# AN EFFICIENT APPROACH TO VALUATION OF CREDIT BASKET PRODUCTS AND RATINGS TRIGGERED STEP-UP BONDS*

Tomasz R. Bielecki, Andrea Vidozzi and Luca Vidozzi
Department of Applied Mathematics
Illinois Institute of Technology
Chicago, IL 60616, USA

## 1 Introduction

In Bielecki et al. [1] a fully Markovian model was presented so to provide a basis for systematic approach to valuation and hedging of basket credit derivatives. In this paper, we adopt the approach of [1] and adapt it to efficiently price some selected basket credit derivatives, such as CDOs and FSTDs. [1]. Although we discuss in this paper other credit related products, such as CDO2s and credit quality triggered corporate step-up bonds, we do not provide any numerical results regarding their calibration and pricing; these will be included in a follow-up paper, which will be submitted for publication. Hedging issues will also be addressed in the follow-up paper.

The pricing models are calibrated to credit data provided by the GFI Group, Citigroup. Bond data is provided by Bloomberg, through the direct feed available at the IIT's Stuart School of Business. The calibration and pricing results presented in Section 6 indicate extreme efficiency and robustness of our approach.

In Section 2 we provide a formal description of the credit products we discuss in the paper, representing relevant cash flows in terms of formulae that we find well suited for calibration and valuation applications. In the following section, we summarize the aspects of the modeling approach presented in [1], which are relevant to practical implementation. A brief description of a simulation algorithm and of the calibration methodology we use, are given in Section 4 and in Section 5, respectively. The final section presents calibration and pricing results for selected credit products.

## 2 Description of relevant credit products

In this section, we describe the cash-flows associated to the main-stream basket credit products, focusing in particular on the recently developed standardized instruments like the Dow Jones Credit

*The authors would like to express their sincere gratitude and appreciation to Matt Woodhams from the GFI Group, and to Youssef Elouerkhaoui from the Citigroup for providing us with data relevant to this paper.

[1]The pricing and calibration libraries are implemented in C++ interfaced through Excel spreadsheet.

Default Swap indices (iTraxx and CDX), and the relative derivative contracts. In particular we will discuss Collateralized Debt Obligations (CDO), CDO squared and First to Default Swaps.

## 2.1 CDS indices

CDS indices are static portfolios of equally weighted credit default swaps (CDSs) with standard maturities of five to ten years. Typically, the index matures few months before the underlying CDSs. The debt obligations underlying the CDSs in the pool are selected from among those with highest CDS trading volume in the respective industry sector. We will typically refer to the underlying debt obligations as reference entities. CDS indices are typically issued by a pool of licensed financial institutions, known as the market makers. At the time of issuance, the market makers determine an annual rate known as *(index) spread*, to be paid out to investors on a periodic basis. By purchasing the index, an investor enters into a binding contract, whose main provisions are summarized below:

(i) The inception time of the contract is time[2] $t = 0$; the maturity time of the contract is $T$. At inception, the pool is referenced by $N$ credit names and its notional value[3] is $N$.

(ii) By purchasing the index, the investor sells protection to the market makers. Thus, the investor assumes the role of a protection seller and the market makers assume the role of protection buyers. In practice, the investors agrees to absorb all losses due to defaults in the reference portfolio, occurring between the time of inception and maturity. In case of default of a reference entity, the protection seller pays to the market makers the protection payment in the amount of $(1-\delta)$, where $\delta \in [0,1]$ is the agreed recovery rate (typically 40%). The notional on which the market maker pays the spread, henceforth referred to as *residual protection*, is then reduced by such amount. For instance, after the first default, the residual protection is updated as follows (recall that, at inception, the notional is $N$):

$$N \to N - (1-\delta).$$

(iii) In exchange, the protection seller receives from the market maker a periodic fixed premium on the residual protection[4] at the annual rate of $\eta$. If, at inception, the market index spread is different from the issuance spread, the present value of the difference is settled through an upfront payment.

We denote by $\tau_i$ the random default time of the $i^{\text{th}}$ name in the index and by $H_t^i$ the right continuous process defined as $H_t^i = \mathbb{1}_{\{\tau_i \le t\}}$, $i = 1, 2, ..., N$. Also, let $\{t_j,\ j = 0, 1, ..., J\}$ with $0 = t_0$ and $t_J \le T$ denote the tenor of the premium leg payments dates. The discounted cumulative cash flows associated to a CDS index are as follows:

$$\textit{Premium Leg} = \eta \sum_{j=0}^{J} \beta_{t_j} \left( \sum_{i=1}^{N} 1 - H_{t_j}^i (1-\delta) \right),$$

$$\textit{Protection Leg} = \sum_{i=1}^{L} \beta_{\tau_i} (1-\delta) H_T^i,$$

where $\beta_t := \exp(-\int_0^t r_s ds)$ is the discount factor.

[2]Throughout the paper we shall set to $t = 0$ the inception dates of various products discussed here. This is done so to simplify the notation; our discussion generalizes in a straightforward manner to any inception date $t \ge 0$.

[3]We henceforth assume that the face value of each reference entity is one. Thus the total notional of the index is $N$.

[4]Whenever a reference entity defaults, its weight in the index is set to zero. By purchasing one unit of index the protection seller owes protection only on those names that have not yet defaulted at time of inception.

## 2.2 Collateralized Debt Obligations

Collateralized Debt Obligations (CDO) are credit derivatives backed by portfolios of assets. If the underlying portfolio is made up of bonds, loans or other securitized receivables, such products are known as *cash* CDOs. Alternatively, the underlying portfolio may consist of credit derivatives referencing a pool of debt obligations. In the latter case, CDOs are said to be *synthetic*. Because of their recently acquired popularity, we focus our discussion on standardized (synthetic) CDO contracts backed by CDS indices. We begin with an overview of the product:

(i) The time of inception of the contract is $t = 0$, the maturity is $T$. The notional of the CDO contract is the residual protection (as defined above) of the underlying CDS index at the time of inception. We shall assume that, at inception, the CDO notional is $N$.

(ii) The credit risk (the potential loss due to credit events) borne by the reference pool is layered into different risk levels. The range in between two adjacent risk levels is called a *tranche*. The lower bound of a tranche is usually referred to as *attachment point* and the upper bound as *detachment point*. The credit risk is sold in these tranches to protection sellers. For instance, in a typical CDO contract on iTraxx, the credit risk is split into equity, mezzanine, and senior tranches, corresponding to $0-3\%, 3-6\%, 6-9\%, 9-12\%$, and $12-22\%$ of the losses, respectively. At inception, the notional value of each tranche is the CDO notional weighted by the respective *tranche width*.

