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Résumé

Nous étudions dans cette thèse des questions liées à l'évaluation et à la couverture des dérivés de crédit sur panier tels que les *Basket Default Swap* et les *CDO*. En particulier, nous nous intéressons, d'une part, à la modélisation de la corrélation de défaut, et d'autre part, à l'incomplétude de marché introduite par le risque de corrélation. Cette thèse contribue également à la littérature sur les méthodes numériques semi-analytiques d'évaluation des produits sur panier.

Dans la première partie, on étudie l'approche des diffusions à sauts conditionnels, ainsi que l'impact du grossissement de filtrations sur la dynamique des processus d'intensité. On établit la formule de l'espérance conditionnelle par rapport à la filtration élargie. Ce résultat est ensuite utilisé pour montrer l'équivalence entre l'approche fonction copule et l'approche diffusion à sauts conditionnels. La deuxième partie est consacrée à la copule de Marshall-Olkin. On effectue une étude détaillée de ces propriétés, ainsi que de la paramétrisation de cette structure de corrélation. Dans la troisième partie, on considère le problème de couverture des produits sur panier. Pour résoudre le problème d'incomplétude dû au risque de corrélation, on utilise un critère de minimisation du risque quadratique et des stratégies auto financées en moyenne. On développe des stratégies de couverture du risque de *spread* de crédit, ainsi que le risque de défaut. Dans la quatrième partie, on développe des méthodes semi-analytiques afin d'évaluer des dérivés de crédit sur panier dans un modèle de Marshall-Olkin. Les méthodes présentées couvrent un champ très large de mathématiques appliquées telles que les transformées de Fourier, les changements de probabilité, les schémas de discrétisation stables, l'intégration de Sobol multidimensionnelle, et les algorithmes récursifs de calcul des produits de convolution. Enfin, dans la cinquième partie, on analyse le risque de corrélation de défaut qu'on trouve dans une nouvelle génération de produits connus sous le nom de CDO au carré.

Mots-Clés. Corrélation de défaut, fonctions copules, swap de défaut sur panier, CDO, grossissement de filtration, sauts contingents, copule de Marshall-Olkin, modèles de chocs Poissoniens, distribution de défaut, méthode de réplication, inversion de transformée de Fourier, récurrence de Panjer, marchés incomplets, minimisation du risque local, risque de défaut, CDO au carré.

Abstract

This thesis investigates the issues related to the pricing and hedging of portfolio credit derivatives such as Basket Default Swaps and Collateralized Debt Obligations. In particular, it addresses the issue of default correlation modelling, and the market incompleteness introduced by the default correlation risk. It also provides a contribution to the computational finance literature on pricing basket credit derivatives with semi-analytical methods.

The first chapter presents the conditional jump diffusion framework and examines the impact of filtration enlargements on the dynamics of the intensity process. A formula of the conditional expectation with respect to the enlarged filtration is derived. This is then used to show the equivalence between the copula approach and the CJD framework. In the second chapter, we introduce the Marshall-Olkin copula. We give a detailed analysis of its properties and we propose a parameterization of this rich correlation structure. The third chapter addresses the problem of hedging basket default swaps with the underlying single name instruments. To handle the market incompleteness due to the default correlation risk, we use a risk-minimization criterion and we allow for mean-self financing strategies. We derive strategies to hedge the credit spread risk and the default risk. The fourth chapter derives semi-analytical methods for the valuation of basket credit derivatives in the Marshall-Olkin framework. The methods presented span a large spectrum of applied mathematics: Fourier transforms, changes of probability measure, numerical stable schemes, high-dimensional Sobol integration and recursive convolution algorithms. Finally, in the fifth chapter, we analyze the default correlation risk in a new generation of products known as CDO-Squareds.

Key Words. Default correlation, copula functions, basket default swaps, CDO, filtration enlargements, default contingent jumps, Marshall-Olkin copula, Poisson shock models, aggregate default distributions, replication method, Fourier transform inversion, Panjer recursion, incomplete markets, local risk minimization, default risk, CDO-Squared.

Introduction et Structure de la Thèse

La présente thèse dont l'objet est l'étude des problèmes de corrélation et d'incomplétude dans les marchés de crédit se décompose en cinq parties distinctes définies comme suit:

- Chapitre 1 : The Conditional Jump Diffusion Framework
- Chapitre 2 : Correlation with a Difference
- Chapitre 3 : Quadratic Hedging
- Chapitre 4 : Basket Asymptotics
- Chapitre 5 : Correlation of Correlation

La première partie de ce travail est consacrée à l'étude des diffusions à sauts conditionnels. La première question, déjà abordée dans quelques travaux récents, est liée au couplement entre l'instant de défaut et l'intensité de défaut. La présence d'un terme de saut est à l'origine de difficultés théoriques et numériques non triviales. En effet, le calcul des probabilités de défaut n'est plus possible avec les formules classiques de Lando (1998). Une construction précise du modèle est nécessaire afin de répondre à ce type de question. Une deuxième question, tout aussi fondamentale, est le lien entre les dynamiques de sauts et les fonctions copules. La notion de fonction copule, introduite dans la littérature de crédit afin de modéliser la dépendance entre les instants de défaut, permet également de calculer l'intensité par rapport à la filtration élargie. Ce qui aboutit à une expression explicite du terme de saut. Nous commençons ici par répondre à la première question en adaptant une méthode de Kusuoka (1999) qui consiste à faire un changement de probabilité judicieux afin d'enlever la dépendance circulaire entre défaut et intensité. Puis, pour le deuxième problème, nous proposons une solution basée sur un résultat qui généralise la formule de Dellacherie (1970) au cas multidimensionnel.

Dans la deuxième partie, nous abordons des questions relatives à la modélisation de la corrélation de défaut à l'aide d'une copule dite de « Marshall-Olkin ». Une des motivations provient du marché des CDO et du phénomène de « skew de corrélation ». On se place dans un cadre statique et on s'intéresse aux distributions de défaut d'un portefeuille d'émetteurs. Le problème posé est celui de la calibration des paramètres de la copule pour reproduire les distributions implicites de marché. Nous montrons que la copule MO est une bonne alternative à la copule Gaussienne standard. Ceci est fait en trois étapes : (1) on introduit le modèle de MO comme l'extension naturelle d'un processus de Poisson unidimensionnel, (2) on traite des problèmes de calibration, (3) on compare MO avec la copule Gaussienne. Le modèle de MO a fait l'objet d'une étude détaillée dans la littérature sur la théorie de la fiabilité dans le cadre de la modélisation de la ruine de systèmes à composantes multiples.

La troisième partie est consacrée au problème de couverture des produits dérivés de crédit de type « *basket* », tels que les *first-to-default swaps* et les CDO. Du fait de l'incomplétude de marché qui est due au risque de corrélation, il est impossible de répliquer parfaitement un profile « *basket* » avec des swaps de défaut. Ici, nous utilisons un critère de minimisation du risque quadratique afin de définir une stratégie de couverture.

Enfin nous consacrons la quatrième et cinquième partie à la dérivation, puis à la mise en œuvre numérique d'un modèle de MO. Notre but est de proposer une alternative à la méthode Monte-Carlo et d'implémenter des méthodes basées sur des solutions semi-analytiques. C'est aussi l'occasion d'apporter une contribution au problème de réplication statique des dérivés de crédit sur paniers. En nous inspirant de diverses techniques numériques, nous établissons ici des méthodes originales basées sur des développements asymptotiques qui réduisent la dimension du problème. Des tests numériques de nos approximations indiquent une bonne adéquation avec la solution exacte. Après avoir présenté les principaux résultats de cette nouvelle méthodologie, nous montrons qu'il est possible d'étendre notre approche à une nouvelle génération de produits structurés : les « CDO au carré ».

Donnons maintenant une description détaillée de chacun des chapitres qui composent cette thèse.

0.1 Chapitre 1 : The Conditional Jump Diffusion Framework

Nous considérons un marché financier construit sur un espace de probabilité $(\Omega, \mathcal{G}, P)$. Soient $(X_t)_{t\geq 0}$ un processus d'Itô d-dimensionnel représentant les variables d'état économiques et $\{\mathcal{F}_t\}$ est la filtration complétée engendrée par $(X_t)_{t\geq 0}$. Les instants de défaut sont modélisés par des temps aléatoires $(\tau_1, ..., \tau_n)$ définis sur cet espace. Pour $i = 1, ..., n$, nous posons

$$D_t^i \triangleq \mathbf{1}_{\{\tau_i \leq t\}},$$

et nous notons $\left\{\mathcal{H}_t^i\right\}$ la filtration engendrée par D_t^i.

Nous introduisons la filtration $\{\mathcal{G}_t\}$ définie par

$$\mathcal{G}_t \triangleq \mathcal{F}_t \vee \mathcal{H}_t^1 \vee ... \vee \mathcal{H}_t^n,$$

ainsi que les filtrations $\left\{\mathcal{G}_t^i\right\}$ et $\left\{\mathcal{G}_t^{-i}\right\}$:

$$\begin{aligned}\mathcal{G}_t^i &\triangleq \mathcal{F}_t \vee \mathcal{H}_t^i, \\ \mathcal{G}_t^{-i} &\triangleq \mathcal{F}_t \vee \mathcal{H}_t^1 \vee ... \vee \mathcal{H}_t^{i-1} \vee \mathcal{H}_t^{i+1} \vee ... \vee \mathcal{H}_t^n.\end{aligned}$$

Supposons, en outre, que l'hypothèse $(\mathcal{H})$ est vérifiée, ce qui implique la conservation de la propriété de martingales quand on passe de la filtration $\{\mathcal{F}_t\}$ à la filtration $\{\mathcal{G}_t\}$.

Nous utiliserons deux types d'intensités : la $\{\mathcal{F}_t\}$-intensité et la $\{\mathcal{G}_t\}$-intensité :

1. Nous appelons $\{\mathcal{F}_t\}$-intensité de τ_i le processus positif $\{\mathcal{F}_t\}$-adapté h^i tel que $D_t^i - \int_0^{t\wedge\tau_i} h_s^i ds$ est une $\left\{\mathcal{G}_t^i\right\}$-martingale.
2. Et nous appelons $\{\mathcal{G}_t\}$-intensité de τ_i, le processus positif $\left\{\mathcal{G}_t^{-i}\right\}$-adapté λ^i tel que $D_t^i - \int_0^{t\wedge\tau_i} \lambda_s^i ds$ est une $\{\mathcal{G}_t\}$-martingale.

Rappelons que sur la filtration $\left\{\mathcal{G}_t^i\right\}$, la probabilité conditionnelle de survie est donnée par

$$\mathbb{P}\left(\tau_i > T \left| \mathcal{G}_t^i\right.\right) = \mathbf{1}_{\{\tau_i > t\}} \mathbb{E}\left[\exp\left(-\int_t^T h_s^i ds\right) | \mathcal{F}_t\right].$$

Dans le cas général, le calcul de l'espérance conditionnelle n'est pas trivial. Pour résoudre cette difficulté, nous utiliserons la méthode de changement de probabilité qui a été développée par Kusuoka (1999).

Dans un premier temps, nous considérons le calcul de l'espérance conditionnelle dans un modèle dans lequel on spécifie la dynamique de l'intensité λ^i et en particulier le couplement avec les instants de défaut. Ensuite, nous effectuons ce même calcul dans le cadre d'un modèle de fonction copule.

Modèles de défauts « circulaires ». Le système de SDEs qui nous intéressera dans la première partie de ce chapitre a été introduit par Kusuoka (1999). Dans sa version bidimensionnelle initiale, il décrit les caractéristiques du défaut de deux obligations dont les intensités sont liées aux instants même de défaut :

$$\begin{aligned}\lambda_t^1 &= \lambda_0^1 + \Delta^{12}\mathbf{1}_{\{\tau_2 \leq t\}}, \\ \lambda_t^2 &= \lambda_0^2 + \Delta^{21}\mathbf{1}_{\{\tau_1 \leq t\}},\end{aligned}$$

où λ_t^1 et λ_t^2 sont les intensités de chaque émetteur ; Δ^{12} et Δ^{21} désignent les amplitudes de saut.

Ici nous nous intéresserons à une version générale qui peut inclure plusieurs obligations :

$$\lambda_t^i = \lambda_0^i + \sum_{\substack{j=1 \\ j\neq i}}^n \Delta^{ij} D_t^j.$$

Signalons que Jarrow et Yu (2001) ont étudié un système similaire où ils supposent que la matrice de couplement est triangulaire, et ce afin de résoudre le problème de la dépendance circulaire.

Précisons également qu'une construction rigoureuse d'un tel modèle n'est pas triviale. Citons parmi les auteurs qui ont abordé cette question : Frey et Backhaus (2004), Yu (2004), Becherer et Schweizer (2005).

Nous privilégierons ici la construction de Kusuoka (1999). L'idée de base est la suivante : On se place dans une espace de probabilité $(\Omega, \mathcal{G}', \{\mathcal{G}'_t\}, P')$ où on considère un ensemble d'instants de défaut $(\tau_1, ..., \tau_n)$ qui sont indépendants et dont les $(P', \{\mathcal{G}'_t\})$-intensités sont égales à 1. Puis on définit une mesure de probabilité P :

$$\frac{dP}{dP'} = \mathcal{E}\left(\sum_{i=1}^{n} \int \left(\lambda_t^i - 1\right)\left(dD_t^i - \left(1 - D_t^i\right) dt\right)\right)_{T^*}.$$

En utilisant le théorème de Girsanov et en définissant $(\Omega, \mathcal{G}, \{\mathcal{G}_t\}, P)$ comme la P-complétude de $(\Omega, \mathcal{G}', \{\mathcal{G}'_t\}, P')$, on peut montrer que ce changement de mesure est tel que λ_t^i est la $(P, \{\mathcal{G}_t\})$-intensité of τ_i.

Nous utiliserons cette technique pour dériver la fonction de densité des instants de défaut que nous établissons dans la proposition suivante.

Proposition (Chapitre 1, page 8)
Soit $(t_1, ..., t_n) \in \mathbb{R}_+^n$ et supposons que

$$t_{\pi(1)} \leq t_{\pi(2)} \leq ... \leq t_{\pi(n)},$$

où $\pi(.) : \{1, ..., n\} \to \{1, ..., n\}$ est une permutation croissante du n-uplet $(t_1, ..., t_n)$. Alors, la densité conjointe des instants de défaut s'écrit

$$f(t_1, ..., t_n) = \prod_{i=1}^{n}\left[\lambda_0^{\pi(i)} + \sum_{j=1}^{i-1} \Delta^{\pi(i)\pi(j)}\right] \exp\left(-\sum_{j=1}^{i}\left[\lambda_0^{\pi(i)} + \sum_{k=1}^{j-1} \Delta^{\pi(i)\pi(k)}\right]\left(t_{\pi(j)} - t_{\pi(j-1)}\right)\right),$$

avec la convention $t_{\pi(0)} = 0$.

Une autre preuve de la proposition s'inspire de la construction de Yu (2004) qui est basée sur les travaux de Norros (1986) et Shaked et Shanthikumar (1987). Partant de l'hypothèse que les défauts simultanés sont exclus, on peut définir la suite ordonnée des instants de défaut $(T_0, T_1, ..., T_n)$: $T_0 = 0 < T_1 < ... < T_n$, ainsi que la suite $(Z_0, Z_1, ..., Z_n)$ identifiant les obligations qui ont fait défaut :

$$\begin{aligned} T_0 &= 0, Z_0 = 0; \\ T_k &= \min\{\tau_i : 1 \leq i \leq n, \ \tau_i > T_{k-1}\}; \\ Z_k &= i \text{ si } T_k = \tau_i. \end{aligned}$$

La suite double $(T_k, Z_k)_{k \geq 0}$ définit un processus ponctuel marqué

$$\begin{aligned} \mu(\omega, dt \times dz) &: (\Omega, \mathcal{G}) \to ((0,\infty) \times E, (0,\infty) \otimes \mathcal{E}), \\ \int_0^t \int_E H(\omega, s, z)\, \mu(\omega, dt \times dz) &= \sum_{k=1}^{n} H(\omega, T_k(\omega), Z_k(\omega))\, \mathbf{1}_{\{T_k(\omega) \leq t\}}, \end{aligned}$$

dont le $(P, \{\mathcal{G}_t\})$-compensateur $\nu(\omega, dt \times dz) = \Phi_t(\omega, dz)\, \lambda_t^\mu dt$ est donné par

$$\begin{aligned}
\lambda_t^\mu &= \sum_{i=1}^{n} \left(1 - D_t^i\right) \lambda_t^i; \\
\Phi_t(i) &= \frac{\left(1 - D_t^i\right) \lambda_t^i}{\lambda_t^\mu}, \text{ pour } i \in \{1, ..., n\}.
\end{aligned}$$

On appelle le couple $(\lambda_t^\mu, \Phi_t(dz))$ les $(P, \{\mathcal{G}_t\})$-caractéristiques locales de la mesure $\mu(dt \times dz)$ (Brémaud (1980)). En utilisant, l'expression de la densité conditionnelle $g^{(k)}(\omega, t, z) \triangleq \mathbb{P}\left(T_k \in dt, Z_k = i \left| \mathcal{G}_{T_{k-1}}\right.\right)(\omega)$, on peut dissocier le problème au calcul de plusieurs probabilités conditionnelles dont Il suffira de faire le produit pour obtenir le résultat.

Généralisation de la formule de Dellacherie. Nous avons mentionné que le calcul des probabilités conditionnelles sur la filtration grossie n'est pas trivial à cause des termes de saut. Ici nous établissons un résultat clé qui généralise la formule de Dellacherie (1970) et qui sera utilisé par la suite pour effectuer des calculs dans un modèle où la dépendance est définie à l'aide d'une copule.

Pour chaque sous-ensemble $\pi \in \mathbf{\Pi_n}$, où $\mathbf{\Pi_n}$ est l'ensemble des sous-ensembles de $\{1, ..., n\}$, on définit l'indicatrice $D_t^{(\pi)}$ et la filtration $\left\{\mathcal{G}_t^{(\pi)}\right\}$:

$$\begin{aligned}
D_t^{(\pi)} &\triangleq \left[\prod_{j\in\pi} \left(D_t^j\right)\right] \times \left[\prod_{j\notin\pi} \left(1 - D_t^j\right)\right], \\
\mathcal{G}_t^{(\pi)} &\triangleq \mathcal{F}_t \vee \left[\bigvee_{j\in\pi} \mathcal{H}_\infty^j\right] = \mathcal{F}_t \vee \left[\bigvee_{j\in\pi} \sigma(\tau_j)\right].
\end{aligned}$$

Théorème (Chapitre 1, page 17)
Soit Y une variable aléatoire $\mathcal{G}$-mesurable. Alors, on a

$$\mathbb{E}[Y \,|\mathcal{G}_t] = \sum_{\pi\in\mathbf{\Pi_n}} D_t^{(\pi)} \frac{\mathbb{E}\left[Y \times \prod_{j\notin\pi}\left(1 - D_t^j\right) \left|\mathcal{G}_t^{(\pi)}\right.\right]}{\mathbb{E}\left[\prod_{j\notin\pi}\left(1 - D_t^j\right) \left|\mathcal{G}_t^{(\pi)}\right.\right]}.$$

La preuve est basée sur un argument de récurrence.

L'approche copule. Nous considérons dans cette section le calcul de la probabilité conditionnelle de survie dans le cadre de l'approche copule. Plus précisément, la structure de dépendance est décrite formellement par le processus $\left(\overline{C}_t^\tau\right)_{t\geq 0}$, $\overline{C}_t^\tau : [0,1]^n \times \Omega \times [0, \infty) \to [0,1]$, où, pour chaque t, $\overline{C}_t^\tau$ représente la fonction copule conditionnelle $\overline{C}_t^\tau(.) \triangleq \overline{C}^\tau(.\,|\mathcal{F}_t)$: pour presque tout $\omega \in \Omega$ et pour tout $(t_1, ..., t_n) \in [0, \infty)^n$,

$$\mathbb{P}(\tau_1 > t_1, ..., \tau_n > t_n \,|\mathcal{F}_t)(\omega) = \overline{C}_t^\tau\left(\mathbb{P}(\tau_1 > t_1 \,|\mathcal{F}_t)(\omega), ..., \mathbb{P}(\tau_n > t_n \,|\mathcal{F}_t)(\omega)\right).$$

Pratiquement, on peut construire le processus $\left(\overline{C}_t^{\tau}\right)_{t\geq 0}$ en partant de l'approche de Lando (1998) où le défaut est défini par

$$\tau_i \triangleq \inf\left\{t : \exp\left(-\int_0^t h_s^i ds\right) \leq U_i\right\},$$

$\left(h_t^i\right)_{t\geq 0}$ est un processus positif $\{\mathcal{F}_t\}$-adapté càdlàg et U_i est une variable aléatoire uniformément répartie sur $[0,1]$ et indépendante de $\mathcal{F}_\infty$, et on suppose que les seuils de défaut sont reliés par une copule statique $\overline{C}^U : [0,1]^n \to [0,1]$,

$$\mathbb{P}\left(U_1 > u_1, ..., U_n > u_n\right) = \overline{C}^U\left(u_1, ..., u_n\right).$$

Nous définissons également les intensités forward $\{\mathcal{F}_t\}$-adapté et $\{\mathcal{G}_t\}$-adapté : $\left(h_{t,T}^i, \lambda_{t,T}^i\right)$. L'application de la formule de Dellacherie généralisé nous permet alors d'établir l'expression de l'intensité $\{\mathcal{G}_t\}$-adapté pour chaque configuration de défaut.

Proposition (Chapitre 1, page 27)
S'il n'y a pas eu de défaut avant l'instant t, alors la $\{\mathcal{G}_t\}$-intensité forward s'exprime

$$\lambda_{t,T}^{i,(\emptyset)} = h_{t,T}^i \exp\left(-\int_0^T h_{t,s}^i ds\right) \frac{\frac{\partial}{\partial x_i}\overline{C}_t^{\tau}\left(e^{-\int_0^t h_{t,s}^1 ds}, ..., e^{-\int_0^T h_{t,s}^i ds}, ..., e^{-\int_0^t h_{t,s}^n ds}\right)}{\overline{C}_t^{\tau}\left(e^{-\int_0^t h_{t,s}^1 ds}, ..., e^{-\int_0^T h_{t,s}^i ds}, ..., e^{-\int_0^t h_{t,s}^n ds}\right)}.$$

Proposition (Chapitre 1, page 28)
Si k obligations indexées par $\pi = \{j_1, ..., j_k\}$ ont déjà fait défaut avant l'instant t, et leurs instants de défaut sont $\{t_{j_1}, ..., t_{j_k}\}$ respectivement, alors la $\{\mathcal{G}_t\}$-intensité forward *s'exprime*

$$\lambda_{t,T}^{i,(\pi)} = h_{t,T}^i \exp\left(-\int_0^T h_{t,s}^i ds\right) \frac{\frac{\partial}{\partial x_i}\frac{\partial^k}{\partial x_{j_1}...\partial x_{j_k}}\overline{C}_t^{\tau}\left(e^{-\int_0^{\Theta_1} h_{t,s}^1 ds}, ..., e^{-\int_0^{\Theta_n} h_{t,s}^n ds}\right)}{\frac{\partial^k}{\partial x_{j_1}...\partial x_{j_k}}\overline{C}_t^{\tau}\left(e^{-\int_0^{\Theta_1} h_{t,s}^1 ds}, ..., e^{-\int_0^{\Theta_n} h_{t,s}^n ds}\right)},$$

où $\Theta_j = t_j$, pour $j \in \pi = \{j_1, ..., j_k\}$; $\Theta_i = T$, pour $j = i$; $\Theta_j = t$, sinon.

Ceci est une extension des résultats de Schönbucher et Schubert (2001).

En outre, nous dérivons la dynamique de la $\{\mathcal{G}_t\}$-intensité en appliquant la formule d'Itô aux expressions précédentes :

$$\begin{aligned}\frac{d\lambda_t^i}{\lambda_{t^-}^i} &= \frac{dh_t^i}{h_t^i} + \left[\left(1 - \frac{\overline{C}^U_{x_i x_i x_{Z_1}...x_{Z_{k-1}}}\overline{C}^U_{x_{Z_1}...x_{Z_{k-1}}}}{\overline{C}^U_{x_i x_{Z_1}...x_{Z_{k-1}}}\overline{C}^U_{x_i x_{Z_1}...x_{Z_{k-1}}}}\right)\lambda_t^i - h_t^i\right] dt \\ &\quad + \int_E \Delta\left(i,\omega,t,z\right)\left(\mu\left(dt\times dz\right) - \nu\left(dt\times dz\right)\right),\end{aligned}$$

où l'amplitude du saut $\Delta\left(i,\omega,t,z\right)$ est donnée par

$$\Delta\left(i,\omega,t,z\right) = \begin{cases} \frac{\overline{C}^U_{x_i x_Z x_{Z_1(\omega)}...x_{Z_{k-1}(\omega)}}\overline{C}^U_{x_{Z_1(\omega)}...x_{Z_{k-1}(\omega)}}}{\overline{C}^U_{x_i x_{Z_1(\omega)}...x_{Z_{k-1}(\omega)}}\overline{C}^U_{x_Z x_{Z_1(\omega)}...x_{Z_{k-1}(\omega)}}} - 1, \text{ si } z \neq i, \\ 0, \text{ si } z = i. \end{cases}$$

La composante de diffusion provient de la contribution de l'intensité $\{\mathcal{F}_t\}$-adapté et les sauts sont causés par les défauts des autres émetteurs. Cette représentation permet ainsi d'établir un lien entre la copule et la dynamique de saut.

0.2 Chapitre 2 : Correlation with a Difference

Pendant les dernières années, plusieurs travaux ont été consacrés au problème de modélisation de la corrélation entres les instants de défauts de plusieurs émetteurs. Puisque les distributions marginales sont déjà connues -obtenues à partir des prix de swaps de défaut- il s'agit alors de choisir une fonction copule pour pouvoir définir la densité conjointe. L'idée d'utiliser les fonctions copules pour le risque de crédit a été introduite par Li (2000). Signalons qu'actuellement, l'approche standard privilégiée par les praticiens est basée sur une copule Gaussienne. Il existe, toutefois, d'autres copules qu'on pourrait utiliser. Citons, par exemple, la t-copule, la copule archimédienne, la copule de Clayton, la copule de Gumbel, etc.

Ici, on s'intéresse à une autre classe de copules connue sous le nom de copule de Marshall-Olkin. Cette approche a été traditionnellement utilisée dans la théorie de la fiabilité pour modéliser les pannes des systèmes composés. La première application au risque de crédit est celle de Duffie (1998).

Notre objectif est principalement de montrer que la copule de Marshall-Olkin pourrait être une bonne alternative au cadre Gaussien classique. Motivé par le phénomène de « skew de corrélation » observé dans les marchés de CDO, nous étudierons les propriétés de ce modèle, et plus particulièrement, ce que ça implique en termes de distribution de portefeuille, nous montrerons par la suite qu'il est possible de calibrer le modèle de façon à reproduire les prix de marché.

On se place sur un espace probabilisé $(\Omega, \mathcal{G}, P)$, soient $(\tau_1, ..., \tau_n)$ les instants de défaut d'un panier d'obligations. $D_t^i \triangleq \mathbf{1}_{\{\tau_i \leq t\}}$ est l'indicatrice de défaut de la i-ième firme. On suppose qu'il existe un ensemble de m processus de Poisson indépendants $(N^{c_j})_{1\leq j\leq m}$, caractérisé par les intensités $(\lambda^{c_j})_{1\leq j\leq m}$, $\lambda^{c_j} \in \mathbb{R}_+$, qui peuvent déclencher des défauts simultanés d'un ou plusieurs émetteurs. Chaque processus de Poisson N^{c_j} peut être représenté de façon équivalente par les instants de saut $\left\{\theta_r^{c_j}\right\}_{r\in\{1,2,...\}}$. Pour chaque évènement de type c_j, et pour tout $t \geq 0$, on définit un vecteur de variables de Bernoulli indépendantes $\left(A_t^{1,j}, ..., A_t^{n,j}\right)$ avec des probabilités $\left(p^{1,j}, ..., p^{n,j}\right)$, $p^{i,j} \in [0,1]$. On suppose que, pour $j \neq k$, les vecteurs $\mathbf{A}_t^j = \left(A_t^{1,j}, ..., A_t^{n,j}\right)$ et $\mathbf{A}_t^k = \left(A_t^{1,k}, ..., A_t^{n,k}\right)$ sont indépendants. Et on suppose que, pour $t \neq s$, les vecteurs $\mathbf{A}_t^j = \left(A_t^{1,j}, ..., A_t^{n,j}\right)$ et $\mathbf{A}_s^j = \left(A_s^{1,j}, ..., A_s^{n,j}\right)$ sont indépendants. A l'instant du r-ième évènement de type c_j, on simule les variables de Bernoulli $\left(A_{\theta_r^{c_j}}^{1,j}, ..., A_{\theta_r^{c_j}}^{n,j}\right)$. La variable $A_{\theta_r^{c_j}}^{i,j}$ indique si l'émetteur i a fait défaut ou pas. Le processus N^i défini par

$$N_t^i \triangleq \sum_{j=1}^{m} \sum_{\theta_r^{c_j} \leq t} A_{\theta_r^{c_j}}^{i,j},$$

est également un processus de Poisson dont l'intensité est

$$\lambda^i = \sum_{j=1}^{m} p^{i,j} \lambda^{c_j}.$$

L'instant de défaut τ_i est défini par le temps du premier saut du processus N^i

$$\tau_i \triangleq \inf \left\{ t : N_t^i > 0 \right\}.$$

Afin d'établir la fonction copule de ce modèle, on a recours à la représentation « en chocs fatals équivalents » de Lindskog et McNeil (2003). Soit $\mathbf{\Pi_n}$ l'ensemble des sous-ensembles de $\{1, ..., n\}$. Pour chaque $\pi \in \mathbf{\Pi_n}$, on introduit le processus N_t^π qui compte le nombre de chocs qui entraînent le défaut des émetteurs appartenant à π :

$$N_t^\pi \triangleq \sum_{j=1}^{m} \sum_{r=1}^{N_t^{c_j}} A_{\theta_r^{c_j}}^{\pi,j},$$

où la variable de Bernoulli $A_t^{\pi,j} \triangleq \prod_{i\in\pi} A_t^{i,j} \prod_{i\notin\pi} \left(1 - A_t^{i,j}\right)$ est égale à 1 si tous les émetteurs $i \in \pi$ font défaut. Notons par $\Lambda_T^\pi \triangleq \int_0^T \lambda^\pi ds = \lambda^\pi T$ le taux de hasard de N^π.

On peut alors énoncer la formule de la probabilité conjointe établie par Marshall-Olkin (1967) :

$$\mathbb{P}\left(\tau_1 > T_1, ..., \tau_n > T_n\right) = \exp\left(-\sum_i \Lambda_{T_i}^{\{i\}} - \sum_{i,j} \Lambda_{\max(T_i,T_j)}^{\{i,j\}} - ... - \Lambda_{\max(T_1,...,T_n)}^{\{1,...,n\}} \right).$$

On se référera à Barlow et Proschan (1981), Joe (1997) ou bien Nelsen (1999) pour plus de détails. Parmi les propriétés fondamentales de cette distribution, notons les « singularités » qui apparaissent sur les diagonales de l'hyper-cube $[0, +\infty]^n$.

Le modèle de Marshall-Olkin offre une structure de corrélation qui est très riche. L'intérêt majeur d'une telle approche est de reproduire la courbe de skew de marché. Il est cependant nécessaire d'avoir une paramétrisation du modèle qui facilite la calibration. Ici, nous supposerons que nous avons quatre types de facteurs communs : (1) un facteur « Beta » représentant le risque cyclique de marché, (2) des facteurs sectoriels, (3) le facteur « World » qui affecte les tranches super senior de CDO, (4) des facteurs idiosyncratiques spécifiques à chaque firme :

$$\lambda^i = \left[\lambda^W\right] + p^{i,B} \left[\lambda^B\right] + \sum_{j=1}^{m_c-2} p^{i,S_j} \left[\lambda^{S_j}\right] + \left[\lambda^{0,i}\right].$$

Nous utiliserons ensuite les prix des tranches de CDO disponibles sur le marché pour entamer notre calibration.

Précisons que le marché des CDO a connu récemment une grande expansion qui a culminé dans la création de tranches standards sur indice qui sont quottées par les banques. Partant des prix de marché, et en inversant la formule de la copule Gaussienne, on trouve qu'il n'existe pas une valeur unique de corrélation qui puisse reproduire tous les prix. Ce phénomène est appelé « skew de corrélation ». Remarquons au passage, l'analogie avec le marché des options et la skew de volatilité.

Après avoir introduit formellement des concepts tels que la « corrélation composée » et la « corrélation de base », nous montrons que la nature multi-modales de la distribution de portefeuille de Marshall-Olkin permet d'introduire une concentration du risque qui peut se traduire par différents niveaux de corrélation implicite. Ainsi nous disposons de plus de flexibilité pour évaluer les tranches equity, mezzanine et senior, ce qui, par conséquent, permet de re-créer cet effet de segmentation de marché.

0.3 Chapitre 3 : Quadratic Hedging

On se place dans un espace de probabilité $(\Omega, \mathcal{G}, P^*)$ et on considère un modèle de Marshall-Olkin dynamique dans lequel on suppose que le vecteur des variables d'états $(X_t)_{t\geq 0}$ est un processus d'Itô

$$dX_t = \alpha(X_t)\,dt + \beta(X_t)\,dW_t,$$

et que l'intensité du facteur de Poisson N^{c_j} est une fonction de X_t : $\lambda^{c_j}(X_t)$, $\lambda^{c_j} : \mathbb{R}^d \to \mathbb{R}_+$, ainsi que les probabilités des variables de Bernoulli $\left(A^{1,j}_{\theta_r^{c_j}}, ..., A^{n,j}_{\theta_r^{c_j}}\right)$:

$$\left(p^{1,j}\left(X_{\theta_r^{c_j}}\right), ..., p^{n,j}\left(X_{\theta_r^{c_j}}\right)\right).$$

Et on se pose la question de la couverture des dérivées de crédit sur panier à l'aide des instruments mono-émetteur (ou *single name*).

Remarquons que la filtration du modèle

$$\widetilde{\mathcal{G}}_t \triangleq \mathcal{F}_t \vee \left[\bigvee_{j=1}^{m} \mathcal{F}_t^{N^{c_j}}\right] \vee \left[\bigvee_{j=1}^{m}\bigvee_{i=1}^{n} \mathcal{F}_t^{A^{i,j}}\right]$$

est beaucoup plus large que la filtration disponible aux agents économiques

$$\mathcal{G}_t \triangleq \mathcal{F}_t \vee \left[\bigvee_{i=1}^{n} \mathcal{H}_t^i\right].$$

On se donne un marché financier où on a $(n+1)$ actifs primaires $S = \left(S^i\right)_{0\leq i\leq n}$. L'actif S^0 représente l'actif sans risque est sera utilisé comme numéraire. Toutes les quantités sont exprimées en unités de S^0. En particulier, S^0 sera est égal à 1 tout le temps. Nous considérons des dérivées de crédit zéro-coupon, aussi appelés des actifs contingents de type européen. L'actif de couverture S^i représentera l'obligation zéro-coupon de maturité T émise par la firme i ; autrement dit son payoff est défini par

$$S_T^i \triangleq 1 - D_T^i.$$

Les obligations zéro-coupons sont rarement traitées sur le marché. On peut néanmoins les créer synthétiquement à partir des swaps de défaut de différentes maturités.

Nous considérons le problème d'évaluation et de couverture des actifs contingents zéro-coupon par réplication dynamique à l'aide des actifs de couverture S.

On suppose que P^* est une mesure de probabilité martingale (risque-neutre). Ici, on utilise l'approche de Föllmer et Sondermann (1986) où une mesure martingale est fixée et la minimisation du risque quadratique est effectuée par rapport a cette mesure.

Définition (Chapitre 3, page 67)

On appelle actif contingent une variable aléatoire $\mathcal{G}_T$-mesurable H_T représentant le payoff à échéance T d'un instrument financier.

On s'intéresse, par exemple, à un k^{th}-*to-default* dont le payoff est défini par

$$H_T^{(k)} \triangleq \mathbf{1}_{\left\{\sum_{i=1}^n D_T^i < k\right\}},$$

ou à une tranche de CDO couvrant les pertes dans un intervalle $[K_1, K_2]$,

$$H_T^{(K_1,K_2)} \triangleq \frac{1}{K_2 - K_1} \min\left(\max\left(\frac{1}{n}\sum_{i=1}^n \left(1 - R^i\right) D_T^i - K_1, 0\right), K_2 - K_1\right),$$

où $0 \leq K_1 < K_2 \leq 1$, et $0 \leq R^i \leq 1$ est le taux de recouvrement de l'émetteur i.

Pour des actifs non-atteignables, une stratégie minimisant le risque quadratique est caractérisée par : (a) le processus de coût est une martingale, (b) le processus de coût est orthogonal à S. Föllmer et Sondermann (1986) ont montré que la stratégie de couverture est obtenue grâce à la décomposition de Kunita-Watanabe de la $\{\mathcal{G}_t\}$-martingale $H_t = \mathbb{E}^* \left[H_T \left|\mathcal{G}_t\right.\right]$:

$$H_T = H_0 + \int_{]0,T]} \left(\alpha_t^{H_T}\right)^{tr} dS_t + L_T^{H_T},$$

où L^{H_T} est une martingale orthogonale à S.

Notre but est d'établir un résultat analytique pour la stratégie $\left(\alpha_t^{H_T}\right)$.

Nous procédons en plusieurs étapes.

1. Nous établissons la représentation en processus ponctuel marqué du modèle $\mu\left(dt \times dz\right)$. Le processus de défaut compensé s'exprime alors

$$M_t^i = \int_0^t \int_E \mathbf{1}_{\{i \in z\}} \left(\mu\left(ds \times dz\right) - \lambda_s\left(dz\right) ds\right).$$

2. Nous établissons la dynamique, sous la mesure de probabilité P^*, des prix d'obligations zéro-coupon sujettes au risque de défaut

$$dS_t^i = S_{t^-}^i \left(\mu_t^i dt + \left(\sigma_t^i\right)^{tr} dW_t - dM_t^i\right).$$

3. Nous appliquons un théorème de représentation des martingales (voir Jacod et Shiryaev (1987)) à la $\{\mathcal{G}_t\}$-martingale $H_t = \mathbb{E}^* \left[H_T \left|\mathcal{G}_t\right.\right]$:

Proposition (Chapitre 3, page 77)

La $\{\mathcal{G}_t\}$-martingale $H_t = \mathbb{E}^* [H_T | \mathcal{G}_t]$, $t \in [0, T^*]$, où H_T est une variable aléatoire $\mathcal{G}_T$-mesurable intégrable par rapport à P^*, admet la représentation intégrale suivante

$$H_t = H_0 + \int_0^t (\xi_s)^{tr} dW_s - \int_0^t \int_E \zeta(s,z) (\mu(ds \times dz) - \lambda_s(dz) ds),$$

où ξ est un processus d-dimensionnel $\{\mathcal{G}_t\}$-prévisible et $\zeta(s,z)$ est un processus E-indexé $\{\mathcal{G}_t\}$-prévisible tel que

$$\int_0^t \|\xi_s\|^2 ds < \infty, \int_0^t \int_E \zeta(s,z) \lambda_s(dz) ds < \infty,$$

presque sûrement.

4. Enfin, nous établissons notre résultat principal que nous énonçons dans le théorème suivant :

Théorème (Chapitre 3, page 78)

La stratégie de minimisation du risque quadratique d'un actif contingent (basket) à l'aide d'instruments mono-émetteur est obtenue par la solution du système d'équations linéaires suivant : pour $1 \leq k \leq n$,

$$\begin{aligned} & \sum_{i=1}^n \alpha_t^i S_{t^-}^i \left[\left(\sigma_t^i\right)^{tr} \sigma_t^k + \int_E \mathbf{1}_{\{i \in z\}} \mathbf{1}_{\{k \in z\}} \lambda_t(dz) \right] \\ = \ & \left(\sigma_t^k\right)^{tr} \xi_t + \int_E \zeta(t,z) \mathbf{1}_{\{k \in z\}} \lambda_t(dz). \end{aligned}$$

Pour illustrer le résultat, une application au cas d'un *first-to-default* est présentée en fin de chapitre.

0.4 Chapitre 4 : Basket Asymptotics

Nous revenons maintenant au cas particulier d'un modèle de Marshall-Olkin statique. On s'intéressera dans ce chapitre aux méthodes numériques semi-analytiques d'évaluation des dérivés de crédit sur panier.

Nous présentons ici nos principales méthodes numériques : (1) l'expansion en FTD, (2) l'expansion homogène, (3) l'expansion homogène asymptotique, (4) l'expansion asymptotique.

On note par $\left(\tau^{[1]}, ..., \tau^{[n]}\right)$ la suite ordonnée des instants de défaut, définie comme suit : $\tau^{[1]} = \min(\tau_1, ..., \tau_n)$, et pour $k = 2, ..., n$,

$$\tau^{[k]} = \min\left(\tau_i : i = 1, ..., n,\ \tau_i > \tau^{[k-1]}\right),$$

et on pose

$$Q^{[k]}(T) \triangleq \mathbb{P}\left(\tau^{[k]} > T\right) = \mathbb{P}(X_T < k),$$

où $X_T \triangleq \sum_{i=1}^{n} D_T^i$ compte le nombre de défauts avant l'horizon de temps T.

L'objectif est de calculer $Q^{[k]}(T)$ à l'aide d'une formule semi-analytique.

Une première possibilité consiste à énumérer tous les états de défauts possibles et à calculer les probabilités de chaque état. Si le calcul du premier temps de défaut est fait à l'aide d'une formule fermée qui est relativement simple, l'implémentation en dimension plus élevée utilise des algorithmes complexes difficiles à mettre en œuvre. Il s'agit là d'introduire une nouvelle représentation dont les coefficients sont calculés en résolvant un système d'équations de récurrence. Plus précisément, la probabilité du k-ième défaut prendra la forme suivante

$$\mathbb{P}\left(\tau^{[k]} > T\right) = \sum_{\{s:d(\widetilde{x_s})<k\}} \sum_{\pi\in\mathbf{\Pi_n}} \alpha_\pi^{\widetilde{x_s}} Q_\pi^{[1]}(T),$$

où $Q_\pi^{[1]}(T) \triangleq \mathbb{E}\left[\prod_{i\in\pi}\left(1 - D_T^i\right)\right]$ et pour chaque configuration de défaut représentée par les vecteurs binaires (constitué de 0's et de 1's) $\widetilde{x_s} = (x_{s,1}, ..., x_{s,2})$, $s = 1, ..., 2^n$, les coefficients $\alpha_\pi^{\widetilde{x_s}} \in \{-1, 0, 1\}$ sont calculés récursivement.

Devant la complexité algorithmique qui résulte de l'approche directe, nous avons choisi de suivre une autre voie : puisque nous disposons déjà d'un *pricer* efficace de FTD (*First-To-Default*), notre idée a été de développer une méthode indépendante du *pricer* FTD, mais qui adapterait ce dernier au cas général des NTD (*Nth-To-Default*). La méthode que nous proposons dite d' « expansion en FTD » repose sur un argument de réplication statique. En effet, en procédant par un raisonnement de récurrence, nous montrons qu'il est possible de répliquer un NTD avec des FTD référençant des sous-ensembles du panier d'origine. Signalons que ce résultat est générique et ne dépend pas de la copule utilisée. En notant par $\pi_s^l(.)$, pour $1 \leq s \leq \binom{n}{l}$, l'ensemble des sous-ensembles de $\{1, ..., n\}$ contenant exactement l éléments :

$$\pi_s^l(.) : \{1, 2, ..., k\} \to \{1, 2, ..., n\},$$

la probabilité de survie de $\tau^{[k]}$ devient

$$\mathbb{P}\left(\tau^{[k]} > T\right) = \sum_{l=n-k+1}^{n} \sum_{s=1}^{\binom{n}{l}} \alpha_n^{[k]}(l)\, Q_{\pi_s^l}^{[1]}(T),$$

où les coefficients sont donnés par $\alpha_n^{[k]}(l) = (-1)^{l-(n-k+1)} \binom{l-1}{n-k}$, pour $n - k + 1 \leq l \leq n$.

On notera que le gain de performance de cette approche est dû à un réarrangement intelligent des termes FTD. Cependant, cette méthode n'est pas toujours satisfaisante, notamment pour des NTD d'ordre élevées.

Abordons maintenant la méthode de la transformation homogène. Remarquons, tout d'abord, que si on applique la formule de réplication en FTD à un panier homogène, le nombre de termes à calculer sera réduit considérablement puisque tous les FTD référençant des sous-paniers de k éléments auront la même valeur. Notre idée a été alors de transformer le panier original en un panier homogène et ce tout en préservant certaines caractéristiques de la distribution des défauts. Nous nous intéressons ici au comportement local au niveau de la queue de distribution. Ainsi, nous avons choisi comme métrique de risque le quantile associé à l'ordre du NTD en question.

Dans le but de calculer les quantiles de distributions, nous utiliserons un algorithme basé sur l'inversion d'une transformée de Fourier. Signalons, cependant, que cet algorithme souffre d'une anomalie intéressante. Les probabilités de défaut de chaque émetteur sont généralement faibles, ce qui entraîne que la distribution de portefeuille est centrée vers la gauche et que les probabilités de queue décroissent exponentiellement. Une fois que la probabilité devient inférieure à la double précision de la machine 10^{-16}, les erreurs d'arrondissement liées aux bruits numériques commencent à dominer. Ceci a pour effet de produire un plateau numérique où les valeurs estimées oscillent de façon erratique. Pour passer outre ce problème, nous proposons un changement de probabilité judicieux qui est adapté à la question. Avant d'effectuer l'inversion de Fourier, on se place sous la nouvelle mesure, ensuite on applique la dérivée de Radon-Nykodim pour retrouver le résultat escompté.

Venons en enfin aux méthodes asymptotiques. Il s'avère que, pour des portefeuilles de tailles importantes, la formule de récurrence de l'expansion homogène est numériquement instable. En effet, cette récurrence est similaire à un schéma de discrétisation instable d'une EDP classique. Au vu des difficultés numériques de ce schéma instable, nous avons choisi d'explorer une autre voie et de nous intéresser au comportement asymptotique du panier homogène. Nous montrons que la solution admet un développement en série que nous énonçons dans la proposition suivante.

Proposition (Chapitre 4, page 111)

Pour un portefeuille homogène, la probabilité de survie du k-ième défaut admet le développement en série suivant :

$$Q^{[k]}(T) = e^{-\Lambda^c(T)} \left[\sum_{n_1=0}^{+\infty} \dots \sum_{n_m=0}^{+\infty} \frac{\Lambda^{c_1}(T)^{n_1}}{n_1!} \dots \frac{\Lambda^{c_m}(T)^{n_m}}{n_m!} B\left(k-1, n, p_{n_1,\dots,n_m}\right) \right],$$

où $\Lambda^c(T) \triangleq \sum_{i=1}^n \Lambda^{c_j}(T)$, $\Lambda^{c_j}(T) \triangleq \int_0^T \lambda^{c_j}(t)\,dt$, *le paramètre* $p_{n_1,\dots,n_m}$ *est*

$$p_{n_1,\dots,n_m} = 1 - e^{-\Lambda^{0,*}(T)} \left(1-p_1^*\right)^{n_1} \dots \left(1-p_m^*\right)^{n_m},$$

et $B(k,n,p)$ *désigne la fonction de probabilité binomiale cumulé de paramètre* p *:* $B(k,n,p) = \sum_{j=0}^k \binom{n}{j} p^j (1-p)^{n-j}$.

Ce développement en série correspond à la formule d'une mixture de binomiales dont les coefficients peuvent être estimés avec une approche analytique ou bien avec une intégration quasi-Monte-Carlo.

Nous établissons également un développement en série de la solution quand le panier sous-jacent est non-homogène. Les distributions conditionnelles qui apparaissent dans la formule sont calculées par un algorithme de convolution récursif. Ce type d'algorithmes a été abondamment utilisé dans la littérature actuarielle.

Pour clore ce chapitre, on effectue des tests numériques pour comparer la précision de l'expansion homogène, l'approximation de Duffie et la méthode de Panjer (Lindskog et McNeil (2003)).

0.5 Chapitre 5 : Correlation of Correlation

Ces dernières années ont connu une forte croissance des émissions de CDO synthétiques dont le portefeuille sous-jacent est lui-même constitué de tranches de CDO. Ces transactions de « CDO de CDO » sont aussi appelées des « CDO au carré ». Une telle transaction comprend typiquement entre 5 et 20 CDO sous-jacents dont le portefeuille de référence compte 300 à 600 entités. Les CDO au carré sont généralement structurés afin d'obtenir une note comprise entre 'A' et 'AA'. Pour des raisons de diversification et de dilution du risque, les primes offertes sont beaucoup plus importantes que celles d'un CDO normal. Ce qui explique l'engouement des investisseurs pour ce produit.

Dans ce chapitre, nous étendons notre approche numérique aux CDO au carré. Dans un premier temps, nous montrons qu'il est possible de répliquer statiquement ce nouveau payoff à l'aide des instruments FTD. Partant de la formule de réplication développée dans le Chapitre 4, et en appliquant des règles de calcul aux sous-ensembles du panier d'obligations de référence, on aboutit à des expressions formelles du portefeuille de réplication. Afin d'illustrer la méthode, il est facile de dériver quelques résultats pour des paniers de petite taille. Cependant, en pratique, le nombre d'instruments de couverture devient important pour des paniers de CDO, ce qui induit des temps de calcul extrêmement longs.

Nous avons donc cherché à réduire la complexité numérique du problème en explorant une autre méthode qui s'inspire de la transformation homogène du Chapitre 4. La méthode que nous proposons dite du « processus individuel équivalent » repose sur l'idée suivante. L'instant du k-ième défaut $\tau^{[k]}$ est caractérisé par une pseudo-intensité (que l'on définira), ainsi qu'une décomposition sur les facteurs communs du modèle de Marshall-Olkin. Il existe alors un processus de Poisson (*single-name*) dont le temps du premier saut admet les mêmes caractéristiques de défaut que $\tau^{[k]}$. Il s'agit donc de définir la transformation suivante :

$$\left([p_{i,j}]_{\substack{1\leq i\leq n \\ 1\leq j\leq m}}, \left(\lambda^{0,i}\right)_{1\leq i\leq n} \right) \rightarrow \left(\left(p_j^{eq}\right)_{1\leq j\leq m}, \lambda^{0,eq} \right),$$

qui relie un panier sous-jacent d'obligations de référence à un processus individuel équivalent préservant les propriétés de défaut de $\tau^{[k]}$. Par un raisonnement analogue, il est possible de définir une transformation équivalente pour une tranche de CDO.

Plus précisément, si on note, pour un horizon de temps T, les pertes du portefeuille

$$L_T \triangleq \sum_{i=1}^{n} L_i D_T^i,$$

où D_T^i et L_i sont l'indicatrice de défaut et la perte de l'émetteur i. La probabilité de survie de la tranche (α, β) est definie

$$Q^{\alpha,\beta}(T) \triangleq \mathbb{E}\left[1 - \frac{M_T^{\alpha,\beta}}{\beta - \alpha}\right],$$

où $M_T^{\alpha,\beta} \triangleq \min\left(\max\left(L_T - \alpha, 0\right), \beta - \alpha\right)$ désigne la perte de la tranche à T.

La transformation équivalente conserve la valeur de $Q^{\alpha,\beta}(T)$ pour un horizon de temps fixé.

Notre algorithme se base sur les théorèmes suivants.

Théorème (Chapitre 5, page 135)

Posons $\psi_i = \exp(iuL_i)$. Pour T fixé, La transformée de Fourier $\phi(u)$ de la variable aléatoire L_T s'écrit

$$\phi(u) = \sum_{\pi \in \mathbf{\Pi_n}} Q_\pi^{[1]}(T) \left[\prod_{i \notin \pi} \psi_i \prod_{i \in \pi} (1 - \psi_i)\right].$$

Théorème (Chapitre 5, page 135)

La transformée de Fourier de la variable de perte $\phi(u)$ admet le développement en série suivant

$$\phi(u) = e^{-\Lambda^c(T)} \left[\sum_{n_1=0}^{+\infty} \dots \sum_{n_m=0}^{+\infty} \frac{\Lambda^{c_1}(T)^{n_1}}{n_1!} \dots \frac{\Lambda^{c_m}(T)^{n_m}}{n_m!} \phi\left(u, \widetilde{p_{n_1,\dots,n_m}}\right)\right],$$

où $\Lambda^c(T) \triangleq \sum_{i=1}^n \Lambda^{c_j}(T)$, $\Lambda^{c_j}(T) \triangleq \int_0^T \lambda^{c_j}(t)\, dt$, et le vecteur des probabilités $\widetilde{p_{n_1,\dots,n_m}} = (p_{n_1,\dots,n_m}(1), \dots, p_{n_1,\dots,n_m}(n))$ est donné par

$$p_{n_1,\dots,n_m}(i) = 1 - e^{-\Lambda_i^0(T)} (1 - p_{i,1})^{n_1} \dots (1 - p_{i,m})^{n_m}.$$

Enfin, pour clore ce chapitre, nous présentons une illustration de la méthode sur un cas test. Cet exemple met en jeu une topographie de corrélation très intéressante. Il permet, en outre, de développer une intuition sur les effets de corrélation et le risque de modèle lié à ces produits.

Part I

Partie Théorique

Chapter 1

The Conditional Jump Diffusion Framework

Enlarging the economic state-variables' filtration by observing the default process of all available credits has some profound implications on the dynamics of intensities. Indeed, the sudden default of one credit triggers jumps in the spreads of all the other obligors. This is referred to as the "Conditional Jump Diffusion" effect. The aim of this chapter is to give a comprehensive and self-contained presentation of the CJD framework, and present some new results. In particular, we derive the density function of default times in the "looping" defaults model; we give a formula of the conditional expectation w.r.t. the enlarged filtration in the n-dimensional case and we study the equivalence between the copula approach and the CJD framework.

1.1 Introduction

The problem of correlating default times has been studied extensively in a number of papers. The main approach used is the Copula approach. The first application of copulas to basket credit derivatives is given in Li (2000). Since then practitioners have used the Gaussian copula as the market standard approach to default correlation. For a comprehensive review of copulas and their applications to finance and risk management you can consult for example the papers by Embrechts, Lindskog, McNeil (2003) or Bouyé, Durrleman, Nikeghbali, Riboulet, Roncalli (2000).

Another approach that was suggested by Duffie (1998) is the correlated intensities method. The default times' intensities, in this model, are assumed to follow some correlated diffusions. Although theoretically appealing and easily implementable, Duffie's approach fails to reproduce some realistic default correlations (see Jouanin, Rapuch, Riboulet and Roncalli (2001) for an empirical study). In fact, a closer look at the formulas shows that correlated intensities introduce an adjustment in the joint default probabilities, which is similar to a quanto adjustment or a convexity adjustment. This is very different, in essence, from default correlations.

One way to improve Duffie's approach is to consider the impact of filtration enlargements. Indeed, the intensity of each default time studied on its natural filtration has a different dynamic from the one observed on the enlarged filtration where the default information of the other obligors is taken into account. This fundamental observation was made in the papers by Jeanblanc and Rutkowski (2000a), Jeanblanc and Rutkowski (2000b) and Elliott, Jeanblanc and Yor (2000). Enlarging the working filtration or, in financial terms, increasing the flow of information available to investors, profoundly alters the intensity dynamics. A sudden default in one credit can create a shock wave in the market, which translates into revising its estimate of the default likelihood of the other obligors; this, in turn, triggers a jump in their credit spreads. This phenomenon will be referred to henceforth as the "Conditional Jump Diffusion" effect.

The first example of CJD dynamics is given in Kusuoka (1999). This example is also studied in Jeanblanc and Rutkowski (2000b). The infectious defaults model of Davis and Lo (2001b) assumes a CJD dynamic, where upon default the intensities jump proportionally to their pre-default levels, then revert back to the pre-default state. Deriving survival probabilities in the general CJD framework is far from being a trivial task. Jarrow and Yu (2001) have studied a simplified version of the model where closed-form formulas can be derived. They assume that the CJD coupling is unidirectional. In other words, the universe of credits is split into two subsets: a default event in the first subgroup triggers jumps in the spreads of the other obligors; the default of an obligor in the second set, on the other hand, has no impact on the spreads of the first.

Copulas and CJD dynamics are two sides of the same coin. A choice of copula implies a specification of a CJD dynamic, and vice-versa. Copulas and CJDs are equivalent in that sense. This was first studied in the paper by Schönbucher and Schubert (2001). This seminal work bridges the gap between the mathematician's approach to default correlation, where it is merely about a choice of copula function, and the trader's approach where default correlation translates into a windfall P&L in the event of default.

The aim of this chapter is to complement the recent literature on the modelling of default correlation in the so-called Conditional Jump Diffusion framework by giving a comprehensive and self-contained presentation of this approach, and presenting new results.

Our first contribution is a general formula of the default times' multivariate distribution in the "looping" defaults model. The second contribution is a generalization of the conditional expectation with respect to the enlarged filtration formula of Dellacherie to the n-dimensional case. And last but not least, we study the equivalence between the copula approach and CJD dynamics along the lines of Schönbucher and Schubert (2001).

The rest of this chapter is organized as follows. Section 1.2 describes the model. Section 1.3 considers interacting Itô and point processes and studies the "looping" defaults' model. In Section 1.4, we derive a generalized Dellacherie formula. This latter is then used in Section 1.5 to establish the equivalence between copulas and CJD dynamics. Section 1.6 gives some numerical examples.

1.2 The Model

We work in an economy represented by a probability space $(\Omega, \mathcal{G}, P)$ and a time horizon $T^* \in (0, \infty)$, on which is given a d-dimensional Brownian motion W. We assume that the probability space $(\Omega, \mathcal{G}, P)$ is rich enough to support a set of n non-negative random variables $(\tau_1, ..., \tau_n)$ representing the default times of the obligors in the economy. Further, we assume, for convenience, that $\mathbb{P}(\tau_i = 0) = 0$ and $\mathbb{P}(\tau_i > t) > 0$ for any $t \in \mathbb{R}_+$.

We also introduce an $\mathbb{R}^d$-valued Itô process $(X_t)_{t \geq 0}$, describing the evolution of the state-variables in the economy, which solves the following SDE

$$dX_t = \alpha(X_t)\,dt + \beta(X_t)\,dW_t,$$

for some Lipschitz functions $\alpha_k : \mathbb{R}^d \to \mathbb{R}$ and $\beta_{kj} : \mathbb{R}^d \to \mathbb{R}^d$, $1 \leq k \leq d$, $1 \leq j \leq d$.

Filtrations. We denote by $\{\mathcal{F}_t\}_{0 \leq t \leq T^*}$ the filtration generated by X and augmented with the P-null sets of $\mathcal{G}$:

$$\mathcal{F}_t = \sigma(X_s : 0 \leq s \leq t) \vee \mathcal{N}.$$

We introduce, for each obligor i, the right-continuous process $D_t^i \triangleq \mathbf{1}_{\{\tau_i \leq t\}}$ indicating whether the firm has defaulted or not. We denote by $\left\{\mathcal{H}_t^i\right\}$ the filtration generated by this process

$$\mathcal{H}_t^i \triangleq \mathcal{F}_t^{D^i} = \sigma\left(D_s^i : 0 \leq s \leq t\right).$$

We define the following filtrations:

1. The collective filtration of the economic state variables and the default processes

$$\mathcal{G}_t \triangleq \mathcal{F}_t \vee \left[\bigvee_{i=1}^{n} \mathcal{H}_t^i\right].$$

2. For each obligor i, the filtration generated by the state variables and the default processes of all other firms other than i

$$\begin{aligned}\mathcal{G}_t^{-i} &\triangleq \mathcal{F}_t \vee \mathcal{H}_t^{-i},\\ \mathcal{H}_t^{-i} &\triangleq \left[\bigvee_{j\neq i} \mathcal{H}_t^j\right] = \mathcal{H}_t^1 \vee ... \vee \mathcal{H}_t^{i-1} \vee \mathcal{H}_t^{i+1} \vee ... \vee \mathcal{H}_t^n.\end{aligned}$$

3. The firm specific information generated by the default filtration of i and the state variables' filtration
$$\mathcal{G}_t^i \triangleq \mathcal{F}_t \vee \mathcal{H}_t^i.$$

We assume that the filtration $\{\mathcal{F}_t\}$ has the martingale invariance property with respect to the filtration $\{\mathcal{G}_t\}$, i.e., that hypothesis $(\mathcal{H})$ holds.

Hypothesis $(\mathcal{H})$**.** Every $\{\mathcal{F}_t\}$-square-integrable martingale is a $\{\mathcal{G}_t\}$-square-integrable martingale.

This implies, in particular, that the $\{\mathcal{F}_t\}$-Brownian motion is a Brownian motion in the enlarged filtration $\{\mathcal{G}_t\}$.

Assumption 1. We assume that the probability of instantaneous joint defaults is equal to zero, i.e., $\mathbb{P}(\tau_i = \tau_j) = 0$, for $i \neq j$.

Intensities. We shall use the following definition of an intensity process for a stopping time τ with respect to a given filtration $\{\mathcal{I}_t\}$. We refer to Brémaud (1980) for details.

