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Hedging with Stochastic Local Volatility

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Abstract

The delta hedging performance of deterministic local volatility models is poor, with most studies showing that even the simple constant volatility Black-Scholes model performs better. But when the local volatility model is extended to capture stochastic dynamics for the spot volatility process the hedge ratios change. Here we derive the local volatility hedge ratios that are consistent with a stochastic spot volatility and show that the stochastic local volatility model is equivalent to the market model for implied volatilities. We also quantify the hedging error that arises from residual hedging uncertainty and provide an empirical example based on a stochastic normal mixture diffusion model for asset returns.

JEL Classification:

Keywords: Local volatility, stochastic volatility, implied volatility, hedging, dynamic delta hedging, volatility dynamics.

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I. Introduction

Before 1987 implied volatilities from market prices of equity index options were reasonably constant by strike and maturity, and so it was believed that pricing European options with the Black-Scholes (1973) model was sufficient for most purpose. However in the twenty or so years since the global equity crash the pattern of implied volatilities has changed dramatically, with a steeply negatively skewed implied volatility smile surface now typical in equity options and other non-constant implied volatility surfaces for other asset classes.[1] To be consistent with the smile an intuitive choice was to allow the spot variance of the underlying process to be a process itself, possibly correlated with the underlying asset.[2] This was the approach of Hull and White (1987) and Heston (1993), among others. By adding a new source of uncertainty to the model it was possible to fit the observed market options prices. But there was a cost. With two sources of uncertainty in the model, delta hedging was not enough and a market with only the underlying asset and a risk-free money market account was incomplete, since it was no longer possible to replicate the payoff of a simple European option.

Then Dupire (1994), Derman and Kani (1994) and Rubinstein (1994) introduced the concept of local volatility by defining a unique 'deterministic' spot volatility consistent with observed market prices.[3] Using Dupire's equation it was possible to fit any continuous market smile exactly. Since no new source of uncertainty was necessary, delta hedging was possible and the market was complete. However, there were also problems. Several papers have tested the hedging performance of local volatility models and the general finding is that they could perform even worse than the Black-Scholes model. Hence, the usual conclusion is that the assumption of a deterministic spot volatility is too restrictive and that stochastic volatility models are more realistic.

Stochastic and local volatility models have been regarded as two alternative and competing approaches to the same unobservable quantity, the volatility of the underlying asset. The former represents the spot variance as a diffusion or jump-diffusion process and the latter derives forward volatilities that are consistent with a 'snap-shot' of implied volatility at a particular time. But while these approaches are not inconsistent with each other, interestingly the few attempts to unify them into a single theory have not been much developed by further research. The heart of the problem is the assumption of a deterministic spot volatility that is imposed by most local volatility models. However such assumption is not actually necessary for a local volatility model. This

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[1] Here we refer to the implied volatility smile for standard European options consistent with the Black and Scholes' (1973) model. See Rubinstein (1994) for some empirical evidence on the behaviour of implied volatilities before and after 1987.
[2] We use the term 'spot' variance for the variance of the underlying asset price process. That is, the spot volatility is the diffusion coefficient in a geometric Brownian motion model for the underlying asset dynamics. Alternative terms are 'instantaneous' or 'process' variance (and volatility).
[3] The spot volatility is said to be deterministic when it is a function of time and asset price level only.
Forward volatility is a forecast of what the spot volatility will be at some future time. See Section II for more details on terminology and notation. A deterministic spot volatility the forward volatility is a function of time and the underlying price only)

was recognized by Dupire (1996) and Kani, Derman and Kamal (1997). These authors define the local variance (i.e. the square of the local volatility) as the expectation of the future spot variance conditional on a given asset price level:

$$\sigma_{LV}^2(t,S) = E^0[\sigma^2(t,S(t),\mathbf{x}(t)) | S(t) = S] \text{ at time } t_0 < t \quad (1)$$

where E^0 denotes the expectation conditional on a filtration $\Im_0$, which includes all information up to time t_0, and $\mathbf{x}(t) = \{x_1, \ldots, x_n\}$ is a vector of all other sources of uncertainty which influence the spot volatility process at time t under the risk-neutral probability.[4] Therefore, even when the spot volatility is stochastic, the local volatility function is still a deterministic function of time t and asset level S in the future.[5] Clearly this definition of local volatility is consistent with any univariate diffusion stochastic volatility model in the literature (e.g. Hull and White, 1987, and Heston, 1993) since $\mathbf{x}(t)$ can be any arbitrage-free set of processes consistent with options prices. Hence Dupire (1996) named model (1) the 'unified theory of volatility'.

So what is the problem with local volatility models? It is precisely the residual uncertainty from $\mathbf{x}(t)$ after taking the expectation in (1), and its influence on the spot volatility. This uncertainty is transferred to the local volatility surface itself. That is, although locally (i.e. at each calibration) the local volatility surface is indeed a deterministic function of t and S, over time that surface moves in an unpredictable manner, i.e. its dynamics are stochastic. The residual uncertainty from $\mathbf{x}(t)$ does not just disappear from the model. In effect, the assumption that the spot volatility is deterministic is inconsistent with any dynamics for local volatility. This explains the poor empirical results on the hedging performance of local volatility models.

In this paper we develop a general model for stochastic local volatility. At each trading date, we assume that a parameterised deterministic local volatility model is calibrated to a smile surface from market prices of Europen calls and puts. We introduce stochastic dynamics for these parameters over time, and hence additional uncertainty into the spot volatility. We then show that the delta, gamma and theta of the stochastic local volatility model are equal to the equivalent deterministic local volatility hedge ratio plus and adjustment factor which depends on the degree of uncertainty in the local volatility parameters and on their correlation with the underlying price. We also show that the stochastic local volatility model is equivalent to the 'market models' of implied volatility that have recently been studied by Schonbucher (1999), Brace *et al* (2001), Ledoit *et al* (2002) and Daglish, Hull and Suo (2003).[6] In particular, the two models have identical hedge ratios. Hence recent advances in both stochastic and local volatility have led to a unified model for the two approaches; the stochastic local volatility model that is introduced in this paper.

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[4] For the moment, assume $\mathbf{x}(t)$ is uncorrelated with $S(t)$. Later we relax this assumption.
[5] The proof of this result can be found in the appendix of Kani, Derman and Kamal (1997).
[6] The use of observable variables, i.e. the implied volatilities, provokes comparison with the 'market models' of interest rates introduced by Brace et al, (1997) and Jamshidian (1997).

The remainder of this paper is as follows: Section II introduces stochastic local volatility (SLV); Section III derives the dynamics of the SLV price of a contingent claim; Section IV derives the new hedge ratios; Section V proposes a simple but intuitive method to estimate SLV parameters; Section VI proves the duality between the SLV model and the market model of implied volatilities; Section VII examines a particular SLV model that is based on the lognormal mixture local volatility model of Brigo and Mercurio (2001); and Section VIII summarizes and concludes.

II. From Deterministic Local Volatility to Stochastic Local Volatility

The popular definition of local volatility that was based on the early work of Dupire (1994), Derman and Kani (1994) and Rubinstein (1994) assumes the underlying asset price process follows a geometric Brownian motion with deterministic spot volatility $\sigma(t, S)$ – i.e. a deterministic function of t and S – as:

$$dS = (r-q)Sdt + \sigma(t,S)SdW_S$$

According to this *effective theory* in the words of Kani, Derman and Kamal (1997) the local volatility $\sigma_{LV}(t, S)$ is equal to the spot volatility $\sigma(t, S)$ at any future time t and asset level S in the future. At the same time, Dupire (1994) showed that the local volatility function is uniquely determined from a surface of market prices $f(T, K)$ of standard European options with different strikes and maturities, using the celebrated 'Dupire's formula':

$$\sigma^2_{LV}(t,S)\Big|_{t=T,S=K} = \frac{2\dfrac{\partial f}{\partial T} + (r-q)K\dfrac{\partial f}{\partial K} + qf}{K^2\dfrac{\partial^2 f}{\partial K^2}}$$

where r denotes the risk-free interest rate and q is the dividend yield (both assumed constant). Interestingly, this early definition of local volatility is consistent with the later definition given by (1) in that case that $\sigma(t, S)$ is not a function of $\mathbf{x}(t)$. Moreover, Dupire's formula is consistent with the general 'forward equation' derived by Kani, Derman and Kamal (1997) and satisfied by all standard European options.

However, direct computation of the local volatility function from Dupire's formula using finite difference methods is problematic. The local volatility surface can be very irregular and sensitive to the interpolation methods used between quoted option prices and their extrapolation to boundary values, requiring some 'regularization method' to obtain the smoothest possible fit to the implied volatility surface.[7] Hence more recent work on local volatility has focussed on the use of parametric forms for local volatility functions. In this case the local volatility function is calibrated by changing parameters so that some distance metric

[7] See Bouchouev and Isakov (1997, 1999) and Avallaneda et. al. (1997) amongst others.

between model prices and market prices is minimized and it may not fit quoted prices exactly.[8] This is the case in virtually all the local volatility models studied in the literature. In these models, at any point in time t_0 a set of values $\mathbf{v}(t_0) = \{v_1(t_0), \ldots, v_n(t_0)\}$ for these parameters is calibrated to the current implied volatility surface and used to price and hedge all sort of options, under the assumption of a deterministic spot variance.[9] The assumed underlying asset price process is then:

$$dS = (r-q)Sdt + \sigma(t,S;\mathbf{v}(t_0))SdW_S \quad \text{for all } t > t_0 \tag{2}$$

where $\mathbf{v}(t_0)$ is known at time t_0. Since the spot volatility is still deterministic, from (1) it must equal the local volatility:[10]

$$\sigma_{LV}^2(t,S;\mathbf{v}(t_0)) = \sigma^2(t,S;\mathbf{v}(t_0)) \quad \text{for all } t > t_0 \tag{3}$$

From henceforth this is referred to as the 'deterministic' local volatility model (DLV), since it assumes a deterministic spot volatility.

Of course, in (3) the local volatility will be sensitive to the calibration, at time t_0. So if at time $t_1 > t_0$ the model is re-calibrated, we have:

$$\sigma_{LV}^2(t,S;\mathbf{v}(t_1)) = \sigma^2(t,S;\mathbf{v}(t_1)) \quad \text{for all } t > t_1$$

and this can of course differ from (3) as long as $\mathbf{v}(t_1) \neq \mathbf{v}(t_0)$. Thus the local volatility surface will be stochastic if the calibrated parameters $\mathbf{v}(t)$ are stochastic, a fact that has not been given much attention in the literature.

We now define the spot variance as $\sigma^2(t, S(t); \mathbf{v}(t))$, i.e. a function of t, S and some *stochastic* vector of parameters $\mathbf{v}(t) = \{v_1(t), \ldots, v_n(t)\}$, calibrated at *future* time t. That is, we assume that all uncertainty on the random variables $\mathbf{x}(t)$ in (1) is captured by the parameters $\mathbf{v}(t)$ of a local volatility model. It is then only under the additional assumption that the parameters $\mathbf{v}(t)$ are constant and equal to $\mathbf{v}(t_0)$ that we have the a deterministic local volatility model such as in (3) above. But if we relax such assumption and allow $\mathbf{v}(t)$ to evolve stochastically, from (1) we have:

$$\sigma_{LV}^2(t,S) = E^0\left[\sigma^2(t,S(t);\mathbf{v}(t)) \middle| S(t) = S\right] = \int_{\Omega_t} \sigma^2(t,S;\mathbf{v}) h_t(\mathbf{v} \mid S) d\mathbf{v} \quad \text{at time } t_0 \tag{4}$$

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[8] See Dumas, Fleming and Whalley (1998), Brown and Randall (1999), McIntyre (2001), Brigo and Mercurio (2001,2002), Alexander (2004), and many others.

[9] Here we have assumed that a single set of parameters $\mathbf{v}(t)$ is calibrated to all available options simultaneously with reasonable accuracy. In practice, for a variety of reasons it may be difficult to fit both the smile and the term structure of implied volatilities with just a few parameters. So it is common practice to restrict the calibration to near the money options and to a few maturities. However, restricting the calibration to very few maturities neglects the importance of the term structure and can lead to the wrong local volatility surface.

[10] (3) is derived from (1) using $\mathbf{x}(t) \equiv \mathbf{v}(t) = \mathbf{v}(t_0)$.

In (4), the expectation is conditional on a filtration $\Im_0$, which includes all information up to time t_0. In particular the past history of $S(t)$ and of every additional stochastic factor in the spot variance is included in $\Im_0$ so that the local volatility surface is well-defined for every pair (t, S) with $t > t_0$. The integration is over Ω_t, the space of all *arbitrage-free* values for $\mathbf{v}(t)$ and $\mathbf{v} \in \Omega_t$ is a realization of $\mathbf{v}(t)$.[11] Finally $h_t(\mathbf{v}\,|\,S)$ denotes the multivariate *conditional* density of the random variables $\mathbf{v}(t)$ at time t for a given S and $\Im_0$.[12]

The local volatility surface is then defined at a given time t_0 conditional on some filtration $\Im_0$. Thus it can evolve over time and there is an implicit dependence of this surface on time t, underlying asset price $S(t)$ and other variables $\mathbf{v}(t)$ and their past histories. That is, the local volatility is *stochastic* and we therefore call (4) the 'stochastic local volatility' (SLV) model

From (4) it is also clear that the local volatility and the spot volatility are not the same when $\mathbf{v}(t)$ is stochastic, which contrasts with the earlier definition. But, in Appendix A we show that, if the calibration of a deterministic local volatility model such as in (3) is accurate, (3) and (4) should produce the same local volatility surface so that:

$$\sigma^2(t,S;\mathbf{v}(t_0)) = E^0\left[\sigma^2(t,S(t);\mathbf{v}(t))\middle|S(t)=S\right] \quad \text{for all } t > t_0 \tag{5}$$

which is an important constraint to the permissible dynamics for $\mathbf{v}(t)$. In fact, applying a standard Taylor's series expansion of $\sigma^2(t, S(t); \mathbf{v}(t))$, Appendix A also shows that:

$$\sigma^2(t,S;\mathbf{v}(t_0)) = \sigma^2(t,S;\tilde{\mathbf{v}}(t_0)) + \tfrac{1}{2}\sum_i\sum_j \frac{\partial^2\sigma^2(t,S;\tilde{\mathbf{v}}(t_0))}{\partial v_i \partial v_j} Cov^0\left[v_i(t),v_j(t)\middle|S(t)=S\right] \tag{6}$$

with $\tilde{\mathbf{v}}(t_0) = E^0\left[\mathbf{v}(t)\middle|S(t)=S\right]$. Hence, it follows that $\mathbf{v}(t_0) \neq E^0\left[\mathbf{v}(t)\middle|S(t)=S\right]$. That is, there is no obvious relationship between the calibrated parameters at time t_0, $\mathbf{v}(t_0)$, and the future parameters $\mathbf{v}(t)$ unless the spot variance is either deterministic or a linear function of $\mathbf{v}(t)$, when the second term of (6) is zero.

