CONVERTIBLE BONDS IN A JUMP-DIFFUSION MODEL OF CREDIT RISK

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1 Introduction

In [7], working in an abstract set-up, we characterized arbitrage prices of generic convertible securities (CS), such as convertible bonds (CB), and we provided a rigorous decomposition of a CB into a bond component and a game option component, in order to give a definite meaning to commonly used terms, such as: "CB spread" and "CB implied volatility". Moreover, in [9], we showed that in the hazard process set-up, the theoretical problem of pricing and hedging CS can essentially be reduced to a problem of doubly reflected Backward Stochastic Differential Equations (R2BSDE, see [9]). Finally, in [8], we established the connection between this R2BSDE problem (assumed to have a solution) and related variational inequalities with double obstacle in a generic Markovian intensity model.

Here, considering a primary market made of a savings account, the stock of the reference entity (firm) underlying a CB, and a further CDS on the firm to 'complete the market', we construct a Markovian jump-diffusion set-up that specifies our generic Markovian model, namely a model with time and stock-dependent default intensity and volatility. In this model, using also results from [18], we show that the R2BSDE associated to a CB has a solution (Theorem 4.1), so that the CB is hedgeable (Theorem 3.2, obtained by application of the general results of [9, 8]). Moreover, we characterize the price of the CB in terms of viscosity solutions of associated variational inequalities, and we give conditions ensuring convergence of deterministic approximation schemes (Theorems 4.3 to 4.6).

Despite its obviously simplistic character and its limitations, the above jump-diffusion model appears as an industry standard for convertible bonds, which is enough to motivate the present study. Beyond the interest of this model as such (see [10] for an alternative model), our motivation is also to demonstrate that the general approach that we adopted in the previous papers [7, 9, 8] is consistent, in the sense that in the industry standard model for convertible bonds, all the working assumptions that we made so far, and in particular the existence of a solution to the related R2BSDEs, can be satisfied. In particular, because of the so-called notice period covenant, a CB typically continues to live some time beyond its call time (see [7]). In the previous papers, we essentially worked under the simplifying assumption that the value U_t^{cb} of a the CB upon a call at time t is a well defined process satisfying some "natural" conditions, and that was considered as an exogenous input to a CB problem. In the specific framework of this paper, we are actually able to prove that this is indeed the case, and we also give ways to compute U_t^{cb} (Theorems 5.1 and 5.2).

As in [7, 9, 8], we use the notion of vector stochastic integral for predictable integrands as developed in Cherny–Shiryaev [16]. In particular, given a standard stochastic basis, any predictable locally bounded integrand is integrable in this sense with respect to any semimartingale. Note that by default we denote by $\int_0^t$ integrals over $(0,t]$, otherwise me mention the domain of integration as a subscript of $\int$.

2 The Standard Market Model for Convertible Bonds

In this section, we provide a Markovian specification of the generic Markovian default intensity set-up in [8], so that we can derive analytical characterizations of the price of a CS as solution to some variational inequalities with double obstacle in later sections. More precisely, we consider a jump-diffusion model with time and stock-dependent *local default intensity* and *local volatility*, referred to in the sequel as *the Standard Market Model* for convertible bonds (SMM for short, see also [2, 1, 21, 22, 30, 15]).

2.1 Canonical Construction

Let us fix a finite horizon date $T > 0$. Given a filtered probability space $(\Omega, \mathbb{F}, \mathbb{Q})$ satisfying the usual conditions, where $\mathbb{F}$ is the filtration of a standard Brownian motion W under $\mathbb{Q}$, we define what will later be interpreted as the *pre-default stock price* $\widetilde{S}$ of the firm underlying a CS, by setting, for $t \in \mathbb{R}^+$,

$$d\widetilde{S}_t = \widetilde{S}_t\Big((r(t) - q(t) + \eta\gamma(t, \widetilde{S}_t))dt + \sigma(t, \widetilde{S}_t)dW_t\Big), \quad \widetilde{S}_0 \in \mathbb{R} \,. \tag{1}$$

Assumption 2.1 We assume that:
(i) the riskless short interest rate $r(t)$, the equity dividend yield $q(t)$ and the local default intensity $\gamma(t,S) \geq 0$, are bounded Borel functions, and η is a non-negative constant,
(ii) the local volatility $\sigma(t,S)$ is a positively bounded Borel function, and
(iii) the functions $\gamma(t,S)S$ and $\sigma(t,S)S$ are Lipschitz in S.

Then SDE (1) admits a unique strong solution $\widetilde{S}$, which does not vanish on $[0,\infty)$.

