On Default Correlation: A Copula Function Approach

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The rapidly growing credit derivatives market has created a new set of financial instruments that can be used to manage the most important dimension of financial risk — credit risk. Besides the standard credit derivatives products such as credit default swaps and total return swaps based upon a single underlying credit risk, many new products are now associated with a portfolio of credit risks.

A typical example is the product whose payment is contingent upon the time and identity of the first- or second-to-default in a given credit risk portfolio. Variations include instruments with payment contingent upon the cumulative loss before a given time in the future. The equity tranche of a collateralized bond obligation (CBO) or a collateralized loan obligation (CLO) is yet another variation, where the holder of the equity tranche incurs the first loss. Deductible and stop-loss insurance product features could also be incorporated into the basket credit derivatives structure.

As more financial firms try to manage their credit risk at the portfolio level, and the CBO/CLO market continues to expand, the demand for basket credit derivatives products will most likely continue to grow. Central to the valuation of credit derivatives written on a credit portfolio is the issue of default correlation. The problem of default correlation arises even in the valuation of a simple credit default swap with one underlying reference asset if we do not assume independence of default between the reference asset and the default swap seller.

Surprising though it may seem, the default correlation has not been well defined or understood in finance. The literature tends to define default correlation on the basis of discrete events divided according to survival or default at a critical period such as one year.

For example, if we denote

$$q_A = \Pr[E_A], \quad q_B = \Pr[E_B], \quad q_{AB} = \Pr[E_A E_B]$$

where E_A and E_B are the default events of two securities A and B over one year, the default correlation ρ between two default events E_A and E_B, based on the standard definition of correlation of two random variables, is defined as:

$$\rho = \frac{q_{AB} - q_A q_B}{\sqrt{q_A(1-q_A)q_B(1-q_B)}} \tag{1}$$

This discrete event approach has been taken by Lucas [1995]. Hereafter we call this definition of default correlation the *discrete default correlation.*

The choice of a specific period like one year is more or less arbitrary. It may correspond with many empirical studies of the default rate over a one-year period, but the dependence of default correlation on a specific time interval has disadvantages.

First, default is a time-dependent event, and so is default correlation. Take the lifetime of a person as an example. The probability of dying within one year for a person aged fifty years today is about 0.6%, but the probability of dying for the same person within fifty years is almost a sure event. Similarly, default correlation is a time-dependent quantity. Consider the survival times of a couple, both aged fifty years today. The correlation between the two discrete events that each dies within one year is very small. But the correlation between the two discrete events that each dies within a hundred years is 1.0.

Second, concentration on a single period of one year wastes important information. Empirical studies show that the default tendency of corporate bonds is linked to their age since issue. Also, there are strong links between the economic cycle and defaults. Arbitrarily focusing on a one-year period neglects this important information.

Third, in the majority of credit derivatives valuations, what we need is not the default correlation of two entities over the next year. We may need to have a joint distribution of survival times for the next ten years. Finally, the calculation of default rates as simple proportions is possible only when no samples are censored during the one-year period.[1]

This article describes a few techniques used in survival analysis. These techniques have been widely applied to other areas, such as life contingencies in actuarial science and industry life testing in reliability studies, which are similar to the credit problems we encounter here. We first introduce a random variable called "time-until-default" to denote the survival time of each defaultable entity or financial instrument. Then, we define the default correlation of two entities as the correlation between their survival times.

In credit derivatives valuation, we need first to construct a credit curve for each credit risk. A credit curve gives all marginal conditional default probabilities over a number of years. This curve is usually derived from the risky bond spread curve or asset swap spreads observed currently from the market. Spread curves and asset swap spreads incorporate information on factors such as default probabilities, recovery rate, and liquidity. Assuming an exogenous recovery rate and a default treatment, we can extract a credit curve from the spread curve or asset swap spread curve.

For two credit risks, we would obtain two credit curves from market observable information. Then, we need to specify a joint distribution for the survival times such that the marginal distributions are the credit curves. Obviously, this problem has no unique solution.

Copula functions used in multivariate statistics provide a convenient way to specify the joint distribution of survival times with given marginal distributions. We describe the concept of copula functions, their basic properties, and some commonly used copula functions, and provide some numerical examples of credit derivatives valuation to demonstrate their use and the impact of default correlation.

I. CHARACTERIZATION OF DEFAULT BY TIME-UNTIL-DEFAULT

Default studies look at a group of individual companies for each of which there is defined a point event, often called default (or survival), occurring after some period of time. We introduce a random variable for a security called the *time-until-default*, or simply survival time, to denote this length of time. This random variable is the basic building block for the valuation of cash flows subject to default.

To precisely determine time-until-default, we need: an unambiguously defined time origin, a time scale for measuring the passage of time, and a clear definition of default.

We choose the current time as the time origin to allow use of current market information to build credit curves. The time scale is defined in terms of years for continuous models, or number of periods for discrete models. The meaning of default is defined by rating agencies such as Moody's.