(iii) The tranche buyer sells partial protection to the pool owner, by agreeing to absorb the pool's losses comprised in between the tranche attachment and detachment point. This is better understood by an example: Assume that, at inception, the protection seller purchases one currency unit worth of the $6-9\%$ tranche. One year later, as a consequence of a series of default events, the cumulative loss breaks through the tranche attachment point, reaching 8%. The protection seller fulfills his/her obligation by paying out two thirds ($= \frac{8\%-6\%}{9\%-6\%}$) of a currency unit to the market maker . The tranche notional is then reduced to one third of its pre-default event value. We refer to the remaining tranche notional as *residual tranche protection*.

(iv) In exchange, up until maturity, the CDO issuer (protection buyer) makes periodic spread payments to the tranche buyer on the residual tranche protection. Returning to our example, after the loss reaches 8%, premium payments are made on $\frac{1}{3}$ ($= \frac{9\%-8\%}{9\%-6\%}$) of the tranche notional, until the next credit event occurs or the contract matures.

We denote by $L_l$ and $U_l$ the lower and upper attachment points of the $l^{\text{th}}$ tranche and by $\kappa_l$ its market spread. It is also convenient to introduce the fractional loss process,

$$\Gamma_t = \frac{1}{N}\sum_{i=1}^{N} H_t^i(1-\delta). \tag{1}$$

Finally define by $C_l = U_l - L_l$ the portion of credit risk assigned to the $l^{\text{th}}$ tranche.

Purchasing one unit of the $l^{\text{th}}$ tranche generates the following discounted cash flows:

$$\textit{Premium Leg} = \kappa_l \sum_{j=0}^{J} \beta_{t_j} N \left(C_l - \min\left(C_l, \max(\Gamma_{t_j} - L_l, 0)\right)\right),$$

$$\textit{Protection Leg} = \sum_{i=1}^{N} \beta_{\tau_i} H_T^i (1-\delta) \mathbb{1}_{\{L_l \le \Gamma_{\tau_i} \le U_l\}}.$$

We remark here, that the equity tranche of the CDO on iTraxx or CDX is quoted as an upfront rate, say $\kappa_0$, on the total tranche notional, in addition to 500 basis points (5% rate) paid annually on the residual tranche protection. The premium leg payment, in this case, is as follows:

$$\kappa_0 C_0 N + \sum_{j=0}^{J} \beta_{t_j}(.05) N \left(C_0 - \min\left(C_0, \max(\Gamma_{t_j} - L_0, 0)\right)\right).$$

## 2.3 CDO squared

Squared CDOs (also denoted as CDO2 or $CDO^2$) have gained considerable popularity in the last twelve to eighteen months. A prototypical synthetic CDO-2 is backed by a portfolio ("outer" CDO) consisting of other synthetic CDO tranches ("inner" CDOs). The outer CDO may be referenced by up to 1000 names, although, in general the number of underlying obligors ranges between 250 and 400. Due to the limited number of liquid CDS in the market, there might be a considerable amount of overlapping among the inner CDOs. This means that a number of the underlying credit names might reference one or more of the inner CDO contracts. As a consequence, a default event might simultaneously affect more than one of the inner CDO tranches, and this leads to the necessity of keeping track of the identity of the defaulted entities. In what follows, we provide a brief description of this very exotic product:

(i) The time of inception of the contract is $t = 0$, the maturity is $T$. Clearly, the outer CDO matures at or before the maturity dates[5] of the inner CDOs. The notional of the outer CDO is the sum of the notionals of the inner CDO tranches (as defined in the previous section).

(ii) The notional of the outer CDO is, again, layered into credit levels, or tranches. We shall call the tranches of the outer and inner CDOs outer and inner tranches, respectively. Each outer tranche is responsible for a portion of the losses suffered by the outer CDO notional, which arise as a consequence of the losses incurred by the inner tranches.

(iii) The buyer of a tranche in the outer CDO sells partial protection, by agreeing to absorb the losses comprised in between the outer tranche attachment and detachment points. This is better understood by a simple example: Consider a CDO squared backed by the mezzanine tranches of three CDO contracts. The protection seller purchases the equity outer tranche (having, for example, attachment points $0 - 5\%$). Assume that credit name XYZ references all of the inner CDOs. Assume, in addition, that at the default time of XYZ, say $\tau_{XYZ}$, the cumulative loss in two out of three inner CDOs breaks through the attachment point of the respective mezzanine tranche. Then, assuming a recovery rate of $\delta$, at $\tau_{XYZ}$ the protection seller pays $2(1 - \delta)$ and the residual protection of the outer equity tranche is reduced by the same amount.

(iv) In exchange, the protection seller makes periodic spread payments on the residual notional of the outer tranche.

We shall need the following notation. Let the outer CDO be backed by $m = 1, \ldots, M$ inner CDO tranches, with respective attachment points $L_{l(m)}, U_{l(m)}$ (note that the attachment points of the inner tranches need not be the same). Let $N_m$ denote the size of the reference pool for the $m^{\text{th}}$ inner CDO. The cumulative (fractional) loss in the $m^{\text{th}}$ pool is defined as (cf. (1)):

$$\Gamma_t^m = \frac{1}{N_m} \sum_{i=1}^{N_m} H_t^i (1 - \delta).$$

In addition, we define the cumulative fractional loss in the outer CDO as:

$$\Gamma_t^{sq} = \frac{\sum_{m=1}^{M} \sum_{i=1}^{N_m} H_t^i (1 - \delta) \mathbb{1}_{\{L_{l(m)} \le \Gamma_{\tau_i}^m \le U_{l(m)}\}}}{\sum_{m=1}^{M} N_m}.$$

[5] The inner CDOs may mature at different dates.

Purchasing one unit of the $p^{\text{th}}$ outer tranche (with attachment points $[L_p\ U_p]$) generates the following discounted cash flows:

$$\textit{Premium Leg} = \upsilon^p \sum_{j=0}^{J} \beta_{t_j} \left( \sum_{m=1}^{M} N_m \right) \left( C_p - \min\left( C_p, \max(\Gamma^{sq}_{t_j} - L_p, 0) \right) \right),$$

$$\textit{Protection Leg} = \sum_{m=1}^{M} \sum_{i=1}^{N_m} \beta_{\tau_i} H^i_T (1-\delta) \mathbb{1}_{\{L_{l(m)} \leq \Gamma^m_{\tau_i} \leq U_{l(m)}\}} \mathbb{1}_{\{L_p \leq \Gamma^{sq}_{\tau_i} \leq U_p\}},$$

where, as in the previous section $C_p = U_p - L_p$ denotes the outer tranche width and $\upsilon^p$ is the spread on the outer $p^{\text{th}}$ tranche [6].