Remarks 2.1 (i) In this work we find it convenient to define (1) for any initial condition $\widetilde{S}_0 \in \mathbb{R}$, even though only the positive values will have a financial interpretation. This will be useful for the variational inequalities approach (see Remark A.1).
(ii) A possible choice for γ is a negative power function of $\widetilde{S}$ capped when $\widetilde{S}$ is close to 0. A possible refinement is to positively floor γ. The lower bound on γ then represents pure credit risk, as opposed to equity-related credit risk.

Introducing a unit exponential random variable ε independent of W on the probability space $(\Omega, \mathcal{G}, \mathbb{Q})$, where $\mathcal{F}_\infty \subset \mathcal{G}$, one defines the random default time τ_d by the so-called canonical construction [12]:

$$\tau_d = \inf\Big\{\, t \in \mathbb{R}_+ \,:\, \int_0^t \gamma(u, \widetilde{S}_u)\, du \geq \varepsilon \Big\} \tag{2}$$

and one defines the underlying stock price process S by:

$$dS_t = S_{t-}((r(t) - q(t))dt + \sigma(t, S_t)dW_t - \eta dM_t^d), \quad S_0 \in \mathbb{R}. \tag{3}$$

Here η is the fractional loss on the stock upon default, assumed to be a constant $0 \leq \eta \leq 1$, and $dM_t^d = dH_t - (1 - H_t)\gamma(t, \widetilde{S}_t)dt$, with $H_t := \mathbb{1}_{\tau_d \leq t}$.

Let $\mathbb{H}$ be the filtration generated by the process H, and $\mathbb{G}$ be given as $\mathbb{F} \vee \mathbb{H}$. Thanks to the previous construction, $\gamma(t, \widetilde{S}_t)$ is the $\mathbb{F}$-intensity of τ_d, and M^d a $\mathbb{G}$-martingale, called the compensated jump martingale. Moreover, as τ_d was obtained by the canonical construction, the optional projection G of $1 - H$ on $\mathbb{F}$ is given by $G_t = exp^{-\int_0^t \gamma(u, \widetilde{S}_u)du}$, a continuous and non-increasing process, and Hypothesis H is satisfied (see for example [12]). Therefore, the $\mathbb{F}$-Brownian motion W is also a $\mathbb{G}$-Brownian motion. Since the process M^d is a also $\mathbb{G}$-martingale, we conclude that the process S, once adjusted for interest rates and dividend yields, is a $\mathbb{G}$-martingale.

2.2 Specification of the Model

We assume that the primary market model is defined on some probability space $(\Omega, \mathbb{G}, \mathbb{P})$, where $\mathbb{G}$ is as before and $\mathbb{P} \sim \mathbb{Q}$ is the statistical probability measure. The primary market is composed of the savings account and of two primary risky assets, such that by assumption:
• the *discount factor* process, that is, the inverse of the savings account, is given by $\beta_t = \exp(-\int_0^t r_u\, du)$, $t \in \mathbb{R}_+$;
• the first primary risky asset is S above, which models the value of the stock of a reference entity (firm) with default time represented by τ_d;
• the second primary risky asset is a CDS contract written on the reference entity.

Thus, the $\mathbb{Q}$-dynamics of the price P of the CDS has to be as follows [11]:

$$dP_t \quad = \quad r_t P_t dt + \mathbb{1}_{t \leq \tau_d}\Big(\Big(s - \nu_t \gamma(t, \widetilde{S}_t)\Big)dt + \alpha_t^{-1} d\widehat{m}_t - P_{t-}\, dM_t^d\Big)\ , \tag{4}$$

where:
• s is the contracted CDS spread,
• ν_t is the protection payment process,
• $\alpha_t = e^{-\int_0^t (r_u + \gamma(u, \widetilde{S}_u))du}$,
• $\widehat{m}$ is an $\mathbb{F}$-local martingale.

Furthermore, it is possible to show that under suitable regularity assumptions, we have:

$$d\widehat{m}_t = \widetilde{S}_t \partial_S m(t, \widetilde{S}_t)\sigma(t, \widetilde{S}_t)dW_t\ , \tag{5}$$

where m is a continuously differentiable in time and twice continuously differentiable in space function. In the sequel we assume (5), and we also assume that the protection payment process ν_t is given by a bounded Borel function of time. For $t \in [0, T]$, define

$$\upsilon_t = e^{\int_0^t q_u du}\ ,\ \widehat{S}_t = \upsilon_t S_t\ ,\ \widehat{P}_t = P_t + \beta_t^{-1} \int_{[0, t \wedge \tau_d]} \beta_u(\nu_u\, dH_u - s\, du)\ ,$$

and

$$\Upsilon_t = \begin{pmatrix} \upsilon_t \\ 1 \end{pmatrix}\ ,\ \widehat{X}_t = \begin{pmatrix} \widehat{S}_t \\ \widehat{P}_t \end{pmatrix}\ .$$

