PRICING PARISIAN-STYLE OPTIONS WITH A LATTICE METHOD

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ABSTRACT. A Parisian-style barrier option expires if the price of the underlying asset remains above or below some level(s) continuously over a specified period of time(the "window"). A trinomial-lattice scheme is developed for calculating the price and the sensitivities of such options. Monte-Carlo simulation of hedging events using the resulting deltas show errors which are of the same magnitude as for hedging vanilla options, confirming the validity of proposed scheme. We use these results to price callable and convertible bonds with this "window" feature.

1. INTRODUCTION

Parisian options are variations on the standard barrier option contracts. They are option contracts that are either "knocked in" or "knocked out" once the asset price remains above or below some level(s) continuously over some pre-specified length of time. For example, a Parisian-style up-and-out put option with 5-day window, with strike 100 and barrier 110, will have terminal payoff $V(S_T, T) = \max(100 - S_T, 0)$ subject to the condition that the asset price S_t over the life of the contract never exceeds 110 continuously for a period of more than 5 days. If the asset price remains above 110 for more than 5 days, the option expires worthless. We shall denote the time-window defining the Parisian option by D, measured in years. At any moment t, the asset path is therefore monitored over the time window $(t - D, t)$ in order to determined whether the option has been knocked out or not. In the extreme cases $D = 0$ and $D = T$, the Parisian put option reduces to the standard knock-out put option and vanilla put option, respectively.

Parisian options can be attractive alternatives to standard knock-out(KO) options. Standard KO barrier options are cheap but they are vulnerable(from the investor's point of view) to "spikes" in the asset price or to market manipulation in the case of thinly traded underliers. The window feature arises as a natural remedy to these disadvantages. On the other hand, it is intuitively clear that the value of a Parisian option increases with the length of the window. The window feature gives a tradeoff between protection(near the barrier) and cost. Over the years there have been strong interests in the options with window feature.

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In Hong Kong, it is quite common for callable convertible bonds traded in the OTC market to carry the window feature in their call provision. The accurate pricing of the window provision(the embedded Parisian option) is a recent practice.

Chesney, Picque and Yor(1995) pioneered the rigorous pricing of Parisian options. They derived a formula for pricing Parisian options using Browning motion excursion theory. The formula involves the density of excursion for a period greater than D, which has to be solved by using numerical inverse Laplace transform. In particular, this method is limited to constant model parameters(volatility, cost-of carry, etc). Cornwall and Kentwell(1995) implemented the results of Chesney *et al* using a semi-analytical approach and extended their results to discrete monitoring as well.

In this paper we model and price Parisian-style options by formulating the problem as a partial differential equation(PDE) which is solved numerically by trinomial lattice method. Our derivation is based on the following observations: (i) if the asset price is beyond the barrier, the option can be viewed as a knock-in option, because it will be "restored" if the asset price crosses the barrier again within a specified period of time. (ii) If the asset price has never crossed the barrier, the option can be regarded as a knock-out option with rebate, with the rebate being equal to the (unknown) option value along the barrier. (iii) The option value along the barrier is determined by imposing continuity of the derivative with respect to the asset price, i.e., by "smoothly pasting" the values on both sides of the barrier. Continuity of delta is justified using standard no-arbitrage arguments.

In a continuous-time formulation, our model consists essentially of a Black-Scholes PDE to be solved in the interior of the domain (before crossing the barrier) and a special boundary condition along the barrier which is simple to compute. This formulation translates easily into a finite-difference scheme which applies without major changes to models with non-constant coefficients and to various window features.

This paper is organized as follows. In section 2, we describe an explicit roll-back scheme on a trinomial lattice for numerical valuation of the option and its sensitivities. In section 3, we derive the PDE formulation and boundary conditions. Experiments of dynamical hedging using the numerical deltas are given in section 4, where option values and deltas are examined for different window duration. In section 5 we model and price a convertible bond with window feature in its call provision. Finally in section 6 we draw conclusion.

2. Risk-Neutral Valuation with Trinomial Tree

Let V denote the value of a Parisian option, (t, S) be time and asset price, T be the time to expiration, X be the strike price, H be the barrier, and D be the length of the window over which the asset price is monitored. To describe the payoff of the Parisian put option in concise terms, we define excursion age associating the asset price as

$$D(S_t, t) = \begin{cases} 0, & \text{if } S_t \leq H, \\ t - g_t, & \text{if } S_t > H, \end{cases} \tag{1}$$

where g_t is the last time before t at which the asset price crossed H from below:

$$g_t = \sup\{\tau \leq t | S_\tau = H\}. \tag{2}$$

The terminal payoff of a Parisian put option can be expressed as

$$V(S_T, T) = \begin{cases} \max(X - S_T, 0), & \text{if } D(S_t, t) < D, \quad D \leq t \leq T, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

