for the fixed point with the same sign as $u^{(0)}$: if ever the value of $u^{(k)}$ changed sign, we simply replaced (23) by $u^{(k)} = u^{(k-1)}/10$. This modification takes care of any instability in the iterations which might otherwise develop, and the resulting sequence converges to u_m^- whenever $u^{(0)} < 0$ and to u_m^+ whenever $u^{(0)} > 0$.

The remaining question concerns the choice of initial value $u^{(0)}$, and hence which of $u_m^\pm$ to be used. Part (1) of Proposition 5.1 implies that NR iteration beginning from $u^{(0)} = u^*$ will lead to u_m^- if $E[\hat{L}] < x < \sum_j l_j$ and u_m^+ if $0 < x < E[\hat{L}]$. In nonhomogeneous cases where u^* is not explicit, we instead take $u^{(0)}$ given by (18) with $l = \sum_j l_j/M$ and $p = \sum_j p_j l_j / \sum_j l_j$, with the same result guaranteed. The resulting NR iterations typically converge in 5 or 6 steps.

6 Numerical results

In this section, we survey the performance of the saddlepoint method over a wide range of models with an aim to test the following criteria: their effectiveness compared to the Edgeworth approximation and Monte Carlo simulation; accuracy and how it varies over the parameter set; computational efficiency; and robustness and sensitivity to roundoff errors of the numerical algorithm; sensitivity to nonhomogeneity of the underlying firms. We will be focusing on the dependence on four key parameters M, t, y, x. The number of firms varies over the values $M = 2^m, m = 2, 3, \ldots, 10$; the time t ranges over $[0, T]$ for a time horizon $T = 5$ years; the conditioning random variable y ranges over $[-3, 3]$ for the normal copula and $[0, 5]$ for the Clayton copula; and finally the tranche variable x ranges over the standard attachment points (0.3, 0.15, 0.10, 0.07, 0.03). All remaining parameters are set by one of the following four fixed specifications of one factor copula models whose marginal default times are exponentially distributed with parameter λ:

1. Problem A (Homogeneous default probabilities, normal copula and loss amounts): $\lambda_j = 0.01$, $a_j = \sqrt{0.3}$, $l_j = 0.6/M$;

2. Problem B (Homogeneous normal copula and loss amounts, nonhomogeneous default probabilities): $a_j = \sqrt{0.3}$, $l_j = 0.6/M$ and either $\lambda_j = 0.01$ or $\lambda_j = 0.04$ with equal likelihood.

3. Problem C (Homogeneous Clayton copula, homogeneous loss amounts, nonhomogeneous default probabilities): $\theta = 0.173$, $l_j = 0.6/M$ and either $\lambda_j = 0.01$ or $\lambda_j = 0.04$ with equal likelihood.

4. Problem D (Homogeneous normal copula, nonhomogeneous default probabilities and loss amounts): Here we take all firms with $a_j = \sqrt{0.3}$, and the firms falling into four equal size groups with parameters (λ_j, l_j) one of $(0.01, 0.6/M)$, $(0.01, 0.15/M)$ $(0.04, 0.6/M)$, $(0.04, 0.15/M)$.

We computed the various model specifications on a grid of (t, y) pairs in four different ways.

1. Method 1 (Exact): Problems A, B, C, D can all be computed exactly using "shortcuts" not applicable to fully nonhomogeneous portfolios. Problem A is computable in $O(M)$ flops in terms of binomial probabilities. Problems B, C, can be computed in $O(M^2)$ flops by a pairwise convolution of binomial distributions. Problem D can be computed in $O(M^4)$ flops by a four-fold convolution of binomial distributions.

2. Method 2 (Edgeworth method); This method approximates the conditional loss distribution by a Gaussian distribution $N(\mu, \sigma^2)$ with first and second moments computed by the formulas:

$$\mu = \sum_j p_j l_j, \qquad \sigma^2 = \sum_j p_j(1 - p_j) l_j^2.$$

3. Method 3 (First order saddle method): This method computes one of the saddlepoints $u_1^{\pm}$ by NR iteration and takes the first term of (20), leading to a relative error which is heuristically $O(M^{-1})$. The rule for selecting which saddlepoint to use is given in the last paragraph of Section 5.

