Basket Default Swaps, CDO's and Factor Copulas

Jean-Paul Laurent[1] & Jon Gregory[2]

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Abstract

We consider a factor copula appoach to the pricing of basket credit derivatives and CDO tranches. Our purpose is to deal in a convenient way with dependent defaults and credit spreads. We provide semi-explicit expressions of the stochastic intensities of default times, credit spreads, and price of basket default swaps involving large number of names. We also consider the explicit pricing of CDO tranches within our framework. Two cases are studied in detail: mean-variance mixture models and Archimedean copulas.

Introduction

We consider a factor copula appoach to the pricing of basket credit derivatives and CDO tranches. Our purpose is to deal in a convenient way with dependent default dates, credit spreads and basket default swap premiums associated with this modelling of dependence between default dates. A typical pattern with a regular copula such as the Gaussian one is the following: credit spreads tend to decrease until the first to default time since no defaults on the other underlyings usually means good news for a given underlying[1], while we observe jumps in credit spreads at default times: the default on one given name usually provide some news for the remaining reference credits. We consider in greater detail the special case of the *factor copulas*. The dimensionality issue is important for theoretical and practical reasons. When pricing counterparty risk on derivatives, one usually needs only the joint distribution of two default dates, but when dealing with synthetic CDO's, we must consider up to one hundred names or more. It may even be uneasy to specify simple models such as the multivariate exponential models of Marshall and Olkin in large dimension. Moreover, it involves a large number of unknown parameters, while the factor approach is usually more parcimonious, thus easing the calibration. Even in the simple Gaussian framework, one must

[1]ISFA Actuarial School, University of Lyon & BNP Paribas.
laurent.jeanpaul@free.fr, http://laurent.jeanpaul.free.fr
[2]BNP Paribas. Jon.Gregory@bnpparibas.com

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[1]By underlying, we mean a reference credit within the basket or the CDO structure.

usually rely on Monte Carlo techniques that prove to be costly and where acceleration techniques often perform poorly. Thanks to the factor approach, we can provide semi-explicit expressions of the stochastic intensities of default times, credit spreads, and basket default swap premiums involving large number of names. We also consider the explicit pricing of CDO tranches within our framework.

The firm value approach to credit (see Bielecki and Rutkowki [2002] for a presentation) has been extended to multi name pricing (see Arvanitis and Gregory [2001], Hull and White [2001]). This approach is suitable for the pricing of hybrid equity credit products. However, it proves to be time-consuming for multi-name basket structures, especially for risk analysis.

The framework of reduced form models has also been considered for the pricing of basket credit derivatives (see Duffie [1998]) and leads to simple theoretical expressions of prices. Dependence of default times has been firstly addressed through correlated stochastic risk intensities (see Duffie and Gârleanu [1998] for an application to the pricing of CDO). However, this usually results in low default times dependence as studied in Andreasen [2001]. Another approach consists in relaxing the independence assumption of the latent uniform random variables involved in the Cox process modelling. This results in a series of models such as the Gaussian copula approach introduced for the pricing of basket credit derivatives by Li [1999, 2000]. The multivariate exponential copula of Marshall and Oklin [1967] (see Duffie and Singleton [1998], Wong [1998], Kijima [2000], Li [2000]) provides another framework which allows for simultaneous defaults and is associated with non smooth joint distribution functions. Schönbucher and Schubert [2001] study the dynamics of default intensities and show that Clayton copulas, a member of the Archimedean copula family, are related to the dependent intensities approaches of Kusuoka [1999], Davis and Lo [1999, 2001], Jarrow and Yu [2001]). A related reference is Giesecke [2001]. Bouyé et al [2000], Durrleman et al [2000], Schmidt and Ward [2002] also consider some applications of copulas for the pricing of basket credit derivatives.

On the other hand, latent factor models have been widely used for the computation of default events and loan loss distributions (see Crouhy, Galai & Mark [2000], Belkin, Suchover and Forest [1998], Finger [1998, 1999], Koyluoglu and Hickman [1998], Lucas, Klaasen, Spreij and Staetmans [1999], Merino and Nyfeler [2002], Schönbucher [2002], Vasicek [1997]). Frey, McNeil and Nyfeler [2001], Wang [1998], relate factor and copula approaches. The new Basel agreement, popular models such as Credit Metrics, Credit Risk+, KMV rely on such approaches. These models have been thoroughly used in the statistical literature (see Junker and Ellis [1998] for some characterizations of one factor models and Gouriéroux and Monfort [2002] for some application to credit risk). Moreover, de Finetti's theorem for exchangable sequences of binary random variables provide some theoretical background for the use of such factor models in the credit risk framework. The main feature of these models is that default events, conditionally on some latent state variables are independent. This eases the computation of aggregate loss distributions through dimensionality reduction. This factor approach is nicely suited for large dimensional problems. Since semi-explicit expressions of most relevant quantities can be obtained, it provides an alternative route to Monte Carlo approaches, while we can still rely on the latter when useful. The main technical assumption in our paper is the smoothness of the joint survival function. On economic grounds, this precludes simultaneous defaults. The smoothness assumption is not fulfilled in some multivariate exponential models (associated with Marshall Olkin [1967] copulas), that have been used by Duffie and Singleton [1998], Wong [2000].

The paper is organized as follows:

- The first section defines marginal and conditional hazard rates of default times. We relate conditional hazard rates from the joint survival function of default times. Eventually, some expressions of conditional hazard rates as conditional probabilities are provided for regular joint distributions of default times. marginal and conditional hazard rates can be related to stochastic intensities of default times under different filtrations. The stated results will provide some building blocks for the pricing of basket credit derivatives and CDO. The proofs are gathered in appendix A. We also recall some standard results on copulas and also relates this approach to multivariate Cox processes.
- The second section considers the special case of factor copulas. We study the conditional survival functions, hazard rates and the number of defaults under this assumption. We consider Gaussian copulas, mean-variance mixtures and Archimedean copulas where semi-explicit expressions can be provided.
- The third section deals with the computation of the various basket default swaps premiums.
- The fourth section considers the pricing of CDO tranches.

1 Survival function and hazard rates

1.1 Marginal hazard rates

We consider n underlying defaultable issuers, with associated default times, τ_i, $i = 1, \dots, n$. For all $t_1, \dots, t_n$ in $\mathbb{R}$, $(t_1, \dots, t_n) \to S(t_1, \dots, t_n)$ will denote the joint survival function of default times: $S(t_1, \dots, t_n) = Q(\tau_1 \geq t_1, \dots, \tau_n \geq t_n)$ where Q denotes the risk-neutral probability[2]. Let us note that we consider here unconditional probabilities. Similarly, the joint distribution function F will be such that $F(t_1, \dots, t_n) = Q(\tau_1 < t_1, \dots, \tau_n < t_n)$. S_i and F_i will be respectively the marginal survival and marginal distribution function, $S_i(t_i) = Q(\tau_i \geq t_i)$ and $F_i(t_i) = Q(\tau_i < t_i) = 1 - S_i(t)$. We denote by $\tau^1 = \min(\tau_1, \dots, \tau_n) = \tau_1 \wedge \dots \wedge \tau_n$ the **first to default time**. The survival function of τ^1 is thus simply given by $Q(\tau^1 \geq t) = S(t, \dots, t)$. In the following, we will make the following convenient and simplifying assumption:

Assumption 1 *The marginal distributions of default times τ_i, $i = 1, \dots, n$, are absolutely continuous (wrt to the Lebesgue measure), i.e. for all $i = 1, \dots, n$, with right continuous densities f_i, i.e. $F_i(t) = \int_0^t f_i(u)du$ for all $t \geq 0$ and f_i are right continuous*[3]*.*

Since the marginal distributions are absolutely continuous, the F_i are continuous. Moreover, assuming right continuity of the densities implies that the distribution functions F_i are right differentiable[4].

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[2] We assume here the use of some pricing measure and do not discuss the existence or uniqueness of such a measure. Q characterizes an arbitrage free pricing model.

[3] it is presumably possible to state a slighly different and possibly weaker assumption, i.e. that the distribution functions F_i are right differentiable.

[4] The right continuity of f_i implies that $\forall \varepsilon > 0, \exists \alpha_\varepsilon > 0$ with $u \in [t, t + \alpha_\varepsilon[\Rightarrow | f_i(u) - f_i(t) | < \varepsilon$. Let us consider α with $0 < \alpha \leq \alpha_\varepsilon$. $\forall u \in [t, t + \alpha[$, we have $f_i(t) - \varepsilon \leq f_i(u) \leq f_i(t) + \varepsilon$, which gives

Definition 1 marginal hazard rates
Under the oustanding assumption (1), we define the following marginal hazard rates of default times τ_i*,* $i = 1, \dots, n$ *as:*

$$\boxed{h_i(t) = \lim_{dt \to 0^+} \frac{Q(\tau_i \in [t, t+dt[\mid \tau_i \geq t)}{dt} = \frac{f_i(t)}{S_i(t)} = -\frac{1}{S_i(t)} \frac{dS_i(t)}{dt^+}.}$$

Since the distribution of τ_i is absolutely continuous and right differentiable, the $h_i(t)$ are well defined. Indeed,

$$\frac{Q(\tau_i \in [t, t+dt[\mid \tau_i \geq t)}{dt} = \frac{Q(\tau_i \geq t) - Q(\tau_i \geq t+dt)}{Q(\tau_i \geq t)dt} = \frac{1}{S_i(t)} \frac{F_i(t+dt) - F_i(t)}{dt}.$$

From the previous footnote, the limit of the previous quantity as dt tends to 0 (by the right) is equal to $\dfrac{f_i(t)}{S_i(t)}$.
Assuming only right continuity of f_i allows to deal with the following modelling: $S_i(t) = \exp \int_0^t -h_i(u)du$, where h_i is piecewise constant of the form $h_i(u) = \sum_{j \in \mathbb{N}} h_{i,j} \mathcal{I}_{[a_j, a_{j+1}[}(u)$ and $a_j, h_{i,j} \in \mathbb{R}$, $j \in \mathbb{N}$ is an increasing to infinity sequence with $a_0 = 0$. This modelling is common when stripping defaultable bonds or credit default swaps[5].

We can notice that since the marginal distributions of default times τ_i are continuous, then a standard probability result states that the joint distribution of default times $S(t_1, \dots, t_n)$ is continuous too. As a consequence, under assumption (1), there is no point with a (strictly) positive probability[6]. However, it can be that the marginal distributions of default times are absolutely continuous while the joint distribution is not absolutely continuous (i.e. there does not exist a joint density)[7].

$f_i(t) - \varepsilon \leq \dfrac{\int_t^{t+\alpha} f_i(u)du}{\alpha} \leq f_i(t) + \varepsilon$. The last equality states that $\lim_{\alpha \downarrow 0} \dfrac{\int_t^{t+\alpha} f_i(u)du}{t + \alpha - t} = f_i(t)$, which means that $F_i(t)$ is right differentiable and the associated derivative is $f_i(t)$. One point to check is the existence of some hazard rate under the sole assumption of absolute continuity

[5] Assumption (1) may have to be further relaxed since the outcomes of some strippers are rather discrete distributions with mass points at tenor dates.

[6] i.e. the joint distribution is atomless.

[7] A typical example is the following. Let us consider three independent exponentially distributed random variables, denoted by $\bar{\tau}_1$, $\bar{\tau}_2$ and τ_c with corresponding parameters $\bar{\lambda}_1, \bar{\lambda}_2, \lambda_c$. Let us define two default times as $\tau_1 = \min(\bar{\tau}_1, \tau_c)$ and $\tau_2 = \min(\bar{\tau}_2, \tau_c)$. It can be seen that τ_1 and τ_2 are exponentially distributed with parameters $\bar{\lambda}_1 + \lambda_c$ and $\bar{\lambda}_2 + \lambda_c$. Thus, the marginal distributions are smooth. Moreover, the first to default time $\min(\tau_1, \tau_2) = \min(\bar{\tau}_1, \bar{\tau}_2, \bar{\tau}_c)$ is also exponentially distributed with parameter $\bar{\lambda}_1 + \bar{\lambda}_2 + \lambda_c$. However, it appears that (τ_1, τ_2) does not admit a joint density. The probability of simultaneous defaults of the two reference credits is provided by $Q(\tau_1 = \tau_2) = Q(\tau_c \leq \min(\bar{\tau}_1, \bar{\tau}_2)) > 0$. It can be checked that if (τ_1, τ_2) has a joint density, then $Q(\tau_1 = \tau_2) = 0$, since the diagonal has zero Lebesgue measure in $\mathbb{R}^2$. It can be easily checked that the joint survival function is given by $S(t_1, t_2) = \exp -(\bar{\lambda}_1 t_1 + \lambda_2 t_2 + \lambda_c \max(t_1, t_2))$, which is indeed continuous but not absolutely continuous. From this example, we can also conclude that a model where simultaneous defaults can occur cannot be associated with a joint density of default times.