It is important to note that although the local volatility surface derived from a DLV model can fit the current smile, the assumption of a deterministic spot volatility is unrealistic. This can affect the price of exotics and other types of derivatives, and hedge ratios for all options. We shall discuss this in Section IV. Thus, in the

[11] Note that, depending on the functional form assumed for the spot variance, some values for $\mathbf{v}(t)$ can introduce arbitrage opportunities if they violate at least one of the no-arbitrage conditions mentioned in Appendix B. Therefore, those values are excluded from Ω_t in an arbitrage-free market.

[12] From probability theory, we know that $h_t(\mathbf{v}\,|\,S)$ is related to the joint density of S and $\mathbf{v}$ by $h_t(\mathbf{v}\;|S) = h_t(\mathbf{v}, S)/g_t(S)$, where $g_t(S)$ is the unconditional density of S at time t. There is an extensive literature on the estimation of $g_t(S)$ – see Breeden and Litzenberger (1978) or Brunner and Hafner (2003) just to mention a few – but there is no easy way to calculate the joint density $h_t(\mathbf{v}, S)$ unless we have a specific model for the spot variance.

same way stochastic volatility models have extended the Black-Scholes model to more realistic volatility processes, deterministic local volatility models must be extended to account for such variability in parameters.[13] The next section formalises the stochastic local volatility model and derives the corresponding dynamics for the price of a contingent claim. These results will be used to derive new hedge ratios and to prove the duality between the SLV model and a 'market model' of stochastic implied volatility.

III. Stochastic Local Volatility Price Dynamics

Assume the asset price follows a geometric Brownian motion under the risk-neutral measure *Q*:[14]

$$dS = (r-q)Sdt + \sigma(t, S; v_1, ... v_n)SdW_S \quad (7)$$

with continuous and adapted spot volatility satisfying the no-arbitrage conditions in Appendix B and, for all $T > t_0$, $\int_{t_0}^{T} \sigma^2(t, S; v_1, ... v_n)dt < \infty$ almost surely. Also assume the continuously-compounded risk-free rate r and dividend yield q are constant, and that $\mathbf{v}(t) = [v_1(t), v_2(t), \ldots v_n(t)]^T$ is an n x 1 vector of parameters of a deterministic local volatility (DLV) model such as in Section II, each of them stochastic and correlated with the asset price S and with each other. The spot volatility $\sigma(t, S; \mathbf{v}(t))$ is therefore stochastic.

Now assume the dynamics for each parameter v_i in $\mathbf{v}(t)$ under the risk-neutral measure follow:[15]

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[13] An analogy with the Heath-Jarrow-Morton model for interest rates is enlightening. The spot variance and local variance can be seen as analogous to the spot interest rate and the forward rate in the HJM model, so that the local volatility surface is the analogue to the forward yield curve. See Kani *et al* (1997).

[14] We can use the risk-neutral measure whenever the numeraire is the money market account and all discounted tradable assets are martingales under that measure. It does not matter how many Brownians we have in the dynamics since it is always possible to create new forward-like contracts to hedge all Brownians. We can always introduce new tradable hedging instruments as long as they can be priced under some martingale measure.

[15] In (8) we allow all coefficients to depend on all variables in the model so that (8) can be as general as possible, including a variety of reasonable implementations for each parameter v_i, e.g. arithmetic or geometric Brownian motions, mean-reverting, etc. There is also an implicit dependence on the filtration $\Im_0$ at time t_0.

with

$$\left.\begin{aligned} dv_i &= \alpha_i(t,S,\mathbf{v})dt + \beta_i(t,S,\mathbf{v})dZ_i \\ dZ_i &= \varrho_{i,S}(t,S,\mathbf{v})dW_S + \sqrt{1-\varrho_{i,S}^2(t,S,\mathbf{v})}\,dW_i \\ dZ_i dZ_j &\overset{a.s.}{\rightarrow} \varrho_{i,j}(t,S,\mathbf{v})dt \quad \text{and} \quad dW_i dW_S \overset{a.s.}{\rightarrow} 0 \quad \text{for } i,j \in \{1, 2\ldots n\} \end{aligned}\right\} \tag{8}$$

satisfying, almost surely, the usual regularity conditions and for all $T > t_0$:

$$\int_{t_0}^{T} \left|\alpha_i(t,S,\mathbf{v})\right| dt < \infty \quad \text{and} \quad \int_{t_0}^{T} \beta_i^2(t,S,\mathbf{v})dt < \infty$$

so that $\varrho_{i,j} \in [-1, 1]$ is the correlation between variations in v_i and v_j and $\varrho_{i,S} \in [-1, 1]$ is the correlation between variations in v_i and S.

Together (7) and (8) provide the full specification of the SLV model. Note that here we actually model the stochastic parameters of a DLV model. The advantage of this approach is that it relates nicely to the extant literature on local volatility, addressing the typical parameter instability and its consequence for pricing and hedging of derivatives. To simplify notation, henceforth the dependence of the α's, β's and ϱ's on t, S and **v** are omitted when they are clear from the context.[16]

Now define the local volatility price for a contingent claim as $f_L = f_L(t, S(t); \mathbf{v}(t) \mid \Im_0)$, calibrated at time $t_0 < t$ (i.e. $\mathbf{v}(t_0)$ is included in the filtration $\Im_0$). Since $\mathbf{v}(t)$ contains the *future* parameters of a DLV model for any $t > t_0$, the price f_L must satisfy *locally* the Black-Scholes pde for all $t > t_0$:[17]

$$\frac{\partial f_L(t,S;\mathbf{v})}{\partial t} + (r-q)S\frac{\partial f_L(t,S;\mathbf{v})}{\partial S} + \tfrac{1}{2}\sigma(t,S;\mathbf{v})^2 S^2 \frac{\partial^2 f_L(t,S;\mathbf{v})}{\partial S^2} = r f_L(t,S;\mathbf{v}) \tag{9}$$

where $S \in \Re^+$ is a realization of $S(t)$, $\mathbf{v} \in \Omega_t$ is a realization of $\mathbf{v}(t)$ and the filtration $\Im_0$ has been omitted for convenience. Note that (9) holds locally, i.e. assuming the DLV model is re-calibrated at each time t. However, since the calibrated parameters are likely to be different at each re-calibration, we assume $\mathbf{v}(t)$ is stochastic and defined such as in (8) above. Hence, that uncertainty will affect the actual dynamics of the claim price f_L, given in the following theorem. The proof of the theorem as well as of most results in this paper is available in Appendix A.

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[16] As before, formally $\mathbf{v} \in \Omega_t$ is a realization of $\mathbf{v}(t)$, where Ω_t is the space of arbitrage-free values for $\mathbf{v}(t)$.

[17] Within a deterministic local volatility model, $\mathbf{v}(t)$ is assumed constant, hence f_L can be expressed as a function of t and S only. Then, (12) follows from a standard application of Ito's lemma and risk-neutrality argument.

Theorem 1

Under assumptions (7) and (8) and satisfying all no-arbitrage conditions above, the evolution of the model price $f_L(t, S(t); \mathbf{v}(t) | \Im_0)$ at every time t, $t_0 < t < T$, under the risk-neutral probability is given by:

$$df_L = rf_L dt + \left(\sigma S \frac{\partial f_L}{\partial S} + \sum_i \beta_i \varrho_{i,S} \frac{\partial f_L}{\partial v_i} \right) dW_S + \sum_i \beta_i \sqrt{1 - \varrho_{i,S}^2} \frac{\partial f_L}{\partial v_i} dW_i \tag{10}$$

satisfying the regularity conditions:

$$\int_{t_0}^{T} \left(\frac{\partial f_L}{\partial S} \right)^2 dt < \infty \qquad \text{and} \qquad \int_{t_0}^{T} \left(\frac{\partial f_L}{\partial v_i} \right)^2 dt < \infty, \ i \in \{1, 2 \dots n\}$$

almost surely, and the coefficients of (8) must satisfy the following drift condition at every t:

$$\sum_i \left(\alpha_i \frac{\partial f_L}{\partial v_i} + \sigma(t, S; \mathbf{v}) S \beta_i \varrho_{i,S} \frac{\partial^2 f_L}{\partial v_i \partial S} + \tfrac{1}{2} \sum_j \varrho_{i,j} \beta_i \beta_j \frac{\partial^2 f_L}{\partial v_i \partial v_j} \right) = 0 \tag{11}$$

Proof: See appendix A.

Therefore, when the parameters $\mathbf{v}(t)$ of a deterministic local volatility model are stochastic, the claim price has a multi-factor dynamics including one Brownian motion from the underlying asset price dynamics (7) and another Brownian motion for each stochastic parameter v_i in the model. This result has important implications for hedging, the focus of the next section.

IV. Hedging with Local Volatility revisited

Since the stochastic local volatility (SLV) model (7) and (8) introduces new sources of randomness, perfect hedging is complex and would require a combination of several traded options. For instance, Kani, Derman and Kamal (1997) define a 'volatility gadget' as a small portfolio of traded options, combined in such a way that it would be possible to hedge any specific region from the local volatility surface. Then, by combining these gadgets a multitude of hedging possibilities are available to the volatility trader. In this paper, however, we do not focus on perfect hedging. Instead, here we show that the uncertainty about the local volatility surface in the future – in terms of its parameters – can explain the poor hedging performance that has been reported in the literature on local volatility models.

Dumas *et al* (1998) and McIntyre (2001) find that delta hedges derived from local volatility models perform worse than even the simple Black-Scholes (1973) delta hedge, but Coleman *et al* (2001) find the opposite for long hedging periods. More importantly, Hagan *et al* (2002) claim that local volatility deltas will be inaccurate since these models fail to capture proper dynamics of the implied volatility. However readers should be very cautious of these findings. The deltas that were applied in these papers did not include an adjustment for the possible movement in local volatility over time. In common with the majority of research literature in this field, the local volatility models studied were incomplete in this respect.

For instance, Dumas *et al* (1998) assume and test several different parametric or semi-parametric forms for the local volatility function. They calibrate the parameters to S&P 500 index options prices on a particular date, repeating this on a weekly basis and hence compare the hedging performance of the local volatility models with that of the Black-Scholes model. Their conclusion is that Black-Scholes hedge ratios appear to be more reliable than those obtained from the local volatility models they tested. However, they do not explore the impact of the instability of the local volatility surface (as implied from the instability of the calibrated parameters) on the hedge ratios. This could be the main reason for their disappointing conclusion. Indeed we now show that the hedge ratios derived from such a view of volatility are biased, and the intuition behind that bias is the implicit and unrealistic assumption of a static local volatility surface.

The next theorem shows that the SLV model requires an adjustment for the deterministic local volatility (DLV) hedge ratios. In this theorem, $\delta_L = \partial f_L/\partial S$, $\gamma_L = \partial^2 f_L/\partial S^2$ and $\Theta_L = \partial f_L/\partial t$ are the DLV hedge ratios calculated at time $t_0 < t$ using a calibrated local volatility surface, i.e. for a given $\mathbf{v}(t_0)$.

Theorem 2

Under the SLV model (7) and (8) the first and second order sensitivities of the claim price $f_L(t, S; \mathbf{v}\,|\,\Im_0)$ at time t_0 with respect to S, and the first order sensitivity to time t, are given by:

$$\delta_{SLV}(t,S;\mathbf{v}) = \delta_L(t,S;\mathbf{v}) + \sum_i \frac{\beta_i \varrho_{i,S}}{\sigma(t,S;\mathbf{v})S} \frac{\partial f_L}{\partial v_i} \tag{12-a}$$

$$\gamma_{SLV}(t,S;\mathbf{v}) = \gamma_L(t,S;\mathbf{v}) + \sum_i \frac{\beta_i \varrho_{i,S}}{\sigma(t,S;\mathbf{v})S} \left(2\frac{\partial^2 f_L}{\partial S \partial v_i} - \frac{1}{S}\frac{\partial f_L}{\partial v_i} + \sum_j \frac{\beta_j \varrho_{j,S}}{\sigma(t,S;\mathbf{v})S} \frac{\partial^2 f_L}{\partial v_i \partial v_j} \right) \tag{12-b}$$

$$\Theta_{SLV}(t,S;\mathbf{v}) = \Theta_L(t,S;\mathbf{v}) + \sum_i \frac{\partial f_L}{\partial v_i} \left(\alpha_i - (r-q)\frac{\beta_i \varrho_{i,S}}{\sigma(t,S;\mathbf{v})} + \tfrac{1}{2}\sigma(t,S;\mathbf{v})\beta_i \varrho_{i,S} \right) \tag{12-c}$$

Proof: See Appendix A.

The intuition behind Theorem 2 is as follows: because the vector $\mathbf{v}(t)$ is stochastic the local volatility surface also evolves stochastically; thus a correction term must be added to hedge ratios to account for correlation between movements of each v_i and the asset price S. In effect we can split each hedge ratio into two parts: a sensitivity derived from the standard 'deterministic' view of local volatility (i.e. calibrated to the smile at a particular time) and a correction factor due to the dynamics of the local volatility parameters $\mathbf{v}(t)$.

From Theorem 2 it follows that not only is the traditional delta hedging not perfect but it is also invalid unless we use the correct delta given by (12-a). Moreover, as shown in Appendix A, it is possible to quantify the hedging error under the assumption of a deterministic spot volatility:

$$\Lambda_{DLV} = \sum_i \int_{t_0}^{T} \beta_i \frac{\partial f_L}{\partial v_i} dZ_i \quad \text{for some } T > t_0 \tag{13}$$

Then, from (13) we imply that the hedging error Λ_{DLV} is the sum of Ito's stochastic integrals, each of them with zero expected value but non-zero variance, so that:

$$E[\Lambda_{DLV}] = 0 \quad \text{and} \quad Var[\Lambda_{DLV}] = \sum_i \sum_j Cov\left[\int_{t_0}^{T} \beta_i \frac{\partial f_L}{\partial v_i} dZ_i, \int_{t_0}^{T} \beta_j \frac{\partial f_L}{\partial v_j} dZ_j\right]$$

Note that the only possibility to have Var[Λ_{DLV}] = 0 above (i.e. perfect hedging) is if $\beta_i = 0$ or $\partial f_L/\partial v_i = 0$ for all *i*, which clearly requires a deterministic spot volatility. Hence, whilst the delta hedge strategy can be unbiased (zero expected hedging error), it cannot be perfect when the spot volatility is stochastic.