Survival Function

Let us consider an existing security A. This security's time-until-default, T_A, is a continuous random variable that measures the length of time from today to the time when default occurs. For simplicity, we just use T, which should be understood as the time-until-default for a specific security A. Let F(t) denote the distribution function of T:

$$F(t) = \Pr(T \le t) \qquad t \ge 0 \qquad (2)$$

and set

$$S(t) = 1 - F(t) = \Pr(T > t) \qquad t \ge 0 \qquad (3)$$

We also assume that $F(0) = 0$, which implies $S(0) = 1$. The function S(t) is called the *survival function*. It gives the probability that a security will attain age t. The distribution of T_A can be defined by specifying either the distribution function F(t) or the survival function S(t). We can also define a probability density function as follows

$$f(t) = F'(t) = -S'(t) = \lim_{\Delta \to 0^+} \frac{\Pr[t < T \le t + \Delta]}{\Delta}$$

To make probability statements about a security that has survived x years, the future lifetime for this security is $T - x \mid T > x$.

Additional notation is:

$$_tq_x = \Pr[T - x \le t \mid T > x] \qquad t \ge 0$$

$$_tp_x = 1 - {_tq_x} = \Pr[T - x > t \mid T > x] \quad t \ge 0 \quad (4)$$

The symbol $_tq_x$ can be interpreted as the conditional probability that the security A will default within the next t years, conditional on its survival for x years. In the special case of x = 0, we have

$$_tp_0 = S(t) \qquad x \ge 0$$

If t = 1, we use the actuarial convention to omit the prefix 1 in the symbols $_tq_x$ and $_tp_x$, and we have

$$p_x = \Pr[T - x > 1 \mid T > x]$$

$$q_x = \Pr[T - x \le 1 \mid T > x]$$

The symbol q_x is usually called the *marginal default probability*, which represents the probability of default in the next year, conditional on survival until the beginning of the year. A credit curve is then simply defined as the sequence of $q_0, q_1, \ldots, q_n$ in discrete models.

Hazard Rate Function

The distribution function F(t) and the survival function S(t) provide two mathematically equivalent ways of specifying the distribution of the random variable time-until-default, and there are many other equivalent functions. The one used most frequently by statisticians is the hazard rate function, which gives the instantaneous default probability for a security that has attained age x:

$$\Pr[x < T \le x + \Delta x \mid T > x] = \frac{F(x + \Delta x) - F(x)}{1 - F(x)} \approx \frac{f(x)\Delta x}{1 - F(x)}$$

The function

$$\frac{f(x)}{1 - F(x)}$$

has a conditional probability density interpretation. It gives the value of the conditional probability density function of T at exact age x, given survival to that time. Let's denote it as h(x), which is usually called the *hazard rate function*. The relationship of the hazard rate function to the distribution function and survival function is as follows:

$$h(x) = \frac{f(x)}{1 - F(x)} = -\frac{S'(x)}{S(x)} \qquad (5)$$

Then, the survival function can be expressed in terms of the hazard rate function:

$$S(t) = e^{-\int_0^t h(s)ds}$$

Now, we can express $_tq_x$ and $_tp_x$ in terms of the hazard rate function as follows:

$$_tp_x = e^{-\int_0^t h(s+x)ds}$$

$$_tq_x = 1 - e^{-\int_0^t h(s+x)ds} \qquad (6)$$

In addition:

$$F(t) = 1 - S(t) = 1 - e^{-\int_0^t h(s)ds}$$

and

$$f(t) = S(t)h(t) \qquad (7)$$

which is the density function for T.

A typical assumption is that the hazard rate is a constant, h, over a certain period, such as $[x, x + 1]$. In this case, the density function is

$$f(t) = he^{-ht}$$

which shows that the survival time follows an exponential distribution with parameter h. Under this assumption, the survival probability over the time interval $[x, x + t]$ for $0 < t \leq 1$ is

$${}_tp_x = 1 - {}_tq_x = e^{-\int_0^t h(s)ds} = e^{-ht} = (p_x)^t$$

where p_x is the probability of survival over a one-year period. This assumption can be used to scale down the default probability over one year to a default probability over a time interval shorter than one year.

Modeling a default process is equivalent to modeling a hazard rate function. There are a number of reasons why modeling the hazard rate function may be a good idea. First, it provides us information on the immediate default risk of each entity known to be alive at exact age t. Second, the comparisons of groups of individuals are most incisively made via the hazard rate function. Third, the hazard rate function-based model can be easily adapted to more complicated situations, such as where there is censoring or there are several types of default, or where we would like to consider stochastic default fluctuations.

Finally, there are many similarities between the hazard rate function and the short rate. Many modeling techniques for the short rate processes can be readily borrowed to model the hazard rate.

We can define the joint survival function for two entities A and B according to their survival times T_A and T_B:

$$S_{T_A T_B}(s, t) = \Pr[T_A > s, T_B > t]$$

The joint distributional function is

$$F(s, t) = \Pr[T_A \leq s, T_B \leq t]$$

$$= 1 - S_{T_A}(s) - S_{T_B}(t) + S_{T_A T_B}(s, t)$$

These concepts and results are described further in some survival analysis books, such as Bowers et al. [1997] and Cox and Oakes [1984].