1.2 Conditional hazard rates up to first to default time

Typically, the pricing of basket default derivatives requires the joint distribution of default times. In this subsection, we define the conditional hazard rates up to first to default time. We relate then to conditional probabilities under some continuity assumption. We thus need to introduce the following assumption:

Assumption 2 *The joint survival function $(t_1, \dots, t_n) \to S(t_1, \dots, t_n)$ is right differentiable in each coordinate for every point of the first diagonal $(t_1, \dots, t_n) = (t, \dots, t)$, $t \geq 0$.*

Definition 2 conditional hazard rates (before first to default time)
Under assumption (2), we define the conditional hazard rates $\lambda_i(t)$, $i = 1, \dots, n$ as:

$$\boxed{\lambda_i(t) = \lim_{dt \to 0^+} \frac{Q(\tau_i \in [t, t+dt[| \, \tau_1 \geq t, \dots, \tau_n \geq t)}{dt} = -\frac{1}{S(t, \dots, t)} \frac{\partial S(t, \dots, t)}{\partial t_i^+}.}$$

In the previous definition, the conditioning information set consists in the joint observation of default times. $\lambda_i(t)$ is interpreted as the probability of name i defaulting in the next small time interval $[t, t+dt[$ provided that none of the reference credits have defaulted prior to t, which means that $\lambda_i(t)$ is defined on $\{\tau_1 \geq t, \dots, \tau_n \geq t\}$. Let us remark that while for notational convenience $\lambda_i(t)$ is indexed on i only, it depends on the whole joint distribution of default times. For instance, if the set of names in a basket is modified, so does the conditional hazard rate of some given name. We can circumvent this difficulty by defining a large set of relevant reference credits including the names in traded baskets, keeping this set unchanged[8]. We consider hazard rates before the first default. We will consider later the effect of some default on the remaining reference credits. Assumption (2) guarantees that the $\lambda_i(t)$ are well defined.

We can provide some interpretation of the marginal hazard rates and the conditional hazard rates as stochastic intensities of default times under different filtrations. We denote by $\mathcal{H}_{i,t}$ the natural filtration of the stopping time τ_i. The marginal hazard rate $h_i(t)$ will be related to the $\mathcal{H}_i$ stochastic intensity of τ_i. More precisely, the stochastic intensity of τ_i is $h_i(t)\mathcal{I}_{\tau_i < t}$. We consider the filtration $\mathcal{H}_t = \bigvee_{i=1}^n \mathcal{H}_{i,t}$. $\lambda_i(t)$ is related to the $\mathcal{H}_t$ stochastic intensity of τ_i. For instance on $\{\tau^1 < t\}$ the $\mathcal{H}_t$ stochastic intensity of τ_i is equal to $\lambda_i(t)$, where τ^1 is the first to default time.

In the special case where default times $\tau_1, \dots, \tau_n$ are independent, we have:

Lemma 1.1 *Under assumption (1), if default times $\tau_1, \dots, \tau_n$ are independent, then $h_i(t) = \lambda_i(t)$, for $i = 1, \dots, n$.*

This is a consequence of:

$$Q(\tau_i \in [t, t+dt[| \, \tau_1 \geq t, \dots, \tau_n \geq t) = \frac{Q(\tau_i \in [t, t+dt[, \tau_1 \geq t, \dots, \tau_n \geq t)}{Q(\tau_1 \geq t, \dots, \tau_n \geq t)} = \frac{Q(\tau_i \in [t, t+dt[)}{Q(\tau_i \geq t)},$$

the latter equality is due to the independence assumption. However, in the following examples, default times will be correlated and the previous property may not hold. When default times are positively correlated,

[8]However, for pricing purpose, the conditional hazard rates associated with the set of reference names in a basket or CDO structure are to be used. Later on, we show that these hazard rates are related to stochastic intensities of τ_i under different filtrations.

we can expect that $\lambda_i(t)$ will be smaller than $h_i(t)$. Indeed if none of the underlyings have defaulted, this provides some extra good news about underlying i. Let us emphasize that we can still have "*positive correlation*" between default times while $h_i(t) = \lambda_i(t)$. If we turn back to the example with simultaneous defaults, we can check that $S(t_1, t_2)$ is right-differentiable in each coordinate on the first diagonal and that $\lambda_1(t) = \bar{\lambda}_1 + \lambda_c = h_1(t)$.

Assumption 3 *The joint survival function $(t_1, \dots, t_n) \to S(t_1, \dots, t_n)$ is differentiable on $\mathbb{R}^{+n}$.*

We recall that if $S(t_1, \dots, t_n)$ is differentiable, then there exist derivatives in all directions. For instance, under assumption (3) the function $t \to S(t, \dots, t)$ is differentiable and:

$$\frac{d}{dt} S(t, \dots, t) = \sum_{i=1}^{n} \frac{\partial S}{\partial t_i}(t, \dots, t),$$

where $\frac{\partial S}{\partial t_i}(t, \dots, t)$ denotes the derivative of the joint survival function with respect to the ith component taken at point $(t, \dots, t)$. A sufficient condition for $S(t_1, \dots, t_n)$ to be differentiable is that there exists continuous partial derivatives $\frac{\partial S}{\partial t_i}$ (in this case, $S(t_1, \dots, t_n)$ is continuously differentiable). We can also state the useful technical lemma:

Lemma 1.2 *Under assumption (3), we have:*

$$\boxed{\frac{\partial S}{\partial t_i}(t_1, \dots, t_n) = -Q\left(\tau_j \geq t_j, \forall j \neq i \mid \tau_i = t_i\right) f_i(t_i).} \tag{1.1}$$

Under the differentiability assumption, we now now write the conditional hazard rate using conditional expectations :

Property 1.1 conditional hazard rate before the first to default time
Under assumption (3), the conditional hazard rates before the first to default time are given by:

$$\boxed{\lambda_i(t) = f_i(t) \times \frac{Q\left(\min_{j \neq i} \tau_j \geq t \mid \tau_i = t\right)}{Q\left(\min_j \tau_j \geq t\right)},} \tag{1.2}$$

This is a direct consequence of previous lemma. The conditional hazard rates can be compared to the marginal hazard rates $h_i(t)$:

$$h_i(t) = \frac{f_i(t)}{S_i(t)} = f_i(t) \times \frac{Q\left(\min_{j \neq i} \tau_j \geq t \mid \tau_i \geq t\right)}{Q\left(\min_j \tau_j \geq t\right)}. \tag{1.3}$$

If default times τ_j are positively correlated, then $\{\tau_i = t\}$ means worse news than $\{\tau_i \geq t\}$. Thus, we can expect that $Q\left(\min_{j \neq i} \tau_j \geq t \mid \tau_i = t\right) < Q\left(\min_{j \neq i} \tau_j \geq t \mid \tau_i \geq t\right)$, which then means that $\lambda_i(t) < h_i(t)$.

1.3 Between first and second to default

The computation of hazard rates at and after the first to default times and the related credit spread curves are useful to study the pricing of second to default swaps or counterparty risk on credit default swaps. We start with a useful technical lemma:

Lemma 1.3 *For given dates, t, t_j, t_i, $0 \le t_j \le t \le t_i$ and a set of default times $\tau_1, \dots, \tau_n$, we can write the following conditional probability of default as:*

$$\boxed{Q\left(\tau_i \geq t_i \mid \min_{k \neq j} \tau_k \geq t, \tau_j = t_j\right) = \frac{Q\left(\tau_i \geq t_i, \min_{k \neq j} \tau_k \geq t \mid \tau_j = t_j\right)}{Q\left(\min_{k \neq j} \tau_k \geq t \mid \tau_j = t_j\right)}.} \tag{1.4}$$

We can remark that $Q\left(\tau_i \geq t_i \mid \min_{k \neq j} \tau_k \geq t, \tau_j = t_j\right)$ is a conditional survival function, i.e. the conditional probability that default on i occurs after time t_i. However the conditioning set is no more $\{\min_i \tau_i \geq t\} = \{\tau^1 \geq t\}$ (that is before the first default on the basket) as was studied in the previous subsection but $\{\min_{k \neq j} \tau_k \geq t, \tau_j = t_j\}$, which can be interpreted as follows for $t_j \leq t \leq t_i$: first default has occured on underlying j at time $t_j < t$, t is the current date and no other default has occured since t_j (we are between the first to default and the second to default time). The conditional survival functions $Q\left(\tau_i \geq t_i \mid \min_{k \neq j} \tau_k \geq t, \tau_j = t_j\right)$ (for $i \neq j$) will be involved in the computation of credit spread curves between the first to default time τ^1 and the second to default time τ^2.

It is possible to go further in this analysis and compute survival functions and hazard rates given a series of past defaults. This usually results in jumps of hazard rates at default times. The derivations involve partial derivatives of the joint survival function[9] and conditional distributions of default times. However, these extra computations are not required for the pricing of basket credit derivatives and CDO.

1.4 First to default time hazard rates

Assumption 4 *The function $t \to S(t, \dots, t)$ is right differentiable for all $t \geq 0$.*

Lemma 1.4 *Under assumption (4), the hazard rate of the first to default time τ^1 defined as:*

$$\lambda^1(t) = \lim_{dt \to 0^+} Q(\tau^1 \in [t, t+dt[\mid \tau^1 \geq t) = \lim_{dt \to 0^+} Q(\tau^1 \in [t, t+dt[\mid \tau_1 \geq t, \dots \tau_n \geq t),$$

is equal to $\boxed{-\dfrac{1}{S(t, \dots, t)} \dfrac{dS(t, \dots, t)}{dt^+}}$.

Lemma 1.5 *Under assumption (3) the hazard rate of the first to default time τ^1 is equal to the sum of the conditional hazard rates:*

$$\lambda^1(t) = \sum_{i=1}^{n} \lambda_i(t).$$

[9] One can also use derivatives of the copula of default times as in Schönbucher and Schubert [2001].

This lemma is a straightforward consequence of the previous lemmas. Let us notice that in the correlated case, $\lambda_F(t) \neq \sum_{i=1}^{n} h_i(t)$. This is rather unfortunate since the $h_i(t)$ are standard outputs from strippers while the $\lambda_i(t)$ are more difficult to obtain. Let us remark that assumption (3) may not be satisfied in practical examples. It can be that $t \to S(t,\dots,t)$ is differentiable while $(t_1,\dots,t_n) \to S(t_1,\dots,t_n)$ is not differentiable. Since when default rates are correlated conditional hazard rates differ from marginal hazard rates, the first to default time differs from the sum of the marginal hazard rates.

1.5 Copula functions and joint distributions

The use of Copula functions[10] will allow to have some easy to implement results on the dynamics of credit spreads and default times. Let us firstly recall the definition of a copula.