Likewise, it is also possible to quantify the hedging error using the correct delta from Theorem 2 (also derived in Appendix A):

$$\Lambda_{SLV} = \sum_i \int_{t_0}^{T} \beta_i \sqrt{1-\varrho_{i,S}^2} \frac{\partial f_L}{\partial v_i} dW_i \quad \text{for some } T > t_0 \tag{14}$$

which, again, is the sum of Ito's stochastic integrals so that:

$$E[\Lambda_{SLV}] = 0 \quad \text{and} \quad Var[\Lambda_{SLV}] = \sum_i \sum_j Cov\left[\int_{t_0}^{T} \beta_i \sqrt{1-\varrho_{i,S}^2} \frac{\partial f_L}{\partial v_i} dW_i, \int_{t_0}^{T} \beta_j \sqrt{1-\varrho_{j,S}^2} \frac{\partial f_L}{\partial v_j} dW_j\right]$$

The delta hedge strategy is again unbiased but not perfect. However, there is an important distinction to be made here. The hedging error Λ_{DLV} can be written in terms of Λ_{SLV} by replacing dZ_i from (8). That is:

$$\Lambda_{DLV} = \sum_i \int_{t_0}^{T} \beta_i \frac{\partial f_L}{\partial v_i} \left(\varrho_{i,S} dW_S + \sqrt{1-\varrho_{i,S}^2} dW_i\right) = \sum_i \int_{t_0}^{T} \beta_i \frac{\partial f_L}{\partial v_i} \varrho_{i,S} dW_S + \Lambda_{SLV}$$

so that the variances are related by:

$$Var[\Lambda_{DLV}] = Var[\Lambda_{SLV}] + Var\left[\sum_i \int_{t_0}^{T} \beta_i \frac{\partial f_L}{\partial v_i} \varrho_{i,S} dW_S\right] \tag{15}$$

where we have used the fact that dW_S and dW_i are uncorrelated by definition.[18] Therefore, although using the correct delta (12-a) does not resolve all uncertainty in the model, it should at least reduce the total variance of the hedging error, improving the overall hedging performance.

Finally, all results above are based on the assumption that (7) and (8) are a good approximation of reality. In particular, the underlying asset price is assumed to follow a continuous process as in (7). Hence, if for instance the price process is discontinuous (with jumps), the expressions for the hedging error above may not hold and the delta hedge strategy may be even biased with non-zero expected value for the hedging error.

V. Model Estimation

This section considers how the model parameters in (8) can be estimated in practice. Proper estimation of the model will entail advanced econometric and optimisation techniques: it involves a calibration over a time series of cross-sectional option prices for several strikes and maturities simultaneously. This is beyond the scope of the present paper. Nevertheless, we would like to illustrate a practical example of the model in Section VII. Here we propose a 'quick-and-dirty' approach that splits the problem into two parts: the calibration of the deterministic local volatility model on a snap-shot of option prices for each day separately, and a daily time series analysis of calibrated parameters.

In the following theorem we assume a parametric local volatility model has been calibrated at m points in time $\{t_1, \ldots, t_m\}$ prior to time t_0. We analyse the time series of the calibrated parameters and in particular the $m \times (n+1)$ matrix $X = [\Delta S\ \Delta v_1\ \Delta v_2 \ldots \Delta v_n]_m$ of variations in each of the risk factors, i.e. the asset price S and the model parameters v_i ($i = 1, \ldots, n$). Hence we obtain an estimate of the sample covariance matrix $X^T X/m$. The elements of this matrix are then used to approximate the correction factors for the SLV delta and gamma in Theorem 2 as follows:

Theorem 3

Given a time series of calibrated deterministic local volatility surfaces up to time t_0 the correction factors for the delta and gamma in Theorem 2 may be approximated as follows:

$$\delta_{SLV}(t,S;\mathbf{v}) \approx \delta_L(t,S;\mathbf{v}) + \sum_i \frac{Cov(\Delta S, \Delta v_i)}{Var(\Delta S)} \frac{\partial f_L}{\partial v_i}$$

$$\gamma_{SLV}(t,S;\mathbf{v}) \approx \gamma_L(t,S;\mathbf{v}) + \sum_i \frac{Cov(\Delta S, \Delta v_i)}{Var(\Delta S)} \left(2\frac{\partial^2 f_L}{\partial S \partial v_i} - \frac{1}{S}\frac{\partial f_L}{\partial v_i} + \sum_j \frac{Cov(\Delta S, \Delta v_j)}{Var(\Delta S)} \frac{\partial^2 f_L}{\partial v_i \partial v_j} \right).$$

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[18] Equation (15) may not be exactly true since the α's, β's and ϱ's are also functions of S and $\mathbf{v}$.

where the α's, β's and ϱ's in (8) are assumed constant. These approximations are accurate provided the following no-arbitrage condition is satisfied:

$$\sum_i \left(\frac{1}{m} \sum_{t=1}^{m} \Delta v_{i,t} \frac{\partial f_L}{\partial v_i} - \frac{Cov(\Delta S, \Delta v_i)}{\sqrt{Var(\Delta S)}} \lambda \sqrt{\Delta t} + Cov(\Delta S, \Delta v_i) \frac{\partial^2 f_L}{\partial v_i \partial S} + \tfrac{1}{2} \sum_j Cov(\Delta v_i, \Delta v_j) \frac{\partial^2 f_L}{\partial v_i \partial v_j} \right) \approx 0$$

where λ is the market price of risk.

Proof: See Appendix A.

Here we assume that sample moments approximate population moments, and this is perhaps rather strong. Nevertheless Theorem 3 provides a pragmatic method to adjust local volatility hedge ratios so that they *do* account for our uncertainty about the future calibrated parameters. But it is only an approximation. While it is quite standard to assume the spot variance is constant over a small time-step Δt, it is clearly poor to approximate it by the historical variance over the sample. So any application of Theorem 3 should be considered with care and justified only when its arbitrage-free condition is at least approximately satisfied.

The main issue here is that normally other volatility models – local volatility and stochastic volatility models – never take account of the uncertainty on the calibrated parameters. Thus whilst the hedge ratio adjustments in Theorem 3 are only an approximation to the exact values stated in Theorem 2, the historical covariance matrix might at least identify the sign of these adjustments. Section VII applies these approximations within the context of a particular local volatility model and investigates whether the adjusted deltas are indeed more accurate than the 'standard' DLV deltas.

VI. Implied Volatility Dynamics

Recent work of Dupire (2003) derives a general relationship between local volatilities and Black-Scholes implied volatilities, in which implied volatilities are nothing but weighted averages of local volatilities:[19]

$$\theta_M^2(T,K) = \frac{\int_{t_0}^{T}\int_{0}^{\infty} \sigma_{LV}^2(t,S) S^2 \gamma_{BS}(T,K;t,S) g_t(S) dS dt}{\int_{t_0}^{T}\int_{0}^{\infty} S^2 \gamma_{BS}(T,K;t,S) g_t(S) dS dt} \text{ at time } t_0 \quad (16)$$

where $\theta_M(T,K)$ is the Black-Scholes implied volatility for strike K and maturity T implied from market prices at time t_0, σ_{LV} is the local volatility given by (4), γ_{BS} is the Black-Scholes gamma of the option and $g_t(S)$ is the

[19] The proof is provided in Appendix A.

unconditional density of S at time t, $t_0 < t < T$.[20] Since implied volatilities are also known to evolve stochastically over time, Equation (16) raises the question of duality between a stochastic local volatility model and a stochastic implied volatility model. That duality motivates the present section.

Equation (16) also indicates which options are more relevant to the calibration of the local volatility parameters. Since gamma is the only variable depending on strike and maturity in the right-hand side of (16), options with high gamma will have the greatest impact on the calibration. But, for standard European options, the gamma is higher near the money and close to maturity. Hence, any calibration which does not take into account at least those options is flawed. Such a requirement is particularly important for the selection of maturities, refuting the common practice of restricting the calibration to fewer maturities when the model fails to fit the term structure of implied volatilities.[21]

We now derive an explicit relationship between the dynamics of the local volatility price for a standard European option and the evolution of the associated implied volatility. For a vanilla European option with strike K and maturity T, the local volatility price of this option at time t when the asset price is S is denoted by $f_L(K, T; t, S, \mathbf{v})$. Similarly, when the Black-Scholes (BS) implied volatility is θ, we denote the BS price of this option at time t when the asset price is S by $f_{BS}(K, T; t, S, \theta)$. We define the *market* implied volatility $\theta_M = \theta_M(K, T; t, S)$ as that θ such that the Black-Scholes model price equals the observed market option price.[22] Since market prices are observable, market implied volatilities are observable.

Now assume that the local volatility model is calibrated to an implied volatility surface at each time t. Then the *local* implied volatility $\theta = \theta(K, T; t, S, \mathbf{v})$ is defined by equating the local volatility price to the BS price:[23]

$$f_L(K,T;t,S,\mathbf{v}) = f_{BS}(K,T;t,S,\theta(K,T;t,S,\mathbf{v})) \tag{17}$$

The following results are derived using local implied volatilities and not market implied volatilities. That is, we seek the relationship between a stochastic local volatility function and the associated local implied volatilities on the assumption that the SLV model can fit market options prices on any day with acceptable accuracy.

To prove the theorem of this section, we need two lemmas that focus on the sensitivities of the local implied volatility surface $\theta(K, T; t, S, \mathbf{v})$ to changes in t, S and $\mathbf{v}$.[24] These depend on the dynamics of the implied

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[20] The unconditional risk-neutral probability of $S(t)$ is discussed by Breeden and Litzenberger (1978), Jackwerth (1999) and Brunner and Hafner (2003), among others.
[21] An explicit local volatility model for the term structure of implied volatilities is introduced by Alexander (2004).
[22] It should not matter whether we use calls or puts, since the implied volatility is the same due to the put-call parity.
[23] Note that both the local implied volatility and the local volatility price are conditional on the filtration $\mathfrak{I}_0$.
[24] The sensitivities to K and T are intuitive but since these are not central to the main theorem we relegate these results to the appendix C.

volatility surface and they cannot be derived from a snap-shot of the surface alone. Many authors have performed empirical investigations of the sensitivity to S (see Derman and Kamal, 1997; Skiadopoulos *et al*, 1999; Alexander, 2001; Cont and da Fonseca, 2001; Fengler *et al*, 2003 and others). In this paper, however, we will focus on a theoretical approach. That is, we assume that the parameters of a local volatility model evolve stochastically as specified in (8) and we derive the implied volatility dynamics that are consistent with this.

Lemma 1

Denote the BS model price sensitivities by $\delta_{BS} = \partial f_{BS}/\partial S$; $\gamma_{BS} = \partial^2 f_{BS}/\partial S^2$; $\Theta_{BS} = \partial f_{BS}/\partial t$; $\Upsilon_{BS} = \partial f_{BS}/\partial\theta$; $\varkappa_{BS} = \partial^2 f_{BS}/\partial\theta^2$; and $\Omega_{BS} = \partial^2 f_{BS}/\partial S\partial\theta$. For the SLV model price the deterministic sensitivities (i.e. for a *fixed* $\mathbf{v}(t)$) are denoted $\delta_L = \partial f_L/\partial S$, $\gamma_L = \partial^2 f_L/\partial S^2$ and $\Theta_L = \partial f_L/\partial t$.[25] Then the local implied volatility function $\theta(K, T; t, S, \mathbf{v})$ has sensitivities to t, S and $\mathbf{v}$ given by:

$$\frac{\partial\theta(K,T;t,S,\mathbf{v})}{\partial t} = \frac{\Theta_L(K,T;t,S,\mathbf{v}) - \Theta_{BS}(K,T;t,S,\theta)}{\Upsilon_{BS}(K,T;t,S,\theta)}$$

$$\frac{\partial\theta(K,T;t,S,\mathbf{v})}{\partial S} = \frac{\delta_L(K,T;t,S,\mathbf{v}) - \delta_{BS}(K,T;t,S,\theta)}{\Upsilon_{BS}(K,T;t,S,\theta)}$$

$$\frac{\partial\theta(K,T;t,S,\mathbf{v})}{\partial v_i} = \frac{1}{\Upsilon_{BS}(K,T;t,S,\theta)}\frac{\partial f_L(K,T;t,S,\mathbf{v})}{\partial v_i}$$

$$\frac{\partial^2\theta(K,T;t,S,\mathbf{v})}{\partial S^2} = \frac{1}{\Upsilon_{BS}}\left[\gamma_L - \gamma_{BS} - \varkappa_{BS}\left(\frac{\delta_L - \delta_{BS}}{\Upsilon_{BS}}\right)^2 - 2\Omega_{BS}\left(\frac{\delta_L - \delta_{BS}}{\Upsilon_{BS}}\right)\right]$$

$$\frac{\partial^2\theta(K,T;t,S,\mathbf{v})}{\partial S\partial v_i} = \frac{1}{\Upsilon_{BS}}\left[\frac{\partial^2 f_L}{\partial S\partial v_i} - \left(\frac{1}{\Upsilon_{BS}}\frac{\partial f_L}{\partial v_i}\right)\left(\Omega_{BS} + \varkappa_{BS}\frac{\delta_L - \delta_{BS}}{\Upsilon_{BS}}\right)\right]$$

$$\frac{\partial^2\theta(K,T;t,S,\mathbf{v})}{\partial v_i\partial v_j} = \frac{1}{\Upsilon_{BS}}\left[\frac{\partial^2 f_L}{\partial v_i\partial v_j} - \frac{1}{(\Upsilon_{BS})^2}\varkappa_{BS}\frac{\partial f_L}{\partial v_i}\frac{\partial f_L}{\partial v_j}\right]$$

Proof: Differentiate (17) with respect to t, S and each v_i and apply the chain rule in the right-hand side whenever necessary. For instance:

$$\frac{\partial f_L}{\partial S} = \frac{\partial f_{BS}}{\partial S} + \frac{\partial f_{BS}}{\partial\theta}\frac{\partial\theta}{\partial S} \Rightarrow \frac{\partial\theta}{\partial S} = \frac{\delta_L - \delta_{BS}}{\Upsilon_{BS}}$$

$$\frac{\partial^2\theta}{\partial S^2} = \frac{\partial}{\partial S}\left(\frac{\delta_L - \delta_{BS}}{\Upsilon_{BS}}\right) = -\frac{\delta_L - \delta_{BS}}{(\Upsilon_{BS})^2}\left(\Omega_{BS} + \varkappa_{BS}\frac{\partial\theta}{\partial S}\right) + \frac{1}{\Upsilon_{BS}}\left(\gamma_L - \gamma_{BS} - \Omega_{BS}\frac{\partial\theta}{\partial S}\right)$$

and so forth. ■

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[25] Note that in the SLV model, the sensitivities δ_L, γ_L and Θ_L are *not* the local volatility hedge ratios delta, gamma and theta. See Section IV.