We make the usual assumptions that the option value depends only on the underlying asset price S and time t, and S follows the usual risk-neutral lognormal process:

$$dS = (r - q)Sdt + \sigma S dz, \tag{4}$$

with constant coefficients. For numerical option valuation we approximate the lognormal process by trinomial discrete random walks. A one-period trinomial tree is sketched in Figure 1,

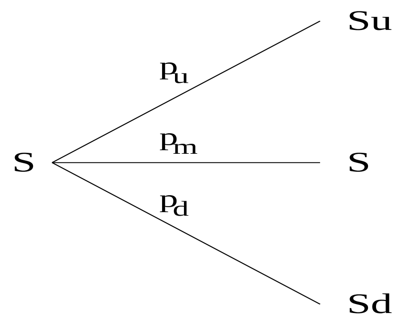

Figure 1. One-period tree of asset

where

$$u = e^{\sigma\sqrt{2\Delta t}}, \qquad d = 1/u; \qquad p = \frac{e^{(r-q)\Delta t/2} - \sqrt{d}}{\sqrt{u} - \sqrt{d}}, \\ p_u = p^2, \quad p_m = 2p(1-p), \quad p_d = (1-p)^2. \tag{5}$$

A multi-period trinomial tree is displayed on Figure 2,

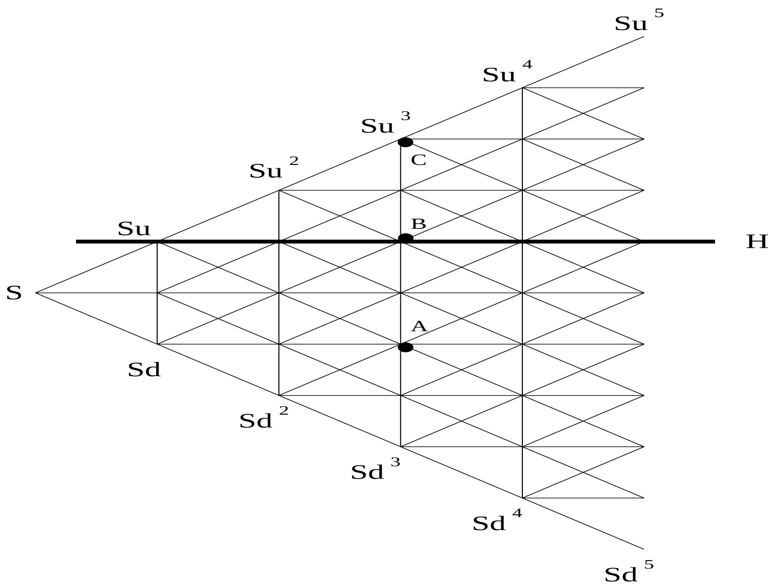

FIGURE 2. Multi-period trinomial Tree

The solid line represents the barrier $S = H$.

The pricing methodology introduced below is based on two observations. First, the principle of risk-neutral valuation is valid as long as the option is alive. This means that the relation of backward induction

$$V(S,t) = e^{-r\Delta t}[p_u V(Su, t+\Delta t) + p_m V(S, t+\Delta t) + p_d V(Sd, t+\Delta t)] \tag{6}$$

holds for option values at all nodes. Such relation is visualized in Figure 3.

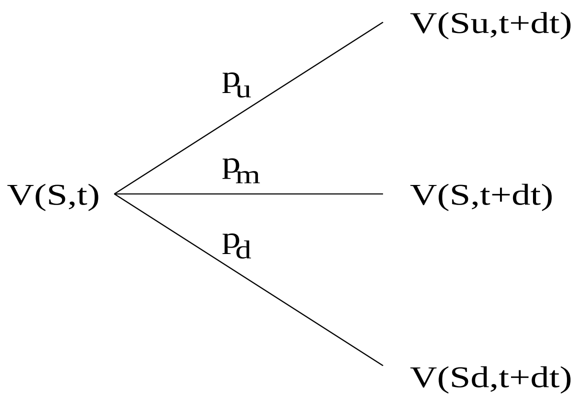

FIGURE 3. One-period tree of option

Second, for any asset price S outside the barrier, the option should be priced as a knock-in option with time to expiration equal to $D - D(S,t)$, since the asset price has drifted outside the barrier for time $D(S,t)$. The interpretation of knock-in option avoid the troubles caused by the path-dependent nature of the Parisian option. These observations suggest a procedure for computing the option value by backward induction.

We classify the nodes into three groups as shown in Figure 2. Namely, nodes inside the barrier, like A; nodes at the barrier, like B; and nodes outside the barrier, like C. For nodes in different groups we apply different schemes.