Definition 3 *A copula function C on $\mathbb{R}^n$, $n \in \mathbb{N}$ is a joint distribution function on $\mathbb{R}^n$ with marginal distribution functions being uniform on $[0,1]$.*

We now state a useful result that relates continuous joint distributions to copulas:

Theorem 1.1 Sklar theorem for continuous marginal distributions
Let F be a joint distribution function on $\mathbb{R}^n$ with continuous marginal distributions F_i. Then there exist a unique copula function C such that for all $x_1,\dots,x_n$ in $\mathbb{R}$,

$$F(x_1,\dots,x_n) = C(F_1(x_1),\dots,F_n(x_n)).$$

While keeping the assumption of continuity, Let us moreover assume that the F_i are strictly increasing. On financial grounds, this means that default can occur at all positive times. In this case, the cdf F_i, $i = 1,\dots,n$, have plain inverses[11] F_i^{-1} we simply get:

$$C(u_1,\dots,u_n) = F(F_1^{-1}(u_1),\dots,F_n^{-1}(u_n)). \tag{1.5}$$

1.6 Multivariate Cox Processes

We can easily relate the copula approach and the multivariate Cox process framework. Let us consider a random vector $U = (U_1,\dots,U_n)$ with marginals uniformly distributed on $[0,1]$ and joint distribution function $C(u_1,\dots,u_n)$. We construct a set of default times $\tau_1,\dots,\tau_n$ as:

$$\tau_i = \inf\left\{u, \int_0^u h_i(v)dv \geq -\log U_i\right\},\ i = 1,\dots,n.$$

The h_i are some positive deterministic functions[12]. It can be checked that the marginal survival functions are given by:

$$S_i(t_i) = Q(\tau_i \geq t_i) = \exp\left(-\int_0^{t_i} h_i(v)dv\right).$$

[10] See Joe [1997], Frees and Valdez [1998], Nelsen [1999] for some general presentations and further details.

[11] For a general univariate distribution function F, we define F^{-1} as the generalized inverse or quantile function by $F^{-1}(u) = \{\sup z, F(z) \leq u\}$. We can check that: $x \leq F^{-1}(u) \Leftrightarrow F(x) \leq u,\ u,x \in \mathbb{R}$. As a consequence, if U is uniformly distributed, $F^{-1}(U)$ has distribution F.

[12] These can be made stochastic but for simplicity, we rely on the deterministic assumption.

Thus, h_i are the marginal hazard rates introduced above. Similarly, the joint survival function is given by:

$$S(t_1, \ldots, t_n) = C\left(\exp - \int_0^{t_1} h_1(v)dv, \ldots, \exp - \int_0^{t_n} h_n(v)dv\right).$$

We can notice that C is the survival copula of default times[13] and thus we can equally think of the dependence structure of the default times and of the underlying uniform random variables.

2 Factor Copula approaches

We now detail various examples with practical importance.

2.1 Gaussian Copulas

The *Gaussian Copula* is given by:

$$C_\rho(u_1, \ldots, u_n) = \Phi_{n,\rho}(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_n)), \tag{2.1}$$

where $\Phi_{n,\rho}$ is the joint distribution function of a multivariate Gaussian vector with mean zero and covariance matrix equal to ρ (where ρ is a correlation matrix) and Φ is the distribution function of a standard Gaussian random variable. It has been introduced by Li [1999, 2000] for the pricing of basket credit derivatives and corresponds to the dependence structure underlying CreditMetrics and the New Basel Agreement. From equations (1.5) and (2.1), we get :

$$F(t_1, \ldots, t_n) = P(\tau_1 < t_1, \ldots, \tau_n < t_n) = \Phi_{n,\rho}(\Phi^{-1}(F_1(t_1)), \ldots, \Phi^{-1}(F_n(t_n))). \tag{2.2}$$

We can rewrite the previous equation as :

$$P\left(\Phi^{-1}(F_1(\tau_1)) < \Phi^{-1}(F_1(t_1)), \ldots\right) = \Phi_{n,\rho}(\Phi^{-1}(F_1(t_1)), \ldots, \Phi^{-1}(F_n(t_n))), \tag{2.3}$$

which states that $\left(\Phi^{-1}(F_1(\tau_1)), \ldots, \Phi^{-1}(F_n(\tau_n))\right)$ is a Gaussian vector with mean zero and covariance matrix ρ. However, even if the computation of default times is fairly easy, let us remark that for good quality reference credits, simulated default times are usually much larger than maturity dates. Thus, Monte Carlo convergence is very slow, especially regarding the computation of the greeks, and this calls for acceleration techniques such as importance sampling. Another route is to provide some semi-explicit results and rely on numerical integration in one or more dimensions.

One factor Gaussian copula: If we consider a Gaussian vector $(X_1, \ldots, X_n)$ where $X_i = \rho_i V + \sqrt{1-\rho_i^2}\bar{V}_i$, where $V, \bar{V}_i,\ i = 1, \ldots, n$ are independent standard Gaussian random variables[14] we get $\mathrm{cov}(X_i, X_j) = \rho_i\rho_j$ for $i \neq j$ and $\mathrm{cov}(X_i, X_i) = 1$ for $i = 1, \ldots, n$. Such a correlation structure is appropriate for computations: it involves only n parameters and provides tractable expressions for survival functions. Let us consider the Gaussian copula in that framework:

$$C(u_1, \ldots, u_n) = Q\left(X_1 < \Phi^{-1}(u_1), \ldots, X_n < \Phi^{-1}(u_n)\right).$$

[13] This depends on the h_i's being deterministic.

[14] Here, we depart from the BIS notations, where $X_i = \sqrt{\rho_i}V + \sqrt{1-\rho_i}\bar{V}_i$. In the BIS settings, if $\rho_i = \rho$ is independent of i, then $\mathrm{cov}(X_i, X_j) = \rho$. Thus ρ can readily be seen as a correlation parameter.

By iterated expectations theorem, we can write the previous term as:

$$E\left(Q\left(X_1 < \Phi^{-1}(u_1), \dots, X_n < \Phi^{-1}(u_n)\right) \middle| V\right),$$

which leads to:

$$\boxed{C(u_1, \dots, u_n) = \int \left(\prod_{i=1}^{n} \Phi\left(\frac{\Phi^{-1}(u_i) - \rho_i v}{\sqrt{1-\rho_i^2}} \right) \right) \varphi(v) dv,}$$

where $\varphi(v) = \frac{1}{\sqrt{2\pi}} e^{-v^2/2}$ is the Gaussian density. Thus, the n-dimensional copula is computed through a one dimensional integral. Similarly, the joint distribution and the joint survival functions are provided by:

$$\boxed{F(t_1, \dots, t_n) = \int \left(\prod_{i=1}^{n} \Phi\left(\frac{\Phi^{-1}(F_i(t_i)) - \rho_i v}{\sqrt{1-\rho_i^2}} \right) \right) \varphi(v) dv,}$$

and by:

$$\boxed{S(t_1, \dots, t_n) = \int \left(\prod_{i=1}^{n} \Phi\left(\frac{\rho_i v - \Phi^{-1}(F_i(t_i))}{\sqrt{1-\rho_i^2}} \right) \right) \varphi(v) dv.}$$

The previous integrals can be easily computed through some quadrature.

2.2 One factor mean variance Gaussian mixtures

The previously described one factor Gaussian copula can be extended to a variety of one factor models with easy implementation. This includes mean variance Gaussian mixtures and Archimedean copulas. In a mean variance or location scale mixture model, we write $X_i = m_i(\theta) + \sigma_i(\theta)\varepsilon_i$, where θ is a one dimensional mixing latent variable with density function f and the ε_i are independent standard Gaussian random variables. A special case to be discussed below is $m_i(\theta) = \rho_i \theta$ and $\sigma_i = \sqrt{1-\rho_i^2}$. Conditionally on θ, the X_i's are independent Gaussian random variables with mean $m_i(\theta)$ and standard deviation $\sigma_i(\theta)$. Let us firstly compute the marginal and joint distributions of the X_i's. $G_i(x_i) = Q(X_i < x_i) = \int \Phi\left(\frac{x_i - m_i(u)}{\sigma_i(u)}\right) f(u) du$ and $G(x_1, \dots, x_n) = Q(X_1 < x_1, \dots, X_n < x_n) = \int \prod_{i=1}^{n} \Phi\left(\frac{x_i - m_i(u)}{\sigma_i(u)}\right) f(u) du$, where the G_i's denote the marginal cdf and G the joint cdf. From this, we derive the copula:

$$\boxed{C(u_1, \dots, u_n) = \int \prod_{i=1}^{n} \Phi\left(\frac{G_i^{-1}(u_i) - m_i(u)}{\sigma_i(u)} \right) f(u) du, \quad \forall u_1, \dots, u_n \in [0,1],}$$

which involves the computation of a one dimensional integral. Let us remark (see below that the ε_i do not need to be Gaussian in order to obtain this dimensionality reduction. By invariance of copulas under monotonic transforms, the previous copula is also that of the default times τ_i. The calibration to the marginal distributions of default times consists in determining increasing real functions g_i, $i = 1, \dots, n$ such that $\boxed{\tau_i = g_i(X_i)}$ and τ_i has distribution function F_i. We have $F_i(g_i(t)) = Q(\tau_i < g_i(t)) = Q(X_i < t) = G_i(t) = \int \Phi\left(\frac{t - m_i(u)}{\sigma_i(u)}\right) f(u) du$. Thus, we have the calibrating equation:

$$\boxed{g_i(t) = F_i^{-1}\left(\int \Phi\left(\frac{t - m_i(u)}{\sigma_i(u)} \right) f(u) du \right),} \tag{2.4}$$

which completes the description of the model. Let us remark that the joint survival function of default times is readily obtained as:

$$\boxed{S(t_1,\ldots,t_n) = Q(\tau_1 \geq t_1,\ldots,\tau_n \geq t_n) = \int \prod_{i=1}^{n} \Phi\left(\frac{m_i(u) - g_i^{-1}(t_i)}{\sigma_i(u)}\right) f(u)du.} \tag{2.5}$$

2.3 One factor structure and Archimedean copulas

Let F_i be the cdf of default time τ_i, f the density of a positive mixing variable θ and $\psi(s) = \int_0^\infty e^{-sx} f(x)dx$, the Laplace transform of f. We define G_i as $\forall t \geq 0$, $G_i(t) = \exp\left(-\psi^{-1}(F_i(t))\right)$. G_i defines a distribution function. Thus $F_i(t) = \psi\left(-\ln G_i(t)\right) = \int_0^\infty G_i^\alpha(t) f(\alpha)d\alpha$. Let us remark that G_i^α is a proper distribution function. Thus conditionally on α, the distribution of τ_i is G_i^α. We define the joint distribution of default times by: $F(t_1,\ldots,t_n) = \int_0^\infty \prod_{i=1}^n G_i^\alpha(t_i) f(\alpha)d\alpha$. $\forall t_1,\ldots,t_n$, $Q\left(\tau_1 < t_1,\ldots,\tau_n < t_n \mid \alpha\right) = \prod_{i=1}^n G_i^\alpha(t_i)$; conditionally on α the default times are independent. Thus $Q\left(\tau_1 \geq t_1,\ldots,\tau_n \geq t_n \mid \alpha\right) = \prod_{i=1}^n \left(1 - G_i^\alpha(t_i)\right)$. By iterated expectations theorem, this leads to the joint survival function:

$$\boxed{S(t_1,\ldots,t_n) = \int \prod_{i=1}^{n} \left(1 - G_i^\alpha(t_i)\right) f(\alpha)d\alpha.}$$

Since $\int_0^\infty \prod_{i=1}^n G_i^\alpha(t_i) f(\alpha)d\alpha = \psi\left(-\sum_{i=1}^n \ln G_i(t_i)\right) = \psi\left(\sum_{i=1}^n \psi^{-1}\left(F_i(t_i)\right)\right)$, we conclude that the joint distribution function can be computed directly as:

$$\boxed{F(t_1,\ldots,t_n) = \psi\left(\sum_{i=1}^n \psi^{-1}\left(F_i(t_i)\right)\right),}$$

and the copula of default times is given by:

$$\boxed{C(u_1,\ldots,u_n) = \psi\left(\psi^{-1}(u_1) + \ldots + \psi^{-1}(u_n)\right).}$$

Thus C is an Archimedean copula with generator $\phi = \psi^{-1}$. We will see further that we can compute the distribution of k-th to default times and loss distribution within this one factor framework. A typical example is the Clayton copula, where the mixing variable has a Gamma distribution with parameter $1/\theta$, where $\theta > 0$. More precisely, we have $f(x) = \frac{1}{\Gamma(1/\theta)} e^{-x} x^{(1-\theta)/\theta}$, $\psi^{-1}(s) = s^{-\theta} - 1$, $\psi(s) = (1+s)^{-1/\theta}$. This leads to $C(u_1,\ldots,u_n) = \left(u_1^{-\theta} + \ldots + u_n^{-\theta} - n + 1\right)^{-1/\theta}$ and $G_i(t) = \exp\left(1 - F_i(t)^{-\theta}\right)$.