Lemma 2

Any local implied volatility $\theta(K, T; t, S, \mathbf{v})$ for an European option with strike K and maturity T, must satisfy the following:

$$\frac{\partial\theta}{\partial t}+\left(r-q-\sigma^2(t,S;\mathbf{v})\frac{d_2}{\theta\sqrt{\tau}}\right)S\frac{\partial\theta}{\partial S}+\tfrac{1}{2}\sigma^2(t,S;\mathbf{v})S^2\left(\frac{\partial^2\theta}{\partial S^2}+\frac{d_1 d_2}{\theta}\left(\frac{\partial\theta}{\partial S}\right)^2\right)+\tfrac{1}{2}\frac{1}{\theta\tau}\left(\sigma^2(t,S;\mathbf{v})-\theta^2\right)=0$$

where $\tau = T - t > 0$ and d_1 and d_2 are such as in the Black-Scholes formula:

$$d_1=\frac{\ln M}{\theta\sqrt{\tau}}+\tfrac{1}{2}\theta\sqrt{\tau}\qquad d_2=d_1-\theta\sqrt{\tau}\qquad M=\frac{Se^{-q\tau}}{Ke^{-r\tau}}.$$

Proof: See Appendix A.

Lemma 2 describes the dynamics of implied volatility that are consistent with any parametric local volatility model. Note that the pde in Lemma 2 has no partial derivative on the elements of **v**. But of course $\theta(K, T; t, S, \mathbf{v})$ is not independent of **v** because it depends on the spot volatility $\sigma(t, S; \mathbf{v})$. In effect, equation (16) must be a solution of Lemma 2 if we replace the risk neutral density $g_t(S)$ by the model density $g_{L,t}(S)$, discussed in Appendix B. Besides, clearly the implied volatility surface *can* move over time even when local volatility parameters are constant over time, which is the implicit assumption in most local volatility models. Thus whilst these models are not inconsistent with movement in implied volatilities over time, the permissible movements in implied volatility are very restricted.

The next theorem now states the general relationship between the stochastic local volatility model and the dynamics of implied volatility. The corollary to this theorem is interesting because it shows that the SLV model is *equivalent* to the dynamic model for implied volatilities introduced by Schonbucher (1999).

Theorem 4

Under assumptions (7) and (8) the dynamics of $\theta = \theta(K, T; t, S, \mathbf{v})$, the local implied volatility for a European option with strike K and maturity T, are given by:

$$d\theta=\left[\tfrac{1}{2}\frac{1}{\theta(T-t)}\left(\theta^2-\sigma^2\right)+\sigma(t,S;\mathbf{v})\frac{d_2}{\theta\sqrt{T-t}}\psi-\tfrac{1}{2}\frac{d_1 d_2}{\theta}\eta^2\right]dt+\sigma(t,S;\mathbf{v})S\frac{\partial\theta}{\partial S}dW_S+\sum_i\frac{\partial\theta}{\partial v_i}\beta_i dZ_i \quad (18)$$

satisfying:

$$\int_{t_0}^{T}\left(\frac{\partial\theta}{\partial S}\right)^2 dt<\infty,\quad \int_{t_0}^{T}\left(\frac{\partial\theta}{\partial v_i}\right)^2 dt<\infty,\quad \int_{t_0}^{T}\psi^2 dt<\infty,\quad \int_{t_0}^{T}\eta^2 dt<\infty \quad \text{and}$$

$$\int_{t_0}^{T}\left|\tfrac{1}{2}\frac{1}{\theta(T-t)}\left(\theta^2-\sigma^2\right)+\sigma\frac{d_2}{\theta\sqrt{T-t}}\psi-\tfrac{1}{2}\frac{d_1 d_2}{\theta}\eta^2\right|dt<\infty$$

with:

$$\psi dt = d\theta dW_S = \left(\sigma(t,S;\mathbf{v})S\frac{\partial\theta}{\partial S} + \sum_i \beta_i \varrho_{i,S}\frac{\partial\theta}{\partial v_i} \right) dt$$

$$\eta^2 dt = d\theta d\theta = \left(\psi^2 + \sum_i\sum_j \beta_i\beta_j\left(\varrho_{i,j} - \varrho_{i,S}\varrho_{j,S}\right)\frac{\partial\theta}{\partial v_i}\frac{\partial\theta}{\partial v_j} \right) dt$$

and all partial derivatives of θ as in Lemma 1.

Proof: See Appendix A.

Theorem 4 shows how the dynamics of each implied volatility are derived from the same stochastic local volatility model (7) and (8). Thus we know the dynamics of the entire implied volatility surface that are consistent with the SLV model. These dynamics are governed by the same stochastic factors as those driving the local volatility and the option price. Finally, clearly the options prices consistent with these implied volatilities must satisfy all no-arbitrage conditions mentioned in Appendix B.

Note that there is a very interesting singularity on the drift of θ when $t \to T$, when it seems to explode, as reported by Schonbucher (1999). However, from Theorem 4, we learn that this is not a problem as long as the following condition holds for all T, $T > t > t_0$:

$$\int_{t_0}^{T} \left| \tfrac{1}{2}\frac{1}{\theta(T-t)}\left(\theta^2 - \sigma^2\right) + \sigma\frac{d_2}{\theta\sqrt{T-t}}\psi - \tfrac{1}{2}\frac{d_1 d_2}{\theta}\eta^2 \right| dt < \infty. \qquad (19)$$

In effect, when $T \to t_0$, i.e. just before expiry, we can assume the integrand of (19) is constant so that, after replacing d_1 and d_2 and cancelling $(T - t_0)$ whenever possible, (19) converges in probability to:

$$\lim_{T\to t_0} \left| \tfrac{1}{2}\frac{1}{\theta}\left(\theta^2 - \sigma^2\right) + \sigma\left(\frac{\ln M}{\theta^2} - \tfrac{1}{2}(T-t_0)\right)\psi - \tfrac{1}{2}\left(\frac{(\ln M)^2}{\theta^3} - \tfrac{1}{4}\theta(T-t_0)^2\right)\eta^2 \right| < \infty$$

or simply:

$$\lim_{T\to t_0} \left| \tfrac{1}{2}\left(\theta^2 - \sigma^2\right) + \sigma\left(\frac{\ln M}{\theta}\right)\psi - \tfrac{1}{2}\left(\frac{\ln M}{\theta}\right)^2 \eta^2 \right| < \infty$$

as long as:

$$\lim_{T\to t_0} |\psi| < \infty \qquad \text{and} \qquad \lim_{T\to t_0} \eta^2 < \infty$$

which also means:

$$\lim_{T\to t_0}\left|\frac{\partial\theta}{\partial v_i}\right| < \infty \Rightarrow \lim_{T\to t_0}\left|\frac{1}{\Upsilon_{BS}}\frac{\partial f_L}{\partial v_i}\right| < \infty \Rightarrow \lim_{T\to t_0}\frac{\partial f_L}{\partial v_i} = 0$$

$$\lim_{T\to t_0}\left|\frac{\partial\theta}{\partial S}\right| < \infty \Rightarrow \lim_{T\to t_0}\left|\frac{\delta_L - \delta_{BS}}{\Upsilon_{BS}}\right| < \infty \Rightarrow \lim_{T\to t_0}(\delta_L - \delta_{BS}) = 0$$

since the Black-Scholes vega $\Upsilon_{BS} \to 0$ close to expiry.

That is, at expiry the deterministic local volatility delta must converge in probability to the BS delta and there should be no sensitivity to the parameters **v**. Note that both limits are sensible. At expiry, both deltas should be either 0 or 1 depending on whether the option is out-of-the-money or in-the-money respectively, and there is no time value in the price of the option, hence no sensitivity to volatility, which is the only variable depending on **v** in this model.

Such behaviour imposes a severe constraint on the dynamics of implied volatilities before expiry and resembles the drift of a Brownian bridge. In effect, as already pointed out by Dupire (2003), the density of the implied volatility consistent with (16) is equivalent to the density of the Brownian bridge from (S, t_0) to (K, T), which concords with our findings.[26]

Finally, note that ψ in Theorem 4 is related to the covariance between implied volatility and asset price movements and η^2 is the annualised variance of implied volatility. Yet, in general it is possible to re-write (18) using only *un*correlated Brownian motions, as shown below.

Corollary 1

Assuming the vector $\boldsymbol{dW} = [dW_1, dW_2 \ldots dW_n]$ has positive definite correlation matrix $\boldsymbol{\Sigma}$, the dynamics of the local implied volatility from Theorem 4 can also be expressed in terms of uncorrelated Brownian motions as:

$$d\theta = \left[\tfrac{1}{2}\frac{1}{\theta\tau}\left(\theta^2 - \sigma^2\right) + \sigma\frac{d_2}{\theta\sqrt{\tau}}\psi - \tfrac{1}{2}\frac{d_1 d_2}{\theta}\eta^2 \right]dt + \psi dW_S + \sum_j \omega_j dW_j^* \tag{20}$$

satisfying the same regularity conditions of Theorem 4 and $dW_S dW_j^* = dW_i^* dW_j^* = 0$ for all $i \neq j$ almost surely, and:

$$\omega_j = \sum_{i=j}^{n} \beta_i \sqrt{1-\varrho_{i,S}^2}\, A_{i,j}\frac{\partial\theta}{\partial v_i}$$

where $A_{i,j}$ are the elements of the Cholesky decomposition **A** of the correlation matrix $\boldsymbol{\Sigma}$ with:

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[26] More information about the asymptotic relationship between local volatility and Black-Scholes implied volatility can be found in Berestycki, Busca and Florent (2002).

$$\Sigma_{i,j} = \frac{\varrho_{i,j} - \varrho_{i,S}\varrho_{j,S}}{\sqrt{(1-\varrho_{i,S}^2)(1-\varrho_{j,S}^2)}} \qquad \text{and} \qquad \eta^2 = \psi^2 + \sum_j \omega_j^2 \,.$$

Proof: See Appendix A.

Apart from minor differences in notation, (20) is precisely the same as equation (2.7) from Schonbucher (1999) for the dynamics of a stochastic implied volatility with the drift term given by equation (3.7).[27],[28] This is an interesting result since Schonbucher models stochastic implied volatilities directly, yet here we assume a stochastic local volatility. Nevertheless there is an important distinction to be made. Whereas Schonbucher models the implied volatility for a *given* strike K and maturity T, the dynamics (20) hold for *all* strikes and maturities simultaneously. In the former, an implied volatility diffusion is defined for each strike K and maturity T. So if there are options for k strikes and m maturities in the market, the market model specifies mk diffusions, one for each traded option. On the other hand, if on every day a conditional local volatility model fits the smile surface with a few parameters, say $n << mk$, then the SLV model has reduced the probability space from $mk + 1$ random variables to only $n + 1$, including the asset price S.

But naturally if the 'market models' of stochastic implied volatilities are equivalent to the SLV model, they should produce the same hedge ratios as in Theorem 2. In fact, the 'market model' (MM) delta is related to the BS delta by the chain rule:

$$\delta_{MM} = \delta_{BS} + \Upsilon_{BS}\frac{d\theta}{dS}$$

where $d\theta/dS$ follows from (7) and (20):

$$\frac{d\theta}{dS} = \frac{Cov(d\theta, dS)}{Var(dS)} = \frac{\psi\sigma S dt}{\sigma^2 S^2 dt} = \frac{\partial\theta}{\partial S} + \sum_i \frac{\beta_i \varrho_{i,S}}{\sigma S}\frac{\partial\theta}{\partial v_i}\,.$$

Then, using Lemma 1 we have:

$$\delta_{MM} = \delta_{BS} + \Upsilon_{BS}\left(\frac{\delta_L - \delta_{BS}}{\Upsilon_{BS}} + \sum_i \frac{\beta_i \varrho_{i,S}}{\sigma S}\frac{1}{\Upsilon_{BS}}\frac{\partial f_L}{\partial v_i}\right) = \delta_L + \sum_i \frac{\beta_i \varrho_{i,S}}{\sigma S}\frac{\partial f_L}{\partial v_i}$$

which is exactly the same as (12-a).

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[27] We believe there was a typo in equation (3.3) from Schonbucher (1999) for the variance of implied volatility, where the term γ^2 appears to be missing. Many thanks to Hyungsok Ahn of Commerzbank, London for drawing our attention to this.

[28] It also possible to derive dynamics consistent with the implied volatility diffusion models of Brace *et al* (2001), Ledoit *et al.* (2002) and Daglish, Hull and Suo (2003). The former extends the BGM interest rate model to account for smiles and skews, the second models implied volatilities for fixed moneyness and time to maturity, while the latter models implied variances for fixed strike and maturity.

Corollary 2

The correlation between the local implied volatility and asset price movements is given by:

$$\varrho_{0,S}(K,T;t,S,\mathbf{v}) = \frac{\psi}{\sqrt{\psi^2 + \sum_j \omega_j^2}} = \frac{\psi}{|\eta|} \tag{21}$$

Proof: The correlation follows from (7) and (20):

$$\varrho_{0,S} = \frac{Cov(d\theta, dS)}{\sqrt{Var(d\theta)Var(dS)}} = \frac{\psi\sigma S dt}{\sqrt{(\psi^2 + \sum_j \omega_j^2)dt(\sigma^2 S^2 dt)}} = \frac{\psi}{|\eta|} \qquad \blacksquare$$

We have used the absolute value $|\eta|$ to stress that the denominator of (21) is strictly positive. For instance, when $\omega_j = 0$ for all j, the denominator is $|\psi|$ and the correlation is ± 1, depending on the sign of ψ. In effect, perfect correlation can be true only when $\omega_j = 0$ for all j, i.e. when:

(a) $\beta_i = 0$ for all i, when the vector **v** is no longer stochastic; or

(b) $\varrho_{i,S} = 1$ for all i, when the vector **v** is a deterministic function of S and t; or

(c) $\partial f_L / \partial v_i = 0$ for all i, when there is no need for the vector **v**.

Note that in all three cases (a) – (c) above, the dynamics (7) and (8) collapse to a deterministic local volatility model, hence the perfect correlation with S. That is, the local volatility is deterministic if and only if variations in implied volatility and asset price are *perfectly* correlated, i.e. when implied volatilities are also deterministic.

VII. An Application: Normal Mixture Local Volatility

In this section we investigate how relevant the correction term in (12-a) may be in the context of normal mixture diffusion (NMD) local volatility models, introduced by Brigo and Mercurio (2001). Naturally we could have chosen any other parametric local volatility model, but we believe the NMD models are more illustrative since their parameters have an economic interpretation and they are easy to calibrate using the analytical solution for the normal mixture option price.