II. DEFINITION OF DEFAULT CORRELATIONS

The default correlation of two entities A and B can then be defined with respect to their survival times T_A and T_B as follows:

$$\rho_{AB} = \frac{\mathrm{Cov}(T_A, T_B)}{\sqrt{\mathrm{Var}(T_A)\mathrm{Var}(T_B)}}$$

$$= \frac{E(T_A T_B) - E(T_A)E(T_B)}{\sqrt{\mathrm{Var}(T_A)\mathrm{Var}(T_B)}} \quad (8)$$

Hereafter we call this definition of default correlation the *survival time correlation*. The survival time correlation is a much more general concept than a discrete default correlation based on one period. If we have the joint distribution $f(s, t)$ of two survival times T_A, T_B, we can calculate the discrete default correlation. For example, if we define

$$E_A = [T_A < 1]$$

$$E_B = [T_B < 1]$$

then the discrete default correlation can be calculated using Equation (1) with calculations as follows:

$$q_{AB} = \Pr[E_A E_B] = \int_0^1 \int_0^1 f(s, t)dsdt$$

$$q_A = \int_0^1 f_A(s)ds$$

$$q_B = \int_0^1 f_B(t)dt$$

Knowing the discrete default correlation over a one-year period does not allow us to specify the survival time correlation, however.

III. CONSTRUCTION OF THE CREDIT CURVE

The distribution of survival time or time-until-default can be characterized by the distribution function, survival function, or hazard rate function. We have shown that all default probabilities can be calculated once a characterization is given. The hazard rate function used to characterize the distribution of survival time can also be called a credit curve due to its similarity to a yield curve. But the basic question is: How do we obtain the credit curve or the distribution of survival time for a given credit?

There are three methods to obtain the term structure of default rates:

1. Obtaining historical default information from rating agencies.
2. Taking the Merton option-theoretical approach.
3. Taking the implied approach using the market prices of defaultable bonds or asset swap spreads.

Rating agencies like Moody's publish historical default rate studies regularly. Besides the commonly cited one-year default rates, agencies also present multiyear default rates. From these rates we can obtain the hazard rate function.

For example, Moody's publishes weighted-average cumulative default rates from one to twenty years (see Carty and Lieberman [1997]). For the B rating, the first five years' cumulative default rates in percentages are 7.27, 13.87, 19.94, 25.03, and 29.45. From these rates we can obtain the marginal conditional default probabilities.

The first marginal conditional default probability in year 1 is simply the one-year default probability, 7.27%. The other marginal conditional default probabilities can be obtained using the formula:

$$ {}_{n+1}q_x = {}_nq_x + {}_np_x q_{x+n} \tag{9} $$

which simply states that the probability of default over time interval [0, n + 1] is the sum of the probability of default over the time interval [0, n], plus the probability of survival to the end of the n-th year and default in the following year. Using Equation (9), we have the marginal conditional default probability:

$$ q_{x+n} = \frac{{}_{n+1}q_x - {}_nq_x}{1 - {}_nq_x} $$

which results in the marginal conditional default probabilities in years 2, 3, 4, and 5 as 7.12%, 7.05%, 6.36%, and 5.90%. If we assume a piecewise constant hazard rate function over each year, we can obtain the hazard rate function using Equation (6). The hazard rate function obtained is given in Exhibit 1.

Using diffusion processes to describe changes in the value of the firm, Merton [1974] demonstrates that a firm's default can be modeled using the Black and Scholes methodology. He shows that stock can be considered a call option on the firm with strike price equal to the face value of a single-payment debt. Using this framework, we can obtain the default probability for the firm over one period, and then translate this default probability into a hazard rate function.

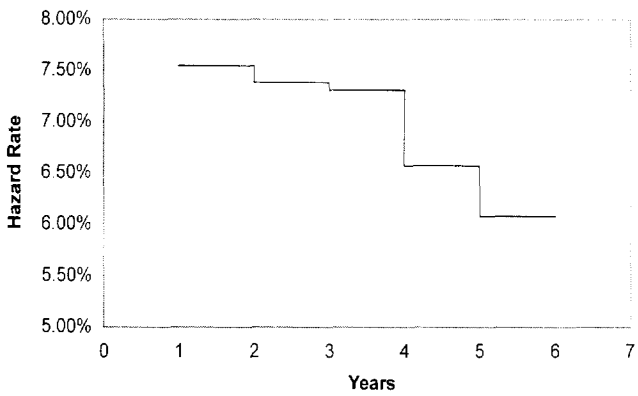
Delianedis and Geske [1998] extend Merton's analysis to produce a term structure of default probabilities. Using the relationship between the hazard rate and the default probabilities, we can obtain a credit curve.

Alternatively, we can take the implicit approach by using market-observable information, such as asset swap spreads or risky corporate bond prices. This is the approach that most credit derivatives trading desks use. The extracted default probabilities reflect the market-agreed perception today about the future default tendency of the underlying credit.