2.4 Survival function of first to default time

We consider here the computation of the survival function of the first to default time that will be further involved in computing basket default swap premiums under various dependence assumptions. The distribution of first to default time can be computed directly, while for more general kth to default time, we use a moment generating function approach as discussed below.

General Gaussian copula: The survival function of the first to default time τ^1, $S(t) = Q(\tau^1 \geq t)$ is equal to:
$S(t) = S(t, \ldots, t) = Q(\tau_1 \geq t, \ldots, \tau_n \geq t) = Q\left(X_1 \geq \Phi^{-1}(F_1(t)), \ldots, X_n \geq \Phi^{-1}(F_n(t))\right)$. Eventually, we get:

$$S(t) = \bar{\Phi}_{n,\rho}\left(\Phi^{-1}(F_1(t)), \ldots, \Phi^{-1}(F_n(t))\right)$$

where ρ is the covariance matrix of $(X_1, \ldots, X_n)$ and $\bar{\Phi}_{n,\hat{\rho}}$ is the n Gaussian joint survival function with covariance matrix ρ.

One factor Gaussian copula: Let us consider the previous one factor assumption for the correlation structure and compute the distribution of the first to default time $S(t) = Q(\tau^1 \geq t)$. Since,

$$S(t) = S(t, \ldots, t) = Q\left(X_1 \geq \Phi^{-1}(F_1(t)), \ldots, X_n \geq \Phi^{-1}(F_n(t))\right),$$

where $X_i = \rho_i V + \sqrt{1-\rho_i^2}\bar{V}_i$, we can write:

$$S(t) = E\left[Q\left(X_1 \geq \Phi^{-1}(F_1(t)), \ldots, X_n \geq \Phi^{-1}(F_n(t)) \mid V\right)\right],$$

from iterated expectations theorem. Using that $\bar{V}_i \geq \dfrac{\Phi^{-1}(F_i(t)) - \rho_i V}{\sqrt{1-\rho_i^2}}$ and the independence assumptions, we obtain:

$$\boxed{S(t) = E\left[\prod_{i=1}^{n} \Phi\left(\frac{\rho_i V - \Phi^{-1}(F_i(t))}{\sqrt{1-\rho_i^2}}\right)\right] = \int \prod_{i=1}^{n} \Phi\left(\frac{\rho_i v - \Phi^{-1}(F_i(t))}{\sqrt{1-\rho_i^2}}\right)\varphi(v)dv,} \tag{2.6}$$

where $\varphi(v) = \frac{1}{\sqrt{2\pi}}e^{-v^2/2}$ is the Gaussian density. This permits to compute the premium leg of a first to homogeneous[15] default swap under the assumption of independence between defaults and interest rates.

One factor mean variance Gaussian mixture copula: We need once again to compute $S(t) = Q(\tau_1 \geq t, \ldots, \tau_n \geq t)$. Since $\tau_i = g_i(X_i)$, $i = 1, \ldots, n$, we get the survival function of first to default time as: $S(t) = Q\left(X_1 \geq g_1^{-1}(t), \ldots, X_n \geq g_n^{-1}(t)\right)$. Conditioning on θ, we get:

$$S(t) = E\left[Q\left(X_1 \geq g_1^{-1}(t), \ldots, X_n \geq g_n^{-1}(t) \middle| \theta\right)\right],$$

which leads to:

$$\boxed{S(t) = \int \left(\prod_{i=1}^{n} \Phi\left(\frac{m_i(v) - g_i^{-1}(t)}{\sigma_i(v)}\right)\right) f(v)dv.} \tag{2.7}$$

Archimedean copula: From the expression of the joint survival function, we get:

$$\boxed{S(t) = \int \left(\prod_{i=1}^{n} (1 - G_i^{\alpha}(t))\right) f(\alpha)d\alpha,}$$

which in the case of the Clayton copula provides:

$$S(t) = \frac{1}{\Gamma(1/\theta)} \int \left(\prod_{i=1}^{n} 1 - e^{\alpha\left(1-F_i(t)^{-\theta}\right)}\right) e^{-\alpha}\alpha^{(1-\theta)/\theta}d\alpha.$$

[15] Equal nominal amounts and recovery rates of reference credits.

2.5 Survival functions at and after the first to default time

We concentrate here on the conditional survival functions just after first to default time; thus $\tau^1 = t$ where t is the current time. This will allow to compute the jumps in the credit spreads at first to default time and to study hedging effects. Moreover, we will obtain some further technical results useful in pricing of first to default swaps. To make notations simpler, we assume that name one has defaulted first. Thus $\tau_1 = \tau^1$. The time t conditional survival function for name i $(i > 1)$ is then given by:

$$\boxed{S_{i,t}(t_i) = Q\left(\tau_i \geq t_i \mid \tau_1 = t, \min_{k>1} \tau_k \geq t\right).}$$

Thanks to lemma 1.3, we can alternatively write $S_{i,t}(t_i)$ as:

$$\boxed{S_{i,t}(t_i) = \frac{Q\left(\tau_i \geq t_i,\ \min_{k>1} \tau_k \geq t \mid \tau_1 = t\right)}{Q\left(\min_{k>1} \tau_k \geq t \mid \tau_1 = t\right)}.}$$

In the Gaussian copula framework, this usually involves computations of $n-1$ dimensional survival Gaussian distributions. The computations can be made simpler under a factor structure assumption. This allows to reduce the dimension of the problem.

One factor Gaussian copula: Let us as before denote by:

$$(X_1, \dots, X_n) = (\Phi^{-1}(F_1(\tau_1)), \dots, \Phi^{-1}(F_n(\tau_n))),$$

the Gaussian vector associated with default times. We assume here that $X_i = \rho_i V + \sqrt{1-\rho_i^2}\bar{V}_i$. Let us now compute the probabilities conditional on $\tau_1 = t$. We start with the following lemma:

Lemma 2.1 conditional Gaussian distribution, one factor case
Under the one factor Gaussian copula assumption, we have:

$$\boxed{Q\left(X_2 \geq x_2, \dots, X_n \geq x_n \mid X_1 = x_1\right) = \int \prod_{i=2}^{n} \Phi\left(\frac{\rho_i\sqrt{1-\rho_1^2}u + \rho_i\rho_1 x_1 - x_i}{\sqrt{1-\rho_i^2}}\right)\varphi(u)du.} \quad (2.8)$$

Using previous lemma 2.1, we get $S_{i,t}(t_i) = Q\left(\tau_i \geq t_i \mid \tau_1 = t, \min_{k>1}\tau_k \geq t\right)$ $(t_2 \geq t)$, the survival function on name i just after default of name one. We have :

$$\boxed{S_{i,t}(t_i) = \frac{Q\left(\tau_i \geq t_i,\ \min_{k>1} \tau_k \geq t \mid \tau_1 = t\right)}{Q\left(\min_{k>1} \tau_k \geq t \mid \tau_1 = t\right)} = \frac{\displaystyle\int \prod_{j=2}^{n} \Phi\left(\frac{\rho_j\sqrt{1-\rho_1^2}u + \rho_j\rho_1 x_1 - x_j}{\sqrt{1-\rho_j^2}}\right)\varphi(u)du}{\displaystyle\int \prod_{j=2}^{n} \Phi\left(\frac{\rho_j\sqrt{1-\rho_1^2}u + \rho_j\rho_1 y_1 - y_j}{\sqrt{1-\rho_j^2}}\right)\varphi(u)du},} \quad (2.9)$$

for $t_i \geq t$, with $x_j = y_j = \Phi^{-1}(F_j(t))$ for $j \neq i$ and $x_i = \Phi^{-1}(F_i(t_i))$, $y_i = \Phi^{-1}(F_i(t))$.

One factor mean-variance Gaussian mixture copula: Simple expressions can also be obtained in that framework. We use here the connection between conditional probabilities and the partial derivatives of the joint survival function (see lemma (1.2)):

$$Q\left(\tau_i \geq t_i,\ \min_{k>1} \tau_k \geq t \mid \tau_1 = t\right) = -\frac{1}{f_1(t)} \frac{\partial S}{\partial t_1}(t, \dots, t_i, \dots, t),$$

for $t_i \geq t$. Let us now compute the partial derivative. We start from:

$$S(t, \dots, t_i, \dots, t) = \int \prod_{j=1}^{n} \Phi\left(\frac{m_j(u) - g_j^{-1}(t_j)}{\sigma_j(u)}\right) f(u)du,$$

with $t_j = t$ for$j \neq i$. This leads to:

$$\frac{\partial S}{\partial t_1}(t, \dots, t_i, \dots, t) = -\frac{dg_1^{-1}(t)}{dt} \int \frac{1}{\sigma_1(u)} \varphi\left(\frac{m_1(u) - g_1^{-1}(t)}{\sigma_1(u)}\right) \prod_{j=2}^{n} \Phi\left(\frac{m_j(u) - g_j^{-1}(t_j)}{\sigma_j(u)}\right) f(u)du,$$

with $t_j = t$ for$j \neq i$. Eventually, the conditional survival function is provided by:

$$\boxed{S_{i,t}(t_i) = \frac{\displaystyle\int \prod_{j=2}^{n} \Phi\left(\frac{m_j(u) - g_j^{-1}(t_j)}{\sigma_j(u)}\right) \varphi\left(\frac{m_1(u) - g_1^{-1}(t)}{\sigma_1(u)}\right) \frac{f(u)}{\sigma_1(u)} du}{\displaystyle\int \prod_{j=2}^{n} \Phi\left(\frac{m_j(u) - g_j^{-1}(t)}{\sigma_j(u)}\right) \varphi\left(\frac{m_1(u) - g_1^{-1}(t)}{\sigma_1(u)}\right) \frac{f(u)}{\sigma_1(u)} du},} \tag{2.10}$$

for $t_i \geq t$, with $t_j = t$ for$j \neq i$.

Archimedean copulas: we assume here that the distributions G_i admit some continuous hazard rates, i.e. that we can write $1 - G_i(t) = \exp - \int_0^t w_i(u)du$, for some positive continuous functions w_i. Since $S(t, \dots, t_i, \dots, t) = \displaystyle\int \prod_{j=1}^{n} (1 - G_j^{\alpha}(t_j)) f(\alpha) d\alpha$ and $\dfrac{dG_1^{\alpha}(t)}{dt} = \alpha w_1(t)(1 - G_1(t)) G_1^{\alpha-1}(t)$ where $t_j = t$ for $j \neq i$, we get:

$$\frac{\partial S}{\partial t_1}(t, \dots, t_i, \dots, t) = -w_1(t)(1 - G_1(t)) \int \alpha G_1^{\alpha-1}(t) \prod_{j=2}^{n} (1 - G_j^{\alpha}(t_j)) f(\alpha) d\alpha,$$

where $t_j = t$ for $j \neq i$. Eventually,

$$\boxed{S_{i,t}(t_i) = \frac{\displaystyle\int \alpha G_1^{\alpha-1}(t) \prod_{j=2}^{n} (1 - G_j^{\alpha}(t_j)) f(\alpha) d\alpha}{\displaystyle\int \alpha G_1^{\alpha-1}(t) \prod_{j=2}^{n} (1 - G_j^{\alpha}(t)) f(\alpha) d\alpha},} \tag{2.11}$$

for $t_i \geq t$, with $t_j = t$ for$j \neq i$.