- For nodes inside the barrier, the scheme is just the standard trinomial backward induction (6).
- For nodes at the barrier, the scheme begins with (See Figure 4)

$$V(H,t) = e^{-r\Delta t}(p_u V_{ki}(Hu, t+\Delta t) + p_m V(H, t+\Delta t) + p_d V(Hd, t+\Delta t)), \tag{7}$$

  where the subindex ki is used to highlight the knock-in nature of the option at node $(Hu, t+\Delta t)$. The window for knock-in is $(t+2\Delta t, t+D]$. Let α_m be the possibility for the underlying asset to cross node $(H, t+m\Delta t)$ for the first time. According to the risk-neutral valuation, we should have

$$\begin{aligned} V_{ki}(Hu, t+\Delta t) = \alpha_2 e^{-r\Delta t} V(H, t+2\Delta t) + \alpha_3 e^{-2r\Delta t} V(H, t+3\Delta t) + \\ \ldots + \alpha_n e^{-(n-1)r\Delta t} V(H, t+n\Delta t), \end{aligned} \tag{8}$$

  where $n = D/\Delta t$ and the nodes involved span the time window $(t+2\Delta t, t+D]$(this is the value of the "knock-in option" alluded in the introduction).

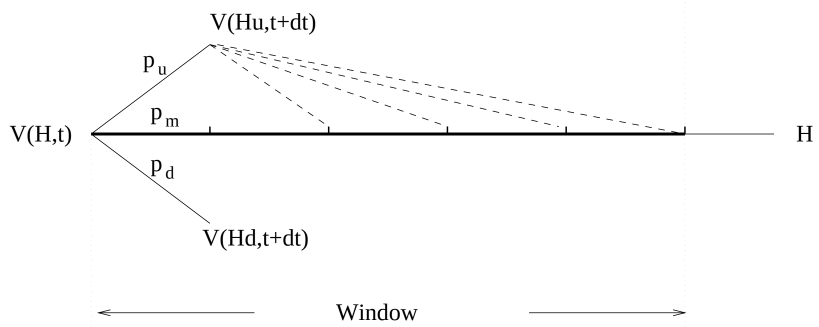

FIGURE 4. Node right at the barrier

- For nodes beyond the barrier, we use backward induction in the partial tree displayed in Figure 5, where the payoff is set to zero at time $t+D-D(S,t)$ and option values are given along the barrier. Notice that V_{ki} in (8) is a special case of this category, which corresponds to a knock-in option with window $(t+2\Delta t, t+D)$.

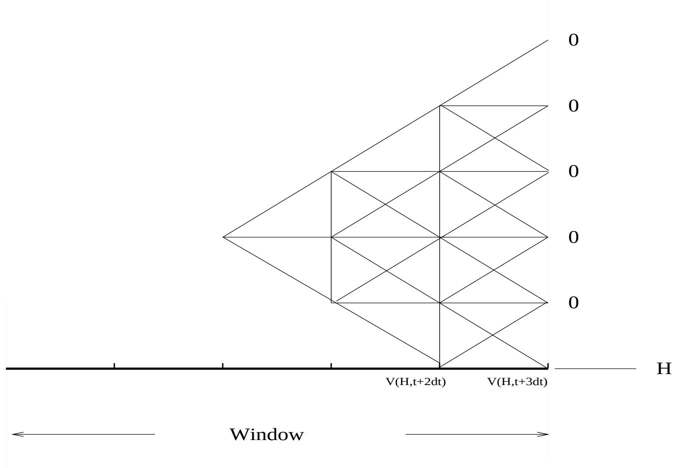

FIGURE 5. Node outside the barrier

This gives a well-defined method for pricing Parisian-style options in a trinomial tree[1]. In the next section we present a PDE formulation that can be used to generate a boundary-value problem for $V(S,t)$ with lateral boundary $S = H$.

3. Continuous-Time Model

We now sketch a derivation of the continuous-time formulation corresponding to the above trinomial schemes. It is well-known that, when taking limit $\Delta t \to 0$, the trinomial backward induction gives rise to the Black-Scholes equation. The value of the knock-in option alluded to in the last section is

$$V_{ki}(S,t) = \int_t^{t+D-D(S,t)} e^{-r(\theta-t)}\Pi(H,\theta;S,t)V(H,\theta)d\theta, \quad S > H, \tag{9}$$

where the kernel $\Pi(H,\theta;S,t)$ represents the density function of first passage time across $S = H$ at time θ from above and $V(H,\theta)$ is the value of the Parisian at the barrier (Rubinstein and Reiner, 1991).

In the special case of a lognormal model, the density function $\Pi(H,\theta;S,t)$ is (see, for instance, Kwok, Wu and Yu, 1997)

$$\Pi(H,\theta;S,t) = \frac{\ln\frac{S}{H}}{\sigma\sqrt{2\pi(\theta-t)^3}} \exp\left(-\frac{\left[\ln\frac{S}{H} + (r-q-\frac{\sigma^2}{2})(\theta-t)\right]^2}{2\sigma^2(\theta-t)}\right). \tag{10}$$

Because the value of the knock-in option in equation (9) involves the boundary value $V(H,\theta)$, we can use this equation to construct a boundary condition for $V(S,t)$ in the region $\{S <$

[1]The reader should note that this backward-induction procedure can be implemented, if necessary, as a semi-implicit Crank-Nicholson scheme.