In the NMD model, the asset returns probability density is defined as a linear combination of normal densities:

$$g(x) = \sum_{i=1}^{n} \lambda_i \varphi(x; \mu_i, \sigma_i^2) \tag{22}$$

where each normal density φ can have different means and variances and the weights λ_i sum to 1. Brigo and Mercurio (2001) show that there is a unique *deterministic* spot volatility function consistent with (22) and that the local volatility option price is given by a mixture of Black-Scholes prices:

$$f_{NM}(S,K,\tau,r,q;\mathbf{v}) = \sum_{i=1}^{n} \lambda_i f_{BS}(S,K,\tau,r,q;\mu_i,\sigma_i^2). \tag{23}$$

In (23), τ is the time to expiry, **v** denotes the vector of parameters $\{\lambda_i, \mu_i, \sigma_i^2\}$ for all i, and each f_{BS} refers to the Black-Scholes price assuming the asset returns density is normal with mean μ_i and variance σ_i^2. To avoid arbitrage opportunities, as specified in Appendix B, the drift of all normal densities in (22) must satisfy the following drift condition:

$$\sum_{i=1}^{n} \lambda_i \exp(\mu_i) = \exp((r-q)\tau). \tag{24}$$

To simplify the analysis, here we adopt the smallest version of (22) consistent with the equity skew. This requires only two normal densities with different means and variances in the mixture:

$$g(x) = \lambda\varphi(x;(r-q+s_H)\tau,\sigma_H^2\tau) + (1-\lambda)\varphi(x;(r-q+s_L)\tau,\sigma_L^2\tau) \tag{25}$$

where the 'shifts' on the drift s_H and s_L are related from (24):

$$s_L = \frac{1}{\tau}\ln\left(\frac{1-\lambda\exp(s_H\tau)}{1-\lambda}\right)$$

In this case, there are only four model parameters to calibrate: the weight λ, the shift s_H (with s_L given as above) and two volatilities σ_H and σ_L. Then for a given $\mathbf{v} = \{\lambda, s_H, \sigma_H, \sigma_L\}$, the local volatility price of a European option is:[29]

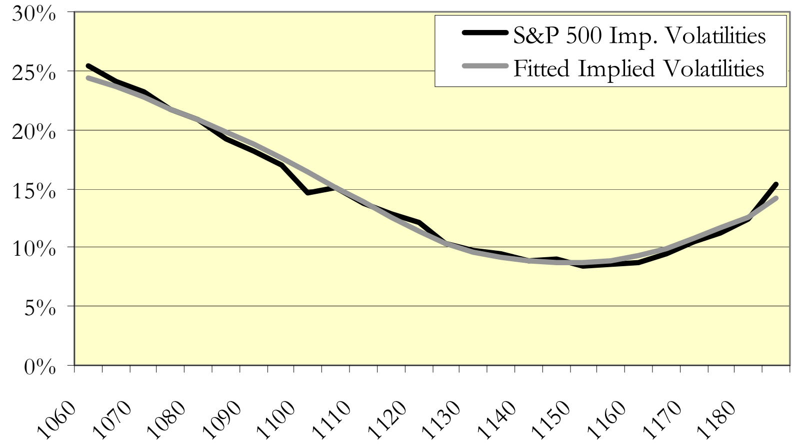
$$f_{NM}(S,K,\tau,r,q;\mathbf{v}) = \lambda f_{BS}(S,K,\tau,r,q-s_H,\sigma_H) + (1-\lambda) f_{BS}(S,K,\tau,r,q-s_L,\sigma_L). \tag{26}$$

Intuitively, we can interpret model (26) as a weighted average over two possible future outcomes: a 'crash' market with low weight λ, strongly negative drift $r - q + s_H$ and high volatility σ_H; and a 'normal' market with high weight $1 - \lambda$, positive drift $r - q + s_L$ and lower volatility σ_L.

Despite its simplicity, model (26) is very powerful for fitting equity skews. Figure 1 shows the typical calibration result on S&P 500 index options, where we have used all the available June 2004 call options between 1060 and 1190 (26 strikes) on 10/06/2004 and minimised the root-mean-square-error (RMSE) of fitted implied volatilities.[30]

[29] 'Shifting' the mean of each normal density is equivalent to shifting the dividend q in the Black-Scholes pricing formula.
[30] We have used only one maturity in this exercise because the NMD model (26) cannot fit several maturities simultaneously, i.e. it is not appropriate to model the term structure of implied volatilities (see also Alexander, 2004). This problem may also jeopardise the calibration since there may be several solutions for $\mathbf{v}(t)$ consistent with a given smile.

Figure 1: June 2004 S&P 500 implied volatilities on 10/06/2004, calibrated with (26). Model parameters: λ = 5.23%, s_H = -1.29, σ_H = 39.36% and σ_L = 7.01%. RMSE = 0.5837%.



However, even for a simple model such as (26), the calibration of **v** is not trivial. It requires a non-linear constrained ($0 < \lambda < 1$ and σ_H, $\sigma_L > 0$) minimization of a given objective function by changing four (possibly correlated) parameters simultaneously. Several minimization algorithms could be used, but gradient-based methods are problematic since they require a 4x4 Hessian matrix of a highly non-linear objective function. In fact we have found that convergence problems are common because the optimum can be heavily influenced by the particular choice of starting values. Instead, here we use a simple and intuitive grid-based approach where at each iteration we reduce the size of the range of possible values for each parameter. The method is similar to the one-dimensional bisection algorithm except that here it is used in a four-dimensional grid. The advantage of this approach is that it always finds a local optimum within a pre-specified region, although not necessarily the global optimum.

We have calibrated (26) to each day separately on a time series of prices of the June 2004 S&P 500 options, from 02/01/2004 to 15/06/2004 (111 business days) with strikes in the region of 10% around the S&P 500 index level. Table 1 reports the calibrated parameters during just the last 20 business days.[31] The RMSE remains below 1% in the whole sample except for the last three days in Table 1. This shows that calibrated prices will remain well within the normal bid-ask spread, but very close to the expiry date the quoted prices for far ITM and OTM options can be very unreliable as they correspond to sale quotes.[32] Hence it is sensible to reduce the range of strikes for very short maturity options. For instance, for 10/06/2004, Table 1 reports a RMSE of 1.198%, twice as big as the RMSE in figure 1. The difference is due to the range of strikes. In Table 1, all strikes from 1025 to 1190 (30 strikes) have been used while in figure 1 we have used only 26 strikes.

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[31] We have constrained $5\% < \lambda < 10\%$ to improve the convergence properties of the algorithm.
[32] In this application, we used the arithmetic average of bid and ask option prices rather than the close price.

Table 1: Calibrated parameters for S&P 500 call options expiring in 19/06/2004 and calibrated with (26).

Date	# of Strikes	λ	s_H	σ_H	σ_L	RMSE
17/05/2004	23	6.084%	-1.828	31.099%	13.787%	0.571%
18/05/2004	24	5.996%	-1.664	24.310%	13.539%	0.568%
19/05/2004	24	5.654%	-1.777	22.017%	13.051%	0.598%
20/05/2004	28	5.772%	-1.679	22.873%	13.078%	0.539%
21/05/2004	29	6.187%	-1.425	39.249%	12.928%	0.668%
24/05/2004	29	6.011%	-1.614	30.977%	12.696%	0.826%
25/05/2004	34	6.231%	-1.399	12.437%	11.065%	0.508%
26/05/2004	34	5.195%	-1.634	13.334%	10.873%	0.525%
27/05/2004	33	6.231%	-1.385	13.054%	10.060%	0.537%
28/05/2004	31	5.308%	-1.617	13.271%	10.383%	0.477%
01/06/2004	33	5.073%	-1.764	14.004%	11.426%	0.638%
02/06/2004	32	5.117%	-1.656	23.287%	11.264%	0.707%
03/06/2004	34	5.938%	-1.809	13.256%	12.301%	0.721%
04/06/2004	32	5.928%	-1.381	34.757%	10.676%	0.680%
07/06/2004	30	5.073%	-1.250	27.607%	10.292%	0.670%
08/06/2004	30	6.563%	-1.182	35.637%	7.714%	0.844%
09/06/2004	32	5.156%	-1.348	36.722%	8.767%	0.645%
10/06/2004	30	5.469%	-1.376	43.366%	6.530%	1.198%
14/06/2004	31	5.078%	-3.039	51.677%	9.238%	1.142%
15/06/2004	28	5.000%	-1.620	45.590%	7.512%	1.930%

The standard deviations and correlations of the calibrated parameters in 15/06/2004, based on all 111 business days are reported in Table 2. In particular, we are interested in the first column of Table 2, which reports the correlation between variations in model parameters and the underlying asset price. Notably ΔS is not significantly correlated with $\Delta\lambda$ and $\Delta\sigma_H$ but, on the other hand, there is significant negative correlation with $\Delta\sigma_L$ and positive correlation with Δs_H, justifying the adjustments in Theorem 2 at least for these two parameters.

Table 2: Standard deviations and correlations of calibrated parameters in 15/06/2004. Standard deviations are reported in the diagonal and the correlations in the remaining cells of the matrix.

	ΔS	Δs_H	$\Delta\sigma_H$	$\Delta\sigma_L$	$\Delta\lambda$
ΔS	8.4748				
Δs_H	0.3961	0.2784			
$\Delta\sigma_H$	-0.0044	0.0748	0.1301		
$\Delta\sigma_L$	-0.5696	-0.6174	-0.2555	0.0075	
$\Delta\lambda$	0.0216	0.4188	-0.3970	-0.1786	0.0105

The empirical correlations in Table 2 are consistent with the expected dynamics of implied volatilities. In particular there is a strong *negative* correlation between the low volatility, which pertains about 95% of the time (i.e. in ordinary market circumstances) and the asset price. This negative correlation is consistent with the negative skew that is ordinarily observed in equity index markets.[33] Hence Table 2 indicates that movements in kurtosis are broadly uncorrelated with changes in S (since λ and σ_H are directly linked to the kurtosis of the risk neutral distribution, i.e. they affect the probability mass in the tails of the distribution) but movements in skewness are strongly correlated with movements in S (it is widely understood that skewness in the risk-neutral distribution is the main reason for the observed skew shape in the implied volatility surface from equity index options). The positive correlation between changes in the 'extreme market' shift s_H and changes in the asset price in Table 2 implies that the range of the equity skew increases when the price jumps down and decreases when the price jumps up. This finding agrees with the conclusions drawn by Alexander (2001) from a principal component analysis of implied volatilities.

The approximation in Theorem 3 is now used to calculate the adjusted normal mixture deltas as:

$$\delta_{SNM}(S,K,\tau,r,q;\mathbf{v}) \approx \delta_{NM}(S,K,\tau,r,q;\mathbf{v}) + \sum_i \frac{Cov(\Delta S,\Delta v_i)}{Var(\Delta S)}\frac{\partial f_{NM}}{\partial v_i} \qquad (27)$$

where $v_i \in \mathbf{v} = \{\lambda, s_H, \sigma_H, \sigma_L\}$ and:

$$\delta_{NM}(S,K,\tau,r,q;\mathbf{v}) = \lambda\delta_{BS}(S,K,\tau,r,q-s_H,\sigma_H) + (1-\lambda)\delta_{BS}(S,K,\tau,r,q-s_L,\sigma_L).$$

The partial derivatives $\partial f_{NM}/\partial v_i$ can be calculated via finite differences on the model price (26).[34] Consider the options on 19/04/2004, about two months before expiry. On that particular day values $\mathbf{v}$ = {5.986%, -0.8827, 46.92%, 11.85%} were estimated. Table 3 reports first each $\partial f_{NM}/\partial v_i$ and then three different deltas calculated with the implied volatility from model price (26), i.e. the Black-Scholes δ_{BS}, the deterministic NM delta δ_{NM} and the stochastic NM delta, i.e. the adjusted delta δ_{SNM}. Only the last of these deltas allows the local volatility surface the move over time.

Whilst only pertaining to a single day in our sample, Table 3 and Figure 2, which plots the three deltas as a function of K/S, exhibit the typical relationship between the deltas. This basic pattern remains valid throughout the sample. First note that the (deterministic) normal mixture delta is always greater than the Black-Scholes delta except for very high strikes. Due to the linearity of the NM price (26) with respect to the BS price, it can be easily shown that the following results hold for standard European calls and puts:

$$f_{BS} = S\frac{\partial f_{BS}}{\partial S} + K\frac{\partial f_{BS}}{\partial K} \quad \text{and} \quad f_{NM} = S\frac{\partial f_{NM}}{\partial S} + K\frac{\partial f_{NM}}{\partial K}$$

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[33] If volatility is uncorrelated with price, uncertainty in volatility makes both tails of the price density heavier symmetrically. But if volatility is negatively correlated with price, it increases when the price falls, so the lower tail of the price density will be heavier than the upper tail – i.e. the price density will have a negative skew.

[34] Of course there is an analytical solution for each of these partial derivatives, but we believe the use of finite differences here is sufficient.

and using (17):

$$f_{NM} = f_{BS} \Leftrightarrow S\left(\frac{\partial f_{NM}}{\partial S} - \frac{\partial f_{BS}}{\partial S}\right) = -K\left(\frac{\partial f_{NM}}{\partial K} - \frac{\partial f_{BS}}{\partial K}\right) \Leftrightarrow \delta_{NM} - \delta_{BS} = -\frac{K}{S}\left(\frac{\partial f_{NM}}{\partial K} - \frac{\partial f_{BS}}{\partial K}\right)$$

Then, using Lemma 1 and the chain rule for K, we have:

$$\delta_{NM} = \delta_{BS} - \frac{K}{S}\Upsilon_{BS}\frac{\partial\theta}{\partial K} \qquad \text{and} \qquad \frac{\partial\theta}{\partial S} = -\frac{K}{S}\frac{\partial\theta}{\partial K} \tag{28}$$

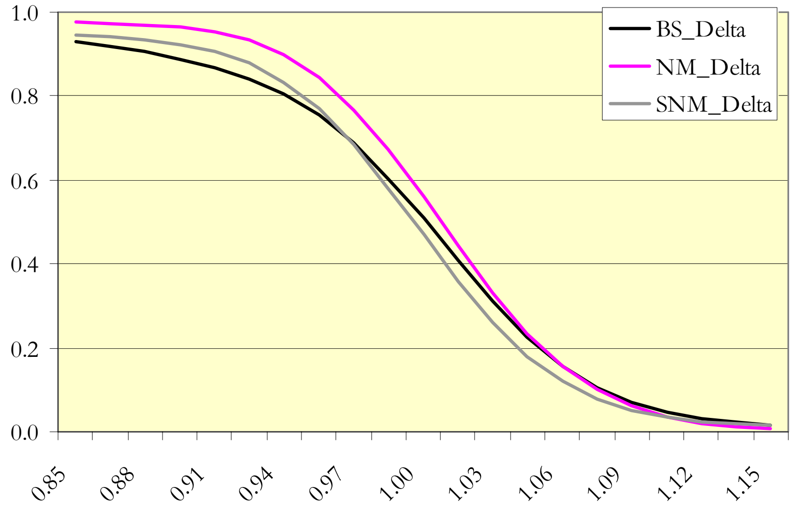
Table 3: Model sensitivities and deltas on 19/04/2004.