2.6 Conditional hazard rates until the first to default time

In the case of a Gaussian copula, the joint survival function is differentiable and thus fulfills assumption (3). We can then use property (1.1) and we recall the expression of the conditional hazard rates before the first to default time:

$$\lambda_i(t) = f_i(t) \times \frac{Q\left(\min_{j \neq i} \tau_j \geq t \mid \tau_i = t\right)}{Q\left(\min_j \tau_j \geq t\right)} \tag{2.12}$$

One factor Gaussian copula: Using lemma (2.1), we get:

$$Q(\min_{j\neq i}\tau_j \geq t \mid \tau_i = t) = \int \prod_{j\neq i} \Phi\left(\frac{\rho_j\sqrt{1-\rho_i^2}u + \rho_j\rho_i x_i - x_j}{\sqrt{1-\rho_j^2}}\right)\varphi(u)du, \tag{2.13}$$

where $x_j = \Phi^{-1}(F_j(t))$ for $j = 1, \dots, n$. Then from property (1.1), we write the following:

$$\boxed{\lambda_i(t)S(t) = h_i(t)S_i(t) \times \int \prod_{j\neq i} \Phi\left(\frac{\rho_j\sqrt{1-\rho_i^2}u + \rho_j\rho_i x_i - x_j}{\sqrt{1-\rho_j^2}}\right)\varphi(u)du,} \tag{2.14}$$

where $h_i(t)$, $S_i(t)$ are the marginal hazard rate and the marginal survival function (taken at time t) for name i and as before where $x_j = \Phi^{-1}(F_j(t))$ for $j = 1, \dots, n$. Such an expression is useful for the computation of non homogeneous[16] first to default swaps.

One factor mean-variance Gaussian mixture copula: in order to get the conditional hazard rates up to first default time, we rely the joint survival function since by definition:

$$\lambda_i(t)S(t) = -\frac{\partial S(t, \dots, t)}{\partial t_i}.$$

From equation (2.5), we have:

$$S(t, \dots, t) = \int \prod_{j=1}^{n} \Phi\left(\frac{m_j(u) - g_j^{-1}(t)}{\sigma_j(u)}\right) f(u)du,$$

and:

$$\lambda_i(t)S(t) = -\frac{\partial S(t, \dots, t)}{\partial t_i} = \frac{dg_i^{-1}(t)}{dt} \int \varphi\left(\frac{m_i(u) - g_i^{-1}(t)}{\sigma_i(u)}\right) \prod_{j\neq i} \Phi\left(\frac{m_j(u) - g_j^{-1}(t)}{\sigma_j(u)}\right) \frac{f(u)}{\sigma_i(u)}du.$$

Let us remark that from the calibrating equation (2.4), we get:

$$h_i(t)S_i(t) = f_i(t) = \frac{dg_i^{-1}(t)}{dt} \int \varphi\left(\frac{m_i(u) - g_i^{-1}(t)}{\sigma_i(u)}\right) \frac{f(u)}{\sigma_i(u)}du,$$

where f_i is the marginal density of default time τ_i. Thus:

$$\boxed{\lambda_i(t)S(t) = f_i(t)\frac{\displaystyle\int \varphi\left(\frac{m_i(u) - g_i^{-1}(t)}{\sigma_i(u)}\right) \prod_{j\neq i} \Phi\left(\frac{m_j(u) - g_j^{-1}(t)}{\sigma_j(u)}\right) \frac{f(u)}{\sigma_i(u)}du}{\displaystyle\int \varphi\left(\frac{m_i(u) - g_i^{-1}(t)}{\sigma_i(u)}\right) \frac{f(u)}{\sigma_i(u)}du}.} \tag{2.15}$$

Archimedean copulas: using the above approach and the expression of the survival function, we get:

$$\boxed{\lambda_i(t)S(t) = -\frac{\partial S(t, \dots, t)}{\partial t_i} = w_i(t)(1 - G_i(t)) \int \alpha G_i^{\alpha-1}(t) \prod_{j\neq i}^{n}(1 - G_j^{\alpha}(t))f(\alpha)d\alpha,} \tag{2.16}$$

[16] Nominal amounts or recovery rates may differ.

where w_i is the hazard rate associated with the baseline distribution G_i.

Lastly, under the assumption that the marginal densities $f_i(t)$ are continuous, in our factor copula framework, the joint survival function is differentiable and the hazard rate of the first to default time is equal to the sum of the conditional hazard rates: $\lambda^1(t) = \sum_{i=1}^{n} \lambda_i(t)$.

2.7 Number of defaults: pgf and FFT approaches

When considering k out of n default swaps, it may be important to compute the probability of k couterparties being in default at time t where $k = 0, \ldots, n$. We thereafter denote by $N(t) = \sum_{i=1}^{n} \mathcal{I}_{\{\tau_i \leq t\}}$, $N(t)$ being the counting process associated to the number of defaults. If we denote by $N_i(t)$ the indicator of default of name i at time t ($N_i(t) = \mathcal{I}_{\{\tau_i \leq t\}}$), we have $N(t) = \sum_{i=1}^{n} N_i(t)$. We will compute the probabilities of k defaults at time t, i.e. $P(N(t) = k)$, $k = 0, \ldots, n$ through the probability generating function (or pgf) and discrete Fourier transform (DFT) or FFT (Fast Fourier Transform).

2.7.1 Computation of the probability generating function of $N(t)$

One factor Gaussian correlation structure: We consider here the simple correlation structure with a single common factor. We denote by $p_t^{i|V}$ the probability of $\{\tau_i \leq t\}$ conditionally on V (i.e. the probability of name i to be in default at time t conditionally on V). We have:

$$p_t^{i|V} = Q\left(\tau_i \leq t \mid V\right) = Q\left(\rho_i V + \sqrt{1-\rho_i^2}\bar{V}_i \leq \Phi^{-1}(F_i(t)) \mid V\right) = \Phi\left(\frac{-\rho_i V + \Phi^{-1}(F_i(t))}{\sqrt{1-\rho_i^2}}\right).$$

Let us compute the moment generating function (or probability generating fucntion) of $N(t)$:

$$\psi_{N(t)}(u) = E\left[u^{N(t)}\right] = \sum_{k=0}^{n} Q(N(t) = k) u^k.$$

Let us remark that $N_i(t)$ is a Bernoulli random variable and $E\left[u^{N_i(t)} \mid V\right] = 1 - p_t^{i|V} + p_t^{i|V} \times u$. Using that $E\left[u^{N(t)}\right] = E\left[E\left[u^{N(t)} \mid V\right]\right]$, and the conditional independence of the $N_i(t)$, we obtain: $\psi_{N(t)}(u) = E\left[\prod_{i=1}^{n}\left(1 - p_t^{i|V} + p_t^{i|V} \times u\right)\right]$. This leads to:

$$\boxed{\psi_{N(t)}(u) = \int \prod_{i=1}^{n}\left(\Phi\left(\frac{\rho_i v - \Phi^{-1}(F_i(t))}{\sqrt{1-\rho_i^2}}\right) + \Phi\left(\frac{-\rho_i v + \Phi^{-1}(F_i(t))}{\sqrt{1-\rho_i^2}}\right) u\right)\varphi(v)dv,} \tag{2.17}$$

Since $\psi_{N(t)}(u)$ can be written as $E\left[u^n \phi_n(V) + \ldots + \phi_0(V)\right]$, where $\phi_k(V)$ is obtained by a formal expansion of:

$$\prod_{i=1}^{n}\left(\Phi\left(\frac{\rho_i V - \Phi^{-1}(F_i(t))}{\sqrt{1-\rho_i^2}}\right) + \Phi\left(\frac{-\rho_i V + \Phi^{-1}(F_i(t))}{\sqrt{1-\rho_i^2}}\right) u\right).$$

We can then obtain the probability of k names being in default at time t as :

$$Q(N(t) = k) = E[\phi_k(V)] = \int \phi_k(v)\varphi(v)dv. \tag{2.18}$$

One factor mean-variance mixture model: We recall that $\tau_i = g_i(X_i)$, $i = 1, \dots, n$ and $X_i = m_i(\theta) + \sigma_i(\theta)\varepsilon_i$ where the mixing variable θ has density f. As a consequence the conditional probability $p_t^{i|\theta}$ can be written as: $p_t^{i|\theta} = Q(\tau_i < t \mid \theta) = Q\left(X_i < g_i^{-1}(t) \mid \theta\right) = \Phi\left(\frac{g_i^{-1}(t) - m_i(\theta)}{\sigma_i(\theta)}\right)$ and $\psi_{N_i(t)}(u) = 1 - p_t^{i|\theta} + p_t^{i|\theta}u$. This leads to the pgf of the number of defaults:

$$\boxed{\psi_{N(t)}(u) = \int \prod_{i=1}^{n} \left(\Phi\left(\frac{m_i(\theta) - g_i^{-1}(t)}{\sigma_i(\theta)}\right) + \Phi\left(\frac{g_i^{-1}(t) - m_i(\theta)}{\sigma_i(\theta)}\right) u \right) f(\theta)d\theta.} \tag{2.19}$$

One factor Archimedean Copula: Here the conditional probability of default $p_t^{i|\alpha}$ is given by $G_i^{\alpha}(t)$ where $G_i(t) = \exp\left(-\psi^{-1}(F_i(t))\right)$ and ψ is the Laplace transform of the density f of the mixing variable α. This leads to the following expression of the pgf of the number of defaults:

$$\boxed{\psi_{N(t)}(u) = \int \prod_{i=1}^{n} \left(1 - \exp\left(-\alpha\psi^{-1}(F_i(t))\right) + \exp\left(-\alpha\psi^{-1}(F_i(t))\right) u\right) f(\alpha)d\alpha.} \tag{2.20}$$

Let us remark that for practical purpose, the formal expansion approach to the computation of the probabilities $P(N(t) = k)$, $k = 0, \dots, n$ is well suited for small dimensional problems. More generally one can use FFT approaches to obtain the distribution function from its pgf.

3 Pricing of basket default swaps

We consider thereafter the pricing of various basket default swaps. In a first to default swap, there is a default payment at the first to default time. The payment corresponds to the non recovered part of bond in default[17]. In a m out of n basket default swap ($m \leq n$, where n is the number of names), there is a default payment a the m-th default time. The payment corresponds to the non recovered part of the corresponding bond in default. There are also some basket default swaps that provide protection for defaults ranking between d_m and d_M, with $1 \leq d_m \leq d_M \leq n$. The default leg here is simply the sum of default legs of m out of n default swaps, with $d_m \leq m < d_M$. We detail below the premium payments of such a basket credit derivative. In an homogeneous basket, the nominals and the recovery rates of the reference credits are assumed to be equal. However, the marginal default probabilities may differ.

We compute separately the price of the premium leg and of the default payment leg. The basket premium is such that the prices of the two legs are equal.

[17]More precisely, the recovery is based on the nominal plus the accrued coupon. In the following, we will make the simplifying assumption of a recovery based on the nominal only. However, this assumption can easily be relaxed by considering a time dependent nominal in the pricing formulas.

- The premium payment leg computations only involve the distribution of the number of defaults $N(t)$. In the case of first to default swaps, computations depend only on the survival function of the first to default time.
- The price of the default leg of an homogeneous m out of n basket default swap only involves the distribution of the number of defaults[18]. Under the homogeneity assumption, first to default swaps do not deserve a special treatment: the only point worth mentioning is that the survival function of first to default time can be easily computed, without the use of the pgf of $N(t)$. In the general case of the default leg of a non homogeneous m out of n basket default swap, the computations are a bit more involved. First to default swaps can also be valuated in a somehow more direct way.

We thereafter compute the price at time 0 as the expected discounted payoff at time 0. For simplicity, we moreover assume independence between default dates and interest rates, since the important issue for basket type credit derivatives is the modelling of dependence between default dates[19]. Since basket derivatives payoffs do not depend on interest rates, we can then only consider the discount bond prices at time 0. Similarly, we assume that the recovery rates on the underlying bonds are independent from default times and interest rates. Since the payoffs of basket default swaps are linear in these recovery rates, only the expected recovery rates are involved. For notational simplicity, we will thus confuse recovery rates and their expectation[20].