Definition 1 *(Intensity process). Let τ be an $\{\mathcal{I}_t\}$-stopping time and let λ be a non-negative $\{\mathcal{I}_t\}$-predictable process such that, for all $t \geq 0$, $\int_0^t \lambda_s ds < \infty$ almost surely. We say that λ is an $\{\mathcal{I}_t\}$-intensity of the stopping time τ if*

$$M_t \triangleq D_t - \int_0^{t\wedge\tau} \lambda_s ds$$

is an $\{\mathcal{I}_t\}$-martingale. The process M_t is called the compensated point process.

Remark 2 *From Definition 1, the intensity is not uniquely defined after the occurrence of the default time. Indeed, if λ_t is an intensity process, then $\lambda_t^1 = \lambda_t \mathbf{1}_{\{t\leq\tau\}}$ and $\lambda_t^2 = \lambda_t^1 + \theta\mathbf{1}_{\{\tau>t\}}$, for all $\theta \geq 0$, are also intensity processes for τ.*

For each obligor i, we define two types of intensities: the first one with respect to the firm-specific filtration $\{\mathcal{G}_t^i\}$ and the second one with respect to the enlarged filtration $\{\mathcal{G}_t\}$.

Firm-Specific-Information Setting. We assume that τ_i has an intensity $\widetilde{h}^i$ with respect to the filtration $\{\mathcal{G}_t^i\}$. We know that there exists an $\{\mathcal{F}_t\}$-adapted process h^i such that $\mathbf{1}_{\{\tau_i>t\}}\widetilde{h}_t^i = \mathbf{1}_{\{\tau_i>t\}}h_t^i$ (see, for instance, Jeanblanc and Rutkowski (2000b) for details). The process h^i is the $\{\mathcal{F}_t\}$-adapted version of the $\{\mathcal{G}_t^i\}$-intensity, i.e.,

$$h^i \text{ is } \{\mathcal{F}_t\}\text{-adapted, and } D_t^i - \int_0^{t\wedge\tau_i} h_s^i ds \text{ is a } \{\mathcal{G}_t^i\}\text{-martingale.}$$

Enlarged-Information Setting. We assume that τ_i has an intensity $\widetilde{\lambda}^i$ with respect to the enlarged filtration $\{\mathcal{G}_t\}$. There exists a $\{\mathcal{G}_t^{-i}\}$-adapted process λ^i such that $\mathbf{1}_{\{\tau_i>t\}}\widetilde{\lambda}_t^i = \mathbf{1}_{\{\tau_i>t\}}\lambda_t^i$. The process λ^i is the $\{\mathcal{G}_t^{-i}\}$-adapted version of the $\{\mathcal{G}_t\}$-intensity, i.e.,

$$\lambda^i \text{ is } \{\mathcal{G}_t^{-i}\}\text{-adapted, and } D_t^i - \int_0^{t\wedge\tau_i} \lambda_s^i ds \text{ is a } \{\mathcal{G}_t\}\text{-martingale.}$$

Expectations. If we work in the firm-specific filtration $\{\mathcal{G}_t^i\}$, it is well-known that the conditional survival probabilities can be computed as

$$\mathbb{P}\left(\tau_i > T \left|\mathcal{G}_t^i\right.\right) = \mathbf{1}_{\{\tau_i>t\}}\mathbb{E}\left[\exp\left(-\int_t^T h_s^i ds\right)|\mathcal{F}_t\right], \text{ for } T \geq t. \tag{1.1}$$

However, when we consider the general case where we work on the enlarged filtration $\{\mathcal{G}_t\}$, the conditional survival probability cannot be computed in a straightforward way. To address this issue, we shall use the change of measure technique introduced by Kusuoka (1999): under the new measure the "circular" nature of this type of "looping" default models is broken and calculations can be easily carried out.

Assumption 2. For practical applications, we make the assumption that the intensities are Markov-functionals of the background process X and the default indicators $\mathbf{D} \triangleq \left(D^1, ..., D^n\right)$. We assume that

$$\lambda_t^i = \lambda^i\left(X_t, D_t^1, ..., D_t^{i-1}, 0, D_t^{i+1}, ..., D_t^n\right),$$

for some bounded continuous functions $\lambda^i(.,.) : \mathbb{R}^d \times \{0,1\}^n \to \mathbb{R}_+$, which are $\mathcal{C}^2$ in the first argument. This is similar to the Markovian setting considered in Frey and Backhaus (2004).

1.3 Interacting Itô and Point Processes

In this section, we consider a general model of "looping" defaults where the default point processes impact intensities, which, in turn, drive the default processes. This type of "circular" dependence between Itô and point processes has been studied in Becherer and Schweizer (2005). They have addressed, in particular, the question of existence and uniqueness of the solution.

Construction via a Change of Measure. To construct this non-standard dependence structure, we use a change of measure method, which extends the argument in Kusuoka (1999). The basic idea is to start with some probability space $(\Omega, \mathcal{G}', \{\mathcal{G}'_t\}, P')$ on which we are given a Brownian motion W and a set of well defined independent default times with constant $(P', \{\mathcal{G}'_t\})$-intensities equal to 1. Assume $\mathcal{G}'_0$ is trivial, $\mathcal{G}'_{T^*} = \mathcal{G}'$, and $\{\mathcal{G}'_t\}$ satisfies the usual conditions. Then, define the probability measure P as

$$\frac{dP}{dP'} = \mathcal{E}\left(\sum_{i=1}^{n} \int \left(\lambda_t^i - 1\right)\left(dD_t^i - \left(1 - D_t^i\right) dt\right)\right)_{T^*}. \tag{1.2}$$

Using Girsanov theorem, we can see that this measure change is such that λ_t^i is the $(P, \{\mathcal{G}'_t\})$-intensity of τ_i. Again, by Girsanov, W is a local $(P, \{\mathcal{G}'_t\})$-martingale; its quadratic covariance process $\langle W \rangle$ is the same under P and P', hence it is a $(P, \{\mathcal{G}'_t\})$-Brownian motion. Define $(\Omega, \mathcal{G}, \{\mathcal{G}_t\}, P)$ as the P-completion of $(\Omega, \mathcal{G}', \{\mathcal{G}'_t\}, P')$, one can verify that $\{\mathcal{G}_t\}$ satisfies the usual conditions under P. Therefore, W is a $(P, \{\mathcal{G}_t\})$-Brownian motion.

We shall use the change of measure construction to derive the default times' density function in Proposition 3.

Non-Standard SDEs. Applying Itô's lemma to the intensity process $\lambda_t^i = \lambda^i(X_t, \mathbf{D}_t)$, one finds that the looping defaults model in a Markovian setting can be described as:

$(\boldsymbol{\lambda}, \mathbf{D})$ is the solution of the following system of SDEs

$$d\lambda_t^i = \alpha^i(X_t, \mathbf{D}_t)\, dt + \beta^i(X_t, \mathbf{D}_t)\, dW_t + \sum_{\substack{j=1 \\ j \neq i}}^{n} \Delta^{ij}(X_t, \mathbf{D}_t)\, dM_t^j, \tag{1.3}$$

the functions $\alpha^i(.,.) : \mathbb{R}^d \times \{0,1\}^n \to \mathbb{R}$, for $i = 1, ..., n$; $\beta^{il}(.,.) : \mathbb{R}^d \times \{0,1\}^n \to \mathbb{R}$, for $i = 1, ..., n$, $l = 1, ..., d$; $\Delta^{ij}(.,.) : \mathbb{R}^d \times \{0,1\}^n \to \mathbb{R}$ for $i = 1, ..., n$, $j = 1, ..., n$, are given by

$$\begin{aligned}\alpha^{i}(\mathbf{x},\mathbf{y}) &= \mathcal{L}^{X}\lambda^{i}(\mathbf{x},\mathbf{y})+\mathcal{L}^{\mathbf{D}}_{[\mathbf{x}]}\lambda^{i}(\mathbf{x},\mathbf{y}),\\ \beta^{il}(\mathbf{x},\mathbf{y}) &= \sum_{k=1}^{d}\frac{\partial\lambda^{i}(\mathbf{x},\mathbf{y})}{\partial x_k}\beta^{kl}(\mathbf{x}),\\ \Delta^{ij}(\mathbf{x},\mathbf{y}) &= \lambda^{i}\left(\mathbf{x},\mathbf{y}^{-j}\right)-\lambda^{i}(\mathbf{x},\mathbf{y}),\end{aligned}$$

where $\mathcal{L}^X$ is the infinitesimal generator of the $\mathbb{R}^d$-valued diffusion process X_t, and $\mathcal{L}^{\mathbf{D}}_{[\mathbf{x}(\omega)]}$ denotes the infinitesimal generator of the Markovian process $\mathbf{D}$, for a given path of the background process $X_t(\omega)=\mathbf{x}(\omega)$.

This is not a standard SDE: the coefficients defining the intensities depend on the default state, and the default state vector depends in turn on the intensities. An example of this class of models is studied next.

The Jarrow and Yu Model[1]**.** The looping defaults model of Jarrow and Yu follows the SDE

$$d\lambda^i_t=\sum_{\substack{j=1\\ j\neq i}}^{n}\Delta^{ij}dD^j_t, \tag{1.4}$$

for some constant jumps $\Delta^{ij}\in\mathbb{R}$. This is a particular case of equation (1.3), where

$$\begin{aligned}\alpha^{i}(\mathbf{x},\mathbf{y}) &= \sum_{i=1}^{n}(1-\mathbf{y}(i))\,\lambda^{i}(\mathbf{x},\mathbf{y})\,\Delta^{ij};\\ \beta^{il}(\mathbf{x},\mathbf{y}) &= 0;\\ \Delta^{ij}(\mathbf{x},\mathbf{y}) &= \Delta^{ij},\ \Delta^{ii}=0.\end{aligned}$$

The intensity process λ^i is, then, given by

$$\lambda^i_t=\lambda^i_0+\sum_{\substack{j=1\\ j\neq i}}^{n}\Delta^{ij}D^j_t. \tag{1.5}$$

Our first result is an analytical expression of the default times' multivariate density.

Proposition 3 *(Default Times' Multivariate Density). Let* $(t_1,t_2,...,t_n)\in\mathbb{R}^n_+$*, and suppose that*

$$t_{\pi(1)}\le t_{\pi(2)}\le ...\le t_{\pi(n)},$$

[1] In the Jarrow and Yu paper, a simplified version of the general model is considered. They assume that the coupling matrix $\left[\Delta^{ij}\right]$ is upper-triangular in order to break the circular dependence.

where the mapping $\pi(.) : \{1, ..., n\} \to \{1, ..., n\}$ *is a monotonic permutation of* $(t_1, t_2, ..., t_n)$. *Then, the default times' multivariate density is given by*

$$f(t_1, ..., t_n) = \prod_{i=1}^{n} \left[\lambda_0^{\pi(i)} + \sum_{j=1}^{i-1} \Delta^{\pi(i)\pi(j)} \right] \exp\left(- \sum_{j=1}^{i} \left[\lambda_0^{\pi(i)} + \sum_{k=1}^{j-1} \Delta^{\pi(i)\pi(k)} \right] \left(t_{\pi(j)} - t_{\pi(j-1)} \right) \right), \tag{1.6}$$

with the convention $t_{\pi(0)} = 0$.

Example. For $n = 3$, suppose $t_2 \le t_1 \le t_3$, then

$$\begin{aligned} f(t_1, t_2, t_3) \;=\; & \lambda_0^2 \left(\lambda_0^1 + \Delta^{12}\right) \left(\lambda_0^3 + \Delta^{32} + \Delta^{31}\right) \\ & \times \exp\left(-\lambda_0^2 t_2\right) \\ & \times \exp\left(-\left(\lambda_0^1 t_2 + \left(\lambda_0^1 + \Delta^{12}\right)(t_1 - t_2)\right)\right) \\ & \times \exp\left(-\left(\lambda_0^3 t_2 + \left(\lambda_0^3 + \Delta^{32}\right)(t_1 - t_2) + \left(\lambda_0^3 + \Delta^{32} + \Delta^{31}\right)(t_3 - t_1)\right)\right). \end{aligned}$$

Proof. Extending the argument of Kusuoka (1999), we use a change of measure technique. We assume that we have some probability measure P' under which the default times $(\tau_1, ..., \tau_n)$ are independent random variables exponentially distributed with parameter 1. So that the P'-density of $(\tau_1, ..., \tau_n)$ is

$$f'(t_1, ..., t_n) \triangleq \mathbb{P}'(\tau_1 \in dt_1, ..., \tau_n \in dt_n) = \exp\left(-(t_1 + ... + t_n)\right).$$

Define the probability measure P (as in equation (1.2))

$$\frac{dP}{dP'} = L_{T^*},\; P'\text{-a.s.},$$

where L satisfies, for $t \in [0, T^*]$,

$$L_t = 1 + \sum_{i=1}^{n} \int_{]0,t]} L_{s^-} \phi_s^i \left(dD_s^i - \left(1 - D_s^i\right) ds \right). \tag{1.7}$$

ϕ^i is given by

$$\phi_t^i \triangleq \lambda_t^i - 1 = \lambda_0^i + \sum_{\substack{j=1 \\ j \neq i}}^{n} \Delta^{ij} D_t^j - 1. \tag{1.8}$$

By Girsanov's theorem, the intensity of the default time τ_i under P' is equal to $\left(1 + \phi_t^i\right) \times 1 = \lambda_t^i$.

The Doléans-Dade martingale L can also be written as the product

$$L_t = \prod_{i=1}^{n} L_t^i, \text{ for } t \in [0, T^*],$$

where L^i, $1 \le i \le n$, is defined as

$$L_t^i = 1 + \int_{]0,t]} L_{s-}^i \phi_s^i \left(dD_s^i - \left(1 - D_s^i\right) ds \right). \quad (1.9)$$

The solution of the SDE (1.9) is

$$\begin{aligned} L_t^i &= \exp\left(- \int_0^{\tau_i \wedge t} \left(\lambda_s^i - 1\right) ds \right) \left[\mathbf{1}_{\{\tau_i > t\}} + \mathbf{1}_{\{\tau_i \le t\}} \lambda_{\tau_i}^i \right] \\ &= \exp\left((\tau_i \wedge t) - \Lambda_{\tau_i \wedge t}^i \right) \left[\mathbf{1}_{\{\tau_i > t\}} + \mathbf{1}_{\{\tau_i \le t\}} \lambda_{\tau_i}^i \right], \end{aligned}$$

where Λ^i denotes the hazard rate process $\Lambda_t^i \triangleq \int_0^t \lambda_s^i ds$.

To compute the density function at $(t_1, ..., t_n)$, we proceed as follows.

We fix an arbitrary positive number $\alpha > 0$, and we compute, for $(\epsilon_1, ..., \epsilon_n) \in [-\alpha, \alpha]^n$, the expression of

$$\mathbb{P}\left(\tau_1 \in (t_1 - \epsilon_1, t_1], ..., \tau_n \in (t_n - \epsilon_n, t_n]\right).$$

To this end, we choose a time horizon T^* such that $T^* > \max\limits_{1 \le i \le n} (t_i) + \alpha$ (e.g., $T^* = 1 + \max\limits_{1 \le i \le n} (t_i) + \alpha$), and we use the change of probability measure

$$\frac{dP}{dP'} = L_{T^*} = \prod_{i=1}^n L_{T^*}^i, \ P'\text{-a.s.}$$

Thus, we have

$$\begin{aligned} & \mathbb{P}\left(\tau_1 \in (t_1 - \epsilon_1, t_1], ..., \tau_n \in (t_n - \epsilon_n, t_n]\right) \\ = \ & \mathbb{E}^{P'}\left[\mathbf{1}\left\{\tau_1 \in (t_1 - \epsilon_1, t_1], ..., \tau_n \in (t_n - \epsilon_n, t_n]\right\} \times L_{T^*} \right] \\ = \ & \int_{u_1 \in (t_1 - \epsilon_1, t_1]} ... \int_{u_n \in (t_n - \epsilon_n, t_n]} \exp\left(-\left(u_1 + ... + u_n\right)\right) \times \left[\prod_{i=1}^n \exp\left(u_i - \Lambda_{u_i}^i\right) \left[\lambda_{u_i}^i\right] \right] du_1 ... du_n \\ = \ & \int_{u_1 \in (t_1 - \epsilon_1, t_1]} ... \int_{u_n \in (t_n - \epsilon_n, t_n]} \left[\prod_{i=1}^n \exp\left(-\Lambda_{u_i}^i\right) \left[\lambda_{u_i}^i\right] \right] du_1 ... du_n. \quad (1.10) \end{aligned}$$

The first equality follows from the change of measure, the second equality uses the fact that on the set $\{(\tau_1, ..., \tau_n) \in (t_1 - \epsilon_1, t_1] \times ... \times (t_n - \epsilon_n, t_n]\}$, we have

$$L_{T^*}^i = \exp\left(\tau_i - \Lambda_{\tau_i}^i\right) \left[\lambda_{\tau_i}^i\right],$$

since $\tau_i \le t_i + \alpha < T^*$.

Consider the monotonic permutation of the default times $\pi : \tau_{\pi(1)} \le \tau_{\pi(2)} \le ... \le \tau_{\pi(n)}$, and let us derive explicitly the expressions of the intensity and hazard rate for $\left\{\tau_{\pi(1)} \le ... \le \tau_{\pi(n)}\right\}$.

By (1.5), the intensity $\lambda^{\pi(i)}_{\tau_{\pi(i)}}$ at time $t = \tau_{\pi(i)}$ is simply

$$\lambda^{\pi(i)}_{\tau_{\pi(i)}} = \lambda^{\pi(i)}_0 + \sum_{j=1}^{i-1} \Delta^{\pi(i)\pi(j)}. \tag{1.11}$$

The hazard rate is computed by (1.5) and using the fact that $\lambda^{\pi(i)}$ is piece-wise constant on the intervals $\left[\tau_{\pi(j-1)}, \tau_{\pi(j)}\right)$, for $1 \leq j \leq i$:

$$\begin{aligned} \Lambda^{\pi(i)}_{\tau_{\pi(i)}} &= \sum_{j=1}^{i} \Lambda^{\pi(i)}_{\tau_{\pi(j)}} - \Lambda^{\pi(i)}_{\tau_{\pi(j-1)}} \\ &= \sum_{j=1}^{i} \lambda^{\pi(i)}_{\tau_{\pi(j-1)}} \left(\tau_{\pi(j)} - \tau_{\pi(j-1)}\right) \\ &= \sum_{j=1}^{i} \left[\lambda^{\pi(i)}_0 + \sum_{k=1}^{j-1} \Delta^{\pi(i)\pi(k)}\right] \left(\tau_{\pi(j)} - \tau_{\pi(j-1)}\right). \end{aligned} \tag{1.12}$$

By introducing the permutation π, (1.10) becomes

$$\begin{aligned} & \mathbb{P}\left(\tau_1 \in (t_1 - \epsilon_1, t_1], ..., \tau_n \in (t_n - \epsilon_n, t_n]\right) \\ = & \int_{u_1 \in (t_1 - \epsilon_1, t_1]} ... \int_{u_n \in (t_n - \epsilon_n, t_n]} \left[\prod_{i=1}^{n} \exp\left(-\Lambda^{\pi(i)}_{u_{\pi(i)}}\right) \left[\lambda^{\pi(i)}_{u_{\pi(i)}}\right]\right] du_1 ... du_n. \end{aligned}$$

Taking the limit $\epsilon_i \to 0$, for $1 \leq i \leq n$, we arrive at the expression of the density under the P-measure

$$f(t_1, ..., t_n) = \left[\prod_{i=1}^{n} \exp\left(-\Lambda^{\pi(i)}_{t_{\pi(i)}}\right) \left[\lambda^{\pi(i)}_{t_{\pi(i)}}\right]\right].$$

Substituting the intensity and the hazard rate by their expressions (1.11) and (1.12) ends the proof. ■

Marked Point Process Representation. An alternative way of constructing the looping defaults' model is to use the total hazard rate construction developed by Norros (1986) and Shaked and Shanthikumar (1987). This approach was used in Yu (2004).

Assumption 1 excludes simultaneous defaults, we can therefore define the sequence of strictly ordered default times $(T_0, T_1, ..., T_n) : T_0 = 0 < T_1 < ... < T_n$, as well as the identity of the defaulted obligor $(Z_0, Z_1, ..., Z_n)$:

$$\begin{aligned} T_0 &= 0,\ Z_0 = 0; \\ T_k &= \min\left\{\tau_i : 1 \leq i \leq n,\ \tau_i > T_{k-1}\right\}; \\ Z_k &= i \text{ if } T_k = \tau_i. \end{aligned}$$

When Assumption 1 is not satisfied, at each default time, multiple defaults can occur, which means that the size of the mark space is 2^n. This is the case, for example, in the Marshall-Olkin copula. The marked point process (T_n, Z_n) is called the failure process associated to $(\tau_1, ..., \tau_n)$ (see Norros (1986)). For each $1 \leq i \leq n$, the point process D_t^i is given by

$$\begin{aligned} \tau_i &= \min \{T_k : Z_k = i\}; \\ D_t^i &= \sum_{k=1}^{n} \mathbf{1}_{\{T_k \leq t\}} \mathbf{1}_{\{Z_k = i\}}. \end{aligned}$$

The internal history of the process $\left(D_t^1, ..., D_t^n\right)$, $\mathcal{G}_t = \bigvee_{i=1}^n \mathcal{F}_t^{D^i}$, satisfies the following properties (see Brémaud (1980) Chap III T2):

$$\begin{aligned} \mathcal{G}_{T_n} &= \sigma\left(T_0, Z_0, ..., T_n, Z_n\right); \\ \mathcal{G}_{T_n^-} &= \sigma\left(T_0, Z_0, ..., T_{n-1}, Z_{n-1}, T_n\right); \\ \mathcal{G}_{T_n} &= \mathcal{G}_{T_n^-} \vee \sigma\left(Z_n\right). \end{aligned}$$

The compensators Λ^i w.r.t. the internal history can be written in regenerative form (Brémaud (1980) Chap III T7)

$$\Lambda_t^i = \sum_{k=1}^{n} \mathbf{1}_{\{T_{k-1} \leq t < T_k\}} A_t^{(k)}\left(i; T_1, Z_1, ..., T_{k-1}, Z_{k-1}\right), \tag{1.13}$$

where each $A_{\cdot}^{(k)}(i; ...)$ is a deterministic function of its arguments. The stochastic compensator Λ^i is a piece-wise deterministic function evaluated at default times. To clarify the notations, we underline that $A_{\cdot}^{(k)}(i; ...)$ are deterministic functions

$$\begin{aligned} A_t^{(k)}(i; ...) &: \left([0, +\infty) \times \{1, ..., n\}\right)^{k-1} \to \mathbb{R}^+ \\ (T_1, Z_1, ..., T_{k-1}, Z_{k-1}) &\to A_t^{(k)}\left(i; T_1, Z_1, ..., T_{k-1}, Z_{k-1}\right), \end{aligned}$$

but the compensator Λ^i is stochastic

$$\Lambda_t^i(\omega) = \sum_{k=1}^{n} \mathbf{1}_{\{T_{k-1}(\omega) \leq t < T_k(\omega)\}} A_t^{(k)}\left(i; T_1(\omega), Z_1(\omega), ..., T_{k-1}(\omega), Z_{k-1}(\omega)\right).$$

It is well known that if: (a) $\mathbb{P}(0 < \tau_i < \infty) = 1$, (b) $\mathbb{P}(\tau_i = \tau_j) = 0$, for $i \neq j$, (c) the compensators Λ^i are continuous, then the random variables $\Lambda_\infty^i = \Lambda_{\tau_i}^i$, for $1 \leq i \leq n$, are independent and exponentially distributed with parameter 1 (see Norros (1986) Th 2.1). Hence, the default times $(\tau_1, ..., \tau_n)$ are mapped to some independent exponential variables $(\theta_1, ..., \theta_n)$. Norros establishes that this mapping is bijective and thus provides a way of constructing the default times in terms of independent exponentials.

The mapping $\psi : \mathbb{R}_+^n \to \overline{\mathbb{R}}_+^n : \psi(\tau_1, ..., \tau_n) = (\theta_1, ..., \theta_n)$ is called the total hazard function of $(\tau_1, ..., \tau_n)$. It is almost inverse ψ^*, in the sense that $\psi^*(\psi(\tau_1, ..., \tau_n)) = (\tau_1, ..., \tau_n)$ a.s., is obtained by the following algorithm.

Define $A_\theta^{(k),*}(i; ...) = \inf\left\{t : A_t^{(k)}(i; ...) > \theta\right\}$. For a given $(\theta_1, ..., \theta_n) \in \mathbb{R}_+^n$, and proceed as follows:

$$\begin{cases} T_1 = \min\left\{A_{\theta_i}^{(1),*}(i; \emptyset) : i \in \{1, ..., n\}\right\}; \\ Z_1 = \min\left\{i : A_{\theta_i}^{(1),*}(i; \emptyset) = T_1,\ i \in \{1, ..., n\}\right\}; \\ \to \tau_{Z_1} = T_1 \end{cases}$$

.

$$\begin{cases} T_{k+1} = \min\left\{A_{\theta_i}^{(k+1),*}\left(i; (T_l, Z_l)_{1 \le l \le k}\right) : i \in \{1, ..., n\} \setminus \{Z_1, ..., Z_k\}\right\}; \\ Z_{k+1} = \min\left\{i : A_{\theta_i}^{(k+1),*}\left(i; (T_l, Z_l)_{1 \le l \le k}\right) = T_{k+1},\ i \in \{1, ..., n\} \setminus \{Z_1, ..., Z_k\}\right\} \\ \to \tau_{Z_{k+1}} = T_{k+1}; \end{cases}$$

.

$$\begin{cases} T_n = A_{\theta_n}^{(n),*}\left(Z_n; (T_l, Z_l)_{1 \le l \le k-1}\right); \\ \to \tau_{Z_n} = T_n; \end{cases}$$

See Algorithm 2.4 in Norros (1986) for further details.

This provides another way of constructing the looping defaults model. Indeed, one can establish the link between the intensity (1.5) and the functions $A_\cdot^{(k)}(i; ...)$ of the regenerative form (1.13). Observe first that the compensator Λ_t^i is given by

$$\Lambda_t^i = \int_0^t \left(1 - D_s^i\right) \lambda_s^i ds,$$

and let us write it down explicitly for an example where obligor i is the third default, i.e. $T_3 = \tau_i$, $Z_3 = i$.

$0 \le t < T_1 : \Lambda_t^i = \lambda_0^i t = A_t^{(1)}(i)$

$T_1 \le t < T_2 : Z_1 \ne i : \Lambda_t^i = \lambda_0^i T_1 + \left(\lambda_0^i + \Delta^{iZ_1}\right)(t - T_1) = A_t^{(2)}(i; T_1, Z_1)$

$T_2 \le t < T_3 : Z_2 \ne i : \Lambda_t^i = \lambda_0^i T_1 + \left(\lambda_0^i + \Delta^{iZ_1}\right)(T_2 - T_1) + \left(\lambda_0^i + \Delta^{iZ_1} + \Delta^{iZ_2}\right)(t - T_2) = A_t^{(3)}(i; T_1, Z_1, T_2, Z_2)$

$T_3 \le t < T_4 : Z_3 = i : \Lambda_t^i = \lambda_0^i T_1 + \left(\lambda_0^i + \Delta^{iZ_1}\right)(T_2 - T_1) + \left(\lambda_0^i + \Delta^{iZ_1} + \Delta^{iZ_2}\right)(T_3 - T_2) = A_t^{(4)}(i; T_1, Z_1, T_2, Z_2, T_3, Z_3)$

$T_4 \le t : \Lambda_t^i = \Lambda_{T_3}^i$

In general, the looping defaults model (1.5) can be expressed in regenerative form

(1.13) as follows:

$$\begin{aligned} A_t^{(k)}(i;T_1,Z_1,...,T_{k-1},Z_{k-1}) &= \sum_{j=1}^{k-1}\left[\lambda_0^i+\sum_{l=1}^{j-1}\Delta^{iZ_l}\right](T_j-T_{j-1})\prod_{l=1}^{j-1}\mathbf{1}_{\{Z_l\neq i\}} \\ &+\left[\lambda_0^i+\sum_{l=1}^{k-1}\Delta^{iZ_l}\right](t-T_{k-1})\prod_{l=1}^{k-1}\mathbf{1}_{\{Z_l\neq i\}}. \end{aligned}$$

The marked point process representation (T_n, Z_n) offers an alternative proof of Proposition 3. One could write, for example,

$$\begin{aligned} & \mathbb{P}(\tau_1\in dt_1,...,\tau_n\in dt_n) \\ = & \mathbb{P}\left(\tau_{\pi(1)}\in dt_{\pi(1)},...,\tau_{\pi(n)}\in dt_{\pi(n)}\right) \\ = & \mathbb{P}\left(T_1\in dt_{\pi(1)},Z_1=\pi(1),...,T_n\in dt_{\pi(n)},Z_n=\pi(n)\right) \\ = & \prod_{k=1}^{n}\mathbb{P}\left(T_k\in dt_{\pi(k)},Z_k=\pi(k)\left|T_1=t_{\pi(1)},Z_1=\pi(1),...,T_{k-1}=t_{\pi(k-1)},Z_{k-1}=\pi(k-1)\right.\right). \end{aligned}$$

The double sequence $(T_k, Z_k)_{1\leq k\leq n}$ defines a marked point process with counting measure

$$\begin{aligned} \mu(\omega,dt\times dz) &: (\Omega,\mathcal{G})\to((0,\infty)\times E,(0,\infty)\otimes\mathcal{E}), \\ \int_0^t\int_E H(\omega,s,z)\,\mu(\omega,dt\times dz) &= \sum_{k=1}^{n}H(\omega,T_k(\omega),Z_k(\omega))\,\mathbf{1}_{\{T_k(\omega)\leq t\}}, \end{aligned}$$

and predictable compensator

$$\nu(\omega,dt\times dz)=\Phi_t(\omega,dz)\,\lambda_t^\mu dt,$$

where λ_t^μ is a non-negative $\{\mathcal{G}_t\}$-predictable process and $\Phi_t(\omega, dz)$ is a probability transition kernel from $(\Omega\times[0,\infty), \mathcal{G}\otimes\mathcal{B}_+)$ into $(E,\mathcal{E})$. The pair $(\lambda_t^\mu, \Phi_t(dz))$ is called the $(P,\mathcal{G}_t)$-local characteristics of $\mu(dt\times dz)$. Here, we have

$$\begin{aligned} \lambda_t^\mu &= \sum_{i=1}^{n}\left(1-D_t^i\right)\lambda_t^i; \\ \Phi_t(i) &= \frac{\left(1-D_t^i\right)\lambda_t^i}{\lambda_t^\mu}, \text{ for } i\in\{1,...,n\}, \end{aligned}$$

with the convention $\Phi_t(.)=0$ if $\lambda_t^\mu=0$. The density of (T_k, Z_k), conditional on $\mathcal{G}_{T_{k-1}}$, is given by (see Brémaud (1980) Chap VIII T7)

$$\begin{aligned} g^{(k)}(\omega,t,z) &: \left(\Omega\times[0,\infty),\mathcal{G}_{T_{k-1}}\otimes\mathcal{B}_+\right)\to(E,\mathcal{E}) \\ g^{(k)}(\omega,t,z) &\triangleq \mathbb{P}\left(T_k\in dt,Z_k=i\left|\mathcal{G}_{T_{k-1}}\right.\right)(\omega)=\Phi_t(\omega,i)\,\lambda_t^\mu(\omega)\exp\left(-\int_{T_{k-1}(\omega)}^t\lambda_t^\mu(\omega)\,ds\right)dt. \end{aligned}$$

Reverting back to the default times' multivariate density and using the expression of the conditional density $g^{(k)}(\omega, t, z)$, we can express each term in the product as

$$\mathbb{P}\left(T_k \in dt_{\pi(k)}, Z_k = \pi(k) \,\middle|\, \mathcal{G}_{T_{k-1}}\right) = \left(1 - D^{\pi(k)}_{t^-_{\pi(k)}}\right) \lambda^{\pi(k)}_{t_{\pi(k)}} \exp\left(-\int_{T_{k-1}(\omega)}^{t_{\pi(k)}} \lambda^\mu_s ds\right) dt_{\pi(k)}.$$

Conditioning on the path $\left\{T_1 = t_{\pi(1)}, Z_1 = \pi(1), ..., T_{k-1} = t_{\pi(k-1)}, Z_{k-1} = \pi(k-1)\right\}$, and using the fact that the intensities are piece-wise constant between jumps, we get

$$\begin{aligned} & \mathbb{P}\left(T_k \in dt_{\pi(k)}, Z_k = \pi(k) \,\middle|\, T_1 = t_{\pi(1)}, Z_1 = \pi(1), ..., T_{k-1} = t_{\pi(k-1)}, Z_{k-1} = \pi(k-1)\right) \\ = \; & \lambda^{\pi(k)}_{t_{\pi(k)}} \exp\left(-\lambda^\mu_{t_{\pi(k-1)}} \left(t_{\pi(k)} - t_{\pi(k-1)}\right)\right) dt_{\pi(k)}. \qquad (1.14) \end{aligned}$$

The intensity $\lambda^{\pi(k)}_{t_{\pi(k)}}$ at time $t = t_{\pi(k)}$ is given by

$$\lambda^{\pi(k)}_{t_{\pi(k)}} = \lambda^{\pi(k)}_0 + \sum_{j=1}^{i-1} \Delta^{\pi(i)\pi(j)}, \qquad (1.15)$$

and the intensity of the failure process $\lambda^\mu_{t_{\pi(k-1)}}$ is given by the sum of the remaining intensities associated with the obligors that have survived after $t_{\pi(k-1)}$

$$\lambda^\mu_{t_{\pi(k-1)}} = \sum_{i=1}^n \left(1 - D^i_{t_{\pi(k-1)}}\right) \lambda^i_{t_{\pi(k-1)}} = \sum_{i=1}^n \left(1 - D^{\pi(i)}_{t_{\pi(k-1)}}\right) \lambda^{\pi(i)}_{t_{\pi(k-1)}} = \sum_{i=k}^n \lambda^{\pi(i)}_{t_{\pi(k-1)}}. \qquad (1.16)$$

Putting (1.14), (1.15) and (1.16) together gives the result after some basic algebra:

$$\begin{aligned} f(t_1, ..., t_n) &= \prod_{k=1}^n \lambda^{\pi(k)}_{t_{\pi(k)}} \exp\left(-\lambda^\mu_{t_{\pi(k-1)}} \left(t_{\pi(k)} - t_{\pi(k-1)}\right)\right) \\ &= \left[\prod_{k=1}^n \lambda^{\pi(k)}_{t_{\pi(k)}}\right] \exp\left(-\sum_{k=1}^n \sum_{i=k}^n \lambda^{\pi(i)}_{t_{\pi(k-1)}} \left(t_{\pi(k)} - t_{\pi(k-1)}\right)\right) \\ &= \left[\prod_{k=1}^n \lambda^{\pi(k)}_{t_{\pi(k)}}\right] \exp\left(-\sum_{i=1}^n \sum_{k=1}^i \lambda^{\pi(i)}_{t_{\pi(k-1)}} \left(t_{\pi(k)} - t_{\pi(k-1)}\right)\right) \\ &= \left[\prod_{k=1}^n \lambda^{\pi(k)}_{t_{\pi(k)}}\right] \exp\left(-\sum_{i=1}^n \Lambda^{\pi(i)}_{t_{\pi(i)}}\right) \\ &= \left[\prod_{i=1}^n \left[\lambda^{\pi(i)}_{t_{\pi(i)}}\right] \exp\left(-\Lambda^{\pi(i)}_{t_{\pi(i)}}\right)\right]; \end{aligned}$$

we have inverted the summation order in the third equality, introduced the cumulative hazard rate in the fourth equality and replaced the index k by i in the last line.

1.4 Generalization of the Dellacherie Formula

In this section, we derive a formula of the conditional expectation with respect to the enlarged filtration. This is a generalization of the Dellacherie formula. We shall use this key result to compute the expectations that we encounter in the CJD framework. In particular, the conditional survival probability can be computed with our formula. We apply this result in Section 1.5 where survival probability calculations are carried out in details.

The Dellacherie Formula. We start with the following result established in Dellacherie (1970).

Lemma 4 *Let Y be a $\mathcal{G}$-measurable random variable. Then, we have*

$$\mathbb{E}\left[Y\left|\mathcal{F}_t\vee\mathcal{H}_t^i\right.\right]=\mathbf{1}_{\{\tau_i\leq t\}}\mathbb{E}\left[Y\left|\mathcal{F}_t\vee\mathcal{H}_\infty^i\right.\right]+\mathbf{1}_{\{\tau_i>t\}}\frac{\mathbb{E}\left[Y\times\mathbf{1}_{\{\tau_i>t\}}\left|\mathcal{F}_t\right.\right]}{\mathbb{E}\left[\mathbf{1}_{\{\tau_i>t\}}\left|\mathcal{F}_t\right.\right]}.$$

In order to compute the conditional expectation with respect to the filtration $\left\{\mathcal{F}_t\vee\mathcal{H}_t^i\right\}$, one needs to consider the two possible default states. On each set, representing whether a default has occurred before t or not, the conditional distribution is different. In general, for n default times $(\tau_1,...,\tau_n)$, we have a set of 2^n default states; one has to compute the conditional expectation with respect to the enlarged filtration $\{\mathcal{G}_t\}=\left\{\mathcal{F}_t\vee\bigvee_{i=1}^n\mathcal{H}_t^i\right\}$ on each default scenario. To be more precise, we introduce the following notations.

Notation. At time t, each default state is represented by $\pi\in\mathbf{\Pi_n}$, where $\mathbf{\Pi_n}$ is the set of all subsets of $\{1,...,n\}$. $\pi=\emptyset$ means that there has been no defaults before t; $\pi=\{1,...,n\}$ means that every obligor has already defaulted. If $\pi=\{j_1,...,j_k\}$ for some indexes $j_m\in\{1,...,n\}$, then these indicate the obligors that have defaulted. To the subset π, we associate the indicator $D_t^{(\pi)}$, which is equal to 1 if we are in the default state (π) or 0 otherwise:

$$D_t^{(\pi)}\triangleq\left[\prod_{j\in\pi}\left(D_t^j\right)\right]\times\left[\prod_{j\notin\pi}\left(1-D_t^j\right)\right].\qquad(1.17)$$

We also define the filtration $\left\{\mathcal{G}_t^{(\pi)}\right\}$ as:

$$\mathcal{G}_t^{(\pi)}\triangleq\mathcal{F}_t\vee\left[\bigvee_{j\in\pi}\mathcal{H}_\infty^j\right]=\mathcal{F}_t\vee\left[\bigvee_{j\in\pi}\sigma\left(\tau_j\right)\right].\qquad(1.18)$$

For instance, if $n = 2$, then we have 4 possible default states: $\mathbf{\Pi_2} = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$; the default state indicators are

$$\begin{aligned} D_t^{(\emptyset)} &\triangleq \mathbf{1}_{\{\tau_1 > t\}}\mathbf{1}_{\{\tau_2 > t\}};\ D_t^{(\{1\})} \triangleq \mathbf{1}_{\{\tau_1 \le t\}}\mathbf{1}_{\{\tau_2 > t\}}; \\ D_t^{(\{2\})} &\triangleq \mathbf{1}_{\{\tau_1 > t\}}\mathbf{1}_{\{\tau_2 \le t\}};\ D_t^{(\{1,2\})} \triangleq \mathbf{1}_{\{\tau_1 \le t\}}\mathbf{1}_{\{\tau_2 \le t\}}. \end{aligned}$$

The filtrations $\left\{\mathcal{G}_t^{(\emptyset)}\right\}$ $\left\{\mathcal{G}_t^{(\{1\})}\right\}$ $\left\{\mathcal{G}_t^{(\{2\})}\right\}$ $\left\{\mathcal{G}_t^{(\{1,2\})}\right\}$ follow immediately from (1.18).

Generalized Dellacherie Formula. Next, we state our conditional expectation result.

Proposition 5 *(Conditional expectation w.r.t. the enlarged filtration). Let Y be a $\mathcal{G}$-measurable random variable. Then, we have*

$$\mathbb{E}\left[Y\left|\mathcal{G}_t\right.\right] = \sum_{\pi \in \mathbf{\Pi_n}} D_t^{(\pi)} \frac{\mathbb{E}\left[Y \times \prod_{j \notin \pi}\left(1 - D_t^j\right)\left|\mathcal{G}_t^{(\pi)}\right.\right]}{\mathbb{E}\left[\prod_{j \notin \pi}\left(1 - D_t^j\right)\left|\mathcal{G}_t^{(\pi)}\right.\right]}.$$

The proof is based on a repeated use of Lemma 4.

Proof. We shall proceed by induction. By Lemma 4, the property is satisfied for $n = 1$.

Assume that the formula is true for n and let us prove that it holds for $n+1$.

Define for $n \ge 1$, the filtration $\left\{\mathcal{F}_t^{(n)}\right\} \triangleq \left\{\mathcal{F}_t \vee \bigvee_{i=1}^n \mathcal{H}_t^i\right\}$.

Applying Lemma 4 to the filtration $\left\{\mathcal{F}_t^{(n+1)}\right\} = \left\{\mathcal{F}_t^{(n)} \vee \mathcal{H}_t^{n+1}\right\}$, we get

$$\mathbb{E}\left[Y\left|\mathcal{F}_t^{(n+1)}\right.\right] = D_t^{n+1}\mathbb{E}\left[Y\left|\mathcal{F}_t^{(n)} \vee \mathcal{H}_\infty^{n+1}\right.\right] + \left(1 - D_t^{n+1}\right)\frac{\mathbb{E}\left[Y \times \left(1 - D_t^{n+1}\right)\left|\mathcal{F}_t^{(n)}\right.\right]}{\mathbb{E}\left[\left(1 - D_t^{n+1}\right)\left|\mathcal{F}_t^{(n)}\right.\right]}. \tag{1.19}$$

Observing that $\left\{\mathcal{F}_t^{(n)} \vee \mathcal{H}_\infty^{n+1}\right\} = \left\{\left[\mathcal{F}_t \vee \mathcal{H}_\infty^{n+1}\right] \vee \left[\bigvee_{i=1}^n \mathcal{H}_t^i\right]\right\}$, the first term of equation (1.19) can be expanded by applying the induction relationship:

$$\begin{aligned} & D_t^{n+1}\mathbb{E}\left[Y\left|\mathcal{F}_t^{(n)} \vee \mathcal{H}_\infty^{n+1}\right.\right] \\ = {} & D_t^{n+1}\mathbb{E}\left[Y\left|\left[\mathcal{F}_t \vee \mathcal{H}_\infty^{n+1}\right] \vee \left[\bigvee_{i=1}^n \mathcal{H}_t^i\right]\right.\right] \\ = {} & D_t^{n+1} \sum_{\pi_n \in \mathbf{\Pi_n}} D_t^{(\pi_n)} \frac{\mathbb{E}\left[Y \times \prod_{j \notin \pi_n}\left(1 - D_t^j\right)\left|\left[\mathcal{F}_t \vee \mathcal{H}_\infty^{n+1}\right] \vee \left[\bigvee_{j \in \pi_n} \mathcal{H}_\infty^j\right]\right.\right]}{\mathbb{E}\left[\prod_{j \notin \pi_n}\left(1 - D_t^j\right)\left|\left[\mathcal{F}_t \vee \mathcal{H}_\infty^{n+1}\right] \vee \left[\bigvee_{j \in \pi_n} \mathcal{H}_\infty^j\right]\right.\right]}. \end{aligned} \tag{1.20}$$

Define a partition of $\mathbf{\Pi_{n+1}}$: $\mathbf{\Pi_{n+1}} = \mathbf{\Pi^+_{n+1}} \cup \mathbf{\Pi^-_{n+1}}$, and $\mathbf{\Pi^+_{n+1}} \cap \mathbf{\Pi^-_{n+1}} = \emptyset$, where $\mathbf{\Pi^+_{n+1}}$ is the set of all subsets containing $(n+1)$, and $\mathbf{\Pi^-_{n+1}}$ is the set of all subsets not containing $(n+1)$. They are constructed as:

$$\begin{aligned}\mathbf{\Pi^+_{n+1}} &= \left\{\pi_n \cup \{n+1\} : \pi_n \in \mathbf{\Pi_n}\right\},\\ \mathbf{\Pi^-_{n+1}} &= \left\{\pi_n \cup \emptyset : \pi_n \in \mathbf{\Pi_n}\right\}.\end{aligned}$$

Equation (1.20), then, becomes

$$\ldots = \sum_{\pi_{n+1} \in \mathbf{\Pi^+_n}} D_t^{(\pi_{n+1})} \frac{\mathbb{E}\left[Y \times \prod_{j \notin \pi_{n+1}} \left(1 - D_t^j\right) \middle| \mathcal{F}_t \vee \left[\bigvee_{j \in \pi_{n+1}} \mathcal{H}^j_\infty\right]\right]}{\mathbb{E}\left[\prod_{j \notin \pi_{n+1}} \left(1 - D_t^j\right) \middle| \mathcal{F}_t \vee \left[\bigvee_{j \in \pi_{n+1}} \mathcal{H}^j_\infty\right]\right]}. \tag{1.21}$$

The second term in equation (1.19),

$$\left(1 - D_t^{n+1}\right) \frac{\mathbb{E}\left[Y \times \left(1 - D_t^{n+1}\right) \middle| \mathcal{F}_t^{(n)}\right]}{\mathbb{E}\left[\left(1 - D_t^{n+1}\right) \middle| \mathcal{F}_t^{(n)}\right]},$$

can also be computed by the induction relationship. To expand the numerator, we apply it to the variable $\left[Y \times \left(1 - D_t^{n+1}\right)\right]$:

$$\mathbb{E}\left[Y \times \left(1 - D_t^{n+1}\right) \middle| \mathcal{F}_t^{(n)}\right] = \sum_{\pi_n \in \mathbf{\Pi_n}} D_t^{(\pi_n)} \frac{\mathbb{E}\left[\left[Y \times \left(1 - D_t^{n+1}\right)\right] \times \prod_{j \notin \pi_n} \left(1 - D_t^j\right) \middle| \mathcal{G}_t^{(\pi_n)}\right]}{\mathbb{E}\left[\prod_{j \notin \pi_n} \left(1 - D_t^j\right) \middle| \mathcal{G}_t^{(\pi_n)}\right]},$$

and to expand the denominator, we apply it to the variable $\left(1 - D_t^{n+1}\right)$:

$$\mathbb{E}\left[\left(1 - D_t^{n+1}\right) \middle| \mathcal{F}_t^{(n)}\right] = \sum_{\pi'_n \in \mathbf{\Pi_n}} D_t^{(\pi'_n)} \frac{\mathbb{E}\left[\left[\left(1 - D_t^{n+1}\right)\right] \times \prod_{j \notin \pi'_n} \left(1 - D_t^j\right) \middle| \mathcal{G}_t^{(\pi'_n)}\right]}{\mathbb{E}\left[\prod_{j \notin \pi'_n} \left(1 - D_t^j\right) \middle| \mathcal{G}_t^{(\pi'_n)}\right]}.$$

The ratio is obtained by observing that $D_t^{(\pi_n)} D_t^{(\pi'_n)} = 0$ if $\pi_n \neq \pi'_n$, so that we are left in the denominator with one term, which corresponds to $\pi_n = \pi'_n$.

$$\begin{aligned}&\frac{\mathbb{E}\left[Y \times \left(1 - D_t^{n+1}\right) \middle| \mathcal{F}_t^{(n)}\right]}{\mathbb{E}\left[\left(1 - D_t^{n+1}\right) \middle| \mathcal{F}_t^{(n)}\right]}\\
= &\sum_{\pi_n \in \mathbf{\Pi_n}} D_t^{(\pi_n)} \frac{\mathbb{E}\left[\left[Y \times \left(1 - D_t^{n+1}\right)\right] \times \prod_{j \notin \pi_n} \left(1 - D_t^j\right) \middle| \mathcal{G}_t^{(\pi_n)}\right]}{\mathbb{E}\left[\prod_{j \notin \pi_n} \left(1 - D_t^j\right) \middle| \mathcal{G}_t^{(\pi_n)}\right] \times \left[D_t^{(\pi_n)} \frac{\mathbb{E}\left[\left[\left(1 - D_t^{n+1}\right)\right] \times \prod_{j \notin \pi_n} \left(1 - D_t^j\right) \middle| \mathcal{G}_t^{(\pi_n)}\right]}{\mathbb{E}\left[\prod_{j \notin \pi_n} \left(1 - D_t^j\right) \middle| \mathcal{G}_t^{(\pi_n)}\right]}\right]}\\
= &\sum_{\pi_n \in \mathbf{\Pi_n}} D_t^{(\pi_n)} \frac{\mathbb{E}\left[\left[Y \times \left(1 - D_t^{n+1}\right)\right] \times \prod_{j \notin \pi_n} \left(1 - D_t^j\right) \middle| \mathcal{G}_t^{(\pi_n)}\right]}{\mathbb{E}\left[\left[\left(1 - D_t^{n+1}\right)\right] \times \prod_{j \notin \pi_n} \left(1 - D_t^j\right) \middle| \mathcal{G}_t^{(\pi_n)}\right]}.\end{aligned} \tag{1.22}$$

Using the $\mathbf{\Pi_{n+1}}$ corresponding default states, equation (1.22) becomes

$$\left(1-D_t^{n+1}\right)\frac{\mathbb{E}\left[Y\times\left(1-D_t^{n+1}\right)\middle|\mathcal{F}_t^{(n)}\right]}{\mathbb{E}\left[\left(1-D_t^{n+1}\right)\middle|\mathcal{F}_t^{(n)}\right]}=\sum_{\pi_{n+1}\in\mathbf{\Pi_n^-}}D_t^{(\pi_{n+1})}\frac{\mathbb{E}\left[[Y]\times\prod_{j\notin\pi_{n+1}}\left(1-D_t^j\right)\middle|\mathcal{G}_t^{(\pi_{n+1})}\right]}{\mathbb{E}\left[\prod_{j\notin\pi_{n+1}}\left(1-D_t^j\right)\middle|\mathcal{G}_t^{(\pi_{n+1})}\right]}. \tag{1.23}$$

Combining (1.21) and (1.23) ends the proof. ■

1.5 The Copula Approach

In this section, we consider the problem of computing the conditional survival probability, $\mathbb{P}(\tau_i > T\,|\mathcal{G}_t)$, in a copula framework. Using our generalized Dellacherie formula, we derive an analytical result depending on the default state that we are in. This approach was first studied in Schönbucher and Schubert (2001). However, our set-up is more general since we consider a "time-dependent" conditional copula to model the default times' multivariate dependence. The theory of conditional copulas is presented next.

Copulas. First, we give the formal definition of a copula function (see Nelsen (1999)).

Definition 6 *(Copula). An n-dimensional copula is any function $C:[0,1]^n\to[0,1]$ with the following properties*

- *C is grounded, i.e., $C(u_1,...,u_n)=0$ for all $(u_1,...,u_n)\in[0,1]^n$ such that $u_k=0$ for at least one k;*
- *C is n-increasing, i.e., the C-volume of all n-boxes whose vertices lie in $[0,1]^n$ is positive:*
$$\sum_{i_1=1}^{2}\cdots\sum_{i_n=1}^{2}(-1)^{i_1+....+i_n}\,C\left(u_1^{i_1},...,u_n^{i_n}\right)\geq 0,$$
for all $\left(u_1^1,...,u_n^1\right)$ and $\left(u_1^2,...,u_n^2\right)$ in $[0,1]^n$ with $u_k^1\leq u_k^2$, $1\leq k\leq n$;
- *C has margins C_k, which satisfy $C_k(u_k)=C(1,...,1,u_k,1...,1)=u_k$ for all u_k in $[0,1]$.*

This definition ensures that C is a multivariate uniform distribution. The link between copulas and the construction of multivariate distributions is given by Sklar's theorem.

Theorem 7 *(Sklar's theorem). Let F be an n-dimensional distribution function with margins $F_1, ..., F_n$. Then, there exists an n-copula C such that for all $(x_1, ..., x_n) \in \overline{\mathbb{R}}^n$,*

$$F(x_1, ..., x_n) = C(F_1(x_1), ..., F_n(x_n)).$$

If $F_1, ..., F_n$ are all continuous, then C is unique. Otherwise, C is uniquely determined on Ran $F_1 \times ... \times$ Ran F_n. Conversely, if C is an n-copula and $F_1, ..., F_n$ are distribution functions, then the function F defined above is an n-dimensional distribution function with margins $F_1, ..., F_n$.

The Theory of Conditional Copulas. Patton (2001) has extended the existing theory of (unconditional) copulas to the conditional case by introducing the so-called "conditional copula". This tool can be used in the modelling of time-varying conditional dependence. Here, we follow Patton's approach by giving the formal definitions of a conditional multivariate distribution and a conditional copula, and Sklar's theorem for conditional distributions.

Let $\mathcal{A}$ be some arbitrary sub-σ-algebra.

Definition 8 *(Conditional multivariate distribution). An n-dimensional conditional distribution function is a function $H(|\mathcal{A}) : \overline{\mathbb{R}}^n \times \Omega \to [0,1]$ with the following properties:*

- *$H(|\mathcal{A})$ is grounded, i.e., for almost every $\omega \in \Omega$, $H(x_1, ..., x_n |\mathcal{A})(\omega) = 0$ for all $(x_1, ..., x_n) \in \overline{\mathbb{R}}^n$ such that $x_k = -\infty$ for at least one k;*

- *$H(|\mathcal{A})$ is n-increasing, i.e., for almost every $\omega \in \Omega$, the H-volume of all n-boxes in $\overline{\mathbb{R}}^n$ is positive:*

$$V_H\left(\left[x_1^1, x_1^2\right] \times ... \times \left[x_n^1, x_n^2\right]\right) = \sum_{i_1=1}^{2} ... \sum_{i_n=1}^{2} (-1)^{i_1+....+i_n} H\left(x_1^{i_1}, ..., x_n^{i_n} |\mathcal{A}\right)(\omega) \geq 0,$$

  for all $\left(x_1^1, ..., x_n^1\right)$ and $\left(x_1^2, ..., x_n^2\right)$ in $\overline{\mathbb{R}}^n$ with $x_k^1 \leq x_k^2$, $1 \leq k \leq n$;

- *$H(\infty, ..., \infty |\mathcal{A})(\omega) = 1$, for almost every $\omega \in \Omega$.*

- *$H(x_1, ..., x_n |\mathcal{A})(.) : \Omega \to [0,1]$ is a measurable function on $(\Omega, \mathcal{A})$, for each $(x_1, ..., x_n) \in \overline{\mathbb{R}}^n$.*

The margins of an n-dimensional conditional distribution function are conditional distribution functions.

Definition 9 *(Conditional copula). An n-dimensional conditional copula is a function $C(|\mathcal{A}) : [0,1]^n \times \Omega \to [0,1]$ with the following properties:*

- *$C(|\mathcal{A})$ is grounded, i.e., for almost every $\omega \in \Omega$, $C(u_1, ..., u_n |\mathcal{A})(\omega) = 0$ for all $(u_1, ..., u_n) \in [0,1]^n$ such that $u_k = 0$ for at least one k;*
- *$C(|\mathcal{A})$ is n-increasing, i.e., for almost every $\omega \in \Omega$, the C-volume of all n-boxes in $[0,1]^n$ is positive:*

$$\sum_{i_1=1}^{2} ... \sum_{i_n=1}^{2} (-1)^{i_1+....+i_n} C\left(u_1^{i_1}, ..., u_n^{i_n} |\mathcal{A}\right)(\omega) \geq 0,$$

  for all $\left(u_1^1, ..., u_n^1\right)$ and $\left(u_1^2, ..., u_n^2\right)$ in $\overline{\mathbb{R}}^n$ with $u_k^1 \leq u_k^2$, $1 \leq k \leq n$;
- *$C(|\mathcal{A})$ has (conditional) margins $C_k(|\mathcal{A})$, which satisfy, for almost every $\omega \in \Omega$, $C_k(u_k |\mathcal{A})(\omega) = C(1, ..., 1, u_k, 1..., 1 |\mathcal{A})(\omega) = u_k$ for all u_k in $[0,1]$.*
- *$C(u_1, ..., u_n |\mathcal{A})(.) : \Omega \to [0,1]$ is a measurable function on $(\Omega, \mathcal{A})$, for each $(u_1, ..., u_n) \in [0,1]^n$.*

Alternatively, a conditional copula can be defined as a conditional distribution function with uniformly distributed conditional margins.

Theorem 10 *(Sklar's theorem for conditional distributions). Let $H(|\mathcal{A})$ be an n-dimensional conditional distribution function with conditional margins $F_1(|\mathcal{A}), ..., F_n(|\mathcal{A})$. Then, there exists a conditional n-copula $C(|\mathcal{A}) : [0,1]^n \times \Omega \to [0,1]$ such that for almost every $\omega \in \Omega$, and for all $(x_1, ..., x_n) \in \overline{\mathbb{R}}^n$,*

$$H(x_1, ..., x_n |\mathcal{A})(\omega) = C(F_1(x_1 |\mathcal{A})(\omega), ..., F_n(x_n |\mathcal{A})(\omega) |\mathcal{A})(\omega).$$

If for almost every $\omega \in \Omega$, the functions $x_i \to F_i(x_i |\mathcal{A})(\omega)$ are all continuous, then $C(|\mathcal{A})(\omega)$ is unique. Otherwise, $C(|\mathcal{A})(\omega)$ is uniquely determined on the product of the values taken by $x_i \to F_i(x_i |\mathcal{A})(\omega)$, $i = 1, ..., n$. Conversely, if $C(|\mathcal{A})$ is a conditional n-copula and $F_1(|\mathcal{A}), ..., F_n(|\mathcal{A})$ are conditional distribution functions, then the function $H(|\mathcal{A})$ defined above is a conditional n-dimensional distribution function with conditional margins $F_1(|\mathcal{A}), ..., F_n(|\mathcal{A})$.

As a corollary to Theorem 10, we can construct the conditional copula from any conditional multivariate distribution.

Definition 11 *(Generalized-inverse). The generalized-inverse of a univariate distribution function F is defined as*

$$F^{-1}(u) = \inf\{x : F(x) \geq u\}, \text{ for all } u \in [0,1].$$

Corollary 12 *Let $H\left(|\mathcal{A}\right)$ be an n-dimensional conditional distribution with continuous conditional margins $F_1\left(|\mathcal{A}\right), ..., F_n\left(|\mathcal{A}\right)$. Then, there exists a unique conditional copula $C\left(|\mathcal{A}\right) : [0,1]^n \times \Omega \to [0,1]$ such that*

$$C\left(u_1, ..., u_n\left|\mathcal{A}\right.\right)(\omega) = H\left(F_1^{-1}\left(u_1\left|\mathcal{A}\right.\right)(\omega), ..., F_n^{-1}\left(u_n\left|\mathcal{A}\right.\right)(\omega)\left|\mathcal{A}\right.\right)(\omega),$$

for almost every $\omega \in \Omega$, and for all $(u_1, ..., u_n) \in [0,1]^n$.

Given a set of conditional marginal distributions and a conditional copula, we can construct a joint conditional distribution, and from any given joint conditional distribution, we can extract the conditional margins and the conditional copula.

Next, we make the following assumptions.

Model. As before, we denote by $\tau_1, ..., \tau_n$ a set of non-negative variables on a probability space $(\Omega, \mathcal{G}, P)$, such that $\mathbb{P}(\tau_i = 0) = 0$ and $\mathbb{P}(\tau_i > t) > 0$ for any $t \in R_+$. We set $D_t^i = \mathbf{1}_{\{\tau_i \leq t\}}$ and we denote by $\left\{\mathcal{H}_t^i\right\}$ the associated filtration: $\mathcal{H}_t^i = \sigma\left(D_s^i : s \leq t\right)$. We are also given an Itô process X and its filtration $\{\mathcal{F}_t\}$ augmented with the P-null sets of $\mathcal{G}$, and $\mathcal{F}_0$ is trivial. The investor filtration, in this model, is $\{\mathcal{G}_t\}$: $\mathcal{G}_t = \mathcal{F}_t \vee \bigvee_{i=1}^n \mathcal{H}_t^i$. The firm-specific information is denoted by $\left\{\mathcal{G}_t^i\right\}$: $\mathcal{G}_t^i = \mathcal{F}_t \vee \mathcal{H}_t^i$.

On the firm-specific filtration $\left\{\mathcal{G}_t^i\right\}$, the default time τ_i has an $\{\mathcal{F}_t\}$-adapted intensity h^i, i.e., $D_t^i - \int_0^{t\wedge\tau_i} h_s^i ds$ is a $\left\{\mathcal{G}_t^i\right\}$-martingale. So that

$$\mathbb{P}\left(\tau_i > T\left|\mathcal{G}_t^i\right.\right) = \left(1 - D_t^i\right)\mathbb{E}\left[\exp\left(-\int_t^T h_s^i ds\right)\left|\mathcal{F}_t\right.\right].$$

On the enlarged filtration $\{\mathcal{G}_t\}$, τ_i has a $\left\{\mathcal{G}_t^{-i}\right\}$-adapted intensity λ^i, i.e., $D_t^i - \int_0^{t\wedge\tau_i} \lambda_s^i ds$ is a $\{\mathcal{G}_t\}$-martingale.