$H\}$. Clearly, we have $V_{ki}(H,t) = V(H,t)$. Therefore, the matching conditions will involve matching the normal derivatives along the boundary. [2]

Using equation (9), we compute exterior normal derivative of the value of the "knock-out" option, *viz.*, (see the Appendix)

$$\frac{\partial V(H,t)}{\partial S} = \frac{C(\mu,\sigma,D)}{H} V(H,t) + \frac{1}{H}\int_t^{t+D} \frac{e^{-\mu(\theta-t)}}{\sqrt{2\pi(\theta-t)^3}} \left(e^{-r(\theta-t)}V(H,\theta) - V(H,t)\right) d\theta, \tag{11}$$

where

$$C(\mu,\sigma,D) = \frac{2}{\sigma\sqrt{2\pi D}} e^{-\frac{\mu^2 D}{2\sigma^2}} - \frac{\mu}{\sigma^2} erf\left(\frac{\mu\sqrt{D}}{\sigma\sqrt{2}}\right), \tag{12}$$

$\mu = r - q - \sigma^2/2$, and $erf(\cdot)$ is the error function

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2} dt. \tag{13}$$

This gives a continuous-time formulation of the computational scheme of the previous section, which corresponds to letting the mesh-size tend to zero and matching the first non-trivial terms in the boundary expansion. Notice that equation (11) corresponds to a *Dirichlet-to-Neumann* boundary condition.

Using (11), we can implement a scheme in the domain $S \leq H$ as opposed to solving the two coupled problems. A similar treatment can be used in the domain $S > H$ to determine the value of the option after the boundary has ben crossed.

4. Examples of Delta Hedging

The main result presented here is the hedging performance of our PDE model, which we regard as a way to validate the model. For the short position of Parisian put options, we form a hedging portfolio consisting of shares and cash in a money market account, and re-balance the portfolio according to the delta until the termination of the option. For simplicity we ignore transaction costs and other market friction. The distribution of hedging error(or profit and loss) is obtained by using large number of simulated asset paths. It will be seen that the hedging error is comparable with that for delta-hedging the vanilla put option. We also tested the effects of window length on the option values and delta with an up-and-out call option and a corridor knock-out option. For the corridor option we observe that the option values increase notably when the window lengthens.

Two Parisian put options are replicated. One has barrier out of money($H > X$), another has barrier in the money. The particulars of the put options are

[2] This corresponds to requiring that "delta" be continuous across the boundary.

- Annualized interest rate $r = 0.1$;
- Annualized dividend rate $q = 0.0$;
- Annualized volatility $\sigma = 0.3$;
- Time to maturity $T = 1$;
- Current asset price $S = 100$;
- Length of window $D = 0.1$;
- (i) Strike $X = 100$, barrier $H = 120$; or
  (ii) Strike $X = 120$, barrier $H = 110$.

Starting with the current asset price, we simulate the asset path down to contract maturity with $N = 50$ discrete lognormal random shocks. For each new asset price, we adjust the asset holding according to $\Delta = \frac{\partial V}{\partial S}$. The adjustment is financed by the money market account. The initial replicating portfolio is $\Delta_0 S_0 + B_0$, where $B_0 = -\Delta_0 S_0 + V_0$ is the initial balance of the money market account. The hedging error is measured by the percentage difference between the terminal values of option and the replicating portfolio:

$$PL = \frac{e^{-rT}(-V_T + \Delta_T S_T + B_T)}{V_0}. \tag{14}$$

Note that the present value is taken. For each option, a total of 200 paths are simulated.

The distribution of hedging errors are illustrated with histograms. Figure 6 shows the hedging errors for the "out-of-the-money" barrier, with $X = 100, H = 120$. The initial premium obtained is $V(100, 0) = 7.0392$. In the distribution, the mean of the percentage errors is $3.5496 \cdot 10^{-4}$, and the standard deviation is 0.0236. The hedging error for the "in-the-money" barrier, with $X = 120, H = 110$, is given in Figure 7. The mean of the errors is $-1.0961 \cdot 10^{-2}$ and the standard deviation is 0.0394, and the option value is $V(100, 0) = 14.1047$. The hedging errors obtained are comparable to that of delta-hedging the corresponding vanilla European put option, which is known to have standard deviation around $\sigma/\sqrt{N} = 0.042$. In our trinomial tree, the resolution of $\Delta t = 1/365$ is used, which is seven times finer than the interval of re-balancing. Such resolution is not needed for the option values, but for the deltas.

Next we will price an up-and-out call option and a corridor option. These options are more popular than the up-and-out put options. We want to know the effect of window length on option value and delta. Such effect is particularly interesting when the underlying is near a barrier. The underlying in these options is the exchange rate between US dollar(USD) and Japanese yen(JPY). For the spot exchange rate and the exchange rate one yen away from the (upper) barrier, we compare the option values and deltas for various window lags. The inputs of the options are taken as the market data of September 15, 1997.