4. Method 4 (Second order saddle method): This method computes one of the saddlepoints $u_1^{\pm}$ by NR iteration and takes the first three terms of (20), leading to a relative error which is heuristically $O(M^{-2})$. The rule for selecting which saddlepoint to use is given in the last paragraph of Section 5.

We repeat the observation that Methods 2, 3, 4 all work for fully nonhomogeneous portfolios in $O(M)$ flops.

6.1 Results on the tranche function

In this section we focus on the tranche function $F(x,t,y) := F^{(1)}(x|t,y)$. Table 1 shows the saddlepoints computed as an intermediate step for problem A for various values of t, x and with $y = 0, M = 128$:

	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
$x = 0.03$	-655.25280476	-460.75355618	-351.31097847	-277.22907362	-223.05280579
$x = 0.07$	-837.83258066	-637.19755394	-521.31324452	-440.07509682	-377.85318316
$x = 0.1$	-923.73264541	-722.10761622	-605.25181996	-522.98398033	-459.63397500
$x = 0.15$	-1030.87663034	-828.62318833	-711.17793620	-628.31017117	-564.32754966
$x = 0.3$	-1263.83462458	-1061.12171124	-943.26415657	-859.99419953	-795.60487253

Table 1: The saddlepoints computed for problem A with $t = 1, 2, \cdots, 5$, $x = [0.03, 0.07, 0.1, 0.15, 0.3]$ and $y = 0$.

The graphs shown in Figures 2, 3, 4, 5 represent the norm (2) of the difference between the exact method and the three approximations as a function of the number of firms M, for each of the five attachment points, for problems A, B, C, D. The norm is computed over a discrete set of 40×20 (t, y) pairs. A number of observations can be made based on these figures:

1. Absolute errors are not very sensitive to tranche level;
2. Method 2 typically shows $O(M^{-p})$ accuracy with p between 1 and 2;
3. The accuracy of both Methods 3,4 tends to flatten out for large M;
4. Method 4 usually outperforms Method 2 for M smaller than 100, but not always when M gets large;
5. Method 4 always outperforms Method 3, and is typically about ten times more accurate;
6. Comparison of Problems B, C, D to Problem A shows that nonhomogeneity of the firms degrades the accuracy of saddlepoint methods somewhat;
7. The NR iterations needed for Methods 3 and 4 typically converge in 4 to 10 steps. Thus Methods 3 and 4 run on average about 7 times slower than Method 2.

Programming methods 3 and 4, the saddlepoint methods, present some difficulties not encountered using the other methods. The Newton–Raphson root finding method at the core of the saddlepoint presents three extra difficulties: initializing the recursion, finding a robust termination condition for the iteration, and the choice of positive or negative saddlepoint. All three difficulties did arise in practise for some parts of the parameter space of interest, but were overcome satisfactorily with the detailed approach described in Section 5. Some further issues were noticed: roundoff error became an important limit on accuracy for Methods 3 and 4 when the firm number exceeded 2^{10}; roundoff error also led to the possibility of (theoretically impossible) cycles in the NR iteration when the firm size exceeded 2^{10}. Beyond these difficulties, the saddlepoint methods were easy to program.

6.2 Results for CDOs

We computed CDO spreads using equations (7), (8), (11), (12) for a wide range of model specifications, and overall found consistency with the numerics of the tranche function itself. Tables 2 and 3 show typical results: these tables compare CDO spreads computed for Problems A and B for various values of M, using Methods 1, 2, 3 and 4.

The one surprising aspect of these tables is that for CDO spreads Method 4 does not outperform Method 3 to the extent one would guess from Figures 2 and 3: it appears that Method 3 does better than one might expect.