Proposition 2.1 *We have:*

$$d\left(\beta_t \widehat{X}_t\right) = \beta_t \Upsilon_t\ \Xi_t\ d\begin{pmatrix} W_t \\ M_t^d \end{pmatrix}\ ,\ t \in [0, T] \tag{6}$$

where the $\mathbb{G}$-predictable dispersion matrix *process Ξ is given by*

$$\Xi_t = \begin{bmatrix} \sigma(t, \widetilde{S}_t)\widehat{S}_{t-} & -\eta \widehat{S}_{t-} \\ \mathbb{1}_{\{t \leq \tau_d\}} \alpha_t^{-1} \sigma(t, \widetilde{S}_t)\widetilde{S}_t \partial_S m(t, \widetilde{S}_t) & \nu_t - \widehat{P}_{t-} \end{bmatrix},\quad t \in [0, T]\ . \tag{7}$$

So, under the following condition:

$$\alpha_t(\nu_t - \widehat{P}_{t-}) + \eta \widetilde{S}_t \partial_S m(t, \widetilde{S}_t) \neq 0 \ , \ t \in [0, \tau_d] \ , \tag{8}$$

the matrix Ξ_t *is invertible,* $t \in [0, \tau_d]$. *Then, for any equivalent local martingale measure* $\widetilde{\mathbb{Q}}$ *on the primary market, we have that* $\frac{d\widetilde{\mathbb{Q}}}{dQ} = 1$ *on* $[0, \tau_d]$.

Proof. Direct computation gives (7). Note that in the financial interpretation, $\widehat{S}$ represents the current value at time t of a buy-and-hold strategy in one share of S at time 0, assuming that all the dividend payments on S are immediately reinvested in the equity S. As for $\widehat{P}$, it denotes, in accordance with our general convention for the notation $\widehat{\cdot}$ in [9, 8], the current value at time t of a buy-and-hold strategy in one CDS contract at time 0, assuming that all the CDS payments are immediately reinvested in the savings account. This explains the factor v in front of S_t above, and the vector Υ in (6).

Invertibility of Ξ_t on $[0, \tau_d]$ is obvious under (8). By the Girsanov theorem, (8) also ensures that $\frac{d\widetilde{\mathbb{Q}}}{dQ} = 1$ on $[0, \tau_d]$, for any equivalent local martingale measure $\widetilde{\mathbb{Q}}$ on the primary market. □

In the sequel we assume (8). Moreover, we assume throughout this work that the following integrability assumption is satisfied (simple sufficient conditions will be given later, cf. Assumption 4.1 and Remark 4.1):

$$\mathbb{E}_{\mathbb{Q}}\Big(\sup_{t\in[0,T]} S_t^2 \,\Big|\, \mathcal{G}_0\Big) < \infty \text{ a.s.} \tag{9}$$

3 Convertible Securities in the Standard Market Model

We now specify to the SMM the notions of convertible securities and convertible bonds that were introduced in a general semi-martingalian set-up in [7].

3.1 Definition

Let 0 (respectively T) stand for the *inception date* (respectively the *maturity date*) of a convertible security (CS) with underlying S. For any $t \in [0, T]$, we write $\mathcal{F}_T^t$ (resp. $\mathcal{G}_T^t$) to denote the set of all $\mathbb{F}$-stopping times (resp. $\mathbb{G}$-stopping times) with values in $[t, T]$. Given the *lifting time of a call protection of a CS*, $\bar{\tau} \in \mathcal{F}_T^0$, let also $\bar{\mathcal{F}}_T^t$ stand for $\{\tau \in \mathcal{F}_T^t \, : \, \tau \geq \bar{\tau}\}$, $\bar{\mathcal{G}}_T^t$ stand for $\{\tau \in \mathcal{G}_T^t \, ; \, \tau \wedge \tau_d \geq \bar{\tau} \wedge \tau_d\}$, and let finally τ denote $\tau_p \wedge \tau_c$, for any $(\tau_p, \tau_c) \in \mathcal{G}_T^t \times \bar{\mathcal{G}}_T^t$.