K/S	$\partial f_{NM}/\partial s_H$	$\partial f_{NM}/\partial\sigma_H$	$\partial f_{NM}/\partial\sigma_L$	$\partial f_{NM}/\partial\lambda$	δ_{BS}	δ_{NM}	δ_{SNM}
0.850	-4.2200	9.4064	0.3221	67.2277	0.9298	0.9754	0.9482
0.865	-4.5678	9.5158	1.0867	76.0776	0.9183	0.9730	0.9422
0.880	-4.9017	9.5470	3.1620	85.3422	0.9047	0.9697	0.9347
0.895	-5.2044	9.5030	7.9920	94.7180	0.8883	0.9640	0.9241
0.910	-5.4454	9.3886	17.6661	103.6394	0.8678	0.9531	0.9071
0.925	-5.5796	9.2097	34.3699	111.2146	0.8408	0.9330	0.8793
0.940	-5.5512	8.9732	59.2067	116.2743	0.8041	0.8985	0.8356
0.955	-5.3074	8.6866	90.8152	117.5989	0.7540	0.8445	0.7722
0.970	-4.8191	8.3575	124.6884	114.2964	0.6875	0.7684	0.6882
0.985	-4.0997	7.9940	154.0023	106.1866	0.6045	0.6715	0.5872
1.000	-3.2115	7.6037	171.9162	93.9860	0.5088	0.5600	0.4773
1.015	-2.2531	7.1941	174.2344	79.1812	0.4079	0.4433	0.3684
1.030	-1.3336	6.7720	160.9930	63.6187	0.3108	0.3321	0.2702
1.045	-0.5439	6.3439	136.1684	49.0062	0.2252	0.2351	0.1890
1.060	0.0634	5.9154	105.8220	36.5196	0.1560	0.1574	0.1274
1.075	0.4774	5.4915	75.8359	26.6748	0.1044	0.1000	0.0840
1.090	0.7196	5.0765	50.2877	19.4128	0.0686	0.0606	0.0556
1.105	0.8292	4.6739	30.9559	14.3189	0.0454	0.0355	0.0379
1.120	0.8482	4.2867	17.7438	10.8511	0.0310	0.0206	0.0274
1.135	0.8129	3.9171	9.4980	8.5011	0.0222	0.0121	0.0211
1.150	0.7498	3.5668	4.7609	6.8721	0.0168	0.0076	0.0173

Now, since the Black-Scholes vega Υ_{BS} is always positive, we have that the deterministic NM delta will be greater than the BS delta whenever $\partial\theta/\partial K$ is negative and vice-versa. But $\partial\theta/\partial K$ is simply the slope of the smile surface in the strike metric and this is clearly negative in equity index options, except perhaps for deep OTM call/ITM put options.[35] Hence, as is clearly seen in Figure 2, the NM deltas are consistently higher than BS deltas. This 'over hedging' is a well-documented feature of local volatility models, and has been attributed to the supposedly poor hedging performance. However, this analysis only applies to the deterministic local volatility deltas.

[35] In effect, note that the relationship (28) is virtually independent of **v** since $\partial\theta/\partial K$ can be approximated from an interpolated implied volatility surface based on observed market option prices. See Appendix C for more information about the estimation of $\partial\theta/\partial K$.

Figure 2: Comparison between deltas from different models on 19/04/2004.



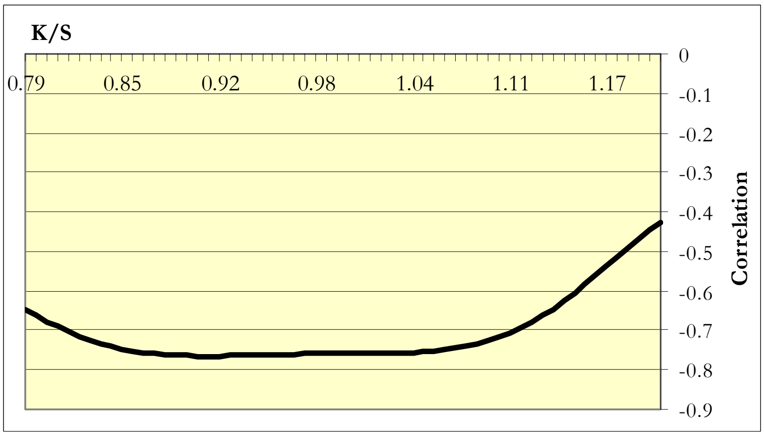
The stochastic NM deltas, in turn, can be greater than or less than the BS deltas depending on the strike of the option. From Table 2, we know that the parameters that are most correlated with movements in the underlying asset price are s_H and σ_L, hence we expect most of the adjustment in delta to result from the terms $Cov(\Delta S, \Delta s_H) * \partial f_{NM}/\partial s_H$ and $Cov(\Delta S, \Delta\sigma_L) * \partial f_{NM}/\partial\sigma_L$ in Theorem 3. We have $Cov(\Delta S, \Delta s_H) > 0$ and $Cov(\Delta S, \Delta\sigma_L) < 0$, and there are good reasons to expect these signs, so the second adjustment is negative but the first one can be positive or negative, depending on the sign of $\partial f_{NM}/\partial s_H$. In Table 3, this is negative except for far OTM call options, so the adjustment in delta due to stochastic local volatility will be downwards, except when K/S is much greater than 1. This downward adjustment of the stochastic NM deltas means they can be greater than or less than the BS deltas, as is clear from Figure 2.

The extensive literature on the hedging performance of deterministic local volatility models has focused on the deterministic delta δ_L (or δ_{NM} in this example). But from Corollary 2 we know that the use of a deterministic spot volatility implies a correlation of ±1 between variations in implied volatilities and asset price, which is not supported by empirical evidence.[36] Figure 3 shows a typical example from the FTSE 100 index call options market. The correlations have been calculated on the daily first differences of the index and 60 interpolated time series of 1-month fixed moneyness (K/S) implied volatilities from 02/01/1998 to 01/10/1999 (a total of 442 business days). Clearly there is strong negative correlation, but it is far from perfect, with a minimum of −0.76 for near the money options, and reducing monotonically in absolute value for far

[36] To be fair, these tests could still be valid if the parameters were stochastic but uncorrelated with S, i.e. $\beta_i \neq 0$ and $\varrho_{i,S} = 0$ for all i, when the correlation (21) is less than perfect but the correction term in (12-a) is not necessary.

ITM and OTM options. This provides empirical evidence of idiosyncratic dynamics for implied volatility and justifies our approach in this paper, by introducing new sources of uncertainty to the SLV model.

Figure 3: Correlation between daily first differences of the FTSE 100 index and 1-month fixed moneyness call options implied volatilities.



VIII. Summary and Conclusions

Two approaches to modelling the Black-Scholes implied volatility smile or skew surface have developed separately even though potential links between them were identified by Dupire (1996) and Kani, Derman and Kamal (1997). In the intervening years most research on stochastic volatility has specified a univariate diffusion or jump-diffusion for the spot variance or volatility of the underlying asset. Likewise, most research on local volatility models has assumed a deterministic spot volatility function for the underlying asset price diffusion at a particular point in time, with no reference to the dynamic evolution of volatility. Both approaches are incomplete, the former capturing the dynamic properties of volatility but only in a one-dimensional space, the latter focussing on the multi-dimensional aspects of volatility but ignoring its time-evolution. However recent developments of multivariate diffusions for implied volatility have extended the stochastic volatility approach to be consistent with the cross-section of implied volatilities as well as their dynamics. To be consistent with this view, the deterministic local volatility model, which implies only a deterministic evolution for implied volatility, requires generalization.

Following Dupire (1996) and Kani, Derman and Kamal (1997) we regard the deterministic local volatility model as merely a special case of a more general stochastic local volatility model. That is, we define local volatility as the square root of the conditional expectation of future spot variance in terms of $n + 1$ stochastic risk factors, viz. the underlying price plus n parameters of the local volatility surface. Hence we explicitly model the stochastic evolution of a locally deterministic volatility surface over time. We have proved that this general stochastic local volatility model is equivalent to the market model for implied volatilities that was introduced by Schonbucher (1999). This important 'duality' result has shown that the stochastic and local volatility approaches can be unified within a single general framework. It is only when these approaches take a restricted view on volatility that they appear to be different.

Several results on the behaviour of implied volatility, which have previously been proved only in the context of specific models, are here proved within this general framework. Moreover, we provide an important insight to the correct derivation of local volatility hedge ratios. Deterministic local volatility models fail to capture dynamics of implied volatilities and as a result the hedge ratios derived from these models are incorrect. Hence some standard critiques of local volatility models no longer apply. Indeed, from the equivalence of the implied volatility market model and the general stochastic local volatility model, we show that these models have identical hedge ratios.

An empirical example of a stochastic local volatility model with local volatilities given by the lognormal mixture local volatility model of Brigo and Mercurio (2001) specifies the evolution of the model parameters as the SP500 index price changes. Contrasting the stochastic local volatility hedge ratios with the standard 'deterministic' local volatility hedge ratios, the latter were shown to be incorrect because the movements in the implied volatility surface were far from perfectly correlated with movements in the underlying. The extant literature on the hedging performance of local volatility models has thus been testing an unrealistic and incomplete model, and this explains the surprising conclusions that have been drawn.

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Appendix A: Mathematical Proofs

Here we provide formal proofs for most of the mathematical results in this paper.

Proof of Equation (5)

Suppose a deterministic local volatility (DLV) model has been calibrated at time t_0 assuming:

$$dS = (r-q)Sdt + \sigma(t,S;\mathbf{v}(t_0))SdW_S \quad \text{for all } t > t_0$$

while the underlying asset process actually follows:

$$dS = (r-q)Sdt + \sigma(t,S;\mathbf{v}(t))SdW_S \quad \text{for all } t > t_0$$

with $\mathbf{v}(t)$ stochastic. Now define the delta-hedged portfolio $\Pi = f_L - \delta_L S$ where $f_L = f_L(t, S; \mathbf{v}(t_0))$ is the value of a standard European option and δ_L is the option delta consistent with the DLV model. Then, from a standard application of Ito's lemma and the Black-Scholes pde (9), we have:

$$d\Pi = df_L - \delta_L\left(dS + qSdt\right) = r\Pi dt + \tfrac{1}{2}\left(\sigma^2(t,S;\mathbf{v}(t)) - \sigma^2(t,S;\mathbf{v}(t_0))\right)S^2\gamma_L dt$$

where γ_L is the option gamma consistent with the DLV model. Next, integrating over $t \in [t_0, T]$, the total hedging error (hence the total pricing error) is:

$$\Lambda = \int_{t_0}^{T} \tfrac{1}{2}\left(\sigma^2(t,S;\mathbf{v}(t)) - \sigma^2(t,S;\mathbf{v}(t_0))\right)S^2\gamma_L dt$$

which is stochastic since S and $\mathbf{v}(t)$ are stochastic. Thus, conditioning on S and taking expectation we have:

$$E^0[\Lambda] = \int_{t_0}^{T}\int_{0}^{\infty} \tfrac{1}{2}\left(E^0\left[\sigma^2(t,S;\mathbf{v}(t))\middle|S\right] - \sigma^2(t,S;\mathbf{v}(t_0))\right)S^2\gamma_L dSdt$$

which must be zero for any arbitrary $T > t_0$ if options are fairly priced by the model, thus:

$$\sigma^2(t,S;\mathbf{v}(t_0)) = E^0\left[\sigma^2(t,S;\mathbf{v}(t))\middle|S\right] \tag{A-1}$$

Finally, since the expectation in (A-1) is precisely the general definition (4) for the local volatility, we conclude that the local volatility surface $\sigma^2(t, S; \mathbf{v}(t_0))$ calibrated by a DLV model is correct if options prices are fitted properly, i.e. the expected pricing error $E^0[\Lambda]$ is zero. ■

Proof of Equation (6)

Define $\tilde{\mathbf{v}}(t_0) = E^0\left[\mathbf{v}(t)\middle|S\right]$. Then, a standard Taylor's series expansion of $\sigma^2(t, S; \mathbf{v}(t))$ gives:

$$\sigma^2(t,S;\mathbf{v}(t)) = \sigma^2(t,S;\tilde{\mathbf{v}}(t_0)) + \sum_i \frac{\partial\sigma^2(t,S;\tilde{\mathbf{v}}(t_0))}{\partial v_i}\left(v_i(t) - \tilde{v}_i(t_0)\right) +$$
$$\tfrac{1}{2}\sum_i\sum_j \frac{\partial^2\sigma^2(t,S;\tilde{\mathbf{v}}(t_0))}{\partial v_i \partial v_j}\left(v_i(t) - \tilde{v}_i(t_0)\right)\left(v_j(t) - \tilde{v}_j(t_0)\right)$$

and taking the expectation at time t_0 conditional on S we have:

$$E^0\left[\sigma^2(t,S;\mathbf{v}(t))\middle|S\right] = \sigma^2(t,S;\tilde{\mathbf{v}}(t_0)) + \tfrac{1}{2}\sum_i\sum_j \frac{\partial^2\sigma^2(t,S;\tilde{\mathbf{v}}(t_0))}{\partial v_i \partial v_j} Cov^0\left(v_i(t), v_j(t)\middle|S\right)$$

where the first order term cancels out in the expectation. Finally, replacing (A-1) we conclude the proof. Note that the covariance above refers to the portion of $\mathbf{v}(t)$ that is *un*correlated with S. Hence if $\mathbf{v}(t)$ is deterministic the second order term above also cancels out. ■

Proof of Equations (13) and (14)

This proof of the hedging error is similar to the proof of equation (5) above, except that it uses the dynamics (10) for the claim price from Theorem 1. The dynamics of the delta-hedged portfolio $\Pi = f_L - \delta_L S$ is:

$$d\Pi = df_L - \delta_L\left(dS + qSdt\right) = r\Pi dt + \left(\sigma S\left(\frac{\partial f_L}{\partial S} - \delta_L\right) + \sum_i \beta_i \varrho_{i,S}\frac{\partial f_L}{\partial v_i}\right)dW_S + \sum_i \beta_i\sqrt{1-\varrho_{i,S}^2}\frac{\partial f_L}{\partial v_i}dW_i$$

so that, replacing $\delta_L = \partial f_L/\partial S$ (i.e. the deterministic local volatility delta), the total hedging error is:

$$\Lambda_{DLV} = \sum_i\left[\int_{t_0}^{T}\beta_i\varrho_{i,S}\frac{\partial f_L}{\partial v_i}dW_S + \int_{t_0}^{T}\beta_i\sqrt{1-\varrho_{i,S}^2}\frac{\partial f_L}{\partial v_i}dW_i\right] = \sum_i\int_{t_0}^{T}\beta_i\frac{\partial f_L}{\partial v_i}dZ_i \quad \text{for some } T > t_0$$

where we have used the definition for dZ_i from (8). Therefore, under the assumption of a deterministic spot volatility, the hedging error is the sum of stochastic integrals related to all the uncertainty around the local volatility parameters.