Our goal is to establish a formula of the conditional expectation

$$\mathbb{P}\left(\tau_i > T\left|\mathcal{G}_t\right.\right).$$

Furthermore, we assume that the multivariate dependence structure is described by the process $\left(\overline{C}_t^\tau\right)_{t\geq 0}$, where

$$\overline{C}_t^\tau : [0,1]^n \times \Omega \times [0,\infty) \to [0,1]$$

is the conditional (survival) copula, $\overline{C}_t^\tau(.) \triangleq \overline{C}^\tau\left(.\left|\mathcal{F}_t\right.\right)$, i.e., for almost every $\omega \in \Omega$, and for all $(t_1, ..., t_n) \in [0,\infty)^n$,

$$\mathbb{P}\left(\tau_1 > t_1, ..., \tau_n > t_n\left|\mathcal{F}_t\right.\right)(\omega) = \overline{C}_t^\tau\left(\mathbb{P}\left(\tau_1 > t_1\left|\mathcal{F}_t\right.\right)(\omega), ..., \mathbb{P}\left(\tau_n > t_n\left|\mathcal{F}_t\right.\right)(\omega)\right)(\omega).$$

One can formally construct the process $\left(\overline{C}_t^\tau\right)_{t\geq 0}$ as follows.

Denote by $G_t(x_1, ..., x_n) = \mathbb{P}(\tau_1 \leq x_1, ..., \tau_n \leq x_n | \mathcal{F}_t)$ the conditional multivariate distribution function, and by $G_t^i(x) = \mathbb{P}(\tau_i \leq x | \mathcal{F}_t)$ the conditional margins. Define the generalized inverse of $G_t^i(.)$, $I_t^i(u) = \inf\left\{x : G_t^i(x) \geq u\right\}$; for almost every $\omega \in \Omega$, the functions $x \to \mathbb{P}(\tau_i \leq x | \mathcal{F}_t)(\omega)$ are continuous. The conditional copula C_t^τ is then given by

$$C_t^\tau(u_1, ..., u_n)(\omega) = G_t\left(I_t^1(u_1)(\omega), ..., I_t^n(u_n)(\omega)\right)(\omega),$$

for almost every $\omega \in \Omega$, and for all $(u_1, ..., u_n) \in [0,1]^n$. The conditional survival copula, which links the marginal survival functions, is given by

$$\overline{C}_t^\tau(u_1, ..., u_n)(\omega) = \sum_{i_1=1}^{2} ... \sum_{i_n=1}^{2} (-1)^{i_1+...+i_n} C_t^\tau\left(v_1^{i_1}, ..., v_n^{i_n}\right)(\omega),$$

where $v_i^1 = 1 - u_i$ and $v_i^2 = 1$, and $(u_1, ..., u_n) \in [0,1]^n$.

Assumption 3. We assume that, for each $t \geq 0$, and for almost every $\omega \in \Omega$, the function

$$\begin{array}{rcl} \overline{C}_t^\tau(., \omega) & : & [0,1]^n \to [0,1] \\ (u_1, ..., u_n) & \to & \overline{C}_t^\tau(u_1, ..., u_n)(\omega) \end{array}$$

is absolutely continuous.

Comparison with the Threshold Copula. In practice, one can construct the copula process by considering a Cox-process approach as in Lando (1998). In this subsection, we compare the conditional time-dependent copula with the threshold copula used in Schönbucher and Schubert (2001). Let $\left(h_t^i\right)_{t\geq 0}$ be an $\{\mathcal{F}_t\}$-adapted nonnegative càdlàg process. Set

$$\tau_i \triangleq \inf\left\{t : \exp\left(-\int_0^t h_s^i ds\right) \leq U_i\right\}, \tag{1.24}$$

where U_i is a random variable uniformly distributed on $[0,1]$ and independent of $\mathcal{F}_\infty$. One can check that h^i is the $\{\mathcal{F}_t\}$-adapted intensity of τ_i with respect to the firm-specific filtration $\left\{\mathcal{G}_t^i\right\}$. Assume that the distribution of the n-dimensional vector $(U_1, ..., U_n)$ is defined by the (static) survival copula function $\overline{C}^U : [0,1]^n \to [0,1]$,

$$\mathbb{P}(U_1 > u_1, ..., U_n > u_n) = \overline{C}^U(u_1, ..., u_n).$$

$\overline{C}^U(.)$ is referred to as the default thresholds (survival) copula. Following a remark in Jouanin et al. (2001), one can relate this copula to the default times' conditional survival probability: for almost every $\omega \in \Omega$, we have

$$\begin{aligned}
& \mathbb{P}\left(\tau_1 > t_1, ..., \tau_n > t_n \,|\mathcal{F}_t\right)(\omega) \\
= \; & \mathbb{E}\left[\mathbb{P}\left(\tau_1 > t_1, ..., \tau_n > t_n \,|\mathcal{F}_\infty\right)(\omega)\,|\mathcal{F}_t\right](\omega) \\
= \; & \mathbb{E}\left[\mathbb{P}\left(U_1 > \exp\left(-\int_0^{t_1} h_s^1(\omega)\,ds\right), ..., U_n > \exp\left(-\int_0^{t_n} h_s^n(\omega)\,ds\right)|\mathcal{F}_\infty\right)(\omega)\,|\mathcal{F}_t\right](\omega) \\
= \; & \mathbb{E}\left[\mathbb{P}\left(U_1 > \exp\left(-\int_0^{t_1} h_s^1(\omega)\,ds\right), ..., U_n > \exp\left(-\int_0^{t_n} h_s^n(\omega)\,ds\right)\right)|\mathcal{F}_t\right](\omega) \\
= \; & \mathbb{E}\left[\overline{C}^U\left(\exp\left(-\int_0^{t_1} h_s^1(\omega)\,ds\right), ..., \exp\left(-\int_0^{t_n} h_s^n(\omega)\,ds\right)\right)|\mathcal{F}_t\right](\omega)\,;
\end{aligned}$$

the first equality follows from the law of iterated expectation; the second equality is from the construction of the default times (1.24); the third equality is due to the independence of the threshold variables $(U_1, ..., U_n)$ from $\mathcal{F}_\infty$ and the fact that the $\int_0^{t_i} h_s^i ds$ are $\mathcal{F}_\infty$-measurable; and the fourth equality follows from the definition of the threshold copula.

The default times' conditional copula process $\left(\overline{C}_t^\tau\right)_{t\geq 0}$ can then be constructed via

$$\overline{C}_t^\tau\left(\mathbb{E}\left[e^{-\int_0^{t_1} h_s^1 ds}\,|\mathcal{F}_t\right], ..., \mathbb{E}\left[e^{-\int_0^{t_n} h_s^n ds}\,|\mathcal{F}_t\right]\right) = \mathbb{E}\left[\overline{C}^U\left(e^{-\int_0^{t_1} h_s^1 ds}, ..., e^{-\int_0^{t_n} h_s^n ds}\right)|\mathcal{F}_t\right]. \tag{1.25}$$

This provides a practical way of generating the conditional copula process that was formally introduced in the previous section. By choosing an absolutely continuous threshold copula, one can ensure that Assumption 3 is verified.

Forward Intensity. A convenient way of parameterizing the conditional survival probabilities $\mathbb{P}\left(\tau_i > T\,\middle|\mathcal{F}_t \vee \mathcal{H}_t^i\right)$ and $\mathbb{P}\left(\tau_i > T\,|\mathcal{G}_t\right)$ is to introduce the forward intensity processes $\left(h_{t,T}^i\right)_{t\geq 0}$ and $\left(\lambda_{t,T}^i\right)_{t\geq 0}$.

Definition 13 *(Forward Intensity). Assume that* $\mathbb{P}\left(\tau_i > 0\,|\mathcal{F}_t\right) = 1$, *and* $\mathbb{P}\left(\tau_i > T\,|\mathcal{F}_t\right) > 0$, *for all* $T \in \mathbb{R}^+$.

Let $F_t^i(T) \triangleq \mathbb{P}\left(\tau_i > T\,|\mathcal{F}_t\right)$, *for all* $T \in \mathbb{R}^+$, *denote the conditional survival probability. We assume that* $F_t^i(T)$ *is a.s. continuously differentiable in the* T*-variable. Then, the* $\{\mathcal{F}_t\}$*-adapted forward intensity at* T *is defined as*

$$h_{t,T}^i = -\frac{\partial \log F_t^i(T)}{\partial T}, \text{ for all } T \in \mathbb{R}^+. \tag{1.26}$$

Integrating (1.26), we obtain

$$\mathbb{P}\left(\tau_i > T\,|\mathcal{F}_t\right) = \mathbb{P}\left(\tau_i > 0\,|\mathcal{F}_t\right)\exp\left(-\int_0^T h_{t,s}^i ds\right) = \exp\left(-\int_0^T h_{t,s}^i ds\right), \text{ for } T \in \mathbb{R}^+.$$

Thus, we can write the conditional survival probability w.r.t. $\mathcal{F}_t \vee \mathcal{H}_t^i$ as

$$\begin{aligned}\mathbb{P}\left(\tau_i > T \left| \mathcal{F}_t \vee \mathcal{H}_t^i \right.\right) &= \left(1 - D_t^i\right) \frac{\mathbb{P}\left(\tau_i > T \left| \mathcal{F}_t \right.\right)}{\mathbb{P}\left(\tau_i > t \left| \mathcal{F}_t \right.\right)} \\ &= \left(1 - D_t^i\right) \exp\left(-\int_t^T h_{t,s}^i ds\right), \text{ for } T \in \mathbb{R}^+.\end{aligned}$$

Similarly, we define the $\left\{\mathcal{G}_t^{-i}\right\}$-adapted forward intensity.

Definition 14 *(Forward Intensity). Assume that* $\mathbb{P}\left(\tau_i > 0 \left| \mathcal{G}_t^{-i} \right.\right) = 1$*, and* $\mathbb{P}\left(\tau_i > T \left| \mathcal{G}_t^{-i} \right.\right) > 0$*, for all* $T \in \mathbb{R}^+$*.*

Set $H_t^i(T) \triangleq \mathbb{P}\left(\tau_i > T \left| \mathcal{G}_t^{-i} \right.\right)$ *, for all* $T \in \mathbb{R}^+$*, and define the* $\left\{\mathcal{G}_t^{-i}\right\}$*-adapted forward intensity as:*

$$\lambda_{t,T}^i \triangleq -\frac{\partial \log H_t^i(T)}{\partial T}, \text{ for all } T \in \mathbb{R}^+. \quad (1.27)$$

Then, we have

$$\begin{aligned}\mathbb{P}\left(\tau_i > T \left| \mathcal{G}_t^{-i} \right.\right) &= \exp\left(-\int_0^T \lambda_{t,s}^i ds\right), \text{ for } T \in \mathbb{R}^+. \\ \mathbb{P}\left(\tau_i > T \left| \mathcal{G}_t \right.\right) &= \left(1 - D_t^i\right) \exp\left(-\int_t^T \lambda_{t,s}^i ds\right), \text{ for } T \in \mathbb{R}^+.\end{aligned}$$

At this point, the forward intensity process is merely a parameterization of the conditional survival probability. Next, we establish the link between this parameterization and the standard "spot" intensities.

Link Between Forward and Spot Intensity. Conditional survival probabilities and intensities are linked via the following result due to Aven (1985).

Proposition 15 *(Aven (1985)). Let* $\left(\Omega, \mathcal{G}, \left\{\widetilde{\mathcal{G}}_t\right\}_{t \in [0,T^*]}, P\right)$ *be a filtered probability space and* $D_t = \mathbf{1}_{\{\tau \leq t\}}$ *with* τ *a* $\left\{\widetilde{\mathcal{G}}_t\right\}$*-stopping time. Let* $\{\varepsilon_n\}_{n=1}^{\infty}$ *be a sequence, which decreases to zero and let* Y_t^n*,* $t \in [0, T^*]$ *be a measurable version of the process*

$$Y_t^n = \frac{1}{\varepsilon_n} \mathbb{E}\left[D_{t+\varepsilon_n} - D_t \left| \widetilde{\mathcal{G}}_t \right.\right].$$

Assume that there are non-negative and measurable processes $\widetilde{g}_t$ *and* y_t*,* $t \in [0, T^*]$ *such that:*

1. *for each* t*,*

$$\lim_{n \to \infty} Y_t^n = \widetilde{g}_t \text{ a.s.;}$$

2. *for each t, there exists for almost all $\omega \in \Omega$, an $n_0 = n_0(t, \omega)$ such that*

$$|Y_s^n(\omega) - \widetilde{g}_s(\omega)| \leq y_s(\omega), \forall s \leq t, \; n \geq n_0;$$

where

$$\int_0^t y_s ds < \infty, \text{ a.s., } t \in [0, T^*];$$

then $\widetilde{M}_t = D_t - \int_0^t \widetilde{g}_s ds$ is a $\left\{\widetilde{\mathcal{G}}_t\right\}$-martingale, and $\int_0^t \widetilde{g}_s ds$ is the compensator of D_t.

The relationship between intensity and conditional survival probability follows directly from this result (see, for instance, Schönbucher and Schubert (2001)).

Lemma 16 *Let $\widetilde{H}_t(T)$ denote the conditional survival probability*

$$\widetilde{H}_t(T) = \mathbb{P}\left(\tau > T \middle| \widetilde{\mathcal{G}}_t\right).$$

Assume that $\widetilde{H}_t(T)$ is differentiable from the right with respect to T at $T = t$, and that the assumptions of Proposition 15 are satisfied. Then, the intensity process of D_t is given by

$$\widetilde{g}_t = -\frac{\partial \widetilde{H}_t(T)}{\partial T}|_{T=t}.$$

Remark 17 *Note that $\widetilde{g}_t = 0$ after τ and that $\widetilde{g}_t$ is $\left\{\widetilde{\mathcal{G}}_t\right\}$-adapted.*

Using our forward intensity parameterization (and assuming that the technical assumption of Proposition 15 are satisfied), we can recover the $\{\mathcal{F}_t\}$-adapted intensity process of τ_i. Set $\widetilde{F}_t^i(T) \triangleq \mathbb{P}\left(\tau_i > T \middle| \mathcal{F}_t \vee \mathcal{H}_t^i\right)$, and $F_t^i(T) \triangleq \mathbb{P}\left(\tau_i > T \middle| \mathcal{F}_t\right)$. The process $\widetilde{h}_t^i$, which is $\{\mathcal{F}_t \vee \mathcal{H}_t^i\}$-adapted, and the $\{\mathcal{F}_t\}$-adapted version h_t^i are related by

$$\widetilde{h}_t^i = \mathbf{1}_{\{\tau_i > t\}} h_t^i, \text{ for } t \geq 0.$$

The expression of $\widetilde{h}_t^i$ can be obtained by Lemma 16:

$$\begin{aligned}
\widetilde{h}_t^i &= -\frac{\partial \widetilde{F}_t^i(T)}{\partial T}|_{T=t} \\
&= -\frac{\partial}{\partial T}\left[\mathbf{1}_{\{\tau_i > t\}} \frac{F_t^i(T)}{F_t^i(t)}\right]|_{T=t} \\
&= -\frac{\partial}{\partial T}\left[\mathbf{1}_{\{\tau_i > t\}} \exp\left(-\int_t^T h_{t,s}^i ds\right)\right]|_{T=t} \\
&= \mathbf{1}_{\{\tau_i > t\}} h_{t,t}^i.
\end{aligned}$$

Thus, we have the following relationship between the $\{\mathcal{F}_t\}$-adapted spot and forward intensities: for all $t < \tau_i$,

$$h_t^i = h_{t,t}^i. \tag{1.28}$$

Similarly, we can recover the $\left\{\mathcal{G}_t^{-i}\right\}$-adapted intensity: for all $t < \tau_i$,

$$\lambda_t^i = \lambda_{t,t}^i. \tag{1.29}$$

Equations (1.29) and (1.28) can be viewed as the equivalent of the well-known relationship between a forward rate and a short rate in interest rate modelling.

Computing the Conditional Survival Probability. Using our generalized Dellacherie formula, the conditional survival probability $H_t^i(T) = \mathbb{P}(\tau_i > T \,|\mathcal{G}_t)$ can be obtained as

$$H_t^i(T) = \sum_{\pi \in \mathbf{\Pi_n}} D_t^{(\pi)} H_t^{i,(\pi)}(T), \tag{1.30}$$

where

$$\begin{aligned}
H_t^{i,(\pi)}(T) &\triangleq \frac{\mathbb{E}\left[\left(1 - D_T^i\right) \times \prod_{j \notin \pi} \left(1 - D_t^j\right) \Big| \mathcal{G}_t^{(\pi)}\right]}{\mathbb{E}\left[\prod_{j \notin \pi} \left(1 - D_t^j\right) \Big| \mathcal{G}_t^{(\pi)}\right]}, \\
D_t^{(\pi)} &\triangleq \left[\prod_{j \in \pi} \left(D_t^j\right)\right] \times \left[\prod_{j \notin \pi} \left(1 - D_t^j\right)\right], \\
\mathcal{G}_t^{(\pi)} &\triangleq \mathcal{F}_t \vee \left[\bigvee_{j \in \pi} \mathcal{H}_\infty^j\right] = \mathcal{F}_t \vee \left[\bigvee_{j \in \pi} \sigma(\tau_j)\right].
\end{aligned}$$

It is clear that all the terms in the sum (1.30) such that $i \in \pi$ are equal to zero. For all the remaining ones, the conditional distribution of default depends on the default state π observed at time t, and will vary from one state to the next. We derive the formula of the conditional expectation in the pre-default state: $\pi = \emptyset$. Then, we do the same when one or more defaults have occurred before $t : |\pi| > 0$.

Pre-default. We have the following result.

Proposition 18 *(Pre-default). If no default has occurred before time t, then the $\{\mathcal{G}_t\}$-forward intensity is given by*

$$\lambda_{t,T}^{i,(\emptyset)} = h_{t,T}^i \exp\left(-\int_0^T h_{t,s}^i ds\right) \frac{\frac{\partial}{\partial x_i}\overline{C}_t^T\left(e^{-\int_0^t h_{t,s}^1 ds}, ..., e^{-\int_0^T h_{t,s}^i ds}, ..., e^{-\int_0^t h_{t,s}^n ds}\right)}{\overline{C}_t^T\left(e^{-\int_0^t h_{t,s}^1 ds}, ..., e^{-\int_0^T h_{t,s}^i ds}, ..., e^{-\int_0^t h_{t,s}^n ds}\right)}. \tag{1.31}$$

Proof. On the set $\prod_{j=1}^{n}\left(\mathbf{1}_{\{\tau_j>t\}}\right)$, the conditional survival probability is given by

$$H_t^{i,(\emptyset)}(T)=\frac{\mathbb{E}\left[\left(1-D_T^i\right)\times\prod_{j=1}^{n}\left(1-D_t^j\right)|\mathcal{F}_t\right]}{\mathbb{E}\left[\prod_{j=1}^{n}\left(1-D_t^j\right)|\mathcal{F}_t\right]}.$$

The numerator can be computed as

$$\begin{aligned}&\mathbb{E}\left[\left(1-D_T^i\right)\times\prod_{j=1}^{n}\left(1-D_t^j\right)|\mathcal{F}_t\right]\\ =&\ \overline{C}_t^{\tau}\left(\mathbb{E}\left[\left(1-D_t^1\right)|\mathcal{F}_t\right],...,\mathbb{E}\left[\left(1-D_T^i\right)|\mathcal{F}_t\right],...,\mathbb{E}\left[\left(1-D_t^n\right)|\mathcal{F}_t\right]\right)\\ =&\ \overline{C}_t^{\tau}\left(e^{-\int_0^t h_{t,s}^1 ds},...,e^{-\int_0^T h_{t,s}^i ds},...,e^{-\int_0^t h_{t,s}^n ds}\right);\end{aligned}$$

the first equality is from the definition of the survival copula process, and the second equality uses the definition of the $\{\mathcal{F}_t\}$-forward intensities. The denominator is computed similarly. Differentiating with respect to T,

$$\lambda_{t,T}^{i,(\emptyset)}\triangleq-\frac{1}{H_t^{i,(\emptyset)}(T)}\frac{\partial H_t^{i,(\emptyset)}(T)}{\partial T},$$

we obtain the result of Proposition 18. ■

Post-default. If one or more defaults have occurred, the conditional distribution of τ_i changes, which results in a different expression of the $\{\mathcal{G}_t\}$-forward intensity.

Proposition 19 *(Post-default). If k obligors indexed by $\pi=\{j_1,...,j_k\}$ have defaulted before time t, and their default times are $\{t_{j_1},...,t_{j_k}\}$ respectively. Then, the $\{\mathcal{G}_t\}$-forward intensity is given by*

$$\lambda_{t,T}^{i,(\pi)}=h_{t,T}^i\exp\left(-\int_0^T h_{t,s}^i ds\right)\frac{\frac{\partial}{\partial x_i}\frac{\partial^k}{\partial x_{j_1}...\partial x_{j_k}}\overline{C}_t^{\tau}\left(e^{-\int_0^{\Theta_1}h_{t,s}^1 ds},...,e^{-\int_0^{\Theta_n}h_{t,s}^n ds}\right)}{\frac{\partial^k}{\partial x_{j_1}...\partial x_{j_k}}\overline{C}_t^{\tau}\left(e^{-\int_0^{\Theta_1}h_{t,s}^1 ds},...,e^{-\int_0^{\Theta_n}h_{t,s}^n ds}\right)},\tag{1.32}$$

where

$$\begin{aligned}\Theta_j&=t_j,\text{ for } j\in\pi=\{j_1,...,j_k\};\\ \Theta_i&=T,\text{ for } j=i;\\ \Theta_j&=t,\text{ otherwise.}\end{aligned}$$

Proof. On the set $\left[\prod_{m=1}^{k}\mathbf{1}_{\{\tau_{j_m}\in dt_{j_m}\}}\right]\left[\prod_{j\notin\{j_1,...,j_k\}}\mathbf{1}_{\{\tau_j>t\}}\right]$, the conditional survival probability is

$$H_t^{i,(\pi)}(T)=\frac{\mathbb{E}\left[\left(1-D_T^i\right)\times\prod_{j\notin\pi}\left(1-D_t^j\right)\times\prod_{m=1}^{k}\mathbf{1}_{\{\tau_{j_m}\in dt_{j_m}\}}|\mathcal{F}_t\right]}{\mathbb{E}\left[\prod_{j\notin\pi}\left(1-D_t^j\right)\times\prod_{m=1}^{k}\mathbf{1}_{\{\tau_{j_m}\in dt_{j_m}\}}|\mathcal{F}_t\right]}.$$

Using the conditional survival copula process and the $\{\mathcal{F}_t\}$-forward intensities, we have

$$\begin{aligned}&\mathbb{E}\left[\left(1-D_T^i\right)\times\prod_{j\notin\pi}\left(1-D_t^j\right)\times\prod_{m=1}^{k}\mathbf{1}_{\{\tau_{j_m}>t_{j_m}\}}\,|\mathcal{F}_t\right]\\ =\;&\overline{C}_t^{\tau}\left(e^{-\int_0^{\Theta_1}h_{t,s}^1ds},...,e^{-\int_0^{\Theta_n}h_{t,s}^nds}\right),\end{aligned}$$

where

$$\Theta_j=t_j,\text{ for }j\in\pi=\{j_1,...,j_k\}\,;\;\Theta_i=T;\text{ and }\Theta_j=t,\text{ otherwise.}$$

Hence, the k-density is given by

$$\begin{aligned}&\mathbb{E}\left[\left(1-D_T^i\right)\times\prod_{j\notin\pi}\left(1-D_t^j\right)\times\prod_{m=1}^{k}\mathbf{1}_{\{\tau_{j_m}\in dt_{j_m}\}}\,|\mathcal{F}_t\right]\\ =\;&\frac{\partial^k}{\partial\Theta_{j_1}...\partial\Theta_{j_k}}\overline{C}_t^{\tau}\left(e^{-\int_0^{\Theta_1}h_{t,s}^1ds},...,e^{-\int_0^{\Theta_n}h_{t,s}^nds}\right)\\ =\;&\left[\prod_{m=1}^{k}h_{t,T}^{j_m}\exp\left(-\int_0^{t_{j_m}}h_{t,s}^{j_m}ds\right)\right]\frac{\partial^k}{\partial x_{j_1}...\partial x_{j_k}}\overline{C}_t^{\tau}\left(e^{-\int_0^{\Theta_1}h_{t,s}^1ds},...,e^{-\int_0^{\Theta_n}h_{t,s}^nds}\right).\end{aligned}$$

The denominator is computed similarly

$$\begin{aligned}&\mathbb{E}\left[\left(1-D_t^i\right)\times\prod_{j\notin\pi}\left(1-D_t^j\right)\times\prod_{m=1}^{k}\mathbf{1}_{\{\tau_{j_m}\in dt_{j_m}\}}\,|\mathcal{F}_t\right]\\ =\;&\left[\prod_{m=1}^{k}h_{t,T}^{j_m}\exp\left(-\int_0^{t_{j_m}}h_{t,s}^{j_m}ds\right)\right]\frac{\partial^k}{\partial x_{j_1}...\partial x_{j_k}}\overline{C}_t^{\tau}\left(e^{-\int_0^{\theta_1}h_{t,s}^1ds},...,e^{-\int_0^{\theta_n}h_{t,s}^nds}\right),\end{aligned}$$

where

$$\theta_j=t_j,\text{ for }j\in\pi=\{j_1,...,j_k\}\,;\;\theta_i=t;\text{ and }\theta_j=t,\text{ otherwise.}$$

Differentiating the conditional survival probability $H_t^{i,(\pi)}(T)$,

$$H_t^{i,(\pi)}(T)=\frac{\frac{\partial^k}{\partial x_{j_1}...\partial x_{j_k}}\overline{C}_t^{\tau}\left(e^{-\int_0^{\Theta_1}h_{t,s}^1ds},...,e^{-\int_0^{\Theta_n}h_{t,s}^nds}\right)}{\frac{\partial^k}{\partial x_{j_1}...\partial x_{j_k}}\overline{C}_t^{\tau}\left(e^{-\int_0^{\theta_1}h_{t,s}^1ds},...,e^{-\int_0^{\theta_n}h_{t,s}^nds}\right)},$$

with respect to T, we get the result of Proposition 19:

$$\lambda_{t,T}^{i,(\pi)}\triangleq-\frac{1}{H_t^{i,(\pi)}(T)}\frac{\partial H_t^{i,(\pi)}(T)}{\partial T}.$$

■

The formulas of Schönbucher and Schubert are obtained in our framework by setting $T=t$, using the relationship between spot and forward intensity, namely $\lambda_t^i=\lambda_{t,t}^i$, $h_t^i=h_{t,t}^i$, and observing that

$$\overline{C}_t^{\tau}\left(e^{-\int_0^{\Theta_1}h_{t,s}^1ds},...,e^{-\int_0^{\Theta_n}h_{t,s}^nds}\right)=\overline{C}^{U}\left(e^{-\int_0^{\Theta_1}h_s^1ds},...,e^{-\int_0^{\Theta_n}h_s^nds}\right),\qquad(1.33)$$

since for $\Theta_j \leq t$, all the $\int_0^{\Theta_j} h_s^j ds$ are $\{\mathcal{F}_t\}$-measurable and equation (1.25) simplifies to (1.33).

Regenerative Form of the Intensity. Equations (1.31) and (1.32) offer another derivation of the regenerative form of the intensity. Indeed, using the notations of Section 1.3 where the marked point process representation was introduced, one can write:

$$\lambda_t^i = \sum_{k=1}^{n} \mathbf{1}_{\{T_{k-1} \leq t < T_k\}} a_t^{(k)} \left(i; T_1, Z_1, ..., T_{k-1}, Z_{k-1}\right), \tag{1.34}$$

where

$$a_t^{(k)} \left(i; T_1, Z_1, ..., T_{k-1}, Z_{k-1}\right) = h_t^i \exp\left(-\int_0^t h_s^i ds\right) \frac{\overline{C}^U_{x_i x_{Z_1} ... x_{Z_{k-1}}}}{\overline{C}^U_{x_{Z_1} ... x_{Z_{k-1}}}}, \tag{1.35}$$

and the lighter notations $\overline{C}^U_{x_i x_{Z_1} ... x_{Z_{k-1}}}$ and $\overline{C}^U_{x_{Z_1} ... x_{Z_{k-1}}}$ are short for

$$\begin{aligned}
\overline{C}^U_{x_i x_{Z_1} ... x_{Z_{k-1}}} &= \frac{\partial}{\partial x_i} \frac{\partial^{k-1}}{\partial x_{Z_1} ... \partial x_{Z_{k-1}}} \overline{C}^U \left(e^{-\int_0^{\Theta_1} h_s^1 ds}, ..., e^{-\int_0^{\Theta_n} h_s^n ds}\right), \\
\overline{C}^U_{x_{Z_1} ... x_{Z_{k-1}}} &= \frac{\partial^{k-1}}{\partial x_{Z_1} ... \partial x_{Z_{k-1}}} \overline{C}^U \left(e^{-\int_0^{\Theta_1} h_s^1 ds}, ..., e^{-\int_0^{\Theta_n} h_s^n ds}\right), \\
\Theta_{Z_j} &= T_j, \text{ for } 1 \leq j \leq k-1; \ \Theta_i = t; \text{ and } \Theta_j = t \text{ otherwise.}
\end{aligned}$$

Dynamics. Assuming that the $\{\mathcal{F}_t\}$-adapted intensities $\left(h_t^1, ..., h_t^n\right)_{t \geq 0}$ follow a diffusion, we can derive the dynamics of $\left(\lambda_t^i\right)_{t \geq 0}$ by applying Itô's lemma to the regenerative form (1.34). At each default time T_k, for $1 \leq k \leq n$ (and given that obligor i has not defaulted), the jump of λ_t^i is given by

$$\begin{aligned}
\Delta \lambda_{T_k}^i &= \lambda_{T_k}^i - \lambda_{T_k^-}^i = a_{T_k}^{(k+1)} (i) - \lim_{t \uparrow T_k} a_t^{(k)} (i) \\
&= h_{T_k}^i e^{-\int_0^{T_k} h_s^i ds} \left[\frac{\overline{C}^U_{x_i x_{Z_1} ... x_{Z_k}}}{\overline{C}^U_{x_{Z_1} ... x_{Z_k}}} - \frac{\overline{C}^U_{x_i x_{Z_1} ... x_{Z_{k-1}}}}{\overline{C}^U_{x_{Z_1} ... x_{Z_{k-1}}}} \right].
\end{aligned} \tag{1.36}$$

Between jump times, $\left(\lambda_t^i\right)_{t \geq 0}$ follows a diffusion. For $T_{k-1} \leq t < T_k$, it coincides with $a_t^{(k)} (i)$ whose dynamics can be obtained by Itô's lemma:

$$\begin{aligned}
\frac{da_t^{(k)} (i)}{a_t^{(k)} (i)} &= \left[\left(\frac{dh_t^i}{h_t^i} - h_t^i dt \right) - \sum_{j \notin \{Z_1, ..., Z_{k-1}\}} \left(\frac{\overline{C}^U_{x_i x_j x_{Z_1} ... x_{Z_{k-1}}}}{\overline{C}^U_{x_i x_{Z_1} ... x_{Z_{k-1}}}} - \frac{\overline{C}^U_{x_j x_{Z_1} ... x_{Z_{k-1}}}}{\overline{C}^U_{x_{Z_1} ... x_{Z_{k-1}}}} \right) h_t^j e^{-\int_0^t h_s^j ds} \right] \\
&= \left[\left(\frac{dh_t^i}{h_t^i} - h_t^i dt \right) - \sum_{j \notin \{Z_1, ..., Z_{k-1}\}} \left(\frac{\overline{C}^U_{x_i x_j x_{Z_1} ... x_{Z_{k-1}}} \overline{C}^U_{x_{Z_1} ... x_{Z_{k-1}}}}{\overline{C}^U_{x_i x_{Z_1} ... x_{Z_{k-1}}} \overline{C}^U_{x_j x_{Z_1} ... x_{Z_{k-1}}}} - 1 \right) a_t^{(k)} (j) \, dt \right].
\end{aligned}$$

Note that the index $j \notin \{Z_1, ..., Z_{k-1}\}$ covers all the non-defaulted obligors including i. Thus, the dynamics of $\left(\lambda_t^i\right)_{t\geq 0}$ are given by:

for $t < \tau_i$,

$$\begin{aligned}\frac{d\lambda_t^i}{\lambda_{t^-}^i} &= \left[\left(\frac{dh_t^i}{h_t^i} - h_t^i dt\right) - \left(\frac{\overline{C}^U_{x_i x_i x_{Z_1}...x_{Z_{k-1}}} \overline{C}^U_{x_{Z_1}...x_{Z_{k-1}}}}{\overline{C}^U_{x_i x_{Z_1}...x_{Z_{k-1}}} \overline{C}^U_{x_i x_{Z_1}...x_{Z_{k-1}}}} - 1\right)\lambda_{t^-}^i dt\right] \\ &\quad + \left[\sum_{\substack{j\notin\{Z_1,...,Z_{k-1}\} \\ j\neq i}} \left(\frac{\overline{C}^U_{x_i x_j x_{Z_1}...x_{Z_{k-1}}} \overline{C}^U_{x_{Z_1}...x_{Z_{k-1}}}}{\overline{C}^U_{x_i x_{Z_1}...x_{Z_{k-1}}} \overline{C}^U_{x_j x_{Z_1}...x_{Z_{k-1}}}} - 1\right)\left(dD_t^j - \lambda_{t^-}^j dt\right)\right].\end{aligned} \tag{1.37}$$

Defining a version of the intensity λ_t^i, which does not jump after τ_i,

$$\lambda_{\tau_i}^i = \lambda_{\tau_i^-}^i,$$

we can express (1.37) using the marked point process $\mu\,(dt \times dz)$:

$$\begin{aligned}\frac{d\lambda_t^i}{\lambda_{t^-}^i} &= \frac{dh_t^i}{h_t^i} + \left[\left(1 - \frac{\overline{C}^U_{x_i x_i x_{Z_1}...x_{Z_{k-1}}} \overline{C}^U_{x_{Z_1}...x_{Z_{k-1}}}}{\overline{C}^U_{x_i x_{Z_1}...x_{Z_{k-1}}} \overline{C}^U_{x_i x_{Z_1}...x_{Z_{k-1}}}}\right)\lambda_t^i - h_t^i\right] dt \\ &\quad + \int_E \Delta\,(i,\omega,t,z)\,(\mu\,(dt\times dz) - \nu\,(dt\times dz))\,,\end{aligned} \tag{1.38}$$

where the jump size $\Delta\,(i,\omega,t,z)$ is given by

$$\Delta\,(i,\omega,t,z) = \begin{cases} \dfrac{\overline{C}^U_{x_i x_Z x_{Z_1(\omega)}\cdots x_{Z_{k-1}(\omega)}} \overline{C}^U_{x_{Z_1(\omega)}\cdots x_{Z_{k-1}(\omega)}}}{\overline{C}^U_{x_i x_{Z_1(\omega)}\cdots x_{Z_{k-1}(\omega)}} \overline{C}^U_{x_Z x_{Z_1(\omega)}\cdots x_{Z_{k-1}(\omega)}}} - 1, \text{ if } z \neq i, \\ 0, \text{ if } z = i. \end{cases} \tag{1.39}$$

The intensity λ^i has a diffusion part, which is defined by the dynamics of the $\{\mathcal{F}_t\}$-adapted intensity h^i, and jumps upon the default of the other obligors. This is an equivalent representation of the copula model as another member of the class of "looping" default models considered in Section 1.3. Note that, if the tresholds $(U_1, ..., U_2)$ are independent, i.e.,

$$C\,(u_1, ..., u_n) = \prod_{i=1}^n u_i, \text{ for all } u_i \in [0,1]\,,$$

then,

$$\begin{aligned}\frac{\overline{C}^U_{x_i x_Z x_{Z_1(\omega)}\cdots x_{Z_{k-1}(\omega)}} \overline{C}^U_{x_{Z_1(\omega)}\cdots x_{Z_{k-1}(\omega)}}}{\overline{C}^U_{x_i x_{Z_1(\omega)}\cdots x_{Z_{k-1}(\omega)}} \overline{C}^U_{x_Z x_{Z_1(\omega)}\cdots x_{Z_{k-1}(\omega)}}} &= 1, \text{ for } z \neq i, \\ \frac{\overline{C}^U_{x_i x_i x_{Z_1(\omega)}\cdots x_{Z_{k-1}(\omega)}} \overline{C}^U_{x_{Z_1(\omega)}\cdots x_{Z_{k-1}(\omega)}}}{\overline{C}^U_{x_i x_{Z_1(\omega)}\cdots x_{Z_{k-1}(\omega)}} \overline{C}^U_{x_i x_{Z_1(\omega)}\cdots x_{Z_{k-1}(\omega)}}} &= 0, \text{ for } z = i,\end{aligned}$$

and equation (1.38) degenerates to

$$\frac{d\lambda_t^i}{\lambda_{t^-}^i} = \frac{dh_t^i}{h_t^i} + \left[\lambda_t^i - h_t^i\right] dt.$$

This is consistent with the fact that, when the default tresholds are independent, the intensities λ^i and h^i coincide.

1.6 Numerical Examples

The goal of this section is to illustrate the relationship between CJD dynamics and the copula approach. We start with the Jarrow and Yu model, for $n = 2$, and we look at the default correlation implied by a given conditional jump size. Then, we consider the inverse problem where we study the default contingent jump implied by a Gaussian copula.

Case Study: n = 2. We consider the example of 2 default times with interacting intensities:

$$\begin{aligned} \lambda_t^1 &= \lambda_0^1 + \Delta^{12}\mathbf{1}_{\{\tau_2 \le t\}}, \\ \lambda_t^2 &= \lambda_0^2 + \Delta^{21}\mathbf{1}_{\{\tau_1 \le t\}}, \end{aligned}$$

with $\Delta^{12} > \lambda_0^1$ and $\Delta^{22} > \lambda_0^2$. And let α_{12} and α_{21} denote the proportional jump ratios: $\Delta^{12} = \alpha_{12}\lambda_0^1$, $\Delta^{21} = \alpha_{21}\lambda_0^2$.

Figure (1.1) shows how the pair-wise default correlation varies with the jump size for different time horizons. When the jump size is zero, the two default times are independent and their default correlation is zero. On the other hand, when the jump size goes to infinity, the default correlation goes to its maximum value. In this example, the two intensities are identical, and the highest achievable default correlation is 1. Intuitively, an infinite intensity corresponds to the default state. In other words, an infinite jump size implies that the default of one obligor triggers the default of the other.

The copula function implied by these CJD dynamics is depicted in Figure (1.2). For convenience, we have plotted the copula in Gaussian coordinates, i.e., we use the transformation: $(\theta_1, \theta_2) \to \left(\Phi^{-1}\left(\mathbb{P}\left(\tau_1 \le \theta_1\right)\right), \Phi^{-1}\left(\mathbb{P}\left(\tau_2 \le \theta_2\right)\right)\right)$.

In order to do a comparison with the standard Gaussian copula (depicted in Figure (1.3)), we calibrate the parameters of both copulas such that the default correlation is the same. In this example, the 5-year default correlation is 15%. Since the default correlation is the same, the "slope" of the copula functions is preserved, but the behaviour in the tail is very different.

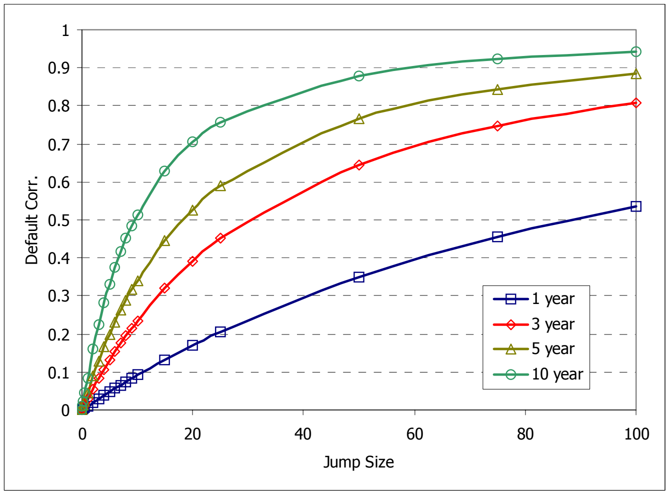

Figure 1.1: Default correlation as a function of the jump size ratio $\alpha = \alpha_{12} = \alpha_{21}$, for different time horizons T . The intensities are $\lambda_0^1 = \lambda_0^2 = 100$ bps.

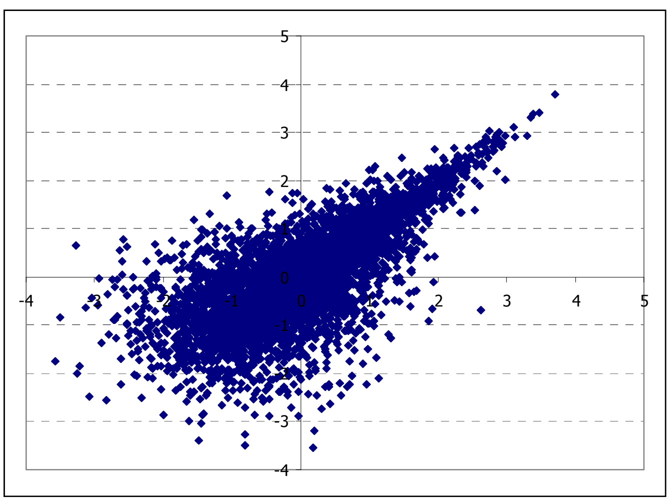

Figure 1.2: Conditional jump diffusion copula, with a jump ratio of $\alpha_{12} = \alpha_{21} = 3.58$, which corresponds to a 5 year-default correlation of 0.15 for $\lambda_0^1 = \lambda_0^2 = 100$ bps.

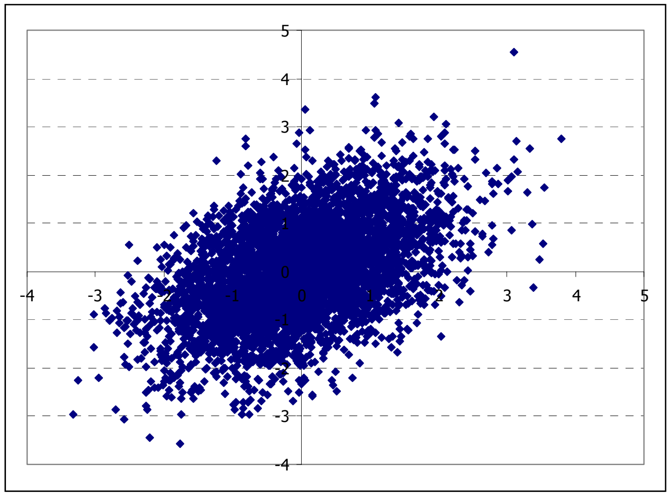

Figure 1.3: Gaussian copula with an asset correlation of 0.497, which corresponds to a 5 year-default correlation of 0.15 for $\lambda_0^1 = \lambda_0^2 = 100$ bps.

Gaussian copula. Now, we investigate the inverse problem. We consider two default times (τ_1, τ_2) with marginals

$$\begin{aligned}\mathbb{P}(\tau_1 > T) &= e^{-h_1 T},\\ \mathbb{P}(\tau_2 > T) &= e^{-h_2 T},\end{aligned}$$

for some fixed $h_1 \in \mathbb{R}_+$, $h_2 \in \mathbb{R}_+$, and whose joint distribution is defined via a Gaussian copula, i.e.,

$$\mathbb{P}(\tau_1 \leq T_1, \tau_2 \leq T_2) = \Phi_2\left(\Phi^{-1}\left(\mathbb{P}(\tau_1 \leq T_1)\right), \Phi^{-1}\left(\mathbb{P}(\tau_2 \leq T_2)\right)\right),$$

where $\Phi_2(.,.)$ is the bivariate normal distribution and $\Phi^{-1}(.)$ is the inverse normal function. And we study the link between the default contingent jump size and correlation.

Figure (1.4) shows how the jump size ratio varies with asset correlation. As correlation increases, the jump size ratio increases as well. For a correlation of 1, the jump size goes to infinity. In other words, the default of one obligor triggers the default of the other one and vice-versa.

1.7 Conclusion

The objective of this chapter was to introduce the conditional jump diffusion framework and its relationship with default correlations and copulas. This was done in two

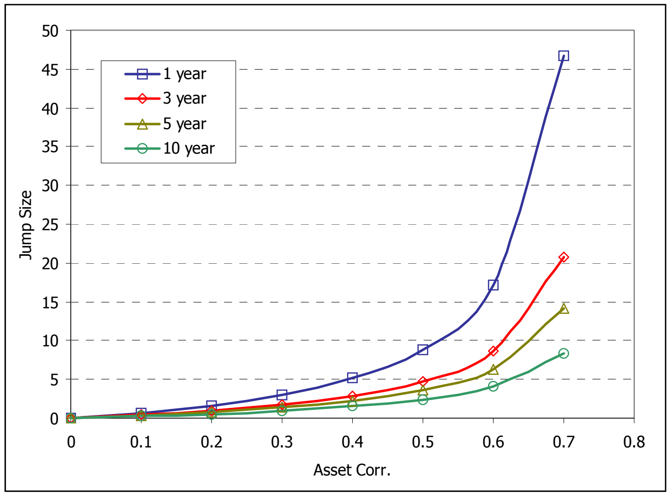

Figure 1.4: Jump size ratio implied by a Gaussian copula as a function of asset correlation. The intensities are $h_1 = h_2 = 100$ bps.

steps. First, we have studied CJD dynamics and their implied default times' multivariate dependence. Then, we have considered the inverse problem and examined the consequences of a choice of copula on the dynamics of intensities. Thus, establishing the equivalence between CJD and Copulas. This equivalence principle has a few useful applications. It can be used as a sanity check, to verify whether a specific calibrated copula function implies reasonable jumps. One example is the standard Gaussian copula calibrated on KMV historical asset correlations. The analysis of implied jumps can be a powerful tool for finding aberrations in the calibration procedure. On the other hand, one can have a view on the jump in default and would rather build a default model, which reflects that view. This is very similar, in spirit, to calibrating a term structure interest rate model, then infer implied forward volatilities, or building a BGM style model, which is consistent with one's view of forward volatilities. Both approaches are useful and complement each other.

Chapter 2

Correlation with a Difference

In this chapter, we analyze the "Marshall-Olkin" copula model in the context of credit risk modelling. This framework was traditionally used in reliability theory to model the failure rate in multi-component systems. The failure of each component is assumed to be contingent on some independent Poisson shocks. Our aim is to show that MO is a viable alternative to the Gaussian copula. This is done in three steps: (1) we introduce the MO model as the natural extension of a univariate Poisson process, (2) we discuss the calibration issues, (3) we compare it with the standard Gaussian copula. Furthermore, we show that the MO model can be used to reproduce the observed correlation skew in the CDO market.

2.1 Introduction

The problem of correlating multiple credits boils down to the specification of a copula function, which links the marginal default distributions. There is a growing literature that addresses this problem. The main approach that has emerged as the market standard is the Gaussian copula approach. The idea of using a Gaussian copula to model the default times' dependence in basket products goes back to Li (2000). It was also used implicitly in the CreditMetrics framework and the KMV firm-value approach. The t-copula is an extension of the Gaussian copula with a higher tail-dependence. It allows for a better modelling of extreme events' risk. A good reference on t-copulas and the pricing of small baskets can be found in Mashal and Naldi (2002).

In this chapter, we study another alternative: the Marshall-Olkin copula. This latter was first used in the context of basket credit derivatives pricing by Duffie (1998), then by Duffie and Garleanu (2001). The Marshall-Olkin copula was traditionally used in reliability theory to model the failure of multi-component systems. In this set-up, the failure of each component is assumed to be contingent on some independent Poisson shocks. This is also known as a multivariate Poisson model. A good description of these models can be found in Barlow and Proschan (1981). The Marshall-Olkin copula can also be viewed as the limiting distribution of a multivariate binary model (Wong (2000)). One practical feature of the Marshall-Olkin approach is the simplicity of its Monte-Carlo implementation. In addition, it has a number of useful analytical results for aggregate portfolio distributions (see Lindskog and McNeil (2003)).

The purpose of this chapter is primarily to show that the Marshall-Olkin model can be a viable alternative to the standard Gaussian copula. This is achieved in three steps. First, we present the MO framework. Second, we propose a parameterization procedure of the model based on market intuition and observed market prices. And third, we compare the Marshall-Olkin copula with its elliptical counterparts: the Gaussian and the t-copula.

The rest of the chapter is structured as follows. In Section 2.2, we introduce the Marshall-Olkin model. In Section 2.3, we derive the copula function of default times. In Section 2.4, we study the aggregate default distribution. In Section 2.5, we discuss the model calibration. In Section 2.6, we compare Marshall-Okin with the Gaussian and t-copula. In Section 2.7, we use the Marshall-Olkin copula to reproduce the correlation skew in the CDO market.

2.2 The Model

We work on probability space $(\Omega, \mathcal{G}, P)$, on which is given a set of n non-negative random variables $(\tau_1, ..., \tau_n)$ representing the default times of a basket of obligors.

We introduce, for each obligor i, the right-continuous process $D_t^i \triangleq \mathbf{1}_{\{\tau_i \leq t\}}$ indicating whether the firm has defaulted or not.

We assume that there exists a set of m independent Poisson processes $\left(N_t^{c_j}\right)_{t \geq 0}$ with intensities $\lambda^{c_j} \in \mathbb{R}_+$, which can trigger simultaneous joint defaults.

Each Poisson process N^{c_j} can be equivalently represented by the sequence of event trigger times $\{\theta_r^{c_j}\}_{r \in \{1,2,...\}}$.

For every event type c_j, and for all $t \geq 0$, we define a set of independent Bernoulli variables $\left(A_t^{1,j}, ..., A_t^{n,j}\right)$ with probabilities $\left(p^{1,j}, ..., p^{n,j}\right)$, $p^{i,j} \in [0,1]$.

We assume that for $j \neq k$, the vectors $\mathbf{A}_t^j = \left(A_t^{1,j}, ..., A_t^{n,j}\right)$ and $\mathbf{A}_t^k = \left(A_t^{1,k}, ..., A_t^{n,k}\right)$ are independent.

We assume that for $t \neq s$, the vectors $\mathbf{A}_t^j = \left(A_t^{1,j}, ..., A_t^{n,j}\right)$ and $\mathbf{A}_s^j = \left(A_s^{1,j}, ..., A_s^{n,j}\right)$ are independent.

At the r^{th} occurrence of an event of type c_j, we draw the set of independent $\{0,1\}$-valued Bernoulli variables $\left(A_{\theta_r^{c_j}}^{1,j}, ..., A_{\theta_r^{c_j}}^{n,j}\right)$. The variable $A_{\theta_r^{c_j}}^{i,j}$ indicates, conditional on an event of type c_j, whether a default of type i has occurred or not.

The process $\left(N_t^i\right)_{t\geq 0}$ defined as

$$N_t^i \triangleq \sum_{j=1}^{m} \sum_{\theta_r^{c_j} \leq t} A_{\theta_r^{c_j}}^{i,j}, \tag{2.1}$$

is also a Poisson process with intensity

$$\lambda^i = \sum_{j=1}^{m} p^{i,j} \lambda^{c_j}. \tag{2.2}$$

It is obtained by superpositioning m independent (thinned) Poisson processes.

The default time τ_i is defined as the first jump time of the Poisson process $\left(N_t^i\right)_{t\geq 0}$:

$$\tau_i \triangleq \inf \left\{ t : N_t^i > 0 \right\}. \tag{2.3}$$

This common shock model can also be described formally by the following SDE

$$dD_t^i = \left(1 - D_{t-}^i\right) \sum_{j=1}^{m} A_t^{i,j} dN_t^{c_j}. \tag{2.4}$$

This description was used, for instance, in Duffie (1998).

2.3 The Copula Function

In this section, we derive the copula function of the shock model described above. To this end, we shall use the "equivalent fatal shock model" of Lindskog and McNeil (2003).

Equivalent Fatal Shock Model. Let $\mathbf{\Pi_n}$ be the set of all subsets of $\{1, ..., n\}$, excluding the empty set $\emptyset$. For each $\pi \in \mathbf{\Pi_n}$, we introduce the point process N_t^π, which counts the number of shocks in $(0, t]$ resulting in joint defaults of the obligors in π only:

$$N_t^\pi \triangleq \sum_{j=1}^{m} \sum_{r=1}^{N_t^{c_j}} A_{\theta_r^{c_j}}^{\pi,j}, \tag{2.5}$$

where, for each trigger time $\theta_r^{c_j}$, $A_{\theta_r^{c_j}}^{\pi,j}$ is a Bernoulli variable, which is equal to 1 if all obligors $i \in \pi$ default and all the others, $i \notin \pi$, survive:

$$A_t^{\pi,j} \triangleq \prod_{i\in\pi} A_t^{i,j} \prod_{i\notin\pi} \left(1 - A_t^{i,j}\right). \tag{2.6}$$

At the occurrence of the r^{th} common shock, of type c_j, at time $\theta_r^{c_j}$, the point process N_t^π gets incremented by $\Delta N^\pi_{\theta_r^{c_j}} = A^{\pi,j}_{\theta_r^{c_j}}$. For example, if $\pi = \{1,2\}$, then the process $N_t^{\{1,2\}}$ counts the shocks, which trigger simultaneous defaults of obligors 1 and 2 but not the other obligors 3 to n.

We have the following fatal shock representation key result. We refer to Lindskog and McNeil (2003) for details (see Proposition 4).

Proposition 20 *(Fatal shock representation). The processes $(N^\pi)_{\pi\in\mathbf{\Pi_n}}$ are independent Poisson processes*[1] *with intensities*

$$\lambda^\pi = \sum_{j=1}^{m} p^{\pi,j}\lambda^{c_j},$$

where

$$p^{\pi,j} = \prod_{i\in\pi} p^{i,j} \prod_{i\notin\pi}\left(1-p^{i,j}\right).$$

This provides a fatal shock representation of the original not-necessarily-fatal shock set-up. It will allow us to analyze the multivariate distribution of the default times.

For $\pi \in \mathbf{\Pi_n}$, let τ_π denote the first jump time of the Poisson process N^π:

$$\tau_\pi = \inf\left\{t : N_t^\pi > 0\right\}.$$

Each obligor i can be equivalently described using the fatal shock representation.

Lemma 21 *(Obligor description using the fatal shock representation).*

1. *The Poisson process N^i can be expressed as*

$$N_t^i = \sum_{\pi\in\mathbf{\Pi_n}} \mathbf{1}_{\{i\in\pi\}} N_t^\pi,$$

and its intensity[2] *is given by*

$$\lambda^i = \sum_{\pi\in\mathbf{\Pi_n}} \mathbf{1}_{\{i\in\pi\}}\lambda^\pi.$$

2. *The default time τ_i is given by*

$$\tau_i = \min\left\{\tau_\pi : i\in\pi,\ \pi\in\mathbf{\Pi_n}\right\}.$$

[1] The processes $(N^\pi)_{\pi\in\mathbf{\Pi_n}}$ do not jump at the same time, i.e., for $\pi_1 \neq \pi_2$, and $t \geq 0$, $\Delta N_t^{\pi_1}\Delta N_t^{\pi_2} = 0$, where the jump process ΔY of a semi-martingale Y is defined as $\Delta Y_t \triangleq Y_t - \lim_{s\uparrow t} Y_s$.

[2] Note that $\lambda^i \neq \lambda^{\{i\}}$. Indeed, for $\pi = \{i\}$, $N_t^i \neq N_t^{\{i\}}$. The process N_t^i counts all shocks affecting obligor i, which may also affect other obligors. $N_t^{\{i\}}$ counts the shocks affecting obligor i only.

Multivariate Exponential Distribution. Since we have

$$\tau_i = \inf\left\{t : N_t^i > 0\right\} = \inf\left\{t : \sum_{\pi\in\mathbf{\Pi_n}} \mathbf{1}_{\{i\in\pi\}} N_t^\pi > 0\right\} = \min_{\pi:i\in\pi} \tau_\pi,$$

the multivariate distribution of $(\tau_1, ..., \tau_n) = \left(\min_{\pi:1\in\pi} \tau_\pi, ..., \min_{\pi:n\in\pi} \tau_\pi\right)$ can be computed as follows.

Proposition 22 *(Multivariate Exponential Distribution). The multivariate distribution of the default times $(\tau_1, ..., \tau_n)$ is*

$$\mathbb{P}(\tau_1 > T_1, ..., \tau_n > T_n) = \prod_{\pi\in\mathbf{\Pi_n}} \exp\left(-\Lambda^\pi_{\max_{i\in\pi} T_i}\right), \tag{2.7}$$

where $\Lambda_T^\pi \triangleq \int_0^T \lambda^\pi ds = \lambda^\pi T$.

This is the Multivariate Exponential Distribution developed by Marshall-Olkin (1967). We refer to Barlow and Proschan (1981), Joe (1997) or Nelsen (1999) for a detailed study of this distribution function:

$$\mathbb{P}(\tau_1 > T_1, ..., \tau_n > T_n) = \exp\left(-\sum_i \Lambda_{T_i}^{\{i\}} - \sum_{i,j} \Lambda^{\{i,j\}}_{\max(T_i,T_j)} - ... - \Lambda^{\{1,...,n\}}_{\max(T_1,...,T_n)}\right).$$

Proof. We proceed as follows:

$$\begin{aligned}
\mathbb{P}(\tau_1 > T_1, ..., \tau_n > T_n) &= \mathbb{P}\left(\min_{\pi:1\in\pi} \tau_\pi > T_1, ..., \min_{\pi:n\in\pi} \tau_\pi > T_n\right) \\
&= \mathbb{P}\left(\bigcap_{\pi\in\mathbf{\Pi_n}} \left\{\tau_\pi > \max_{i\in\pi} T_i\right\}\right) \\
&= \mathbb{P}\left(\bigcap_{\pi\in\mathbf{\Pi_n}} \left\{N^\pi_{\max_{i\in\pi} T_i} = 0\right\}\right) \\
&= \prod_{\pi\in\mathbf{\Pi_n}} \mathbb{P}\left(\left\{N^\pi_{\max_{i\in\pi} T_i} = 0\right\}\right) \\
&= \prod_{\pi\in\mathbf{\Pi_n}} \exp\left(-\Lambda^\pi_{\max_{i\in\pi} T_i}\right),
\end{aligned}$$

the third equality is from the definition of τ_π, the fourth equality is due to the independence of the Poisson processes $(N^\pi)_{\pi\in\mathbf{\Pi_n}}$. ■

The Multivariate Exponential distribution is "memoryless", i.e., it has the property that

$$\mathbb{P}(\tau_1 > T_1, ...\tau_n > T_n \,|\tau_1 > t_1, ...\tau_n > t_n) = \mathbb{P}(\tau_1 > T_1 - t_1, ...\tau_n > T_n - t_n),$$

for all $T_1 > t_1, ..., T_n > t_n$. This is the multidimensional version of the well-known property for the exponential distribution.

Example. For $n = 2$, if we set $u_1 \triangleq \mathbb{P}(\tau_1 > t_1)$, $u_2 \triangleq \mathbb{P}(\tau_2 > t_2)$ and $\alpha_1 \triangleq \frac{\lambda^{\{1,2\}}}{\lambda^{\{1\}}}$, $\alpha_2 \triangleq \frac{\lambda^{\{1,2\}}}{\lambda^{\{2\}}}$, we get the bivariate Marshall-Olkin survival copula[3]

$$\overline{\mathbf{C}}(u_1, u_2) = u_1 u_2 \min\left(u_1^{-\alpha_1}, u_2^{-\alpha_2}\right) = \min\left(u_1^{1-\alpha_1} u_2, u_1 u_2^{1-\alpha_2}\right); \tag{2.8}$$

The copula function (2.8) has an absolutely continuous part on the upper and lower triangles: $\{u_1 < u_2\}$ and $\{u_2 < u_1\}$, and has a singular component on the diagonal $\{u_1 = u_2\}$.

2.4 The Aggregate Default Distribution

The central question in credit portfolio modelling is the study of the aggregate default distribution of a given portfolio. Let X_t denote the total number of defaults, for a fixed time horizon t:

$$X_t \triangleq \sum_{i=1}^{n} D_t^i. \tag{2.9}$$

The distribution of X_t is referred to as the aggregate default distribution at time t. In this section, we derive the aggregate default distribution in the Marshall-Olkin model.

Poisson Approximation. In Lindskog and McNeil (2003), the default indicators D_t^i are approximated by their corresponding Poisson counters N_t^i. For low default probabilities this is a reasonable approximation. For $t \geq 0$, the total number of defaults X_t is then approximated by the random variable Z_t defined as

$$X_t \simeq Z_t \triangleq \sum_{i=1}^{n} N_t^i. \tag{2.10}$$

This is known in the actuarial literature as the approximation of the individual model with the collective model; for low individual default probabilities, the likelihood of multiple jumps in the Poisson process is small, and is neglected for the purposes of estimating the aggregate portfolio distribution.