## 2.4 N$^{\text{th}}$-to-default Swaps

N$^{\text{th}}$-to-default swaps (NTDS) are basket credit instruments backed by portfolios of single name CDSs. Since the growth in popularity of CDS indices and their associated derivatives, NTDS have become rather illiquid. Currently, such products are typically customized bank to client contracts, and hence relatively bespoke to the client's credit portfolio. For this reason, we focus our attention on First to Default Swap contracts issued on the iTraxx index, which are the only ones with a certain degree of liquidity[7]. Standardized FTDS are now issued on each of the iTraxx sector sub-indices. Each FTDS is backed by an equally weighted portfolio of five single name CDSs in the relative sub-index, chosen according to some liquidity criteria. The main provisions contained in a FTDS contract are the following:

(i) The time of inception of the contract is $t = 0$, the maturity is $T$.

(ii) By investing in a FTDS, the protection seller agrees to absorb the loss produced by the first default in the reference portfolio

(iii) In exchange, the protection seller is paid a periodic premium, known as FTDS spread, computed on the residual protection. We denote the FTDS spread by $\varphi$.

Recall that $\{t_j, j = 0, 1, ..., J\}$ with $0 = t_0$ and $t_J \leq T$ denotes the tenor of the premium leg payments dates. Also, denote by $\tau^{(1)}$ the (random) time of the first default in the pool. The discounted cumulative cash flows associated to a FTDS on an iTraxx sub-index containing $N$ names are as follows (again we assume that each name in the basket has notional equal to one):

$$\textit{Premium Leg} = \sum_{j=0}^{J} \varphi \beta_{t_j} \mathbb{1}_{\{\tau^{(1)} \geq t_j\}},$$

$$\textit{Protection Leg} = \beta_{\tau_1} (1-\delta) \mathbb{1}_{\{\tau^{(1)} \leq T\}}.$$

## 2.5 Ratings triggered corporate Step-Up bonds.

These bonds were issued by some European telecom companies in the recent 5-6 years. As of now, to our knowledge, these products are not traded in baskets , however they are of interest because they offer protection against credit events other than defaults. In particular, step-up bonds are corporate coupon issues for which the coupon payment depends on the issuer's credit quality: in

[6] We make the remark that, if there is overlapping among the inner CDOs, the same credit name might be indexed by different $i$ subscripts. If for instance, credit name XYZ is the $10^{\text{th}}$ obligor in the first inner CDO and the $20^{\text{th}}$ obligor in the second inner CDO, XYZ is indexed by $i = 10$ and $i = N_1 + 20$.

[7] Thanks to Matt Woodhams from GFI Group for his valuable comments in this regard.

principle, the coupon payment increases when the credit quality of the issuer declines. In practice, for such bonds, credit quality is reflected in credit ratings assigned to the issuer by at least one credit ratings agency (Moody's-KMV or Standard&Poor's). The provisions linking the cash flows of the step-up bonds to the credit rating of the issuer have different step amounts and different rating event triggers. In some cases, a step-up of the coupon requires a downgrade to the trigger level by both rating agencies. In other cases, there are step-up triggers for actions of each rating agency. Here, a downgrade by one agency will trigger an increase in the coupon regardless of the rating from the other agency. Provisions also vary with respect to step-down features which, as the name suggests, trigger a lowering of the coupon if the company regains its original rating after a downgrade. In general, there is no step-down below the initial coupon for ratings exceeding the initial rating. Next, we give a brief summary of the most common provisions characterizing the payoff of a step-up bond (typically, a step-up bond is subject to a selection of the provisions listed below):

(i) Step-up: The coupon increases if the rating decreases and hits the rating-trigger.

(ii) Step-down: The coupon decreases if the rating increases over the rating-trigger after the trigger level was previously hit.

(iii) One-off: The coupon increases only once, even if the rating falls further below the rating-trigger; for bonds that are not one-off, each further decrease in the rating, causes a further increase in the coupon.

(iv) And/or: Determines whether the coupon is adjusted if both Moody's and S&P ratings hit the trigger, or whether the adjustment occurs if either Moody's or S&P ratings hit the trigger level.

(v) Accrual: the coupon increases may be enforced either starting from the next coupon payment or immediately following a rating action.

Let $X_t$ stand for some indicator of credit quality at time $t$ (note that in this case, the process $X$ may denote two distinct rating processes). Assume that $t_i$, $i = 1, 2, \ldots, n$ are coupon payment dates. In this paper we assume the convention that coupon paid at date $t_n$ depends only on the rating at date $t_{n-1}$, that is: $c_n = c(X_{t_{n-1}})$ be the coupon payments. In other words, we assume that no accrual convention is in force.

Assuming that the bond's notional amount is 1, the cumulative discounted cash flow of the step-up bond is (as usual we assume that the current time is 0):

$$(1 - H_T)\beta_T + \int_{(0,T]} (1 - H_u)\beta_u \, dC_u + \beta_\tau Z_\tau H_T, \tag{2}$$

where $C_t = \sum_{t_i \le t} c_i$, $\tau$ is the bond's default time, $H_t = \mathbb{1}_{\tau \le t}$, and where $Z_t$ is a (predictable) recovery process.

# 3 Markovian Market Model

In this section, we give a brief description of the Markovian market model that we implement for evaluating and hedging basket credit instruments. This framework is a special case of the more general model introduced in Bielecki et al.[1], which allows to incorporate information relative to the dynamic evolution of credit ratings and credit migration processes in the pricing of basket instruments. We begin with some notation.

Let the underlying probability space be denoted by $(\Omega, \mathcal{G}, \mathbb{G}, \mathbf{P})$, where $\mathbf{P}$ is a risk neutral measure inferred from the market (we shall discuss this in further detail when addressing the issue of model calibration), $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$ is a filtration containing all information available to market agents. The filtration $\mathbb{H}$ carries information about the evolution of credit events, such as changes in credit ratings or defaults of respective credit names. The filtration $\mathbb{F}$ is a reference filtration containing information pertaining to the evolution of relevant macroeconomic variables.

We consider $N$ obligors (or credit names) and we assume that the current credit quality of each reference entity can be classified into $\mathcal{K} := \{1, 2, \ldots, K\}$ rating categories. By convention, the category $K$ corresponds to default. Let $X^l$, $l = 1, 2, \ldots, N$ be processes on $(\Omega, \mathcal{G}, \mathbf{P})$ taking values in the finite state space $\mathcal{K}$. The processes $X^l$ represent the evolution of credit ratings of the $l^{\text{th}}$ reference entity. We define the *default time* $\tau_l$ of the $l^{\text{th}}$ reference entity by setting

$$\tau_l = \inf\{\, t > 0 \,:\, X^l_t = K\} \tag{3}$$

We assume that the default state $K$ is absorbing, so that for each name the default event can only occur once.