Example 1: Up-and-out Japanses Yen call option. The characteristics of the option are

- US interest rate $r = 0.056$;
- Japan interest rate $q = 0.007$;
- Volatility of spot exchange rate $\sigma = 13\%$;
- Strike exchange rate $X = 1/125$;
- Knock-out level $H = 1/110$;
- Time to maturity $T = 0.5yr = 180days$;
- Spot exchange rate $S = 1/120.5$,

and the payoff will be $\max(S - X, 0)$ if the option is never knocked out. The option value and delta for different time lags are summerized in Table 1. As expected, the option becomes more valuable for larger time lag, in consistence with the increased protectiveness of the option. Meanwhile, the delta changes sign from negative to positive, indicating the change from long position to short position of yen in the replicating portfolio. Note that analytical formula exists for the case lag=0(Rubinstein and Reiner, 1991), and the analytical solution is $1.4060e - 04$.

The results for exchange rate $S = 1/111$ are displayed in Figure 8 and Figure 9, where option values and deltas are plotted against time to maturity for different time lags. As is seen in Figure 8, an option is more valuable for bigger lag, and the values are decreasing function of time to maturity, admitting the bigger probability to be knocked out during the rest of the life of the option. In Figure 9, we see that the deltas are negative for all lags, indicating the short-asset position for hedge. Yet, delta has sharp turn when time to maturity is comparable with the time lag. In fact, when the time to maturity is close to the time lag, the probability of knock-out reduces significantly, so does the need for short asset hedging position.

lags	values	delta
0 day	1.4068e-04	-7.1689e-02
5 days	2.1537e-04	-2.1307e-02
10 days	2.5052e-04	1.0961e-02
15 days	2.7907e-04	4.0949e-02

TABLE 1. Option Values and Deltas

Example 2: This is a corridor option of in USD/JPY exchange rate. The payoff of such option is one million Japanese yen, subjected to the provision that the exchange rate stays between 1/115 and 1/125 for the next six months (monitored continuously). Other relevant parameters are the same as those in Example 1. The option values and deltas for $S = 1/120.5$ with different time lag are shown in Table 2. Note that, unlike what we have seen in Example 1, the delta is not a monotonic function of the time lag.

lags	values	delta
0 day	2.9775e+03	1.7402e+06
5 days	7.4550e+04	8.9374e+06
10 days	1.4230e+05	4.2076e+06
15 days	2.0249e+05	-3.5378e+06

TABLE 2. Option Values and Deltas

Again, the option value and delta for $S = 1/116$, one yen away from the upper barrier $S = 1/115$, are calculated. The results are plotted in Figure 10 and Figure 11. The patterns are similar to those of Figure 8 and 9: bigger option values for larger time lags, the values are decreasing function of time to maturity, and turn of delta occurs when time to maturity is comparable to lag.

Note that we use adaptive trees in both Example 1 and Example 2, which automatically place branch(es) at the barrier(s). The resolution of the trees is $\Delta t = T/360$.

5. Extension to Callable Convertible Bonds

A callable convertible bond is a portfolio consisting of (i) a straight bond, (ii) a call option for the holder to exchange bond for stocks, (iii) and a call option for the issuer to buy back the bond. The holder is entitled to the scheduled coupon payments and principle at maturity. Yet, at any time before maturity the holder can convert the bond to certain number(namely, "conversion ratio") of shares of the stock. The issuer, meanwhile, can buy back("call") the bond with the so-called call price, provided that the stock price stays above some call level continuously for some duration of time. Once the bond is called, holders can still choose to convert. Hence in practice, call provision is often a way to force conversion. Most existing callable bonds have the one-touch call term, corresponding to the window of zero duration. A positive window duration makes the call-back harder to be triggered, and thus add more protection against possible price manipulation aiming at forcing conversion. Bonds with one-touch trigger level have been well understood and priced, but that is not the case for bonds with window feature. Here we will show that the value added by window feature can be priced with the technique developed for Parisian options.

A number of simplifications are made in order to make the model simpler and to highlight the analysis of window feature. We assume that the only source of price dynamics is the stochastic movement of the underlying stock. Both risk-free interest rate and the credit spread of the straight bond are assumed deterministic, and the latter impounds all information of default risk. Also, conversion is made immediately once the bond is called. These simplifications lead to a one-factor model with stock price and time to be the only variables. Let P denote the price of a bond, S the underlying stock, t for calendar time, and d the discount rate. Under the above assumptions, the discount rate is a deterministic function of S and t as well. Further, let r be the risk-free interest rate, y the yield of the straight bond, Z the principle, K the annual coupon(paid semi-annually), n the conversion ratio, H the call level, and D the duration of the window, and $D(S,t)$ the excursion age. Assume, as usual, that the stock price follows the risk-neutral lognormal process

$$dS = (r-q)Sdt + \sigma S dZ, \tag{15}$$

then we have the following governing equations for bond price

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + (r-q)S\frac{\partial P}{\partial S} - dP + \sum \delta(t-t_i)\frac{K}{2} = 0, \tag{16}$$

$$\frac{\partial d}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 d}{\partial S^2} + (r-q)S\frac{\partial d}{\partial S} = 0. \tag{17}$$