[3] The bivariate survival copula of two randoms variables (X_1, X_2) is defined as the function $\overline{\mathbf{C}}$: $[0,1] \times [0,1] \to [0,1]$ such that

$$\overline{\mathbf{C}}(\mathbb{P}(X_1 > x_1), \mathbb{P}(X_2 > x_2)) = \mathbb{P}(X_1 > x_1, X_2 > x_2).$$

The total number of losses $Z_t \triangleq \sum_{i=1}^{n} N_t^i$ is a compound Poisson process. It is the sum of m independent compound Poisson processes $Z_t^{c_j}$:

$$Z_t^{c_j} \triangleq \sum_{r=1}^{N_t^{c_j}} \sum_{i=1}^{n} A_{\theta_r^{c_j}}^{i,j}. \tag{2.11}$$

Next, we derive the distribution of Z_t, first, using its moment generating function, then using Panjer's algorithm.

Moment Generating Function. The aggregate portfolio counter Z_t is a compound Poisson process, which is obtained as the sum of m independent compound Poisson processes:

$$Z_t = \sum_{j=1}^{m} Z_t^{c_j}. \tag{2.12}$$

The distribution of each compound Poisson $Z_t^{c_j}$ is not available in closed-form, but one can compute the moment generating function of Z_t,$\mathcal{L}_{Z_t}(\alpha) \triangleq \mathbb{E}\left[e^{-\alpha Z_t}\right]$. Since the processes $Z_t^{c_j}$ are independent the m.g.f. of Z_t is given by

$$\mathcal{L}_{Z_t}(\alpha) = \prod_{j=1}^{M} \mathcal{L}_{Z_t^{c_j}}(\alpha).$$

$Z_t^{c_j}$ is defined by the Poisson counter $N_t^{c_j}$ and its compounding distribution X^{c_j}, i.e.,

$$Z_t^{c_j} \overset{d}{=} \sum_{r=1}^{N_t^{c_j}} X_r^{c_j}, \tag{2.13}$$

where $X_1^{c_j}, X_2^{c_j}, ..., X_r^{c_j} \left(\overset{d}{=} X^{c_j}\right)$ are i.i.d. independent of $N_t^{c_j}$. When the jump sizes $X_r^{c_j}$ are discrete random variables taking values in $\{a_1, a_2, ...\}$, one can write

$$Z_t^{c_j} = \sum_k a_k N_t^{a_k},$$

where $N_t^{a_k}$ are independent Poisson processes[4] with intensities

$$\lambda^{a_k} = \lambda^{c_j} \mathbb{P}\left(X^{c_j} = a_k\right),$$

and the m.g.f. of the compound Poisson process is obtained immediately as

$$\mathcal{L}_{Z_t^{c_j}}(\alpha) = \mathbb{E}\left[e^{-\alpha Z_t^{c_j}}\right] = \mathbb{E}\left[\exp\left(-\alpha \sum_k a_k N_t^{a_k}\right)\right] = \exp\left(-t \sum_k \left(1 - e^{-\alpha a_k}\right) \lambda^{a_k}\right).$$

Here, the jump sizes take values in $\{0, 1, .., n\}$ and the distribution of $X^{c_j} = \sum_{i=1}^{n} A^{i,j}$, where $A^{i,j}$ is a Bernoulli variable with probability $p^{i,j}$, can be computed by inverting its Fourier transform, which is given by the product

$$\mathcal{F}_{X^{c_j}}(\alpha) = \mathbb{E}\left[e^{-i\alpha X^{c_j}}\right] = \prod_{i=1}^{n} \left(p^{i,j} e^{-i\alpha} + \left(1 - p^{i,j}\right)\right).$$

The moment generating function of Z_t is then given by

$$\mathcal{L}_{Z_t}(\alpha) = \prod_{j=1}^{m} \exp\left(-t \sum_{k=0}^{n} \left(1 - e^{-\alpha k}\right) \lambda^{c_j} \mathbb{P}\left(X^{c_j} = k\right)\right). \tag{2.14}$$

Panjer's Recursion. As shown in Lindskog and McNeil (2003), the distribution of the compound Poisson Z_t can also be derived using Panjer's recursion. The total number of losses Z_t has the following representation (see Proposition 6 in Lindskog and McNeil (2003)):

$$Z_t \stackrel{d}{=} \sum_{r=1}^{\widetilde{N}_t} \widetilde{X}_r, \tag{2.15}$$

where $\widetilde{N}_t$ is a Poisson process with intensity

$$\widetilde{\lambda} = \sum_{j=1}^{m} \lambda^{c_j} \left[1 - \prod_{i=1}^{n} \left(1 - p^{i,j}\right)\right].$$

[4]Indeed, it suffices to write

$$\begin{aligned} Z_t^{c_j} &= \sum_{r=1}^{N_t^{c_j}} X_r^{c_j} = \sum_{r=1}^{N_t^{c_j}} \sum_k a_k \mathbf{1}_{\left\{X_r^{c_j} = a_k\right\}} \\ &= \sum_k a_k \left[\sum_{r=1}^{N_t^{c_j}} \mathbf{1}_{\left\{X_r^{c_j} = a_k\right\}}\right] = \sum_k a_k N_t^{a_k}. \end{aligned}$$

The process $N_t^{a_k}$ is a Poisson process obtained by thinning $N_t^{c_j}$; its intensity is given by $\lambda^{a_k} = \mathbb{E}\left[\mathbf{1}_{\left\{X^{c_j} = a_k\right\}}\right] \lambda^{c_j}$. The $N_t^{a_k}$'s are independent since they cannot jump at the same time, $\Delta N_t^{a_k} \Delta N_t^{a_l} = 0$, for $k \neq l$.

It counts any loss-causing shock in $(0,t]$. $\widetilde{X}_1, \widetilde{X}_2, ..., \widetilde{X}_r \left(\overset{d}{=} \widetilde{X}\right)$ are i.i.d. and independent of $\widetilde{N}_t$. The distribution of $\widetilde{X}$ is given by

$$\begin{aligned}
\mathbb{P}\left(\widetilde{X}=0\right) &= 0, \\
\mathbb{P}\left(\widetilde{X}=k\right) &= \frac{1}{\widetilde{\lambda}}\sum_{j=1}^{m}\lambda^{c_j}\mathbb{P}\left(X^{c_j}=k\right), \text{ for } k=1,...,n, \\
\mathbb{P}\left(\widetilde{X}=k\right) &= 0, \text{ for } k>n.
\end{aligned}$$

The distribution of Z_t can then be computed with Panjer's algorithm (see Panjer (1981)):

$$\begin{aligned}
\mathbb{P}\left(Z_t=0\right) &= \exp\left(-\widetilde{\lambda}t\right), \\
\mathbb{P}\left(Z_t=l\right) &= \frac{\widetilde{\lambda}t}{l}\sum_{k=1}^{l}k\mathbb{P}\left(\widetilde{X}=k\right)\mathbb{P}\left(Z_t=l-k\right), \text{ for } l>0.
\end{aligned}$$

This recursive algorithm offers a more efficient method for computing the probabilities $\mathbb{P}(Z_t=l)$, than the inversion of the moment generating function.

Duffie's Approximation. Duffie and Pan (2001) have suggested another approximation of the aggregate default distribution. They have neglected the probability of multiple jumps of the common market factor events[5] and they have assumed that the solution of the SDE (2.4) can be approximated as

$$D_t^i \overset{d}{\simeq} \sum_{j=1}^{m} A^{i,j} N_t^{c_j}. \tag{2.16}$$

Using equation (2.16), we can easily approximate the Laplace transform of $X_t \triangleq \sum_{i=1}^{n} D_t^i$ as a product of conditional market factor Laplace transforms:

$$\begin{aligned}
\mathcal{L}_{X_t}(\alpha) &= \mathbb{E}\left[e^{-\alpha X_t}\right] \\
&\simeq \prod_{j=1}^{m}\left[\exp\left(-\Lambda_T^{c_j}\right)+\left(1-\exp\left(-\Lambda_T^{c_j}\right)\right)\prod_{i=1}^{n}\left(p^{i,j}e^{-\alpha}+\left(1-p^{i,j}\right)\right)\right],
\end{aligned}$$

where $\Lambda_T^{c_j} \triangleq \int_0^T \lambda^{c_j} ds = \lambda^{c_j} T$.

If $m>n$, so that we have n idiosyncratic factors and $m-n$ common market factors, i.e.,

$$D_t^i \simeq \sum_{j=1}^{m-n} A^{i,j} N_t^{c_j} + N_t^{0,i},$$

[5] Refer to Duffie and Pan (2001) for a discussion of the domain of validity of the approximation. A numerical comparison of Duffie's approximation, the Panjer approximation and the exact method can be found in Chapter 4.

where $N_t^{0,i} \triangleq N_t^{c_{m-n+i}}$ is the idiosyncratic factor of obligor i. Then, the Laplace transform collapses to

$$\mathcal{F}_{X_t}(\alpha) \simeq \varphi^0(\alpha) \prod_{j=1}^{m} \left[\exp\left(-\Lambda_T^{c_j}\right) + \left(1 - \exp\left(-\Lambda_T^{c_j}\right)\right) \varphi_j^c(\alpha)\right], \tag{2.17}$$

where

$$\begin{aligned}
\varphi^0(\alpha) &= \prod_{i=1}^{n} \left(\left(1 - \exp\left(-\Lambda_T^{0,i}\right)\right) e^{-\alpha} + \exp\left(-\Lambda_T^{0,i}\right)\right), \\
\varphi_j^c(\alpha) &= \prod_{i=1}^{n} \left(p^{i,j} e^{-\alpha} + \left(1 - p^{i,j}\right)\right).
\end{aligned}$$

A direct inversion of the Laplace transform (2.17) gives the aggregate default distribution.

Monte Carlo. The aggregate default distribution can also be estimated using a Monte-Carlo method. A good reference on simulating Multivariate-Exponential Default Times can be found, for instance, in Duffie and Singleton (1998).

We are interested in simulating the correlated default times $(\tau_1, ..., \tau_n)$ only if they occur before a fixed time horizon T. The basic algorithm proceeds as follows:

1. Simulate the jump times of the market factor Poisson processes $\left(N_T^{c_1}, ..., N_T^{c_m}\right)$: $\left\{\theta_r^{c_j}\right\}_{r \in \{1,2,...\}}$, for $1 \leq j \leq m$,
   (a) Initialize $\theta_0^{c_j} = 0$,
   (b) While $\theta_r^{c_j} \leq T$, simulate a uniformly distributed variable U, find the inter-jump time S such that $1 - \exp\left(-\left(\Lambda_S^{c_j} - \Lambda_{\theta_{r-1}^{c_j}}^{c_j}\right)\right) = U$, set $\theta_r^{c_j} = \theta_{r-1}^{c_j} + S$;
2. For each market factor jump time $\theta_r^{c_j}$, simulate the individual default Bernoulli variables $\left(A_{\theta_r^{c_j}}^{1,j}, ... A_{\theta_r^{c_j}}^{n,j}\right)$:
   (a) Simulate a set of n independent uniformly distributed variables $(U_1, ..., U_n)$, set $A_{\theta_r^{c_j}}^{i,j} = \mathbf{1}_{\{U_i \leq p^{i,j}\}}$, for $1 \leq i \leq n$;
3. Set the individual default times:

$$\tau_i = \min\left(A_{\theta_r^{c_j}}^{i,j} \theta_r^{c_j} + \left(1 - A_{\theta_r^{c_j}}^{i,j}\right) T : 1 \leq j \leq m, \theta_r^{c_j} \leq T\right).$$

A variant of this simulation uses the fact that the process $N_t^c \triangleq N_t^{c_1} + ... + N_t^{c_m}$ is a Poisson process and its intensity is given by the sum of intensities $\lambda^c \triangleq \lambda^{c_1} + ... + \lambda^{c_m}$.

The probability of having a market factor jump of type $\{c_j\}$ is given by the ratio $p_j^c = \frac{\lambda^{c_j}}{\lambda^c}$. Conditional on a market factor jump at θ_r^c, the identity of the market factor that triggered follows a multinomial distribution with parameters $\left(\frac{\lambda^{c_1}}{\lambda^c}, ..., \frac{\lambda^{c_m}}{\lambda^c}\right)$. We have the following algorithm:

1. Simulate the jump times of "any type" of market factor events in the interval $(0, T]$: $\{\theta_r^c\}_{r\in\{1,2,...\}}$,
   (a) Initialize $\theta_0^c = 0$,
   (b) While $\theta_r^c \le T$, simulate a uniformly distributed variable U, find the inter-jump time S such that $1 - \exp\left(-\left(\Lambda_S^c - \Lambda_{\theta_{r-1}^c}^c\right)\right) = U$, set $\theta_r^c = \theta_{r-1}^c + S$;
2. For each jump time θ_r^c, simulate the identity of the market factor that has triggered J_r^c:
   (a) Simulate a uniformly distributed variable U, find the index J_r^c in $\{1, ..., m\}$ such that
   $$\sum_{j=1}^{J_r^c-1} p_j^c < U \le \sum_{j=1}^{J_r^c} p_j^c;$$
3. For each pair (θ_r^c, J_r^c), simulate the individual Bernoulli variables $\left(A_r^{1,J_r^c}, ...A_r^{n,J_r^c}\right)$:
   (a) Simulate a set of n independent uniformly distributed variables $(U_1, ..., U_n)$, set $A_r^{i,J_r^c} = \mathbf{1}_{\left\{U_i \le p^{i,J_r^c}\right\}}$, for $1 \le i \le n$;
4. Set the individual default times:
$$\tau_i = \min\left(A_r^{i,J_r^c}\theta_r^c + \left(1 - A_r^{i,J_r^c}\right)T : \theta_r^c \le T\right).$$

The second algorithm is clearly more efficient than the first since we restrict the simulation to a single vector of market factor default times. The intensity of the "any-type" market factor event is potentially m times larger than the individual market factor intensities, hence would produce more jump times. The conditional probabilities $\left(p_j^c\right)_{1\le j\le m}$ provide a much better way to simulate the identity of the market event.

2.5 Calibration

The Marshall-Olkin copula model offers a very rich correlation structure, which can be used to reproduce some observable measures of interdependence such as estimates of default correlations or basket credit derivative prices. In this section, we discuss the calibration of the model parameters.

In general, one needs to specify $m\,(n+1)$ parameters, corresponding to the vector of market factor intensities and the matrix of factor loadings. Market factor events can be classified in two categories: (a) "common" market factors affecting a subset of credits, which share some common characteristics; (b) idiosyncratic factors specific to individual obligors. Suppose $m > n$, so that we have n idiosyncratic factors $N_t^{0,i}$, with intensities $\lambda^{0,i}$, and $m_c \triangleq m - n$ common factors. The decomposition (2.2) becomes

$$\lambda^i = \sum_{j=1}^{m_c} p^{i,j} \lambda^{c_j} + \lambda^{0,i}. \tag{2.18}$$

Each idiosyncratic factor $N^{0,i}$ triggers the default of obligor i only with probability 1. The number of unknown parameters is $m_c\,(n+1)$; the individual idiosyncratic terms are obtained directly by the residuals $\left[\lambda^i - \sum_{j=1}^{m_c} p^{i,j} \lambda^{c_j}\right]$.

The aim of the calibration procedure is threefold:

1. define the common market factor events that constitute the backbone of the correlation structure;

2. specify a parametric form of the market factor loadings;

3. calibrate the intensity levels of the pre-specified market factors in the model.

Next, we explore each point in turn.

Choice of Common Market Factors. The first step consists in specifying the common market factors that explain joint default events. Clearly, this choice is market-specific and depends on macro and micro economic factors, which prevail at a particular point in time. The set of economic drivers that explain the default correlation "sentiment" for investment grade credits, for example, are different from the ones that affect emerging market or high-yield credits. On one hand, one would find that the behaviour of investment-grade credits is most likely to be explained by industry-sector events; on the other hand, in emerging markets the joint behaviour of credit entities is better explained by regional and country factors. Taking investment-grade credits as an example, one can highlight three distinct types of market behaviours:

- **Intra-sector segment:** it is commonly accepted that credit spreads of reference entities, belonging to the same industry-sector, have a tendency to move in tandem. This would seemingly imply the existence of a sector factor, which is generally stable in time and jumps occasionally. Sector factor shocks are observed through the joint co-movements of credit spreads in this particular sector. The sector factor itself cannot be observed but we can observe its effect.

- **Inter-sector segment:** historically, we observe that credit spreads in different industries have also a tendency to move together. The dependence between credits from different sectors is less important than the one observed intra-sector. This would correspond to general economy-wide events such as economic cycles, recessions, etc. Using the equity market terminology, we refer to the inter-sector driver as the "Beta" driver.

- **Super senior risk:** using the market-standard Gaussian copula model, one finds that the value attributed to a super senior CDO tranche is equal to zero. This is due to the "zero-tail-dependence" property of the Gaussian copula. The credit CDO market, however, has a different view. Indeed, super senior tranches are priced and traded at a premium of the order of a few basis points. This suggests that the market is pricing the highly unlikely global Armageddon risk (a situation where everyone defaults), and attributes an insurance-like premium to this "catastrophe" risk. Unlike the Gaussian copula, the Marshall-Olkin model is capable of capturing this effect. By assuming a low-probability global "World" driver and letting every credit in our universe have a factor-loading equal to 1, we ensure that the premiums of super senior CDO tranches are floored at the world driver spread. We can view this as a background radiation effect: the world driver sits silently in the background, and would never be active (under the real probability measure), but if the event occurs, every credit entity would default almost surely.

Summary 23 *In this specification of the Marshall-Olkin model, we have the following decomposition:*

$$\lambda^i = \left[\lambda^W\right] + p^{i,B}\left[\lambda^B\right] + \sum_{j=1}^{m_c-2} p^{i,S_j}\left[\lambda^{S_j}\right] + \left[\lambda^{0,i}\right], \qquad (2.19)$$

where
λ^W *is the intensity of the "World" driver;*
λ^B *is the intensity of the "Beta" driver, and* $p_{i,B}$ *is the loading on that driver;*
λ^{S_j} *is the intensity of the "Sector" driver* S_j*, and* p_{i,S_j} *is the loading on that sector*[6]*;*
$\lambda^{0,i}$ *is the intensity of the idiosyncratic event.*

[6] If $i \in S_j$ then $p_{i,S_j} > 0$ otherwise $p_{i,S_j} = 0$.

Parametric Form of the Factor Loadings. The second step consists in fixing the matrix of factor loadings $\left[p^{i,j}\right]$. It is clear that given the large number of credits that one has to deal with, it is crucial to reduce the dimensionality of the parameters to be specified. A natural approach, as suggested in Lindskog and McNeil (2003) and Duffie and Pan (2001), is to assume that the contribution of each market factor component $\left(p^{i,j}\lambda^{c_j}\right)$, in equation (2.19), is a fixed percentage α_j of the total intensity λ^i, i.e., for all $1 \leq i \leq n$,

$$\frac{p^{i,j}\lambda^{c_j}}{\lambda^i} = \alpha_j < 1.$$

The market factor contributions $(\alpha_1, ..., \alpha_m)$ are chosen such that the residual idiosyncratic intensities are positive:

$$\lambda^i - \sum_{j=1}^{m_c} p^{i,j}\lambda^{c_j} \geq 0 \Leftrightarrow 1 \geq \sum_{j=1}^{m_c} \alpha_j.$$

One consequence of this choice is that the loadings are completely specified by the individual intensities and the corresponding market factor intensities

$$p^{i,j} = \alpha_j \frac{\lambda^i}{\lambda^{c_j}}.$$

Unfortunately, as the intensity λ^i increases, the loading $p^{i,j}$ increases and can breach the condition that it is a probability:

$$p^{i,j} \leq 1. \tag{2.20}$$

A suggested parameterization is to impose condition (2.20) by writing $p^{i,j}$ as a conditional probability function

$$p^{i,j} = 1 - \exp\left(-\gamma_{i,j}\right), \tag{2.21}$$

where the "hazard" rate $\gamma_{i,j}$ is defined as

$$\gamma_{i,j} \triangleq \alpha_j \frac{\lambda^i}{\lambda^{c_j}}.$$

Expanding the exponential to first-order, we find, for $\alpha_j \frac{\lambda^i}{\lambda^{c_j}} << 1$,

$$p^{i,j} = 1 - \exp\left(-\alpha_j \frac{\lambda^i}{\lambda^{c_j}}\right) \simeq \alpha_j \frac{\lambda^i}{\lambda^{c_j}},$$

and as λ^i goes to $+\infty$, $p^{i,j}$ converges asymptotically to 1.

Calibration of the Market Factor Intensities. Finally, given the parametric form (2.21), the only unknown parameters left are the common market factor intensities, which can be recovered from benchmark basket instruments such as first-to-default swaps, or CDO tranches. Another possibility for calibrating the driver intensities is to use empirical default correlations. For example, one can use the average inter-sector default correlation to fit the Beta driver, then for each sector driver use the average intra-sector default correlation.

Note that the calibration method presented in this section resembles the one used for an HJM model of yield curve dynamics,

$$df\left(t,T\right)=\left(...\right)dt+\sum_{i=1}^{n}\Sigma_{i}\left(t,T\right)dW_{t}^{i},$$

which is also done in three steps:

1. define the number of drivers that explain the dynamics of the yield curve, e.g. $n=3$;
2. specify a parameterization of the volatility curve for each driver:
$$\Sigma_{i}\left(t,T\right)=\sigma_{i}\left(t\right)F_{i}\left(T-t\right);$$
3. calibrate the instantaneous volatility levels $\sigma_{i}\left(t\right)$ on a set of benchmark swaption or cap instruments.

2.6 Gauss vs Marshall-Olkin

In this section, we compare the Marshall-Olkin copula with the standard Gaussian and Student copulas.

Overview. The Gaussian copula is defined as

$$\mathbf{C}\left(u_{1},...,u_{n}\right)=\mathbf{\Phi^{n}}\left(\mathbf{\Phi^{-1}}\left(u_{1}\right),...,\mathbf{\Phi^{-1}}\left(u_{2}\right)\right),$$

where $\mathbf{\Phi^{n}}\left(.\right)$ is the joint distribution function of a normally distributed random vector with unit variances and zero means, and $\mathbf{\Phi^{-1}}\left(.\right)$ is the inverse function of the univariate standard normal distribution. The Gaussian copula can be viewed as the copula function of a set of correlated Gaussian variables transposed back into "uniform" space with the inverse normal function.

The t-copula is defined as

$$\mathbf{C}\left(u_{1},...,u_{n}\right)=t_{\nu}^{n}\left(t_{\nu}^{-1}\left(u_{1}\right),...,t_{\nu}^{-1}\left(u_{2}\right)\right),$$

where $t_\nu^n(.)$ is the joint distribution function of the multivariate student distribution and $t_\nu^{-1}(.)$ is the inverse function of the univariate student distribution. The t-copula is obtained from the multivariate dependence of a set of correlated student variables.

Some key differences between Marshall-Olkin, Gaussian and t-copula are summarized below.

- Tail dependence: the upper tail dependence is defined as

$$\lambda_U = \lim_{u \nearrow 1} \mathbb{P}\left(X > F_X^{-1}(u) \,\middle|\, Y > F_Y^{-1}(u)\right) \tag{2.22}$$

  (see Embrechts, Lindskog and McNeil (2003)). The expression of the tail dependence (2.22) for the three copula functions, Gaussian, t-copula and MO, is given by:

$$\begin{aligned} \lambda_U^{Gaussian} &= 0, \\ \lambda_U^{T-copula} &= 2\left(1 - t_{\nu+1}\left(\sqrt{\nu+1}\frac{\sqrt{1-\rho}}{\sqrt{1+\rho}}\right)\right), \\ \lambda_U^{M.O.} &= \min\left(\frac{\lambda^{\{1,2\}}}{\lambda^{\{1\}}}, \frac{\lambda^{\{1,2\}}}{\lambda^{\{2\}}}\right) \geq \left[\lambda^W\right] \min\left(\frac{1}{\lambda^{\{1\}}}, \frac{1}{\lambda^{\{2\}}}\right). \end{aligned}$$

  The tail dependence for a Gaussian copula is always equal to zero. The t-copula and the Marshall-Olkin copula can be parameterized to fit non-zero tail dependence and to capture more extreme tail events.

- Elliptical copulas do not allow for multiple defaults in the interval $[t, t+dt)$, i.e. $\mathbb{P}(\tau_i = \tau_j) = 0$, for $i \neq j$. In a Marshall-Olkin model, the probability of instantaneous joint defaults can be non-zero. In fact, the foundation of the correlation profile in MO is based on joint instantaneous defaults.

- The mixed partial derivatives of a copula function, $\frac{\partial^k \mathbf{C}}{\partial u_1 ... \partial u_k}$, exist for almost all $\mathbf{u} \in [0,1]^n$. The copula function can then be decomposed into its absolutely continuous part $\mathbf{A}(u_1, ..., u_n)$ and its singular part $\mathbf{S}(u_1, ..., u_n)$:

$$\begin{aligned} \mathbf{C}(u_1, ..., u_n) &= \mathbf{A}(u_1, ..., u_n) + \mathbf{S}(u_1, ..., u_n), \\ \mathbf{A}(u_1, ..., u_n) &= \int_0^{u_1} ... \int_0^{u_n} \frac{\partial^k C(x_1, ..., x_n)}{\partial x_1 ... \partial x_n} dx_1 ... dx_n, \\ \mathbf{S}(u_1, ..., u_n) &= \mathbf{C}(u_1, ..., u_n) - \mathbf{A}(u_1, ..., u_n). \end{aligned}$$

  Elliptical copulas, by construction, are absolutely continuous. The Marshall-Olkin copula has a singular part and a continuous part (see, for example, Embrechts, Lindskog and McNeil (2003)).

- Many analytical results are available for the Marshall-Olkin copula, and the pricing of credit derivatives such as first-to-default swaps can be implemented in closed form. In general, for the Gaussian and t-copula, one needs to use a Monte-Carlo simulation, except in simplified one-factor models (as in Schönbucher (2000) or Frey and McNeil (2003)), where semi-analytical results are available by using the conditional independence property and integrating over values of the conditioning latent variable.

Modes of the Aggregate Default Distribution. Consider a portfolio of 100 obligors with 10 credits per sector. Set the intensities of the individual credits to $\lambda^i = 2\%$, the intensity of the World driver to $\lambda^W = 0.05\%$, the intensity of the Beta driver to $\lambda^B = 5\%$, and all the sector intensities to $\lambda^{S_j} = 2.5\%$. Attribute 60% of the credit intensity to the Beta factor, 20% to the sector factor, and the remaining 20% to the idiosyncratic factor. This implies the following values of the factor loadings: $p^{i,B} = 0.24$, and $p^{i,S_j} = 0.16$. The 5-year default correlation in this model is 19.25% intra-sector, and 16.16% inter-sector[7].

To begin with let us consider the aggregate default distribution at the 5-year time horizon.

We plot the distribution of the portfolio specified here with $\left(p^{i,B} = 0.24;\ p^{i,S_j} = 0.16\right)$, and we compare it with the distribution of a portfolio with similar marginals but a different multivariate dependence $\left(p^{i,B} = 0;\ p^{i,S_j} = 0\right)$.

In Figure (2.1), we observe that the default distribution has four different modes: the first big hump corresponds to the idiosyncratic component, the second hump corresponds to the Beta contribution, the third hump is the sector contribution, and the last spike at the far end of the distribution is due to the world driver.

In the second model depicted in Figure (2.2), the Beta and Sector factors are turned off. The joint dependence is built-in via the world driver. Thus, the default distribution has only a single idiosyncratic mode and the world driver spike. Compared with Figure (2.1), the idiosyncratic mode has shifted to the right since the idiosyncratic default probabilities are higher in this case.

Next, we compare the distribution in Figure (2.1) with the ones of a Gaussian copula and a t-copula. In order to do a meaningful comparison, we impose that the 5-year default correlation is the same for the various models. Since the marginal distributions are unchanged, the mean of the aggregate default distribution is fixed independently from the copula function. The additional requirement to have the same

[7] The numerical values in this example are chosen arbitrarily to exhibit the shape of the distributions and compare the various copulas. For empirical studies of default correlation, we refer the reader to the article by Nagpal and Bahar (2001) and the paper by Servigny and Renault (2002).

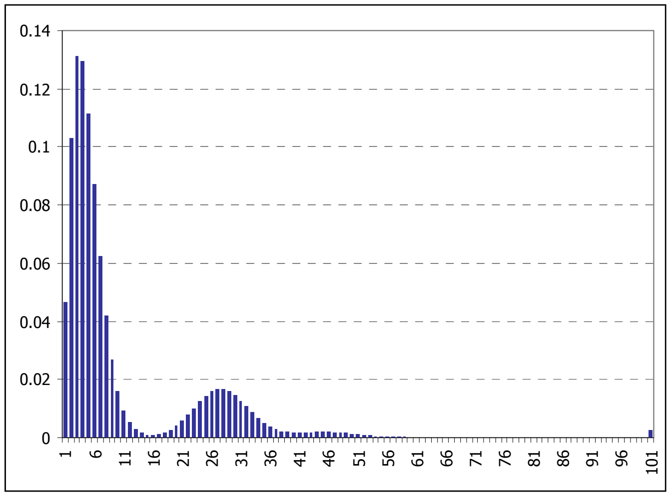

Figure 2.1: Default distribution for a portfolio of 100 credits: $\lambda^i = 2\%$, $\lambda^W = 0.05\%$, $\lambda^B = 5\%$, $\lambda^{S_j} = 2.5\%$, $p^{i,B} = 0.24$ and $p^{i,S_j} = 0.16$.

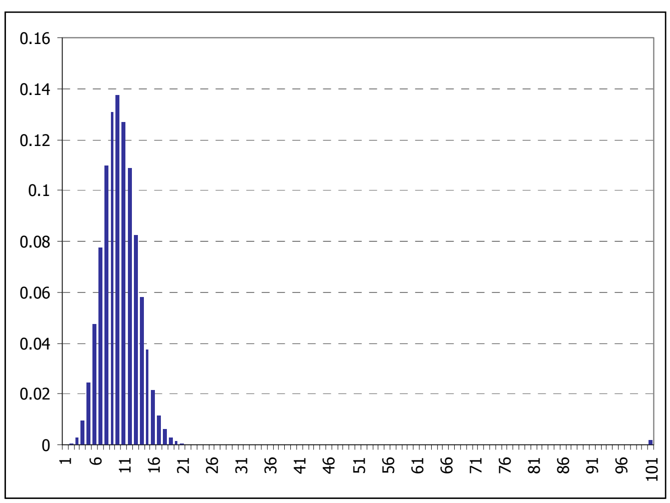

Figure 2.2: Default distribution for a portfolio of 100 credits: $\lambda^i = 2\%$, $\lambda^W = 0.05\%$, $\lambda^B = 5\%$, $\lambda^{S_j} = 2.5\%$, $p^{i,B} = 0$ and $p^{i,S_j} = 0$.

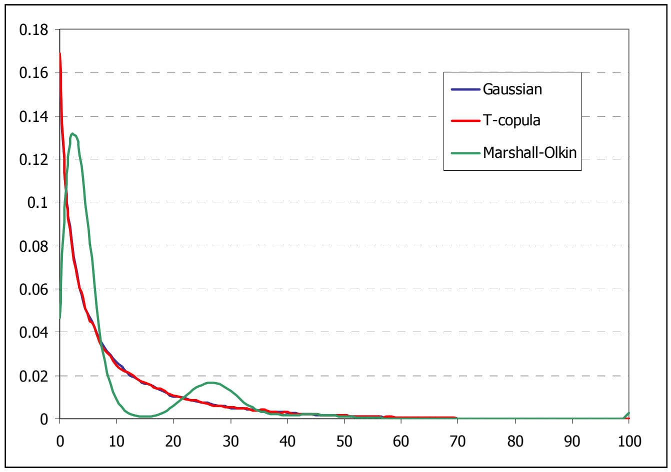

Figure 2.3: Comparison of the default distributions for the calibrated Marshall-Olkin, Gaussian and t-copula.

pair-wise default correlations corresponds to keeping the variance fixed as well. In this example, the 5-year default correlations of the Marshall-Olkin model are $\rho^D_{i,j} = 19.25\%$ intra-sector, and $\rho^D_{i,j} = 16.16\%$ inter-sector. A direct inversion of the default correlation formula, with a Gaussian copula dependence, gives the following values of the Gaussian asset correlation $\rho^A_{i,j} = 41.68\%$ intra-sector, and $\rho^A_{i,j} = 36.39\%$ inter-sector. Doing a similar calibration for a t-copula with a parameter $\nu = 9$, we get $\rho^A_{i,j} = 35.92\%$ intra-sector, and $\rho^A_{i,j} = 30.12\%$ inter-sector. Note that to arrive at the same level of default correlation, the equivalent asset correlation in the t-copula is lower than the one in the Gaussian copula. This is natural since the t-copula has higher tail dependence. In fact, as pointed out in Mashal and Naldi (2002a), even with zero asset correlation, the implied default correlation with the t-copula is non-zero.

Figure (2.3) depicts the default distributions of the three calibrated copula models. Having matched the first two moments of the default distribution, the key difference between the Marshall-Olkin copula and the elliptical copulas is the shape of the distribution function. Marshall-Olkin implies a multi-modal distribution. Gaussian and Student copulas imply uni-modal distributions.

Tail of the Distribution. The Marshall-Olkin and the Gaussian copulas have very different behaviours in the tail of the portfolio distribution. To highlight this difference, we plot the cumulative default distribution of the example portfolio in log-space. We introduce the re-scaled log variable h_k, for $k = 1, 2, ..., n$,

$$h_k \triangleq \frac{-\log\left[\mathbb{P}\left(X_T < k\right)\right]}{T}.$$

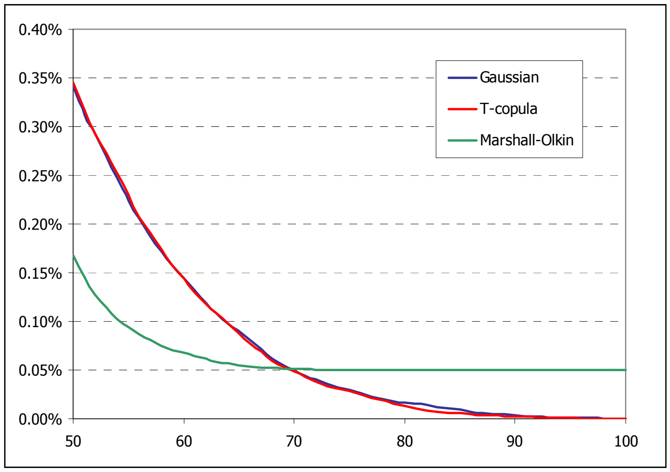

Figure 2.4: Tail of the portfolio default distribution for the Marshall-Olkin, Gaussian and T-copula.

The re-scaled tail measure h_k can be interpreted as the "hazard" rate of the k^{th}-to-default time $\tau^{[k]}$,

$$\mathbb{P}\left(X_T < k\right) = \mathbb{P}\left(\tau^{[k]} > T\right) = \exp\left(-h_k T\right).$$

Figure (2.4) shows that, for the Gaussian and t-copula, h_k converges to zero as we move further into the tail of the distribution. In the Marshall-Olkin model, the values of h_k are floored at 0.05%. This is the effect of the World driver: it suffices to observe that

$$\begin{aligned}\mathbb{P}\left(X_T < k\right) &\leq \mathbb{P}\left(X_T < n\right) = 1 - \mathbb{P}\left(X_T = n\right) \\ &= 1 - \mathbb{P}\left(D_T^W = 1\right) \\ &= \exp\left(-\lambda^W T\right).\end{aligned}$$

Here, we have used the property that all credits default if and only if the world driver triggers, i.e., $\left\{D_T^W = 1\right\}$. This part of the distribution is precisely the one that determines the value of the extreme events and catastrophe risk. The world driver plays a unique role since it can be used to match insurance premiums of super senior risk. This cannot be achieved with a Gaussian copula.

Time Invariance. Another major difference between Marshall-Olkin and the Gaussian copula is the "time" behaviour. Consider an example with two obligors, and a one-factor MO model:

$$\lambda^i = p^{i,c}\lambda^c + \lambda^{0,i}.$$

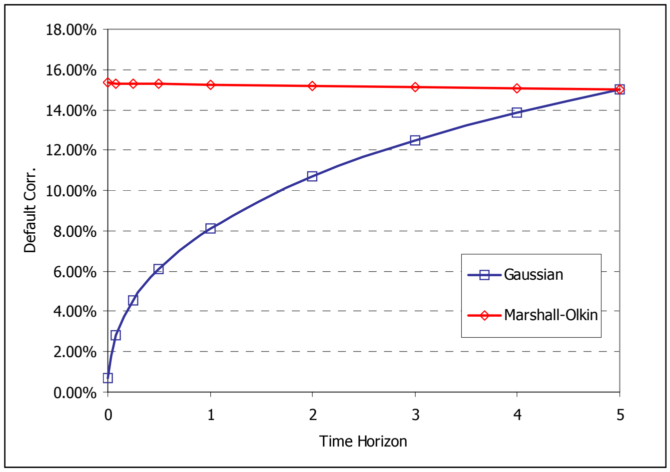

Figure 2.5: Default correlation as a function of the time horizon for the Gaussian and Marshall-Olkin copulas.

Set the intensities to $\lambda^1 = \lambda^2 = \lambda^c = 1\%$, and the factor loadings to $p^{1,c} = p^{2,c} = 0.3915$. The 5-year default correlation is equal to 15%. For the equivalent Gaussian copula, set the asset correlation to $\rho^A = 41.04\%$ in order to match the 15% default correlation at the 5-year time horizon. With this specification, compute the default correlation at other time horizons between 0 and 5 years, and compare the implied term structures.

Figure (2.5) shows that the Marshall-Olkin default correlation is stable through time. This is not surprising, since, as mentioned before, the multivariate exponential distribution is memoryless, therefore the T-default correlation estimated at time t would be the same as the $(T - t)$-default correlation at time 0. The Gaussian copula, on the other hand, is highly time-dependent. The upward sloping shape of its default correlation term structure means that a first-to-default swap, for example, would become cheaper as time goes by, even if the underlying credit spreads remain unchanged. At time $t = 0$, the 5-year FTD basket would be priced at 15% default correlation. Then, after one year, the maturity of the FTD becomes 4 years, which corresponds to a default correlation of 13.8%. And at time $t = 4$ year, the same basket becomes a 1-year trade and would be marked at 8% default correlation. Rogge and Schönbucher (2003) point out the same deficiency of the Gaussian copula by analyzing the size of the default contagion as a function of time.

2.7 Correlation Skew

In this section, we discuss how the Marshall-Olkin copula can be used to match the correlation skew of the CDO market.

Overview. Over the last few years, we have seen an increased liquidity in CDO tranche trading, which resulted in an observable market of default correlation. Dealers are starting to quote a two-way market on a pre-specified set of tranches referenced to a given index portfolio. By inverting the Gaussian copula formula, one finds the implied level of correlation that would match the quoted tranche premiums. As with the Black-Scholes option model, there is not a single correlation number that would match all tranches at various attachement points. Supply and demand factors combined with the credit views and risk appetite of the market participants would explain the discrepancy of correlations across the capital structure. The example below gives the market bid/offer premiums of the European iTraxx index tranches.

0-3%	23.3	24.3
3-6%	134	137
6-9%	44	47
9-12%	28	32.3
12-22%	14.2	15.5

The index level is 37 bps. The (0-3%) tranche is quoted in points upfront for a tranche paying 500 bps running.

Next, we explain some concepts such as base correlation and compound correlation in a formal manner.

One-Factor Gaussian Copula. We give a formal definition of the one-factor Gaussian copula function.

Definition 24 *(One-factor Gaussian Copula). The one-factor Gaussian copula with parameter $\rho \in [0,1)$ is defined as*

$$C(u_1, ..., u_n) \triangleq \int_{-\infty}^{+\infty} \left[\prod_{i=1}^{n} \Phi\left(\frac{\Phi^{-1}(u_i) - \sqrt{\rho} y}{\sqrt{1-\rho}} \right) \right] \phi(y)\, dy,$$

where $\Phi(.)$, $\Phi^{-1}(.)$ and $\phi(.)$ are the standard normal distribution function, its inverse and its density function respectively.

This formal definition can be understood by considering a simplified firm value model as in Schönbucher (2000) for example. The default of obligor i is triggered when the asset value of the firm, denoted V_i are below a given threshold. V_i is assumed to be normally distributed. The relationship between default and the asset value is given by

$$\{\tau_i \leq T\} \Longleftrightarrow \left\{V_i \leq \Phi^{-1}\left(\mathbb{P}\left(\tau_i \leq T\right)\right)\right\}.$$

The asset values of different obligors are correlated. Their joint dependence is defined via a common factor Y, which follows a standard normal distribution, and idiosyncratic standard normal noises $\epsilon_1, ..., \epsilon_n$:

$$V_i \triangleq \sqrt{\rho} Y + \sqrt{1-\rho}\epsilon_i,$$

where Y and $\epsilon_1, ..., \epsilon_n$ are i.i.d. standard normally distributed. The linear correlation between the asset values of two obligors is ρ. This coefficient, which is used to parameterize the family of one-factor Gaussian copulas, is sometimes called an asset correlation. Conditional on a given value of the systemic factor Y, the asset values are independent; hence, the default times are independent as well. This is the set-up of a conditionally independent defaults model.

One can write down the default times' copula function by conditioning on Y and using the law of iterated expectations:

$$\begin{aligned}
& \mathbb{P}\left(\tau_1 \leq T_1, ..., \tau_n \leq T_n\right) \\
= & \int_{-\infty}^{+\infty} \mathbb{P}\left(\tau_1 \leq T_1, ..., \tau_n \leq T_n \left| Y = y\right.\right) \phi\left(y\right) dy \\
= & \int_{-\infty}^{+\infty} \mathbb{P}\left(V_1 \leq \Phi^{-1}\left(\mathbb{P}\left(\tau_1 \leq T_1\right)\right), ..., V_n \leq \Phi^{-1}\left(\mathbb{P}\left(\tau_n \leq T_n\right)\right) \left| Y = y\right.\right) \phi\left(y\right) dy \\
= & \int_{-\infty}^{+\infty} \mathbb{P}\left(\epsilon_1 \leq \frac{\Phi^{-1}\left(\mathbb{P}\left(\tau_1 \leq T_1\right)\right) - \sqrt{\rho}y}{\sqrt{1-\rho}}, ..., \epsilon_n \leq \frac{\Phi^{-1}\left(\mathbb{P}\left(\tau_n \leq T_n\right)\right) - \sqrt{\rho}y}{\sqrt{1-\rho}}\right) \phi\left(y\right) dy \\
= & \int_{-\infty}^{+\infty} \left[\prod_{i=1}^{n} \Phi\left(\frac{\Phi^{-1}\left(u_i\right) - \sqrt{\rho}y}{\sqrt{1-\rho}}\right)\right] \phi\left(y\right) dy.
\end{aligned}$$

The one-factor Gaussian copula is the standard model used to quote CDO tranches in the market.

Pricing CDOs. Let us consider the pricing of a CDO tranche, which covers the losses of a given portfolio between two thresholds $0 \leq K_1 < K_2 \leq 1$.

Letting δ_i denote the recovery rate of obligor i, we define the portfolio loss process as

$$L_t \triangleq \frac{1}{n} \sum_{i=1}^{n} \left(1 - \delta_i\right) D_t^i.$$

The loss on the tranche (K_1, K_2) is defined as

$$M_t^{K_1,K_2} = \min\left(\max\left(L_t - K_1, 0\right), K_2 - K_1\right).$$

The processes L_t and $M_t^{K_1,K_2}$ are pure jump processes. The CDO payments correspond to the increments of $M_t^{K_1,K_2}$, i.e., there is a payment when the process $M_t^{K_1,K_2}$ jumps, which happens at every default time. The payoff of the protection leg of a CDO is therefore defined as the Stieljes integral

$$\text{protection\_leg} \triangleq \int_{]0,T]} \exp\left(-\int_0^t r_s ds\right) dM_t^{K_1,K_2}.$$

Letting $(T_0 = 0, T_1, ..., T_N)$ denote the cashflow dates, $\Delta T_i \triangleq T_i - T_{i-1}$, the payment fractions and S the tranche premium, the payoff of the premium leg is defined as:

$$\text{premium\_leg} \triangleq S \times \sum_{i=1}^{n} \exp\left(-\int_0^{T_i} r_s ds\right)\left[(K_2 - K_1) - M_{T_i}^{K_1,K_2}\right]\Delta T_i.$$

The value of the CDO tranche is given by the expected value of the discounted payoff under a risk neutral measure.

Assume deterministic interest rates and let $B(0,T) \triangleq \exp\left(-\int_0^T r_s ds\right)$ denote the discount factor maturing at time T. Using the integration by part formula and Fubini's theorem to interchange the order of integration, we can re-write the protection integral as

$$\mathbb{E}\left[\int_{]0,T]} \exp\left(-\int_0^t r_s ds\right) dM_t^{K_1,K_2}\right] = B(0,t)\,\mathbb{E}\left[M_T^{K_1,K_2}\right] - \int_0^T \frac{\partial B(0,t)}{\partial t}\mathbb{E}\left[M_t^{K_1,K_2}\right] dt,$$

Similarly, to compute the value of the premium leg, we need to know the expected tranche losses at times T_i: $\mathbb{E}\left[M_{T_i}^{K_1,K_2}\right]$.

The pricing of CDO tranches boils down to computing the values of all "tranchelets":

$$C_t(K_1, K_2) \triangleq \mathbb{E}\left[M_t^{K_1,K_2}\right], \text{ for } 0 \le t \le T. \tag{2.23}$$

For $t \ge 0$, if we know the density function $f_t(.)$ of the portfolio loss L_t:

$$f_t(x) \triangleq \mathbb{P}\left(L_t \in dx\right), \tag{2.24}$$

then, the expectation (2.23) is given by

$$\mathbb{E}\left[M_t^{K_1,K_2}\right] = \int_{K_1}^{K_2} (x - K_1) f_t(x)\,dx + (K_2 - K_1)(1 - F_t(x)), \tag{2.25}$$

where $F_t(x) = \int_{-\infty}^{x} f_t(z)\,dz$ is the cumulative probability function of L_t. With a given copula, such as the one-factor copula, it is easy to compute the density function $f_t(.)$

using techniques such as the FFT (see Laurent and Gregory (2002)) or the convolution recursion (see Andersen, Sidenius, Basu (2003)).

Compound Correlation. As mentioned earlier, the one-factor copula has been used by dealers to quote the standardized CDO tranches traded in the market. Since the prices of various tranches are driven by supply and demand, a single correlation parameter is not sufficient to reproduce market prices. Inverting the pricing formula of the one-factor Gaussian copula, one would find the implied correlation, which matches the market price of each tranche. This implied correlation is referred to as "Compound Correlation".

Definition 25 *(Compound Correlation). For a given CDO tranche with attachment points (K_1, K_2) and quoted premium S^{K_1,K_2}, let $G^{K_1,K_2}(S, \rho)$ denote the model price using the one-factor Gaussian copula with parameter ρ. We call compound correlation, the value of the parameter ρ such that*

$$G^{K_1,K_2}\left(S^{K_1,K_2}, \rho\right) = 0. \tag{2.26}$$

If a compound correlation exists, i.e., if the mapping (2.26) is invertible, then this offers a way to compare different tranches on a relative value basis. Unfortunately, it turns out that the model price is not a monotonic function of compound correlation. Therefore, it is not guaranteed that we can always find a solution. Moreover, in some instances, we can find more than one value of correlation, which satisfies (2.26). This usually happens with Mezzanine tranches, which are not correlation sensitive. This behaviour is well documented (see McGinty, Beinstein, Ahluwalia, Watts (2004)) and has motivated the base correlation approach that we describe next.

Solving for compound correlations in the previous example, we get the following results.

0-3%	20.08%	18.57%
3-6%	5.92%	6.17%
6-9%	13.56%	14.19%
9-12%	20.82%	22.42%
12-22%	29.54%	30.43%

Base Correlation. One can view each CDO tranche with attachment points (K_1, K_2) as the difference between two equity tranches: $(0, K_2)$ and $(0, K_1)$. This can be checked easily from the definition of the payoff:

$$M_T^{K_1,K_2} = M_T^{0,K_2} - M_T^{0,K_1}, \text{ for all } T \geq 0.$$

Therefore, to price any CDO tranche it suffices to have the whole continuum of equity tranches $(0, K)$, for $K \in [0, 1]$. Each one of these equity tranches can be valued with a different one-factor Gaussian copula correlation $\rho(0, K)$. The function $\rho(0, K)$: $[0, 1] \to [0, 1]$ is called the "Base Correlation" curve.

Definition 26 *(Base Correlation). The base correlation curve is a function* $\rho(0, K)$: $[0, 1] \to [0, 1]$, *which parameterizes the prices of all equity tranches* $(0, K)$. *In other words, the price of the* $(0, K)$*-tranche is given by the one-factor Gaussian copula model with parameter* $\rho(0, K)$.

Furthermore, the value of any tranche with attachment points (K_1, K_2) *and quoted premium* S^{K_1,K_2}, *is given by*

$$G^{0,K_2}\left(S^{K_1,K_2}, \rho(0, K_2)\right) - G^{0,K_1}\left(S^{K_1,K_2}, \rho(0, K_1)\right). \tag{2.27}$$

Using the standard tranches quoted in the market, one would proceed with a bootstrapping algorithm to find the base correlation curve, which reproduces the market quotes. The popularity of this method lies in the fact that the function

$$h(x) = G^{0,K}(S, x), \text{ for a given } K \in [0, 1] \text{ and } S \in \mathbb{R}_+,$$

is monotonic. Hence, we can always invert the relationship (2.27) for each attachment point.

Mathematically, base correlation is just another way of parameterizing the density function $f_T(.)$ of the portfolio loss L_T. Indeed, given a base correlation curve $(\rho(0, K))_{0 \leq K \leq 1}$, one can compute the value of all "tranchelets" $(C_T(0, K))_{0 \leq K \leq 1}$:

$$C_T(0, K) \triangleq \mathbb{E}\left[M_T^{0,K}\right].$$

Assuming that, for $T \geq 0$, the function $K \to C_T(0, K)$ is $\mathcal{C}^2$, we can recover the density function as:

$$f_T(K) = -\frac{\partial^2 C_T(0, K)}{\partial K^2}. \tag{2.28}$$

This follows directly from equation (2.25). This is similar to the Breeden & Litzenberger (1978) formula in options theory where the implied density of the forward stock price is obtained from the continuum of call prices at different strikes.

Solving for base correlations in the previous example, we get the following results.

0-3%	20.08%	18.57%
0-6%	29.60%	27.43%
0-9%	37.10%	34.12%
0-12%	42.54%	38.50%
0-22%	56.04%	49.28%

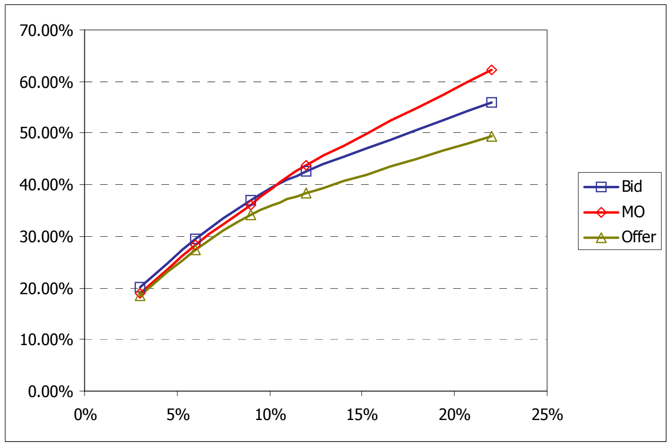

Figure 2.6: Base Correlation Skew.

Marshall-Olkin Skew. As mentioned earlier, because of the multi-modality of the Marshall-Olkin loss distribution, it is possible to use each mode of the distribution to match various parts of the capital structure. Figure (2.1), for example, suggests that the idiosyncratic hump can be used to match the equity tranche (0-3%), the Beta hump can be used to match the mezzanine tranches (3-6%, 6-9% and 9-12%), and the World driver can be used to match the senior tranche (12-22%). Additional tweaking of the calibration can also be done with sector drivers.

Figure (2.6) shows the results of the calibration using a MO model with one common Beta driver and the World driver.

Using the Beta driver, we can match accurately most of the equity and mezzanine tranches. The senior tranches are more sensitive to extreme events and require additional common factors to have a better market fit. Here our intent is solely to show that the multi-modality feature of the MO copula generates a correlation skew curve, which mirrors the one observed in the market. A precise study of the market calibration is outside the scope of this chapter.

2.8 Conclusion

We have presented in this chapter the Marshall-Olkin copula in the context of default correlation modelling. We have proposed a calibration procedure to fit this rich correlation structure to an intuitively sound market dynamic. And we have shown that MO offers some desirable features that make it an eligible alternative to the Gaussian copula. The comparison between MO and the Gaussian copula is similar in many ways to the evolution from Black-Scholes to term structure models in fixed income markets. Black-Scholes has been used as the model of choice by a lot of traders because of its simplicity. It converts one volatility number to a price. However, there is no guarantee that an exogenous BS swaption matrix is arbitrage-free or at least self-consistent. On the other hand, a calibrated HJM model, which is built upon a defined set of yield curve deformations or drivers, is self-consistent by construction. The Gaussian copula can be viewed as the Black-Scholes of default correlation. The Marshall-Olkin approach corresponds to an HJM framework. Once the market factors are calibrated, all combinations of sub-baskets can be priced consistently in this calibrated term-structure of default inter-dependence.

Chapter 3

Quadratic Hedging

In this chapter, we present a methodology for hedging basket credit derivatives with single name instruments. Because of the market incompleteness due to the residual correlation risk, perfect replication cannot be achieved. We allow for mean self-financing strategies and use a risk-minimization criterion to find the hedge.

3.1 Introduction

In this chapter, we address the problem of hedging basket credit derivatives with single name instruments. Typical basket products such as first-to-default swaps and CDOs reference a pool of underlying credit entities and their payoff is dependent on the joint default behaviour of the underlying basket. This introduces a default correlation risk, which makes the multi-credit market incomplete: a basket product cannot be completely replicated with single name instruments. Other correlation sensitive instruments are required to offset the residual correlation risk. Furthermore, credit securities have two potential sources of risk: spread risk and default risk. In general, one cannot hedge both at the same time. The hedger would use his judgement and focus mainly on one source of risk depending on the prevailing market conditions. This bi-modal nature of the credit markets introduces another level of complexity to the default correlation incompleteness.

One approach to handle market incompleteness is to use quadratic optimality criteria. Quadratic hedging approaches, such as local risk-minimization and mean-variance hedging, have been developed in a series of papers by Föllmer, Schweizer and Sondermann (see, for example, Föllmer and Sondermann (1986), Föllmer and Schweizer (1991), Schweizer (1993), Schweizer (1994)). Other criteria for determining an optimal hedging strategy have also been developed in the literature. See, for example, El Karoui and Quenez (1995), Davis (1997), Cvitanic (1997). Here, we use the risk-minimization approach of Föllmer and Sondermann to find the optimal hedging strategy for basket products. This analysis is done in the Marshall-Olkin copula framework, where each individual default process is decomposed on a basis of independent Cox processes. The so-called common market factors can trigger joint defaults in the basket, whereas idiosyncratic factors, on the other hand, can only trigger individual defaults. The market incompleteness is the result of the M.O. model, which has simultaneous defaults and hence a particularly large mark space for the point process representing defaults.

This chapter is organized as follows. In Section 3.2, the model is described. In Section 3.3, we formulate the problem to be addressed. In Section 3.4, we present the "Equivalent Fatal Shock" model. In Section 3.5, we derive the dynamics of the zero-coupon defaultable bonds used for hedging. In Section 3.6, we give the martingale representation of the basket contingent claim price process. In Section 3.7, we derive the risk-minimizing hedging strategy.

3.2 The Model

We work in a financial market represented by a probability space $(\Omega, \mathcal{G}, P^*)$ and a time horizon $T^* \in (0, \infty)$, on which is given a d-dimensional Brownian motion W and n non-negative random variables $(\tau_1, ..., \tau_n)$ representing the default times of the obligors in the economy.

Assumption 1. We assume that P^* is a (risk-neutral) martingale measure. Throughout, we shall work under this martingale measure. We follow the approach of Föllmer and Sondermann (1986) where a "good" martingale measure is chosen, then the minimization of the risk is done with respect to this measure.

Assumption 2. We introduce an $\mathbb{R}^d$-valued Itô process X, describing the state variables in the economy, which solves the following SDE:

$$dX_t = \alpha(X_t)\, dt + \beta(X_t)\, dW_t,$$

for some Lipschitz functions $\alpha_k : \mathbb{R}^d \to \mathbb{R}$ and $\beta_{kj} : \mathbb{R}^d \to \mathbb{R}^d$, $1 \leq k \leq d$, $1 \leq j \leq d$.

We denote by $\{\mathcal{F}_t\}$ the filtration generated by X and augmented with the P-null sets of $\mathcal{G}$:

$$\mathcal{F}_t \triangleq \sigma(X_s : 0 \leq s \leq t) \vee \mathcal{N}.$$

We introduce, for each obligor i, the right-continuous process $D_t^i \triangleq \mathbf{1}_{\{\tau_i \leq t\}}$ indicating whether the firm has defaulted or not. We denote by $\{\mathcal{H}_t^i\}$ the filtration generated by this process:

$$\mathcal{H}_t^i \triangleq \mathcal{F}_t^{D^i} \triangleq \sigma\left(D_s^i : 0 \leq s \leq t\right).$$

The agents' filtration is the one generated by the economic state variables and the default processes

$$\mathcal{G}_t \triangleq \mathcal{F}_t \vee \left[\bigvee_{i=1}^{n} \mathcal{H}_t^i\right]. \tag{3.1}$$

Assumption 3. We assume that the default times are correlated and we allow for multiple instantaneous joint defaults. The multivariate dependence is defined by a Marshall-Olkin copula.

More precisely, we assume that there exists a set of m independent Cox processes $\left(N_t^{c_j}\right)_{t\geq 0}$ with continuous bounded intensities $\lambda^{c_j}(X_t)$, $\lambda^{c_j} : \mathbb{R}^d \to \mathbb{R}_+$, which can trigger simultaneous joint defaults.

Each Cox process N^{c_j} can be equivalently represented by the sequence of event trigger times $\left\{\theta_r^{c_j}\right\}_{r\in\{1,2,\ldots\}}$.

For every event type c_j, and for all $t \geq 0$, we define a set of independent Bernoulli variables $\left(A_t^{1,j}, \ldots, A_t^{n,j}\right)$ with probabilities $\left(p^{1,j}(X_t), \ldots, p^{n,j}(X_t)\right)$, where $p^{i,j} : \mathbb{R}^d \to [0,1]$ are some continuous functions of the background process.

We assume that for $j \neq k$, the vectors $\mathbf{A}_t^j = \left(A_t^{1,j}, \ldots, A_t^{n,j}\right)$ and $\mathbf{A}_t^k = \left(A_t^{1,k}, \ldots, A_t^{n,k}\right)$ are independent.

We assume that for $t \neq s$, the vectors $\mathbf{A}_t^j = \left(A_t^{1,j}, \ldots, A_t^{n,j}\right)$ and $\mathbf{A}_s^j = \left(A_s^{1,j}, \ldots, A_s^{n,j}\right)$ are independent.

At the r^{th} occurrence of an event of type c_j, at time $\theta_r^{c_j}$, we draw the set of independent $\{0,1\}$-valued variables $\left(A_{\theta_r^{c_j}}^{1,j}, \ldots, A_{\theta_r^{c_j}}^{n,j}\right)$; the variable $A_{\theta_r^{c_j}}^{i,j}$ indicates whether a default of type i has occurred or not.

The process $\left(N_t^i\right)_{t\geq 0}$ defined as

$$N_t^i \triangleq \sum_{j=1}^{m} \sum_{\theta_r^{c_j} \leq t} A_{\theta_r^{c_j}}^{i,j}, \tag{3.2}$$

is also a Cox process with intensity

$$\lambda^i(X_t) \triangleq \sum_{j=1}^{m} p^{i,j}(X_t)\, \lambda^{c_j}(X_t).$$

It is obtained by superpositioning m independent (thinned) Cox processes.

The default time τ_i is defined as the first jump time of the Cox process $\left(N_t^i\right)_{t\geq 0}$:

$$\tau_i \triangleq \inf\left\{t : N_t^i > 0\right\}. \tag{3.3}$$

This common shock model can also be described formally by the following SDE

$$dD_t^i = \left(1 - D_{t^-}^i\right) \sum_{j=1}^{m} A_t^{i,j} dN_t^{c_j}. \tag{3.4}$$

The information flow in the Marshall-Olkin model is much larger that the one accessible to agents. It contains the evolution of the common trigger events and the "conditional" Bernoulli events:

$$\widetilde{\mathcal{G}}_t = \mathcal{F}_t \vee \left[\bigvee_{j=1}^{m} \mathcal{F}_t^{N^{c_j}}\right] \vee \left[\bigvee_{j=1}^{m} \bigvee_{i=1}^{n} \mathcal{F}_t^{A^{i,j}}\right]. \tag{3.5}$$

3.3 The Problem

In our economy, we assume that we have $(n+1)$ primary assets available for hedging with price processes $S^i = \left(S_t^i\right)_{0 \le t \le T^*}$. The first asset S^0 is the money-market account, i.e., $S_t^0 \triangleq \exp\left(\int_0^t r_s ds\right)$ for some $\mathcal{F}_t$-adapted process r. S^0 will be used as numeraire and all quantities will be expressed in units of S^0. In particular, S^0 will be equal to 1 at all times. We shall consider only zero-coupon credit derivatives or contingent claims of the European type. The hedging asset S^i will represent the zero-coupon defaultable bond maturing at T linked to obligor i; i.e., it pays 1 if obligor i survives until time T, or 0 otherwise. The payoff at maturity is defined as:

$$S_T^i \triangleq 1 - D_T^i.$$

In practice, zero-coupon defaultable bonds are not traded in the market. They can, however, be extracted from the prices of liquid default swap instruments with different maturities. Given some recovery rate assumption and an interpolation method between maturities, a bootstrapping algorithm can be used to extract the value of zero-coupon bonds.