We denote by $X = (X^1, X^2, \ldots, X^N)$ the joint credit rating process of the portfolio of $N$ credit names. The state space of $X$ is $\mathcal{X} := \mathcal{K}^N$ and the elements of $\mathcal{X}$ will be denoted by $x$. We postulate that the filtration $\mathbb{H}$ is the natural filtration of the process $X$ and that the filtration $\mathbb{F}$ is generated by a $\mathbb{R}^n$ valued factor process, $Y$, representing the evolution of relevant economic variables, like short rate or equity price processes.

We assume that the process $M = (X, Y)$ is jointly Markov under $\mathbf{P}$, so that we have, for every $0 \le t \le s$, $x \in \mathcal{X}$, and any set $\mathcal{Y}$ from the state space of $Y$,

$$\mathbf{P}(X_s = x, Y_s \in \mathcal{Y} \,|\, \mathcal{H}_t \vee \mathcal{F}^Y_t) = \mathbf{P}(X_s = x, Y_s \in \mathcal{Y} \,|\, X_t, Y_t). \tag{4}$$

The process $M$ is constructed as a Markov chain modulated by a Lévy process. We shall refer to $X$ ($Y$, respectively) as the *Markov chain component* of $M$ (the *Lévy component* of $M$, respectively). We provide the following structure to the generator of the process $M$.

$$\begin{aligned} \mathbf{A} f(x,y) = {} & (1/2)\sum_{i,j=1}^{n} a_{ij}(y)\partial_i\partial_j f(x,y) + \sum_{i=1}^{n} b_i(y)\partial_i f(x,y) \\ & + \int_{\mathbb{R}^n} \big(f(x, y + g(y,y')) - f(x,y)\big)\nu(dy') \\ & + \sum_{l=1}^{L} \sum_{x^{l\prime} \in \mathcal{K}} \lambda^l(x, x'_l; y) f(x'_l, y), \end{aligned} \tag{5}$$

where we write $x'_l = (x^1, x^2, \ldots, x^{l-1}, x'^l, x^{l+1}, \ldots, x^L)$. Given $X_t = x$ and $Y_t = y$, the intensity matrix of the Markov chain component is given by $\Lambda_t = [\lambda(x, x'; y)]_{x' \in \mathcal{X}}$. The Lévy component satisfies the SDE:

$$dY_t = b(Y_t)\, dt + \sigma(Y_t)\, dW_t + \int_{\mathbb{R}^n} g(Y_{t-}, y')\, N(dy', dt),$$

where, for a fixed $y \in \mathbb{R}^n$, $N(dy', dt)$ is a counting process with Lévy measure $\nu(dy')$ and $\sigma(y)$ satisfies $\sigma(y)\sigma(y)^{\mathsf{T}} = a(y)$.

Note that the model specified by (5) does not allow for simultaneous jumps of the components $X^l$ and $X^{l'}$ for $l \neq l'$. In other words, the ratings of different credit names may not change simultaneously. Nevertheless, this is not a serious lack of generality, as the ratings of both credit names may still change in an arbitrarily small time interval. The advantage is that, for the purpose of simulation of paths of process $X$, rather than dealing with $\mathcal{X} \times \mathcal{X}$ intensity matrix $[\lambda(x, x'; y)]$, we shall deal with $N$ intensity matrices $[\lambda^l(x, x'_l; y)]$, each of dimension $\mathcal{K} \times \mathcal{K}$ (for any fixed $y$). We stress that, within the present set-up, the current credit rating of the credit name $l$ directly impacts the intensity of transition of the rating of the credit name $l'$, and vice versa. This property, known as *frailty*, may contribute to default contagion.

## 3.1 Valuation of Basket Credit Derivatives in the Markovian Framework

We now discuss the pricing of the basket instruments introduced in section two of the paper. In particular, computing the fair spreads of such products involves evaluating the conditional expectation, under the risk neutral measure $\mathbf{P}$, of some quantities related to the cash flows associated to each instrument. In the case of CDS indices, CDOs, CDO2s and FTDS, the fair spread is such that, at inception, the value of the contract is exactly zero, i.e the risk neutral expectations of the fixed leg and protection leg payments are identical. The following expressions can be easily derived from the discounted cumulative cash flows given in Section 2. They represent initial (at time $t = 0$) values of spreads and prices, given the state of the market at inception, $(X_0, Y_0) = (x, y)$:

- the fair spread of a single name CDS is:

$$\eta^l = \frac{\mathbf{E}_{\mathbf{P}}^{x,y}\left(\beta_{\tau_l} H_T^l\right)(1-\delta)}{\mathbf{E}_{\mathbf{P}}^{x,y}\left(\sum_{j=0}^{J} \beta_{t_j}(1 - H_{t_j}^l)\right)}$$

- the fair spread of a CDS index is:

$$\eta = \frac{\mathbf{E}_{\mathbf{P}}^{x,y} \sum_{i=1}^{L} \beta_{\tau_i}(1-\delta) H_T^i}{\mathbf{E}_{\mathbf{P}}^{x,y} \sum_{j=0}^{J} \beta_{t_j} \left(\sum_{i=1}^{N} 1 - H_{t_j}^i (1-\delta)\right)}$$

- the fair spread of the CDO equity tranche is:

$$\kappa_0 = \frac{1}{C_0 N} \mathbf{E}_{\mathbf{P}}^{x,y} \Big( \sum_{i=1}^{N} \beta_{\tau_i} H_T^i (1-\delta) \mathbb{1}_{\{L_l \le \Gamma_{\tau_i} \le U_l\}}$$

$$- \sum_{j=0}^{J} \beta_{t_j} (.05) N \left(C_0 - \min\left(C_0, \max(\Gamma_{t_j} - L_0, 0)\right)\right) \Big)$$

- the fair spread of the $l^{\text{th}}$ CDO tranche is:

$$\kappa_l = \frac{\mathbf{E}_{\mathbf{P}}^{x,y}\left(\sum_{i=1}^{N} \beta_{\tau_i} H_T^i (1-\delta) \mathbb{1}_{\{L_l \le \Gamma_{\tau_i} \le U_l\}}\right)}{\mathbf{E}_{\mathbf{P}}^{x,y}\left(\sum_{j=0}^{J} \beta_{t_j} N \left(C_l - \min\left(C_l, \max(\Gamma_{t_j} - L_l, 0)\right)\right)\right)}$$

- the fair spread of the $p^{\text{th}}$ tranche of the CDO squared is:

$$\upsilon_p = \frac{\mathbf{E}_{\mathbf{P}}^{x,y}\left(\sum_{m=1}^{M} \sum_{i=1}^{N_m} \beta_{\tau_i} H_T^i (1-\delta) \mathbb{1}_{\{L_{l(m)} \le \Gamma_{\tau_i}^m \le U_{l(m)}\}} \mathbb{1}_{\{L_p \le \Gamma_{\tau_i}^{sq} \le U_p\}}\right)}{\mathbf{E}_{\mathbf{P}}^{x,y}\left(\sum_{j=0}^{J} \beta_{t_j} \left(\sum_{m=1}^{M} N_m\right) \left(C_p - \min\left(C_p, \max(\Gamma_{t_j}^{sq} - L_p, 0)\right)\right)\right)}$$