In Li [1998] I present one approach to building the credit curve from market information based on the Duffie and Singleton [1997] default treatment. I assume that there are a series of bonds with maturity 1, 2, ..., n years that are issued by the same company and have the same seniority. All the bonds have observable market prices. From the market prices of these bonds we can calculate their yields to maturity.

Using the yield to maturity of corresponding Treasury bonds, we obtain a yield spread curve over Treasury (or asset swap spreads for a yield spread curve over LIBOR). The credit curve construction is based on this yield spread curve and an exogenous assumption about the

EXHIBIT 1
Hazard Rate Function



recovery rate depending on the seniority and the rating of the bonds and the industry of the corporation.

This approach is contrary to the use of historical default experience information provided by rating agencies such as Moody's. We use market information rather than historical information for several reasons:

- The calculation of profit and loss for a trading desk can be based only on current market information. This current market information reflects the market-agreed perception about the evolution of the market in the future, on which the actual profit and loss depend. The default rate derived from current market information may be much different from historical default rates.
- Rating agencies use classification variables in the hope of expressing homogeneous risks through classification. The technique is used elsewhere, such as in pricing automobile insurance. Unfortunately, classification techniques often omit some firm-specific information. Constructing a credit curve for each credit allows us to use more firm-specific information.
- Rating agencies react much more slowly than the market in anticipation of future credit quality. A typical example is rating agencies' reactions to the 1997-1998 Asian crisis.
- Ratings are primarily used to calculate default frequency instead of default severity, but much of a credit derivative's value depends on both default frequency and severity.
- The information available from a rating agency is usually the one-year default probability for each rating group and the rating migration matrix. Neither the transition matrices nor the default probabilities are necessarily stable over long periods of time. In addition, many credit derivatives products have maturities well beyond one year, which requires the use of long-term marginal default probability.

It is shown under the Duffie and Singleton approach that a defaultable instrument can be valued as if it is a default-free instrument by discounting the defaultable cash flow at a *credit risk-adjusted discount factor.* The credit risk-adjusted discount factor or the total discount factor is the product of the risk-free discount factor and the pure credit discount factor if the underlying factors affecting default and those affecting the interest rate are independent. Under this framework, and assuming a piecewise constant hazard rate function, we can derive a credit curve or specify the distribution of the survival time.

IV. DEPENDENT MODELS — COPULA FUNCTIONS

Let us study some problems of an n-credit portfolio. Using either the historical approach or the market-implicit approach, we can construct the marginal distribution of survival time for each of the credit risks in the portfolio. If we assume mutual independence among the credit risks, we can study any problem associated with the portfolio.

In fact, the independence assumption for credit risks is obviously not realistic; in reality, the default rate for a group of credits tends to be higher in a recession and lower when the economy is booming. This implies that each credit is subject to the same set of macroeconomic influences, and that there is some form of positive dependence among the credits.

To introduce a correlation structure into the portfolio, we must determine how to specify a joint distribution of survival times, with given marginal distributions. Obviously, this problem has no unique solution. Generally speaking, knowing the joint distribution of random variables allows us to derive the marginal distributions and the correlation structure among the random variables, but not vice versa.

There are many different techniques in statistics that allow us to specify a joint distribution function with given marginal distributions and a correlation structure. Among them, the copula function is a simple and convenient approach.

Definition and Basic Properties of Copula Function

A copula function is a function that links or marries univariate marginals to their full multivariate distribution. For n uniform random variables, U_1, U_2, ..., U_n, the joint distribution function C, defined as

$$C(u_1, u_2, \ldots, u_n, \rho) = \Pr[U_1 \leq u_1, U_2 \leq u_2, \ldots, U_n \leq u]$$

can also be called a *copula function.*

Copula functions can be used to link marginal distributions with a joint distribution. For given univariate marginal distribution functions $F_1(x_1)$, $F_2(x_2)$, ..., $F_n(x_n)$, the function

$$C[F_1(x_1), F_2(x_2), ..., F_n(x_n)] = F(x_1, x_2, ..., x_n)$$

which is defined using a copula function C, results in a multivariate distribution function with univariate marginal distributions as specified $F_1(x_1), F_2(x_2), ..., F_n(x_n)$.