We shall consider here the problem of pricing and hedging zero-coupon contingent claims by dynamically trading the hedging assets S. The contingent claims in this context include credit derivatives of the basket type.

Definition 27 *(Contingent Claim). A contingent claim is a $\mathcal{G}_T$-measurable random variable $H_T \in L^2(P^*)$ describing the payoff at maturity T of a financial instrument.*

A well-known example is a k^{th}-to-default (zero-coupon note) maturing at T. Its payoff is defined as:

$$H_T^{(k)} \triangleq \mathbf{1}_{\left\{\sum_{i=1}^{n} D_T^i < k\right\}},$$

it will pay 1 if there are less than k defaults in the basket or 0 otherwise. The most common structure in this category is a first-to-default, $H_T^{(1)}$, which pays 1 if no obligor in the basket defaults before T. Another popular example of contingent claims is a CDO (zero-coupon note). If we assume that the recovery rate for obligor i is a constant proportion $0 \le R^i \le 1$, then the payoff of a CDO tranche covering the portfolio losses, which fall in some range $[K_1, K_2]$, where $0 \le K_1 < K_2 \le 1$, is

$$H_T^{(K_1,K_2)} \triangleq \frac{1}{K_2 - K_1} \min\left(\max\left(\frac{1}{n}\sum_{i=1}^{n}\left(1 - R^i\right) D_T^i - K_1, 0\right), K_2 - K_1\right).$$

As usual, the problem of the seller of this contract is twofold: (a) how much should he charge for this security at time 0, (b) how can he cover his position with the available hedging instruments.

This is formalized by considering dynamic strategies $(\alpha, \eta) = (\alpha_t, \eta_t)_{0 \le t \le T^*}$, where α is a n-dimensional predictable process and η is adapted. α_t^i represents the number of units of the single-name hedge instrument i held at time t, and η_t is the amount of money invested in the cash account. The value of the portfolio is given by $V_t = (\alpha_t)^{tr} S_t + \eta_t$, and the cumulative gains up to time t are $G_t(\alpha) = \int_{]0,t]} (\alpha_u)^{tr} dS_u$. The cost process is $C_t = V_t - G_t(\alpha)$.

For non-attainable claims, a risk-minimizing strategy is characterized by: (a) its cost process must be a martingale, (b) the cost process is orthogonal to S. As shown in Föllmer and Sondermann (1986), the hedging strategy is obtained by the Kunita-Watanabe decomposition of the $\{\mathcal{G}_t\}$-martingale $H_t = \mathbb{E}^*[H_T | \mathcal{G}_t]$:

$$H_T = H_0 + \int_{]0,T]} \left(\alpha_t^{H_T}\right)^{tr} dS_t + L_T^{H_T},$$

where L^{H_T} is a martingale orthogonal to S.

Our goal is to find an analytical result for the strategy $\left(\alpha_t^{H_T}\right)$.

3.4 Marked Point Process Representation

In this section, we present the "Equivalent Fatal Shock Model" (see Lindskog and McNeil (2003)) used to equivalently describe the Marshall-Olkin set up introduced in Section 3.2. This provides an explicit representation of the marked point process, which will be used throughout.

Equivalent Fatal Shock Model. Let $\mathbf{\Pi_n}$ be the set of all subsets of $\{1, ..., n\}$. For each $\pi \in \mathbf{\Pi_n}$, we introduce the point process N_t^π, which counts the number of shocks in $(0, t]$ resulting in joint defaults of the obligors in π only. For example, if

$\pi = \{1,2\}$, then the process $N_t^{\{1,2\}}$ counts the shocks which trigger simultaneous defaults of obligor 1 and 2 but not the other obligors 3 to n. The process N_t^π is, then, formally defined as

$$N_t^\pi \triangleq \sum_{j=1}^{m} \sum_{r=1}^{N_t^{c_j}} A_{\theta_r^{c_j}}^{\pi,j}, \tag{3.6}$$

where

$$A_t^{\pi,j} \triangleq \prod_{i\in\pi} A_t^{i,j} \prod_{i\notin\pi} \left(1 - A_t^{i,j}\right). \tag{3.7}$$

For $t \geq 0$, $A_t^{\pi,j}$ is a Bernoulli variable, which is equal to 1 if all obligors $i \in \pi$ default and all the others survive. So, at the occurrence of the r^{th} common shock, of type c_j, at time $\theta_r^{c_j}$, the point process N_t^π gets incremented by $\Delta N_{\theta_r^{c_j}}^\pi = A_{\theta_r^{c_j}}^{\pi,j}$.

Note that for $\pi = \{i\}$, $N_t^{\{i\}}$ should not be confused with N_t^i:

$$N_t^i \neq N_t^{\{i\}};$$

N_t^i counts all the shocks that affect obligor i and may affect other obligors as well, on the other hand $N_t^{\{i\}}$ counts the shocks affecting obligor i only and do not affect the others.

Next, we state the key result of the fatal shock representation. We refer to Lindskog and McNeil (2003) for details (see Proposition 4).

Proposition 28 *(Fatal shock representation).* $(N^\pi)_{\pi\in\mathbf{\Pi_n}}$ *are Cox processes, with intensities*

$$\lambda^\pi (X_t) = \sum_{j=1}^{m} p^{\pi,j} (X_t)\, \lambda^{c_j} (X_t),$$

where

$$p^{\pi,j} (X_t) = \prod_{i\in\pi} p^{i,j} (X_t) \prod_{i\notin\pi} \left(1 - p^{i,j} (X_t)\right).$$

Moreover, conditional on $\mathcal{F}_\infty$*, the processes* $(N^\pi)_{\pi\in\mathbf{\Pi_n}}$ *are independent.*

We summarize in the following lemma the equivalent description of each obligor i using the fatal shock representation.

Lemma 29 *(Obligor description using the fatal shock representation).*

The Cox process N^i *is given by*

$$N_t^i = \sum_{\pi\in\mathbf{\Pi_n}} \mathbf{1}_{\{i\in\pi\}} N_t^\pi, \tag{3.8}$$

and its intensity can be expressed as the sum[1]

$$\lambda^i(X_t) = \sum_{\pi\in\mathbf{\Pi_n}} \mathbf{1}_{\{i\in\pi\}}\lambda^\pi(X_t). \tag{3.9}$$

Marked Point Process Representation. The Marshall-Olkin model is defined on the filtration $\left\{\widetilde{\mathcal{G}}_t\right\}$, which is larger than the one available to investors, namely $\{\mathcal{G}_t\}$. We shall use the generic tools of the MO model, however, the local characteristics of the MPP representation are derived for $\{\mathcal{G}_t\}$.

We define the sequence of ordered default times $(T_0, T_1, ..., T_n) : T_0 = 0 \leq T_1 \leq ... \leq T_n$, and identities of the defaulted obligors as:

$$\begin{aligned} T_0 &= 0,\ Z_0 = \emptyset; \\ T_k &= \min\{\tau_i : 1 \leq i \leq n,\ \tau_i > T_{k-1}\}; \\ Z_k &= \pi \text{ if } T_k = \tau_i \text{ for all } i \in \pi, \text{ and } \pi \in \mathbf{\Pi_n}; \end{aligned}$$

The mark space of this point process is $E \triangleq \mathbf{\Pi_n}$, the set of all subsets of $I_n \triangleq \{1, ..., n\}$.

The double sequence $(T_k, Z_k)_{k\geq1}$ defines a marked point process with counting measure

$$\begin{aligned} \mu(\omega, dt\times dz) &: (\Omega, \mathcal{G}) \to ((0,\infty)\times E, (0,\infty)\otimes\mathcal{E}), \\ \int_0^t \int_E H(\omega, t, z)\,\mu(\omega, dt\times dz) &= \sum_{k\geq1} H(\omega, T_k(\omega), Z_k(\omega))\,\mathbf{1}_{\{T_k(\omega)\leq t\}}. \end{aligned}$$

The MPP $(T_k, Z_k)_{k\geq1}$ can also be described through a family of counting processes $(D_t(\pi))_{\pi\in\mathbf{\Pi_n}}$ defined as: $D_t(\pi) \triangleq \sum_{k\geq1}\mathbf{1}_{\{T_k\leq t, Z_k=\pi\}}$; $D_t(\pi)$ counts the number of events on $(0,t]$ matching a mark equal to the subset π. We can also define $\tau(\pi) \triangleq \sum_{k\geq1} T_k\mathbf{1}_{\{Z_k=\pi\}}$ as the default time where all obligors in π default simultaneously. If we denote by $(\lambda_t^\pi)_{t\geq0}$ the $(P^*, \{\mathcal{G}_t\})$-intensity of the default time $\tau(\pi)$, then the $(P^*, \{\mathcal{G}_t\})$-intensity kernel of the counting measure $\mu(dt\times dz)$ is given by

$$\lambda_t(\omega, dz)\,dt = \lambda_t(\omega)\,\Phi_t(\omega, dz)\,dt,$$

where $(\lambda_t)_{t\geq0}$ is the non-negative $\{\mathcal{G}_t\}$-predictable process

$$\lambda_t = \sum_{\pi\in\mathbf{\Pi_n}} \lambda_t^\pi, \tag{3.10}$$

and $\Phi_t(\omega, dz)$ is the probability transition kernel from $(\Omega\times[0,\infty), \mathcal{G}\otimes\mathcal{B}_+)$ into $(E, \mathcal{E})$

$$\Phi_t(\omega, \pi) = \frac{\lambda_t^\pi}{\lambda_t}, \text{ for } \pi\in\mathbf{\Pi_n}, \tag{3.11}$$

[1]Here, we have used the fact that if N^i has intensity λ^i, then $\sum a_i N^i$ has intensity $\sum a_i\lambda^i$ when the intensities are calculated in the filtration generated by all N^i.

with $\Phi_t(.) = 0$ if $\lambda_t = 0$. The pair $(\lambda_t, \Phi_t(dz))$ is the $(P^*, \{\mathcal{G}_t\})$-local characteristics of the counting measure $\mu(dt \times dz)$.

For each subset $\pi \in \mathbf{\Pi_n}$, the intensity $(\lambda_t^\pi)_{t\geq 0}$ can be computed as

$$\lambda_t^\pi = \lim_{h\downarrow 0} \frac{1}{h} \mathbb{P}^* \left(D_{t+h}(\pi) - D_t(\pi) \left| \mathcal{G}_t \right. \right) = -\frac{\partial}{\partial T} \mathbb{P}^* \left(\tau(\pi) > T \left| \mathcal{G}_t \right. \right)|_{T=t} .$$

Lemma 30 *The process* $(\lambda_t^\pi)_{t\geq 0}$ *is given by*

$$\lambda_t^\pi = \left[\prod_{i\in\pi} \left(1 - D_t^i\right) \right] \times \left[\sum_{x\subset(I_n\backslash\pi)} \left[\prod_{i\in x} D_t^i \right] \lambda^{(\pi\cup x)}(X_t) \right].$$

Proof. Recall that the (conditional) multivariate distribution function of default times in the M.O. model is

$$\mathbb{P}^* \left(\tau_1 > t_1, ..., \tau_n > t_n \left| \mathcal{F}_t \right. \right) = \mathbb{E}^* \left[\prod_{\pi\in\mathbf{\Pi_n}} \exp\left(-\Lambda^\pi_{\max_{i\in\pi} t_i} \right) |\mathcal{F}_t \right],$$

where $\Lambda_T^\pi \triangleq \int_0^T \lambda^\pi(X_s)\, ds$, for all $\pi \in \mathbf{\Pi_n}$. By defining, for each n-uplet $(t_1, ..., t_n) \in \mathbb{R}_+^n$, the mapping

$$\begin{aligned}
(t_1, ..., t_n) &\rightarrow \{(\theta_1, \pi_1), ..., (\theta_k, \pi_k), ...\}, \\
\theta_0 &= 0,\ \pi_0 = \emptyset, \\
\theta_k &= \min\{t_i : 1 \leq i \leq n,\ t_i > \theta_{k-1}\}, \\
\pi_k &= \pi \text{ if } \theta_k = t_i \text{ for all } i \in \pi, \text{ and } \pi \in \mathbf{\Pi_n},
\end{aligned}$$

which sorts and groups the times $(t_1, ..., t_n)$ in a strictly increasing order, we can express the (conditional) density function,

$$f_t(t_1, ..., t_n) \triangleq \mathbb{P}^* \left(\tau_1 \in dt_1, ..., \tau_n \in dt_n \left| \mathcal{F}_t \right. \right),$$

as follows

$$f_t(t_1, ..., t_n) = \mathbb{E}^* \left[\prod_{k\geq 1} \left(\sum_{S\subset\{\pi_1\cup...\cup\pi_{k-1}\}} \lambda^{\pi_k\cup S}(X_{\theta_k}) \right) \exp\left(- \sum_{S\subset\{\pi_1\cup...\cup\pi_{k-1}\}} \Lambda_{\theta_k}^{\pi_k\cup S} \right) |\mathcal{F}_t \right],$$

where S spans the set of all subsets of $\{\pi_1 \cup ... \cup \pi_{k-1}\}$, which contains all the obligors who have defaulted prior to θ_k.

In order to compute the conditional expectation $\mathbb{P}^* \left(\tau(\pi) > T \left| \mathcal{G}_t \right. \right)$, we use the generalized Dellacherie formula

$$\mathbb{P}^* \left(\tau(\pi) > T \left| \mathcal{G}_t \right. \right) = \sum_{x\in\mathbf{\Pi_n}} D_t^{(x)} \frac{\mathbb{E}^* \left[\mathbf{1}_{\{\tau(\pi)>T\}} \times \prod_{i\notin x} \left(1 - D_t^i\right) \left| \mathcal{G}_t^{(x)} \right. \right]}{\mathbb{E}^* \left[\prod_{i\notin x} \left(1 - D_t^i\right) \left| \mathcal{G}_t^{(x)} \right. \right]}. \tag{3.12}$$

The only subsets, $x \in \mathbf{\Pi_n}$, for which the conditional expectation is not equal to zero are the ones of the form $x \subset (I_n \setminus \pi)$, i.e., at least all the obligors in π are alive; here, the notation $(I_n \setminus \pi)$ represents the complement of π in I_n The default state indicator can be expressed as

$$D_t^{(x)} = \left[\prod_{i \in x} D_t^i\right]\left[\prod_{i \in \pi}\left(1 - D_t^i\right)\right]\left[\prod_{i \in (I_n \setminus \pi)\setminus x}\left(1 - D_t^i\right)\right],$$

and equation (3.12) becomes

$$\begin{aligned}
\mathbb{P}^*\left(\tau(\pi) > T \,|\mathcal{G}_t\right) &= \left[\prod_{i \in \pi}\left(1 - D_t^i\right)\right]\sum_{x \subset (I_n \setminus \pi)}\left[\prod_{i \in x} D_t^i \prod_{i \in (I_n \setminus \pi)\setminus x}\left(1 - D_t^i\right)\right] H_t^{\pi,(x)}(T), \\
H_t^{\pi,(x)}(T) &\triangleq \frac{\mathbb{E}^*\left[\mathbf{1}_{\{\tau(\pi) > T\}} \times \prod_{i \notin x}\left(1 - D_t^i\right)\Big|\mathcal{G}_t^{(x)}\right]}{\mathbb{E}^*\left[\prod_{i \notin x}\left(1 - D_t^i\right)\Big|\mathcal{G}_t^{(x)}\right]}.
\end{aligned}$$

We shall compute each term separately.

On the set $\left[\prod_{i \in x}\mathbf{1}_{\{\tau_i \in ds_i\}}\right]\left[\prod_{i \in \pi}\mathbf{1}_{\{\tau_i > t\}}\right]\left[\prod_{i \in (I_n \setminus \pi)\setminus x}\mathbf{1}_{\{\tau_i > t\}}\right]$, the conditional probability is equal to

$$H_t^{\pi,(x)}(T) = \frac{\mathbb{E}^*\left[\mathbf{1}_{\{\tau(\pi) > T\}} \times \left[\prod_{i \in \pi}\left(1 - D_t^i\right)\right] \times \left[\prod_{i \in (I_n \setminus \pi)\setminus x}\left(1 - D_t^i\right)\right] \times \left[\prod_{i \in x}\mathbf{1}_{\{\tau_i \in ds_i\}}\right]|\mathcal{F}_t\right]}{\mathbb{E}^*\left[\left[\prod_{i \in \pi}\left(1 - D_t^i\right)\right] \times \left[\prod_{i \in (I_n \setminus \pi)\setminus x}\left(1 - D_t^i\right)\right] \times \left[\prod_{i \in x}\mathbf{1}_{\{\tau_i \in ds_i\}}\right]|\mathcal{F}_t\right]}.$$

Using the expression of the (conditional) density function, we find that the numerator is given by

$$\begin{aligned}
&\mathbb{E}^*\left[\mathbf{1}_{\{\tau(\pi) > T\}} \times \left[\prod_{i \in \pi}\left(1 - D_t^i\right)\right] \times \left[\prod_{i \in (I_n \setminus \pi)\setminus x}\left(1 - D_t^i\right)\right] \times \left[\prod_{i \in x}\mathbf{1}_{\{\tau_i \in ds_i\}}\right]|\mathcal{F}_t\right] \\
&= \mathbb{E}^*\left[\exp\left(-\sum_{S \subset x}\Lambda_T^{\pi \cup S}\right) \times \left[\prod_{q \subset (I_n \setminus \pi)\setminus x}\exp\left(-\sum_{S \subset x}\Lambda_t^{q \cup S}\right)\right] \times \left[f^{(x)}\left(\left(s_i\right)_{i \in x}\right)\right]|\mathcal{F}_t\right],
\end{aligned}$$

where $f^{(x)}\left(\left(s_i\right)_{i \in x}\right) \triangleq \mathbb{P}^*\left(\bigcap_{i \in x}\{\tau_i \in ds_i\}\,|\mathcal{F}_\infty\right)$ is the density function of the first $d = |x|$ defaulted obligors before time t.

Similarly, we have for the denominator

$$\begin{aligned}
&\mathbb{E}^*\left[\left[\prod_{i \in \pi}\left(1 - D_t^j\right)\right] \times \left[\prod_{i \in (I_n \setminus \pi)\setminus x}\left(1 - D_t^j\right)\right] \times \left[\prod_{i \in x}\mathbf{1}_{\{\tau_i \in ds_i\}}\right]|\mathcal{F}_t\right] \\
&= \mathbb{E}^*\left[\exp\left(-\sum_{S \subset x}\Lambda_t^{\pi \cup S}\right) \times \left[\prod_{q \subset (I_n \setminus \pi)\setminus x}\exp\left(-\sum_{S \subset x}\Lambda_t^{q \cup S}\right)\right] \times \left[f^{(x)}\left(\left(s_i\right)_{i \in x}\right)\right]|\mathcal{F}_t\right].
\end{aligned}$$

Using the fact that $\left[\prod_{q\subset(I_n\backslash\pi)\backslash x}\exp\left(-\sum_{S\subset x}\Lambda_t^{q\cup S}\right)\right]\times\left[f^{(x)}\left((s_i)_{i\in x}\right)\right]$, for all $s_i\leq t$, $i\in x$, is $\mathcal{F}_t$-measurable, we obtain

$$H_t^{\pi,(x)}(T)=\mathbb{E}^*\left[\exp\left(-\sum_{S\subset x}\left(\Lambda_T^{\pi\cup S}-\Lambda_t^{\pi\cup S}\right)\right)|\mathcal{F}_t\right].$$

Differentiating with respect to T, and evaluating at $T=t$, we get

$$\lambda_t^\pi=\left[\prod_{i\in\pi}\left(1-D_t^i\right)\right]\times\left[\sum_{x\subset(I_n\backslash\pi)}\left[\prod_{i\in x}D_t^i\prod_{i\in(I_n\backslash\pi)\backslash x}\left(1-D_t^i\right)\right]\sum_{S\subset x}\lambda^{(\pi\cup S)}\left(X_t\right)\right]. \quad (3.13)$$

After some basic algebra, we find that equation (3.13) can be expressed as

$$\lambda_t^\pi=\left[\prod_{i\in\pi}\left(1-D_t^i\right)\right]\times\left[\sum_{x\subset(I_n\backslash\pi)}\left[\prod_{i\in x}D_t^i\right]\lambda^{(\pi\cup x)}\left(X_t\right)\right].$$

■

For $1\leq i\leq n$, the compensated point process $M^i : M_t^i\triangleq D_t^i-\int_0^{t\wedge\tau_i}\lambda^i\left(X_s\right)ds$, is given by

$$M_t^i=\int_0^t\int_E\mathbf{1}_{\{i\in z\}}\left(\mu\left(ds\times dz\right)-\lambda_s\left(dz\right)ds\right), \quad (3.14)$$

which can also be written as

$$M_t^i=\sum_{\pi\in\mathbf{\Pi_n}}\mathbf{1}_{\{i\in z\}}M_t^\pi, \quad (3.15)$$

where, for each $\pi\in\mathbf{\Pi_n}$, M^π is the compensated point process:

$$M_t^\pi\triangleq D_t\left(\pi\right)-\int_0^t\lambda_s^\pi ds. \quad (3.16)$$

This MPP representation makes formal the idea that the mark space of the default times $(\tau_1,...,\tau_n)$ is $\mathbf{\Pi_n}$ since joint defaults are allowed. Here, we have fixed the mark space, but as default events occur, we put zero probability mass for the states of $\mathbf{\Pi_n}$, which cannot occur anymore.

3.5 Dynamics of the Zero-coupon Defaultable Bonds

In this section, we derive the dynamics of the zero-coupon defautable bonds under the martingale measure P^*. First, we give an explicit formula of the zero-coupon defaultable bond price. Then, we apply Itô's lemma to compute the martingale representation of the price process.

Lemma 31 *(Explicit formula of the zero-coupon defaultable bond).*

$$S_t^i = \left(1 - D_t^i\right) \mathbb{E}^* \left[\exp\left(- \int_t^T \lambda^i \left(X_s\right) ds \right) |\mathcal{F}_t \right].$$

Proof. The value of the (discounted) zero-coupon single-name bond is

$$S_t^i = \mathbb{E}^* \left[1 - D_T^i \,|\mathcal{G}_t \right].$$

Since τ^i is the first jump time of a Cox process,

$$\left\{\tau^i > T\right\} \Longleftrightarrow \left\{N_T^i = 0\right\},$$

we can write, for $t = 0$,

$$S_0^i = \mathbb{P}^* \left(N_T^i = 0\right) = \mathbb{E}^* \left[\exp\left(- \int_0^T \lambda^i \left(X_s\right) ds \right) \right].$$

In the Marshall-Olkin copula framework, the conditional expectation formula holds as well, i.e.,

$$S_t^i = \mathbb{E}^* \left[1 - D_T^i \,|\mathcal{G}_t \right] = \left(1 - D_t^i\right) \mathbb{E}^* \left[\exp\left(- \int_t^T \lambda^i \left(X_s\right) ds \right) |\mathcal{F}_t \right].$$

To verify this property, we need to check that the survival probability does not jump upon the default of the other obligors. For clarity, we shall do the calculations for $n = 2$, the general case is a straightforward extension.

Using the generalized Dellacherie formula, we have

$$\mathbb{P}^* \left(\tau_1 > T \,|\mathcal{G}_t\right) = \mathbf{1}_{\{\tau_1 > t\}} \mathbf{1}_{\{\tau_2 > t\}} \frac{\mathbb{P}^* \left(\tau_1 > T, \tau_2 > t \,|\mathcal{F}_t\right)}{\mathbb{P}^* \left(\tau_1 > t, \tau_2 > t \,|\mathcal{F}_t\right)} + \mathbf{1}_{\{\tau_1 > t\}} \mathbf{1}_{\{\tau_2 \leq t\}} \frac{\mathbb{P}^* \left(\tau_1 > T \,|\mathcal{F}_t \vee \sigma\left(\tau_2\right)\right)}{\mathbb{P}^* \left(\tau_1 > t \,|\mathcal{F}_t \vee \sigma\left(\tau_2\right)\right)}.$$

Recall that the M.O. (conditional) multivariate probability function is given by

$$\mathbb{P}^* \left(\tau_1 > T_1, \tau_2 > T_2 \,|\mathcal{F}_t\right) = \mathbb{E}^* \left[\exp\left(-\Lambda_{T_1}^{\{1\}} - \Lambda_{T_2}^{\{2\}} - \Lambda_{\max(T_1,T_2)}^{\{1,2\}} \right) |\mathcal{F}_t \right], \text{ for all } T_1, T_2 \geq 0,$$

where $\Lambda_T^\pi \triangleq \int_0^T \lambda^\pi \left(X_t\right) dt$, for $\pi \in \mathbf{\Pi_n}$. We shall compute each term in turn. We start with the survival case.

$$\begin{aligned}
\frac{\mathbb{P}^* \left(\tau_1 > T, \tau_2 > t \,|\mathcal{F}_t\right)}{\mathbb{P}^* \left(\tau_1 > t, \tau_2 > t \,|\mathcal{F}_t\right)} &= \frac{\mathbb{E}^* \left[\exp\left(-\Lambda_T^{\{1\}} - \Lambda_t^{\{2\}} - \Lambda_T^{\{1,2\}} \right) |\mathcal{F}_t \right]}{\mathbb{E}^* \left[\exp\left(-\Lambda_t^{\{1\}} - \Lambda_t^{\{2\}} - \Lambda_t^{\{1,2\}} \right) |\mathcal{F}_t \right]} \\
&= \frac{\mathbb{E}^* \left[\exp\left(-\Lambda_T^{\{1\}} - \Lambda_t^{\{2\}} - \Lambda_T^{\{1,2\}} \right) |\mathcal{F}_t \right]}{\exp\left(-\Lambda_t^{\{1\}} - \Lambda_t^{\{2\}} - \Lambda_t^{\{1,2\}} \right)} \\
&= \mathbb{E}^* \left[\frac{\exp\left(-\Lambda_T^{\{1\}} - \Lambda_t^{\{2\}} - \Lambda_T^{\{1,2\}} \right)}{\exp\left(-\Lambda_t^{\{1\}} - \Lambda_t^{\{2\}} - \Lambda_t^{\{1,2\}} \right)} |\mathcal{F}_t \right] \\
&= \mathbb{E}^* \left[\exp\left(- \left(\Lambda_T^{\{1\}} - \Lambda_t^{\{1\}}\right) - \left(\Lambda_T^{\{1,2\}} - \Lambda_t^{\{1,2\}}\right) \right) |\mathcal{F}_t \right] \\
&= \mathbb{E}^* \left[\exp\left(- \left(\Lambda_T^1 - \Lambda_t^1\right) \right) |\mathcal{F}_t \right],
\end{aligned}$$

the first equality follows from the multivariate probability function; the second and third equalities are due to the fact that $\Lambda_t^{\{1\}}$, $\Lambda_t^{\{2\}}$, and $\Lambda_t^{\{1,2\}}$ are $\mathcal{F}_t$-measurable; the last equality is due to the fatal shock representation: $\lambda^1(X_t) = \lambda^{\{1\}}(X_t) + \lambda^{\{1,2\}}(X_t)$.

For the default case, let us compute the conditional probability on the set $\mathbf{1}_{\{\tau_1 > t\}}\mathbf{1}_{\{\tau_2 \in ds\}}$, for $s \leq t$,

$$\frac{\mathbb{P}^*(\tau_1 > T, \tau_2 \in ds \,|\mathcal{F}_t)}{\mathbb{P}^*(\tau_1 > t, \tau_2 \in ds \,|\mathcal{F}_t)}$$

The numerator is computed as follows: for $\epsilon > 0$,

$$\begin{aligned}
& \mathbb{P}^*(\tau_1 > T, \tau_2 \in (s-\epsilon, s] \,|\mathcal{F}_t) \\
= \;& \mathbb{P}^*(\tau_1 > T, \tau_2 > s \,|\mathcal{F}_t) - \mathbb{P}^*(\tau_1 > T, \tau_2 > s-\epsilon \,|\mathcal{F}_t) \\
= \;& \mathbb{E}^*\left[\exp\left(-\Lambda_T^{\{1\}} - \Lambda_s^{\{2\}} - \Lambda_T^{\{1,2\}}\right)|\mathcal{F}_t\right] - \mathbb{E}^*\left[\exp\left(-\Lambda_T^{\{1\}} - \Lambda_{s-\epsilon}^{\{2\}} - \Lambda_T^{\{1,2\}}\right)|\mathcal{F}_t\right] \\
= \;& \mathbb{E}^*\left[\exp\left(-\Lambda_T^{\{1\}} - \Lambda_s^{\{2\}} - \Lambda_T^{\{1,2\}}\right)\left(1 - \exp\left(\Lambda_s^{\{2\}} - \Lambda_{s-\epsilon}^{\{2\}}\right)\right)|\mathcal{F}_t\right] \\
= \;& \mathbb{E}^*\left[\left[-\lambda^{\{2\}}(X_s)\,\epsilon\right]\exp\left(-\Lambda_T^{\{1\}} - \Lambda_s^{\{2\}} - \Lambda_T^{\{1,2\}}\right)|\mathcal{F}_t\right] + o(\epsilon).
\end{aligned}$$

Similarly, we have for the denominator

$$\begin{aligned}
\mathbb{P}^*(\tau_1 > t, \tau_2 \in (s-\epsilon, s] \,|\mathcal{F}_t) &= \mathbb{E}^*\left[\left[-\lambda^{\{2\}}(X_s)\,\epsilon\right]\exp\left(-\Lambda_t^{\{1\}} - \Lambda_s^{\{2\}} - \Lambda_t^{\{1,2\}}\right)|\mathcal{F}_t\right] + o(\epsilon). \\
&= \left[-\lambda^{\{2\}}(X_s)\,\epsilon\right]\exp\left(-\Lambda_t^{\{1\}} - \Lambda_s^{\{2\}} - \Lambda_t^{\{1,2\}}\right) + o(\epsilon).
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{\mathbb{P}^*(\tau_1 > T, \tau_2 \in ds \,|\mathcal{F}_t)}{\mathbb{P}^*(\tau_1 > t, \tau_2 \in ds \,|\mathcal{F}_t)} &= \frac{\mathbb{E}^*\left[-\lambda^{\{2\}}(X_s)\exp\left(-\Lambda_T^{\{1\}} - \Lambda_s^{\{2\}} - \Lambda_T^{\{1,2\}}\right)|\mathcal{F}_t\right]}{\left[-\lambda^{\{2\}}(X_s)\right]\exp\left(-\Lambda_t^{\{1\}} - \Lambda_s^{\{2\}} - \Lambda_t^{\{1,2\}}\right)} \\
&= \mathbb{E}^*\left[\frac{\exp\left(-\Lambda_T^{\{1\}} - \Lambda_T^{\{1,2\}}\right)}{\exp\left(-\Lambda_t^{\{1\}} - \Lambda_t^{\{1,2\}}\right)}|\mathcal{F}_t\right] \\
&= \mathbb{E}^*\left[\exp\left(-\left(\Lambda_T^1 - \Lambda_t^1\right)\right)|\mathcal{F}_t\right].
\end{aligned}$$

Therefore, the conditional survival probability does not jump upon the default of other obligors:

$$S_t^i = \left(1 - D_t^i\right)\mathbb{E}^*\left[\exp\left(-\int_t^T \lambda^i(X_s)\,ds\right)|\mathcal{F}_t\right].$$

■

Applying Itô's lemma and using the Markovian property of X, we find an explicit expression of the martingale representation of the price process S^i.

Proposition 32 *(Single-name price process representation). We have*

$$\begin{aligned} S_t^i &= S_0^i - \int_0^t \int_E s^i(s, X_s) \mathbf{1}_{\{i \in z\}} (\mu(ds \times dz) - \lambda_s(dz) ds) \\ &\quad + \int_0^t \left(1 - D_s^i\right) \sum_{j=1}^d \sum_{k=1}^d \frac{\partial s^i(s, X_s)}{\partial x_j} \beta_{jk}(X_s) dW_s^k, \end{aligned}$$

where $s^i(t, x) : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ *is defined as*

$$s^i(t, x) \triangleq \mathbb{E}_{(t,x)}^* \left[\exp\left(-\int_t^T \lambda^i(X_s) ds \right) \right].$$

Proof. The value of the (discounted) zero-coupon single-name bond is

$$S_t^i = \left(1 - D_t^i\right) \mathbb{E}^* \left[\exp\left(-\int_t^T \lambda^i(X_s) ds \right) | \mathcal{F}_t \right].$$

Given the Markovian property of the background process X, we have

$$S_t^i = \left(1 - D_t^i\right) s^i(t, X_t),$$

where $s^i(t, x) : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ is

$$s^i(t, x) \triangleq \mathbb{E}_{(t,x)}^* \left[\exp\left(-\int_t^T \lambda^i(X_s) ds \right) \right].$$

We assume that the function s^i is sufficiently smooth, in particular, that it is continuous, $\mathcal{C}^1$ in the first argument and $\mathcal{C}^2$ in the second argument. X is an $\mathbb{R}^d$-valued diffusion process with drift vector $\alpha(x)$, diffusion matrix $\beta(x)$, and infinitesimal generator

$$\mathcal{A}_t F(x) \triangleq \frac{1}{2} \sum_{j=1}^d \sum_{k=1}^d \sum_{l=1}^d \beta_{jl}(x) \beta_{kl}(x) \frac{\partial^2 F(x)}{\partial x_j \partial x_k} + \sum_{j=1}^d \alpha_j(x) \frac{\partial F(x)}{\partial x_j}.$$

Using the fact that $L_t \triangleq \mathbb{E}^* \left[\exp\left(-\int_0^T \lambda^i(X_s) ds \right) | \mathcal{F}_t \right]$ is an $\{\mathcal{F}_t\}$-martingale, we find that the function s^i is solution of the Feyman-Kac equation

$$\begin{aligned} -\lambda^i(x) s^i(t, x) + \frac{\partial s^i(t, x)}{\partial t} + \mathcal{A}_t s^i(t, x) &= 0, \text{ for } (t, x) \in [0, T] \times \mathbb{R}^d, \\ s^i(T, x) &= 1, \text{ for } x \in \mathbb{R}^d. \end{aligned}$$

Applying Itô's lemma to S_t^i, we get

$$\begin{aligned} dS_t^i &= -dD_t^i s^i(t, X_t) + \left(1 - D_t^i\right) ds^i(t, X_t) \\ &= -dD_t^i s^i(t, X_t) + \left(1 - D_t^i\right) \left[\left(\frac{\partial s^i(t, X_t)}{\partial t} + \mathcal{A}_t s^i(t, X_t) \right) dt + \sum_{j=1}^d \sum_{k=1}^d \frac{\partial s^i(t, X_t)}{\partial x_j} \beta_{jk}(X_t) dW_t^k \right]. \end{aligned}$$

Replacing the term in dt with the Feyman-Kac equation, we get

$$dS_t^i = -dM_t^i s^i(t, X_t) + \left(1 - D_t^i\right) \left[\sum_{j=1}^{d} \sum_{k=1}^{d} \frac{\partial s^i(t, X_t)}{\partial x_j} \beta_{jk}(X_t)\, dW_t^k \right],$$

where M^i is the compensated martingale $M_t^i \triangleq D_t^i - \int_0^{t \wedge \tau_i} \lambda^i(X_s)\, ds$. Using the marked point process representation,

$$M_t^i = \int_0^t \int_E \mathbf{1}_{\{i \in z\}} \left(\mu(ds \times dz) - \lambda_s(dz)\, ds\right),$$

we arrive at the result. ■

Corollary 33 *The dynamics of the zero-coupon defaultable bonds are given by*

$$dS_t^i = S_{t-}^i \left(\mu_t^i dt + \left(\sigma_t^i\right)^{tr} dW_t - dM_t^i \right),$$

where the drift and volatility processes μ_t^i and σ_t^i are

$$\begin{aligned} \mu_t^i &= 0, \\ \left(\sigma_t^i\right)^k &= \left[\sum_{j=1}^{d} \frac{\partial \log s^i(t, X_t)}{\partial x_j} \beta_{jk}(X_t) \right], \text{ for } k = 1, ..., d. \end{aligned}$$

3.6 Martingale Representation

In this section, we derive a martingale representation of the $\{\mathcal{G}_t\}$-martingale $H_t = \mathbb{E}^*[H_T | \mathcal{G}_t]$.

To this end, we use the fatal shock representation of our model in conjunction with the martingale representation result for marked point processes.

The agents' information structure is modelled by the filtered probability space $(\Omega, \mathcal{G}, \{\mathcal{G}_t\}, P^*)$, where $\{\mathcal{G}_t\}$ is the natural filtration generated by the d-dimensional Brownian motion W and the marked point process $\mu(dt \times dz)$ with the $(P^*, \{\mathcal{G}_t\})$-intensity kernel $\lambda_t(dz)$. The Martingale Representation Theorem (see Jacod and Shiryaev (1987) Chap III Corollary 4.31) shows that the martingale generator in this economy is $\left(W, (\mu(dt \times \{z\}) - \lambda_t(\{z\}))_{z \in \mathbf{\Pi_n}}\right)$.

Proposition 34 *(Martingale representation of H_t). The $\{\mathcal{G}_t\}$-martingale $H_t = \mathbb{E}^*[H_T | \mathcal{G}_t]$, $t \in [0, T^*]$, where H_T is a $\mathcal{G}_T$-measurable random variable, integrable with respect to P^*, admits the following integral representation*

$$H_t = H_0 + \int_0^t (\xi_s)^{tr} dW_s - \int_0^t \int_E \zeta(s, z) \left(\mu(ds \times dz) - \lambda_s(dz)\, ds\right), \tag{3.17}$$

where ξ is a d-dimensional $\{\mathcal{G}_t\}$-predictable process and $\zeta(s,z)$ is an E-indexed $\{\mathcal{G}_t\}$-predictable process $\zeta(s,z)$ such that

$$\int_0^t \|\xi_s\|^2 ds < \infty, \quad \int_0^t \int_E \zeta(s,z)\,\lambda_s(dz)\,ds < \infty,$$

almost surely.

This can be written as

$$H_t = H_0 + \int_0^t (\xi_s)^{tr} dW_s - \sum_{\pi \in \mathbf{\Pi_n}} \int_{]0,t]} \zeta(s,\pi)\,dM_s^\pi. \tag{3.18}$$

In order to replicate the claim H_T, one needs to match the diffusion terms ξ_s^i, $1 \le i \le d$, and the jump-to-default terms $[-\zeta(s,\pi)]$ for each possible default state $\pi \in \mathbf{\Pi_n}$.

3.7 Computing the Hedging Strategy: The Main Result

In this section, we use the martingale representation of Proposition 34 to derive the risk-minimizing hedging strategy. This is equivalent to finding the Kunita-Watanabe decomposition of H_t:

$$H_T = H_0 + \int_{]0,T]} (\alpha_t)^{tr} dS_t + L_T. \tag{3.19}$$

Our goal is to establish an analytical result, which derives single-name hedges $(\alpha^i)_{1\le i\le n}$ in terms of the martingale representation predictable processes ξ and $\zeta(.,\pi)$, $\pi \in \mathbf{\Pi_n}$.

As shown in Föllmer and Schweizer (1991), the strategy $(\alpha_t)_{t\ge 0}$ can be computed as

$$\alpha_t = d\langle S\rangle_t^{-1} d\langle S, V(\alpha)\rangle_t, \tag{3.20}$$

where the value process is given by

$$V_t(\alpha) = H_t = \mathbb{E}^*[H_T|\mathcal{G}_t], \text{ for } t \in [0,T]. \tag{3.21}$$

This follows from the Kunita-Watanabe decomposition of H and the projection of $V_t(\alpha)$ on the martingale $\int_{]0,t]} (\alpha_s)^{tr} dS_s$.

Theorem 35 *(Risk-minimizing hedging strategy). The risk-minimizing hedging strategy of a general (basket) contingent claim with single name instruments is given by the solution of the following linear system,*

for $1 \le k \le n$,

$$\begin{aligned} &\sum_{i=1}^n \alpha_t^i S_{t-}^i \left[\left(\sigma_t^i\right)^{tr} \sigma_t^k + \int_E \mathbf{1}_{\{i\in z\}} \mathbf{1}_{\{k\in z\}} \lambda_t(dz)\right] \\ = \; &\left(\sigma_t^k\right)^{tr} \xi_t + \int_E \zeta(t,z)\,\mathbf{1}_{\{k\in z\}} \lambda_t(dz). \end{aligned}$$

Proof. Using the single-name instrument representation of Corollary 33, we have

$$\left(S_t^i\right) = \int_0^t S_{s-}^i \left(\left(\sigma_s^i\right)^{tr} dW_s - \int_E \mathbf{1}_{\{i\in z\}} \left(\mu\left(ds\times dz\right) - \lambda_s\left(dz\right)ds\right)\right),$$

and the predictable covariance is

$$d\left\langle S\right\rangle_t^{i,j} = d\left\langle S^i, S^j\right\rangle_t = S_{t-}^i S_{t-}^j \left(\left(\sigma_t^i\right)^{tr} \sigma_t^j + \int_E \mathbf{1}_{\{i\in z\}}\mathbf{1}_{\{j\in z\}}\lambda_t\left(dz\right)\right) dt.$$

The value process $V_t\left(\alpha\right) = \mathbb{E}^*\left[H_T \left|\mathcal{G}_t\right.\right]$ is given by the martingale representation

$$V_t\left(\alpha\right) = \mathbb{E}^*\left[H_T \left|\mathcal{G}_t\right.\right] = H_0 + \int_0^t \left(\xi_s\right)^{tr} dW_s - \int_0^t \int_E \zeta\left(s,z\right)\left(\mu\left(ds\times dz\right) - \lambda_s\left(dz\right)ds\right).$$

Hence, we have

$$d\left\langle S, V\left(\alpha\right)\right\rangle_t^i = S_{t-}^i \left(\left(\sigma_t^i\right)^{tr} \xi_t + \int_E \mathbf{1}_{\{i\in z\}}\zeta\left(t,z\right)\lambda_t\left(dz\right)\right) dt,$$

and the strategy $\left(\alpha_t\right)_{t\geq 0}$ is given by the solution of the following system

$$\begin{aligned}
&\sum_{i=1}^n \alpha_t^i \left[S_{t-}^i S_{t-}^k \left[\left(\sigma_t^i\right)^{tr} \sigma_t^k + \int_E \mathbf{1}_{\{i\in z\}}\mathbf{1}_{\{k\in z\}}\lambda_t\left(dz\right)\right]\right] \\
= \;& S_{t-}^k \left(\sigma_t^k\right)^{tr} \xi_t + \int_E \zeta\left(t,z\right) S_{t-}^k \mathbf{1}_{\{k\in z\}}\lambda_t\left(dz\right).
\end{aligned}$$

■

Proposition 32 establishes the martingale representation for the single-name securities whose payoff is $H_T = 1 - D_T^i$:

$$\begin{aligned}
\zeta^{1-D_T^i}\left(t,z\right) &= \mathbf{1}_{\{i\in z\}} s^i\left(t, X_t\right), \text{ for } z\in\mathbf{\Pi_n}, \\
\left(\xi_t^{1-D_T^i}\right)^k &= \left(1 - D_t^i\right)\sum_{j=1}^d \frac{\partial s^i\left(t,X_t\right)}{\partial x_j}\beta_{jk}\left(X_t\right).
\end{aligned}$$

The hedging strategy is solution of
for $1\leq k\leq n$

$$\begin{aligned}
&\sum_{i=1}^n \alpha_t^i \left[\int_E \zeta^{1-D_T^k}\left(t,z\right)\zeta^{1-D_T^i}\left(t,z\right)\lambda_t\left(dz\right) + \left(\xi_t^{1-D_T^k}\right)^{tr} \xi_t^{1-D_T^i}\right] \\
= \;& \int_E \zeta\left(t,z\right)\zeta^{1-D_T^k}\left(t,z\right)\lambda_t\left(dz\right) + \left(\xi_t^{1-D_T^k}\right)^{tr} \xi_t.
\end{aligned}$$

Note that this problem combines both default risk and spread risk.

Application. We consider a first-to-default (basket) contingent claim whose payoff is

$$H_T^{(1)} \triangleq \prod_{i=1}^{n} \left(1 - D_T^i\right).$$

The price of this claim at time t is

$$H_t^{(1)} = \mathbb{E}^* \left[\prod_{i=1}^{n} \left(1 - D_T^i\right) | \mathcal{G}_t \right].$$

We can show that it can be expressed as (see Chapter 4)

$$H_t^{(1)} = \left[\prod_{i=1}^{n} \left(1 - D_t^i\right) \right] h^{(1)} (t, X_t),$$

where the function $h^{(1)} (t, x) : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ is defined as

$$\begin{aligned} h^{(1)} (t, X_t) &= \mathbb{E}^*_{(t,x)} \left[\exp \left(- \int_t^T \lambda^{(1)} (X_s) \, ds \right) \right], \\ \lambda^{(1)} (X_t) &= \sum_{j=1}^{m} \left[1 - \prod_{i=1}^{n} \left(1 - p^{i,j} (X_t)\right) \right] \lambda^{c_j} (X_t). \end{aligned}$$

Using Itô's lemma and some algebra, we find

$$\begin{aligned} dH_t^{(1)} &= - \int_E h^{(1)} (t, X_t) \left(\mu (dt \times dz) - \lambda_t (dz) \, dt \right) \\ &\quad + \left[\prod_{i=1}^{n} \left(1 - D_t^i\right) \right] \sum_{j=1}^{d} \sum_{k=1}^{d} \frac{\partial h^{(1)} (t, X_t)}{\partial x_j} \beta_{jk} (X_t) \, dW_t^k. \end{aligned}$$

This gives the processes of the martingale representation

$$\begin{aligned} \zeta^{H_T^{(1)}} (t, z) &= h^{(1)} (t, X_t), \text{ for all } z \in \mathbf{\Pi_n}, \\ \left(\xi_s^{H_T^{(1)}} \right)^k &= \left[\prod_{i=1}^{n} \left(1 - D_t^i\right) \right] \sum_{j=1}^{d} \frac{\partial h^{(1)} (t, X_t)}{\partial x_j} \beta_{jk} (X_t), \end{aligned}$$

which can be plugged into the linear system of Theorem 35. Inverting this latter gives the single-name hedge ratios of the first-to-default basket claim.

3.8 Conclusion

The problem of hedging basket credit derivatives with single name instruments is a very interesting challenge both for academics and practitioners. In this chapter, we have presented a solution based on a risk-minimization approach. We have seen that, in the credit market, we have two sources of uncertainty: spread risk and default risk. We have addressed both types of risk and we have shown how to derive the single name hedge ratios by solving a quadratic minimization problem.

The explosive nature of the default space representation will probably be one of the limiting factors that need to be addressed. In the pricing problem, questions of numerical efficiency were handled by Fourier transform techniques and recursion methods borrowed from actuarial mathematics. These techniques might also prove to be very useful for the hedging problem. Another area of research would be to consider other hedging approaches, such as quantile hedging (see Föllmer and Leukert (1999)), and to do a numerical comparison of the effectiveness of each strategy.

Part II

Partie Numérique

Chapter 4

Basket Asymptotics

In this chapter, we provide some efficient numerical methods for the valuation of large basket credit derivatives. The approaches are presented in the Marshall-Olkin copula model. Some formulas are specific to MO, but most of the numerical techniques are generic and could be used with other copulas. The methods presented span a large spectrum of applied mathematics: Fourier transforms, changes of probability measure, numerical stable schemes, high-dimensional Sobol integration, recursive convolution algorithms.

4.1 Introduction

Over the past few years, basket credit derivatives have grown in popularity. They offer investors a new way to take leveraged credit views and to earn higher yields compared with similarly rated bond investments. One example of such products is a first-to-default swap. This structure a default swap where the credit event is defined as the first default in a basket of underlying references. The first-to-default event triggers a payment equal to the par amount minus the recovery value of the defaulted security. In return, the protection buyer pays a periodic default-contingent premium until the maturity of the swap or the time of default, whichever is first. For more risk-averse investors who seek investments in low-risk assets, higher-order default swaps, such as a second-to-default, a third-to-default, ... or an n^{th}-to-default, offer an attractive alternative. Another popular class of basket credit derivatives is synthetic CDO tranches (Collateralized Debt Obligations). CDOs usually reference large portfolios of credits, typically 50 to 100 names, and the credit event is defined in relation to the aggregate loss of the portfolio. The protection seller commits to cover all the losses incurred within a pre-defined range. In return, he receives a periodic premium on a notional, which amortizes with the portfolio losses.

Basket credit derivatives are default-correlation instruments. Their value is mainly driven by the likelihood of joint defaults in the portfolio. For a first-to-default, for example, as correlation changes between 0 and 1, the break-even spread varies between the sum of the individual default swap spreads and the widest default swap spread. The usual approach used to model default correlation is the copula function framework. A comprehensive review of copulas and their applications to risk management can be found in Embrechts, Lindskog, McNeil (2003) or Bouyé, Durrleman, Nikeghbali, Riboulet, Roncalli (2000). The Gaussian copula, often used in credit risk modelling, goes back to Li (2000). Other studies of elliptical copulas with higher tail dependence, such as the t-copula, can be found in Mashal and Naldi (2002). The Marshall-Olkin copula is yet another class of copula functions, which stems from the multivariate compound Poisson process. In this model, individual defaults are constructed from a series of independent common shock. Each common shock is assumed to be a Poisson process. This set-up has been traditionally used in the reliability theory literature (see Barlow and Proschan (1981)). Previous work on the use of the Marshall-Olkin copula in the context of credit risk modelling includes Duffie (1998), Duffie and Singleton (1999), Duffie and Pan (2001), Wong (2000), Lindskog and McNeil (2003).

The objective of this chapter is to develop efficient numerical methods for pricing basket default swaps in the Marshall-Olkin framework. The valuation of large basket products is usually done with Monte-Carlo simulations. The advantage of Monte-Carlo is its simplicity and generality. Its main drawbacks, however, are the quality of the convergence, especially when one computes sensitivities, such as deltas and gammas. A good convergence is particularly hard to achieve for credit products since default events are usually rare, and probabilities in the tail of the distribution are difficult to estimate. On the other hand, the direct implementation in closed form is very accurate but is less trivial to implement; it is also very expensive computationally. Indeed, this method is based on enumerating the 2^n default configurations of the basket and computing the probability of each configuration. This algorithm explodes exponentially as the number of credits increases. To address this problem, we use a collection of techniques from numerical analysis and actuarial mathematics, and we develop a suite of semi-analytic numerical methods based on the asymptotic behaviour of the portfolio. Although the implementation is done in the Marshall-Olkin framework, most of the numerical techniques are generic and could be used with other copula models. Other analytic approximations of the Marshall-Olkin model have been developed by Duffie and Pan (2001), and by Lindskog and McNeil (2003).

This chapter is organized as follows.

In Section 4.2, we present briefly the Marshall-Olkin copula model.

In Section 4.3, we derive the pricing formulas for basket default swaps with the brute force method. The exploding nature of the direct approach will motivate the other methods described in the rest of the chapter.

We show, in Section 4.4, that we can replicate any nth-to-default swap with a portfolio of first-to-default swaps referenced to subsets of the original basket. This replication technique is referred to as the "Expansion" method.

In Section 4.5, we develop the "Homogeneous Expansion" method, where we transform the original reference portfolio to an equivalent homogeneous one. This transformation keeps invariant certain properties of the aggregate default distribution. This invariance principle ensures, in turn, the invariance of the prices of nth-to-default swaps. We compute the default distributions by a Fourier transform inversion; and we show how to resolve the problem with machine precision, when one estimates small tail probabilities, by an importance sampling technique.

For large reference portfolios, we find that the recursion relationship of the homogeneous expansion becomes numerically instable. To address this issue, we propose, in Section 4.6, an asymptotic implementation of the homogeneous expansion. In particular, we show that the density function of the nth-to-default random time admits an infinite asymptotic series expansion. For homogeneous portfolios this series expansion corresponds to the formula of a "Binomial Mixture". We show how to compute the dominant orders in the expansion in closed form, and how to estimate the higher-order terms numerically with a Quasi Monte Carlo integration technique.

In Section 4.7, we describe the infinite series expansion for non-homogeneous portfolios; and we show how to compute the conditional distributions that appear in the expansion by the convolution recursion method. This type of recursions is often used in actuarial mathematics.

4.2 Set-up and Notations

The Marshall-Olkin copula is based on the Multivariate Poisson Process construction. Here, we give a brief overview of this model as described in Lindskog and McNeil (2003).

We consider a set of non-negative random variables $(\tau_1, ..., \tau_n)$, defined on a probability space $(\Omega, \mathcal{G}, P)$, representing the default times of n obligors. For each firm i, we denote by $D_t^i \triangleq \mathbf{1}_{\{\tau_i \leq t\}}$ the default indicator process.

We assume that there exists a set of $(m+n)$ independent Poisson processes $\left(N_t^{c_j}\right)_{t\geq 0}$ with intensities $\lambda^{c_j}(t)$, which can trigger joint defaults.

For every event type c_j, and for all $t \geq 0$, we define a set of independent Bernoulli variables $\left(A_t^{1,j}, ..., A_t^{n,j}\right)$ with probabilities $(p_{1,j}, ..., p_{n,j})$, $p_{i,j} \in [0,1]$.

We assume that for $j \neq k$, the vectors $\mathbf{A}_t^j = \left(A_t^{1,j}, ..., A_t^{n,j}\right)$ and $\mathbf{A}_t^k = \left(A_t^{1,k}, ..., A_t^{n,k}\right)$ are independent.

We assume that for $t \neq s$, the vectors $\mathbf{A}_t^j = \left(A_t^{1,j}, ..., A_t^{n,j}\right)$ and $\mathbf{A}_s^j = \left(A_s^{1,j}, ..., A_s^{n,j}\right)$ are independent.

At the r^{th} occurrence of the common market event of type j, at time $\theta_r^{c_j}$, we draw the set of n independent Bernoulli variables $\left(A_{\theta_r^{c_j}}^{1,j}, ..., A_{\theta_r^{c_j}}^{n,j}\right)$. The variable $A_{\theta_r^{c_j}}^{i,j}$ indicates if obligor i has defaulted or not.

The process $\left(N_t^i\right)_{t \geq 0}$, defined as

$$N_t^i \triangleq \sum_{j=1}^{m+n} \sum_{\theta_r^{c_j} \leq t} A_{\theta_r^{c_j}}^{i,j}, \tag{4.1}$$

is also a Poisson process obtained by superpositionning a set of independent (thinned) Poisson processes. Its intensity is given by

$$\lambda_i(t) = \sum_{j=1}^{m+n} p_{i,j} \lambda^{c_j}(t). \tag{4.2}$$

We define the default time τ_i as the first jump time of the Poisson process N_t^i:

$$\tau_i \triangleq \min\left\{t : N_t^i > 0\right\}. \tag{4.3}$$

We denote by $Q_i(T)$ the survival probability of τ_i:

$$Q_i(T) \triangleq \mathbb{P}(\tau_i > T) = \mathbb{P}\left(N_T^i = 0\right) = \exp\left(-\int_0^T \lambda_i(s)\, ds\right), \tag{4.4}$$

which will be referred to as the Q-factor associated with τ_i.

To specify the model further, we assume that we have two types of market factors: (1) the first m drivers are common market factors which affect more than one obligor; they can represent global economic factors, industry sector factors, or regional and country factors; (2) the n remaining ones are idiosynctratic issuer-specific shocks; they will be denoted as: $N_t^{0,i} \triangleq N_t^{c_{m+i}}$ (and $\lambda^{0,i}(t) \triangleq \lambda^{c_{m+i}}(t)$), for $1 \leq i \leq n$. Their corresponding factor loadings are: $p_{i,m+i} = 1$ and $p_{i,m+k} = 0$, for $1 \leq k \neq i \leq n$.

Equation (4.2) then becomes

$$\lambda_i(t) = \sum_{j=1}^{m} p_{i,j} \lambda^{c_j}(t) + \lambda^{0,i}(t). \tag{4.5}$$

The copula function of the default times $(\tau_1, ..., \tau_n)$, implied by this multivariate Poisson process, is known as the Marshall-Olkin copula (see Proposition 6 in Lindskog and McNeil (2003)).

Furthermore, we associate with the set $(\tau_1, ..., \tau_n)$ the ordered sequence of random times $\tau^{[1]} \leq \tau^{[2]} \leq ... \leq \tau^{[n]}$, defined as $\tau^{[1]} = \min(\tau_1, ..., \tau_n)$, and for $k = 2, ..., n$,

$$\tau^{[k]} = \min\left(\tau_i : i = 1, ..., n,\ \tau_i > \tau^{[k-1]}\right). \tag{4.6}$$

Now, we consider a k^{th}-to-default swap (also called a k out of n default swap in Laurent and Gregory (2002)), which matures at time T, and is specified by the following contractual obligations:

- if k default events occur before maturity, $\tau^{[k]} \leq T$, and credit i is the one that has last defaulted, $\tau^{[k]} = \tau_i$, then the protection seller makes a payment equal to $(1 - \delta_i)$, where δ_i is the recovery value of issuer i;
- in return, the buyer makes a series of periodic premium payments $(C_1, ..., C_N)$,on the cash-flow dates $(T_1, ..., T_N)$, as long as the total number of defaults is less than k. Each cash flow C_i is equal to the product of the premium and the day-count fraction $(T_i - T_{i-1})$.

The value of the premium leg is given by the expectation of the discounted payoff

$$\text{premium\_leg} = \sum_{i=1}^{N} B(0, T_i) C_i \mathbb{P}\left(\tau^{[k]} > T_i\right),$$

where $B(0, T)$ is the risk-free zero-coupon bond maturing at time T. Interest rates and credit processes are assumed to be independent.

The value of the protection leg is given by the following recovery integral:

$$\text{protection\_leg} = \sum_{i=1}^{n} (1 - \delta_i) \left[\int_0^T B(0, t) \mathbb{P}\left(\tau^{[k]} = \tau_i, \tau^{[k]} \in dt\right) dt\right]. \tag{4.7}$$

$Q^{[k]}(T)$, the k^{th}-to-default Q-factor, is the survival probability of the default time $\tau^{[k]}$; it can be written as:

$$Q^{[k]}(T) \triangleq \mathbb{P}\left(\tau^{[k]} > T\right) = \mathbb{P}(X_T < k),$$

where X_T counts the portfolio aggregate defaults at time T

$$X_T \triangleq \sum_{i=1}^{n} D_T^i. \tag{4.8}$$

If we have a basket with homogeneous recoveries, the value of the protection leg simplifies to

$$\text{protection\_leg} = -(1 - \delta) \int_0^T B(0, t)\, dQ^{[k]}(t);$$

otherwise, one needs to compute the density:

$$\mathbb{P}\left(\tau^{[k]}=\tau_i,\tau^{[k]}\in dt\right).$$

We refer to Laurent and Gregory (2002) or Bielecki and Rutkowski (2001) for more details.

It should now be clear that to evaluate basket default swaps, we need to generate the survival probability curve $Q^{[k]}(t)$, for $0\leq t\leq T$. This latter is discretized on a fine mesh and used to estimate the recovery integral of the protection leg. In the next section, we show how to compute this with a direct method; then, in the rest of the chapter, we shall study the properties of the aggregate default distribution X_T by using its probability generating function

$$\varphi(x)=\sum_{k=0}^{n}\mathbb{P}(X_T=k)\ x^k.$$

The "Homogeneous Transformation", the "Asymptotic Homogeneous Expansion", and the "Asymptotic Expansion" are all based on the fundamental "probability generating function representation" of Theorem 44, Section 4.5.

4.3 Direct Approach

To motivate the numerical methods developed in this chapter, we show, in this section, how to compute the key component $Q^{[k]}(T)$ with the direct approach, which is based on enumerating all default combinations in the basket, and mixing their probabilities by using some simple algebraic rules. This combinatorial recipe is only needed for higher-order baskets, where $k\geq 2$. For first-to-default swaps, the algebra is very simple, and closed form solutions are readily available. We begin with the simple FTD case; then we build up the algorithm for the more burdensome cases. We shall use the equivalent fatal shock representation of Lindskog and McNeil

4.3.1 Equivalent Fatal Shock Representation

Let $\mathbf{\Pi_n}$ be the set of all subsets of $\{1,...,n\}$. For each $\pi\in\mathbf{\Pi_n}$, we introduce the point process N_t^{π}, which counts the number of shocks in $(0,t]$ resulting in joint defaults of the obligors in π only:

$$N_t^{\pi}\triangleq\sum_{j=1}^{m+n}\sum_{r=1}^{N_t^{c_j}}A_{\theta_r^{c_j}}^{\pi,j},\tag{4.9}$$

where, for each trigger time $\theta_r^{c_j}$, $A_{\theta_r^{c_j}}^{\pi,j}$ is a Bernoulli variable, which is equal to 1 if all obligors $i \in \pi$ default and all the others, $i \notin \pi$, survive:

$$A_t^{\pi,j} \triangleq \prod_{i\in\pi} A_t^{i,j} \prod_{i\notin\pi} \left(1 - A_t^{i,j}\right). \tag{4.10}$$

For example, if $\pi = \{1, 2\}$, then the process $N_t^{\{1,2\}}$ counts the shocks, which trigger simultaneous defaults of obligors 1 and 2 but not the other obligors 3 to n.