The final and boundary conditions prescribed according to $S \le H$ and $S > H$:

For $S \le H$,

$$V(S,T) = \max(nS, Z), \qquad V(0,t) = Ze^{-y(T-t)}; \tag{18}$$

$$d(S,T) = \begin{cases} r, & \text{for } nS \ge Z, \\ y, & \text{for } nS < Z. \end{cases} \tag{19}$$

For $S > H$,

$$V(S, t+D-D(S,t)) = nS, \tag{20}$$

$$d(S, t+D-D(S,t)) = r. \tag{21}$$

Except the discount rate used, equation (16) is the same as the Black-Scholes equation. For the discount rate d, equations (17), (19) and (21) point to the evolution according to the probability of the bond to end up in redemption or conversion. This idea for the determination of discount rate is borrowed from a research note of Goldman Sachs[3], where binomial method is used to valuate the one-touch callable bond. As part of the modelling, V and $\frac{\partial V}{\partial S}$ are required to be continuous across $S = H$. When $D = 0$, we have solution for $S \geq H$ such as $V(S, t) = nS$, implying that conversion occurs with certainty. Note that the above model does not involve the call price, since, as commented earlier, calling-back only enforces conversion.

The above initial-boundary value problem can be solved with trinomial method in the way similar to that for Parisian options. Yet, at each node we have three instead of two values, namely, stock price, bond price and discount rate. In each step of backward induction the discount rate is calculated first and immediately used in the calculation of bond price for discounting. Let us now look at the values of a callable convertible bonds for different time lags. The particulars of the bonds are

- Par value $Z = 100$;
- Maturity $T = 1$;
- Coupon rate $K = 0$;
- Conversion ratio $n = 2$;
- Call price $M = 103$;
- Call level $H = 52.5$,
- Credit spread $y - r = 5\%$.

Assume the current stock price is $S = \$50$, risk-free interest rate is $r = 10\%$ and dividend rate is zero. This gives the yield of straight bond as 15%. The bond values for different time lags are given in Table 3.

lags	values	delta
0 yr	102.28	1.2042
0.1 yr	104.65	1.3886
0.2 yr	105.01	1.4298
0.4 yr	105.24	1.4617
0.8 yr	105.31	1.4750
1 yr	105.31	1.4752

TABLE 3. Bond values and deltas for different window length

One can see that, from lag=0 to lag=0.2, the value of the bond increases by about 2.6%. Beyond that, the increase of value is insignificant. Note that the call level is usually quoted

[3]The binomial tree approach is described in "Valuing Convertible Bonds as Derivatives," Quantitative Strategies Research Notes, Goldman Sachs, November 1994.

as 150% of the conversion price, defined as the principle divided by conversion ratio(Z/n). In our case this corresponds to $H = 75$. For this call level the bond value increases from 105.2996 to 105.3056 for the increasing time lag as in Table 3. This result indicates that the window feature is almost valueless for high call level. In fact, when stock price is considerably higher than the conversion price, the embedded call option is deep in the money, and the holder is indifferent as to convert or not.

6. Conclusion

We showed that Parisian-style options can easily be modelled using a boundary condition which arises by assuming continuity of delta across the barrier. We present an explicit scheme that uses a trinomial lattice. This procedure can also be used to generate a semi-implicit Crank-Nicholson finite-difference scheme with the same template. We also characterized the value function in the continuous limit. It is shown that the value of the Parisian option before hitting the barrier satisfies a boundary-value problem in which the boundary condition is of Dirichlet-to–Neumann type. Therefore, an alternative PDE method would be to solve the equation in the domain limited by the barrier, by implementing the integral equation using a quadrature. We believe that the first method is more robust and easy to implement in practice, especially in the case of one-factor models where the computational time is negligible. Finally, we presented numerical results supporting this numerical scheme and an application to analyze the value of the window feature in pricing callable bonds.