Definition 3.1 A *convertible security* with underlying S is a game option (see [29, 28, 7, 9, 8]) with the *ex-dividend cumulative discounted cash flows* $\pi(t; \tau_p, \tau_c)$ given by the formula, for any $t \in [0, T]$ and $(\tau_p, \tau_c) \in \mathcal{G}_T^t \times \bar{\mathcal{G}}_T^t$,

$$\beta_t \pi(t; \tau_p, \tau_c) = \int_t^{\tau} \beta_u \, dD_u + \mathbb{1}_{\{\tau_d > \tau\}} \beta_\tau \Big(\mathbb{1}_{\{\tau = \tau_p < T\}} L_{\tau_p} + \mathbb{1}_{\{\tau = \tau_c < \tau_p\}} U_{\tau_c} + \mathbb{1}_{\{\tau = T\}} \xi \Big) \ ,$$

where:
- the *dividend process* $D = (D_t)_{t\in[0,T]}$ equals

$$D_t = \int_{[0,t]} (1 - H_u) \, dC_u + \int_{[0,t]} R_u \, dH_u$$

for some *coupon process* $C = (C_t)_{t\in[0,T]}$, which is a $\mathbb{G}$-adapted càdlàg process with finite variation, and some real-valued, $\mathbb{G}$-predictable *recovery process* $R = (R_t)_{t\in[0,T]}$;
- the *put payment* L is given as a $\mathbb{G}$-adapted, real-valued, càdlàg process on $[0, T]$,
- the *call payment* U is a $\mathbb{G}$-adapted, real-valued, càdlàg process on $[0, T]$, such that

$$L_t \leq U_t \quad \text{for} \quad t \in [\tau_d \wedge \bar{\tau}, \tau_d \wedge T), \tag{10}$$

• the *payment at maturity* ξ is a $\mathcal{G}_T$-measurable real random variable, such that $L_T \leq \xi \leq U_T$. In addition, the processes R, L and the random variable ξ are assumed to satisfy the following inequalities, for some positive real numbers a, b, c:

$$\begin{aligned}
-c &\leq R_t \leq a \vee bS_t, \quad t \in [0,T], \\
-c &\leq L_t \leq a \vee bS_t, \quad t \in [0,T], \\
-c &\leq \xi \leq a \vee bS_T\ .
\end{aligned} \tag{11}$$

Theorem 3.1 *If the $\mathbb{Q}$-Dynkin game related to a CS admits a value Π, specifically,*

$$\begin{aligned}
&\mathrm{esssup}_{\tau_p \in \mathcal{G}_T^t} \mathrm{essinf}_{\tau_c \in \bar{\mathcal{G}}_T^t} \mathbb{E}_{\mathbb{Q}}\big(\pi(t;\tau_p,\tau_c) \,\big|\, \mathcal{G}_t\big) = \Pi_t \\
&\quad = \mathrm{essinf}_{\tau_c \in \bar{\mathcal{G}}_T^t} \mathrm{esssup}_{\tau_p \in \mathcal{G}_T^t} \mathbb{E}_{\mathbb{Q}}\big(\pi(t;\tau_p,\tau_c) \,\big|\, \mathcal{G}_t\big), \quad t \in [0,T],
\end{aligned} \tag{12}$$

and that Π is a $\mathbb{G}$-semimartingale, then Π is the unique arbitrage (ex-dividend) price of the CS.

Proof. Except for the uniqueness statement, this follows by aplication of the general results in [7]. Moreover, under (9), the general results of [7] also imply that any arbitrage price of a CS is given by the value of the related Dynkin Game under an equivalent local martingale measure $\widetilde{\mathbb{Q}}$ on the primary market. Now, for any such $\widetilde{\mathbb{Q}}$, we have that $\frac{d\widetilde{\mathbb{Q}}}{dQ} = 1$ on $[0,\tau_d]$, by Proposition 2.1 (recall that we assume (8)). Therefore, as the cash flows $\pi(t;\tau_p,\tau_c)$ of a convertible security in the SMM are given by a function of $\tau \wedge \tau_d, W_{|[0,\tau\wedge\tau_d]}$ and $M^d_{|[0,\tau\wedge\tau_d]}$, this implies that

$$\mathbb{E}_{\widetilde{\mathbb{Q}}}\big(\pi(t;\tau_p,\tau_c) \,\big|\, \mathcal{G}_t\big) = \mathbb{E}_{\mathbb{Q}}\big(\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}}_{|\tau\wedge\tau_d} \pi(t;\tau_p,\tau_c) \,\big|\, \mathcal{G}_t\big) = \mathbb{E}_{\mathbb{Q}}\big(\pi(t;\tau_p,\tau_c) \,\big|\, \mathcal{G}_t\big), \tag{13}$$

for any $t \in [0,T], \tau_p \in \mathcal{G}_T^t, \tau_c \in \bar{\mathcal{G}}_T^t$. Therefore, if both the $\mathbb{Q}$- and the $\widetilde{\mathbb{Q}}$-related Dynkin Game do have values, these have to coincide. □

Definition 3.2 A *non-callable CS* is a convertible security with $\bar{\tau} = T$, or, equivalently, $\bar{U} = \infty$. An *Elementary Security* (ES) is a non-callable CS with bounded variation dividend process D over $[0,T]$, bounded payment at maturity ξ, and such that

$$\int_{[0,t]} \beta_u \, dD_u + \mathbb{1}_{\{\tau_d > t\}} \beta_t L_t \ \leq \int_{[0,T]} \beta_u \, dD_u + \mathbb{1}_{\{\tau_d > T\}} \beta_T \xi \text{ for } \ t \in [0,T)\ , \tag{14}$$

and with bounded variation dividend process D over $[0,T]$, and bounded payment at maturity ξ.