Further, let $\widetilde{N}_t$ count all shocks which results in any kind of loss, i.e.,

$$\widetilde{N}_t \triangleq \sum_{\substack{\pi\in\mathbf{\Pi_n} \\ \pi\neq\emptyset}} N_t^{\pi}. \tag{4.11}$$

We have the following fatal shock representation key result. We refer to Lindskog and McNeil (2003) for details (see Proposition 4).

Proposition 36 *(Fatal shock representation).*

1. *The processes $(N^{\pi})_{\pi\in\mathbf{\Pi_n}}$ are independent Poisson processes with intensities*

$$\lambda^{\pi}(t) = \sum_{j=1}^{m+n} p_{\pi,j} \lambda^{c_j}(t),$$

where

$$p_{\pi,j} = \prod_{i\in\pi} p_{i,j} \prod_{i\notin\pi} (1 - p_{i,j}).$$

2. *$\widetilde{N}$ is a Poisson process with intensity*

$$\widetilde{\lambda}(t) = \sum_{j=1}^{m+n} \left[1 - \prod_{i=1}^{n} (1 - p_{i,j})\right] \lambda^{c_j}(t).$$

This provides a fatal shock representation of the original not-necessarily-fatal shock set-up. Each obligor i can be represented as

$$N_t^i = \sum_{\pi\in\mathbf{\Pi_n}} \mathbf{1}_{\{i\in\pi\}} N_t^{\pi}.$$

Each default configuration, represented by the subset π, can be alternatively be defined with a n-dimensional vector $\widetilde{x_s} = (x_{s,1}, ..., x_{s,2})$ of zeros and ones such as

$$i \in \pi \Longleftrightarrow x_{s,i} = 1.$$

If $n = 3$, for instance, we have the following mappings

$$\begin{aligned} \{1,2\} &\iff (1,1,0) \\ \{1,3\} &\iff (1,0,1) \\ \{2\} &\iff (0,1,0) \end{aligned}$$

Denoting the fatal shocks N_t^π (and its intensity $\lambda^\pi(t)$) by

$$N_t^{[\widetilde{x_s}]} \triangleq N_t^\pi \text{ and } \lambda^{[\widetilde{x_s}]}(t) \triangleq \lambda^\pi(t),$$

we can express each obligor i as

$$N_t^i = \sum_{s=1}^{2^n} x_{s,i} N_t^{[\widetilde{x_s}]}. \tag{4.12}$$

We shall see that the notations of (4.12) will be useful in the derivation of pricing formulas for basket default swaps.

4.3.2 First-to-Default Swap: $k = 1$

To derive the formula of the FTD Q-factor $Q^{[1]}(T)$, it suffices to observe

$$\begin{aligned} Q^{[1]}(T) &\triangleq \mathbb{P}\left(\tau^{[1]} > T\right) = \mathbb{P}\left(\tau_1 > T, ..., \tau_n > T\right) \\ &= \mathbb{P}\left(\widetilde{N}_T = 0\right) = \exp\left(-\int_0^T \widetilde{\lambda}(t)\, dt\right). \end{aligned}$$

Proposition 37 *The survival probability of the first-to-default time is given by*

$$\mathbb{P}\left(\tau^{[1]} > T\right) = \exp\left(-\int_0^T \lambda^{[1]}(t)\, dt\right),$$

where

$$\lambda^{[1]}(t) = \sum_{j=1}^m \left[1 - \prod_{i=1}^n (1 - p_{i,j})\right] \left[\lambda^{c_j}(t)\right] + \left[\sum_{i=1}^n \lambda^{0,i}(t)\right]. \tag{4.13}$$

Now, we turn to evaluating the density $\mathbb{P}\left(\tau^{[1]} = \tau_i, \tau^{[1]} \in dt\right)$, which is needed for the protection leg when the recovery rates are non-homogeneous. The formulas in Laurent and Gregory (2002), are based on the assumption that the probability of simultaneous defaults is equal to zero; in the Marshall-Olkin model, this is not the case; we need to examine the instantaneous joint defaults case a bit closer. Indeed, if joint defaults occur, we can choose which reference obligation to deliver. One market convention is to deliver the bond with the lowest recovery value. We assume, without loss of generality, that the underlying references in the basket are ordered such that

$\delta_1 \leq \delta_2 \leq ... \leq \delta_n$. This means that the i^{th} reference is delivered if $\tau_i \in (t-\epsilon, t]$ and all the other references with lower recoveries $(\tau_1, \tau_2, ..., \tau_{i-1})$ are alive. In other words, the density is given by

$$\begin{aligned}
\mathbb{P}\left(\tau_i = \tau^{[1]}, \tau^{[1]} \in dt\right) &= \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \mathbb{P}\left(\tau^{[1]} > t-\epsilon, \tau_i \in (t-\epsilon, t], \bigcap_{l=1}^{i-1} \tau_l \notin (t-\epsilon, t]\right) \\
&= \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \mathbb{P}\left(\widetilde{N}_{t-\epsilon} = 0, N_t^i - N_{t-\epsilon}^i = 1, \bigcap_{l=1}^{i-1} N_t^l - N_{t-\epsilon}^l = 0\right) \\
&= \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \mathbb{P}\left(\widetilde{N}_{t-\epsilon} = 0\right) \mathbb{P}\left(N_t^i - N_{t-\epsilon}^i = 1, \bigcap_{l=1}^{i-1} N_t^l - N_{t-\epsilon}^l = 0\right).
\end{aligned}$$

To show that

$$\lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \mathbb{P}\left(N_t^i - N_{t-\epsilon}^i = 1, \bigcap_{l=1}^{i-1} N_t^l - N_{t-\epsilon}^l = 0\right) = \sum_{j=1}^{m} \left[p_{i,j} \prod_{l=1}^{i-1} (1-p_{l,j})\right] \left[\lambda^{c_j}(t)\right] + \left[\lambda^{0,i}(t)\right],$$

we argue as follows. The probability to have more than one common market event in $(t-\epsilon, t]$ is of order $o(\epsilon)$. Hence

$$\begin{aligned}
&\mathbb{P}\left(N_t^i - N_{t-\epsilon}^i = 1, \bigcap_{l=1}^{i-1} N_t^l - N_{t-\epsilon}^l = 0\right) \\
&= \sum_{j=1}^{m+n} \left[\begin{array}{c} \mathbb{P}\left(N_t^i - N_{t-\epsilon}^i = 1, \bigcap_{l=1}^{i-1} N_t^l - N_{t-\epsilon}^l = 0 \,\middle|\, N_t^{c_j} - N_{t-\epsilon}^{c_j} = 1, \bigcap_{k \neq j} N_t^{c_k} - N_{t-\epsilon}^{c_k} = 0\right) \\ \times \mathbb{P}\left(N_t^{c_j} - N_{t-\epsilon}^{c_j} = 1, \bigcap_{k \neq j} N_t^{c_k} - N_{t-\epsilon}^{c_k} = 0\right) \end{array}\right] + o(\epsilon).
\end{aligned}$$

But, since we have

$$N_t^i - N_{t-\epsilon}^i = \sum_{j=1}^{m+n} \sum_{\theta_r^{c_j} \in (t-\epsilon, t]} A_{\theta_r^{c_j}}^{i,j},$$

Then,

$$\begin{aligned}
&\mathbb{P}\left(N_t^i - N_{t-\epsilon}^i = 1, \bigcap_{l=1}^{i-1} N_t^l - N_{t-\epsilon}^l = 0\right) \\
&= \sum_{j=1}^{m+n} \mathbb{P}\left(A_{t-\epsilon}^{i,j} = 1, \bigcap_{l=1}^{i-1} A_{t-\epsilon}^{l,j} = 0\right) \mathbb{P}\left(N_t^{c_j} - N_{t-\epsilon}^{c_j} = 1\right) + o(\epsilon) \\
&= \sum_{j=1}^{m+n} \left[p^{i,j} \prod_{l=1}^{i-1} \left(1 - p^{l,j}\right)\right] \left[\epsilon \lambda^{c_j}(t)\right] + o(\epsilon).
\end{aligned}$$

Taking the limit, we arrive at the following expression:

$$\frac{\mathbb{P}\left(\tau_i = \tau^{[1]}, \tau^{[1]} \in dt\right)}{Q^{[1]}(t)} = \sum_{j=1}^{m} \left[p_{i,j} \prod_{l=1}^{i-1} (1-p_{l,j})\right] \left[\lambda^{c_j}(t)\right] + \left[\lambda^{0,i}(t)\right]. \tag{4.14}$$

If we had a different convention for the bonds to be delivered in the case of joint defaults, we would need to re-order the default times according to the recovery rate delivery rule, and formula (4.14) would hold for the re-ordered basket.

4.3.3 k^{th}-to-Default Swap: $k > 1$

In order to evaluate the Q-factor of the random time $\tau^{[k]}$, for $k > 1$, we have to enumerate all the possible basket default configurations, and to compute their corresponding probabilities. We shall represent each default configuration with a n-dimensional vector of zeros and ones, $\widetilde{x_s} = (x_{s,1}, ..., x_{s,2})$ for $s = 1, ..., 2^n$: if $x_{s,i} = 1$, then the i^{th} reference has defaulted. By $d(\widetilde{x_s}) = \sum_{i=1}^{n} x_{s,i}$, we denote the number of defaults in this configuration. For example, if we have $n = 3$ underlying credits, the basket default configurations are: $\widetilde{x_1} = (0,0,0)$, $\widetilde{x_2} = (1,0,0)$, $\widetilde{x_3} = (0,1,0)$, $\widetilde{x_4} = (1,1,0)$, $\widetilde{x_5} = (0,0,1)$, $\widetilde{x_6} = (1,0,1)$, $\widetilde{x_7} = (0,1,1)$, $\widetilde{x_8} = (1,1,1)$.

Let $Q^{[\widetilde{x_s}]}(T)$ denote the probability of the default configuration $\widetilde{x_s}$ on the interval $(0, T]$:

$$Q^{[\widetilde{x_s}]}(T) \triangleq \mathbb{E}\left[\prod_{i=1}^{n} \left(D_T^i\right)^{x_{s,i}} \left(1 - D_T^i\right)^{1-x_{s,i}}\right], \tag{4.15}$$

with the convention $0^0 = 1$. By definition, summing up the probabilities of all configurations, such that $d(\widetilde{x_s}) < k$, gives the value of the Q-factor $Q^{[k]}(T)$:

$$Q^{[k]}(T) = \sum_{\{s:d(\widetilde{x_s})<k\}} Q^{[\widetilde{x_s}]}(T).$$

It is easy to see, from equation (4.15), that we can write $Q^{[\widetilde{x_s}]}(T)$ as a linear combination of FTD Q-factors on subsets of $\{\tau_1, ..., \tau_n\}$. We introduce the notation $\mathbf{\Pi_n}$ to represent the set of all subsets of $\{1, ..., n\}$. For each subset $\pi \in \mathbf{\Pi_n}$, we define the first-to-default random time $\tau_\pi^{[1]} = \min\{\tau_i : i \in \pi\}$, and its Q-factor $Q_\pi^{[1]}(T)$:

$$Q_\pi^{[1]}(T) = \mathbb{P}\left(\tau_\pi^{[1]} > T\right) = \mathbb{E}\left[\prod_{i\in\pi} \left(1 - D_T^i\right)\right].$$

In the example with $n = 3$, we have the following *basis* of first-to-default Q-factors: $\left\{1, Q_{\{1\}}^{[1]}(T), Q_{\{2\}}^{[1]}(T), Q_{\{3\}}^{[1]}(T), Q_{\{1,2\}}^{[1]}(T), Q_{\{1,3\}}^{[1]}(T), Q_{\{2,3\}}^{[1]}(T), Q_{\{1,2,3\}}^{[1]}(T)\right\}$. Note that $Q_\pi^{[1]}(T)$ for the empty set $\pi = \emptyset$ is equal to 1. For the first configuration $\widetilde{x_1} = (0,0,0)$, the Q-factor $Q^{[\widetilde{x_1}]}(T)$ is given by

$$\mathbb{P}((0,0,0)) = \mathbb{E}\left[\left(1 - D_T^1\right)\left(1 - D_T^2\right)\left(1 - D_T^3\right)\right] = Q_{\{1,2,3\}}^{[1]}(T).$$

For the second configuration $\widetilde{x_2} = (1,0,0)$, we have

$$\mathbb{P}((0,0,0)) = \mathbb{E}\left[D_T^1\left(1 - D_T^2\right)\left(1 - D_T^3\right)\right] = Q_{\{2,3\}}^{[1]}(T) - Q_{\{1,2,3\}}^{[1]}(T).$$

In general, the default configuration Q-factors can be written as

$$Q^{[\widetilde{x_s}]}(T) = \sum_{\pi \in \mathbf{\Pi_n}} \alpha_\pi^{\widetilde{x_s}} Q_\pi^{[1]}(T), \tag{4.16}$$

where the coefficients $\alpha_\pi^{\widetilde{x_s}}$ take values in $\{-1, 0, 1\}$. For $n = 3$, the α_π-coefficients for each configuration are:

				1	$Q^{[1]}_{\{1\}}$	$Q^{[1]}_{\{2\}}$	$Q^{[1]}_{\{3\}}$	$Q^{[1]}_{\{1,2\}}$	$Q^{[1]}_{\{1,3\}}$	$Q^{[1]}_{\{1,3\}}$	$Q^{[1]}_{\{1,2,3\}}$
0	0	0		0	0	0	0	0	0	0	1
1	0	0		0	0	0	0	0	0	1	−1
0	1	0		0	0	0	0	0	1	0	−1
1	1	0	→	0	0	0	0	1	−1	−1	1
0	0	1		0	0	0	0	1	0	0	−1
1	0	1		0	0	1	0	−1	0	−1	1
0	1	1		0	1	0	0	−1	−1	0	1
1	1	1		1	−1	−1	−1	1	1	1	−1

In the general case, this table can be constructed recursively as follows.

We define a partition of $\mathbf{\Pi_{n+1}}$: $\mathbf{\Pi_{n+1}} = \mathbf{\Pi^+_{n+1}} \cup \mathbf{\Pi^-_{n+1}}$, and $\mathbf{\Pi^+_{n+1}} \cap \mathbf{\Pi^-_{n+1}} = \emptyset$. By $\mathbf{\Pi^+_{n+1}}$, we denote the set of all subsets that contain $(n+1)$, and by $\mathbf{\Pi^-_{n+1}}$, the set of all subsets that do not contain $(n+1)$:

$$\begin{aligned} \mathbf{\Pi^+_{n+1}} &= \left\{\pi_{n+1} : \pi_{n+1} \in \mathbf{\Pi_{n+1}},\ (n+1) \in \pi_{n+1}\right\}, \\ \mathbf{\Pi^-_{n+1}} &= \left\{\pi_{n+1} : \pi_{n+1} \in \mathbf{\Pi_{n+1}},\ (n+1) \notin \pi_{n+1}\right\}. \end{aligned}$$

For a $(n+1)$-dimensional basket, the default configurations are either $\left(\widetilde{x^n_s}, 1\right)$ or $\left(\widetilde{x^n_s}, 0\right)$, where $\widetilde{x^n_s}$ is the default configuration for a n-dimensional basket:

$$\begin{aligned} Q^{\left[\left(\widetilde{x^n_s},0\right)\right]}(T) &= \mathbb{E}\left[\left(1 - D_T^{n+1}\right) \prod_{i=1}^{n} \left(D_T^i\right)^{x^n_{s,i}} \left(1 - D_T^i\right)^{1-x^n_{s,i}}\right] \\ &= \mathbb{E}\left[\left(1 - D_T^{n+1}\right) \sum_{\pi_n \in \mathbf{\Pi_n}} \alpha_{\pi_n} \prod_{i \in \pi_n} \left(1 - D_T^i\right)\right] \\ &= \mathbb{E}\left[\sum_{\pi_{n+1} \in \mathbf{\Pi^+_{n+1}}} \alpha_{\pi_n} \prod_{i \in \pi_{n+1}} \left(1 - D_T^i\right)\right], \end{aligned}$$

$$
\begin{aligned}
Q^{\left[\left(\widetilde{x_s^n},1\right)\right]}(T) &= \mathbb{E}\left[\left(D_T^{n+1}\right)\prod_{i=1}^{n}\left(D_T^i\right)^{x_{s,i}^n}\left(1-D_T^i\right)^{1-x_{s,i}^n}\right] \\
&= \mathbb{E}\left[\left(D_T^{n+1}\right)\sum_{\pi_n\in\mathbf{\Pi_n}}\alpha_{\pi_n}\prod_{i\in\pi_n}\left(1-D_T^i\right)\right] \\
&= \mathbb{E}\left[\sum_{\pi_n\in\mathbf{\Pi_n}}\alpha_{\pi_n}\prod_{i\in\pi_n}\left(1-D_T^i\right)\right]-\mathbb{E}\left[\left(1-D_T^{n+1}\right)\sum_{\pi_n\in\mathbf{\Pi_n}}\alpha_{\pi_n}\prod_{i\in\pi_n}\left(1-D_T^i\right)\right] \\
&= \mathbb{E}\left[\sum_{\pi_{n+1}\in\mathbf{\Pi^-_{n+1}}}\alpha_{\pi_n}\prod_{i\in\pi_{n+1}}\left(1-D_T^i\right)\right]-\mathbb{E}\left[\sum_{\pi_{n+1}\in\mathbf{\Pi^+_{n+1}}}\alpha_{\pi_n}\prod_{i\in\pi_{n+1}}\left(1-D_T^i\right)\right].
\end{aligned}
$$

In summary, we have the following recursion:

$$
\begin{aligned}
\alpha_{\pi_{n+1}}^{\left(\widetilde{x_s^n},0\right)} &= 0, \text{ for } \pi_{n+1}\in\mathbf{\Pi^-_{n+1}}, \\
\alpha_{\pi_{n+1}}^{\left(\widetilde{x_s^n},0\right)} &= \alpha_{\pi_n}^{\widetilde{x_s^n}}, \text{ for } \pi_{n+1}\in\mathbf{\Pi^+_{n+1}} \text{ and } \pi_{n+1}=\pi_n\cup\{n+1\},
\end{aligned}
$$

and

$$
\begin{aligned}
\alpha_{\pi_{n+1}}^{\left(\widetilde{x_s^n},1\right)} &= \alpha_{\pi_n}^{\widetilde{x_s^n}}, \text{ for } \pi_{n+1}\in\mathbf{\Pi^-_{n+1}} \text{ and } \pi_{n+1}=\pi_n\cup\emptyset, \\
\alpha_{\pi_{n+1}}^{\left(\widetilde{x_s^n},1\right)} &= -\alpha_{\pi_n}^{\widetilde{x_s^n}}, \text{ for } \pi_{n+1}\in\mathbf{\Pi^+_{n+1}} \text{ and } \pi_{n+1}=\pi_n\cup\{n+1\}.
\end{aligned}
$$

Once we have generated the α_π-representation (equation (4.16)) for each default configuration, and we have computed the subset FTD Q-factors (from Proposition 37), we are in a position to evaluate the k^{th}-to-default Q-factor $Q^{[k]}(T)$.

Proposition 38 *Using the notations in this section, the survival probability of the k^{th}-to-default random time is given by*

$$
\mathbb{P}\left(\tau^{[k]}>T\right)=\sum_{\{s:d(\widetilde{x_s})<k\}}\sum_{\pi\in\mathbf{\Pi_n}}\alpha_\pi^{\widetilde{x_s}}Q_\pi^{[1]}(T),
$$

where the $\{-1,0,1\}$-valued coefficients $\alpha_\pi^{\widetilde{x_s}}$ are computed recursively.

As mentioned in Section 4.2, for non-homogeneous recovery rates, we need to evaluate the density $\mathbb{P}\left(\tau_i=\tau^{[k]},\tau^{[k]}\in dt\right)$. First, let us remark that credit i would be the k^{th}-to-default obligor in an infinitesimal interval $(t-\epsilon,t]$ iff:

$$
\left\{\tau_i=\tau^{[k]},\tau^{[k]}\in(t-\epsilon,t]\right\}\Leftrightarrow\left\{D_t^i-D_{t-\epsilon}^i=1,\sum_{l=1}^{n}D_{t-\epsilon}^l+\sum_{l=1}^{i}D_t^l-D_{t-\epsilon}^l=k\right\}.
$$

This corresponds to the default of credit i being in the interval $(t-\epsilon,t]$, and the total number of defaults is equal to k. If there are instantaneous joint defaults, only the credits in the set $\{1,...,i\}$ define the default-rank for credit i (the references are assumed to be ordered according to the pre-specified recovery-delivery rule). In other words, a credit l would be considered to be in default before credit i if $\tau_l < \tau_i$, or $\tau_l = \tau_i$ and $l < i$. Thus, we need to compute

$$\begin{aligned}\mathbb{P}\left(\tau_i = \tau^{[k]}, \tau^{[k]} \in dt\right) &= \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \mathbb{P}\left(D_t^i - D_{t-\epsilon}^i = 1, \sum_{l=1}^{n} D_{t-\epsilon}^l + \sum_{l=1}^{i} D_t^l - D_{t-\epsilon}^l = k\right) \\ &= \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \mathbb{E}\left[\left(D_t^i - D_{t-\epsilon}^i\right) \mathbf{1}_{\left\{\sum_{l=1}^{n} D_{t-\epsilon}^l + \sum_{l=1}^{i} D_t^l - D_{t-\epsilon}^l = k\right\}}\right]\end{aligned}$$

But we have

$$\mathbb{P}\left(D_t^i - D_{t-\epsilon}^i = 1\right) = \left(1 - D_{t-\epsilon}^i\right) \mathbb{P}\left(N_t^i - N_{t-\epsilon}^i > 0\right).$$

Hence

$$\begin{aligned}&\mathbb{E}\left[\left(D_t^i - D_{t-\epsilon}^i\right) \mathbf{1}_{\left\{\sum_{l=1}^{n} D_{t-\epsilon}^l + \sum_{l=1}^{i} D_t^l - D_{t-\epsilon}^l = k\right\}}\right] \\ = \;&\mathbb{E}\left[\left(1 - D_{t-\epsilon}^i\right)\left(N_t^i - N_{t-\epsilon}^i\right) \mathbf{1}_{\left\{\sum_{l=1}^{n} D_{t-\epsilon}^l + \sum_{l=1}^{i} \left(1 - D_{t-\epsilon}^l\right)\left(N_t^l - N_{t-\epsilon}^l\right) = k\right\}}\right] + o(\epsilon).\end{aligned}$$

Using the equivalent fatal shock representation, we know that

$$N_t^i - N_{t-\epsilon}^i = \sum_{\pi \in \mathbf{\Pi_n}} x_{s,i} \left(N_t^{[\widetilde{x_s}]} - N_{t-\epsilon}^{[\widetilde{x_s}]}\right),$$

and

$$D_t^i = \sum_{s=1}^{2^n} x_{s,i} D_t^{[\widetilde{x_s}]}.$$

Combining the probabilities of the default configurations $\widetilde{x_{s_1}}$, in the interval $(0, t-\epsilon]$, and the probabilities of the configurations $\widetilde{x_{s_2}}$, in the instantaneous interval $(t-\epsilon, t]$, we arrive at

$$\begin{aligned}&\mathbb{E}\left[\left(1 - D_{t-\epsilon}^i\right)\left(N_t^i - N_{t-\epsilon}^i\right) \mathbf{1}_{\left\{\sum_{l=1}^{n} D_{t-\epsilon}^l + \sum_{l=1}^{i} \left(1 - D_{t-\epsilon}^l\right)\left(N_t^l - N_{t-\epsilon}^l\right) = k\right\}}\right] \\ = \;&\sum_{s_1=1}^{2^n} \sum_{s_2=1}^{2^n} \left[\left(1 - x_{s_1,i}\right) x_{s_2,i} \mathbf{1}_{\left\{d\left(\widetilde{x_{s_1}}\right) + \sum_{l=1}^{i} \left(1 - x_{s_1,l}\right) x_{s_2,l} = k\right\}}\right] Q^{[\widetilde{x_{s_1}}]}(t) \left[\epsilon \lambda^{[\widetilde{x_{s_2}}]}(t)\right] + o(\epsilon),\end{aligned}$$

which is a weighted average over all possible scenarios. This is the convolution between the basket default distributions on the intervals $(0, t-\epsilon]$ and $(t-\epsilon, t]$, weighted by the indicator function of the default scenarios that contribute to the recovery payoff $(1 - \delta_i)$.

The density follows by taking the limit

$$\mathbb{P}\left(\tau_i = \tau^{[k]}, \tau^{[k]} \in dt\right) = \sum_{s_1=1}^{2^n} \sum_{s_2=1}^{2^n} Q^{[\widetilde{x_{s_1}}]}(t)\, \lambda^{[\widetilde{x_{s_2}}]}(t) \left[\left(1 - x_{s_1,i}\right) x_{s_2,i} \mathbf{1}_{\left\{d\left(\widetilde{x_{s_1}}\right) + \sum_{l=1}^{i} \left(1 - x_{s_1,l}\right) x_{s_2,l} = k\right\}}\right]. \tag{4.17}$$

4.4 Expanding the Baskets

In principle, the direct method requires, for each time step, 2^{n+1} values, corresponding to the possible combinations of s_1 and s_2 in equation (4.17); as the size of the underlying basket increases, the number of default configurations explodes exponentially. This significant limitation restricts the applicability of the method to baskets under 10 or 11 credits. As an alternative, we propose a different approach, which is based on a static replication idea. We show that each k^{th}-to-default swap can be synthetically generated with a portfolio of first-to-default swaps referenced to subsets of the original basket. This result is not very surprising, since sub-FTDs may be viewed as a type of Arrow-Debreu prices of the multivariate default space. Given the set of all sub-FTD prices, one can compute the value of any higher-order basket default swap. The key building block is the FTD evaluator; a fast implementation of FTD prices ensures, in turn, a reasonable run time for k^{th}-to-default swaps.

In this section, we describe how this static FTD replication is done: first, we show the relationship between k^{th}-to-default and $(k-1)^{\text{th}}$-to-default swaps; then, we apply this recursion step-by-step until we arrive at the complete *FTD expansion*.

4.4.1 The Recursive Formula

To illustrate the recursive relationship, we start with a simple example, where we have a basket of $n = 3$ credits: A, B and C. We observe that the $\{A, B, C\}$-second-to-default swap can be replicated with the first-to-default swaps referenced to the sub-baskets $\{A, B\}$, $\{A, C\}$, $\{B, C\}$ and $\{A, B, C\}$ as:

$$STD\,(A, B, C) = FTD\,(A, B) + FTD\,(A, C) + FTD\,(B, C) - 2FTD\,(A, B, C)\,. \quad (4.18)$$

To see this, it suffices to verify that the payoffs of the two sides of the equation match in all default scenarios. Since equation (4.18) is symmetric in A, B and C, we suppose, for instance, that credit A defaults, and we denote its recovery rate by δ_A. On the left-hand side, $STD\,(A, B, C)$ becomes $FTD\,(B, C)$, and makes no cash payments; on the right-hand side, $FTD\,(A, B)$ pays $(1 - \delta_A)$ and terminates, $FTD\,(A, C)$ pays $(1 - \delta_A)$ and terminates, $FTD\,(B, C)$ is unchanged and does not make any cash payments, and finally $FTD\,(A, B, C)$ pays $(1 - \delta_A)$ and terminates. In summary, we have

PV		PV	cash balance
$FTD\,(A,B)$	$\rightarrow$	0	$(1-\delta_A)$
$FTD\,(A,C)$	$\rightarrow$	0	$(1-\delta_A)$
$FTD\,(B,C)$	$\rightarrow$	$FTD\,(B,C)$	0
$-2FTD\,(A,B,C)$	$\rightarrow$	0	$-2\,(1-\delta_A)$
Total	$\rightarrow$	$FTD\,(B,C)$	0

PV		PV	cash balance
$STD\,(A,B,C)$	$\rightarrow$	$FTD\,(B,C)$	0

For the premium leg, all the trades pay one unit of premium before default. If A defaults, $STD\,(A,B,C)$ and $FTD\,(B,C)$ keep paying one unit of premium each, and the premium payments stop for all the other FTDs which contain the reference A: $FTD\,(A,B)$, $FTD\,(A,C)$, $FTD\,(A,B,C)$,

	Before default	After A defaults
$FTD\,(A,B)$	1	0
$FTD\,(A,C)$	1	0
$FTD\,(B,C)$	1	1
$-2FTD\,(A,B,C)$	-2	0
Total	1	1

	Before default	After A defaults
$STD\,(A,B,C)$	1	1

A similar analysis shows that we can replicate the $\{A,B,C\}$-third-to-default by second-to-defaults on the sub-baskets $\{A,B\}$, $\{A,C\}$, $\{B,C\}$ and $\{A,B,C\}$ as follows:

$$TTD\,(A,B,C)=\frac{1}{2}\left[STD\,(A,B)+STD\,(A,C)+STD\,(B,C)-STD\,(A,B,C)\right].$$

Now, we turn to the general case, and we establish the result for a basket of n credit. We denote by $V_n^{[k]}\,(A_1,...,A_n)$ the value of a k^{th}-to-default swap referenced to the n-dimensional basket $\{A_1,...,A_n\}$. We also define the l-subsets of $\{1,2,...,n\}$ (i.e. subsets containing exactly l elements) by the mappings $\pi_s^l\,(.)$, for $1\leq s\leq\binom{n}{l}$:

$$\pi_s^l\,(.):\{1,2,...,k\}\rightarrow\{1,2,...,n\}\,.$$

Furthermore, it is notationaly convenient to assume that each l-subset π_s^l is an ordered sequence of indices $\pi_s^l\,(1)\leq\pi_s^l\,(2)\leq...\leq\pi_s^l\,(k)$.

Example 39 *We consider $n = 3$ credits; for subsets of $l = 2$ elements, we have:*

$$\left\{\pi_1^2 : \pi_1^2(1) = 1;\, \pi_1^2(2) = 2\right\},\ \left\{\pi_2^2 : \pi_2^2(1) = 1;\, \pi_2^2(2) = 3\right\},\ \left\{\pi_3^2 : \pi_1^2(1) = 2;\, \pi_3^2(2) = 3\right\}.$$

For a second-to-default $V_n^{[2]}$, we can easily check that the replicating portfolio of FTD swaps contains the $\{A_1, ..., A_n\}$-FTD and all the $(n-1)$-element-sub-FTDs,

$$V_n^{[2]}(A_1, ..., A_n) = \sum_{s=1}^{n} V_{n-1}^{[1]}\left(A_{\pi_s^{n-1}(1)}, ..., A_{\pi_s^{n-1}(n-1)}\right) - (n-1)\, V_n^{[1]}(A_1, ..., A_n).$$

For higher-orders $k \geq 2$, we have a recursion of the form

$$a_n^{[k]} V_n^{[k]}(A_1, ..., A_n) = \underbrace{\sum_{s=1}^{n} V_{n-1}^{[k-1]}\left(A_{\pi_s^{n-1}(1)}, ..., A_{\pi_s^{n-1}(n-1)}\right)}_{S} - b_n^{[k]} V_n^{[k-1]}(A_1, ..., A_n). \tag{4.19}$$

Equation (4.19) is symmetric in $A_1, ..., A_n$. Suppose, for example, that A_1 goes into default. The subsets $\left(\pi_s^{n-1}(.)\right)_{1 \leq s \leq n}$ are ordered such that the last set is $\pi_n^{n-1} = \{A_2, ..., A_n\}$ and all the other sets contain credit A_1. In the sum S, the first $(n-1)$ subsets contain the defaulted reference A_1, hence each sub-basket $V_{n-1}^{[k-1]}\left(A_{\pi_s^{n-1}(1)}, ..., A_{\pi_s^{n-1}(n-1)}\right)$, for $1 \leq s \leq n-1$, becomes $V_{n-2}^{[k-2]}\left(A_{\pi_s^{n-2}(2)}, ..., A_{\pi_s^{n-2}(n-1)}\right)$. The sub-basket $V_{n-1}^{[k-1]}(A_2, ..., A_n)$ remains unchanged and the last term $V_n^{[k-1]}(A_1, ..., A_n)$ becomes $V_{n-1}^{[k-2]}(A_2, ..., A_n)$. It follows that, after default, equation(4.19) becomes

$$\begin{aligned} a_n^{[k]} V_{n-1}^{[k-1]}(A_2, ..., A_n) &= \left[\sum_{s=1}^{n-1} V_{n-2}^{[k-2]}\left(A_{\pi_s^{n-2}(2)}, ..., A_{\pi_s^{n-2}(n-1)}\right) + V_{n-1}^{[k-1]}(A_2, ..., A_n)\right] \\ &\quad - b_n^{[k]} V_{n-1}^{[k-2]}(A_2, ..., A_n); \end{aligned}$$

re-arranging the terms, we obtain

$$\begin{aligned} \left(a_n^{[k]} - 1\right) V_{n-1}^{[k-1]}(A_2, ..., A_n) &= \sum_{s=1}^{n-1} V_{n-2}^{[k-2]}\left(A_{\pi_s^{n-1}(2)}, ..., A_{\pi_s^{n-1}(n-1)}\right) \\ &\quad - b_n^{[k]} V_{n-1}^{[k-2]}(A_2, ..., A_n). \end{aligned} \tag{4.20}$$

On the other hand, applying equation (4.19) to the $(n-1)$-element-basket $\{A_2, ..., A_n\}$ yields

$$\begin{aligned} a_{n-1}^{[k-1]} V_{n-1}^{[k-1]}(A_2, ..., A_n) &= \sum_{s=1}^{n-1} V_{n-2}^{[k-2]}\left(A_{\pi_s^{n-1}(2)}, ..., A_{\pi_s^{n-1}(n-1)}\right) \\ &\quad - b_{n-1}^{[k-1]} V_{n-1}^{[k-2]}(A_2, ..., A_n), \end{aligned} \tag{4.21}$$

which, compared with equation (4.20), gives

$$a_n^{[k]} - 1 = a_{n-1}^{[k-1]}, \text{ and } b_n^{[k]} = b_{n-1}^{[k-1]}. \tag{4.22}$$

Applying the double-recursion (4.22), we arrive at the values of $a_n^{[k]}$:

$$a_n^{[k]} = a_{n-1}^{[k-1]} + 1 = ... = a_{n-(k-2)}^{[2]} + (k-2) = 1 + (k-2) = k-1,$$

and the values of $b_n^{[k]}$:

$$b_n^{[k]} = b_{n-1}^{[k-1]} = ... = b_{n-(k-2)}^{[2]} = n - (k-2) - 1 = n - k + 1.$$

Proposition 40 *The replication of a k^{th}-to-default swap referencing a basket of n credits $\{A_1, ..., A_n\}$, is done with $(n+1)$ $(k-1)^{th}$-to-default swap instruments: the replicating swaps reference the original basket $\{A_1, ..., A_n\}$ and all its $(n-1)$-element-sub-baskets,*

$$V_n^{[k]}(A_1, ..., A_n) = \frac{1}{a_n^{[k]}} \left[\sum_{s=1}^{n} V_{n-1}^{[k-1]} \left(A_{\pi_s^{n-1}(1)}, ..., A_{\pi_s^{n-1}(n-1)} \right) - b_n^{[k]} V_n^{[k-1]}(A_1, ..., A_n) \right], \tag{4.23}$$

with $a_n^{[k]} = k-1$ and $b_n^{[k]} = n-k+1$.

4.4.2 The Complete Expansion

Using the relationship (4.23), it is easy to see that we can proceed recursively until we arrive at the complete expansion in terms of the basic FTD instruments, each FTD swap references a sub-basket of l elements, where l varies between $(n-k+1)$ and n:

$$V_n^{[k]}(A_1, ..., A_n) = \sum_{l=n-k+1}^{n} \sum_{s=1}^{\binom{n}{l}} \alpha_n^{[k]}(l)\, V_l^{[1]} \left(A_{\pi_s^l(1)}, ..., A_{\pi_s^l(k)} \right). \tag{4.24}$$

Note that $\alpha_n^{[k]}(l)$, the instrument contribution to the replicating portfolio, depends solely on the size of the subset π_s^l. This is a consequence of the symmetry with respect to the references $A_1, ..., A_n$. In order to derive the expression of the coefficients $\alpha_n^{[k]}(l)$, we replace each term, on the right-hand side of the recursive relationship (4.23), by its FTD representation (4.24),

$$
\begin{aligned}
V_n^{[k]}(A_1,...,A_n) \;=\;& \frac{1}{a_n^{[k]}}\left[\underbrace{\sum_{t=1}^{n}\sum_{l=n-k+1}^{n-1}\sum_{s=1}^{\binom{n-1}{l}} \alpha_{n-1}^{[k-1]}(l)\, V_l^{[1]}\left(A_{\pi_t^{n-1}\left(\pi_s^l(1)\right)},...,A_{\pi_t^{n-1}\left(\pi_s^l(l)\right)}\right)}_{S_1}\right] \\
& -\frac{b_n^{[k]}}{a_n^{[k]}}\left[\underbrace{\sum_{l=n-k+2}^{n}\sum_{s=1}^{\binom{n}{l}} \alpha_n^{[k-1]}(l)\, V_l^{[1]}\left(A_{\pi_s^l(1)},...,A_{\pi_s^l(l)}\right)}_{S_2}\right]. \qquad (4.25)
\end{aligned}
$$

We simplify the triple sum S_1 by interchanging the summation order and using the following lemma.

Lemma 41

$$
\sum_{t=1}^{n}\sum_{s=1}^{\binom{n-1}{l}} V_l^{[1]}\left(A_{\pi_t^{n-1}\left(\pi_s^l(1)\right)},...,A_{\pi_t^{n-1}\left(\pi_s^l(l)\right)}\right) = (n-l)\sum_{s=1}^{\binom{n}{l}} V_l^{[1]}\left(A_{\pi_s^l(1)},...,A_{\pi_s^l(l)}\right).
$$

Proof. To generate all the subsets of the double mapping $\pi_t^{n-1}\left(\pi_s^l(.)\right)$, for $1 \le t \le n$, $1 \le s \le \binom{n-1}{l}$, first we enumerate all the $(n-1)$-subsets of $\{1,...,n\}$, then for each $(n-1)$-subset, we enumerate all of its l-subsets; in total, we have generated $n \times \binom{n-1}{l}$ subsets of $\{1,...,n\}$ containing exactly l elements. This means that we have spanned the set of all l-subsets a number of times d, where d verifies $n \times \binom{n-1}{l} = d \times \binom{n}{l}$; hence $d = n-l$. ■

The sum S_1, then, becomes

$$
S_1 = \sum_{l=n-k+1}^{n-1} \alpha_{n-1}^{[k-1]}(l)\left[(n-l)\sum_{s=1}^{\binom{n}{l}} V_l^{[1]}\left(A_{\pi_s^l(1)},...,A_{\pi_s^l(l)}\right)\right].
$$

On the other hand, we can re-write the double sum S_2 as

$$
S_2 = \sum_{l=n-k+2}^{n} \alpha_n^{[k-1]}(l)\sum_{s=1}^{\binom{n}{l}} V_l^{[1]}\left(A_{\pi_s^l(1)},...,A_{\pi_s^l(l)}\right).
$$

By splitting the k-sums in S_1 and S_2, equation (4.25) becomes

$$
\begin{aligned}
& V_n^{[k]}\left(A_1,...,A_n\right) \\
= & \sum_{l=n-k+1}^{n-k+1}\left(\frac{n-l}{a_n^{[k]}}\right)\alpha_{n-1}^{[k-1]}\left(l\right)\sum_{s=1}^{\binom{n}{l}}V_l^{[1]}\left(A_{\pi_s^l(1)},...,A_{\pi_s^l(l)}\right) \\
& +\sum_{l=n-k+2}^{n-1}\left(\left(\frac{n-l}{a_n^{[k]}}\right)\alpha_{n-1}^{[k-1]}\left(l\right)-\left(\frac{b_n^{[k]}}{a_n^{[k]}}\right)\alpha_n^{[k-1]}\left(l\right)\right)\sum_{s=1}^{\binom{n}{l}}V_l^{[1]}\left(A_{\pi_s^l(1)},...,A_{\pi_s^l(l)}\right) \\
& -\sum_{l=n}^{n}\left(\frac{b_n^{[k]}}{a_n^{[k]}}\right)\alpha_n^{[k-1]}\left(l\right)\sum_{s=1}^{\binom{n}{l}}V_l^{[1]}\left(A_{\pi_s^l(1)},...,A_{\pi_s^l(l)}\right). \qquad (4.26)
\end{aligned}
$$

Comparing equation (4.26) with the original expansion (4.24), we obtain the following double-recursion:

$$
\begin{cases}
\alpha_n^{[k]}\left(l\right)=\left(-\frac{b_n^{[k]}}{a_n^{[k]}}\right)\alpha_n^{[k-1]}\left(l\right), \text{ for } l=n, \\
\alpha_n^{[k]}\left(l\right)=\left(\frac{n-l}{a_n^{[k]}}\right)\alpha_{n-1}^{[k-1]}\left(l\right)-\left(\frac{b_n^{[k]}}{a_n^{[k]}}\right)\alpha_n^{[k-1]}\left(l\right), \text{ for } n-k+1<l<n, \\
\alpha_n^{[k]}\left(l\right)=\left(\frac{n-l}{a_n^{[m]}}\right)\alpha_{n-1}^{[k-1]}\left(l\right), \text{ for } l=n-k+1.
\end{cases}
$$

Let us solve this double-recursion step-by-step.

We start with the case $l=n$,

$$
\begin{aligned}
\alpha_n^{[k]}\left(n\right) &= \left(-\frac{n-k+1}{k-1}\right)\left(-\frac{n-k}{k-2}\right)...\left(-\frac{n-1}{1}\right)\underbrace{\alpha_n^{[1]}\left(n\right)}_{=1} \\
&= \left(-1\right)^{k-1}\frac{\left(n-1\right)!}{\left(k-1\right)!\left(n-k\right)!}=\left(-1\right)^{k-1}\binom{n-k}{n-1}.
\end{aligned}
$$

Then, for $l=n-k+1$, we have

$$
\begin{aligned}
\alpha_n^{[k]}\left(n-k+1\right) &= \frac{n-\left(n-k+1\right)}{k-1}\alpha_{n-1}^{[k-1]}\left(n-k+1\right) \\
&= \alpha_{n-1}^{[k-1]}\left(n-k+1\right)=...=\alpha_{n-k+1}^{[1]}\left(n-k+1\right)=1.
\end{aligned}
$$

And finally, for $n-k+1<l<n$,

$$
\alpha_n^{[k]}\left(l\right)=\left(\frac{n-l}{k-1}\right)\alpha_{n-1}^{[k-1]}\left(l\right)-\left(\frac{n-k+1}{k-1}\right)\alpha_n^{[k-1]}\left(l\right). \qquad (4.27)
$$

From the shape of equation (4.27), we postulate that the solution is of the form

$$
\alpha_n^{[k]}\left(l\right)=\left(-1\right)^{\beta}\binom{\gamma}{\delta},
$$

and using some elementary algebra, we find that the solution is:

$$
\alpha_n^{[k]}\left(l\right)=\left(-1\right)^{l-(n-k+1)}\binom{l-1}{n-k}.
$$

To summarize, we have the following proposition.

Proposition 42 *The replication of a k^{th}-to-default referencing a basket of n credits $\{A_1, ..., A_n\}$, is done with the first-to-default swaps that reference all the l-subsets of the original basket, where $n-k+1 \leq l \leq n$:*

$$V_n^{[k]}(A_1, ..., A_n) = \sum_{l=n-k+1}^{n} \sum_{s=1}^{\binom{n}{l}} \alpha_n^{[k]}(l)\, V_l^{[1]}\left(A_{\pi_s^l(1)}, ..., A_{\pi_s^l(l)}\right),$$

where $\alpha_n^{[k]}(l) = (-1)^{l-(n-k+1)} \binom{l-1}{n-k}$, for $n-k+1 \leq l \leq n$.

Remark 43 *Note that the first-to-default expansion is generic and is applicable to all copula models. The replication itself is independent of the choice of the multivariate dependence.*

In order to compare the FTD expansion with the direct approach, we apply equation (4.24) to an k^{th}-to-default credit linked zero-coupon cash flow, and we establish the expression of the Q-factor associated with the random time $\tau^{[k]}$

$$\mathbb{P}\left(\tau^{[k]} > T\right) = \sum_{l=n-k+1}^{n} \sum_{s=1}^{\binom{n}{l}} \alpha_n^{[k]}(l)\, Q_{\pi_s^l}^{[1]}(T). \tag{4.28}$$

Recall the expression of $Q^{[k]}(T)$ with the direct method (equation (4.16))

$$\mathbb{P}\left(\tau^{[k]} > T\right) = \sum_{\{s : d(\widetilde{x_s}) < k\}} \sum_{\pi \in \mathbf{\Pi_n}} \alpha_\pi^{\widetilde{x_s}} Q_\pi^{[1]}(T). \tag{4.29}$$

The key difference lies in the fact that instead of generating the entire set of subsets $\mathbf{\Pi_n}$, and add up many terms whose coefficients are $\alpha_\pi^{\widetilde{x_s}} = 0$, we restrict ourselves to those sub-FTDs $Q_{\pi_s^l}^{[1]}(T)$ that contribute to the value of the sum. Indeed, looking back at the 2^n-by-2^n matrix $\left[\alpha_\pi^{\widetilde{x_s}}\right]$, in the example of Subsection 4.3.2, one notices that it is actually very sparse. The only extreme case where all the sub-FTDs are required is for the last-to-default swap $k = n$.

4.5 The Homogeneous Transformation

In general, the number of sub-FTDs in the replication formula (4.24) is a function of n, the size of the basket $\{A_1, ..., A_n\}$ and k, the order of the basket default swap. For instance, a STD swap is replicated with $(1+n)$ sub-FTD instruments, a TTD swap is replicated with $\left(1+n+\frac{n(n+1)}{2}\right)$ sub-FTD instruments, and so on. A k^{th}-to-default swap, in general, is replicated with all the FTD instruments referencing the l-sub-baskets, where l varies between $(n-m+1)$ and n; the size of this replicating portfolio is given by the truncated binomial expansion $N(k,n) = 1 + \binom{n-1}{n} + ... + \binom{n-k+1}{n}$.

Clearly, for small values of k and n, the total number of sub-FTDs is reasonable, but when one considers higher default orders, the combinatorial coefficients are larger, and may be impossible to handle numerically in extreme cases: a last-to-default swap, for example, requires a total of $N(n,n) = 2^n - 1$ sub-FTDs.

The most time-consuming step in the evaluation is the generation of the sub-FTDs $Q^{[1]}_{\pi^l_s}(T)$, for all combinations $\left\{\pi^l_s : \pi^l_s \in \mathbf{\Pi_n},\ \left|\pi^l_s\right| = l\right\}$. If we had a homogeneous basket, then, for a given subset size l, all the FTD instruments would have exactly the same value; and equation (4.28) would then collapse to

$$\mathbb{P}\left(\tau^{[k]} > T\right) = \sum_{l=n-k+1}^{n} \alpha^{[k]}_n(l) \binom{n}{l} Q^{[1]}_{\pi^l}(T),$$

where π^l is one l-subset of the homogeneous basket. The number of sub-FTDs to compute, would reduce to one evaluation per l-subset, hence a total of $N(k,n) = k$ FTD evaluations for the whole k^{th}-to-default swap. So, our first approximation is to transform the original non-homogeneous basket to a homogeneous one, and to preserve some properties of the aggregate default distribution. The homogeneous transformation idea goes back to Moody's binomial expansion and diversity-score approach: they consider a transformation that preserves the mean and the variance of the portfolio loss distribution. It is a well-known fact, however, that the first and second moments are not enough to represent heavy-tailed loss distributions. In our approach, for each default order, we consider the corresponding percentile of the aggregate default distribution, and we require that this quantity remains invariant with respect to the homogeneous approximation. We shall see that this transformation is exact for a FTD swap, and that for higher-order defaults our approximation appears to give a very good accuracy.

4.5.1 An Illustrative Example

We begin with a first-to-default swap, and we work out the homogeneous transformation that preserves its present value. We consider the non-homogeneous basket $\{A_1, ..., A_n\}$, which is represented with its MO decomposition: its market factor loadings matrix $[p_{i,j}]_{\substack{1\le i\le n \\ 1\le j\le m}}$, and its idiosyncratic intensities vector $\left(\lambda^{0,i}\right)_{1\le i\le n}$. Our objective is to find an equivalent portfolio $\{A^*_1, ..., A^*_n\}$, where all the underlying credits A^*_i have the same MO representation: $\left(\left(p^*_j\right)_{1\le j\le m}, \lambda^{0,*}\right)$, and the value of the FTD swap referencing this new portfolio is equal to the value of the original non-homogeneous FTD. As discussed in Subsection 4.3.1, the FTD random time $\tau^{[1]}$, in a MO model, is exponentially distributed and its hazard rate $\lambda^{[1]}(t)$ is given by equation (4.14):

$$\lambda^{[1]}(\{A_1,...,A_n\}) = \sum_{j=1}^{m}\left[1-\prod_{i=1}^{n}(1-p_{i,j})\right][\lambda^{c_j}(t)] + \left[\sum_{i=1}^{n}\lambda^{0,i}(t)\right].$$

To ensure that the two FTD swaps have the same value, we need to equate the hazard rates of the FTD times $\tau^{[1]}(\{A_1,...,A_n\}) = \min(\tau_1,...,\tau_n)$ and $\tau^{[1]}(\{A_1^*,...,A_n^*\}) = \min(\tau_1^*,...,\tau_n^*)$:

$$\lambda^{[1]}(\{A_1,...,A_n\}) = \lambda^{[1]}(\{A_1^*,...,A_n^*\}),\ \forall [\lambda^{c_j}(t)] \in \mathbb{R}^+,\ 1 \le j \le m. \tag{4.30}$$

Substituting the FTD hazard rates with their formulas, and because the equality (4.30) holds for all values of the factor intensities $\lambda_j^C(t)$, one gets

$$1-\prod_{i=1}^{n}(1-p_{i,j}) = 1-\left(1-p_j^*\right)^n, \text{ for } 1 \le j \le m; \text{ and } \sum_{i=1}^{n}\lambda^{0,i}(t) = n\lambda^{0,*}(t).$$

This completes the definition of the equivalent portfolio transformation. The FTD equivalent homogeneous portfolio is obtained by taking the arithmetic average of the idiosyncratic intensities, and the geometric average the market factor conditional survival probabilities $q_{i,j} = 1 - p_{i,j}$:

$$\begin{aligned} p_j^* &= 1-\exp\left(\frac{1}{n}\sum_{i=1}^{n}\log(1-p_{i,j})\right), \\ \lambda^{0,*}(t) &= \frac{1}{n}\sum_{i=1}^{n}\lambda^{0,i}(t). \end{aligned}$$

4.5.2 The Homogeneous Transformation

Now, we turn to the general case of higher-order default swaps. We want to define the transformation

$$\left([p_{i,j}]_{\substack{1\le i\le n\\1\le j\le m}}, \left(\lambda^{0,i}\right)_{1\le i\le n}\right) \to \left(\left(p_j^*\right)_{1\le j\le m}, \lambda^{0,*}\right),$$

such that the value of the k^{th}-to-default is the same for the two portfolios $\{A_1,...,A_n\}$ and $\{A_1^*,...,A_n^*\}$. The key is to match the survival probabilities of the random default time $\tau^{[k]}$. Recall that the Q-factor $Q^{[k]}(T)$ and the portfolio aggregate defaults X_T are related as:

$$Q^{[k]}(T) = \mathbb{P}\left(\tau^{[k]} > T\right) = \mathbb{P}(X_T < k).$$

In order to study the properties of the random time $\tau^{[k]}$, we work with the aggregate default distribution X_T, and its probability generating function $\varphi(x)$. We show that $\varphi(x)$ can be conveniently expressed in terms of the basis FTD Q-factors $Q_\pi^{[1]}(T)$ of Subsection 4.3.2.

Theorem 44 *The probability generating function $\varphi(x)$ of the random variable X_T is given by:*

$$\varphi(x) = \sum_{\pi \in \mathbf{\Pi_n}} Q_{\pi}^{[1]}(T) \; x^{n-d(\pi)} (1-x)^{d(\pi)}. \tag{4.31}$$

Proof. See Appendix A.1. ■

Remark 45 *Observing that the survival probability $Q^{[k]}(T)$ can be recovered from the probability generating function $\varphi(x)$, one gets*

$$Q^{[k]}(T) = \sum_{l=0}^{k-1} \frac{\varphi^{(l)}(x)}{l!}.$$

Theorem 44 provides another proof of the FTD expansion formula (4.28) in Subsection 4.4.2.

Letting $\Lambda^{c_j}(T)$ and $\Lambda^{0,i}(T)$ denote the cumulative market factor and idiosyncratic intensities respectively, i.e., $\Lambda^{c_j}(T) = \int_0^T \lambda^{c_j}(t)\,dt$, for $j = 1,...,m$, and $\Lambda^{0,i}(T) = \int_0^T \lambda^{0,i}(t)\,dt$, for $i = 1,...,n$, the value of each sub-FTD Q-factor is given by equation (4.13),

$$Q_{\pi}^{[1]}(T) = \exp\left(-\left[\sum_{j=1}^{m}\left(1-\prod_{i\in\pi}(1-p_{i,j})\right)\Lambda^{c_j}(T)\right] - \left[\sum_{i\in\pi}\Lambda^{0,i}(T)\right]\right). \tag{4.32}$$

In order to remove the non-linear dependence on the market factor intensities, we expand the exponential function to first-order[1]: $\exp\left(\sum_{j=1}^{m}\left(\prod_{i\in\pi}(1-p_{i,j})\right)\Lambda^{c_j}(T)\right) \simeq 1 + \sum_{j=1}^{m}\left(\prod_{i\in\pi}(1-p_{i,j})\right)\Lambda^{c_j}(T)$, and we define a new set of adjusted market factor loadings $\widetilde{p_{i,j}} = 1 - e^{-\Lambda^{0,i}(T)}(1-p_{i,j})$. Then, equation (4.32) becomes

$$Q_{\pi}^{[1]}(T) \simeq \exp\left(-\sum_{j=1}^{m}\Lambda^{c_j}(T)\right)\left[\exp\left(-\sum_{i\in\pi}\Lambda^{0,i}(T)\right) + \sum_{j=1}^{m}\left(\prod_{i\in\pi}(1-\widetilde{p_{i,j}})\right)\Lambda^{c_j}(T)\right]. \tag{4.33}$$

In the spirit of equation (4.14), used in the FTD case, this new formula expresses the survival probability as a linear combination of market factor intensities and a separate idiosyncratic contribution. Direct substitution of equation (4.33) in the expression of the probability generating function of Theorem 44 shows that we can write $\varphi(x)$ as a

[1] This expansion is a very good approximation since the values of the intensities $\left(\Lambda_j^c(T)\right)_{1\le j\le m}$ are usually small.

linear combination of functions $\varphi_j^c(x)$ and $\varphi_0(x)$, defined as:

$$\varphi_j^c(x) = \sum_{\pi\in\mathbf{\Pi_n}} \left(\prod_{i\in\pi}(1-\widetilde{p_{i,j}})\right) x^{n-d(\pi)}(1-x)^{d(\pi)},$$

$$\varphi_0(x) = \sum_{\pi\in\mathbf{\Pi_n}} \exp\left(-\sum_{i\in\pi}\Lambda^{0,i}(T)\right) x^{n-d(\pi)}(1-x)^{d(\pi)},$$

and which can be interpreted as the market factor conditional probability generating functions and the idiosyncratic probability generating function, respectively. Indeed, using Theorem 44, we can easily see that $\varphi_0(x)$ is the p.g.f. of a sum of independent Bernoulli variables X_i^0, where $\mathbb{P}\left(X_i^0=1\right)=1-e^{-\Lambda^{0,i}(T)}$, and $\varphi_j^c(x)$ is the p.g.f. of a sum of independent Bernoulli variables X_i^j, where $\mathbb{P}\left(X_i^j=1\right)=\widetilde{p_{i,j}}$. Inverting the probability generating function $\varphi(x)$,

$$\varphi(x) \simeq \exp\left(-\sum_{j=1}^m \Lambda^{c_j}(T)\right)\left[\varphi_0(x)+\sum_{j=1}^m \Lambda^{c_j}(T)\left[\varphi_j^c(x)\right]\right],$$

gives the value of the k^{th}-to-default Q-factor $\mathbb{P}(X_T<k)$ as a linear combination of market factor contributions and an idiosyncratic term, which allows a one-to-one mapping between the two portfolios,

$$\mathbb{P}(X_T<k) \simeq \exp\left(-\sum_{j=1}^m \Lambda^{c_j}(T)\right)\left[Q_0^{[k]}(T)+\sum_{j=1}^m Q_j^{[k]}(T)\Lambda^{c_j}(T)\right], \tag{4.34}$$

where

$$Q_0^{[k]}(T)=\mathbb{P}\left(\sum_{i=1}^n X_i^0<k\right), \text{ and } Q_j^{[k]}(T)=\mathbb{P}\left(\sum_{i=1}^n X_i^j<k\right), \text{ for } j=1,...,m.$$

In light of equation (4.34), the homogeneous transformation for the k^{th}-to-default payoff is defined as the one that keeps invariant the probability of the idiosyncratic mode $Q_0^{[k]}(T)$, and the probabilities of the market factor modes $Q_j^{[k]}(T)$. For the homogenous basket $\{A_1^*,...,A_n^*\}$, the idiosyncratic and market factor probabilities $Q_0^{[k]}(T)$ and $Q_j^{[k]}(T)$ are computed directly as truncated sums of the binomial expansion: $Q_0^{[k]}(T)=B\left(k-1,n,1-e^{-\Lambda^{0,i}(T)}\right)$, and $Q_j^{[k]}(T)=B\left(k-1,n,\widetilde{p_j^*}\right)$, where $B(k,n,p)=\sum_{i=0}^k\binom{n}{i}(1-p)^{n-i}(p)^i$ is the binomial distribution with parameter p. For the non-homogeneous basket $\{A_1,...,A_n\}$, these probabilities can be computed by an inversion of the Fourier transform[2] $\mathcal{F}_0(s) = \varphi_0(e^s)$ and $\mathcal{F}_j^c(s) = \varphi_j^c(e^s)$,

$$\begin{aligned}
\mathbb{P}\left(\sum_{i=1}^{n} X_i^0 = k\right) &= \mathcal{F}^{-1}\left(\mathcal{F}_0(s)\right)|_{s=k}, \\
\mathbb{P}\left(\sum_{i=1}^{n} X_i^j = k\right) &= \mathcal{F}^{-1}\left(\mathcal{F}_j^c(s)\right)|_{s=k}.
\end{aligned}$$

Since the variables $\left(X_i^0\right)_{1\le i\le n}$ and $\left(X_i^j\right)_{1\le i\le n}$ are independent, the p.g.f. $\varphi_0(x)$ and $\varphi_j^c(x)$ are given by the products $\varphi_0(x) = \prod_{i=1}^n \varphi_{X_i^0}(x)$ and $\varphi_j^c(x) = \prod_{i=1}^n \varphi_{X_i^j}(x)$, where $\varphi_X(x) = (px + (1-p))$ is the p.g.f. of a Bernoulli variable with parameter p. Other methods to generate the distribution of a sum of independent variables, such as the convolution recursion method (see Subsection 4.6.2), are available, and are more efficient than the Fourier transform inversion.

In summary, the homogeneous transformation can be described algorithmically through the following steps:

1. Find $\Lambda^{0,*}(T)$:
   (a) generate the idiosyncratic default distribution $\left[\mathbb{P}\left(\sum_{i=1}^n X_i^0 = l\right)\right]_{0\le l\le n}$ by Fourier inversion;
   (b) use a Brent search to find $\Lambda^{0,*}(T)$ such that $B\left(1 - e^{-\Lambda^{0,i}(T)}, k, n\right) = \mathbb{P}\left(\sum_{i=1}^n X_i^0 < k\right)$.
2. For each market factor, find p_j^*:
   (a) transform the loadings $p_{i,j}$ to the adjusted ones: $\widetilde{p_{i,j}} = 1 - e^{-\Lambda_i^0(T)}\left(1 - p_{i,j}\right)$;
   (b) generate the conditional market factor default distribution $\left[\mathbb{P}\left(\sum_{i=1}^n X_i^j = l\right)\right]_{0\le l\le n}$ by Fourier inversion;
   (c) use a Brent search to find $\widetilde{p_j^*}$ such that $B\left(\widetilde{p_j^*}, k, n\right) = P\left(\sum_{i=1}^n X_i^j < k\right)$;
   (d) transform back $\widetilde{p_j^*}$ to p_j^*: $p_j^* = 1 - e^{\Lambda^{0,*}(T)}\left(1 - \widetilde{p_j^*}\right)$.