7 Conclusion

The present paper has shown that the second order saddlepoint method offers generally superior performance in computing the loss tranche function and CDO prices, for large credit portfolios. Compared to the alternatives, the normal proxy and Edgeworth expansions, the saddlepoint method adds little extra computational cost or programming difficulty, and usually yields better results. The second order method usually outperforms the first order method. The saddlepoint methods are applicable to fully nonhomogeneous portfolios where no exact computations are possible. Although no general theoretical bounds are known for saddlepoint methods, their accuracy is apparently in line with the heuristics of asymptotic expansions.

References

[1] Andersen, L., Sidenius, J., and Basu, S. All your hedges in one basket. *Risk 16* (2003), 67–72.

[2] Antonov, A., Mechkov, S., and Misirpashaev, T. Analytical techniques for synthetic CDOs and credit default risk measures. Numerix Preprint `www.defaultrisk.com/pp_crdrv_77.htm`, 2005.

[3] Chen, L., and Filipović, D. Credit derivatives in an affine framework. working paper, 2004.

[4] Dembo, A., Deuschel, J.-D., and Duffie, D. Large portfolio losses. *Finance Stoch. 8*, 1 (2004), 3–16.

[5] Dembo, A., and Zeitouni, O. *Large deviations techniques and applications*, vol. 38 of *Applications of Mathematics (New York).* Springer-Verlag, New York, 1998.

[6] Duffie, D., and Garleanu, N. Risk and valuation of collateralized debt obligations. *Financial Analysts Journal 57* (2001), 41–59.

[7] Glasserman, P. Tail approximations for portfolio credit risk. *J. Derivatives 12* (2004), 24–42.

[8] Gordy, M. Saddlepoint approximation of credit risk. *J. Banking Finance 26* (2002), 1335–1353.

[9] Hall, P. *The Bootstrap and Edgeworth Expansion*. Springer-Verlag, 1992.

[10] Hull, J., and White, A. Valuation of a CDO and an nth to default CDS without Monte Carlo simulation. *Jour. of Derivatives* (2004), 8–23.

[11] Hurd, T., and Kuznetsov, A. Fast CDO computations in the Affine Markov Chain model. Preprint available at `http://www.defaultrisk.com/pp_crdrv_65.htm`, 2005.

[12] Iscoe, I., and Kreinin, A. Pricing and valuation of synthetic CDOs. Working Paper. Algorithmics Inc., 2004.

[13] Laurent, J.-P., and Gregory, J. Basket default swaps, CDOs and factor copulas. *Jour. of Risk* (2003), 21.

[14] Martin, R., Thompson, K., and Browne, C. Taking to the saddle. In *Credit Risk Modelling: The Cutting-edge Collection*, M. Gordy, Ed. Riskbooks, London, 2003.

[15] Vasicek, O. Probability of loss on loan portfolio. KMV Working Paper `www.defaultrisk.com/pp_model_60.htm`, 1987.

