15.2 A stochastic volatility model with jumps: the Bates model

As noted above, diffusion based stochastic volatility models cannot generate sufficient variability and asymmetry in short-term returns to match implied volatility skews for short maturities. The jump-diffusion stochastic volatility model introduced by Bates [41] copes with this problem by adding proportional log-normal jumps to the Heston stochastic volatility model. In the original formulation the model has the following form:

$$\frac{dS_t}{S_t} = \mu dt + \sqrt{V_t} dW_t^S + dZ_t, \tag{15.11}$$
$$dV_t = \xi(\eta - V_t)dt + \theta\sqrt{V_t} dW_t^V,$$

where (W_t^S) and (W_t^V) are Brownian motions with correlation ρ, driving price and volatility, and Z_t is a compound Poisson process with intensity λ and log-normal distribution of jump sizes such that if k is its jump size then $\ln(1+k) \sim N(\ln(1+\bar{k}) - \frac{1}{2}\delta^2, \delta^2)$. The no-arbitrage condition fixes the drift of the risk neutral process: under the risk-neutral probability $\mu = r - \lambda\bar{k}$. Applying Itô's lemma to Equation (15.11) we obtain the equation for the log-price $X_t = \ln S_t$:

$$dX_t = (r - \lambda\bar{k} - \frac{1}{2}V_t)dt + \sqrt{V_t} dW_t^S + d\tilde{Z}_t,$$

where $(\tilde{Z}_t)$ is a compound Poisson process with intensity λ and Gaussian distribution of jump sizes. This model can also be viewed as a generalization of the Merton jump-diffusion model (see Chapter 4) allowing for stochastic volatility. Although the no arbitrage condition fixes the drift of the price process, the risk-neutral measure is not unique, because other parameters of the model (for example, intensity of jumps and parameters of jump size distribution) can be changed without leaving the class of equivalent probability measures. Jumps in the log-price do not have to be Gaussian in this model. One can replace the Gaussian distribution by any other convenient distribution for the jump size without any loss of tractability, provided that the characteristic function is computable.

Option pricing In this stochastic volatility model as well as in other models of this chapter the characteristic function of the log-price is known in closed form (see below). Therefore, European options can be priced using one of the Fourier transform methods of Chapter 11. For path-dependent options closed-form expressions are not available and one must turn to numerical methods (e.g., Monte Carlo).

Characteristic function of the log-price Let us first compute the characteristic function of the continuous component X^c_t of X_t, following [196]. Let

$$f(x,v,t) = E\{e^{iuX^c_T} | X^c_t = x, V_t = v\}.$$

Applying Itô's formula to $M_t = f(X^c_t, V_t, t)$ yields

$$dM_t = \Big(\frac{1}{2}V_t\frac{\partial^2 f}{\partial x^2} + \rho\theta V_t\frac{\partial^2 f}{\partial x\partial v} + \frac{1}{2}\theta^2 V_t\frac{\partial^2 f}{\partial v^2} + (r - \lambda\bar{k} - \frac{V_t}{2})\frac{\partial f}{\partial x} \\ + \xi(\eta - V_t)\frac{\partial f}{\partial v} + \frac{\partial f}{\partial t}\Big)dt + \sqrt{v}\frac{\partial f}{\partial x}dW^S + \theta\sqrt{v}\frac{\partial f}{\partial v}dW^V.$$

Since $f(X^c_t, V_t, t)$ is a martingale we obtain, by setting the drift term to zero:

$$\frac{1}{2}v\frac{\partial^2 f}{\partial x^2} + \rho\theta v\frac{\partial^2 f}{\partial x\partial v} + \frac{1}{2}\theta^2 v\frac{\partial^2 f}{\partial v^2} + (r - \lambda\bar{k} - \frac{1}{2}v)\frac{\partial f}{\partial x} \\ + \xi(\eta - v)\frac{\partial f}{\partial v} + \frac{\partial f}{\partial t} = 0. \quad (15.12)$$

Together with the terminal condition $f(x,u,T) = e^{iux}$ this equation allows to compute the characteristic function of log-price. To solve it, we guess the functional form of f:

$$f(x,u,t) = \exp\{C(T-t) + vD(T-t) + iux\}, \quad (15.13)$$

where C and D are functions of one variable only. Substituting this into Equation (15.12), we obtain ordinary differential equations for C and D:

$$D'(s) = \frac{1}{2}\theta^2 D^2(s) + (i\rho\theta u - \xi)D(s) - \frac{u^2 + iu}{2}, \\ C'(s) = \xi\eta D(s) + iu(r - \lambda\bar{k})$$

with initial conditions $D(0) = C(0) = 0$. These equations can be solved explicitly:

$$D(s) = -\frac{u^2 + iu}{\gamma\coth\frac{\gamma s}{2} + \xi - i\rho\theta u}, \\ C(s) = ius(r - \lambda\bar{k}) + \frac{\xi\eta s(\xi - i\rho\theta u)}{\theta^2} \\ - \frac{2\xi\eta}{\theta^2}\ln\left(\cosh\frac{\gamma s}{2} + \frac{\xi - i\rho\theta u}{\gamma}\sinh\frac{\gamma s}{2}\right),$$

where $\gamma = \sqrt{\theta^2(u^2 + iu) + (\xi - i\rho\theta u)^2}$. The characteristic function without the jump term can now be found from Equation (15.13). To incorporate the jump term, since jumps are homogeneous and independent from the continuous part, we need only to multiply the characteristic function that we have obtained by the characteristic function of the jumps which in this case is simply

$$\phi_t^J(u) = \exp\{t\lambda(e^{-\delta^2 u^2/2 + i(\ln(1+\bar{k}) - \frac{1}{2}\delta^2)u} - 1)\}.$$

Finally, the characteristic function of the price process in the model of Bates is

$$\phi_t(u) = \phi_t^J(u) \frac{\exp\left(\frac{\xi\eta t(\xi - i\rho\theta u)}{\theta^2} + iut(r - \lambda\bar{k}) + iux_0\right)}{\left(\cosh\frac{\gamma t}{2} + \frac{\xi - i\rho\theta u}{\gamma}\sinh\frac{\gamma t}{2}\right)^{\frac{2\xi\eta}{\theta^2}}} \times \exp\left(-\frac{(u^2 + iu)v_0}{\gamma\coth\frac{\gamma t}{2} + \xi - i\rho\theta u}\right) \quad (15.14)$$

with γ defined above.