Instead, if we had used the correct delta from (12-a) and followed the same argument as above, the total hedging error associated with the delta-hedged portfolio $\Pi = f_L - \delta_{SLV}S$ would be:

$$\Lambda_{SLV} = \sum_i\int_{t_0}^{T}\beta_i\sqrt{1-\varrho_{i,S}^2}\frac{\partial f_L}{\partial v_i}dW_i \quad \text{for some } T > t_0$$

which is stochastic if the correlation between S and at least one parameter v_i is less than perfect, i.e. $\varrho_{i,S} \neq \pm 1$. That is, the delta hedge will not be perfect. ■

Proof of Equation (16)

The proof of (16) is similar to the proof of (5), but assuming a *constant* spot volatility $\theta(K,T)$ for each option (i.e. a different implied volatility for each strike K and maturity T) such as in the Black-Scholes model:

$$dS = (r-q)Sdt + \theta(K,T)SdW_S \quad \text{for all } t > t_0$$

Hence, following the same steps as above, the expected total hedging error is:

$$E^0[\Lambda] = \int_{t_0}^{T}\int_{0}^{\infty} \tfrac{1}{2}\left(E^0\left[\sigma^2(t,S;\mathbf{v}(t))\middle|S\right] - \theta^2(T,K)\right)S^2\gamma_{BS}dSdt$$

where $\theta(T,K)$ is *not* a function of t and S, and $E^0[\Lambda] = 0$ if $\theta = \theta_M(T,K)$, the market implied volatility, so that:

$$\theta_M^2(T,K) = \frac{\int_{t_0}^{T}\int_{0}^{\infty} E^0\left[\sigma^2(t,S;\mathbf{v}(t))\middle|S\right]S^2\gamma_{BS}dSdt}{\int_{t_0}^{T}\int_{0}^{\infty} S^2\gamma_{BS}dSdt}$$

Finally, using $\sigma_{LV}^2(t,S) = E^0\left[\sigma^2(t,S;\mathbf{v}(t))\middle|S\right]$ from (4), we conclude the proof. ■

Proof of Theorem 1

From Ito's lemma, the dynamic of the claim price $f_L(t, S; \mathbf{v})$, defined as a function of t, S and a set of parameters $\mathbf{v}$, under the risk-neutral measure, is:

$$df_L = \frac{\partial f_L}{\partial t}dt + \frac{\partial f_L}{\partial S}dS + \tfrac{1}{2}\frac{\partial^2 f_L}{\partial S^2}dS^2 + \sum_i\left(\frac{\partial f_L}{\partial v_i} + \frac{\partial^2 f_L}{\partial v_i \partial S}dS\right)dv_i + \tfrac{1}{2}\sum_i\sum_j \frac{\partial^2 f_L}{\partial v_i \partial v_j}dv_i dv_j$$

Using (7) and (8):

$$df_L = \xi dt + \left(\sigma S\frac{\partial f_L}{\partial S} + \sum_i \beta_i \varrho_{i,S}\frac{\partial f_L}{\partial v_i}\right)dW_S + \sum_i \beta_i\sqrt{1-\varrho_{i,S}^2}\frac{\partial f_L}{\partial v_i}dW_i$$

with

$$\xi = \frac{\partial f_L}{\partial t} + (r-q)S\frac{\partial f_L}{\partial S} + \tfrac{1}{2}\sigma^2 S^2\frac{\partial^2 f_L}{\partial S^2} + \sum_i\left(\alpha_i\frac{\partial f_L}{\partial v_i} + \sigma S\beta_i\varrho_{i,S}\frac{\partial^2 f_L}{\partial v_i\partial S} + \tfrac{1}{2}\sum_j \varrho_{i,j}\beta_i\beta_j\frac{\partial^2 f_L}{\partial v_i \partial v_j}\right)$$

Then, using the Black-Scholes pde (9) and since under the risk-neutral probability the drift of f_L must be the risk-free rate – to satisfy condition (9-c) – the following drift condition must hold:

$$\sum_i \left(\alpha_i \frac{\partial f_L}{\partial v_i} + \sigma(t,S;\mathbf{v}) S \beta_i \varrho_{i,S} \frac{\partial^2 f_L}{\partial v_i \partial S} + \tfrac{1}{2} \sum_j \varrho_{i,j} \beta_i \beta_j \frac{\partial^2 f_L}{\partial v_i \partial v_j} \right) = 0 .$$

Now, in order to f_L be a proper Ito's process, the variance of each Ito's stochastic integral in f_L must be bounded almost surely for every finite time $T > t_0$. This requires:

$$\int_{t_0}^{T} \left(\sigma S \frac{\partial f_L}{\partial S} + \sum_i \beta_i \varrho_{i,S} \frac{\partial f_L}{\partial v_i} \right)^2 dt < \infty \quad \text{and} \quad \int_{t_0}^{T} \beta_i^{\,2} \left(1 - \varrho_{i,S}^2\right) \left(\frac{\partial f_L}{\partial v_i} \right)^2 dt < \infty$$

Finally, using $\int_{t_0}^{T} \sigma^2 dt < \infty$ and $\int_{t_0}^{T} \beta_i^{\,2} dt < \infty$ from Section III, it is easy to show that:

$$\int_{t_0}^{T} \left(\frac{\partial f_L}{\partial S} \right)^2 dt < \infty \qquad \int_{t_0}^{T} \left(\frac{\partial f_L}{\partial v_i} \right)^2 dt < \infty$$ ∎

Proof of Theorem 2

When $S(t)$ and $\mathbf{v}(t)$ are correlated, we can express each $v_i(t)$ as a function of t, S and W_i so that from Ito's lemma:

$$dv_i = \left(\frac{\partial v_i}{\partial t} + (r-q) S \frac{\partial v_i}{\partial S} + \tfrac{1}{2} \sigma^2(t,S;\mathbf{v}) S^2 \frac{\partial^2 v_i}{\partial S^2} + \tfrac{1}{2} \frac{\partial^2 v_i}{\partial W_i^{\,2}} \right) dt + \sigma(t,S;\mathbf{v}) S \frac{\partial v_i}{\partial S} dW_S + \frac{\partial v_i}{\partial W_i} dW_i$$

and equating coefficients with (8):

$$\frac{\partial v_i}{\partial S} = \frac{\beta_i \varrho_{i,S}}{\sigma S} \Rightarrow \frac{\partial^2 v_i}{\partial S^2} = -\frac{\beta_i \varrho_{i,S}}{\sigma S^2}$$

$$\frac{\partial v_i}{\partial W_i} = \beta_i \sqrt{1 - \varrho_{i,S}^2} \Rightarrow \frac{\partial^2 v_i}{\partial W_i^{\,2}} = 0$$

$$\frac{\partial v_i}{\partial t} = \alpha_i - (r-q) \frac{\beta_i \varrho_{i,S}}{\sigma} + \tfrac{1}{2} \sigma \beta_i \varrho_{i,S}$$

Then, the chain rule gives the first order price sensitivity, delta, as:

$$\delta_{SLV} = \frac{d}{dS}\left(f_L(t,S;\mathbf{v}) \right) = \frac{\partial f_L}{\partial S} + \sum_i \frac{\partial f_L}{\partial v_i} \frac{\partial v_i}{\partial S} \Rightarrow \delta_{SLV} = \delta_L + \sum_i \frac{\beta_i \varrho_{i,S}}{\sigma S} \frac{\partial f_L}{\partial v_i}$$

Note that using the claim price dynamics of Theorem 1 it can be readily verified that the same delta can be derived using a delta-hedged portfolio $\Pi = f_L - \delta_{SLV} S$, with dynamics $d\Pi = df_L - \delta_{SLV}(dS + qSdt)$. Similarly it is easy to show that (15-b) and (15-c) hold for gamma and theta using:

$$\gamma_{SLV} = \frac{d}{dS}\left(\delta_{SLV}(t,S;\mathbf{v})\right) \qquad \Theta_{SLV} = \frac{d}{dt}\left(f_L(t,S;\mathbf{v})\right) \qquad \blacksquare$$

Proof of Theorem 3

In the physical measure, dynamics (7) can be written as:

$$dS = (\mu - q)Sdt + \sigma(t,S;\mathbf{v})SdW_S^P \qquad \text{(A-2a)}$$

with the associated Girsanov transformation:

$$dW_S^P = dW_S - \lambda dt = dW_S - \frac{\mu - r}{\sigma(t,S;\mathbf{v})}dt \qquad \text{(A-2b)}$$

where $\lambda = \dfrac{\mu - r}{\sigma(t,S;\boldsymbol{v})}$ is the market price of risk and $\sigma(t, S; \boldsymbol{v})$ is assumed constant over the infinitesimal time-step dt. Now if we assume (A-2a) and (A-2b) also hold over a *small* time-step Δt, the (observable) discrete price process under the physical measure can be described as:[37]

$$\Delta S = (\mu - q)S\Delta t + \sigma(t,S;\mathbf{v})S\Delta W_S^P\ . \qquad \text{(A-3)}$$

$$\Delta v_i = \alpha_i^P \Delta t + \beta_i \Delta Z_i^P \qquad \text{(A-4)}$$

$$\Delta Z_i^P = \varrho_{i,S}\Delta W_S^P + \sqrt{1-\varrho_{i,S}^2}\,\Delta W_i = \Delta Z_i - \varrho_{i,S}\lambda\Delta t$$

satisfying

$$E\left(\Delta Z_i \Delta Z_j\right) \xrightarrow{a.s.} \varrho_{i,j}\Delta t \quad \text{and} \quad E\left(\Delta W_i \Delta W_S\right) \xrightarrow{a.s.} 0 \quad \text{for } i, j \in \{1, 2\ldots n\}$$

so that

$$\alpha_i^P = \alpha_i + \beta_i \varrho_{i,S}\lambda\ .$$

Using (A-3) and (A-4), the elements of $X^T X/m$ are approximately:

$$Var(\Delta S) \approx \sigma^2 S^2 \Delta t$$

$$Var(\Delta v_i) \approx \beta_i^2(\varrho_{i,S}^2\Delta t + (1-\varrho_{i,S}^2)\Delta t) = \beta_i^2\Delta t\text{, for } 1 \le i \le n.$$

$$Cov(\Delta S, \Delta v_i) \approx \sigma S \beta_i \varrho_{i,S}\Delta t$$

[37] This is not true for *long* time-steps, when the volatility cannot be assumed constant over Δt, and so the Girsanov transformation may not apply. However, over sufficiently small time-steps, e.g. daily, this should not be a problem.

$$Cov(\Delta v_i, \Delta v_j) \approx \beta_i \beta_j \varrho_{i,j} \Delta t$$

That is:

$$\frac{X^T X}{m} \approx \begin{bmatrix} \sigma^2 S^2 & \sigma S \beta_1 \varrho_{1,S} & \dots & \sigma S \beta_n \varrho_{n,S} \\ \sigma S \beta_1 \varrho_{1,S} & \beta_1^2 & \dots & \beta_1 \beta_n \varrho_{1,n} \\ \dots & \dots & \dots & \dots \\ \sigma S \beta_n \varrho_{n,S} & \beta_1 \beta_n \varrho_{1,n} & \dots & \beta_n^2 \end{bmatrix} \Delta t$$

It follows that:

$$\beta_i \approx \sqrt{\frac{Var(\Delta v_i)}{\Delta t}}\,,\ \varrho_{i,S} \approx \frac{Cov(\Delta S, \Delta v_i)}{\sqrt{Var(\Delta S) Var(\Delta v_i)}} \text{ and } \varrho_{i,j} \approx \frac{Cov(\Delta v_i, \Delta v_j)}{\sqrt{Var(\Delta v_i) Var(\Delta v_j)}}$$

Now, consider the expected value of Δv_i. From (A-4):

$$E[\Delta v_i] = \alpha_i^P \Delta t \Rightarrow \alpha_i \approx \frac{1}{m\Delta t} \sum_t \Delta v_{i,t} - \beta_i \varrho_{i,S} \lambda$$

Finally, replacing each approximation into Theorems 1 and 2, we derive the approximations for the no-arbitrage condition and for the delta and gamma, concluding the proof. ■

Proof of Lemma 2:

From the Black-Scholes differential equation we have:

$$\frac{\partial f_L}{\partial t} + (r-q) S \frac{\partial f_L}{\partial S} + \tfrac{1}{2} \sigma^2(t, S; \mathbf{v}) S^2 \frac{\partial^2 f_L}{\partial S^2} = r f_L \quad \text{and} \quad \frac{\partial f_{BS}}{\partial t} + (r-q) S \frac{\partial f_{BS}}{\partial S} + \tfrac{1}{2} \theta^2 S^2 \frac{\partial^2 f_{BS}}{\partial S^2} = r f_{BS}$$

Since $f_L(t, S; \mathbf{v}) = f_{BS}(t, S; \theta)$ by the definition of the local implied volatility θ in (17), we have:

$$\left(\frac{\partial f_L}{\partial t} - \frac{\partial f_{BS}}{\partial t} \right) + (r-q) S \left(\frac{\partial f_L}{\partial S} - \frac{\partial f_{BS}}{\partial S} \right) + \tfrac{1}{2} \sigma^2(t, S; \mathbf{v}) S^2 \left(\frac{\partial^2 f_L}{\partial S^2} - \frac{\partial^2 f_{BS}}{\partial S^2} \right) + \tfrac{1}{2} \left(\sigma^2(t, S; \mathbf{v}) - \theta^2 \right) S^2 \frac{\partial^2 f_{BS}}{\partial S^2} = 0\,.$$

Then, using Lemma 1:

$$\Upsilon_{BS} \frac{\partial \theta}{\partial t} + (r-q) S \Upsilon_{BS} \frac{\partial \theta}{\partial S} + \tfrac{1}{2} \sigma^2(t, S; \mathbf{v}) S^2 \left(\Upsilon_{BS} \frac{\partial^2 \theta}{\partial S^2} + \varkappa_{BS} \left(\frac{\partial \theta}{\partial S} \right)^2 + 2 \Omega_{BS} \frac{\partial \theta}{\partial S} \right) + \tfrac{1}{2} \left(\sigma^2(t, S; \mathbf{v}) - \theta^2 \right) S^2 \gamma_{BS} = 0\,.$$