Equivalently to (14) (given Theorem 3.1), the cash flows $\bar{\pi}(t)$ of a non-callable CS (resp. $\phi(t)$ of an ES) can be redefined by, for $t \in [0,T]$:

$$\beta_t \bar{\pi}(t)(\tau_p) = \int_t^{\tau_p} \beta_u \, dD_u + \mathbb{1}_{\{\tau_d > \tau_p\}} \beta_{\tau_p} \big(\mathbb{1}_{\{\tau_p < T\}} L_{\tau_p} + \mathbb{1}_{\{\tau_p = T\}} \xi\big) \text{ for } \tau_p \in \mathcal{G}_T^t\ ,$$

resp.

$$\beta_t \phi(t) = \int_t^T \beta_u \, dD_u + \mathbb{1}_{\{\tau_d > T\}} \beta_T \xi.$$

Moreover, in the present Markovian set-up, we postulate from now on that:

• the *coupon process* $C_t = C(t) := \int_{[0,t]} c_u du + \sum_{0 \leq T_i \leq t} c^i$, for a bounded Borel *continuous coupon rate* c and deterministic *discrete coupon* dates and amounts, with $T_0 = 0$ and $T_{K-1} < T \leq T_K$;

• the recovery process R_t is of the form $R(t, S_{t-})$, for a Borel function R;

• $\xi = \xi(S_T)$, $L_t = L(t, S_t)$, $U_t = U(t, S_t)$, for a Borel function ξ and Borel functions L, U such that for any t, S, we have $L(t,S) \leq U(t,S)$ and $L(T,S) \leq \xi \leq U(T,S)$.

Definition 3.3 We define the *accrued interest* at time t by

$$A_t = \frac{t - T_{i_t - 1}}{T_{i_t} - T_{i_t - 1}}\, c^{i_t},$$

where i_t is the integer satisfying $T_{i_t-1} \le t < T_{i_t}$, and we denote by $\rho_t = \frac{c^{i_t}}{T_{i_t} - T_{i_t-1}}$. We set $\gamma_t = \gamma(t, \widetilde{S}_t)$, $\mu(t,S) = r + \gamma(t,S)$, $\mu_t = \mu(t, \widetilde{S}_t)$. Finally, let G stand for $c + \rho - \mu A$. To a CS with data (functions) C, R, ξ, L, U, and lifting time of call protection $\bar{\tau} \in \mathcal{F}^0_T$, we associate Borel functions $f = f(t,S,\Theta)$ (Θ real), $g = g(S)$, $\ell = \ell(t,S)$ and $h = h(t,S)$, by

$$g = \xi - A_T\ ,\ \ell = L - A\ ,\ h = U - A\ ,$$

and $f(t,S,\Theta) = \gamma(t,S)R(t,S) + G(t,S) - \mu(t,S)\Theta$. In the case of a non-callable CS, the process U is irrelevant, and we redefine in this case $h(t,S) = +\infty$. In the special case of an ES, we redefine $\widehat{L}$ as $\bar{L}$ such that

$$\alpha \bar{L} = -(c+1)\ , \tag{15}$$

where $-c$ is a lower bound for $\alpha_T \widehat{\xi}$. Finally, we define the processes and random variables associated to a CS or an ES as

$$f_t(\Theta) = f(t, \widetilde{S}_t, \Theta)\ ,\ g = g(\widetilde{S}_T)\ ,\ \ell_t = \ell(t, \widetilde{S}_t)\ ,\ h_t = h(t, \widetilde{S}_t)\ .$$

Note in particular that for $t \in [0,T]$, we have, with the convention that $(A_-)_0 = 0$:

$$\alpha_t A_t = \int_{[0,t]} d(\alpha A)_u = \int_0^t \alpha_u \big(\rho_u - \mu_u A_u\big) du - \sum_{0 \le T_i \le t} \alpha_{T_i} c^i\ . \tag{16}$$

3.2 Doubly Reflected BSDEs Approach

We define:

$\mathcal{H}^2(0,T)$ – the set of real-valued, $\mathbb{F}$-predictable processes Θ such that $\mathbb{E}_{\mathbb{Q}} \int_{t=0}^T \Theta_t^2 dt < \infty$,

$\mathcal{S}^2(0,T)$ – the set of real-valued, $\mathbb{F}$-adapted, continuous processes Θ such that $\mathbb{E}_{\mathbb{Q}} \sup_{t \in [0,T]} \Theta_t^2 < \infty$,

$\mathcal{A}^2(0,T) \subseteq \mathcal{S}^2(0,T)$ – the set of non decreasing processes $\Theta \in \mathcal{S}^2(0,T)$ such that $\Theta_0 = 0$.