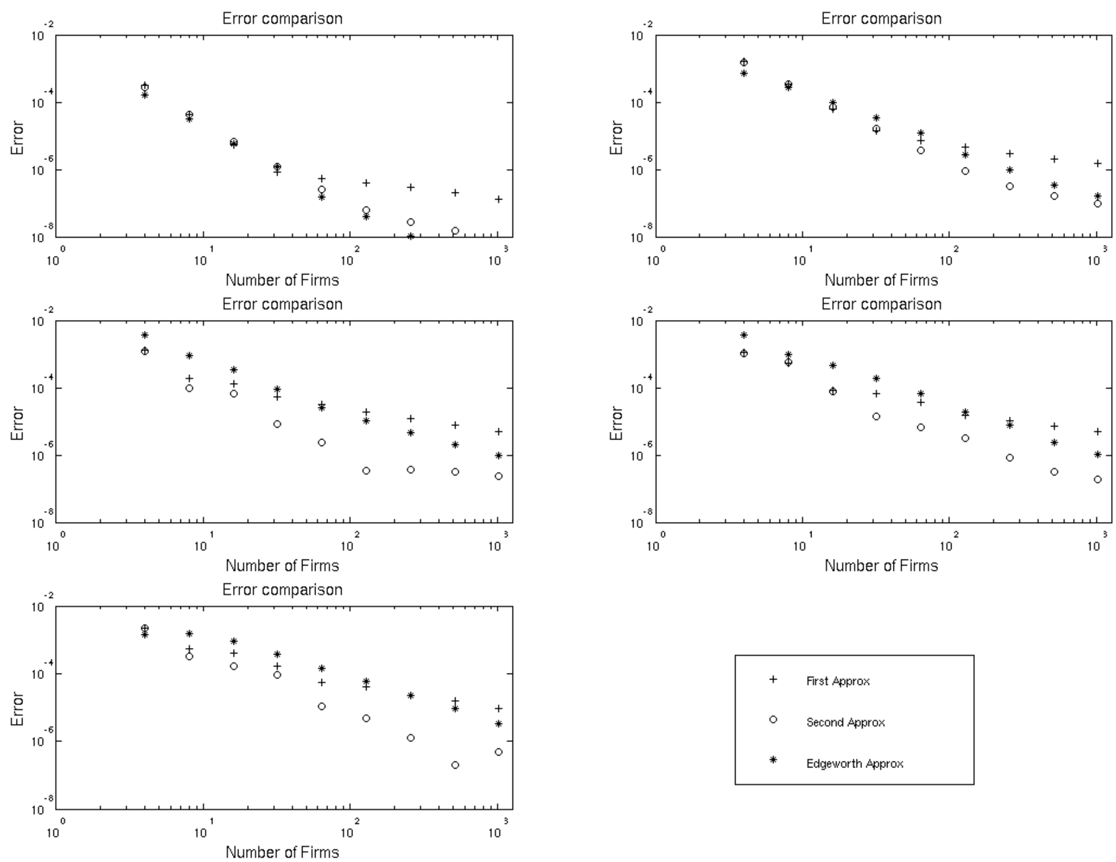

Figure 2: Performance of the three approximation schemes on Problem A, showing averaged error E versus number of firms M. Subgraphs one to five show the five tranche levels $0.3, 0.15, 0.10, 0.07, 0.03$.