3.1 m out of n basket default swaps: premium leg

We consider here a basket default swap on a set of n reference credits, with protection payment arising between defaults of rank d_m (included) and d_M (excluded). We denote by t_i, $i = 1, \dots, I$ the premium payments dates (with $t_I = T$ where T is the maturity date of the basket default swap) and by X the periodic premium. $\Delta_{i-1,i}$ represents the length of period $[t_{i-1}, t_i]$ and $B(0, t_i)$ is the discount factor for maturity t_i. For simplicity, we do not take into account accrued premium payments due to defaults between premium payments dates. Let us firstly detail the premium payments and consider some payment date t_i. If $N(t_i) \geq d_M$, the basket payments are exhausted[21]. If $N(t_i) < d_m$, the premium is due on a full nominal of $d_M - d_m$. In between, if $d_m \leq N(t_i) < d_M$, the premium is payed on the oustanding protected nominal, i.e. $d_M - N(t_i)$.

From the distribution function of $N(t)$ we can compute the premium payment leg for m out of n basket default swaps. The discounted expectation of premium payment of date t_i can then be written as:

$$\Delta_{i-1,i} X B(0, t_i) \times \left((d_M - d_m) Q(N(t_i) < d_m) + \sum_{k=d_m}^{d_M} (d_M - k) Q(N(t_i) = k) \right),$$

where the probabilities of k names being in default at time t, $Q(N(t) = k)$ have already been computed. We can eventually write the price of the premium payment leg by summing over possible premium payment

[18] More precisely the survival function of m-th default time.

[19] However, for such products as quanto default swaps, defaultable interest rate swaps, credit spread options, the dependence between defaults and interest rates is an important issue.

[20] Let us remark that CDO tranches do not fulfill that linearity in the recovery rates. The distribution of recovery rates can have some effect on the price of such tranches (see below).

[21] We recall that $N(t)$ is the number of names in default at time t.

dates:

$$\sum_{i=1}^{n} \Delta_{i-1,i} X B(0,t_i) \times \left((d_M - d_m) \sum_{k=0}^{d_m - 1} Q(N(t_i) = k) + \sum_{k=d_m}^{d_M} (d_M - k) Q(N(t_i) = k) \right).$$

This price only involves semi explicit probabilities $Q(N(t_i) = k)$.

3.2 m out of n homogeneous default swaps: default leg

Similarly, let us consider the default payment leg of an homogeneous basket default swap: we denote by 1, the nominal of a given reference credit and by δ the unique recovery rate. The homogeneity assumption allows the computation of the price of the default payment leg knowing the distribution of the number of defaults only. We denote by τ^k the time of the k-th default. We can write the distribution function of τ^k as:

$$Q(\tau^k \leq t) = Q(N(t) \geq k).$$

Thus, the distribution function of τ^k can be computed. We will consider the default payments at dates $\tau^{d_m+1}, \ldots, \tau^k, \ldots, \tau^{d_M}$ provided that these dates are before the maturity of the basket default swap $T = t_I$. Straightforward algebra shows that the payoff of the default leg is equal to the sum of the payoffs of default legs paying $1 - \delta$ at the k-th default, $d_m \leq k < d_M$, provided that the k-th default is before $T = t_I$. Then, we simply have to compute the current price of a k-th to default payment and sum over possible k. We denote by $F^k(t) = Q(\tau^k \leq t)$ and by $S^k(t)$ the distribution and the survival functions of the k-th to default time. We have:

$$S^k(t) = Q(\tau^k > t) = Q(N(t) < k) = \sum_{l=0}^{k-1} Q(N(t) = l),$$

which involves only the known $Q(N(t) = k)$, $k = 0, 1, \ldots, n$. Under the previous assumptions independence assumptions on interest rates and recovery rates, we can then write the price of the k-th to default payment leg as:

$$\boxed{E\left[(1-\delta) \mathcal{I}_{[0,T]}(\tau^k) \exp\left(-\int_0^{\tau^k} r(s) ds \right) \right] = -(1-\delta) \int_0^T B(0,t) dS^k(t),} \tag{3.1}$$

where $r(s)$ denotes the short rate and $-dS^k(t) = S^k(t) - S^k(t + dt)$ is the probability of the k-th default to occur in $[t, t + dt[$. The previous pricing equation has straightforward financial interpretation and corresponds to equation (5.18) in Bielecki and Rutkowksi [2002]. The price of the default leg of the m out of n homogeneous default swap is obtained by summing up over the relevant ranks:

$$\boxed{-(1-\delta) \sum_{k=d_m}^{d_M - 1} \int_0^T B(0,t) dS^k(t),} \tag{3.2}$$

Integrating by parts, we can write the price of the payment leg of the k-th to default swap as:

$$\boxed{(1-\delta) \times \left(1 - S^k(T) B(0,T) + \int_0^T S^k(t) dB(0,t) \right).} \tag{3.3}$$

Under the usual smoothness assumptions we have that $f(0,t)B(0,t) = -\dfrac{dB(0,t)}{dt}$ where $f(0,t)$ is the spot forward rate. Thus, we can also write the previous price as:

$$(1-\delta) \times \left(1 - S^k(T)B(0,T) - \int_0^T f(0,t)B(0,t)S^k(t)dt\right).$$

We now consider some boundaries of the price of the default payment leg. Since $B(0,t)$ and $S^k(t)$ are decreasing, it is possible to bound the integral $\int_0^T B(0,t)dS^k(t)$ by discrete sums. We have:

$$B(0,t_j) \times \left(S^k(t_{j-1}) - S^k(t_j)\right) \le -\int_{t_{j-1}}^{t_j} B(0,t)dS^k(t) \le B(0,t_{j-1}) \times \left(S^k(t_{j-1}) - S^k(t_j)\right).$$

This provides the following boundaries for the price of the default leg of the k-th to default swap:

$$\boxed{\text{Upper boundary } = (1-\delta) \times \sum_{j=1}^{J} B(0,t_{j-1}) \times \left(S^k(t_{j-1}) - S^k(t_j)\right),} \tag{3.4}$$

$$\boxed{\text{Lower boundary } = (1-\delta) \times \sum_{j=1}^{J} B(0,t_j) \times \left(S^k(t_{j-1}) - S^k(t_j)\right),} \tag{3.5}$$

where $\boxed{S^k(t_{j-1}) - S^k(t_j) = \sum_{l=0}^{k-1} Q(N(t_{j-1}) = l) - Q(N(t_j) = l)}$ involves only known quantities, $t_0 = 0$ (thus $B(0,t_0) = 1$) and $t_J = T$. Using these boundaries, we can easily control the size of the grid t_j, $j = 1, \ldots, J$ in order to get a good accuracy[22].

As an example let us consider a First to Default swap (we drop the $1-\delta$ term):

$$1 - S(T)B(0,T) - \int_0^T f(0,t)B(0,t)S(t)dt,$$

where $S(t)$ is the survival function of first to default date, $B(0,t)$ is the discount factor for maturity t and $f(0,t)$ is the corresponding spot forward rate. From subsection (2.4), we can specialize the previous expression to the one factor Gaussian, one factor mean-variance Gaussian mixture and Archimedean copulas. For instance, in the one factor Gaussian copula case, the price of the default payment leg is provided by:

$$1 - B(0,T)\int \prod_{i=1}^{n} \Phi\left(\frac{\Phi^{-1}(F_i(T)) - \rho_i v}{\sqrt{1-\rho_i^2}}\right)\varphi(v)dv - \int_0^T \int f(0,t)B(0,t)\prod_{i=1}^{n} \Phi\left(\frac{\Phi^{-1}(F_i(t)) - \rho_i v}{\sqrt{1-\rho_i^2}}\right)\varphi(v)dvdt, \tag{3.6}$$

[22] We might have worked through the second expression involving integrals with respect to $B(0,t)$, which leads to the same approximations.

3.3 Default leg of first to default swap: non homogeneous case

We consider a series of n names with nominal N_i and recovery rates δ_i, $i = 1, \dots, n$. We denote by $M_i = N_i \times (1 - \delta_i)$ the payment in case of default[23]. We consider a first to default swap with maturity T. As before, let us denote by τ^1 the first to default time. If $\tau^1 \leq T$, there is a default payment at that time that depends on the name in default: if name i is in default, the payment is equal to M_i. Thus, the default payment can be decomposed as the sum of n default payments, each of them corresponding to a specific name being the first to default. We denote by $N_i(t) = \mathcal{I}_{\{\tau_i \leq t\}}$, $i = 1, \dots, n$, the indicator function of default time of name i. Let us consider the price of the default payment leg as the limit of the price in a discrete model[24] We denote by π_k, $k \in \mathbb{N}$, a sequence of partitions of $[0, T]$ with mesh converging to zero. The time zero price of the default leg is given by:

$$\boxed{\lim_{k\to\infty} \sum_{i=1}^{n} \sum_{t_l \in \pi_k} M_i B(0, t_l) Q\left(\tau_i \in [t_l, t_{l+1}[, \tau_j \geq t_l, j \neq i\right) = -\sum_{i=1}^{n} M_i \int_0^T \partial_i S(t, \dots, t) B(0, t) dt,} \tag{3.7}$$

where $\partial_i S(t, \dots, t) = \dfrac{\partial S(t, \dots, t)}{\partial t_i}$ denotes the partial derivative of the joint survival function with respect to the ith component at point $(t, \dots, t)$. Indeed $-\partial_i S(t, \dots, t) dt \approx Q(\tau_j \geq t, j \neq i, \tau_i \in [t, t + dt[)$[25]. From the definition of the conditional hazard rates, we have $\lambda_i(t) S(t) = -\partial_i S(t, \dots, t)$, where $S(t)$ denotes the survival function of first to default time. Thus, the price of the default payment leg can also be written as:

$$\boxed{\sum_{i=1}^{n} M_i \int_0^T \lambda_i(t) S(t) B(0, t) dt.}$$

Fortunately, we can use the expressions of $\lambda_i S(t)$ for the one factor Gaussian, mean-variance Gaussian mixture and Archimedean copulas stated in subsection (2.6), to readily obtain the price of the default payment leg of a first to default swap in the non homogeneous case. For example, in the one factor Gaussian copula case, using equation (2.14), we obtain:

$$\boxed{\int_0^T \int \sum_{i=1}^{n} M_i h_i(t) S_i(t) \times \prod_{j \neq i} \Phi\left(\frac{\rho_j \sqrt{1 - \rho_i^2} u + \rho_j \rho_i x_i - x_j}{\sqrt{1 - \rho_j^2}}\right) \varphi(u) du dt,} \tag{3.8}$$

where $x_j = \Phi^{-1}(F_j(t))$ for $j = 1, \dots, n$. These expressions only involve simple numerical quadratures.

3.4 default leg of m out of n default swaps: general case

We consider a series of n names with nominal N_i and recovery rates δ_i, $i = 1, \dots, n$ and the default leg of a m out of n basket default swap ($1 \leq m \leq n$) with maturity T. We consider here a single default payment ; more

[23] The nominals will normally be equal but the estimated recoveries may well differ.

[24] See below, default leg of a m out of n basket default swap, for a more detailed discussion.

[25] We can provide a more rigorous, while more abstract, derivation of this statement. Let us denote by $\tau^{(-i)} = \min_{j \neq i} \tau_j$ the first to default time for the set of reference credits, i excluded. Under assumption (3), there are no simultaneous defaults and $\mathcal{I}_{\tau^{(-i)} > \tau_i} = \mathcal{I}_{\tau^1 \geq \tau_i}$ Q-a.s. where $\tau^1 = \min_j \tau_j$ is the first to default date. Thus, the discounted first to default basket payoff can be written as $\sum_{i=1}^{n} M_i B(0, \tau_i) \mathcal{I}_{\tau^1 \geq \tau_i} \mathcal{I}_{\tau_i \leq T}$. From Fubini's theorem, $E\left[B(0, \tau_i) \mathcal{I}_{\tau^1 \geq \tau_i} \mathcal{I}_{\tau_i \leq T}\right] = \int_0^T B(0, t) Q(\tau^1 \geq t \mid \tau_i = t) f_i(t) dt$. From lemma (1.2), we get $Q(\tau^1 \geq t \mid \tau_i = t) f_i(t) = -\partial_i S(t, \dots, t)$, which allows to conclude.

general cases can be treated straightforwardly by summing up (see subsection on homogeneous baskets). We denote by $M_i = N_i \times (1 - \delta_i)$ the payment in case of default. τ^m denotes the m-th default time. If $\tau^m \leq T$, there is a default payment at that time that depends on the name in default: if name i is in default, the payment is equal to M_i. We recall that $N_i(t) = \mathcal{I}_{\{\tau_i \leq t\}})$ and $N(t) = \sum_{j=1}^{n} N_j(t)$. We define $N^{(-i)}(t) = \sum_{j \neq i} N_j(t)$ and $N^m(t) = \mathcal{I}_{\{\tau^m \leq t\}}$.