- the fair spread of a First To Default Swap is:

$$\varphi = \frac{\mathbf{E}_{\mathbf{P}}^{x,y}\left(\beta_{\tau_i}(1-\delta) \mathbb{1}_{\{\tau^{(1)} \le T\}}\right)}{\mathbf{E}_{\mathbf{P}}^{x,y}\left(\sum_{j=0}^{J} \beta_{t_j} \mathbb{1}_{\{\tau^{(1)} \ge t_j\}}\right)}$$

- fair value of the step-up bond is:

$$D = \mathbf{E}_{\mathbf{P}}^{x,y}\left((1 - H_T)\beta_T + \int_{(0,T]} (1 - H_u)\beta_u \, dC_u + \beta_\tau Z_\tau H_T\right)$$

Depending on the dimensionality of the problem, the above conditional expectations will be evaluated either by means of Monte Carlo simulation, or by means of some other numerical method and, in the low dimensional cases, even analytically . In the next sections we address the practical issues of implementing the proposed theoretical framework.

# 4 Model Implementation

In this section, we discuss the practical implementation of our model. In particular we provide further structure to the generator of the Markov chain component of the joint process $(X, Y)$ and specify a general functional form for its transition intensities. We then briefly describe a recursive procedure for simulating the evolution of the process $X$.

## 4.1 Specification of Credit Ratings Transition Intensities

Because we need to simulate the joint process $(X, Y)$, it is important to specify its form in such a way to avoid unnecessary computational complexity. As noted earlier, the structure of the generator $\mathbf{A}$ that we postulate makes it so that simulation of the evolution of process $X$ reduces to recursive simulation of the evolution of processes $X^l$, whose state spaces are only of size $K$ each. In order to facilitate simulations even further, we also postulate that each migration process $X^l$ behaves like a birth-and-death process with absorption at default, and with possible jumps to default from every intermediate state. Conditional upon $(X_t, Y_t) = (x, y)$, the infinitesimal generator governing the evolution of the credit ratings of the $l^{\text{th}}$ name is the sub-stochastic matrix:

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \\ \vdots \\ K-1 \\ K \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & \cdots & K-1 & K \end{array} \\ \begin{pmatrix} \lambda^l(1,1) & \lambda^l(1,2) & 0 & \cdots & 0 & \lambda^l(1,K) \\ \lambda^l(2,1) & \lambda^l(2,2) & \lambda^l(2,3) & \cdots & 0 & \lambda^l(2,K) \\ 0 & \lambda^l(3,2) & \lambda^l(3,3) & \cdots & 0 & \lambda^l(3,K) \\ & & & & & \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda^l(K-1,K-1) & \lambda^l(K-1,K) \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \end{array},$$

where $\lambda^l(x^l, x'^l) = \lambda^l(x^l, x'^l; X_t = x, Y_t = y)$.
The functional form of the transition intensities should reflect the specific characteristics of the instruments we need to price and should be chosen to obtain the best possible fit in the calibration phase.

## 4.2 Simulation Algorithm

In general, a simulation of the evolution of the process $X$ entails high computational costs, as the the cardinality of the state space of $X$ is equal to $K^N$. Thus, for example, in case of $K = 18$ rating categories, as in Moody's ratings, and in case of a portfolio of $N = 100$ credit ratings, the state space has $18^{100}$ elements. However, the specific assumptions on the structure of the generator allow to simulate the process in a recursive fashion, which has a relatively low computational complexity. We consider here simulations of sample paths over a generic time interval, $[t_1, t_2]$, where $0 \leq t_1 < t_2$, and assume that the time $t_1$ state of the process $(X, Y)$ is $(x, y)$. Generating one sample path will, in general, involve the following steps:

**Step 1:** in Step 1, a sample path of the process $Y$ is simulated. Recall that the dynamics of the factor process are described by the SDE

$$\begin{aligned} dY_t &= b(Y_t)\,dt + \sigma(Y_t)\,dW_t + \int_{\mathbb{R}^n} g(Y_{t-}, y')\,N(dy', dt) \\ Y_{t_1} &= y \end{aligned}$$

Any standard procedure can be used to simulate a sample path of $Y$ (the reader is referred, for example, to Kloeden and Platen [2]). We denote by $\widehat{Y}$ the simulated sample path of $Y$.

**Step 2:** generate a sample path of $X$ on the interval $[t_1, t_2]$.

**Step 2.1:** simulate the first jump time of the process $X$ in the time interval $[t_1, t_2]$. Towards this end, draw from a unit exponential distribution. We denote by $\widehat{\eta}_1$ the value of the first draw. The simulated value of the first jump time, $\tau_1^X$, is then given by:

$$\widehat{\tau}_1^X = \inf\left\{ t \in [t_1, t_2] \,:\, \int_{t_1}^{t} \lambda(x, \widehat{Y}_u)\, du \geq \widehat{\eta}_1 \right\},$$

where

$$\lambda(x, \widehat{Y}_t) := -\sum_{i=1}^{N} \lambda^i(x^i, x^i; \widehat{Y}_t)$$

If $\tau_1^X > t_2$ return to step 1, otherwise go to Step 2.2.

**Step 2.2:** simulate which component of the vector process $X$ jumped at time $\widehat{\tau}_1^X$ by drawing from the conditional distribution

$$q(X^l|\tau_1^X) = \frac{\lambda^l(x^l, x^l; \widehat{Y}_{\tau_1^X})}{\lambda(x, \widehat{Y}_{\tau_1^X})}$$

where

$$\lambda^l(x^l, x^l; \widehat{Y}_t) = \lambda^l(x^l, x^l - 1; \widehat{Y}_t) + \lambda^l(x^l, x^l + 1; \widehat{Y}_t) + \lambda^l(x^l, K; \widehat{Y}_t)$$

Recall that $\lambda^l(x^l, x^l; \widehat{Y}_t) = 0$ if $x^l = K$ since $K$ is an absorbing state.