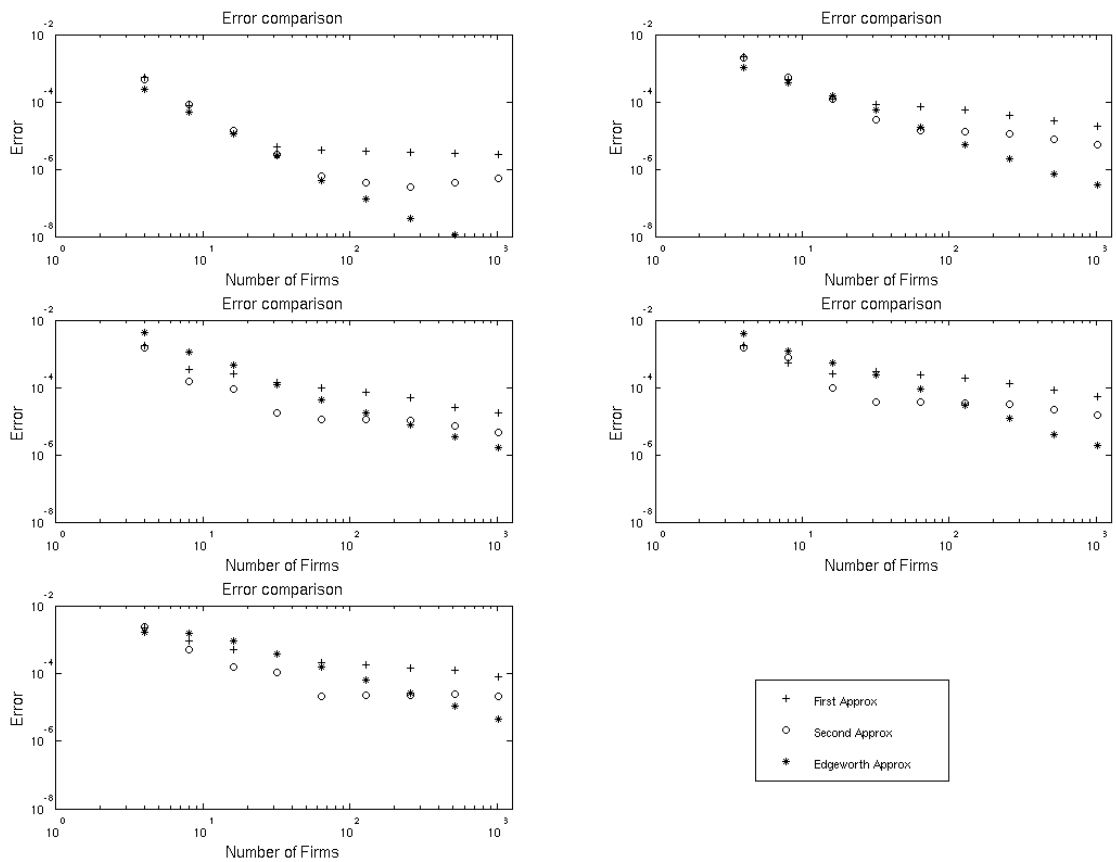

Figure 3: Performance of the three approximation schemes on Problem B, showing averaged error E versus number of firms M. Subgraphs one to five show the five tranche levels $0.3, 0.15, 0.10, 0.07, 0.03$.