Let us firstly compute the probability of name i being the m-th to default time and that default time being in the interval $]t, t']$, $t' > t$. Let us remark that:

$$\{\tau^m = \tau_i, \tau^m \in]t, t']\} = \{N(t) = m - 1, \tau_i \in]t, t']\},$$

The latter set corresponding to $m-1$ names being in default at time t and default date of name i being in the interval $]t, t']$. Since $\{\tau_i \in]t, t']\} = \{N_i(t') - N_i(t) = 1\}$, we can write:

$$\{N(t) = m - 1, \tau_i \in]t, t']\} = \{N(t) = m - 1, N_i(t') - N_i(t) = 1\}.$$

Lastly, for events such that τ_i is after t, $N(t) = N^{(-i)}(t)$. Thus, we need to compute:

$$Q\left(N^{(-i)}(t) = m - 1, N_i(t') - N_i(t) = 1\right).$$

We consider the slightly more general issue of computing $Q\left(N^{(-i)}(t^*) = m - 1, N_i(t') - N_i(t) = 1\right)$, where $t^* \leq t \leq t'$. This can be for done by using the joint pgf of $\left(N^{(-i)}(t^*), N_i(t') - N_i(t)\right)$ defined by $\psi(u, v) = E\left[u^{N_i(t') - N_i(t)} v^{N^{(-i)}(t^*)}\right]$. We compute ψ by conditioning on the latent variable V. Conditionally on V, $N_i(t') - N_i(t)$ is a Bernoulli random variable with $Q\left(N_i(t') - N_i(t) = 1 \mid V\right) = Q(\tau_i \leq t' \mid V) - Q(\tau_i \leq t \mid V) = p_{t'}^{i|V} - p_t^{i|V}$. We can write $\psi(u, v)$ as:

$$\psi(u, v) = \sum_{k=1}^{n} Q\left(N_i(t') - N_i(t) = 0, N^{(-i)}(t^*) = k\right) v^k + \sum_{k=1}^{n-1} Q\left(N_i(t') - N_i(t) = 1, N^{(-i)}(t^*) = k\right) u v^k.$$

On the other hand,

$$\psi(u, v) = E\left[E\left[u^{N_i(t') - N_i(t)} v^{N^{(-i)}(t^*)} \mid V\right]\right].$$

By conditional independence:

$$\psi(u, v) = E\left[E\left[u^{N_i(t') - N_i(t)} \mid V\right] \times E\left[v^{N^{(-i)}(t^*)} \mid V\right]\right],$$

which leads to:

$$\psi(u, v) = E\left[\left(1 - p_{t'}^{i|V} + p_t^{i|V} + \left(p_{t'}^{i|V} - p_t^{i|V}\right) u\right) \times \prod_{j \neq i} \left(1 - p_{t^*}^{j|V} + p_{t^*}^{j|V} v\right)\right].$$

As a consequence, we obtain:

$$\sum_{k=1}^{n-1} Q\left(N_i(t') - N_i(t) = 1, N^{(-i)}(t^*) = k\right) v^k = E\left[\left(p_{t'}^{i|V} - p_t^{i|V}\right) \times \prod_{j \neq i} \left(1 - p_{t^*}^{j|V} + p_{t^*}^{j|V} v\right)\right],$$

where the term within the expectation can be computed by formal expansion.

The price of the default payment leg is given by:

$$\boxed{E\left[\int_0^T B(0,t)\sum_{i=1}^n M_i \mathcal{I}_{N^{(-i)}(t)=m-1} dN_i(t)\right],} \tag{3.9}$$

where $B(0,t)$ is the maturity t discount factor. We can see $N^{(-i)}(t) = m-1$ as an activating condition. When $m-1$ names apart from i are in default, then default of name i triggers a payment of M_i. We have thus decomposed the default payments into n payoffs, each of them being similar to a plain CDS default payment activated upon some event being satisfied. Let us now turn to the computation of the different terms:

$$E\left[\int_0^T B(0,t) M_i \mathcal{I}_{N^{(-i)}(t)=m-1} dN_i(t)\right],\ i = 1, \dots, n.$$

$\int_0^T B(0,t)\mathcal{I}_{N^{(-i)}(t)=m-1} dN_i(t)$ is a plain stochastic integral with respect to the pure jump process $N_i(t)$. Let us consider a given sequence of partitions of $[0,T]$, π_k with mesh converging to zero. We define the processes:

$$V_{i,k}(t) = \sum_{t_l \in \pi_k} \mathcal{I}_{N^{(-i)}(t_l)=m-1} \mathcal{I}_{]t_l, t_{l+1}]}(t).$$

$V_{i,k}$ is an adapted process (with respect to the filtration generated by the set of default times) with càglàd paths. The sequence of processes $V_{i,k}$ converges uniformly on compacts in probability[26] towards $\mathcal{I}_{N^{(-i)}(.)=m-1}$. By continuity properties of stochastic integrals,

$$\int_0^T B(0,t)\mathcal{I}_{N^{(-i)}(t)=m-1} dN_i(t) = \lim_{k\to\infty} \sum_{t_l \in \pi_k} B(0,t_l)\mathcal{I}_{N^{(-i)}(t_{l-1})=m-1}\left(N_i(t_{l+1}) - N_i(t_l)\right),$$

where the limit is taken in probability[27]. The random variables:

$$\int_0^T B(0,t)\mathcal{I}_{N^{(-i)}(t)=m-1} dN_i(t),\ \sum_{t_l \in \pi_k} B(0,t_l)\mathcal{I}_{N^{(-i)}(t_{l-1})=m-1}\left(N_i(t_{l+1}) - N_i(t_l)\right),$$

are uniformly integrable[28]. Therefore, we conclude:

$$E\left[\int_0^T B(0,t)\mathcal{I}_{N^{(-i)}(t)=m-1} dN_i(t)\right] = \lim_{k\to\infty} E\left[\sum_{t_l \in \pi_k} B(0,t_l)\mathcal{I}_{N^{(-i)}(t_{l-1})=m-1}\left(N_i(t_{l+1}) - N_i(t_l)\right)\right],$$

or equivalently as $\lim_{k\to\infty} \sum_{t_l \in \pi_k} B(0,t_l) Q\left(N^{(-i)}(t_{l-1}) = m-1, N_i(t_{l+1}) - N_i(t_l) = 1\right)$. Let us for instance consider the partitions of $[0,T]$ given by $\pi_k = \{0, \frac{T}{k}, \dots, \frac{lT}{k}, \dots, T\}$. We can write:

$$\sum_{k=1}^{n-1} \left[Q\left(N_i(t_{l+1}) - N_i(t_l) = 1, N^{(-i)}(t_l) = k\right) - Q\left(N_i(t_{l+1}) - N_i(t_l) = 1, N^{(-i)}(t_{l-1}) = k\right)\right] v^k,$$

[26] Under the standing assumption of a smooth joint survival function. This implies that there exists some hazard rate for the different rank statistics.

[27] We do not need ucp here.

[28] They take value in [0,1].

as:

$$\sum_{k=1}^{n-1} E\left[\left(p_{t_{l+1}}^{i|V} - p_{t_l}^{i|V}\right) \times \left(\prod_{j\neq i}\left(1 - p_{t_l}^{j|V} + p_{t_l}^{j|V} v\right) - \prod_{j\neq i}\left(1 - p_{t_{l-1}}^{j|V} + p_{t_{l-1}}^{j|V} v\right)\right)\right] v^k = o\left(\frac{1}{k}\right),$$

for smooth conditional default probabilities $p_t^{i|V}$. As a consequence, we can consider the limit:

$$\lim_{k\to\infty} \sum_{t_l\in\pi_k} B(0,t_l) Q\left(N^{(-i)}(t_l) = m-1, N_i(t_{l+1}) - N_i(t_l) = 1\right).$$

For smooth conditional default probabilities $p_t^{i|V}$, we define:

$$Z_k^i(t) = \lim_{t'\to t} \frac{1}{t'-t} Q\left(N_i(t') - N_i(t) = 1, N^{(-i)}(t) = k\right), \quad k = 1,\dots,n-1,\ i = 1,\dots,n.$$

The $Z_k^i(t)$ are given by: $\sum_{k=1}^{n-1} Z_k^i(t) v^k = E\left[\frac{dp_t^{i|V}}{dt} \times \prod_{j\neq i}\left(1 - p_t^{j|V} + p_t^{j|V} v\right)\right]$. The price of the default payment leg is then given by:

$$\boxed{\int_0^T \sum_{i=1}^n B(0,t) M_i Z_{m-1}^i(t) dt.} \tag{3.10}$$

We can provide some simple alternative expressions. Let us denote by:

$$Z_k^{i|V}(t) = \lim_{t'\to t} \frac{1}{t'-t} Q\left(N_i(t') - N_i(t) = 1, N^{(-i)}(t) = k \mid V\right).$$

Then,

$$\sum_{k=1}^{n-1} Z_k^{i|V}(t) v^k = \frac{dp_t^{i|V}}{dt} \times \prod_{j\neq i}\left(1 - p_t^{j|V} + p_t^{j|V} v\right),$$

and $E\left[Z_k^{i|V}(t)\right] = Z_k^i(t)$, where the expectation is taken upon V. Moreover,

$$Z_k^{i|V}(t) = \frac{dp_t^{i|V}}{dt} \times Q\left(N^{(-i)}(t) = k \mid V\right).$$

As a consequence the price of the default payment leg can be written as:

$$\boxed{E\left[\int_0^T \sum_{i=1}^n B(0,t) M_i Q\left(N^{(-i)}(t) = m-1 \mid V\right) dp_t^{i|V}\right],} \tag{3.11}$$

where the expectation is taken over V and $Q\left(N^{(-i)}(t) = m-1 \mid V\right)$ is obtained from the formal expansion of the polynomial $\prod_{j\neq i}\left(1 - p_t^{j|V} + p_t^{j|V} v\right)$.

As can be seen from the previous equations, one can readily compute the price of the default payment leg of a general m out of n default swap, once the conditional (on the latent variable) probabilities of default are given. Putting in the relevant probabilities provides the price of the default payment leg for the one factor Gaussian model, the mean variance mixture model and the Archimedean copula model.

4 Loss distributions and the pricing of CDO's

Under the factor copula framework, it is easy to compute the characteristic function of the cumulative loss at a given time. Here, we consider the losses due to defaults only and not the losses due to changes in credit spreads. The knowledge of the loss distribution is useful in some CDO computations. We can obtain it by inverting the characteristic function and we can also provide some explicit computations of moments of the Loss distribution that can be useful for cross-checking the Monte Carlo or tree approaches.

We consider n reference credits, with nominal N_i, $i = 1, \dots, n$ and recovery rate δ_i. $M_i = (1 - \delta_i)N_i$ will denote the Loss Given Default. In a first step, we will consider the recovery rates as determistic and further show how the framework can easily be extended to stochastic recovery rates. We denote by τ_i the default time of name i and by $N_i(t)$ the counting processes $N_i(t) = \mathcal{I}_{\tau_i \le t}$ which jumps from 0 to 1 at default time of name i. $L(t)$ will denote the cumulative loss on the credit portfolio at time t:

$$L(t) = \sum_{i=1}^{n} M_i N_i(t), \tag{4.1}$$

which is thus a pure jump process.