This property can be easily shown as follows:

$$C(F_1(x_1), F_2(x_2), ..., F_n(x_n), \rho)$$

$$= \Pr[U_1 \le F_1(x_1), U_2 \le F_2(x_2), ..., U_n \le F_n(x_n)]$$

$$= \Pr[F_1^{-1}(U_1) \le x_1, F_2^{-1}(U_2) \le x_2, ..., F^{-1}(U_n) \le x_n]$$

$$= \Pr[X_1 \le x_1, X_2 \le x_2, ..., X_n \le x_n]$$

$$= F(x_1, x_2, ..., x_n)$$

The marginal distribution of X_i is

$$C(F_1(+\infty), F_2(+\infty), ..., F_i(x_i), ..., F_n(+\infty), \rho)$$

$$= \Pr[X_1 \le +\infty, X_2 \le +\infty, ..., X_i \le x_i, X_n \le +\infty]$$

$$= \Pr[X_i \le x_i]$$

$$= F_i(x_i)$$

Sklar [1973] establishes the converse. He shows that any multivariate distribution function F can be written in the form of a copula function. He proves that if $F(x_1, x_2, ..., x_n)$ is a joint multivariate distribution function with univariate marginal distribution functions $F_1(x_1), F_2(x_2), ..., F_n(x_n)$, there exists a copula function $C(u_1, u_2, ..., u_n)$ such that

$$F(x_1, x_2, ..., x_n) = C[F_1(x_1), F_2(x_2), ..., F_n(x_n)]$$

If each F_i is continuous, then C is unique. Thus, copula functions provide a unifying and flexible way to study multivariate distributions.

For simplicity's sake, we discuss only the properties of bivariate copula functions $C(u, v, \rho)$ for uniform random variables U and V, defined over the area $[(u, v) \mid 0 < u \le 1, 0 < v \le 1]$, where ρ is a correlation parameter. We call ρ simply a correlation parameter since it does not necessarily equal the usual correlation coefficient defined by Pearson (nor Spearman's rho, nor Kendall's tau).

The bivariate copula function has properties as follows:

- Since U and V are positive random variables, $C(0, v, \rho) = C(u, 0, \rho) = 0$.
- Since U and V are bounded above by 1, the marginal distributions can be obtained by $C(1, v, \rho) = v$, $C(u, 1, \rho) = u$.
- For independent random variables U and V, $C(u, v, \rho) = uv$.

Frechet [1951] shows there exist upper and lower bounds for a copula function:

$$\max(0, u + v - 1) \le C(u, v) \le \min(u, v)$$

The multivariate extension of Frechet bounds is given by Dall'Aglio [1972].

Some Common Copula Functions

We describe a few copula functions commonly used in biostatistics and actuarial science.

Frank Copula. The Frank copula function is defined as

$$C(u, v) = \frac{1}{\alpha} \ln\left[1 + \frac{(e^{\alpha u} - 1)(e^{\alpha v} - 1)}{e^{\alpha} - 1}\right] \quad \text{for } -\infty < \alpha < \infty$$

Bivariate Normal. The bivariate normal is

$$C(u, v) = \Phi_2(\Phi^{-1}(u), \Phi^{-1}(v), \rho) \quad \text{for } -1 \le \rho \le 1 \quad (10)$$

where Φ_2 is the bivariate normal distribution function with correlation coefficient ρ, and Φ^{-1} is the inverse of a univariate normal distribution function. As we shall see later, this is the copula function used in CreditMetrics.

Bivariate Mixture Copula Function. We can form a new copula function using existing copula functions. If the two uniform random variables u and v are independent, we have a copula function $C(u, v) = uv$. If the two random variables are perfectly correlated, we have the copula function $C(u, v) = \min(u, v)$.

Mixing the two copula functions by a mixing coefficient ($\rho > 0$), we obtain a new copula function as follows

$$C(u, v) = (1 - \rho)uv + \rho \min(u, v) \quad \text{if } \rho > 0$$

If $\rho \le 0$ we have

$$C(u, v) = (1 + \rho)uv - \rho(u - 1 + v)\Theta(u - 1 + v)$$

where

$$\Theta(x) = 1 \text{ if } x \geq 0$$
$$= 0 \text{ if } x < 0$$

Copula Function and Correlation Measurement

To compare different copula functions, we need to have a correlation measurement independent of marginal distributions. The usual Pearson's correlation coefficient, however, depends on the marginal distributions (see Lehmann [1966]). Both Spearman's rho and Kendall's tau can be defined using a copula function only as follows

$$\rho_s = 12 \iint [C(u, v) - uv]dudv$$

$$\tau = 4 \iint C(u, v)dC(u, v) - 1$$

Comparisons between results using different copula functions should be based on either a common Spearman's rho or Kendall's tau.

Copula functions are further described in Frees and Valdez [1988] and Nelsen [1999].

Calibration of Default Correlation in Copula Functions

Having chosen a copula function, we need to compute the pairwise correlation of survival times. Using the Gupton, Finger, and Bhatia [1997] asset correlation approach, we can obtain the default correlation of two discrete events over a one-year period. As it happens, they use the normal copula function in their default correlation formula even though not explicitly.

First let us summarize the calculation of the joint default probability of two credits A and B. Suppose the one-year default probabilities for A and B are q_A and q_B. The steps would be:

- Obtain Z_A and Z_B such that

$$q_A = \Pr[Z < Z_A]$$

$$q_B = \Pr[Z < Z_B]$$

where Z is a standard normal random variable.