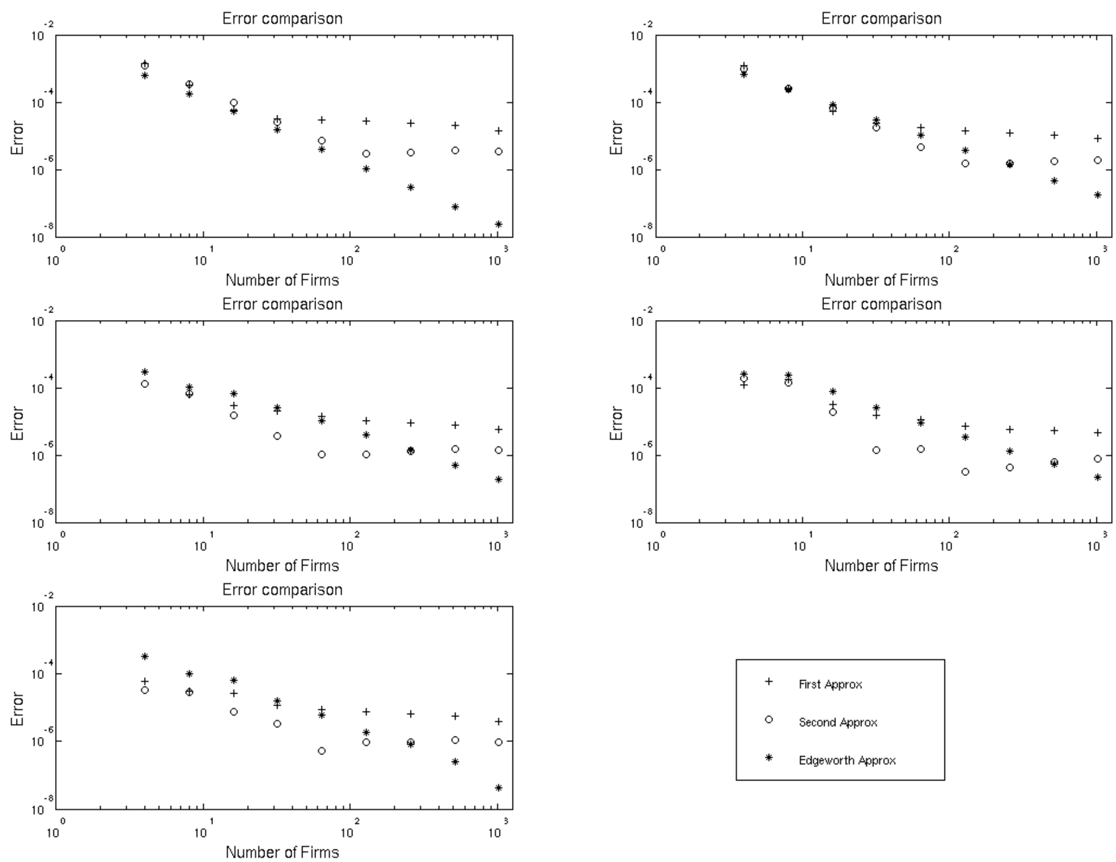

Figure 4: Performance of the three approximation schemes on Problem C with Clayton Copula where $\theta = 0.173$, showing averaged error E versus number of firms M. Subgraphs one to five show the five tranche levels $0.3, 0.15, 0.10, 0.07, 0.03$,

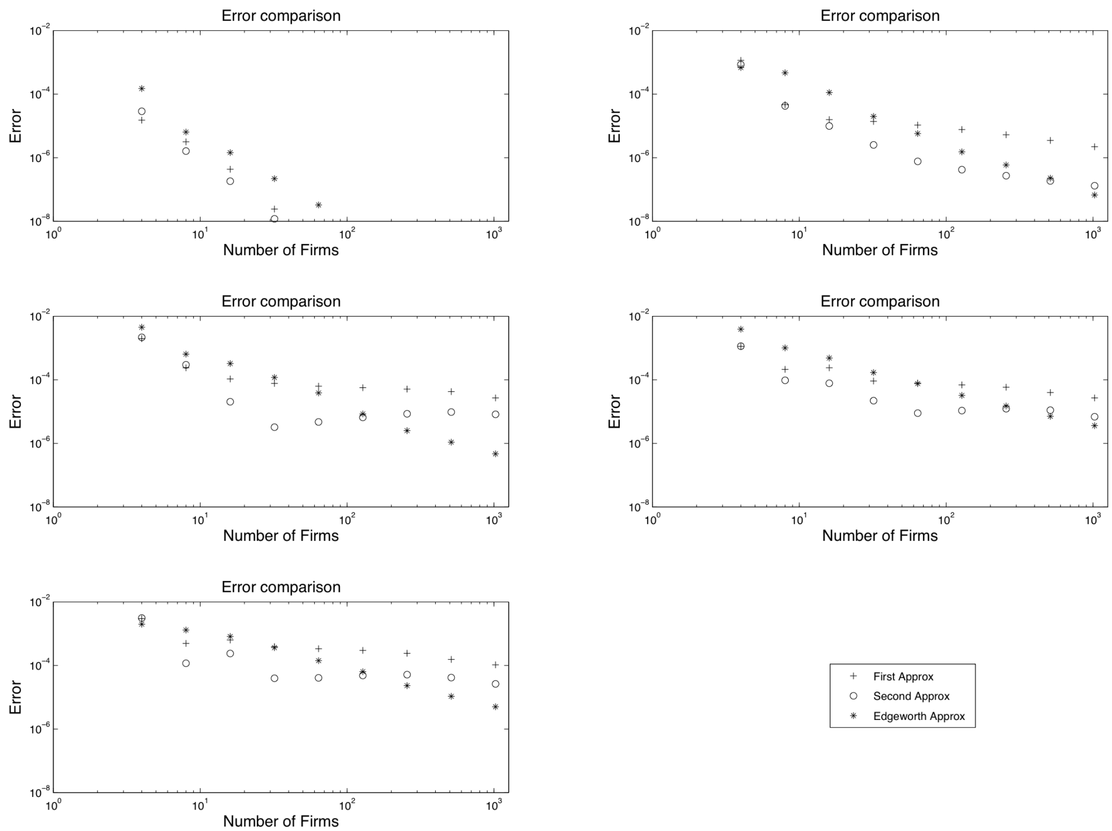

Figure 5: Performance of the three approximation schemes on Problem D, showing averaged error E versus number of firms M. Subgraphs one to five show the five tranche levels $0.3, 0.15, 0.10, 0.07, 0.03$.