4.1 Probability Generating Function of Loss Distribution

Let us now compute the loss distribution of $L(t)$ assuming a discrete grid for the values of M_i, $i = 1, \dots, n$. We proceed with the moment generating function $\psi_{L(t)}(u)$:

$$\psi_{L(t)}(u) = E\left[u^{L(t)}\right].$$

As a starting point, we model the default times along the one factor Gaussian copula where V denotes the Gaussian factor, for the sake of simplifying the presentation. We can thus write from iterated expectations theorem: $\psi_{L(t)}(u) = E\left[E\left[u^{L(t)} \mid V\right]\right]$. From the independence of $N_i(t)$ conditionally on V, we get $E\left[u^{L(t)} \mid V\right] = \Pi_{i=1}^{n} E\left[u^{M_i N_i(t)} \mid V\right]$. This gives : $E\left[u^{L(t)} \mid V\right] = \prod_{i=1}^{n}\left(1 - p_t^{i|V} + p_t^{i|V} u^{M_i}\right)$, where $p_t^{i|V} = Q(\tau_i \le t \mid V) = \Phi\left(\frac{-\rho_i V + \Phi^{-1}(F_i(t))}{\sqrt{1-\rho_i^2}}\right)$. The conditional moment generationg function can be computed recursively as follows : let us define $\psi_{L(t)}^{k|V}(u) = \prod_{i=1}^{k}\left(1 - p_t^{i|V} + p_t^{i|V} u^{M_i}\right)$. Then, $\psi_{L(t)}^{k+1|V}(u) = \psi_{L(t)}^{k|V}(u) \times \left(1 - p_t^{k+1|V} + p_t^{k+1|V} u^{M_{k+1}}\right)$. We eventually get $\psi_{L(t)}(u) = E\left[u^N \phi_N(V) + \ldots + \phi_0(V)\right]$, where $N = \sum_{i=1}^{n} M_k$ and the $\phi_k(V)$ are obtained as the coefficients of the polynomial expansion of $E\left[u^{L(t)} \mid V\right]$. Thus, we eventually obtain the probability distribution of $L(t)$ as : $Q(L(t) = k) = E[\phi_k(V)) = \int \phi_k(v)\varphi(v)dv$, where $\varphi(v) = \frac{1}{\sqrt{2\pi}} e^{-v^2/2}$ denotes the Gaussian density. Let us also remark that the probability of no loss occuring is given by: $Q(L(t) = 0) = E\left[\prod_{i=1}^{n}\left(1 - p_t^{i|V}\right)\right]$.

4.2 Pricing the default payment leg of a CDO tranche

Let us consider a tranche of a CDO, where the default payment leg pays all losses that occur on the credit portfolio, above a threshold A and below a threshold B where $0 \le A \le B \le \sum_{i=1}^{n} N_i$. When $A = 0$, we usually speak of the equity tranche. If $A > 0$ and $B < \sum_{i=1}^{n} N_i$, we consider mezzanine tranches and when $B = \sum_{i=1}^{n} N_i$, we consider senior or super-senior tranches. In order to simplify notations, we use a unique terminology $M(t)$ to denote the cumulative losses on a given tranche. These losses are equal to zero if $L(t) \le A$, to $L(t) - A$ if $A \le L(t) \le B$ and to $B - A$ if $L(t) \ge B$. This can be summerized as : $M(t) = (L(t) - A)\mathcal{I}_{[A,B]}(L(t)) + (B - A)\mathcal{I}_{]B,\sum N_i]}(L(t))$. We can notice that as $L(t)$, $M(t)$ is a pure jump process. The default payments are simply the increments of $M(t)$. In other words, there is a payment of $M(t^+) - M(t)$ at every jump time of $M(t)$ (which is such that $M(t^+) - M(t) > 0$). Since $M(t)$ is an increasing process, we can define Stieltjes integrals with respect to $M(t)$. Here, since $M(t)$ is constant apart from jump times, any integral with respect to $M(t)$, $\int g(t)dM(t)$ (where g is some function) turns out to be a discrete sum $\sum_i g(t_i)\left(M(t_i^+) - M(t_i)\right)$, where the t_i denotes the jump times. From the previous remarks, we can write the price of the default payment leg of the given tranche as:

$$E^Q\left[\int_0^T B(0,t)dM(t)\right],$$

where $B(0,t)$ stands for the discount factor for maturity t and T is the maturity of the CDO. We assume here deterministic interest rates. Let us again insist that the term within the brackets is the sum of discounted default payments on the tranche. Using the Stieltjes framework allows to use the integration by parts formula. This allows to write $\int_0^T B(0,t)dM(t) = B(0,T)M(T) + \int_0^T f(0,t)B(0,t)M(t)dt$, where $f(0,t)$ denotes the spot forward rate. Using Fubini theorem, we then have:

$$E^Q\left[\int_0^T B(0,t)dM(t)\right] = B(0,T)E^Q[M(T] + \int_0^T f(0,t)B(0,t)E^Q[M(t)]dt.$$

Let us remark that we only need the first moment of the cumulative loss on the tranche. This can be computed knowing the distribution of total losses. Indeed, we have:

$$E^Q[M(t)] = \sum_{k=A}^{B}(k - A)Q(L(t) = k) + (B - A)Q(L(t) > B),$$

or equivalently as:

$$E^Q[M(t)] = \sum_{k=A}^{B}(k - A)Q(L(t) = k) + (B - A)\sum_{k=B+1}^{\sum N_i} Q(L(t) = k),$$

Here, we have assumed that A and B were corresponding to some possible values of the cumulative loss at time t, $L(t)$. We then can compute the price of the default payment leg of the tranche, since we already know how to compute the probabilites $Q(L(t) = k)$ for $k = 1, \ldots, \sum N_i$.

Appendix A: Proofs related to section 1

Proof of lemma (1.2): $Q(\tau_1 \geq t_1, \dots, \tau_n \geq t_n) = E\left[E\left[\mathcal{I}_{\tau_j \geq t_j, \forall j} \mid \tau_i\right]\right] = E\left[\mathcal{I}_{\tau_i \geq t_i} E\left[\mathcal{I}_{\tau_j \geq tj, \forall j \neq i} \mid \tau_i\right]\right]$. Let us remark that $E\left[\mathcal{I}_{\tau_j \geq t_j, \forall j \neq i} \mid \tau_i\right] = Q\left(\tau_j \geq t_j, \forall j \neq i \mid \tau_i\right)$ is a measurable function of τ_i. Thus, $S(t_1, \dots, t_n) = \int_{t_i}^{\infty} Q\left(\tau_j \geq t_j, \forall j \neq i \mid \tau_i = u_i\right) f_i(u_i) du_i$. The previous equation can also be seen as a direct consequence of Fubini's theorem. Assuming that $Q\left(\tau_j \geq t_j, \forall j \neq i \mid \tau_i = u_i\right)$ is continuous at t_i, we obtain the stated result by differentiation ■

Proof of lemma (1.3): $Q\left(\tau_i \geq t_i, \min_{k \neq j} \tau_k \geq t \mid \tau_j = t_j\right) = E\left[\mathcal{I}_{\{\tau_i \geq t_i, \min_{k \neq j} \tau_k \geq t\}} \middle| \tau_j = t_j\right]$. From iterated expectation theorem:

$$E\left[\mathcal{I}_{\{\tau_i \geq t_i, \min_{k \neq j} \tau_k \geq t\}} \middle| \tau_j = t_j\right] = E\left[E\left[\mathcal{I}_{\{\tau_i \geq t_i, \min_{k \neq j} \tau_k \geq t\}} \middle| \min_{k \neq j} \tau_k \geq t, \tau_j\right] \middle| \tau_j = t_j\right].$$

The previous term can be expressed as: $E\left[\mathcal{I}_{\{\min_{k \neq j} \tau_k \geq t\}} E\left[\mathcal{I}_{\{\tau_i \geq t_i\}} \middle| \min_{k \neq j} \tau_k \geq t, \tau_j\right] \middle| \tau_j = t_j\right]$. It can be checked that $E\left[\mathcal{I}_{\{\tau_i \geq t_i\}} \middle| \min_{k \neq j} \tau_k \geq t, \tau_j\right] = h(\tau_j)$ where h is a measurable function. Thus:

$$E\left[\mathcal{I}_{\{\min_{k \neq j} \tau_k \geq t\}} E\left[\mathcal{I}_{\{\tau_i \geq t_i\}} \middle| \min_{k \neq j} \tau_k \geq t, \tau_j\right] \middle| \tau_j = t_j\right] = h(t_j) E\left[\mathcal{I}_{\{\min_{k \neq j} \tau_k \geq t\}} \mid \tau_j = t_j\right],$$

which can be written as:

$$Q\left(\tau_i \geq t_i \mid \min_{k \neq j} \tau_k \geq t, \tau_j = t_j\right) \times Q\left(\min_{k \neq j} \tau_k \geq t \mid \tau_j = t_j\right),$$

which proves the stated result ■

Proof of lemma (1.4): Let us firstly remark that $\{\tau^1 \geq t\} = \{\tau_1 \geq t, \dots \tau_n \geq t\}$. Thus $Q(\tau^1 \geq t) = S(t, \dots, t)$. Then, we simply remark that:

$$Q(\tau^1 \in [t, t + dt[| \tau^1 \geq t) = \frac{Q(\tau^1 \geq t) - Q(\tau^1 \geq t + dt)}{Q(\tau^1 \geq t)} = -\frac{S(t + dt, \dots, t + dt) - S(t, \dots, t)}{S(t, \dots, t)} \ ■$$

Proof of lemma (2.1): It can be seen that $X_i = \rho_1 \rho_i X_1 + \varepsilon_i$, where ε_i is Gaussian and independent from X_1. A quick expansion shows in turn that $\varepsilon_i = \rho_i \sqrt{1 - \rho_1^2}\left(\sqrt{1 - \rho_1^2} V - \rho_1 \bar{V}_1\right) + \sqrt{1 - \rho_i^2} \bar{V}_i$, where $W_1 = \sqrt{1 - \rho_1^2} V - \rho_1 \bar{V}_1$ and $\bar{V}_i$ are standard Gaussian independent random variables and also independent from X_1. As a consequence, we get:

$$Q\left(X_2 \geq x_2, \dots, X_n \geq x_n \mid X_1 = x_1\right) = Q\left(\varepsilon_2 \geq x_2 - \rho_1 \rho_2 x_1, \dots, \varepsilon_n \geq x_n - \rho_1 \rho_n x_1\right).$$

Let us denote by $\tilde{x}_i = x_i - \rho_1 \rho_i x_1$. From iterated expectations theorem, we get:

$$Q\left(\varepsilon_2 \geq \tilde{x}_2, \dots \varepsilon_n \geq \tilde{x}_n\right) = E\left[Q\left(\varepsilon_2 \geq \tilde{x}_2, \dots \varepsilon_n \geq \tilde{x}_n \mid W_1\right)\right].$$

Using the independence properties, we have:

$$Q\left(\varepsilon_2 \geq \tilde{x}_2, \dots \varepsilon_n \geq \tilde{x}_n \mid W_1\right) = \prod_{i=2}^{n} \bar{\Phi}\left(\frac{\tilde{x}_i - \rho_i \sqrt{1 - \rho_1^2} W_1}{\sqrt{1 - \rho_i^2}}\right),$$

where $\bar{\Phi}(x) = 1 - \Phi(x)$ is the survival Gaussian distribution function. We can thus write:

$$Q(\varepsilon_2 \geq \tilde{x}_2, \ldots \varepsilon_n \geq \tilde{x}_n) = \int \prod_{i=2}^{n} \Phi\left(\frac{\rho_i\sqrt{1-\rho_1^2}u - \tilde{x}_i}{\sqrt{1-\rho_i^2}}\right)\varphi(u)du,$$

Eventually, we obtain the stated result:

$$Q(X_2 \geq x_2, \ldots, X_n \geq x_n \mid X_1 = x_1) = \int \prod_{i=2}^{n} \Phi\left(\frac{\rho_i\sqrt{1-\rho_1^2}u + \rho_i\rho_1 x_1 - x_i}{\sqrt{1-\rho_i^2}}\right)\varphi(u)du \quad \blacksquare$$

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