- If ρ is the asset correlation, the joint default probability for credits A and B is calculated as follows:

$$\Pr[Z < Z_A, Z < Z_B] =$$

$$\int_{-\infty}^{Z_A} \int_{-\infty}^{Z_B} \phi_2(x, y \mid \rho)dxdy = \Phi_2(Z_A, Z_B, \rho) \qquad (11)$$

where $\phi_2(x, y \mid \rho)$ is the standard bivariate normal density function with a correlation coefficient ρ, and Φ_2 is the bivariate accumulative normal distribution function.

If we use a bivariate normal copula function with a correlation parameter γ, and denote the survival times for A and B as T_A and T_B, the joint default probability can be calculated as follows

$$\Pr[T_A < 1, T_B < 1] =$$

$$\Phi_2(\Phi^{-1}[F_A(1)], \Phi^{-1}[F_B(1)], \gamma) \qquad (12)$$

where F_A and F_B are the distribution functions for the survival times T_A and T_B.

If we notice that

$$q_i = \Pr[T_i < 1] = F_i(1)$$

and

$$Z_i = \Phi^{-1}(q_i) \qquad \text{for } i = A, B$$

we see that Equation (12) and Equation (11) give the same joint default probability over a one-year period if $\rho = \gamma$.

We can conclude that this process uses a bivariate normal copula function, with the asset correlation as the correlation parameter in the copula function. Thus, to generate survival times of two credit risks, we use a bivariate normal copula function with correlation parameter equal to this asset correlation.

Note that this correlation parameter is not the correlation coefficient between the two survival times. The correlation coefficient between the survival times is much smaller than the asset correlation. Conveniently, the marginal distribution of any subset of an n-dimensional normal distribution is still a normal distribution. Using asset correlations, we can construct high-dimensional normal copula functions to model a credit portfolio of any size.

V. NUMERICAL ILLUSTRATIONS

Assume there are two credit risks, A and B, which have flat spread curves of 300 bp and 500 bp over LIBOR. These spreads are usually given in the market as asset swap spreads. Using these spreads and a constant recovery assumption of 50%, we build two credit curves for the two credit risks. For details, see Li [1998].

The two credit curves are given in Exhibits 2 and 3. These two curves are used in the numerical illustrations.

Illustration 1. Default Correlation versus Length of Time Period

The first example examines the relationship between the discrete default correlation (1) and the survival time correlation (8). The survival time correlation is a much more general concept than the discrete default correlation defined for two discrete default events at an arbitrary period of time, such as one year. Knowing the former allows us to calculate the latter over any time interval in the future, but not vice versa.

Using two credit curves, we can calculate all marginal default probabilities up to any time t in the future:

$$_t q_0 = \Pr[\tau < t] = 1 - e^{-\int_0^t h(s)ds}$$

EXHIBIT 2

Credit Curve A: Instantaneous Default Probability (spread = 300 bp, recovery rate = 50%)

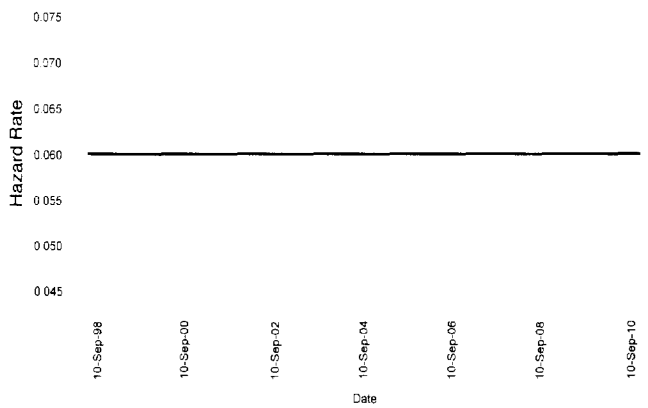
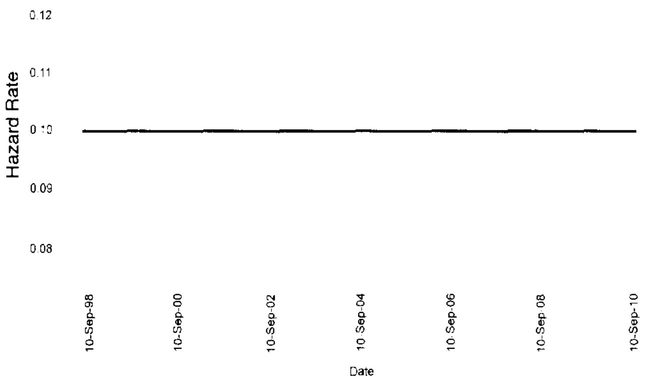


EXHIBIT 3

Credit Curve B: Instantaneous Default Probability (spread = 500 bp, recovery rate = 50%)



where h(s) is the instantaneous default probability given by a credit curve. If we have the marginal default probabilities $_t q_0^A$ and $_t q_0^B$ for both A and B, we can also obtain the joint probability of default over the time interval [0, t] by a copula function C(u, v):

$$\Pr[T_A < t, T_B < t] = C(_t q_0^A, {}_t q_0^B)$$

Of course we need to specify a correlation parameter ρ in the copula function. Knowing ρ would allow us to calculate the survival time correlation between T_A and T_B.