M=32				
Tranche (%)	Exact	1st order	2nd order	Edgeworth
0-3	1269.4	1287.0(17.6)	1281.7(12.3)	1509.3(-239.9)
3-7	460.0	456.0(-4.0)	454.7(-5.3)	453.0(-7.0)
7-10	203.4	199.0(-4.4)	203.0(-0.4)	200.8(-2.6)
10-15	96.9	95.7(-1.2)	95.9(-1)	94.1(-2.8)
15-30	20.3	20.4(0.1)	20.5(0.2)	20.1(-0.2)
30-60	0.7	0.7(0.002)	0.7(0.008)	0.7(-0.002)
M=128				
0-3	1496.8	1500.1(3.3)	1497.4(0.6)	1539.6(42.8)
3-7	420.6	420.4(-0.2)	420.0(-0.6)	419.4(-1.2)
7-10	173.2	171.7(-1.5)	173.5(0.3)	173.5(0.3)
10-15	82.0	82.1(0.1)	82.0(-0.06)	81.8(-0.2)
15-30	15.8	15.7(-0.1)	15.8(0.02)	15.8(-0.02)
30-60	0.5	0.5(-3e4)	0.5(5e-4)	0.5(-2e-4)
M=512				
0-3	1559.5	1560.1(0.6)	1559.5(0)	1565.9(6.4)
3-7	412.8	413.3(0.5)	412.8(-0.02)	412.5(-0.3)
7-10	158.3	157.5(-0.8)	158.4(0.02)	158.3(-0.07)
10-15	81.2	81.3(0.1)	81.2(-0.01)	81.2(0.02)
15-30	14.1	14.1(-0.05)	14.2(0.003)	14.1(0.002)
30-60	0.5	0.5 (-3e-5)	0.5(0)	0.5(-2e-5)

Table 2: CDO Spreads for Problem A for different tranches and $M = 32, 128, 512$. Numbers in parentheses are absolute spread errors in bps.

M=32				
Tranche (%)	Exact	1st order	2nd order	Edgeworth
0-3	2938.1	3146.2(208.1)	2986.6(48.5)	3316.7(-378.6)
3-7	1302.9	1283.5(-19.4)	1294.5(-8.4)	1303.3(0.4)
7-10	698.3	669.2(-29.1)	692.9(-5.4)	698.2(-0.1)
10-15	388.4	374.3(-14.1)	383.5(-4.9)	382.8(-5.6)
15-30	103.0	101.7(-1.3)	103.6(0.6)	102.9(-0.1)
30-60	4.5	4.4(-0.1)	4.5(0.03)	4.5(0.01)
M=128				
0-3	3546.9	3674.0(127.1)	3572.3(25.4)	3605.9(59.0)
3-7	1297.6	1281.3(-16.3)	1294.1(-3.5)	1296.6(-1.0)
7-10	658.9	653.4(-5.5)	657.8(-1.1)	660.3(1.4)
10-15	361.3	354.6(-6.7)	359.4(-1.9)	360.6(-0.7)
15-30	88.5	87.5(-1.0)	88.4(-0.01)	88.5(0.005)
30-60	3.3	3.2(-0.1)	3.2(-0.01)	3.3(0)
M=512				
0-3	3730.5	3747.9(17.4)	3741.5(11.0)	3737.8(7.3)
3-7	1300.0	1290.0(-10.0)	1299.0(-1.0)	1299.7(-0.3)
7-10	634.5	644.2(9.7)	633.9(-0.6)	634.6(0.1)
10-15	363.0	362.9(-0.1)	362.3(-0.7)	363.0(0.05)
15-30	83.0	82.8 (-0.2)	82.9(-0.1)	83.0(0)
30-60	3.0	3.0(0.004)	3.0(-0.005)	3.0(1e-4)

Table 3: CDO Spreads for Problem B for different tranches and $M = 32, 128, 512$. Numbers in parentheses are absolute spread errors in bps.