4.5.3 Fourier Inversion and Importance Sampling

One issue that arises in the Fourier inversion procedure is that its failure to estimate accurately small probabilities in the tail of the distribution. Indeed, since underlying default probabilities are usually small, the portfolio aggregate default distribution is

[2] See Appendix B.2 for the Fourier transform inversion formula.

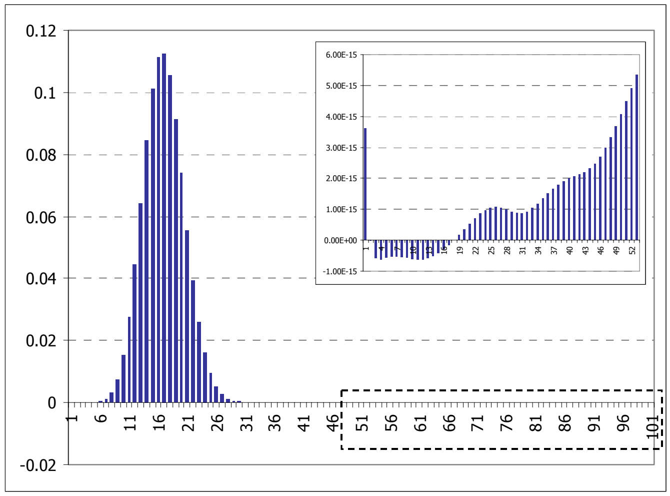

Figure 4.1: Fourier inversion round-off plateau.

centred to the left, and the tail probabilities decay exponentially as we move further from the mean. When the tail probabilities are smaller that the machine double-precision $\sim 10^{-16}$, the numerical round-off errors dominate. This results in a well-documented oscillating behaviour, which is known in the numerical analysis literature as a round-off plateau[3].

To address this problem, we apply a technique inspired by the importance sampling method, which is often used in Monte Carlo simulations to improve the speed of the convergence. The idea is to consider the aggregate default distribution under a new probability measure. Provided that the change of measure is chosen appropriately, such that the new target distribution is centred in the tail, the percentile to be estimated will be significantly larger than the machine precision, and we are in a better position to estimate it accurately with the Fourier inversion algorithm. Once the percentile is evaluated, it can then be converted back to the original measure with the Radon-Nykodim derivative (which, in fact, is a fast-decaying-exponential).

To define the appropriate measure change, we use the concepts of tilted distributions, Esscher transforms and saddle points [4].

For a given probability density function $f(x)$, we call tilted densities of f, the family of density functions $f_\theta(x)$ defined by

$$f_\theta(x) = f(x) \frac{e^{\theta x}}{M(\theta)} = f(x) e^{\theta x - K(\theta)}, \tag{4.35}$$

[3] See Boyd (2000) for a discussion on machine precision and round-off plateaus.
[4] A good reference on exponential tilting can be found in Ross (1997).

where $M(t) = \mathbb{E}\left[e^{tX}\right] = \int e^{tx} f(x)\, dx$ is its moment generating function, and $K(t) = \log(M(t))$ is its cumulant generating function. The mean and the variance of the tilted distribution are given by: $\mu_\theta = K'_\theta(0) = K'(\theta)$, and $\sigma^2_\theta = K''_\theta(0) = K''(\theta)$.

The change of probability measure corresponding to the unique solution of the equation

$$K'\left(\widehat{\theta}\right) = x \tag{4.36}$$

is called an Esscher transform. The parameter $\widehat{\theta}$ is known as the saddle-point. This, basically, defines a change of probability measure such that the new distribution is centred around x.

In our setting, we have a set of independent Bernoulli variables X_i with parameter p_i, and we need to estimate the percentile of the aggregate distribution $\mathbb{P}(X_1 + ... + X_n \geq k)$, for a given k.

The probability mass function of the Bernoulli variable X_i is given by:

$$f_i(x) = p_i^x (1 - p_i)^{1-x}, \; x = 0, 1,$$

its moment generating function is:

$$M_i(\theta) = p_i e^\theta + 1 - p_i,$$

and we can write its tilted probability mass as:

$$f_i^\theta(x) = \frac{e^{\theta x} p_i^x (1 - p_i)^{1-x}}{M_i(\theta)} = \widehat{p_i}^x (1 - \widehat{p_i})^{1-x}, \text{ for } x = 0, 1,$$

with

$$\widehat{p_i} = \frac{p_i e^\theta}{p_i e^\theta + 1 - p_i}.$$

$f_i^\theta(x)$ corresponds to the probability mass function of a Bernoulli variable with the tilted probability $\widehat{p_i}$.

For the aggregate defaults r.v. $X_1 + ... + X_n$, the tilted probability mass is:

$$f^\theta(x) = \frac{e^{\theta x} f(x)}{M(\theta)} = \frac{e^{\theta x} f(x)}{\prod_{i=1}^n M_i(\theta)}, \; x = 0, 1, ..., n,$$

where, since the variables X_i are independent, the moment generating function $M(\theta)$ is given by the product of the individual m.g.f. $M_i(\theta)$. The Radon-Nykodim derivative used to transform $f^\theta(x)$ back to the original density $f(x)$ is:

$$f(x) = f^\theta(x)\, e^{K(\theta) - \theta x}.$$

Using the tilted probabilities $(\widehat{p_1}, ..., \widehat{p_n})$, we generate the distribution of the aggregate defaults under $\mathbb{P}^{\widehat{\theta}}$; then multiplying by the Radon-Nykodim derivative $\frac{d\mathbb{P}}{d\mathbb{P}^{\theta}} = e^{K(\theta)-\theta x}$, we arrive at the probability distribution of the random variable $X_1 + ... + X_n$ under the original probability measure $\mathbb{P}$.

To summarize, the algorithm to compute the tail probability $\mathbb{P}\left(\sum_{i=1}^{n} X_i < k\right)$ is as follows:

1. Step 1: use a Brent method to solve for the saddle point $\widehat{\theta}$, such that $K'\left(\widehat{\theta}\right) = k$. $K'(t)$ is given by
$$K'(t) = \sum_{i=1}^{n} \frac{p_i e^t}{p_i e^t + 1 - p_i}.$$
2. Step 2: compute the tilted probabilities $\widehat{p_i}$
$$\widehat{p_i} = \frac{p_i e^{\widehat{\theta}}}{p_i e^{\widehat{\theta}} + 1 - p_i}.$$
3. Step 3: generate the aggregate default distribution, in the new measure $\mathbb{P}^{\widehat{\theta}}$, $\left[\mathbb{P}^{\widehat{\theta}}\left(\sum_{i=1}^{n} X_i = j\right)\right]_{0\leq j\leq n}$ by Fourier inversion.
4. Step 4: compute the default distribution in the original measure:
$$\mathbb{P}\left(\sum_{i=1}^{n} X_i = j\right) = \mathbb{P}^{\widehat{\theta}}\left(\sum_{i=1}^{n} X_i = j\right) \exp\left(K\left(\widehat{\theta}\right) - \widehat{\theta} j\right).$$

4.6 The Asymptotic Homogeneous Expansion

The transformation, described in the previous section, produces a homogeneous portfolio, which mimics some properties of the aggregate default distribution, and can be used pari-pasu for the purposes of basket default swap valuation. By using this homogeneous portfolio, the numerical burden that comes with the pricing of large baskets is eased, and the valuation algorithm is significantly speeded up. From equation (4.23), the k^{th}-to-default survival probability $Q_n^{[k]} = \mathbb{P}\left(\tau^{[k]} > T\right)$ for the n-dimensional homogeneous portfolio $\{A_1^*, ..., A_n^*\}$ can be computed recursively as:

$$Q_n^{[k]} = \left(\frac{n}{k-1}\right)\left(Q_{n-1}^{[k-1]} - Q_n^{[k-1]}\right) + Q_n^{[k-1]}. \tag{4.37}$$

This simple-looking recursion hides a nasty numerical problem: it is numerically unstable. As one moves up the recursion tree, the numerical round-off errors (which originate mainly from the difference $\left(Q_{n-1}^{[k-1]} - Q_n^{[k-1]}\right)$) propagate rapidly, and the resulting prices are completely erroneous. In fact, a closer look at equation (4.37) shows that this latter resembles the discretized finite-difference scheme of the wave equation PDE, with time-dependent and space-dependent velocity:

$$\begin{aligned} Q_n^{[k]} - Q_n^{[k-1]} &= -\left(\frac{n}{k-1}\right)\left(Q_n^{[k-1]} - Q_{n-1}^{[k-1]}\right), \\ \frac{\partial Q_{x,t}}{\partial t} &= -v_{x,t}\frac{\partial Q_{x,t}}{\partial x}. \end{aligned} \tag{4.38}$$

Equation (4.38) is a well-known unstable difference scheme[5].

To address this issue, we take a different route: rather than using the recursive approach, we study the asymptotic behaviour of the homogeneous portfolio. We show, in this section, that the solution $Q_n^{[k]}$ admits an asymptotic series expansion, and we explain how to compute each term in the expansion.

4.6.1 Asymptotic Series Expansion

We fix a time horizon T, and we consider the k^{th}-to-default Q-factor: $Q^{[k]}(T) = \mathbb{P}\left(\tau^{[k]} > T\right)$.

Proposition 46 *Asymptotic Series Expansion*

The homogeneous portfolio k^{th}-to-default Q-factor has the following series expansion:

$$Q^{[k]}(T) = e^{-\Lambda^c(T)}\left[\sum_{n_1=0}^{+\infty} ... \sum_{n_m=0}^{+\infty} \frac{\Lambda^{c_1}(T)^{n_1}}{n_1!} ... \frac{\Lambda^{c_m}(T)^{n_m}}{n_m!} B\left(k-1, n, p_{n_1,...,n_m}\right)\right], \tag{4.39}$$

where

$$p_{n_1,...,n_m} = 1 - e^{-\Lambda^{0,*}(T)}\left(1-p_1^*\right)^{n_1} ... \left(1-p_m^*\right)^{n_m},$$

and $B(k,n,p)$ is the cumulative Binomial probability with parameter p:

$$B(k,n,p) = \sum_{j=0}^{k} \binom{n}{j} p^j (1-p)^{n-j}.$$

Proof. As in Subsection 4.5.2, we consider the p.g.f. of the defaults counting variable X_T. By virtue of Theorem 44, $\varphi(x)$ is given by

$$\varphi(x) = \sum_{\pi\in\mathbf{\Pi_n}} Q_\pi^{[1]}(T)\ x^{n-d(\pi)}(1-x)^{d(\pi)},$$

where the sub-FTD Q-factors of the homogeneous basket are

$$Q_\pi^{[1]}(T) = \exp\left(-\left[\sum_{j=1}^{m}\left(1-\left(1-p_j^*\right)^{d(\pi)}\right)\Lambda^{c_j}(T)\right] - \left[d(\pi)\Lambda^{0,*}(T)\right]\right). \tag{4.40}$$

[5] We refer the reader to Press, Teukolsky, Vetterling, Flannery (1992).

Replacing each exponential $\exp\left(\left(1-p_j^*\right)^{d(\pi)}\Lambda^{c_j}(T)\right)$ by its series expansion,

$$\exp\left(\left(1-p_j^*\right)^{d(\pi)}\Lambda^{c_j}(T)\right)=\sum_{n_j=0}^{+\infty}\frac{\left[\left(1-p_j^*\right)^{d(\pi)}\Lambda^{c_j}(T)\right]^{n_j}}{n_j!},$$

equation (4.40) becomes

$$\begin{aligned}
&Q\left(\widetilde{x_s}\right)\\
=\;& e^{-\Lambda^c(T)-\left[d(\pi)\Lambda^{0,*}(T)\right]}\sum_{n_1=0}^{+\infty}\dots\sum_{n_m=0}^{+\infty}\frac{\left[\left(1-p_j^*\right)^{d(\pi)}\Lambda^{c_1}(T)\right]^{n_1}}{n_1!}\dots\frac{\left[\left(1-p_j^*\right)^{d(\pi)}\Lambda^{c_m}(T)\right]^{n_m}}{n_m!}\\
=\;& e^{-\Lambda^c(T)}\sum_{n_1=0}^{+\infty}\dots\sum_{n_m=0}^{+\infty}\frac{\Lambda^{c_1}(T)^{n_1}}{n_1!}\dots\frac{\Lambda^{c_m}(T)^{n_m}}{n_m!}e^{-d(\pi)\Lambda^*_{n_1,\dots,n_m}(T)}, \qquad (4.41)
\end{aligned}$$

where

$$\Lambda^*_{n_1,\dots,n_m}(T)=\Lambda^{0,*}(T)+\log\left(\left(1-p_1^*\right)^{n_1}\right)+\dots+\log\left(\left(1-p_m^*\right)^{n_m}\right).$$

Substituting (4.41) in the expression of the probability generating function gives

$$\varphi(x)=e^{-\Lambda^c(T)}\sum_{n_1=0}^{+\infty}\dots\sum_{n_m=0}^{+\infty}\frac{\Lambda^{c_1}(T)^{n_1}}{n_1!}\dots\frac{\Lambda^{c_m}(T)^{n_m}}{n_m!}\varphi_{n_1,\dots,n_m}(x),$$

where

$$\begin{aligned}
\varphi_{n_1,\dots,n_m}(x) &= \sum_{\pi\in\mathbf{\Pi_n}}Q_{n_1,\dots,n_m}(\pi)\;x^{n-d(\pi)}(1-x)^{d(\pi)},\\
Q_{n_1,\dots,n_m}(\pi) &= \exp\left(-d(\pi)\Lambda^*_{n_1,\dots,n_m}(T)\right).
\end{aligned}$$

Using Theorem 44 again, it is easy to see that $\varphi_{n_1,\dots,n_m}(x)$ is, in fact, the probability generating function of a set of n independent identically distributed variables $(Z_1,\dots,Z_n)$, with an idiosyncratic intensity $\Lambda^*_{n_1,\dots,n_m}(T)$, or a default probability $\mathbb{P}(Z_i=1)=1-e^{-\Lambda^*_{n_1,\dots,n_m}(T)}$, for $i=1,\dots,n$. Indeed, $Q_{n_1,\dots,n_m}(\pi)$ is the sub-FTD Q-factor of the π-subset of $(Z_1,\dots,Z_n)$,

$$Q_{n_1,\dots,n_m}(\pi)=\prod_{i\in\pi}\exp\left(-\Lambda^*_{n_1,\dots,n_m}(T)\right)=\prod_{i\in\pi}\mathbb{P}(Z_i=0).$$

The p.g.f. $\varphi_{n_1,\dots,n_m}(x)$ is, then, given by the product of the p.g.f. of the individual independent variables Z_i:

$$\varphi_{n_1,\dots,n_m}(x)=\left(x+\left(e^{-\Lambda^0_*(T)}\left(1-p_1^*\right)^{n_1}\dots\left(1-p_m^*\right)^{n_m}\right)(1-x)\right)^n.$$

In other words, $\varphi_{n_1,...,n_m}(x)$ is the probability generating function of a Binomial distribution with parameter $p_{n_1,...,n_m} = 1 - e^{-\Lambda^{0,*}(T)}(1-p_1^*)^{n_1}...(1-p_m^*)^{n_m}$, and $\varphi(x)$ is a weighted average of the Binomial p.g.f. $\varphi_{n_1,...,n_m}(x)$:

$$\begin{aligned}\varphi(x) &= \sum_{n_1=0}^{+\infty}...\sum_{n_m=0}^{+\infty} w_{n_1,...,n_m}\cdot\varphi_{n_1,...,n_m}(x),\\ w_{n_1,...,n_m} &= e^{-\Lambda^c(T)}\frac{\Lambda^{c_1}(T)^{n_1}}{n_1!}...\frac{\Lambda^{c_m}(T)^{n_m}}{n_m!}.\end{aligned}$$

A direct inversion of the probability generating function completes the proof. ■

Remark 47 *Note that the weights $w_{n_1,...,n_m}$ in front of each Binomial function sum up to 1:*

$$e^{-\Lambda^c(T)}\left[\sum_{n_1=0}^{+\infty}\sum_{n_2=0}^{+\infty}...\sum_{n_m=0}^{+\infty}\frac{\Lambda^{c_1}(T)^{n_1}}{n_1!}\frac{\Lambda^{c_2}(T)^{n_2}}{n_2!}...\frac{\Lambda^{c_m}(T)^{n_m}}{n_m!}\right] = 1.$$

Equation (4.39) can be seen as the formula of a "Binomial Mixture" i.e., a weighted average of Binomial distributions. The first mode $(w_{0,...,0}, p_{0,...,0})$ is the "independent" mode, or the pure idiosyncratic mode; then the "correlation" modes are added on top. Each correlation mode corresponds to a market factor contribution, or a combination thereof.

As n goes to infinity, each Binomial mode converges to a Gaussian distribution, and equation (4.39) becomes the formula of a "Gaussian Mixture".

Figure (4.2) depicts an example of a typical default distribution in a M.O. model with three types of market factors: "World" driver, Beta driver and Sector driver. In Figure (4.3), we give examples of the Binomial mixture modes with different values of the exponents (n_W, n_B, n_S).

In practice, rather than treating the market factor Poisson jumps separately, it is more efficient to consider the process $\{N^c(t), t \geq 0\}$:

$$N_t^c \triangleq \sum_{j=1}^{m} N_t^{c_j},$$

which counts the total number of market factor events of any type[6]. Since N_t^c is the sum of independent Poisson processes, it is a Poisson process as well, and its intensity is given by the sum of intensities: $\lambda^c(t) = \lambda^{c_1}(t) + ... + \lambda^{c_m}(t)$. For a fixed time

[6] See Duffie and Pan (2001).

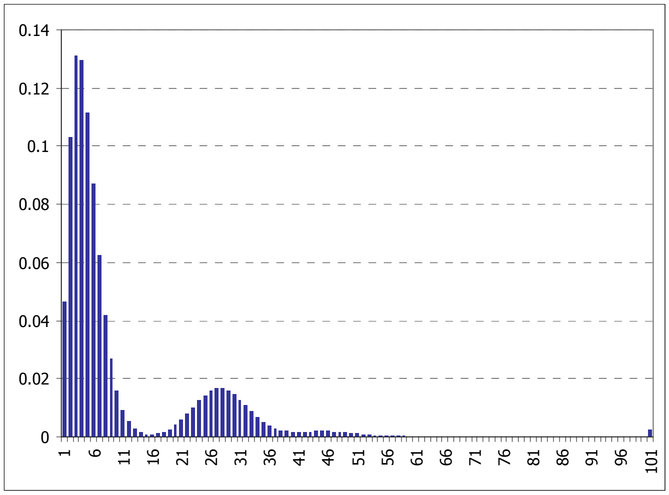

Figure 4.2: Binomial mixture modes. There are four modes in this example: the first one corresponds to the idiosyncratic defaults; the second one is the Beta mode, representing joint defaults in different sectors; the third hump, which is less pronounced, is the sector mode; and finally, the last peak, at the end of the distribution, is a state of the world where all credits defaults.

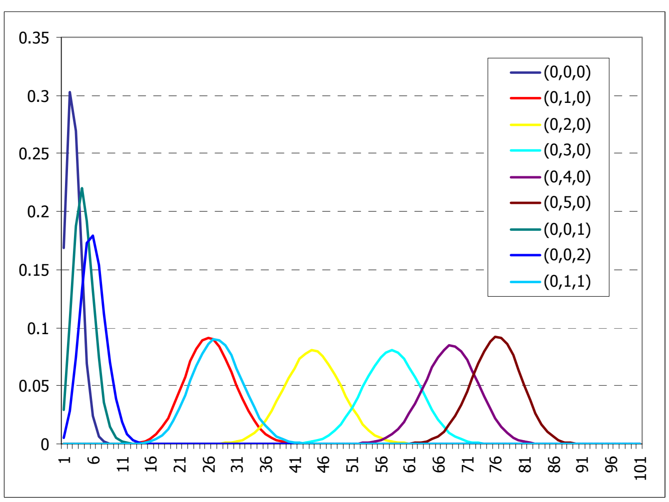

Figure 4.3: Conditional Binomial Distributions in the Asymptotic Homogeneous Expansion.

horizon T, conditional on the total number of jumps, the probability of a jump of type j is given by $\pi_j = \frac{\Lambda^{c_j}(T)}{\Lambda^c(T)}$, for $1 \leq j \leq m$. Re-writing the series expansion (4.39) by conditioning on the total number of jumps, we get the following result.

Corollary 48 *The asymptotic series expansion (4.39) can be re-written as*

$$Q^{[k]}(T) = \sum_{n=0}^{+\infty} e^{-\Lambda^c(T)} \frac{\Lambda^c(T)^n}{n!} \left[\sum_{n_1+...+n_m=n} \frac{n!}{n_1!...n_m!} \pi_1^{n_1}...\pi_m^{n_m} B\left(k-1, n, p_{n_1,...,n_m}\right) \right]. \tag{4.42}$$

The conditional terms $\frac{n!}{n_1!...n_m!}\pi_1^{n_1}...\pi_m^{n_m}$ are the probabilities of a multinomial distribution with parameters $(\pi_1, ..., \pi_m)$. For each expansion order n, the number of terms in the multinomial expansion (i.e., the number of sets $(n_1, ..., n_m)$ such that $n_1 + ... + n_m = n$) is equal to $\binom{n+m-1}{m-1}$. For high values of m and n, the number of terms in the multinomial expansion can be very large. For a typical diversified portfolio of 100 to 125 underlying credits, we find that the number of market factors, representing mostly various industry sectors, is of the order of $m \sim 25$. In order to reduce the truncation error in (4.42), this type of portfolios usually requires the series expansion cut off point to be of the order of $N \sim 15$. For $m = 25$ market factors, as soon as we exceed order $n = 6$, the number of terms in the multinomial expansions becomes significantly large, and very time-consuming to compute. To improve this part of the algorithm, we use a n-dimensional Quasi-Monte-Carlo integration technique.

A few comments are in order.

- It should be emphasized that the need to go into high orders is driven by the values of the cumulative intensity $\Lambda^c(T) = \Lambda^{c_1}(T) + ... + \Lambda^{c_m}(T)$; the truncation order is determined so that
$$\left| 1 - \sum_{n=0}^{N} e^{-\Lambda^c(T)} \frac{\Lambda^c(T)^n}{n!} \right| < \varepsilon_N,$$
where ε_N is a given accuracy level. Longer maturities, higher market factor intensities, and larger number of market factors require more orders in the expansion to achieve the same level of accuracy.

- The conditional jump type probabilities $(\pi_1, .., \pi_m)$ are, by construction, significantly higher than zero, therefore the convergence properties of the Quasi-Monte-Carlo sample are very good, which in turn implies that only a reduced number of paths is usually needed (typically, number of samples is ~ 1000 paths).

- Estimating the probabilities of the multinomial distribution is a well-posed integration problem on the n-dimensional hypercube, which is particularly suited for Quasi-Monte-Carlo.

4.6.2 Quasi-Monte Carlo Integration of the Conditional Distribution

Re-writing equation (4.42) in terms of expectations, the k^{th}-to-default Q-factor $Q^{[k]}(T)$ is:

$$Q^{[k]}(T) = \sum_{n=0}^{+\infty} e^{-\Lambda^c(T)} \frac{\Lambda^c(T)^n}{n!} \mathbb{P}\left(B\left(k-1, n, p_{N_T^{c1},\ldots,N_T^{cm}}\right) |N_T^c = n\right).$$

The conditional expectation $\mathbb{P}(.\,|N_T^c = n)$ is given analytically by summing over the weights of the multinomial distribution,

$$\mathbb{P}\left(B\left(k-1, n, p_{N_T^{c1},\ldots,N_T^{cm}}\right) |N_T^c = n\right) = \sum_{n_1+\ldots+n_m=n} \frac{n!}{n_1!\ldots n_m!} \pi_1^{n_1}\ldots\pi_m^{n_m} B\left(k-1, n, p_{n_1,\ldots,n_m}\right).$$

Here, we estimate this expectation numerically using Quasi-Monte Carlo integration[7].

We use an n-dimensional Sobol sequence to simulate a vector of n indices $(x_1, \ldots, x_n)$, where the x_i's are discrete random variables with the following distribution:

$$P(x_i = j) = \pi_j, \text{ for } 1 \leq j \leq m.$$

For each path, the parameter of the conditional Binomial distribution $p_{N_T^{c1},\ldots,N_T^{cm}}$ is given by:

$$p_{N_T^{c1},\ldots,N_T^{cm}} = 1 - e^{-\Lambda^{0,*}(T)} \left(1 - p_{x_1}^*\right) \ldots \left(1 - p_{x_n}^*\right) = p_{x_1,\ldots,x_n},$$

and the conditional expectation is evaluated as:

$$\begin{aligned} & \mathbb{P}\left(B\left(k-1, n, p_{N_T^{c1},\ldots,N_T^{cm}}\right) |N_T^c = n\right) \\ = & \ \mathbb{P}\left(B\left(k-1, n, p_{x_1,\ldots,x_n}\right)\right) \\ \simeq & \ \frac{1}{N} \sum_{path=1}^{N} B\left(k-1, n, p_{x_1^{path},\ldots,x_n^{path}}\right). \end{aligned}$$

In practice, we compute the dominant terms of the series expansion in closed form, and we estimate the higher-order terms, needed for the convergence of the asymptotic series (i.e., to reduce the truncation error ε_n), by numerical integration. For a typical basket of 25 market factors, the orders 1 through 6 use the closed form formula; orders 7 to 15 use Quasi-Monte Carlo. Sobol integration is particularly appropriate for this type of high dimensional integration problems, it benefits from the space filling properties of this low-discrepancy sequence. We can also use other low-discrepancy sequences, such as Faure, Halton, Haselgrove[8], etc.

[7] A similar approach is used in Merino and Nyfeler (2002).

[8] See Niederreiter (1992) for a discussion of low-discrepancy sequences and their application to high-dimensional integration.

4.7 The Asymptotic Expansion

In this section, we relax the homogeneous portfolio assumption, and we derive an asymptotic series expansion of the k^{th}-to-default Q-factor in the non-homogeneous case. We also show how to compute the conditional aggregate default distributions that appear in the expansion, using the convolution recursion algorithm. This latter and other recursive methods have been traditionally used in actuarial mathematics to evaluate ruin probabilities and insurance premia.

4.7.1 The Asymptotic Series Expansion

We consider the k^{th}-to-default Q-factor, at a fixed time horizon T, for the non-homogeneous portfolio $\{A_1, ..., A_n\}$. By $B(k, \widetilde{p})$, we denote the cumulative probability of a random variable $Y_1 + ... + Y_n$,

$$B(k, \widetilde{p}) = \sum_{j=0}^{k} \mathbb{P}\left(\sum_{i=1}^{n} Y_i = j\right),$$

where $(Y_1, ..., Y_n)$ are n independent Bernoulli variables with parameters $\widetilde{p} = (p_1, ..., p_n)$.

Proposition 49 *Asymptotic Series Expansion*

The k^{th}-to-default Q-factor $Q^{[k]}(T)$ for the non-homogeneous portfolio $\{A_1, ..., A_n\}$ has the following series expansion:

$$Q^{[k]}(T) = e^{-\Lambda^c(T)}\left[\sum_{n_1=0}^{+\infty} ... \sum_{n_m=0}^{+\infty} \frac{\Lambda^{c_1}(T)^{n_1}}{n_1!} ... \frac{\Lambda^{c_m}(T)^{n_m}}{n_m!} B\left(k-1, \widetilde{p_{n_1,...,n_m}}\right)\right], \quad (4.43)$$

where the probability vector $,\widetilde{p_{n_1,...,n_m}} = (p_{n_1,...,n_m}(1), ..., p_{n_1,...,n_m}(n))$ *is given by:*

$$p_{n_1,...,n_m}(i) = 1 - e^{-\Lambda^{0,i}(T)}(1-p_{i,1})^{n_1} ... (1-p_{i,m})^{n_m}.$$

Proof. As before, we proceed as follows:

1. we consider the probability generating function of the aggregate defaults counting variable $\varphi(x)$,

2. we expand the exponentials $\left[\exp\left(\prod_{i\in\pi}(1-p_{i,j})\Lambda^{c_j}(T)\right)\right]$ that appear in the sub-FTD Q-factors $Q_{\pi}^{[1]}(T)$,

3. which is then substituted in the formula of the p.g.f. $\varphi(x)$,

4. and we arrive at the expression of $\varphi(x)$ as a weighted average of conditional independent probability generating functions $\varphi_{n_1,...,n_m}(x)$:

$$\begin{aligned}\varphi(x) &= \sum_{n_1=0}^{+\infty} ... \sum_{n_m=0}^{+\infty} w_{n_1,...,n_m} \cdot \varphi_{n_1,...,n_m}(x), \\ w_{n_1,...,n_m} &= e^{-\Lambda^c(T)} \frac{\Lambda^{c_1}(T)^{n_1}}{n_1!} ... \frac{\Lambda^{c_m}(T)^{n_m}}{n_m!},\end{aligned}$$

where

$$\varphi_{n_1,...,n_m}(x) = \prod_{i=1}^{n} \left(x + \left(e^{-\Lambda^{0,i}(T)} (1-p_{i,1})^{n_1} ... (1-p_{i,m})^{n_m} \right) (1-x) \right),$$

is the probability generating function of the sum of n independent Bernoulli variables $(Y_1, ..., Y_n)$ with parameters $\widetilde{p_{n_1,...,n_m}} = (p_{n_1,...,n_m}(1), ..., p_{n_1,...,n_m}(n))$:

$$p_{n_1,...,n_m}(i) = 1 - e^{-\Lambda^{0,i}(T)} (1-p_{i,1})^{n_1} ... (1-p_{i,m})^{n_m}.$$

Equation (4.43) follows from the inversion of the probability generating function $\varphi(x)$.

■

Conditioning on $N_t^c \triangleq N_t^{c_1} + ... + N_t^{c_m}$, the total number of market factor shocks in $(0, t]$, we obtain the compact version of equation (4.43).

Corollary 50 *The asymptotic series (4.43), for the non-homogeneous portfolio* $\{A_1, ..., A_n\}$*, can be written as the follows:*

$$Q^{[k]}(T) = \sum_{n=0}^{+\infty} e^{-\Lambda^c(T)} \frac{\Lambda^c(T)^n}{n!} \left[\sum_{n_1+...+n_m=n} \frac{n!}{n_1!...n_m!} \pi_1^{n_1} ... \pi_m^{n_m} B\left(k-1, \widetilde{p_{n_1,...,n_m}}\right) \right]. \tag{4.44}$$

As with the AHX method, the dominant terms in equation (4.44) are computed in closed form, and the higher-order terms are estimated with Sobol numerical integration.

4.7.2 Recursion Methods for the Computation of Aggregate Distributions

To compute the aggregate default distributions $B\left(k, \widetilde{p_{n_1,...,n_m}}\right)$ that appear in the series expansion (4.44), we can use the standard FFT algorithm; we can also use other recursive methods, which were studied extensively in the actuarial literature. Here, we present the convolution recursion and we refer the reader to the literature for other approximate recursions by Panjer, Kornya, Hipp and DePril.

The general problem is the following: for a set of n independent discrete random variables $(Y_1, ..., Y_n)$, we want to derive the distribution of the sum

$$S_n = Y_1 + ... + Y_n. \tag{4.45}$$

The generating function of the random variable S_n,

$$\varphi_{S_n}(x) = \sum_{k=0}^{\infty} \mathbb{P}(S_n = k)\ x^k,$$

is given by the product of the generating functions of the variables Y_i (since they are all independent):

$$\varphi_S(x) = \prod_{i=1}^{n} \varphi_{Y_i}(x),$$

and its distribution $p_{S_n}(s) = \mathbb{P}(S_n = s)$ is given by the n-fold convolution of the distributions p_{Y_i}:

$$p_{S_n}(s) = \bigotimes_{i=1}^{n} p_{Y_i}(s). \tag{4.46}$$

We compute this convolution product by applying the following recursion:

$$\begin{aligned} p_{S_{k+1}}(s) &= p_{S_k} \otimes p_{Y_{k+1}}(s) \\ &= \sum_{y=0}^{s} p_{Y_{k+1}}(y)\, p_{S_k}(s-y), \text{ for } 1 \leq k \leq n-1. \end{aligned} \tag{4.47}$$

In our set-up, the aggregate default distributions of equation (4.44) are sums of independent Bernoulli variables. The distribution density of the sum S_n, when $Y_1, ..., Y_n$ are Bernoulli variables with parameters $\widetilde{p} = (p_1, ..., p_n)$, is computed recursively as follows:

for $0 \leq k \leq n-1$,

$$p_{S_{k+1}}(s) = p_{k+1} p_{S_k}(s-1) + (1 - p_{k+1})\, p_{S_k}(s),\ 0 \leq s \leq k+1,$$

with the convention

$$p_{S_k}(-1) = 0,$$

and the distribution of the empty sum S_0 being defined as:

$$\begin{cases} p_{S_0}(0) = 1 \\ p_{S_0}(s) = 0 \text{ for } s > 0 \end{cases}.$$

4.7.3 Numerical Comparisons

We consider a portfolio of $n = 100$ underlying credits, where we have 10 credits per sector, and all the individual intensities are equal to $\lambda_i = 200$ bps. We assume that the MO representation of each credit is given by:

$$\lambda_i = \lambda^W + p_{i,B}\lambda^B + p_{i,S_\cdot}\lambda^{S_\cdot} + \lambda^{0,i},$$

where the world driver is $\lambda^W = 5$ bps, the Beta driver is $\lambda^B = 500$ bps, and the sector drivers are $\lambda^{S_j} = 250$ bps. The loadings are fixed at $p_{i,B} = 0.24$ and $p_{i,S_j} = 0.16$, so that 60% of the intensity is due to the Beta component $\left[p_{i,B}\lambda^B\right]$ and 20% is due to the sector component $\left[p_{i,S_\cdot}\lambda^{S_\cdot}\right]$.

We compute the break-even spreads $\left(s^{[1]}, ..., s^{[n]}\right)$ of the k^{th}-to-default swaps referencing this portfolio; and we compare the the accuracy of various approximations with the exact asymptotic solution (4.44). We estimate the accuracy of the asymptotic homogeneous expansion, as well as the approximation suggested by Duffie and Pan, and the Panjer recursion based solution developed in Lindskog and McNeil.

In Duffie and Pan (2001), the default indicator D_T^i is approximated by:

$$\begin{aligned} D_T^i &\simeq \sum_{j=1}^{m} A_{i,j} D_T^{c_j} + D_T^{0,i}, \qquad (4.48) \\ X_T &= \sum_{i=1}^{n} D_T^i. \end{aligned}$$

where $D_T^{c_j} \triangleq \mathbf{1}_{\left\{N_T^{c_j} > 0\right\}}$ and $D_T^{0,i} \triangleq \mathbf{1}_{\left\{N_T^{0,i} > 0\right\}}$.

In Lindskog and McNeil (2003), the aggregate defaults counter is approximated by the sum of Poisson variables, and then Panjer's recursion is used to derive its distribution:

$$\begin{aligned} N_T^i &= \sum_{j=1}^{m} \sum_{\theta_r^{c_j} \leq t} A_{\theta_r^{c_j}}^{i,j} + N_t^{0,i}, \\ X_T &\simeq \sum_{i=1}^{n} N_T^i. \qquad (4.49) \end{aligned}$$

By construction, approximation (4.48) underestimates the number of joint defaults since it considers only the first shock of the common Poisson processes $N_T^{c_j}$. On the other hand, approximation (4.49) overestimates the number of defaults since it accounts for all the Poisson events N_T^i.

Figure (4.4) shows the relative error of the three approximations.

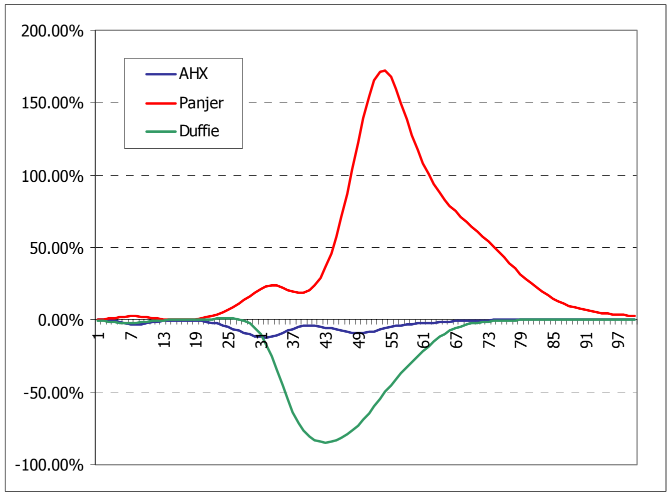

Figure 4.4: Comparison between the accuracy of the AHX method, the Panjer approximation and Duffie's approximation.

4.8 Conclusion

In this chapter, we have considered the problem of pricing large basket credit derivatives in the Marshall-Okin framework with semi-analytic approaches. We have shown that the direct approach explodes exponentially as the basket size increases. As an alternative, we have explored a wide spectrum of methods that evolved from the replicating portfolio method to the asymptotic series expansion of the basket. We have presented how to compute each conditional distribution in the series expansion with a Fourier transform method, or a recursive convolution method. Recursion methods are very popular in insurance mathematics, and have been studied extensively in the actuarial literature. Combining the tools of stochastic calculus, numerical analysis and actuarial mathematics offers a very powerful platform for addressing the issues raised by large portfolio credit derivatives. This may prove to be a very promising area for further research.

Chapter 5

Correlation of Correlation

In this chapter, we analyze the “correlation of correlation” risk in the Marshall-Olkin copula framework. The valuation of compounded correlation products such as “CDOs of CDOs” (also known as “CDO-Squared”) or “baskets of baskets” is mainly driven by correlation of correlation effects. First, We extend the first-to-default replication method to baskets of basket products. Then, we develop an intuitive methodology for analyzing this type of structures. The idea is to model each underlying basket security as a single name process, and to derive its equivalent intensity process and its equivalent decomposition on the MO common market factors. This, in turn, defines the multivariate dependence between the underlying basket securities in the portfolio.

5.1 Introduction

In this chapter, we study the default correlation risk for a new credit derivatives asset class known as CDOs of CDOs. The payoff of a traditional CDO tranche depends on the performance of a pool of underlying single name assets. For a portfolio of n obligors, with notionals $(n_i)_{1\leq i\leq n}$ and recovery rates $(\delta_i)_{1\leq i\leq n}$, the aggregate loss at time t is defined by

$$L_t \triangleq \sum_{i=1}^{n} n_i \left(1-\delta_i\right) \mathbf{1}_{\{\tau_i \leq t\}}.$$

The CDO tranche with attachment points $0 \leq K_1 < K_2 \leq 1$ covers all the losses between a lower bound $\alpha \triangleq K_1 \sum_{i=1}^{n} n_i$ and an upper bound $\beta \triangleq K_2 \sum_{i=1}^{n} n_i$. Its present value is given by the Stieljes integral

$$\mathbb{E}\left[\int_0^T \exp\left(-\int_0^t r_s ds\right) dM_t^{\alpha,\beta}\right], \tag{5.1}$$

where r_t is the risk-free rate and the tranche loss $M_t^{\alpha,\beta}$ is defined as a the payoff of a "call-spread" on the portfolio loss L_t:

$$M_t^{\alpha,\beta} \triangleq \min\left(\max\left(L_t - \alpha, 0\right), \beta - \alpha\right). \tag{5.2}$$

$M_t^{\alpha,\beta}$ is a pure jump process; it jumps each time a default occurs. The integral (5.1) is a discrete sum over the portfolio default times

$$\mathbb{E}\left[\int_0^T \exp\left(-\int_0^t r_s ds\right) dM_t^{\alpha,\beta}\right] = \mathbb{E}\left[\sum_{\tau\leq T} \exp\left(-\int_0^{\tau^-} r_s ds\right)\left(M_\tau^{\alpha,\beta} - M_{\tau^-}^{\alpha,\beta}\right)\right]. \tag{5.3}$$

Using the integration by parts formula and interchanging the order of integration, we can re-write the integral (5.1) as

$$\mathbb{E}\left[\int_0^T \exp\left(-\int_0^t r_s ds\right) dM_t^{\alpha,\beta}\right] = B_{0,T}\mathbb{E}\left[M_t^{\alpha,\beta}\right] - \int_0^T \mathbb{E}\left[M_t^{\alpha,\beta}\right] dB_{0,t}, \tag{5.4}$$

where $B_{0,t}$ is the risk-free discount factor: $B_{0,t} \triangleq \mathbb{E}\left[\exp\left(-\int_0^t r_s ds\right)\right]$; and the interest rate and credit processes are assumed to be independent.

Summary 51 *The value of a CDO tranche is the sum of call spread oplets on the pure jump process L_t.*

For CDOs of CDOs the underlying portfolio is a basket of CDO tranche securities. And the payoff of the structure depends on the aggregated losses of the underlying tranches. CDOs of CDOs are also referred to in the market as CDO-Squared. For a universe of n credits, we can define p portfolios with a matrix of notionals $\left[n_i^j\right]_{\substack{1\leq j\leq p,\\ 1\leq i\leq n}}$ if a credit i is not included in portfolio j, then $n_i^j = 0$. For each portfolio, we define a CDO tranche with the lower and upper loss bounds $[\alpha_j]_{1\leq j\leq p}$ and $\left[\beta_j\right]_{1\leq j\leq p}$.

The loss for each portfolio $1 \leq j \leq m$ is

$$L_t^j \triangleq \sum_{i=1}^n n_i^j \left(1-\delta_i\right) \mathbf{1}_{\{\tau_i\leq t\}}.$$

Each CDO tranche $1 \leq j \leq p$, with attachment points $0 \leq K_1^j < K_2^j \leq 1$, covers the losses from $\alpha_j \triangleq K_1^j \sum_{i=1}^n n_i^j$ to $\beta_j \triangleq K_2^j \sum_{i=1}^n n_i^j$. The tranche loss variable is given by

$$M_t^{j,\alpha_j,\beta_j} \triangleq \min\left(\max\left(L_t^j - \alpha_j, 0\right), \beta_j - \alpha_j\right).$$

Now, we can define the aggregate loss on the portfolio of CDO tranches as the sum of the tranche losses

$$L_t \triangleq \sum_{j=1}^p M_t^{j,\alpha_j,\beta_j},$$

and the (α, β)-tranche of the CDO of CDOs is given by

$$M_t^{\alpha,\beta} \triangleq \min\left(\max\left(L_t - \alpha, 0\right), \beta - \alpha\right).$$

The valuation of CDOs of CDOs depends on a compounded type of default correlation that we call "correlation of correlation" risk. In the same way that "options on options" are sensitive to the "volatility of volatility" parameter, CDO-Squareds are sensitive to correlation of correlation. The choice of the copula function used to model the default times' multivariate dependence and its calibration becomes particularly relevant to the pricing problem. While the mis-calibration of a copula model may have a first order effect on standard CDOs, this error can be much larger for a CDO-Squared.

Similarly, we can also define baskets of baskets i.e., nth-to-default swaps referencing a basket of nth-to-default securities.

CDO-Squareds are very popular with investors seeking to boost their portfolio returns in a tight-spread, low-default environment. Historically, the first CDOs of structured notes were introduced for ABS (Asset Backed Securities), CMBS (Commercial Mortgage Backed Securities) and RMBS (Residential Mortgage Backed Securities). The rationale for this type of products is mainly spread enhancement and diversification through portfolio-based pooling. Recently, pure synthetic CDO-Squareds, where the underlying pool is a portfolio of synthetic CDO tranches, gained a lot of popularity. They offer a higher diversification than simple CDOs, and they have a higher spread compared with equally rated CDO tranches. From an investor point of view, this provides a better alternative for taking a moderately bullish credit view. Typical CDO-Squareds use a master pool of 250 to 350 single-name credits. The names in the master pool are used to reference a number of CDO tranches (typically, 6 to 10 Mezzanine tranches). Then, these CDO tranches are used as the underlying portfolio for the CDO-Squared structure. Figure (5.1) illustrates the mechanics of the CDO-Squared.

The aim of this chapter is to provide an intuitive method for analyzing the correlation of correlation risk in the Marshall-Olkin copula framework. Our first contribution is to extend our FTD replication method (see Chapter 4) to basket of basket products. Secondly, we develop a simple intuitive approach to modelling basket of basket structures. The idea is to look at each underlying basket as a single name security and to derive its intensity process and its multivariate representation.

The rest of the chapter is organized as follows. Section 5.2 describes the modelling framework. In Section 5.3, we show how the FTD replication method applies to baskets of baskets. In Section 5.4, we develop the "Equivalent Single Name Process" approach for baskets of baskets. And we conclude in Section 5.5.

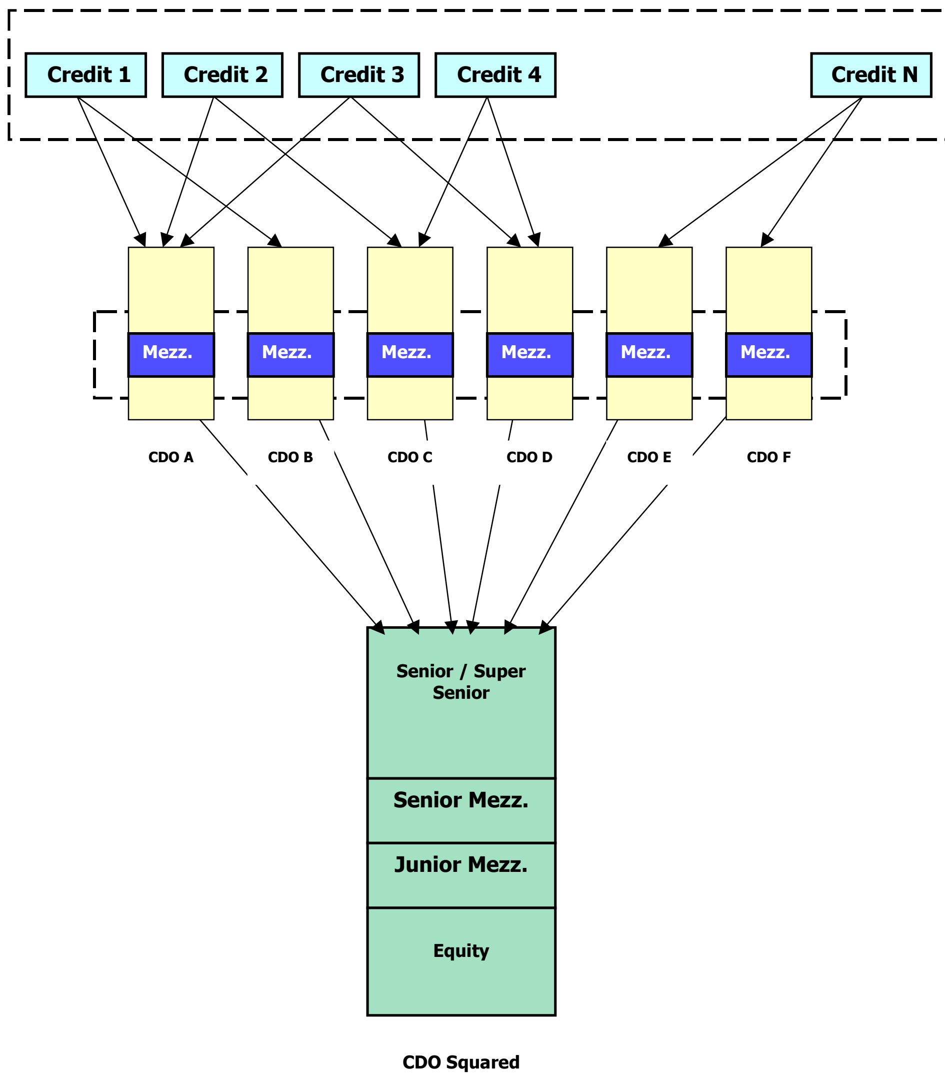

Figure 5.1: Typical CDO Squared Structure

5.2 Set-up

We consider a portfolio of n obligors with default times $(\tau_1, ..., \tau_n)$. All random variables are defined on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$.

We assume that we have a set of $(m+n)$ independent Poisson processes $\left(N_t^{c_j}\right)_{t\geq 0}$ with intensities $\lambda^{c_j}(t)$, which can trigger joint defaults. To each Poisson process $\left(N_t^{c_j}\right)_{t\geq 0}$, we associate the sequence of jump times $\left\{\theta_r^{c_j}\right\}_{r\in\{1,2,...\}}$.

For every event type c_j, and for all $t \geq 0$, we define a set of independent Bernoulli variables $\left(A_t^{1,j}, ..., A_t^{n,j}\right)$ with probabilities $(p_{1,j}, ..., p_{n,j})$, $p_{i,j} \in [0,1]$.

We assume that for $j \neq k$, the vectors $\mathbf{A}_t^j = \left(A_t^{1,j}, ..., A_t^{n,j}\right)$ and $\mathbf{A}_t^k = \left(A_t^{1,k}, ..., A_t^{n,k}\right)$ are independent.

We assume that for $t \neq s$, the vectors $\mathbf{A}_t^j = \left(A_t^{1,j}, ..., A_t^{n,j}\right)$ and $\mathbf{A}_s^j = \left(A_s^{1,j}, ..., A_s^{n,j}\right)$ are independent.

At time $\theta_r^{c_j}$ (i.e., at the r^{th} occurrence of the market event of type j), we draw the set of n independent Bernoulli variables $\left(A_{\theta_r^{c_j}}^{1,j}, ..., A_{\theta_r^{c_j}}^{n,j}\right)$. The variable $A_{\theta_r^{c_j}}^{i,j}$ indicates if obligor i has defaulted or not. We classify the market factors $\left(N_t^{c_j}\right)_{t\geq 0}$ so that the first m factors are common market events, which impact more than two obligors, and the last n factors are idiosynratic events. We shall denote the latter by $N_t^{0,i} \triangleq N_t^{c_{m+i}}$ (and $\lambda^{0,i}(t) \triangleq \lambda^{c_{m+i}}(t)$), for $1 \leq i \leq n$. The corresponding factor loadings are: $p_{i,m+i} = 1$ and $p_{i,m+k} = 0$, for $1 \leq k \neq i \leq n$.

We define the process $\left(N_t^i\right)_{t\geq 0}$ as

$$N_t^i \triangleq \sum_{j=1}^{m+n} \sum_{r=1}^{N_t^{c_j}} A_{\theta_r^{c_j}}^{i,j}. \tag{5.5}$$

$\left(N_t^i\right)_{t\geq 0}$ is also a Poisson process since it is obtained as the superposition of independent (thinned) Poisson processes. Its intensity is given by

$$\begin{aligned} \lambda_i(t) &= \sum_{j=1}^{m+n} p_{i,j} \lambda^{c_j}(t) \\ &= \sum_{j=1}^{m} p_{i,j} \lambda^{c_j}(t) + \lambda^{0,i}(t). \end{aligned}$$

The single-name survival probability of obligor i is given by

$$Q_i(T) \triangleq \mathbb{P}(\tau_i > T) = \mathbb{P}\left(N_T^i = 0\right) = \exp\left(-\int_0^T \lambda_i(t)\, dt\right); \tag{5.6}$$

We denote by $D_t^i = \mathbf{1}_{\{\tau_i \leq t\}}$ the default indicator process of the i^{th} credit. The multivariate dependence structure can be alternatively described by the following SDE (as in Duffie (1998))

$$dD_t^i = \left(1 - D_{t^-}^i\right) \sum_{j=1}^{m+n} A_t^{i,j} dN_t^{c_j}.$$

The copula function implied by this Poisson shock model is called a Marshall-Olkin copula.

5.3 Replication Method

In Chapter 4, we have shown that any nth-to-default payoff can be replicated with a set of first-to-defaults referencing sub-baskets of the original pool of credits. For example, the second-to-default on a three-name basket $\{A, B, C\}$ can be replicated by first-to-defaults on the sub-baskets $\{A, B\}$, $\{A, C\}$, $\{B, C\}$ and $\{A, B, C\}$ as:

$$STD(A, B, C) = FTD(A, B) + FTD(A, C) + FTD(B, C) - 2FTD(A, B, C).$$

The third-to-default on the basket $\{A, B, C\}$ can be replicated by second-to-defaults on the sub-baskets $\{A, B\}$, $\{A, C\}$, $\{B, C\}$ and $\{A, B, C\}$ as:

$$TTD(A, B, C) = \frac{1}{2}\left[STD(A, B) + STD(A, C) + STD(B, C) - STD(A, B, C)\right].$$

The STDs can then be decomposed into first-to-defaults on smaller sub-baskets

$$TTD(A, B, C) = \frac{1}{2}\left\{\begin{array}{c} \left[FTD(A) + FTD(B) - FTD(A, B)\right] \\ +\left[FTD(A) + FTD(C) - FTD(A, C)\right] \\ +\left[FTD(B) + FTD(C) - FTD(B, C)\right] \\ -\left[FTD(A, B) + FTD(A, C) + FTD(B, C) - 2FTD(A, B, C)\right] \end{array}\right\}.$$

In general, we can show that an nth-to-default payoff can be replicated by $(n-1)$-th-to-default payoffs, which in turn can be replicated by $(n-2)$-th-to-default payoffs, and so forth. Applying this replication recursively, we get the complete FTD expansion.

The generic FTD replication result is given in the following proposition.

Let $V_n^{[k]}(A_1, ..., A_n)$ denote the value of a k^{th}-to-default on the n-name basket $\{A_1, ..., A_n\}$. And let us define the l-subsets of $\{1, 2, ..., n\}$ (i.e. the subsets containing exactly l elements) by the mappings $\pi_s^l(.)$ for $1 \leq s \leq \binom{n}{l}$:

$$\pi_s^l(.) : \{1, 2, ..., l\} \rightarrow \{1, 2, ..., n\}.$$

We also assume, without loss of generality, that the elements of each l-subset are sorted in an increasing order

$$\pi_s^l(1) \leq \pi_s^l(2) \leq ... \leq \pi_s^l(l).$$

Proposition 52 *(First-to-Default Expansion). A k^{th}-to-default on n names $V_n^{[k]}$ can be replicated with first-to-defaults on sub-baskets of l names $V_l^{[1]}$, for $n-k+1 \leq l \leq n$, as follows*

$$V_n^{[k]}\left(A_1,...,A_n\right) = \sum_{l=n-k+1}^{n} \sum_{s=1}^{\binom{n}{l}} \alpha_n^{[k]}\left(l\right) V_l^{[1]}\left(A_{\pi_s^l(1)},...,A_{\pi_s^l(l)}\right), \tag{5.7}$$

where

$$\alpha_n^{[k]}\left(l\right) = \left(-1\right)^{l-(n-k+1)} \binom{l-1}{n-k}, \text{ for } n-k+1 \leq l \leq n.$$

To apply the FTD replication to baskets of baskets, we need to introduce some additional notations.

Suppose we have a master pool of credits $\{A_1,...,A_N\}$ which contains all the names in the underlying basket securities. And we have n underlying basket securities $\{S_1,...,S_n\}$ defined as

$$S_i = V_{n_i}^{k_i}\left(A_{\Sigma_i(1)},...,A_{\Sigma_i(n_i)}\right),$$

where n_i specifies the number of credits in the underlying basket S_i, k_i specifies the order of the nth-to-default (e.g. FTD, STD, ...) and the mapping

$$\Sigma_i\left(.\right) : \{1,2,...,n_i\} \rightarrow \{1,2,...,N\}$$

specifies the names included in the underlying basket S_i.

The objective is to find the FTD replication of the master basket of baskets $V_n^{[k]}\left(S_1,...,S_n\right)$.

To start with, we apply the expansion (5.7) formally to the basket $V_n^{[k]}$

$$V_n^{[k]}\left(S_1,...,S_n\right) = \sum_{l=n-k+1}^{n} \sum_{s=1}^{\binom{n}{l}} \alpha_n^{[k]}\left(l\right) V_l^{[1]}\left(S_{\pi_s^l(1)},...,S_{\pi_s^l(l)}\right);$$

then, we compute each first-to-default term

$$V_l^{[1]}\left(S_{i_i},...,S_{i_l}\right).$$

To this end, we expand the securities S_i into their FTD representation and we apply the following algebraic rules to the FTD operator $V^{[1]} = V_l^{[1]}(...)$

$$V^{[1]}\left(S_1,S_2,...,S_k\right) = V^{[1]}\left(S_1,V^{[1]}\left(S_2,...,S_k\right)\right), \tag{5.8}$$

$$V^{[1]}\left(\alpha S_1+\beta S_2,S_3\right) = \alpha V^{[1]}\left(S_1,S_3\right)+\beta V^{[1]}\left(S_2,S_3\right), \tag{5.9}$$

$$V^{[1]}\left(S_1,S_2\right) = V^{[1]}\left(S_2,S_1\right). \tag{5.10}$$

For example, consider two baskets $\{A,B,C\}$ and $\{A,D,E\}$. And let us generate the FTD expansions of the following baskets of baskets

1. $V^{[1]}\left(V^{[1]}\left(A,B,C\right),V^{[1]}\left(A,D,E\right)\right)$
2. $V^{[2]}\left(V^{[1]}\left(A,B,C\right),V^{[1]}\left(A,D,E\right)\right)$
3. $V^{[1]}\left(V^{[2]}\left(A,B,C\right),V^{[2]}\left(A,D,E\right)\right)$

The first example is simple. It suffices to observe that a first-to-default of first-to-defaults is a first-to-default on the union of the two baskets

$$V^{[1]}\left(V^{[1]}\left(A,B,C\right),V^{[1]}\left(A,D,E\right)\right)=V^{[1]}\left(A,B,C,D,E\right).$$

To do the second example, we expand the master STD, then we apply our algebra

$$\begin{aligned}
& V^{[2]}\left(V^{[1]}\left(A,B,C\right),V^{[1]}\left(A,D,E\right)\right)\\
= \; & V^{[1]}\left(V^{[1]}\left(A,B,C\right)\right)+V^{[1]}\left(V^{[1]}\left(A,D,E\right)\right)-V^{[1]}\left(V^{[1]}\left(A,B,C\right),V^{[1]}\left(A,D,E\right)\right)\\
= \; & V^{[1]}\left(A,B,C\right)+V^{[1]}\left(A,D,E\right)-V^{[1]}\left(A,B,C,D,E\right).
\end{aligned}$$

To expand the third example, we apply the FTD expansion to the STD sub-securities, then do the algebra

$$\begin{aligned}
& V^{[1]}\left(V^{[2]}\left(A,B,C\right),V^{[2]}\left(A,D,E\right)\right)\\
= \; & V^{[1]}\left(\begin{array}{c} V^{[1]}\left(A,B\right)+V^{[1]}\left(B,C\right)+V^{[1]}\left(A,C\right)-V^{[1]}\left(A,B,C\right), \\ V^{[1]}\left(A,D\right)+V^{[1]}\left(D,E\right)+V^{[1]}\left(A,E\right)-V^{[1]}\left(A,D,E\right) \end{array}\right)\\
= \; & V^{[1]}\left(A,B,D\right)+V^{[1]}\left(A,B,D,E\right)+V^{[1]}\left(A,B,E\right)-V^{[1]}\left(A,B,D,E\right)\\
& +V^{[1]}\left(A,B,C,D\right)+V^{[1]}\left(B,C,D,E\right)+V^{[1]}\left(A,B,C,E\right)-V^{[1]}\left(A,B,C,D,E\right)\\
& +V^{[1]}\left(A,C,D\right)+V^{[1]}\left(A,C,D,E\right)+V^{[1]}\left(A,C,E\right)-V^{[1]}\left(A,C,D,E\right)\\
& -V^{[1]}\left(A,B,C,D\right)-V^{[1]}\left(A,B,C,D,E\right)-V^{[1]}\left(A,B,C,E\right)+V^{[1]}\left(A,B,C,D,E\right).
\end{aligned}$$

For CDO-Squared structures, where we have the same notional n and same recovery δ for all the names in the master pool, we can easily convert each CDO payoff to a portfolio of NTD baskets.

The portfolio loss is given by

$$L_t \triangleq \sum_{i=1}^{n} n\left(1-\delta\right)\mathbf{1}_{\{\tau_i\leq t\}},$$

and the portfolio aggregated defaults counter is

$$X_t \triangleq \sum_{i=1}^{n} \mathbf{1}_{\{\tau_i\leq t\}}.$$

A CDO tranche that covers the losses between a lower bound α and an upper bound β, is equivalent to the default slices between

$$\frac{\alpha}{n(1-\delta)} \leq X_t \leq \frac{\beta}{n(1-\delta)}.$$

If we define $m_\alpha = \left[\frac{\alpha}{n(1-\delta)}\right]$ and $m_\beta = \left[\frac{\beta}{n(1-\delta)}\right]$, where $[a]$ denotes the integer part of a, then the value of the CDO tranche $V^{(\alpha,\beta)}$ will be expressed in terms of the default slices $V^{[k]}$ as follows

$$V^{(\alpha,\beta)} = \left(1 - \frac{\alpha}{n(1-\delta)} + m_\alpha\right) V^{[m_\alpha]} + \sum_{k=m_\alpha+1}^{m_\beta} V^{[k]} + \left(\frac{\beta}{n(1-\delta)} - m_\beta\right) V^{[m_\beta]}. \tag{5.11}$$

We can then apply the replication for baskets of baskets as explained before.

5.4 Equivalent Single Name Process

The idea of the "Equivalent Single Name Process" method is to consider each basket security in the underlying portfolio as a single name security whose default time is driven by a Poisson process, and to derive the probabilistic properties of this process. We shall use a similar technique to our "Homogeneous Portfolio Approach"[1].