We can now obtain the discrete default correlation coefficient ρ_t between the two discrete possibilities that A and B default over the time interval [0, t] according to Equation (1). Intuitively, the discrete default correlation ρ_t should be an increasing function of t since the two underlying credits should have a higher tendency of joint default over longer periods. Using the bivariate normal copula function (10) and $\rho = 0.1$, we obtain Exhibit 4.

In this graph we see explicitly that the discrete default correlation over time interval [0, t] is a function of t. For example, the default correlation coefficient goes from 0.021 to 0.038 when t goes from six months to twelve months. The increase slows down as t becomes large.

Illustration 2: Default Correlation and Credit Swap Valuation

Suppose that credit A is the credit swap seller, and credit B is the underlying reference asset. If we buy a

EXHIBIT 4

Discrete Default Correlation versus Length of Time Interval

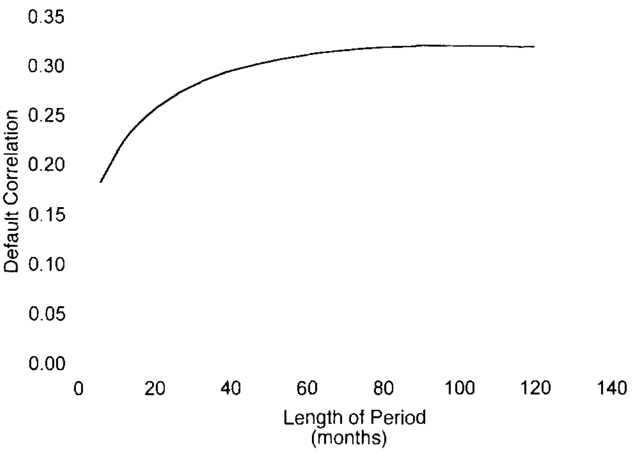
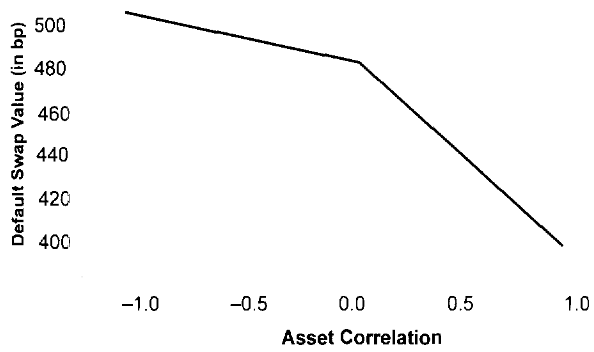


EXHIBIT 5

Asset Correlation versus Value of Credit Swap



default swap of three years with a reference asset of credit B from a risk-free counterparty, we should pay 500 bp since holding the underlying asset and having a long position on the credit swap would create a riskless portfolio. But if we buy the default swap from a risky counterparty, the amount we should pay depends on the credit quality of the counterparty and the default correlation between the underlying reference asset and the counterparty.

Knowing only the discrete default correlation over one year, we cannot value any credit swaps with a maturity longer than one year. Exhibit 5 shows the impact of asset correlation (or implicitly default correlation) on the credit swap premium. From the graph, we see that the annualized premium decreases as the asset correlation between the counterparty and the underlying reference asset increases. Even at zero default correlation, the credit swap has a value less than 500 bp since the counterparty is risky.

Illustration 3: Default Correlation and First-to-Default Valuation

The third example shows how to value a first-to-default contract. Suppose we have a portfolio of n credits. Let us assume that for each credit i in the portfolio we have constructed a credit curve or a hazard rate function for its survival time T_i. The distribution function of T_i is $F_i(t)$. Using a copula function C, we also obtain the joint distribution of the survival times as follows

$$F(t_1, t_2, \ldots, t_n) = C[F_1(t_1), F_2(t_2), \ldots, F_n(t_n)]$$

If we use a normal copula function, we have

$$F(t_1, t_2, \ldots, t_n) =$$

$$\Phi_n(\Phi^{-1}[F_1(t_1)], \Phi^{-1}[F_2(t_2)], \ldots, \Phi^{-1}[F_n(t_n)])$$

where Φ_n is the n-dimensional normal cumulative distribution function with correlation coefficient matrix Σ.

To simulate correlated survival times, we introduce another series of random variables $Y_1, Y_2, \ldots, Y_n$ such that

$$Y_1 = \Phi^{-1}[F_1(T_1)],\ Y_2 = \Phi^{-1}[F_2(T_2)], \ldots,$$

$$Y_n = \Phi^{-1}[F_n(T_n)]) \tag{13}$$

Then there is a one-to-one mapping between Y and T. Simulating $[T_i \mid i = 1, 2, \ldots, n]$ is equivalent to simulating $[Y_i \mid i = 1, 2, \ldots, n]$. The correlation between the Ys is the asset correlation of the underlying credits.