Implied volatility patterns in the Bates model Some implied volatility smiles attainable in the Bates model are shown in Figure 15.2. In this model there are two ways to generate an implied volatility skew. The first way is to introduce a (negative) correlation between returns and volatility movements, as in diffusion-based stochastic volatility models. Alternatively, an implied volatility skew for short-term options can be generated by asymmetric jumps (as in exp-Lévy models) even if the noise sources driving volatility and returns are independent.

Therefore the "intuition" — originating from bivariate diffusion models — that the implied volatility skew is systematically linked to a "leverage effect" is groundless: it is simply due to the symmetry (stemming from normality) of Brownian increments and gives yet another example of a property specific to the Brownian universe.

We have seen that correlation and jumps have similar effect on the implied volatility smile; is there any feature which allows to distinguish them? The answer is yes: jumps and correlation both induce an implied volatility skew but they influence the term structure of volatility differently. It is clear from Figure 15.2 that the smiles that are due to jumps are stronger at short maturities and flatten out much faster as the time to maturity increases. In contrast, smiles due to correlation can be used to obtain a skew at longer maturities but are not sufficient to explain the prices of short-maturity options. Also, introducing jumps increases the overall level of implied volatility while correlation has little effect on it. Finally, at-the-money volatility stays roughly the same in absence of jumps and tends to increase for longer maturities when the jumps are present.

These remarks shed light on a very nice feature of the Bates model: here the implied volatility patterns for long-term and short-term options can be adjusted separately. One can start by calibrating the jumps on one or two shortest maturities and then fix the jump parameters and calibrate the other ones on longer maturities. This approach will not give the best possible calibration quality but yields reasonable results. If instead of using this procedure all parameters are fitted at the same time with least-squares method, the cost function will not be convex and one may end up in a local minimum with strange parameter values [97].

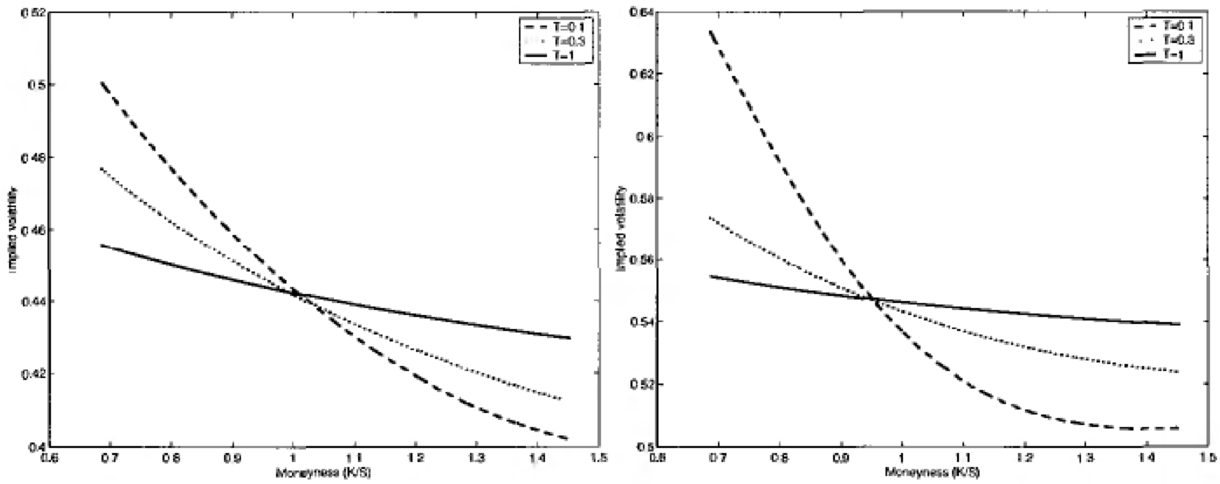

FIGURE 15.2: Implied volatility patterns in the Bates model. Left graph shows the smiles for various maturities when the jumps are absent but there is a correlation between the volatility process and the price process ($\rho = -0.5$). On the right graph the correlation is absent and the skew is due to jumps (with intensity $\lambda = 5$). The other parameters are the same for both graphs: $S_0 = 1$, $r = 0$, $V_0 = 0.2$, $\theta = 0.7$ (volatility of volatility), $\xi = 10$ (inverse correlation length), $\eta = 0.2$ (mean square volatility), $\delta = 0.1$ (jump dispersion) and $\bar{k} = -0.1$ (average relative jump size).

15.3 Non-Gaussian Ornstein-Uhlenbeck processes

Using Lévy processes as driving noise, one can construct a large family of mean-reverting jump processes with linear dynamics on which various properties such as positiveness or the choice of a marginal distribution can be imposed. We will call these processes, which are non-Gaussian generalizations of the Gaussian Ornstein-Uhlenbeck process, non-Gaussian Ornstein-Uhlenbeck processes or simply OU processes. These processes are not only convenient to model "volatility" but also have an independent interest for modelling stationary time series of various kinds such as commodity prices or interest rates [36, 307].