Given a CS with data $C, R, \xi, L, U, \bar{\tau}$ and associated processes and random variables (f_t, g, ℓ_t, h_t), where ℓ_t and h_t are supposed to be *time-continuous* processes, we introduce the following doubly reflected Backward Stochastic Differential Equation $(\mathcal{E})$ with data $(f_t, g, \ell_t, h_t, \bar{\tau})$ (R2BSDE for short, see [8, 9]), such that for $t \in [0,T]$:

$$\left.\begin{array}{c} -d\Theta_t = f_t(\Theta_t)\, dt + dk_t - z_t dW_t, \quad \Theta_T = g \\ \ell_t \le \Theta_t \le \bar{h}_t \\ (\Theta_t - \ell_t)\, dk_t^+ = (\bar{h}_t - \Theta_t)\, dk_t^- = 0\ , \end{array}\right\} (\mathcal{E})$$

where $\bar{h}_t := \mathbb{1}_{\{t<\bar{\tau}\}}\infty + \mathbb{1}_{\{t \ge \bar{\tau}\}} h_t$. By a *solution* to $(\mathcal{E})$, we mean a triple of processes

$$(\Theta, z, \mathbf{k}) \in \mathcal{S}^{\mathbf{2}}(\mathbf{0}, \mathbf{T}) \times \mathcal{H}^{\mathbf{2}}(\mathbf{0}, \mathbf{T}) \times (\mathcal{A}^{\mathbf{2}}(\mathbf{0}, \mathbf{T}))^{\mathbf{2}}$$

that satisfies all conditions in $(\mathcal{E})$ for any $0 \le t \le T$, with $k = k^+ - k^-$ where $(k^+, k^-) = \mathbf{k}$, using the convention that $0 \times \pm\infty = 0$. So, in particular, Θ and $\mathbf{k}$ have to be continuous processes.

Note that in the case where $\bar{\tau} = T$, we have $k^- = 0$, so that $(\mathcal{E})$ reduces to a simply reflected BSDE with data (f, g, ℓ) and no process k^- involved in the solution. In the special case of an elementary security (ES), In the special case of an ES with $\widehat{L}$ redefined as $\bar{L}$ in (15), it is easy to see that $\mathbf{k}=\mathbf{0}$ in any solution $(\Theta, z, \mathbf{k}$ to $(\mathcal{E})$, so that $(\mathcal{E})$ reduces to an elementary bsde with data (f,g) and no process $\mathbf{k}$ involved in the solution. In the latter case, $(\mathcal{E})$ will be renoted $(\mathcal{E}')$.

By application of the general results in [8], we have the following hedging result, in which the bivariate process ζ represents a $\mathbb{G}$-predictable trading strategy (number of units held) in the primary risky assets S and P. For the involved notions of *issuer hedge*, resp. *holder hedge* of a CS (with no cost, here), we refer the reader to [8].

Theorem 3.2 (See [8]) *Let $(\widehat{\Theta}, z, \mathbf{k})$ be a solution to $(\mathcal{E})$, assumed to exist, and let Θ_t denote $\mathbb{1}_{\{t<\tau_d\}}\widetilde{\Theta}_t$, with $\widetilde{\Theta} := \widehat{\Theta} + A$. Then Θ is the unique arbitrage price process of the CS, and for any $t \in [0,T]$:*
(i) *An issuer hedge with initial wealth Θ_t for the CS is furnished by*

$$\tau_c^* = \inf\Big\{ u \in [\bar{\tau} \vee t, T]\,;\, \widehat{\Theta}_u = h_u \Big\} \wedge T \in \bar{\mathcal{F}}_T^t \;,$$

and

$$\zeta_u^* := \mathbb{1}_{u \le \tau_d} \Big[z_u \;,\; R_u - \breve{\Theta}_{u-} \Big] \, \Xi_u^{-1} \;,\; t \le u \le T \tag{17}$$

where $\breve{\Theta}_t = \widetilde{\Theta}_t$ and $d\breve{\Theta}_u - \mu_u \breve{\Theta}_u du = d\widetilde{\Theta}_u - \mu_u \widetilde{\Theta}_u du + dk_u \;,\; t \le u \le T$. Moreover, Θ_t is the least initial wealth of an issuer hedge for the CS, and the corresponding wealth process is bounded from below.
(ii) *A holder hedge with initial wealth $-\Theta_t$ for the CS is furnished by*

$$\tau_p^* = \inf\Big\{ u \in [t, T]\,;\, \widehat{\Theta}_u = \ell_u \Big\} \wedge T \in \mathcal{F}_T^t \;,$$

and $\zeta = -\zeta^*$ above.