Appendix A. Derivation of boundary condition

Immediately (Δt) after the last exit from the barrier, $V(S, t+\Delta t)$ satisfies

$$V(S,t+\Delta t) = \int_{t+\Delta t}^{t+D} e^{-r(\theta-t)}\Pi(H,\theta;S,t)V(H,\theta)d\theta. \tag{A.1}$$

For geometric Brownian motion, there is

$$\begin{aligned}\int_{t+\Delta t}^{t+D}\Pi(H,\theta;S,t)d\theta &= Prob.\{\tau < t+D|S(t+\Delta t)=S\}\\ &= 1 - C(\mu,\sigma,D)\ln(S/H) + o(\ln(S/H)).\end{aligned} \tag{A.2}$$

The constant $C(\mu,\sigma,D)$, given in (12), is obtained from direct calculation. Now we multiply $V(H,t)$ to (A.2), and then subtract the equation from (A.1). The equation so obtained is

$$\begin{aligned}V(S,t) - V(H,t) = &\int_t^{t+D-\Delta t}\Pi(H,\theta;S,t)\left(e^{-r(\theta-t)}V(H,\theta) - V(H,t)\right)d\theta\\ &+ C(\mu,\sigma,D)\ln(S/H)V(H,t) + o(\ln(S/H)).\end{aligned}$$

Dividing both sides by $S-H$ and passing to the limit gives

$$\frac{\partial V(H,t)}{\partial S} - \frac{C(\mu,\sigma,D)}{H}V(H,t) = \int_t^{t+D} Q(H,\theta,t)\left(e^{-r(\theta-t)}V(H,\theta) - V(H,t)\right)d\theta. \tag{A.3}$$

Here

$$Q(H,\theta,t) = \frac{\partial\Pi(H,\theta;S,t)}{\partial S}\Bigg|_{S=H} = \frac{1}{H}\frac{e^{-\mu(\theta-t)}}{\sqrt{2\pi(\theta-t)^3}}.$$

This completes the derivation.