We start with a first-to-default payoff to illustrate the method, then we tackle the general k^{th}-to-default case, where $k > 1$.

5.4.1 First-to-Default Case

We consider a portfolio of n names $\{A_1, ..., A_n\}$ with intensities $(\lambda_i(t))_{1 \leq i \leq n}$.

The decomposition of each credit $\{A_i\}$ in the Marshall-Olkin model is given by a vector of loadings $[p_{i,j}]_{1 \leq j \leq m}$ and an idiosyncratic intensity $\left(\lambda^{0,i}(t)\right)$. We need to find an equivalent single name process (τ^{eq}), which has the same intensity and the same MO multivariate representation as the FTD basket.

In the first-to-default case, the default time $\left(\tau^{ftd} \triangleq \min(\tau_1, ..., \tau_n)\right)$ is a Poisson process and its intensity is given by

$$\lambda^{ftd}(\{\tau_1, ..., \tau_n\}) = \sum_{j=1}^{m} \left[1 - \prod_{i=1}^{n} (1 - p_{i,j})\right] \left[\lambda^{c_j}(t)\right] + \left[\sum_{i=1}^{n} \lambda^{0,i}(t)\right].$$

On the other hand, the intensity of the equivalent process is

$$\lambda^{eq}(t) = \sum_{j=1}^{m} p_j^{eq} \left[\lambda^{c_j}(t)\right] + \left[\lambda^{0,eq}(t)\right].$$

[1]See Chapter 4.

To ensure that the values of the FTD basket and the single name security are the same, we need to have

$$\lambda^{ftd}(t)\left(\{\tau_1,...,\tau_n\}\right)=\lambda^{eq}(t)\,,\ \forall\left[\lambda^{c_j}(t)\right]\in\mathbb{R}^+,\ 1\le j\le m,$$

which implies for each market factor, $1\le j\le m$

$$p_j^{eq}=1-\prod_{i=1}^{n}\left(1-p_{i,j}\right), \tag{5.12}$$

and for the idiosyncratic term

$$\lambda^{0,eq}(t)=\sum_{i=1}^{n}\lambda^{0,i}(t)\,. \tag{5.13}$$

This defines the equivalence transformation for first-to-default payoffs.

5.4.2 The Equivalence Transformation

In the general case, we want to define the transformation

$$\left(\left[p_{i,j}\right]_{\substack{1\le i\le n\\1\le j\le m}},\left(\lambda^{0,i}\right)_{1\le i\le n}\right)\to\left(\left(p_j^{eq}\right)_{1\le j\le m},\lambda^{0,eq}\right),$$

such that the intensity of a k^{th}-to-default time $\tau^{[k]}$ of the portfolio $(\tau_1,...,\tau_n)$ and its multivariate properties are the same as the equivalent single name default time τ^{eq}.

As usual, we fix a time horizon T and we consider the distribution of the random variable

$$X_T\triangleq\sum_{i=1}^{n}D_T^i.$$

D_T^i is the default indicator for issuer i, and $X_T$counts the number of defaults in the basket that occurred before time T. The Q-factor associated with the k^{th}-to-default time $\tau^{[k]}$ is given by

$$Q^{[k]}(T)\triangleq\mathbb{P}\left(\tau^{[k]}>T\right)=\mathbb{P}\left(X_T<k\right). \tag{5.14}$$

For fixed T, we consider the probability generating function of the r.v. X_T

$$\varphi(x)\triangleq\sum_{k=0}^{n}\mathbb{P}\left(X_T=k\right)\ x^k. \tag{5.15}$$

We denote by $\mathbf{\Pi_n}$ the set of all subsets of $\{1,...,n\}$. For each subset $\pi\in\mathbf{\Pi_n}$, we define the first-to-default random time $\tau_\pi^{[1]}=\min\{\tau_i:i\in\pi\}$, and its associated Q-factor $Q_\pi^{[1]}(T)$:

$$Q_\pi^{[1]}(T)\triangleq\mathbb{P}\left(\tau_\pi^{[1]}>T\right)=\mathbb{E}\left[\prod_{i\in\pi}\left(1-D_T^i\right)\right]. \tag{5.16}$$

We know that the probability generating function $\varphi(x)$ can be expressed in terms of the first-to-default Q-factors $\left[Q_{\pi}^{[1]}(T)\right]_{\pi\in\mathbf{\Pi_n}}$ as follows[2].

Proposition 53 *The probability generating function $\varphi(x)$ of the random variable X_T is given by*

$$\varphi(x) = \sum_{\pi\in\mathbf{\Pi_n}} Q_{\pi}^{[1]}(T)\ x^{n-d(\pi)}(1-x)^{d(\pi)}. \tag{5.17}$$

We also know that the FTD Q-factor $Q_{\pi}^{[1]}(T)$ is given by

$$Q_{\pi}^{[1]}(T) = \exp\left(-\left[\sum_{j=1}^{m}\left(1-\prod_{i\in\pi}(1-p_{i,j})\right)\Lambda_j^c(T)\right]-\left[\sum_{i\in\pi}\Lambda^{0,i}(T)\right]\right), \tag{5.18}$$

where $\Lambda^{c_j}(T)$ and $\Lambda^{0,i}(T)$ are the cumulative intensities

$$\begin{aligned}\Lambda^{c_j}(T) &\triangleq \int_0^T \lambda^{c_j}(t)\,dt,\\ \Lambda^{0,i}(T) &\triangleq \int_0^T \lambda_i^0(t)\,dt.\end{aligned}$$

We expand the exponential to first order and we define a new set of loadings $\widetilde{p_{i,j}} = 1-e^{-\Lambda^{0,i}(T)}(1-p_{i,j})$

$$Q_{\pi}^{[1]}(T) \simeq \exp\left(-\sum_{j=1}^{m}\Lambda^{c_j}(T)\right)\left[\exp\left(-\sum_{i\in\pi}\Lambda^{0,i}(T)\right)+\sum_{j=1}^{m}\left(\prod_{i\in\pi}(1-\widetilde{p_{i,j}})\right)\Lambda^{c_j}(T)\right]. \tag{5.19}$$

Substituting equation (5.19) in the expression of the p.g.f. yields

$$\begin{aligned}\exp\left(\sum_{j=1}^{m}\Lambda^{c_j}(T)\right)\varphi(x) &= \left[\sum_{s=1}^{2^n}\exp\left(-\sum_{i\in\pi}\Lambda^{i,0}(T)\right)\ x^{n-d(\pi)}(1-x)^{d(\pi)}\right]\\ &\quad+\sum_{j=1}^{m}\Lambda^{c_j}(T)\left[\sum_{s=1}^{2^n}\left(\prod_{i\in\pi}(1-\widetilde{p_{i,j}})\right)\ x^{n-d(\pi)}(1-x)^{d(\pi)}\right].\end{aligned}$$

We define the idiosyncratic p.g.f. $\varphi_0(x)$ and the market factor conditional p.g.f. $\varphi_j^c(x)$ as

$$\varphi_0(x) \triangleq \sum_{s=1}^{2^n}\exp\left(-\sum_{i\in\pi}\Lambda^{i,0}(T)\right)\ x^{n-d(\pi)}(1-x)^{d(\pi)}, \tag{5.20}$$

$$\varphi_j^c(x) \triangleq \sum_{s=1}^{2^n}\left(\prod_{i\in\pi}(1-\widetilde{p_{i,j}})\right)\ x^{n-d(\pi)}(1-x)^{d(\pi)}. \tag{5.21}$$

[2] See Chapter 4.

Using the result in Proposition 53, we know that $\varphi_0(x)$ is the characteristic function of a sum of independent Bernoulli variables X_i^0, where $\mathbb{P}\left(X_i^0 = 1\right) = 1 - \exp\left(-\Lambda^{i,0}(T)\right)$ and $\mathbb{P}\left(X_i^0 = 0\right) = \exp\left(-\Lambda^{i,0}(T)\right)$. Similarly, $\varphi_j^c(x)$ is the characteristic function of a sum of independent Bernoulli variables X_i^j, where $\mathbb{P}\left(X_i^j = 1\right) = \widetilde{p_{i,j}}$ and $\mathbb{P}\left(X_i^j = 0\right) = 1 - \widetilde{p_{i,j}}$. Thus

$$\varphi_0(x) = \prod_{i=1}^{n}\left(\left(1 - \exp\left(-\Lambda^{i,0}(T)\right)\right)x + \exp\left(-\Lambda^{i,0}(T)\right)\right), \tag{5.22}$$

$$\varphi_j^c(x) = \prod_{i=1}^{n}\left(\widetilde{p_{i,j}}x + \left(1 - \widetilde{p_{i,j}}\right)\right). \tag{5.23}$$

Using the fact that

$$\mathbb{P}(X = l) = \frac{\varphi^{(l)}(x)}{l!},$$

the k^{th}-to-default Q-factor $\mathbb{P}(X_T < k)$ can be expressed as a linear combination of market factor contributions and the idiosyncratic term

$$\mathbb{P}(X_T < k) \simeq \exp\left(-\sum_{j=1}^{m}\Lambda^{c_j}(T)\right)\left[\mathbb{P}\left(\sum_{i=1}^{n}X_i^0 < k\right) + \sum_{j=1}^{m}\mathbb{P}\left(\sum_{i=1}^{n}X_i^j < k\right)\Lambda^{c_j}(T)\right]. \tag{5.24}$$

On the other hand, the Q-factor of the equivalent single name default time τ^{eq} is given by

$$Q^{eq}(T) \triangleq \mathbb{P}(\tau^{eq} > T) = \exp\left(-\left[\sum_{j=1}^{m}p_j^{eq}\Lambda^{c_j}(T)\right] - \left[\Lambda^{0,eq}(T)\right]\right), \tag{5.25}$$

where $\Lambda^{0,eq}(T)$ is the cumulative idiosyncratic intensity

$$\Lambda^{0,eq}(T) \triangleq \int_0^T \lambda_{eq}^0(t)\,dt.$$

We expand the exponential and we replace with the adjusted set of loadings $\widetilde{p_j^{eq}} = 1 - e^{-\Lambda_{eq}^0(T)}\left(1 - p_j^{eq}\right)$

$$Q^{eq}(T) \simeq \exp\left(-\sum_{j=1}^{m}\Lambda^{c_j}(T)\right)\left[\exp\left(-\Lambda^{0,eq}(T)\right) + \sum_{j=1}^{m}\left(1 - \widetilde{p_j^{eq}}\right)\Lambda^{c_j}(T)\right]. \tag{5.26}$$

The equivalence transformation will be defined as the one that keeps the Q-factors invariant

$$Q^{[k]}(T)(\{\tau_1, ..., \tau_n\}) = Q^{eq}(T), \forall\left[\Lambda^{c_j}(T)\right] \in \mathbb{R}^+, 1 \leq j \leq m. \tag{5.27}$$

We substitute (5.24) and (5.26) in (5.27). This implies that the market factor components and idiosyncratic terms are equal

$$\begin{aligned}\mathbb{P}\left(\sum_{i=1}^{n} X_i^0 < k\right) &= \exp\left(-\Lambda^{0,eq}(T)\right),\\ \mathbb{P}\left(\sum_{i=1}^{n} X_i^j < k\right) &= 1-\widetilde{p_j^{eq}}.\end{aligned}$$

The probabilities $\mathbb{P}\left(\sum_{i=1}^{n} X_i^0 < k\right)$ and $\mathbb{P}\left(\sum_{i=1}^{n} X_i^j < k\right)$ can be computed as usual with the standard methods: Fourier Inversion or Convolution Recursion.

The equivalent single name process is therefore completely defined by its idiosyncratic intensity and its market factor loadings as

$$\Lambda^{0,eq}(T) = -\log\left[\mathbb{P}\left(\sum_{i=1}^{n} X_i^0 < k\right)\right]. \tag{5.28}$$

$$p_j^{eq} = 1-\left[\exp\left(\Lambda^{0,eq}(T)\right)\mathbb{P}\left(\sum_{i=1}^{n} X_i^j < k\right)\right]. \tag{5.29}$$

In summary, the equivalence transformation can be described algorithmically by the following steps:

1. Find $\lambda^{0,eq}$
   (a) Generate the idiosyncratic default distribution $\left[\mathbb{P}\left(\sum_{i=1}^{n} X_i^0 = j\right)\right]_{0\le j\le n}$ by Fourier inversion or Convolution recursion
   (b) Solve for $\lambda^{0,eq}$ using equation (5.28)
2. For each market factor, find p_j^{eq}
   (a) Transform the $p_{i,j}$'s to the $\widetilde{p_{i,j}}$'s: $\widetilde{p_{i,j}} = 1-e^{-\Lambda^{0,i}(T)}\left(1-p_{i,j}\right)$
   (b) Generate the conditional market factor default distribution $\left[\mathbb{P}\left(\sum_{i=1}^{n} X_i^j = j\right)\right]_{0\le j\le n}$ by Fourier inversion or Convolution recursion
   (c) Solve for p_j^{eq} using equation (5.29)

5.4.3 Equivalence Transformation for CDOs

In this section, we construct the "Equivalence Transformation" for CDO tranches.

We fix a time horizon T and we consider the aggregate portfolio loss variable L_T

$$L_T \triangleq \sum_{i=1}^{n} L_i D_T^i.$$

D_T^i and L_i are the default indicator and the loss variable for issuer i.

The Q-factor associated with the (α, β)-tranche is given by

$$Q^{\alpha,\beta}(T) \triangleq \mathbb{E}\left[1 - \frac{M_T^{\alpha,\beta}}{\beta - \alpha}\right], \tag{5.30}$$

where $M_T^{\alpha,\beta}$ is the tranche loss

$$M_T^{\alpha,\beta} \triangleq \min\left(\max\left(L_T - \alpha, 0\right), \beta - \alpha\right).$$

We consider the Fourier transform $\phi : \mathbb{R} \to \mathbb{C}$ of the r.v. L_T, defined by

$$\phi(u) = \mathbb{E}\left[\exp\left(iuL_T\right)\right]. \tag{5.31}$$

First, we derive a similar result to Proposition 53. We express $\phi(u)$ in terms of the first-to-default Q-factors $\left[Q_\pi^{[1]}(T)\right]_{\pi \in \mathbf{\Pi_n}}$.

Proposition 54 *The Fourier transform $\phi(u)$ of the random variable L_T is given by*

$$\phi(u) = \sum_{\pi \in \mathbf{\Pi_n}} Q_\pi^{[1]}(T) \left[\prod_{i \notin \pi} \psi_i \prod_{i \in \pi} (1 - \psi_i)\right], \tag{5.32}$$

where $\psi_i = \exp(iuL_i)$

Proof. See Appendix B.1. ■

If we denote by $\phi\left(u, \widetilde{p_{n_1,...,n_m}}\right)$ the Fourier transform of the random variable $\sum_{i=1}^n L_i Y_i$, where $(Y_1, ..., Y_n)$ are n independent Bernoulli variables with parameters $\widetilde{p} = (p_1, ..., p_n)$

$$\phi\left(u, \widetilde{p_{n_1,...,n_m}}\right) \triangleq \prod_{i=1}^n \left((1 - p_i) + p_i \exp\left(iuL_i\right)\right), \tag{5.33}$$

we can show that $\phi(u)$ admits the following asymptotic series expansion.

Proposition 55 *(Asymptotic Series Expansion). The loss Fourier transform function $\phi(u)$ has the following series expansion*

$$\phi(u) = e^{-\Lambda^c(T)} \left[\sum_{n_1=0}^{+\infty} ... \sum_{n_m=0}^{+\infty} \frac{\Lambda^{c_1}(T)^{n_1}}{n_1!} ... \frac{\Lambda^{c_m}(T)^{n_m}}{n_m!} \phi\left(u, \widetilde{p_{n_1,...,n_m}}\right)\right],$$

where $\Lambda^c(T) \triangleq \sum_{i=1}^n \Lambda^{c_j}(T)$ and the probability vector, $\widetilde{p_{n_1,...,n_m}} = (p_{n_1,...,n_m}(1), ..., p_{n_1,...,n_m}(n))$ is given by

$$p_{n_1,...,n_m}(i) = 1 - e^{-\Lambda_i^0(T)} (1 - p_{i,1})^{n_1} ... (1 - p_{i,m})^{n_m}.$$

Proof. See Appendix B.2. ■

We define the idiosyncratic loss variable $L_T^0 \triangleq \sum_{i=1}^n L_i X_i^0$, where $\left(X_i^0\right)_{1\leq i\leq n}$ is a vector of independent Bernoulli variables, given by $\mathbb{P}\left(X_i^0 = 1\right) = 1 - \exp\left(-\Lambda^{i,0}(T)\right)$ and $\mathbb{P}\left(X_i^0 = 0\right) = \exp\left(-\Lambda^{i,0}(T)\right)$. Similarly, for each market factor, we define the loss variable $L_T^{c_j} \triangleq \sum_{i=1}^n L_i X_i^j$, where $\left(X_i^j\right)_{1\leq i\leq n}$ is a vector of independent Bernoulli variables, given by $\mathbb{P}\left(X_i^j = 1\right) = \widetilde{p_{i,j}}$ and $\mathbb{P}\left(X_i^j = 0\right) = 1 - \widetilde{p_{i,j}}$. $\widetilde{p_{i,j}}$ is the duration adjusted loading $\widetilde{p_{i,j}} = 1 - e^{-\Lambda^{0,i}(T)}\left(1 - p_{i,j}\right)$.

We also define the (α,β)-tranche loss variables M_T^0 and $M_T^{c_j}$

$$\begin{aligned} M_T^0 &\triangleq \min\left(\max\left(L_T^0 - \alpha, 0\right), \beta - \alpha\right), \\ M_T^{c_j} &\triangleq \min\left(\max\left(L_T^{c_j} - \alpha, 0\right), \beta - \alpha\right). \end{aligned}$$

Using the order-one truncated expansion, we have

$$\mathbb{E}\left[M_T^{\alpha,\beta}\right] \simeq \exp\left(-\sum_{j=1}^m \Lambda^{c_j}(T)\right)\left[\mathbb{E}\left[M_T^0\right] + \sum_{j=1}^m \mathbb{E}\left[M_T^{c_j}\right]\Lambda^{c_j}(T)\right],$$

and the (α,β)-tranche Q-factor is given by

$$Q^{\alpha,\beta}(T) \simeq \exp\left(-\sum_{j=1}^m \Lambda^{c_j}(T)\right)\left[\mathbb{E}\left[1 - \frac{M_T^0}{\beta-\alpha}\right] + \sum_{j=1}^m \mathbb{E}\left[1 - \frac{M_T^{c_j}}{\beta-\alpha}\right]\Lambda^{c_j}(T)\right]. \tag{5.34}$$

On the other hand, we have from equation (5.19) the expression of the Q-factor for the single name equivalent process

$$Q^{eq}(T) \simeq \exp\left(-\sum_{j=1}^m \Lambda^{c_j}(T)\right)\left[\exp\left(-\Lambda^{0,eq}(T)\right) + \sum_{j=1}^m \left(1 - \widetilde{p_j^{eq}}\right)\Lambda^{c_j}(T)\right].$$

Hence, by matching the idiosyncratic term and the market factor components, we get

$$\begin{aligned} \exp\left(-\Lambda^{0,eq}(T)\right) &= \mathbb{E}\left[1 - \frac{M_T^0}{\beta-\alpha}\right], \\ \left(1 - \widetilde{p_j^{eq}}\right) &= \mathbb{E}\left[1 - \frac{M_T^{c_j}}{\beta-\alpha}\right]. \end{aligned}$$

the (α,β)-tranche equivalence transformation is completely defined by

$$\Lambda^{0,eq}(T) = -\log\left[\mathbb{E}\left[1 - \frac{M_T^0}{\beta-\alpha}\right]\right]. \tag{5.35}$$

$$p_j^{eq} = 1 - \left[\exp\left(\Lambda^{0,eq}(T)\right)\mathbb{E}\left[1 - \frac{M_T^{c_j}}{\beta-\alpha}\right]\right]. \tag{5.36}$$

In summary, the (α,β)-tranche equivalence transformation can be described algorithmically as follows:

1. Find $\lambda^{0,eq}$

   (a) Generate the idiosyncratic loss distribution $\left[\mathbb{P}\left(L_T^0 = l\right)\right]$ by Fourier inversion[3] or Convolution recursion[4]

   (b) Compute the Expected value of the idiosyncratic (α, β)-tranche loss

$$\mathbb{E}\left[M_T^0\right] = \sum_{l=\alpha}^{\beta} (l-\alpha)\,\mathbb{P}\left(L_T^0 = l\right) + (\beta - \alpha)\,\mathbb{P}\left(L_T^0 > \beta\right).$$

   (c) Solve for $\lambda^{0,eq}$ using equation (5.35)

2. For each market factor, find p_j^{eq}

   (a) Transform the $p_{i,j}$'s to the $\widetilde{p_{i,j}}$'s: $\widetilde{p_{i,j}} = 1 - e^{-\Lambda^{0,i}(T)}\left(1 - p_{i,j}\right)$

   (b) Generate the conditional market factor default distribution $\left[\mathbb{P}\left(L_T^{c_j} = l\right)\right]$ by Fourier inversion or Convolution recursion

   (c) Compute the Expected value of the market factor (α, β)-tranche loss

$$\mathbb{E}\left[M_T^{c_j}\right] = \sum_{l=\alpha}^{\beta} (l-\alpha)\,\mathbb{P}\left(L_T^{c_j} = l\right) + (\beta - \alpha)\,\mathbb{P}\left(L_T^{c_j} > \beta\right).$$

   (d) Solve for p_j^{eq} using equation (5.36)

5.4.4 Numerical Examples

In this section, we use the results derived in the previous section to study the default correlation properties of basket securities.

We use a 100-name investment grade diversified portfolio. The average credit spread is 120 bps, the maximum spread is 500 bps and the minimum spread is 30 bps. The portfolio is diversified across 19 industry sectors, where the industry concentrations vary from 2% to 11%. The portfolio spread distribution is given in Figure (5.2).

The portfolio industry concentrations is given in Figure (5.3).

[3] If $\phi(u) = \mathbb{E}\left[\exp\left(iuL(T)\right)\right]$ is the characteristic function of the r.v. $L(T)$, then its probability distribution is given by the standard Fourier-inversion formula

$$\mathbb{P}\left[L(T) \le \alpha\right] = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{\operatorname{Im}\left[\phi(u)\exp(-iu\alpha)\right]}{u}du$$

where Im[z] denotes the imaginary part of a complex number z. See, for example, Duffie and Pan (1999) for a discussion of the numerical integration of the Fourier-inversion formula and control of the discretization error.

[4] See Appendix C.3.

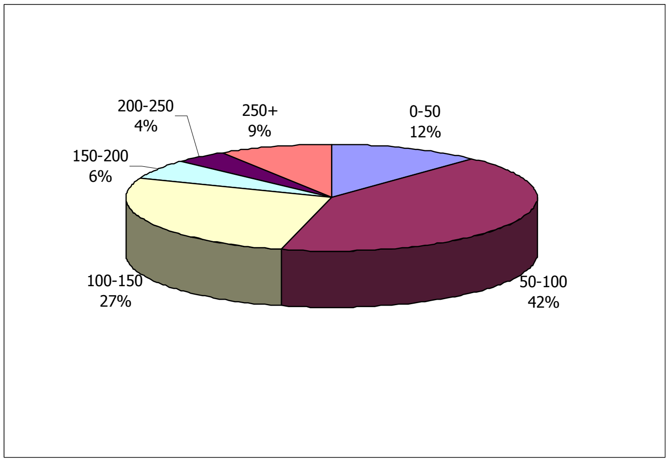

Figure 5.2: Portfolio Spread Distribution

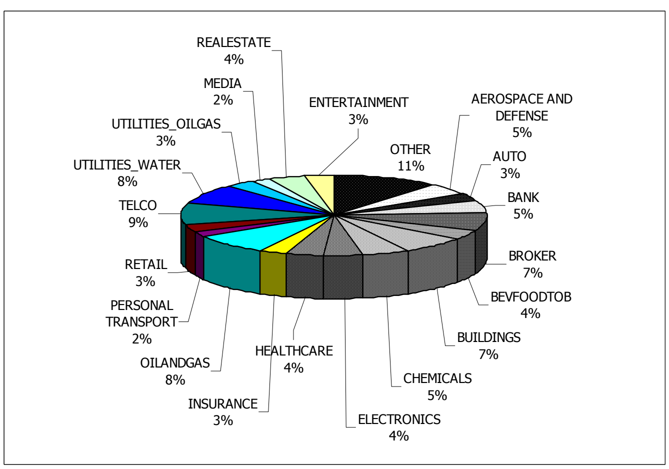

Figure 5.3: Portfolio Industry Concentration

We assume that the intensity of each issuer is has the following Marshall-Olkin decomposition:

$$\lambda_i(t) = \left[\lambda^W(t)\right] + p_{i,B}\left[\lambda^B(t)\right] + \sum_{j=1}^{m} p_{i,S_j}\left[\lambda^{S_j}(t)\right] + \left[\lambda^{0,i}(t)\right],$$

where

$\lambda_W(t)$ is the intensity of the "World" driver

$\lambda_B(t)$ is the intensity of the "Beta" driver, and $p_{i,B}$ is the loading on that driver

$\lambda_{S_j}(t)$ is the intensity of the "Sector" driver S_j, and p_{i,S_j} is the loading on that sector[5]

$\lambda_i^0(t)$ is the intensity of the idiosyncratic events

The "World" driver represents the global Armageddon risk, which triggers the joint defaults of all the credits in the universe. The world driver event is a very low-probability event. However, the loading of each credit on this driver is equal to 1. The world driver is used to calibrate to the super AAA risk in the CDO market. The "Beta" driver is responsible for the correlation between names in different sectors. And the "Sector" drivers make names in the same sector more correlated than the rest of the universe.

The intensity of the World driver is $\lambda^W = 2.5$ bps. The intensity of the Beta driver is $\lambda^B = 400$ bps. The Sector driver intensities vary from 100 bps to 300 bps. We assume that 50% of the spread is Beta, 25% is sector and 25% is idiosyncratic. On average, the implied 5 year default correlation in this model is 7% inter-sector and 14% intra-sector.

Using the equivalence transformation in Subsection 5.4.2, we generate the equivalent single name process and its MO decomposition for all the default slices on this portfolio i.e. FTD, STD, ... The MO representation for the intensity of a k^{th}-to-default time $\tau^{[k]}$ of the portfolio $(\tau_1, ..., \tau_n)$ is then given by

$$\lambda^{[k]}(t) = \left[\lambda^W(t)\right] + p_B^{[k]}\left[\lambda^B(t)\right] + \sum_{j=1}^{m} p_{S_j}^{[k]}\left[\lambda^{S_j}(t)\right] + \left[\lambda^{[k],0}(t)\right]. \tag{5.37}$$

Obviously, the loading of the k^{th}-to-default on the world driver will also be equal to 1. The world driver triggers the defaults of all the names in the portfolio, therefore $\tau^{[k]}$ will also trigger for all default slices $1 \leq k \leq n$.

Figure (5.4) shows how the loading on the Beta driver $p_B^{[k]}$ and the loading on one of the Sector drivers $p_{S_j}^{[k]}$ vary across the default slices $k = 1, 2, ..., n$.

[5] If $i \in S_j$ then $p_{i,S_j} > 0$ otherwise $p_{i,S_j} = 0$

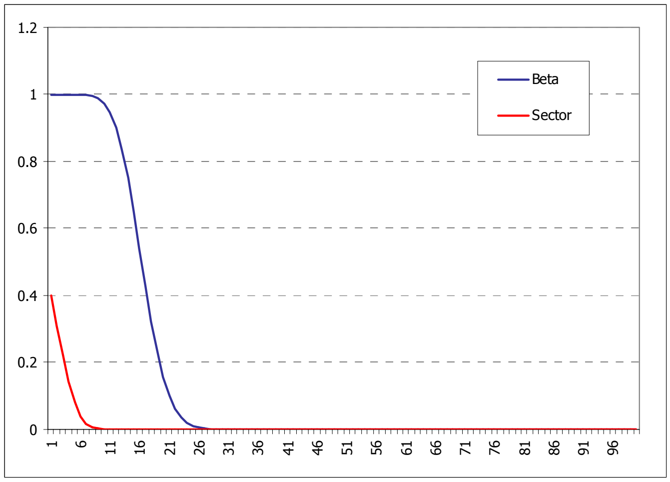

Figure 5.4: Loading of the default time $\tau^{[k]}$ on the Beta driver and one of the sector drivers as a function of the slice index k

Clearly, for the high-order tranches (i.e. super-senior risk) from $k = 30$ to $k = 100$, the default event $\tau^{[k]}$ becomes a pure World driver event. It is very unlikely that a super-senior tranche will be hit by defaults unless there is a global meltdown where everyone defaults. So, the intensity of $\tau^{[k]}$ reduces to $\lambda^{[k]}(t) = \lambda^{W}(t)$ and $p_B^{[k]} = 0$, $p_{S_j}^{[k]} = 0$.

On the other side of the default spectrum (low values of k), we know that for a FTD intensity the loadings are given by

$$p_B^{[1]} = 1 - \prod_{i=1}^{n} (1 - p_{i,B}) \leq 1 - \left(1 - \max_{1 \leq i \leq n} (p_{i,B})\right)^n .$$

For large portfolios $n \to \infty$,

$$\prod_{i=1}^{n} (1 - p_{i,B}) \to 0 \text{ as } n \to \infty,$$

and the order of magnitude of $p_B^{[1]}$ will be around 1

$$p_B^{[1]} \simeq 1.$$

For the loadings on the sector drivers, $p_{S_j}^{[1]}$ is equal to $1 - \prod_{i \in S_j} \left(1 - p_{i,S_j}\right)$. Since the

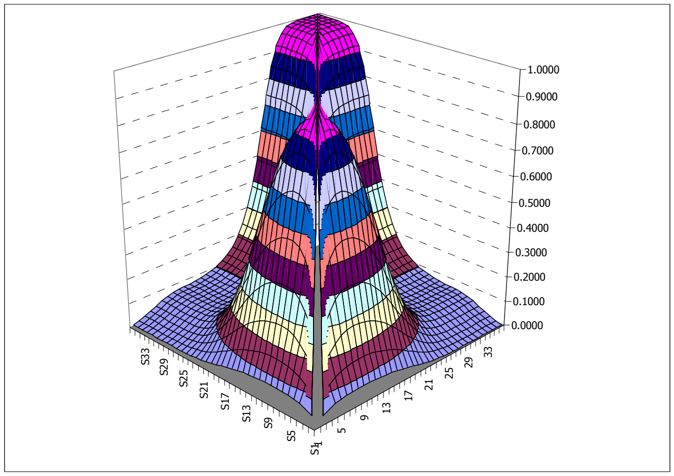

Figure 5.5: 5-year default correlation surface of the k^{th}-to-default times $\left(\tau^{[1]}, \tau^{[2]}, ..., \tau^{[n]}\right)$

portfolio is diversified, the loading $p_{S_j}^{[1]}$ will be of the order of

$$\begin{aligned} p_{S_j}^{[1]} &= 1 - \prod_{i \in S_j} \left(1 - p_{i,S_j}\right) \\ &\simeq 1 - \left(1 - \overline{p_{i,S_j}}\right)^{|S_j|} \\ &\simeq 1 - \left(1 - |S_j| * \overline{p_{i,S_j}}\right) \\ &\simeq \text{ number of names in } S_j \ * \text{ average sector loading.} \end{aligned}$$

Now, if we look at the implied 5y default correlation of the default times $\left(\tau^{[1]}, \tau^{[2]}, ..., \tau^{[n]}\right)$, we get the surface depicted in Figure (5.5).

The default correlation between the low-default slices is close to zero. The correlation between the super-senior slices is equal to 1. And we have a hump in the middle where the correlation increases to 0.95 and drops again to 0.15. This can be seen more easily on Figure (5.6) where we plot the upper-diagonal (i.e. the pairs $\rho_{1,2}, \rho_{2,3}, \rho_{3,4}, ..., \rho_{n-1,n}$).

For the higher slices, the default event $\tau^{[k]}$ degenerates to a pure world driver event, therefore, by construction the default correlation between all senior slices will be a perfect 1. For the lowest slices, as one would expect we have exactly the opposite effect. Equity slices are mostly driven by idiosyncratic events, therefore the default correlation between these events is close to zero. The hump that we observe for the middle Mezzanine slices can be explained by the Beta driver. The slices $k = 8, 9, 10, 11, 12$ have a high probability of triggering almost simultaneously if a Beta

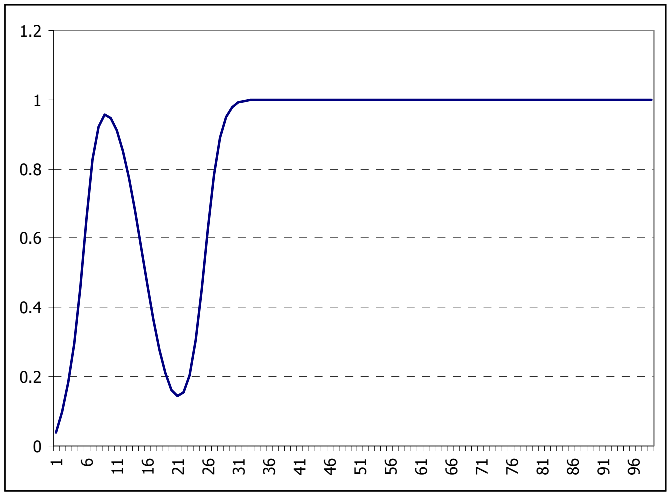

Figure 5.6: Upper-diagonal correlation curve $[\rho_{1,2}, \rho_{2,3}, \rho_{3,4}, ..., \rho_{n-1,n}]$

event occurs. Therefore, their default correlation is exceptionally high. This effect can also be exhibited if we superpose on our correlation plot the ATM spreads of the corresponding slices (Figure (5.7)). We can immediately spot that there are two plateaus in the graph:

1. The super-senior plateau where all the slices $k \geq 30$ have an ATM spread of 2.5 bps i.e. the world driver spread.

2. The Beta plateau where the slices $k = 8, 9, 10, 11, 12$ have roughly the same spread $\simeq$ 400 - 450 bps.

If we remove the Beta driver and Sector driver dependencies, the Mezzanine hump will disappear and the default correlation plot will vary from 0 for low slices to 1 for high slices (Figure (5.8)). The speed of switching from the 0-correlation regime to the 1-correlation regime will depend on the level of the world driver.

5.5 Conclusion

The new generation of portfolio credit derivatives such as CDOs of CDOs and baskets of baskets offer a new modelling challenge in default correlation space. In this chapter, we have presented a simple approach to this problem, which provides a better intuitive understanding of the compound correlation effects. First, we have shown that the first-to-default replication techniques can easily be extended to this type of products.

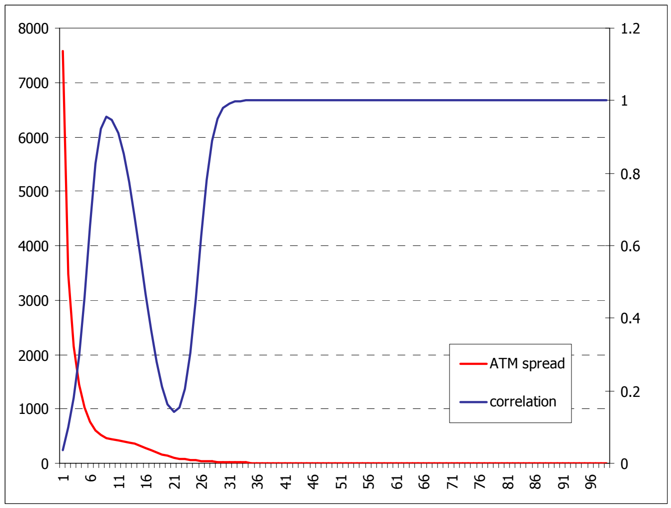

Figure 5.7: Upper-diagonal correlation and ATM spreads for the corresponding tranches

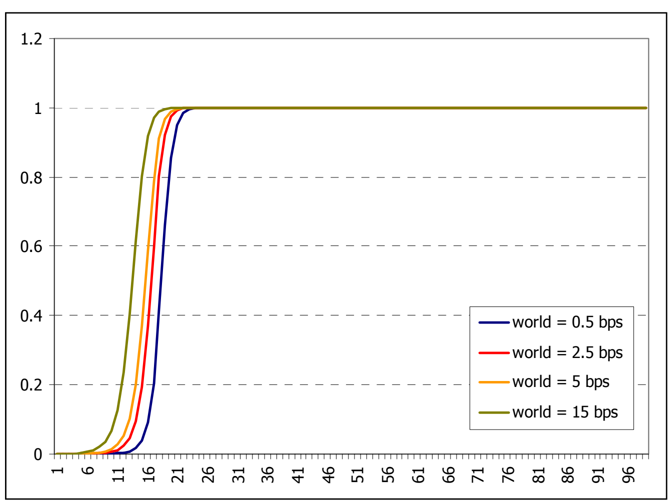

Figure 5.8: Upper-diagonal correlation curves for different values of the world driver

Then, we have shown that each underlying basket security could be viewed as a single-name process. Deriving the probabilistic characteristics of the single-name process, i.e., its intensity and its Marshall-Olkin decomposition, has allowed us to study the basket correlation behaviour and its dependence on the choice of the copula function parameters. One extension to this work would be to apply the equivalent single-name technique to other copula models such as the often-used Gaussian or t-copula. A second extension would be to study the effects of the basket securities portfolio overlaps on the correlation properties of the basket of baskets.

Appendix A

Additional Proofs of Chapter 4

A.1 Proof of Theorem 44

Proof. We proceed by induction: we assume that the property is verified for n, and we prove that it holds for $n+1$.

The probability generating function for a basket of $(n+1)$ underlying credits is defined by:

$$\varphi_{n+1}(x) = \sum_{k=0}^{n+1} \mathbb{P}\left(\sum_{i=1}^{n+1} D_T^i = k\right) x^k.$$

Conditioning on D_T^{n+1}, we can re-write the p.g.f. as:

$$\varphi_{n+1}(x) = \mathbb{E}\left[x^{D_T^1+...+D_T^{n+1}}\right] = \mathbb{E}\left[\mathbb{E}\left[x^{D_T^1+...+D_T^{n+1}}\left|D_T^{n+1}\right.\right]\right]. \tag{A.1}$$

D_T^{n+1} takes two values: 0 or 1, hence,

$$\begin{aligned}\varphi_{n+1}(x) &= \mathbb{P}\left(D_T^{n+1}=0\right)\mathbb{E}\left[x^{D_T^1+...+D_T^n}\left|D_T^{n+1}=0\right.\right] \\ &+\mathbb{P}\left(D_T^{n+1}=1\right)\cdot x\cdot\mathbb{E}\left[x^{D_T^1+...+D_T^n}\left|D_T^{n+1}=1\right.\right].\end{aligned} \tag{A.2}$$

Applying the induction relationship to the conditional p.g.f. of the n-sum $D_T^1+...+D_T^n$, we have

$$\mathbb{E}\left[x^{D_T^1+...+D_T^n}\left|D_T^{n+1}\right.\right] = \sum_{\pi_n\in\mathbf{\Pi_n}} \mathbb{P}\left(\prod_{i\in\pi_n}\left(1-D_T^i\right)\left|D_T^{n+1}\right.\right) x^{n-d(\pi_n)}(1-x)^{d(\pi_n)}.$$

Replacing in (A.2), and observing that

$$\begin{aligned}\mathbb{P}\left(D_T^{n+1}=0\right)\mathbb{P}\left(\prod_{i\in\pi_n}\left(1-D_T^i\right)\left|D_T^{n+1}=0\right.\right) &= \mathbb{P}\left(\left[\prod_{i\in\pi_n}\left(1-D_T^i\right)\right]\left(1-D_T^{n+1}\right)\right), \\ \mathbb{P}\left(D_T^{n+1}=1\right)\mathbb{P}\left(\prod_{i\in\pi_n}\left(1-D_T^i\right)\left|D_T^{n+1}=1\right.\right) &= \mathbb{P}\left(\left[\prod_{i\in\pi_n}\left(1-D_T^i\right)\right]D_T^{n+1}\right),\end{aligned}$$

we get

$$\begin{aligned}\varphi_{n+1}(x) &= \sum_{\pi_n \in \mathbf{\Pi_n}} \mathbb{P}\left(\left[\prod_{i\in\pi_n}\left(1-D_T^i\right)\right]\left(1-D_T^{n+1}\right)\right) x^{n-d(\pi_n)}\left(1-x\right)^{d(\pi_n)} \\ &+ \sum_{\pi_n \in \mathbf{\Pi_n}} \mathbb{P}\left(\left[\prod_{i\in\pi_n}\left(1-D_T^i\right)\right] D_T^{n+1}\right) x^{1+n-d(\pi_n)}\left(1-x\right)^{d(\pi_n)}. \quad \text{(A.3)}\end{aligned}$$

Observing that

$$\mathbb{P}\left(\left[\prod_{i\in\pi_n}\left(1-D_T^i\right)\right] D_T^{n+1}\right) = \mathbb{P}\left(\prod_{i\in\pi_n}\left(1-D_T^i\right)\right) - \mathbb{P}\left(\left[\prod_{i\in\pi_n}\left(1-D_T^i\right)\right]\left(1-D_T^{n+1}\right)\right), \quad \text{(A.4)}$$

equation (A.3) becomes

$$\begin{aligned}\varphi_{n+1}(x) &= \sum_{\pi_n \in \mathbf{\Pi_n}} \mathbb{P}\left(\left[\prod_{i\in\pi_n}\left(1-D_T^i\right)\right]\left(1-D_T^{n+1}\right)\right) x^{n-d(\pi_n)}\left(1-x\right)^{1+d(\pi_n)} \\ &+ \sum_{\pi_n \in \mathbf{\Pi_n}} \mathbb{P}\left(\prod_{i\in\pi_n}\left(1-D_T^i\right)\right) x^{1+n-d(\pi_n)}\left(1-x\right)^{d(\pi_n)}. \quad \text{(A.5)}\end{aligned}$$

Recall that $\mathbf{\Pi_{n+1}}$ is partitioned into two sets $\mathbf{\Pi^+_{n+1}}$ and $\mathbf{\Pi^-_{n+1}}$:

$$\begin{aligned}\mathbf{\Pi^+_{n+1}} &= \left\{\pi_{n+1} : \pi_{n+1} \in \mathbf{\Pi_{n+1}},\ (n+1) \in \pi_{n+1}\right\}, \\ \mathbf{\Pi^-_{n+1}} &= \left\{\pi_{n+1} : \pi_{n+1} \in \mathbf{\Pi_{n+1}},\ (n+1) \notin \pi_{n+1}\right\}.\end{aligned}$$

For $\pi_{n+1} \in \mathbf{\Pi^+_{n+1}}$: $\pi_{n+1} = \pi_n \cup \{n+1\}$, we have

$$Q^{[1]}_{\pi_{n+1}}(T) = \mathbb{E}\left(\prod_{i\in\pi_{n+1}}\left(1-D_T^i\right)\right) = \mathbb{E}\left(\left[\prod_{i\in\pi_n}\left(1-D_T^i\right)\right]\cdot\left(1-D_T^{n+1}\right)\right),$$

and

$$d(\pi_{n+1}) = d(\pi_n) + 1.$$

For $\pi_{n+1} \in \mathbf{\Pi^-_{n+1}}$: $\pi_{n+1} = \pi_n \cup \emptyset$, we have

$$Q^{[1]}_{\pi_{n+1}}(T) = \mathbb{E}\left(\prod_{i\in\pi_{n+1}}\left(1-D_T^i\right)\right) = \mathbb{E}\left(\prod_{i\in\pi_n}\left(1-D_T^i\right)\right),$$

and

$$d(\pi_{n+1}) = d(\pi_{n+1}).$$

Substituting in equation (A.5) yields

$$\begin{aligned}\varphi_{n+1}(x) &= \sum_{\pi_{n+1} \in \mathbf{\Pi^+_{n+1}}} Q^{[1]}_{\pi_{n+1}}(T)\ x^{n+1-d(\pi_{n+1})}\left(1-x\right)^{d(\pi_{n+1})} \\ &+ \sum_{\pi_{n+1} \in \mathbf{\Pi^-_{n+1}}} Q^{[1]}_{\pi_{n+1}}(T)\ x^{n+1-d(\pi_{n+1})}\left(1-x\right)^{d(\pi_{n+1})},\end{aligned}$$

which ends the proof. ■

A.2 Fourier Transform Inversion

In this appendix, we give the Fourier transform inversion formula for a discrete random variable X. Let $\widetilde{p} = \{p_0, p_1, ..., p_{n-1}\}$ be its probability function, and $\varphi(s)$ its discrete Fourier Transform,

$$\varphi(s) = \sum_{i=0}^{n-1} e^{isk} p_k.$$

Evaluating the Fourier transform $\varphi(s)$ at the points $s_j = \frac{2\pi j}{n}$ for $j = 0, 1, ..., n-1$, gives the following system of equations:

$$\left\{ \varphi(s_j) = \sum_{i=0}^{n-1} e^{iks_j} p_k \right\}_{0 \le j \le n-1}.$$

By letting $\widetilde{\varphi} = \{\varphi(s_0), \varphi(s_1), ..., \varphi(s_{n-1})\}$ denote the vector of Fourier transform values, the system of equations can be written in matrix form as:

$$\widetilde{\varphi} = F\widetilde{p},$$

where F is the $n \times n$ matrix

$$F = \left(e^{is_jk}\right)_{\substack{0 \le j \le n-1 \\ 0 \le k \le n-1}}.$$

Since we have

$$\sum_{k=0}^{n-1} e^{is_jk} e^{-is_lk} = \begin{cases} \frac{e^{i\left(s_j - s_l\right)n} - 1}{e^{i\left(s_j - s_l\right)} - 1} = 0 \text{ for } j \neq l \\ n \qquad\qquad\qquad \text{for } j = l \end{cases},$$

then, the inverse matrix F^{-1} is given by:

$$F^{-1} = \left(\frac{1}{n} e^{-is_kj}\right)_{\substack{0 \le j \le n-1 \\ 0 \le k \le n-1}},$$

and the probability function $\widetilde{p}$ is recovered from the Fourier transform values $\varphi(s_j)$ as:

$$p_k = \frac{1}{n} \sum_{j=0}^{n-1} \varphi(s_j) e^{-is_jk}.$$

Appendix B

Additional Proofs of Chapter 5

B.1 Proof of Proposition 54

Proof. We proceed by induction. We assume the property is verified for n, and we prove that it holds for $n+1$.

For $n+1$ names, the Fourier transform of the loss variable

$$L_T^{n+1} = \sum_{i=1}^{n+1} L_i D_T^i, \tag{B.1}$$

is given by

$$\phi^{n+1}(u) = \mathbb{E}\left[\exp\left(-iuL_T^{n+1}\right)\right].$$

Conditioning on D_T^{n+1}, we have

$$\begin{aligned} \phi^{n+1}(u) &= \mathbb{E}\left[\mathbb{E}\left[\exp\left(-iuL_T^{n+1}\right)\left|D_T^{n+1}\right.\right]\right] \\ &= \mathbb{E}\left[\exp\left(-iuL^{n+1}D_T^{n+1}\right)\mathbb{E}\left[\exp\left(-iuL_T^{n}\right)\left|D_T^{n+1}\right.\right]\right]. \end{aligned} \tag{B.2}$$

Using the induction relationship, we can write the conditional characteristic function as

$$\mathbb{E}\left[\exp\left(-iuL_T^{n}\right)\left|D_T^{n+1}\right.\right] = \sum_{\pi_n \in \mathbf{\Pi_n}} \mathbb{E}\left(\prod_{i\in\pi_n}\left(1-D_T^i\right)\left|D_T^{n+1}\right.\right)\left[\prod_{i\notin\pi_n}\psi_i \prod_{i\in\pi_n}\left(1-\psi_i\right)\right],$$

and equation (B.2) becomes

$$\phi^{n+1}(u) = \mathbb{E}\left[\psi_{n+1}^{D_T^{n+1}}\left[\sum_{\pi_n \in \mathbf{\Pi_n}} \mathbb{E}\left(\prod_{i\in\pi_n}\left(1-D_T^i\right)\left|D_T^{n+1}\right.\right)\left[\prod_{i\notin\pi_n}\psi_i \prod_{i\in\pi_n}\left(1-\psi_i\right)\right]\right]\right].$$

Writing the expectation explicitly, we get

$$\begin{aligned}
&\phi^{n+1}(u) \\
= \; & \mathbb{P}\left(D_T^{n+1}=0\right) \sum_{\pi_n \in \mathbf{\Pi_n}} \mathbb{E}\left(\prod_{i\in\pi_n} \left(1-D_T^i\right) \middle| \left\{D_T^{n+1}=0\right\} \right) \left[\prod_{i\notin\pi_n} \psi_i \prod_{i\in\pi_n} (1-\psi_i) \right] \quad \text{(B.3)} \\
&+\mathbb{P}\left(D_T^{n+1}=1\right) \psi_{n+1} \sum_{\pi_n \in \mathbf{\Pi_n}} \mathbb{E}\left(\prod_{i\in\pi_n} \left(1-D_T^i\right) \middle| \left\{D_T^{n+1}=1\right\} \right) \left[\prod_{i\notin\pi_n} \psi_i \prod_{i\in\pi_n} (1-\psi_i) \right].
\end{aligned}$$

Observing that

$$\mathbb{P}\left(D_T^{n+1}=0\right) \mathbb{E}\left(\prod_{i\in\pi_n} \left(1-D_T^i\right) \middle| \left\{D_T^{n+1}=0\right\} \right) = \mathbb{E}\left(\left[\prod_{i\in\pi_n} \left(1-D_T^i\right) \right] \left(1-D_T^{n+1}\right) \right),$$

and

$$\begin{aligned}
&\mathbb{P}\left(D_T^{n+1}=1\right) \mathbb{E}\left(\prod_{i\in\pi_n} \left(1-D_T^i\right) \middle| \left\{D_T^{n+1}=1\right\} \right) \\
= \; & \mathbb{E}\left(\left[\prod_{i\in\pi_n} \left(1-D_T^i\right) \right] D_T^{n+1} \right) \\
= \; & \mathbb{E}\left(\left[\prod_{i\in\pi_n} \left(1-D_T^i\right) \right] \right) - \mathbb{E}\left(\left[\prod_{i\in\pi_n} \left(1-D_T^i\right) \right] \left(1-D_T^{n+1}\right) \right),
\end{aligned}$$

equation (B.3) becomes

$$\begin{aligned}
\phi^{n+1}(u) \; = \; & \sum_{\pi_n \in \mathbf{\Pi_n}} \mathbb{E}\left(\left[\prod_{i\in\pi_n} \left(1-D_T^i\right) \right] \left(1-D_T^{n+1}\right) \right) \left(1-\psi_{n+1}\right) \left[\prod_{i\notin\pi_n} \psi_i \prod_{i\in\pi_n} (1-\psi_i) \right] \\
&+ \sum_{\pi_n \in \mathbf{\Pi_n}} \mathbb{E}\left(\left[\prod_{i\in\pi_n} \left(1-D_T^i\right) \right] \right) \psi_{n+1} \left[\prod_{i\notin\pi_n} \psi_i \prod_{i\in\pi_n} (1-\psi_i) \right]. \qquad \text{(B.4)}
\end{aligned}$$

The set of subsets $\mathbf{\Pi_{n+1}}$ is partitioned into $\mathbf{\Pi^+_{n+1}}$ and $\mathbf{\Pi^-_{n+1}}$:

$$\begin{aligned}
\mathbf{\Pi^+_{n+1}} &= \left\{\pi_{n+1} : \pi_{n+1} \in \mathbf{\Pi_{n+1}},\ (n+1) \in \pi_{n+1}\right\}, \\
\mathbf{\Pi^-_{n+1}} &= \left\{\pi_{n+1} : \pi_{n+1} \in \mathbf{\Pi_{n+1}},\ (n+1) \notin \pi_{n+1}\right\}.
\end{aligned}$$

For $\pi_{n+1} \in \mathbf{\Pi^+_{n+1}}$: $\pi_{n+1} = \pi_n \cup \{n+1\}$, we have

$$Q^{[1]}_{\pi_{n+1}}(T) = \mathbb{E}\left(\prod_{i\in\pi_{n+1}} \left(1-D_T^i\right) \right) = \mathbb{E}\left(\left[\prod_{i\in\pi_n} \left(1-D_T^i\right) \right] \cdot \left(1-D_T^{n+1}\right) \right),$$

and

$$n+1 \in \pi_{n+1}.$$

For $\pi_{n+1} \in \mathbf{\Pi}^{-}_{\mathbf{n+1}}$: $\pi_{n+1} = \pi_n \cup \emptyset$, we have

$$Q^{[1]}_{\pi_{n+1}}(T) = \mathbb{E}\left(\prod_{i\in\pi_{n+1}}\left(1-D^i_T\right)\right) = \mathbb{E}\left(\prod_{i\in\pi_n}\left(1-D^i_T\right)\right),$$

and

$$n+1 \notin \pi_{n+1}.$$

Substituting in equation (B.4) yields

$$\begin{aligned}\phi^{n+1}(u) \quad = \quad & \sum_{\pi_{n+1}\in\mathbf{\Pi}^{+}_{\mathbf{n+1}}} Q^{[1]}_{\pi_{n+1}}(T)\left[\prod_{i\notin\pi_{n+1}}\psi_i\prod_{i\in\pi_{n+1}}(1-\psi_i)\right] \\ & + \sum_{\pi_{n+1}\in\mathbf{\Pi}^{-}_{\mathbf{n+1}}} Q^{[1]}_{\pi_{n+1}}(T)\left[\prod_{i\notin\pi_{n+1}}\psi_i\prod_{i\in\pi_{n+1}}(1-\psi_i)\right],\end{aligned}$$

which ends the proof. ■

B.2 Proof of Proposition 55

Proof. Using Proposition 54, we express the probability generating function of the r.v. L_T in terms of the FTD Q-factors $Q^{[1]}_{\pi}(T)$

$$\phi(u) = \sum_{\pi\in\mathbf{\Pi_n}} Q^{[1]}_{\pi}(T)\left[\prod_{i\notin\pi}\psi_i\prod_{i\in\pi}(1-\psi_i)\right]. \tag{B.5}$$

Each FTD Q-factor $Q^{[1]}_{\pi}(T)$ is given by

$$Q^{[1]}_{\pi}(T) = \exp\left(-\left[\sum_{j=1}^{m}\left(1-\prod_{i\in\pi}(1-p_{i,j})\right)\Lambda^{c_j}(T)\right] - \left[\sum_{i\in\pi}\Lambda^{0,i}(T)\right]\right). \tag{B.6}$$

We expand the exponentials

$$\exp\left(\prod_{i\in\pi}(1-p_{i,j})\Lambda^{c_j}(T)\right) = \sum_{n_j=0}^{+\infty}\frac{\left[\prod_{i\in\pi}(1-p_{i,j})\Lambda^{c_j}(T)\right]^{n_j}}{n_j!}.$$

We substitute in equation (B.6)

$$\begin{aligned} & Q^{[1]}_{\pi}(T) \\ = \quad & e^{-\Lambda^c(T)-\left[\sum_{i\in\pi}\Lambda^{0,i}(T)\right]}\sum_{n_1=0}^{+\infty}\ldots\sum_{n_m=0}^{+\infty}\frac{\left[\prod_{i\in\pi}(1-p_{i,1})\Lambda^{c_1}(T)\right]^{n_1}}{n_1!}\ldots\frac{\left[\prod_{i\in\pi}(1-p_{i,m})\Lambda^{c_m}(T)\right]^{n_m}}{n_m!} \\ = \quad & e^{-\Lambda^c(T)}\sum_{n_1=0}^{+\infty}\ldots\sum_{n_m=0}^{+\infty}\frac{\Lambda^{c_1}(T)^{n_1}}{n_1!}\ldots\frac{\Lambda^{c_m}(T)^{n_m}}{n_m!}e^{-\sum_{i\in\pi}\left(\Lambda^{0,i}(T)+\log((1-p_{i,1})^{n_1})+\ldots+\log((1-p_{i,m})^{n_m})\right)}.\end{aligned}$$

Substituting in the expression of the characteristic function yields

$$\phi(u) = e^{-\Lambda^c(T)} \sum_{n_1=0}^{+\infty} ... \sum_{n_m=0}^{+\infty} \frac{\Lambda^{c_1}(T)^{n_1}}{n_1!} ... \frac{\Lambda^{c_m}(T)^{n_m}}{n_m!} \sum_{\pi \in \mathbf{\Pi_n}} Q_{n_1,...,n_m}(\pi) \left[\prod_{i \notin \pi} \psi_i \prod_{i \in \pi} (1-\psi_i) \right],$$

where

$$Q_{n_1,...,n_m}(\pi) = \exp\left(-\sum_{i \in \pi} \left(\Lambda^{0,i}(T) + \log\left((1-p_{i,1})^{n_1}\right) + ... + \log\left((1-p_{i,m})^{n_m}\right) \right) \right).$$

We define the function $\phi_{n_1,...,n_m}(u)$

$$\phi_{n_1,...,n_m}(u) = \sum_{\pi \in \mathbf{\Pi_n}} Q_{n_1,...,n_m}(\pi) \left[\prod_{i \notin \pi} \psi_i \prod_{i \in \pi} (1-\psi_i) \right]. \tag{B.7}$$

$\phi_{n_1,...,n_m}(u)$ is the Fourier transform of n independent idiosycratic terms with intensities $\left(\Lambda^i_{n_1,...,n_m}(T)\right)_{1 \le i \le n}$

$$\Lambda^i_{n_1,...,n_m}(T) = \Lambda^{0,i}(T) + \log\left((1-p_{i,1})^{n_1}\right) + ... + \log\left((1-p_{i,m})^{n_m}\right).$$

Hence

$$\phi_{n_1,...,n_m}(u) = \prod_{i=1}^{n} \left(\psi_i + \left(e^{-\Lambda^{0,i}(T)} (1-p_{i,1})^{n_1} ... (1-p_{i,m})^{n_m} \right) (1-\psi_i) \right).$$

Thus, $\phi_{n_1,...,n_m}(u)$ is the Fourier transform of the random variable $\sum_{i=1}^{n} Y_i$, where $(Y_1,...,Y_n)$ are n independent Bernoulli variables with parameters $\widetilde{p_{n_1,...,n_m}} = (p_{n_1,...,n_m}(1),...,p_{n_1,...,n_m}(n))$

$$p_{n_1,...,n_m}(i) = 1 - e^{-\Lambda^{0,i}(T)} (1-p_{i,1})^{n_1} ... (1-p_{i,m})^{n_m}.$$

$\phi(u)$ is a weighted average of the conditional independent characteristic functions $\varphi_{n_1,...,n_m}(x)$

$$\begin{aligned} \varphi(x) &= \sum_{n_1=0}^{+\infty} ... \sum_{n_m=0}^{+\infty} w_{n_1,...,n_m} \cdot \phi_{n_1,...,n_m}(u), \\ w_{n_1,...,n_m} &= e^{-\Lambda^c(T)} \frac{\Lambda^{c_1}(T)^{n_1}}{n_1!} ... \frac{\Lambda^{c_m}(T)^{n_m}}{n_m!}, \end{aligned} \tag{B.8}$$

which ends the proof. ■

B.3 Convolution Recursion for Computing the Loss Distribution

For a set of n independent discrete random variables $(Y_1,...,Y_n)$, we want to derive the distribution of the sum

$$S_n \triangleq Y_1 + ... + Y_n.$$

The generating function of the random variable S_n,

$$\varphi_{S_n}(x) \triangleq \sum_{k=0}^{\infty} \mathbb{P}(S_n = k)\ x^k,$$

is given by the product of the generating functions of the variables Y_i (since they are all independent)

$$\varphi_S(x) = \prod_{i=1}^{n} \varphi_{Y_i}(x),$$

and its distribution $p_{S_n}(s) = \mathbb{P}(S_n = s)$ is given by the n-fold convolution of the distributions p_{Y_i}

$$p_{S_n}(s) = \bigotimes_{i=1}^{n} p_{Y_i}(s). \tag{B.9}$$

This convolution product is computed by applying the following formula recursively

$$\begin{aligned} p_{S_{k+1}}(s) &= p_{S_k} \otimes p_{Y_{k+1}}(s) \\ &= \sum_{y=0}^{s} p_{Y_{k+1}}(y)\, p_{S_k}(s-y),\ \text{for } 1 \leq k \leq n-1. \end{aligned} \tag{B.10}$$

In our case, we need to generate the aggregate loss distribution $S_n = \sum_{i=1}^{n} L_i X_i$, where $(X_1, ..., X_n)$ are Bernoulli variables with parameters $\widetilde{p} = (p_1, ..., p_n)$.

The sum S_n will be computed recursively as follows:

for $0 \leq k \leq n-1$

$$p_{S_{k+1}}(s) = p_{k+1} p_{S_k}(s - L_{k+1}) + (1 - p_{k+1})\, p_{S_k}(s),\ 0 \leq s \leq \sum_{i=1}^{k+1} L_i,$$

with the convention

$$p_{S_k}(-x) = 0, x \geq 0,$$

and the distribution of the empty sum S_0 is defined as

$$\begin{cases} p_{S_0}(0) = 1, \\ p_{S_0}(s) = 0 \text{ for } s > 0. \end{cases}$$

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