Proof. By application of the general results of [9], Θ satisfies all the assumptions for Π in Theorem 3.1, therefore it is the unique arbitrage price process of the CS. Moreover, given Proposition 2.1, (i) and (ii) result by application of the general results of [9]. Note that in [9] (see also the proof of Proposition 2.1), $\widehat{X}$ denotes the vector of the values of a buy-and-hold strategy in one unit of every risky asset at time 0 assuming that all the dividend payments are immediately reinvested in the savings account, whereas here, $\widehat{S}$ represents the current value at time t of a buy-and-hold strategy in one share of S at time 0 assuming that all the dividend payments on S are immediately reinvested in the equity S. This explains that ζ^* is indeed given by (17), which is the same form as in [9], though the forms of $\widehat{X}$ differ in the present paper and in [9]: the forms of $\widehat{X}$ have to be different because the meaning of $\widehat{X}$ is different, but (17) is the correct expression for ζ^* in both cases, as it can be seen by a simple change of variables. □

So in the SMM, any CS has a bilateral hedging price, which is also the unique arbitrage price, provided the related BSDE has a solution.

3.3 Variational Inequalities Approach

Let $\mathcal{L}$ denote the linear operator

$$\mathcal{L} \equiv \partial_t + (r - q + \eta\gamma) S \partial_S + \frac{\sigma^2 S^2}{2} \partial^2_{S^2}$$

and let be given a CS with data $C, R, \xi, L, U, \bar{\tau}$ such that the associated functions (f, g, ℓ, h) are continuous and real-valued (or $h = +\infty$, in the case where $\bar{\tau} = T$, and also $\ell = \bar{L}$, in the special case of an ES), except maybe for possible left jumps in time of f at the discrete coupon dates T_i of the CS (if there are any discrete coupons involved). We refer to the special case where f is continuous (which typically corresponds to purely continuously paid coupons), as *the continuous coupon case.*

A function Θ on $\mathcal{D}$ is said to have *polynomial growth*, if it is bounded by $C(1 + S^p)$ for suitable real C and integers p (that may depend on Θ). We refer the reader to the Appendix for the definition of viscosity solutions [17] of double obstacle variational inequalities.

Definition 3.4 Given $\bar{S} \leq \infty$, let $\mathcal{D} := [0,T] \times \mathrm{Adh}(-\infty, \bar{S})$, and let

$$\overset{\circ}{\mathcal{D}}_p = [0,T) \times (-\infty, \bar{S}) \text{ and } \partial_p \mathcal{D} := \mathcal{D} \setminus \overset{\circ}{\mathcal{D}}_p$$

stand for the *parabolic interior* and the *parabolic boundary* of $\mathcal{D}$, respectively. We denote by $(\mathcal{P})$ the following variational inequality with double obstacle on $\mathcal{D}$:

$$\max\left(\min\left(-\mathcal{L}\Theta(t,S) - f(t,S,\Theta(t,S)), \Theta(t,S) - \ell(t,S)\right), \Theta(t,S) - h(t,S)\right) = 0 \,.$$

Given a continuous boundary condition b, where b is a continuous function on $\partial_p\mathcal{D}$ such that $b = g$ at T:
(i) By a $\mathcal{S}$-*subsolution* Θ of $(\mathcal{P})$ on $\mathcal{D}$ for the boundary condition b, we mean a viscosity subsolution with polynomial growth on $\mathcal{D}$, and such that $\Theta \leq b$ pointwise on $\partial_p\mathcal{D}$.
(ii) By a $\mathcal{S}$-*supersolution* Θ of $(\mathcal{P})$ on $\mathcal{D}$ for the boundary condition b, we mean a viscosity supersolution with polynomial growth on $\mathcal{D}$, and such that $\Theta \geq b$ pointwise on $\partial_p\mathcal{D}$.
(iii) By a $\mathcal{S}$-*solution* u, we mean a function that is both a $\mathcal{S}$-subsolution and a $\mathcal{S}$-supersolution—hence $\Theta = b$ on $\partial_p\mathcal{D}$.