Finally, since the Black-Scholes sensitivities are related by:

$$\gamma_{BS} = \frac{1}{\theta \tau S^2} \Upsilon_{BS} \qquad \varkappa_{BS} = \frac{d_1 d_2}{\theta} \Upsilon_{BS} \qquad \Omega_{BS} = -\frac{d_2}{\theta S \sqrt{\tau}} \Upsilon_{BS}$$

and since Υ_{BS} is strictly positive for finite $\tau > 0$, we can divide by Υ_{BS} concluding the proof. ■

Proof of Theorem 4:

From Ito's lemma and using (7) and (8), the dynamics of $\theta(K, T; t, S, \mathbf{v})$ are given by:

$$d\theta = \begin{pmatrix} \dfrac{\partial\theta}{\partial t} + (r-q)S\dfrac{\partial\theta}{\partial S} + \tfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2\theta}{\partial S^2} + \sum_i \alpha_i \dfrac{\partial\theta}{\partial v_i} + \\ \sum_i \dfrac{\partial^2\theta}{\partial S \partial v_i}\sigma S \beta_i \varrho_{i,S} + \tfrac{1}{2}\sum_i\sum_j \dfrac{\partial^2\theta}{\partial v_i \partial v_j}\beta_i\beta_j\varrho_{i,j} \end{pmatrix} dt + \sigma S\frac{\partial\theta}{\partial S}dW_S + \sum_i \frac{\partial\theta}{\partial v_i}\beta_i dZ_i$$

Using Lemma 1, the drift expands to:

$$\frac{\partial\theta}{\partial t} + (r-q)S\frac{\partial\theta}{\partial S} + \tfrac{1}{2}\sigma^2 S^2\frac{\partial^2\theta}{\partial S^2} + \frac{1}{\Upsilon_{BS}}\sum_i\left(\alpha_i\frac{\partial f_L}{\partial v_i} + \sigma S\beta_i\varrho_{i,S}\frac{\partial^2 f_L}{\partial S\partial v_i} + \tfrac{1}{2}\sum_j\frac{\partial^2 f_L}{\partial v_i \partial v_j}\beta_i\beta_j\varrho_{i,j}\right) -$$
$$\sum_i \frac{1}{\Upsilon_{BS}}\left[\frac{\partial\theta}{\partial v_i}\left(\Omega_{BS} + \varkappa_{BS}\frac{\partial\theta}{\partial S}\right)\right]\sigma S\beta_i\varrho_{i,S} - \tfrac{1}{2}\sum_i\sum_j\frac{1}{\Upsilon_{BS}}\left[\varkappa_{BS}\frac{\partial\theta}{\partial v_i}\frac{\partial\theta}{\partial v_j}\right]\beta_i\beta_j\varrho_{i,j}$$

Using Theorem 1 and Lemmas 1 and 2, this re-arranges to:

$$\tfrac{1}{2}\frac{1}{\theta\tau}\left(\theta^2 - \sigma^2\right) + \left(\sigma S\frac{\partial\theta}{\partial S} + \sum_i \beta_i\varrho_{i,S}\frac{\partial\theta}{\partial v_i}\right)\sigma\frac{d_2}{\theta\sqrt{\tau}} -$$
$$\tfrac{1}{2}\frac{d_1 d_2}{\theta}\left(\sigma^2 S^2\left(\frac{\partial\theta}{\partial S}\right)^2 + 2\sum_i \sigma S\beta_i\varrho_{i,S}\frac{\partial\theta}{\partial S}\frac{\partial\theta}{\partial v_i} + \sum_i\sum_j\beta_i\beta_j\varrho_{i,j}\frac{\partial\theta}{\partial v_i}\frac{\partial\theta}{\partial v_j}\right)$$

with $\tau > 0$ and $\theta > 0$. Now, defining ψ and η^2 as:

$$\psi = \sigma S\frac{\partial\theta}{\partial S} + \sum_i \beta_i\varrho_{i,S}\frac{\partial\theta}{\partial v_i} \quad \text{and} \quad \eta^2 = \psi^2 + \sum_i\sum_j \beta_i\beta_j\left(\varrho_{i,j} - \varrho_{i,S}\varrho_{j,S}\right)\frac{\partial\theta}{\partial v_i}\frac{\partial\theta}{\partial v_j}$$

it is easy to show that the drift simplifies to:

$$\tfrac{1}{2}\frac{1}{\theta\tau}\left(\theta^2 - \sigma^2\right) + \sigma\frac{d_2}{\theta\sqrt{\tau}}\psi - \tfrac{1}{2}\frac{d_1 d_2}{\theta}\eta^2$$

Finally, in order to θ be a proper Ito's process, we must have:

$$\int_{t_0}^{T}\sigma^2 S^2\left(\frac{\partial\theta}{\partial S}\right)^2 dt < \infty \Rightarrow \int_{t_0}^{T}\left(\frac{\partial\theta}{\partial S}\right)^2 dt < \infty$$

$$\int_{t_0}^{T} \beta_i^2 \left(\frac{\partial \theta}{\partial v_i} \right)^2 dt < \infty \Rightarrow \int_{t_0}^{T} \left(\frac{\partial \theta}{\partial v_i} \right)^2 dt < \infty$$

and
$$\int_{t_0}^{T} \left| \tfrac{1}{2} \frac{1}{\theta (T-t)} \left(\theta^2 - \sigma^2 \right) + \sigma \frac{d_2}{\theta \sqrt{T-t}} \psi - \tfrac{1}{2} \frac{d_1 d_2}{\theta} \eta^2 \right| dt < \infty$$

which also imply that $\int_{t_0}^{T} \psi^2 dt < \infty$ and $\int_{t_0}^{T} \eta^2 dt < \infty$ since $\int_{t_0}^{T} \sigma^2 dt < \infty$ and $\int_{t_0}^{T} \beta_i^{\,2} dt < \infty$ by definition. ■

Proof of Corollary 1:

Since **A** is the Cholesky decomposition of **Σ,** we have $\mathbf{A}\mathbf{A}^{\mathrm{T}} = \mathbf{\Sigma}$ so that:

$$dW_i dW_j = \Sigma_{i,j} dt = \left(\sum_k A_{i,k} A_{j,k} \right) dt$$

and
$$dW_i = \sum_j A_{i,j} dW_j^*$$

Substituting dZ_i from (8) into (18) and using the above, the diffusion term in (18) becomes:

$$\psi dW_S + \sum_j \sum_i \frac{\partial \theta}{\partial v_i} \beta_i \sqrt{1 - \varrho_{i,S}^2} A_{i,j} dW_j^*$$

But since $A_{i,j} = 0$ for all $i < j$, the diffusion term simplifies to:

$$\psi dW_S + \sum_j \omega_j dW_j^* \qquad \text{with} \qquad \omega_j = \sum_{i=j}^{n} \beta_i \sqrt{1 - \varrho_{i,S}^2} A_{i,j} \frac{\partial \theta}{\partial v_i}$$

Finally, from (8):

$$dZ_i dZ_j = \left(\varrho_{i,S} \varrho_{j,S} + \sqrt{1 - \varrho_{i,S}^2} \sqrt{1 - \varrho_{j,S}^2} \Sigma_{i,j} \right) dt = \varrho_{i,j} dt$$

and considering the variance of $d\theta$ we have:

$$\eta^2 dt = d\theta d\theta = \left(\psi^2 + \sum_j \omega_j^2 \right) dt \, .$$ ■

Appendix B: No Arbitrage Conditions

Whatever the functional form we assume for the spot volatility $\sigma(t, S; \mathbf{v})$ in (7), the underlying risk-neutral density consistent with (7), say $g_{L,t}(S)$, must satisfy the following conditions:

$$g_{L,t}(S) \geq 0 \text{ for all } S \geq 0 \quad \text{and} \quad \int_0^\infty g_{L,t}(S)dS = 1 \qquad \text{(B-1a)}$$

$$f_L = e^{-r(t-t_0)} \int_0^\infty G(t,S)g_{L,t}(S)dS \quad \text{for every } t > t_0 \qquad \text{(B-1b)}$$

where $G(t, S)$ is the value of any tradable asset at some time $t > t_0$. Conditions (B-1a) define $g_{L,t}(S)$ as a proper probability density function of S, while (B-1b) requires Q to be a martingale measure.

Although rather obvious, these conditions add an important constraint when pricing options. For instance, if $C(K,T; t, S)$ is the price of a vanilla European call at time t with $K \geq 0$ and $T > t$, then these conditions imply: [38]

$$S \geq C(K,T;t,S) \geq max\left\{S - Ke^{-r(T-t)};0\right\} \qquad \text{(B-2a)}$$

$$C(0,T;t,S) = S \quad \text{and} \quad \lim_{K\to\infty} C(K,T;t,S) = 0 \qquad \text{(B-2b)}$$

$$-1 \leq \frac{\partial C(K,T;t,S)}{\partial K} \leq 0 \quad \text{and} \quad \frac{\partial^2 C(K,T;t,S)}{\partial K^2} \geq 0 \qquad \text{(B-2c)}$$

Whilst (B-2a) and (B-2b) are intuitive, (B-2c) tells an interesting story. It requires the call option price to be a convex and monotonically decreasing function of K, otherwise there will be an arbitrage opportunity.

Finally, Brunner and Hafner (2003) also prove two more necessary but not sufficient conditions for no arbitrage on the term structure of call prices. For $T_1, T_2 \in (t, T)$, $T_1 < T_2$, they require:

$$C(K,T_2;t,S) \geq C(Ke^{-r(T_2-T_1)},T_1;t,S) \qquad \text{(B-3a)}$$

$$\int_0^\infty max\{S-K;0\}\left(e^{-r(T_2-T_1)}g_{L,T_2}(S) - g_{L,T_1}(Se^{-r(T_2-T_1)})\right)dS \geq 0 \qquad \text{(B-3b)}$$

In effect, even when $g_{L,t}(S)$ is a martingale measure satisfying (B-1a) and (B-1b), there is an arbitrage opportunity between different maturities if either (B-3a) or (B-3b) is violated – e.g. using a calendar spread arbitrage strategy.

Finally note that we have been careful to distinguish between the model risk-neutral density $g_{L,t}(S)$ (i.e. consistent with a certain local volatility model) and the market risk-neutral density $g_t(S)$ (i.e. consistent with

[38] See Carr (2001) and Brunner and Hafner (2003).

observed market options prices). Clearly whilst we expect these two densities to share similar properties, they are unlikely to be the same, since parametric local volatility models can only approximate observed options prices in general. Nevertheless, a valid calibration of the local volatility surface must satisfy *all* conditions outlined above, hence the definition that Ω_t denotes the space of permissible arbitrage-free values for $\mathbf{v}(t)$.

Appendix C: Implied Volatility Sensitivities to *K* and *T*.

The sensitivities to K and T are intuitive and easy to derive since prices are quoted for many discrete pairs (K, T). These can be estimated directly, using finite differences based on any reasonable interpolation method for market implied volatility surfaces.[39] For instance, for a given time t and asset price S interpolate the market implied volatilities using a functional form of the type:[40]

$$ln\,\theta_M(K,T;t,S) = b_0 + b_1\, ln\frac{K}{S} + b_2\left(ln\frac{K}{S}\right)^2 + b_3(T-t) + b_4(T-t)^2 + b_5\, ln\frac{K}{S}(T-t) + \varepsilon \qquad \text{(C-1)}$$

This is a second-order Taylor expansion of the log implied volatility about moneyness, $ln\,(K/S)$ and time to expiry $(T - t)$.[41] Now $\partial\theta/\partial K$ and $\partial\theta/\partial T$ and higher order sensitivities follow from simple differentiation, assuming $\partial\varepsilon/\partial K = \partial\varepsilon/\partial T = 0$ almost surely:

$$\frac{\partial\theta_M(K,T;t,S)}{\partial K} = \frac{\theta_M}{K}\left[b_1 + 2b_2\left(ln\frac{K}{S}\right) + b_5(T-t)\right] \qquad \text{(C-2)}$$

$$\frac{\partial\theta_M(K,T;t,S)}{\partial T} = \theta_M\left[b_3 + 2b_4(T-t) + b_5\, ln\frac{K}{S}\right] \qquad \text{(C-3)}$$

Alternatively, the sensitivities to K and T may be estimated from local implied volatilities. Differentiating (17) yields:

$$\frac{\partial\theta(K,T;t,S,\mathbf{v})}{\partial K} = \left(\frac{\partial f_{BS}(K,T;t,S,\theta)}{\partial\theta}\right)^{-1}\left(\frac{\partial f_L(K,T;t,S,\mathbf{v})}{\partial K} - \frac{\partial f_{BS}(K,T;t,S,\theta)}{\partial K}\right) \qquad \text{(C-4)}$$

$$\frac{\partial\theta(K,T;t,S,\mathbf{v})}{\partial T} = \left(\frac{\partial f_{BS}(K,T;t,S,\theta)}{\partial\theta}\right)^{-1}\left(\frac{\partial f_L(K,T;t,S,\mathbf{v})}{\partial T} - \frac{\partial f_{BS}(K,T;t,S,\theta)}{\partial T}\right) \qquad \text{(C-5)}$$

where $\partial f_{BS}/\partial\theta$, $\partial f_{BS}/\partial K$ and $\partial f_{BS}/\partial T$ are the partial derivatives of the Black-Scholes price based on the *local* implied volatility θ. So to calculate (C-4) and (C-5), we only need to estimate the partial derivatives of the model price f_L with respect to K and T, which should not be difficult, especially if we have an analytical solution for f_L such as in Section VII. If the local volatility has been properly calibrated, the sensitivities derived from (C-4) and (C-5) should be approximately the same as those derived from (C-2) and (C-3).

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[39] Smoothing the implied volatility surface is advisable since market quotes for option prices are discrete by definition and liable to problems such as stale prices (poor liquidity), bid-ask spreads that include hedging costs and so forth. See Andersen and Brotherton-Radcliffe (1997), Avellaneda et al. (1997), Bouchouev and Isakov (1997, 1999) among others.

[40] Note that the coefficients b_1, b_2, …, b_5 of (C-1) must satisfy all no-arbitrage conditions mentioned in Appendix B.

[41] The coefficients could normally be estimated using a standard regression technique such as ordinary least squares. Clearly different versions of model (C-1) may produce quite different sensitivities to K and T depending on how smooth the fitted implied volatility surface is. This example is only illustrative.