**Step 2.3:** given that the $i^{th}$ name jumped, simulate the direction of the jump by drawing from the conditional distribution

$$p^i(x'^i|x^i, \tau_1^X) = \frac{\lambda^i(x^i, x'^i; \widehat{Y}_{\tau_1^X})}{\lambda^i(x^i, x^i; \widehat{Y}_{\tau_1^X})}$$

where

$$x'^i = \{x^i - 1; x^i + 1; K\}$$

**Step 2.4:** Repeat Steps 2.1-2.3 on the interval $[\widehat{\tau}_i^X, t_2]$, $i = 1, 2, ...$ until $\widehat{\tau}_i^X > t_2$

**Step 3:** Calculate the simulated value of a relevant functional. For instance, assume that $Y$ represents the short rate process, and is used as a discount factor, i.e $\int_0^t Y_t = -\ln B_t$. In order to compute the protection leg of a CDS index, one would evaluate

$$\sum_{i=1}^{L} \frac{B_{\tau_i}}{B_t}(1 - \delta)(H_T^i - H_t^i)(\omega)$$

at each run $\omega$, and obtain the Monte Carlo estimate by averaging over all sample paths.

# 5 Model Calibration

In the previous sections we assumed a risk neutral pricing measure as given. Arbitrage free pricing, in fact, requires the existence of a risk neutral measure, under which the price processes in the underlying market are martingales. In our market model, relevant assets are the single name CDSs contained in the indices, the indices themselves, and the relative derivative products. It is a standing assumption that financial markets actually are arbitrage free, and a risk neutral measure can thus be inferred from the prevailing market prices. Choosing a risk-neutral probability measure such as to reproduce the prices of traded derivative prices is known as model calibration.

This "inverse" problem is seldom feasible thus, typically, calibrating reduces to achieving the best approximation to market prices within a given model class. The quality of such approximation is a good indicator of the ability of the model to reproduce and explain the workings of the underlying markets. The main-stream copula models are known not to provide a good fit to market data, giving rise to the concept of correlation skews and other market inconsistencies, which really reflect the inability of these models to capture the market risk factors. Calibration provides an important test for our proposed framework.

### 5.1 Calibration Procedure

Calibration of the risk neutral parameters of the model, that is, the parameters corresponding to the risk neutral measure, can be split into two separate problems: calibration of the dynamics of the factor process[8] $Y$, and calibration of the transition intensities of the process $X$.

We discuss in some detail the calibration of the dynamics of the Markov chain component $X$. If analytical formulae, or quasi-analytical formulae, for theoretical prices of financial assets, which are of interest to us, are available, then, in principle, the calibration procedure is straightforward. Such will be the case with regard to step-up bonds, for example.

Otherwise, calibration will be done by simulation. We denote the output of the simulation (which may include the spread on the single name CDS in the basket, the index spread, the CDO tranche spreads, etc.) by[9]

$$E^{\Theta}\left(f\left(X_t^{sim}, Y_t^{sim};\ _{t\leq T}\right)\right),$$

where $\Theta \in \mathbf{\Theta} \subset \mathbb{R}^n$ is a vector of coefficients parameterizing the transition intensities of the Markov chain component $X$. Calibrating the model is done by means of solving the following minimization problem:

$$\inf_{\Theta \in \mathbf{\Theta}} \left|\left|E^{\Theta}\left(f\left(X_t^{sim}, Y_t^{sim};\ _{t\leq T}\right)\right) - \mathbf{M}\right|\right|$$

where $\mathbf{M} \in \mathbb{R}^n$ is a vector of market data corresponding to the simulation output and $\|\cdot\|$ is a norm in $\mathbb{R}^n$.

Since the map $\Theta \to E^{\Theta}\left(f\left(X_t^{sim}, Y_t^{sim};\ _{t\leq T}\right)\right)$ is not known, and is likely to be non-smooth, we use the downhill simplex method (also known as the Nelder-Mead algorithm), to perform the minimization. This algorithm does not require computation of gradients and is particularly effective for high dimensional, non smooth problems. The need for computational speed becomes evident in this phase, since each function evaluation requires simulating a large number of sample paths.

## 6 Applications of the Markovian Set Up to Pricing Basket Credit Derivatives and Step-up Bonds

In the following, we specialize the general Markovian framework to pricing selected credit derivatives. First, in Section 6.1 we shall calibrate a Markovian model to market quotes on individual CDSs, CDS indices, and synthetic CDOs derived from CDS indices. Then, in Sections 6.2.2 and 6.2.3 we shall apply this framework and the calibration results for pricing FTDSs, customized CDOs, and CDO2. In Section 6.3 we shall calibrate another version of our model to market quotes on step-up bonds and price a step-up bond option.

[8] Calibration of Levy models is discussed in [3].

[9] Function $f$ represents the simulation output that is relevant to the given application (such as, CDO spread, etc.). $X^{sim}$, $Y^{sim}$ denote simulated paths of processes $X$ and $Y$.

## 6.1 Calibration of a Two State Markov Model for Pricing Credit Derivatives

In this section, we consider a special version of the general Markov model described above and we calibrate its parameters to market data.

Recall that $X_t = (X_t^1, X_t^2, \dots, X_t^N)$ denotes the joint credit ratings process of the portfolio of $N$ credit names. We assume that the current credit quality of each name in the pool can be classified into two rating categories, i.e. $\mathcal{K} := \{0, K\}$, where, 0 is a pre-default state[10], and as usual, $K$ denotes the default state. Under the term default, we encompass all credit events that warrant a protection payment. In this case, these include actual bankruptcy of the obligor, as well as situations of extreme financial distress (i.e. the obligor files for chapter 11 ).

Note that we have decided to reduce the state space of each ratings process $X^i$ to two states only. This modeling decision rests upon empirical reasons, and appears to be adequate with regard to (basket) credit instruments whose cash flows are not explicitly tied to changes in credit quality of underlying credit names, but rather depend only on occurrence of default. It needs to be stressed though that credit quality of an obligor, as reflected in the value of the corresponding CDS spread, provides a useful implicit quantification of credit ratings of the obligor, and as such is in some way used in our specialized model.[11]

We postulate that, under the risk neutral measure $\mathbf{P}$, the jump intensities (default intensities, in this case) of the Markov chain component $X$ are as follows):

$$\lambda_t^l(0, K) = h(\eta^l, X_t; \Theta), \tag{6}$$

where $f$ is a judiciously chosen function, $\Theta$ is a vector of model parameters and $\eta^l$ denotes the spread of the CDS referenced by the $l^{\text{th}}$ name, at inception. Recall that state $K$ is absorbing, so that $\lambda_t^l(K, 0) = 0$. The discount factor $\beta_t$ is obtained by interpolation of the term structure of T-Bonds at inception.