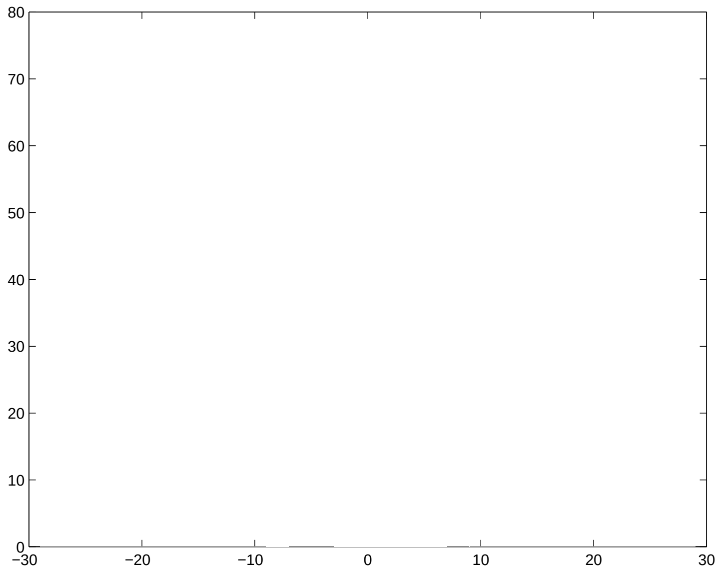

FIGURE 6. Histogram for percentage hedging errors

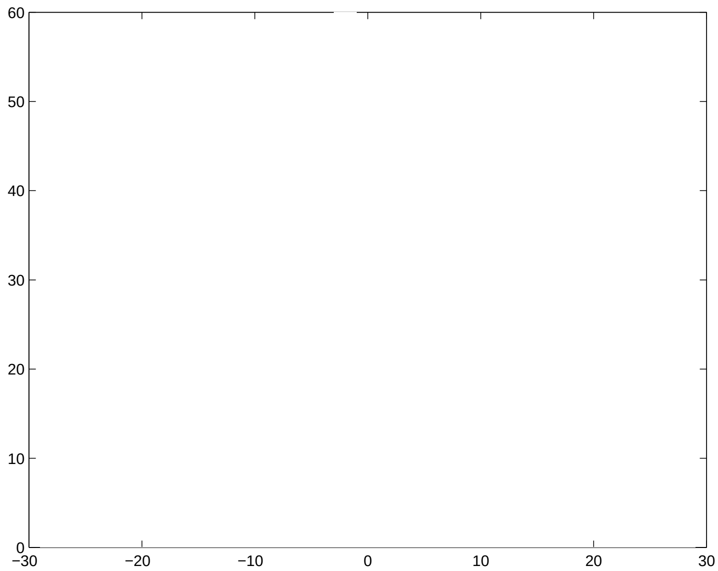

FIGURE 7. Histogram for percentage hedging errors

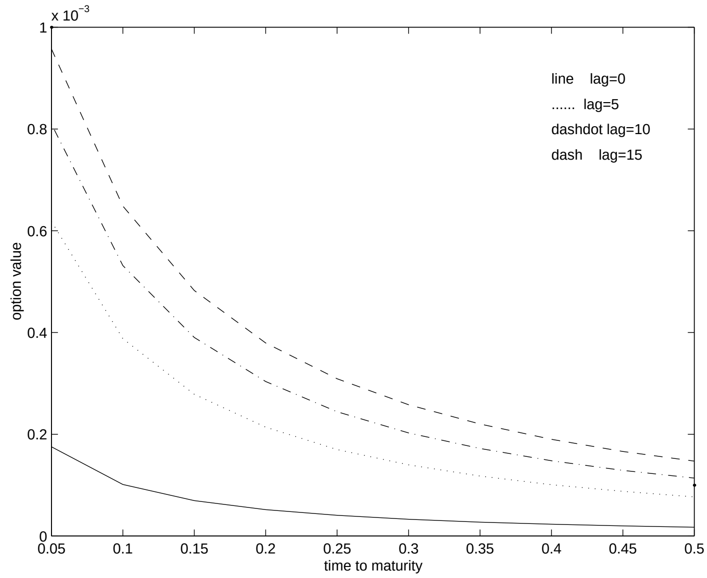

FIGURE 8. Value vs. time to maturity

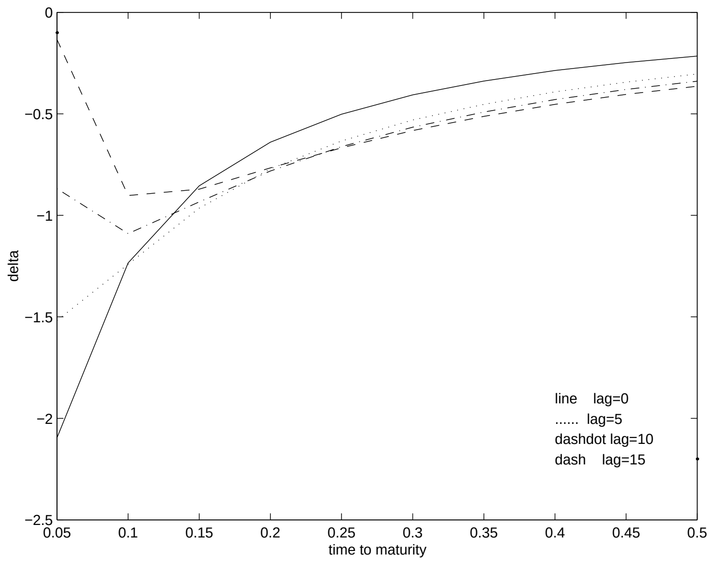

FIGURE 9. Delta vs. time to maturity

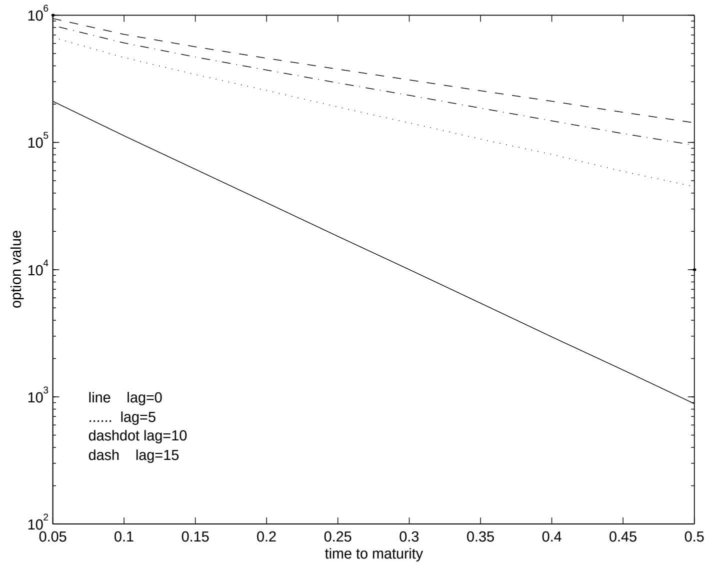

FIGURE 10. Value vs. time to maturity

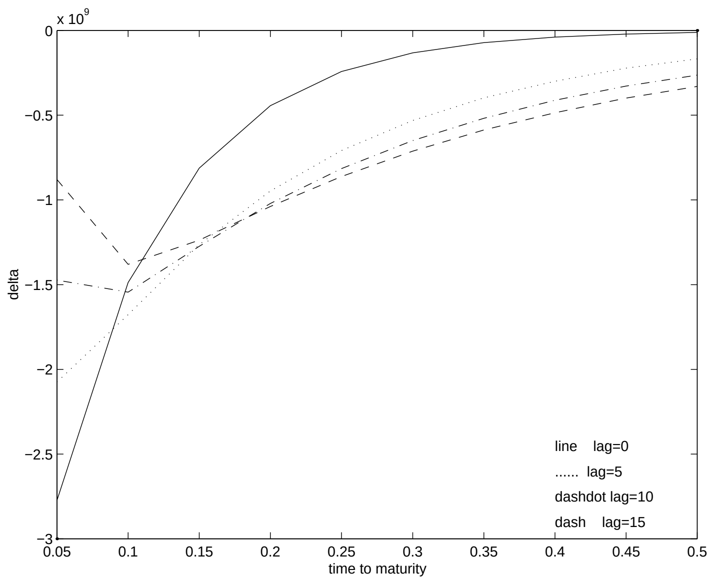

FIGURE 11. Delta vs. time to maturity

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