We have the simulation scheme:

- Simulate $Y_1, Y_2, \ldots, Y_n$ from an n-dimension normal distribution with correlation coefficient matrix Σ.
- Obtain $T_1, T_2, \ldots, T_n$ using $T_i = F_i^{-1}[N(Y_i)]$, $i = 1, 2, \ldots, n$.

With each simulation run, we generate the survival times for all the credits in the portfolio. With this information, we can value any credit derivative structure written on the portfolio. We use a simple structure for illustration. The contract is a two-year transaction that pays one dollar if the first default occurs during the first two years.

We assume each credit has a constant hazard rate of $h = 0.1$ for $0 < t < +\infty$. From Equation (7), we know the density function for the survival time T is he^{-ht}. This shows that the survival time is exponentially distributed with mean $1/h$. We also assume that every pair of credits in the portfolio has a constant asset correlation σ.[2]

Suppose we have a constant interest rate $r = 0.1$. If all the credits in the portfolio are independent, the hazard rate of the minimum survival time $T = \min(T_1, T_2, ..., T_n)$ is easily shown to be

$$h_T = h_1 + h_2 + ... + h_n = nh$$

If $T < 2$, the present value of the contract is $1e^{-rT}$. The survival time for the first-to-default has a density function $f(t) = h_T e^{-h_T t}$, so the value of the contract is given by

$$\begin{aligned} V &= \int_0^2 1e^{-rt} f(t)dt \\ &= \int_0^2 1e^{-rt} h_T e^{-h_T t}dt \\ &= \frac{h_T}{r + h_T}(1 - e^{-2.0(r+h_T)}) \end{aligned} \tag{14}$$

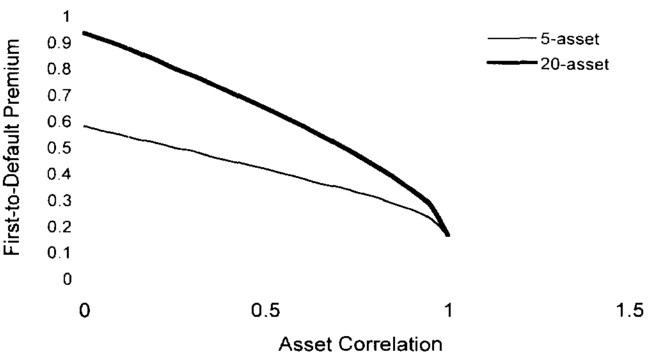
In the general case, we use the Monte Carlo simulation approach and the normal copula function to obtain the distribution of T. For each simulation run, we have one scenario of default times $t_1, t_2, ..., t_n$, from which we have the first-to-default time simply as $t = \min(t_1, t_2, ..., t_n)$.

Let us examine the impact of the asset correlation on the value of the first-to-default contract of five assets. If $\sigma = 0$, the expected payoff function, based on Equation (14), should give a value of 0.5823. Our simulation of 50,000 runs gives a value of 0.5830. If all five assets are perfectly correlated, then the first-to-default of five assets should be the same as the first-to-default of one asset since any one default induces all the others to default. In this case, the contract should be worth 0.1648. Our simulation of 50,000 runs produces a result of 0.1638.

Exhibit 6 shows the relationship between the value of the contract and the constant asset correlation coefficient. We see that the value of the contract decreases as the correlation increases. We also examine the impact of correlation on the value of the first-to-default of twenty assets in Exhibit 6. As expected, the first-to-default of five assets has the same value as the first-to-default of twenty assets when the asset correlation approaches 1.0.

EXHIBIT 6

Value of First-to-Default versus Asset Correlation



VI. SUMMARY AND CONCLUSIONS

This article introduces a few standard techniques used in survival analysis to study the problem of default correlation. We introduce a random variable called the *time-until-default* or survival time to characterize the default, and define the default correlation between two credit risks as the correlation coefficient between their survival times. In practice, we usually use market spread information to derive the distribution of survival times. When it comes to credit portfolio studies, we need to specify a joint distribution with given marginal distributions. The problem cannot be solved uniquely.

The copula function approach provides one way of specifying a joint distribution with known marginals. This article introduces the basic concepts of copula functions into credit studies, and shows how to calibrate the parameter in the normal copula function using CreditMetrics' asset correlation approach to default correlation. It is shown that CreditMetrics essentially uses the normal copula function in its default correlation formula, though not explicitly. We have presented several numerical examples to illustrate the use of copula functions in the valuation of credit derivatives, such as credit default swaps and first-to-default contracts.

ENDNOTES

The author thanks Christopher C. Finger for helpful discussions and comments.

[1]A company that is observed, default-free, by Moody's for five years and then withdrawn from the Moody sample must have a survival time exceeding five years. Another company may enter into Moody's study in the middle of a year, which means that Moody's observes the company for only half of the one-year observation period. In the survival analysis of statistics, such incomplete observation of default time is called *censoring*. According to Moody's studies, such incomplete observations do occur in Moody's credit default samples.

[2]To have a positive-definite correlation matrix, the constant correlation coefficient has to satisfy the condition $\sigma > -[1/(n-1)]$.

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