In order to price consistently the underlying CDSs and the CDO tranches, we calibrate the intensity parameter vector $\Theta$ to univariate and multivariate default information, provided by the average of the single name CDS spreads and by the CDO tranche spreads, respectively.[12] The model fits both single name CDS spreads and CDO tranche spreads. The simulation scheme converges rather quickly, with the (relative) standard error of the estimate ranging between 1% and 4%, after 100000 simulation runs. As for computational speed, 100000 paths can be generated in few seconds on a 1.5 mhz computer. In addition, the calibration of the model takes only few minutes of computing time, provided that the optimization algorithm starts from a sensible initial guess. A comparison of market and model generated spreads are illustrated in Table 1., Table 2. and Table 3.. As usual, the equity tranche is quoted as an up-front premium, in addition to the contractual 500 bps p.a..

| Tranche | Model Spreads | Market Spreads |
|---|---|---|
| 0-3% | 23.79 % | 24 % |
| 3-6% | 83.83 | 83 |
| 6-9% | 24.55 | 27 |
| 9-12% | 14.58 | 14 |
| 12-22% | 8.43 | 9 |

Table 1: *Fit of two state Markov model to iTraxx market data on 31-August-2005. All values are quoted in bps*

Also the individual CDS spreads can be fitted very accurately to market data, with a two percent maximum relative error.[13] In Table 3., we show the fit of the Markov model to DJ CDX data relative

[10] Thus, the "state" 0 represents all the ratings $1, 2, \dots, K-1$.

[11] In order to deal with instruments whose cash flows explicitly depend on changes in credit ratings, such as credit quality triggered step-up bonds (cf. Section 6.3), explicit quantification of relevant credit ratings will be needed.

[12] The market data is relative to CDO on iTraxx as quoted on August 31 2005 and November 5 2005 and on DJ CDX as quoted on November 10 2005. The data was courteously provided by GFI

[13] A table, showing the fit of the model to individual CDS prices is available upon request to the authors.

| Tranche | Model Spreads | Market Spreads |
|---|---|---|
| 0-3% | 24.52 % | 24.9 % |
| 3-6% | 71.75 | 71.5 |
| 6-9% | 22.04 | 24 |
| 9-12% | 12.68 | 11.5 |
| 12-22% | 6.41 | 6.65 |

Table 2: *Fit of two state Markov model to iTraxx market data on 05-November-2005.*

to November 10 2005.

| Tranche | Model Spreads | Market Spreads |
|---|---|---|
| 0-3% | 42.46 % | 43.5 % |
| 3-7% | 123.15 | 123 |
| 7-10% | 27.07 | 30 |
| 10-15% | 14.04 | 13 |
| 15-30% | 6.28 | 6.5 |

Table 3: *Fit of two state Markov model to DJ CDX market data on 10-November-2005.*

Once the risk neutral parameters are calibrated, one can price other credit instruments referenced by some or all of the names in the iTraxx index. In the next section, we use the calibrated model to price a CDO on iTraxx with customized tranche attachments, a CDO squared and an FTDS.

## 6.2 Pricing of selected basket credit derivatives via simulation

### 6.2.1 Pricing of CDOs

Using relevant calibrated data, we priced a CDO on the iTraxx S3 index with customized attachment points. The pricing results are shown in Table 4.

| Tranche | Model Spreads |
|---|---|
| 0-3% | 24.52 % |
| 3-9% | 40.75 |
| 9-16% | 9.35 |
| 16-21% | 4.15 |
| 21-35% | 1.35 |

Table 4: *Pricing of customized CDO tranches on iTraxx index S3, November 5 2005.*

### 6.2.2 Pricing of FTDSs on CDS indices

Using relevant calibration results, we priced an FTDS on a portfolio of credit names referencing the iTraxx S3 index and listed in Table 5. along with market data and our pricing results[14]:

The table shows that the version of our model used here prices consistently FTDS when calibrated to CDO and iTraxx data, and it suggests that the model is able to capture the dynamics of the credit market in a realistic fashion.

[14]The market data is relative to diversified FTDS contract as quoted on November 5 2005. The data was courteously provided by ...

| Sector | Entity | Bid/Ask Market Spread | Model Spreads |
|---|---|---|---|
| Autos | VOLKSWAGEN | 44 / 46 | 45.39 |
| Energy | SUEZ | 26 / 27 | 25.01 |
| Financials | Bayerische Hypo | 18 / 20 | 18.75 |
| Industrials | Bayer | 24 / 26 | 25.23 |
| TMT | FRANCE TELECOM | 42 / 44 | 43.87 |
| Consumer | MARKS AND SPENCER | 63 / 65 | 65.82 |
| FTDS Spread | | 78%/86% | 85% |

Table 5: *November 5, 2005: Diversified FTDS composition and market quote v. model output. The FTDS spread is quoted as a percentage of the sum of the underlying CDS spreads.*

#### 6.2.3 Pricing of CDO2s

Numerical results for this section will be available in a later version of the paper.

### 6.3 Pricing of ratings triggered Step-Up bonds

The Markovian framework proposed in Bielecki et al. and adopted in this paper is ideally suited for pricing ratings triggered step-up bonds. Since we consider a single bond, we may resort to methods other than simulation for pricing such instruments. In this section we shall develop an analytical solutions for the price of a step-up bond in a .

As before, we denote by $X$ the credit ratings process of a step-up bond, and we assume that the current credit quality of $X$ can be classified into $K$ rating categories[15], i.e. $\mathcal{K} := \{1, 2..., K\}$, where, as usual, $K$ is the default state. As the only factor $Y$ we take here the short rate process, so that $Y_t = r_t$. Consequently, $\beta_t = e^{-\int_0^t Y_s\,ds}$.

For the purpose of this paper we additionally assume that processes $X$ and $Y$ are independent under the pricing probability $\mathbf{P}$ (this assumption will be relaxed in a future paper). Finally, to simplify things even more, we shall take the recovery process $Z$ to be constant, and we shall denote its value as $\delta \in [0, 1)$. Again, this assumption will be relaxed in the future.

Let us fix $t \in [0, T]$, and let us denote by $i(t)$ the smallest integer $i \in \{1, 2, \ldots, n\}$ such that $t \le t_i$. Obviously, we are only interested in the price of those bonds that are not defaulted. This means, that we shall only consider the time-t price of a step-up bond on the set $\{t < \tau\} = \{X_t \ne K\}$. Take $\bar{\xi}, \xi \in \{1, 2, ..., K-1\}$. Then, on the set $\{X_{t_{i(t)}-1} = \bar{\xi}, X_t = \xi, Y_t = y\}$, the time-t price of a step up bond is (cf. (2))

$$
\begin{aligned}
\phi(\bar{\xi}, \xi, y, t) \quad &:= \quad \beta_t^{-1} \mathbf{E_P}\Big(\beta_T(1 - H_T) + \int_t^T (1 - H_s)\beta_s\, dC_s + \delta H_T \beta_\tau \Big| \mathcal{F}_t^X \vee \mathcal{F}_t^Y\Big) \\
&= \quad \beta_t^{-1} \mathbf{E_P}\Big(\beta_T(1 - H_T) + \sum_{i=i(t)}^{n} (1 - H_{t_i})\beta_{t_i} c(X_{t_{i-1}}) + \delta H_T \beta_\tau \Big| \mathcal{F}_t^X \vee \mathcal{F}_t^Y\Big) \\
&= \quad \beta_t^{-1} \mathbf{E_P}\Big(\beta_T(1 - H_T) + \sum_{i=i(t)+1}^{n} (1 - H_{t_i})\beta_{t_i} c(X_{t_{i-1}}) + \delta H_T \beta_\tau \Big| \mathcal{F}_t^X \vee \mathcal{F}_t^Y\Big) \\
&\qquad + \beta_t^{-1} \mathbf{E_P}\Big((1 - H_{t_{i(t)}})\beta_{t_{i(t)}} c(X_{t_{i(t)-1}}) \Big| \mathcal{F}_t^X \vee \mathcal{F}_t^Y\Big) \\
&= \quad \beta_t^{-1} \mathbf{E_P}\Big(\beta_T(1 - H_T) + \sum_{i=i(t)+1}^{n} (1 - H_{t_i})\beta_{t_i} c(X_{t_{i-1}}) + \delta H_T \beta_\tau \Big| X_t = \xi, Y_t = y\Big) \\
&\qquad + \beta_t^{-1} \mathbf{E_P}\Big((1 - H_{t_{i(t)}})\beta_{t_{i(t)}} c(\bar{\xi}) \Big| X_t = \xi, Y_t = y\Big). \qquad (7)
\end{aligned}
$$

[15] Note that the rating process is a vector process when the "and" provision is in force.

Using independence between processes $X$ and $Y$ we obtain that

$$\begin{aligned}\phi(\bar{\xi},\xi,y,t) &= B(t,T)(1-p_{t,\xi,K}(T)) \\ &\quad + \sum_{i=i(t)+1}^{n} B(t,t_i)\left(\sum_{x\neq K} p_{t,\xi,x}(t_{i-1})\left(1-p_{t_{i-1},x,K}(t_i)\right)c(x)\right) \\ &\quad + \delta\int_t^T B(t,u)\,d\,p_{t,\xi,K}(u) + B(t,t_{i(t)})\left(1-p_{t,x,K}(t_{i(t)})\right)c(\bar{\xi}), \end{aligned} \tag{8}$$

where for $s \le u$ we let

$$p_{s,x,x'}(u) = \mathbf{P}(X_u = x'|X_s = x),$$

and where $B(u,s)$ is the time-$u$ price of a discount bond that expires at time $s$, that is, $B(u,s) = \mathbf{E_P}(\beta_s\beta_u^{-1}|\mathcal{F}_s^Y)$.

As discussed in Sec 2.5, step-up(down) provisions are often triggered by Moody and S&P joint rating actions. In what follows, we suggest a convenient construction for the transition matrix of the joint rating process, which we denote by $(S_t, M_t)$, where $S_t$ and $M_t$ denote the ratings assigned by S&P and Moody, respectively. We assume $S_t$ is Markovian w.r.t its natural filtration and lives on the state space $\mathcal{K} = \{1,2,\ldots,K\}$. Also, we construct $M_t$ as follows:

$$M_t = f(S_t,\xi_t)$$

where

$$f(S_t,\xi_t) = S_t\mathbb{1}_{\{S_t=1\}} + (S_t+\xi_t)\mathbb{1}_{\{2\le S_t\le K-2\}} + S_t\mathbb{1}_{\{S_t\ge K-1\}}$$

and $\xi_t$ is, itself, Markovian w.r.t its natural filtration, and lives on the state space $\mathcal{S} = \{-1,0,1,\}$. In addition, we assume that $\xi_t$ is independent from $S_t$. It is easy to see that the joint process $(M_t, S_t)$ is also Markovian w.r.t the joint filtration $(\mathcal{F}_t^M \vee \mathcal{F}_t^S)$. In fact,

$$\begin{aligned}\mathbf{P}(S_{t+s}=k, M_{t+s}=l\,|\,\mathcal{F}_t^M\vee\mathcal{F}_t^S) &= \mathbf{P}(S_{t+s}=k, f(S_{t+s},\xi_{t+s})=l\,|\,\mathcal{F}_t^\xi\vee\mathcal{F}_t^S) \\ &= \mathbf{P}(S_{t+s}=k, \xi_{t+s}=\widetilde{f}(k,l)\,|\,\mathcal{F}_t^\xi\vee\mathcal{F}_t^S) \\ &= \mathbf{P}(S_{t+s}=k\,|\,\mathcal{F}_t^S)P(\xi_{t+s}=\widetilde{f}(k,l)\,|\,\mathcal{F}_t^\xi) \\ &= \mathbf{P}(S_{t+s}=k\,|\,\sigma(S_t))\mathbf{P}(\xi_{t+s}=\widetilde{f}(k,l)\,|\,\sigma(\xi_t)) \\ &= \mathbf{P}(S_{t+s}=k, \xi_{t+s}=\widetilde{f}(k,l)\,|\,\sigma(S_t)\vee\sigma(\xi_t)) \\ &= \mathbf{P}(S_{t+s}=k, M_{t+s}=l\,|\,\sigma(S_t)\vee\sigma(M_t))\end{aligned}$$

By the above construction, the *effective* state space of the joint rating process $(S_t, M_t)$ has cardinality $3K$ and is such that $M_t$ and $S_t$ disagree by at most one rating class at any given point in time, and coincide at the default state $K$. If the default state $K$ is absorbing for $S_t$, then the default state, $(K,K)$, is also an absorbing state for the joint process. The default time can be conveniently defined as follows:

$$\tau_{(M,S)} := \inf\{t\in(0,T] : M_t = K, S_t = K\} = \inf\{t\in(0,T] : S_t = K\}$$

In addition, it is easy to verify that the transition matrix of $(S_t, M_t)$ has the following form:

$$\mathbf{P}(M_{t+u}=m', S_{t+u}=s'|M_t=m, S_t=s) := \frac{\mathbf{P}(S_{t+u}=s', S_t=s)\mathbf{P}(\xi_{t+u}=\widetilde{f}(m',s'), \xi_t=\widetilde{f}(m,s))}{\mathbf{P}(S_t=s)\mathbf{P}(\xi_t=\widetilde{f}(m,s))}$$

# References

[1] T.R. Bielecki, S. Crepey, M. Jeanblanc and M. Rutkowski (2006) "Valuation of basket credit derivatives in the credit migrations environment," *Handbook on Financial Engineering*, J. Birge and V. Linetsky eds., Elsevier, forthcoming.

[2] P. E. Kloeden and E. Platen (1995) *Numerical Solution of Stochastic Differential Equations (Applications of Mathematics, Vol 23)*, Springer-Verlag, Corr. 2nd printing.

[3] R. Cont and P. Tankov (2003) *Financial modelling with Jump Processes*, Chapman & Hall/CRC Press.