Theorem 3.3 *Assume furthermore that:*
(i) *f is Lipschitz in S, locally uniformly in t, Θ,*
(ii) *f and g have polynomial growth on $\mathcal{D}$,*
(iii) *ℓ is an increasing pointwise limit of $\mathcal{C}^{1,2}$-functions $\ell_n < h$ such that the related partial derivatives of ℓ_n have polynomial growth on $\mathcal{D}$,*
(iv) *h is bounded from below.*
Then the R2BSDE $(\mathcal{E})$ has a solution $(\Theta, z, \mathbf{k})$, and:
1. ***Cauchy problem.*** *in case where $\bar{\tau} = 0$, the related state-process $\widehat{\Theta}^0_t$ can be written as $\widehat{\Theta}^0(t, \widetilde{S}_t)$, where the function $\widehat{\Theta}^0$ is a $\mathcal{S}$-solution of the Cauchy problem $(\mathcal{P})$ on $\mathcal{D} := [0,T] \times \mathbb{R}$ with terminal condition g at T. Moreover, in the continuous coupon case, $\widehat{\Theta}^0$ is the* unique *$\mathcal{S}$-solution of the problem, as well as its maximal $\mathcal{S}$-subsolution, as well as its minimal $\mathcal{S}$-supersolution;*
2. ***Cauchy–Dirichlet problem.*** *in case where $\bar{\tau} = \inf\{t > 0\,;\, \widetilde{S}_t \geq \bar{S}\} \wedge T$, for some $\bar{S} \in \mathbb{R}^\star_+$, denote $\bar{\Theta}_t = \widehat{\Theta}^0_{t\wedge\bar{\tau}}$, the state-process in point 1 stopped at $\bar{\tau}$. Then $\bar{\Theta}_t$ can be written as $\bar{\Theta}(t, \widetilde{S}_t)$, where the function $\bar{\Theta}$ is a $\mathcal{S}$-solution of the Cauchy-Dirichlet problem $(\mathcal{P})$ on $\bar{\mathcal{D}} := [0,T] \times (-\infty, \bar{S}]$ with terminal condition g at T and Dirichlet condition $\widehat{\Theta}^0(\cdot, \bar{S})$ at level $\bar{S}$. Moreover, in the continuous coupon case, $\bar{\Theta}$ is the* unique *$\mathcal{S}$-solution of this problem, as well as its maximal $\mathcal{S}$-subsolution, as well as its minimal $\mathcal{S}$-supersolution.*

Proof. Note that under our assumptions f is also Lipschitz in Θ, uniformly in t, S. So the theorem results by application of the general results in [18]. □

We refer the reader to the Appendix for the definition of stable, monotonous and consistent approximation schemes of viscosity solutions of non linear PDEs (Barles et al [6]).

Corollary 3.1 *Under the assumptions and in the notation of Theorem 3.3, in the continuous coupon case:*
1. ***Cauchy problem.*** *Let $(\Theta_h)_{h>0}$ denote a stable, monotonous and consistent approximation scheme for the function $\widehat{\Theta}^0$. Then $\Theta_h \to \widehat{\Theta}^0$ locally uniformly on $\mathcal{D}$ as $h \to 0^+$, provided $\Theta_h \to \widehat{\Theta}^0$ at T.*
2. ***Cauchy–Dirichlet problem.*** *Let $(\Theta_h)_{h>0}$ denote a stable, monotonous and consistent approximation scheme for the function $\bar{\Theta}$. Then $\Theta_h \to \bar{\Theta}$ locally uniformly on $\bar{\mathcal{D}}$ as $h \to 0^+$, provided $\Theta_h \to \bar{\Theta} (= \widehat{\Theta}^0)$ at T and at $\bar{S}$.*

Proof. The argument is standard and the same in both cases. Let us give it for (i). By Lemma A.2, the upper and lower limits $\overline{\Theta} \leq \underline{\Theta}$ of Θ_h as $h \to 0^+$ are respectively viscosity subsolutions and supersolutions of $(\mathcal{P})$ on $\overset{\circ}{\mathcal{D}}_p$. If $\widehat{\Theta}_h \to \widehat{\Theta}^0$ at T, they are in fact $\mathcal{S}$-subsolutions and $\mathcal{S}$-supersolutions of $(\mathcal{P})$ on $\mathcal{D}$. So they coincide, by the semi-continuous comparison principle in Theorem 3.3. This implies the local uniform convergence on $\mathcal{D}$, by Lemma A.1(iii). □

4 Application to Reduced Convertible Bonds

4.1 Convertible Bonds and Reduced Convertible Bonds

To describe the covenants of a typical convertible bond (CB), we need to introduce the following additional notation (see [7] for a thorough description and discussion of the associated convertible bonds covenants):

$\bar{N}$: the par (nominal) value,

η: the fractional loss on the underlying equity upon default ($0 \leq \eta \leq 1$),

$\bar{R}_t$: the recovery process on the CB upon default of the issuer at time t, given by $\bar{R}_t = \bar{R}(t, S_{t-})$, for a bounded Borel function $\bar{R}$.

κ : the conversion factor,

$R^{cb}_t = R^{cb}(t, S_{t-}) = (1-\eta)\kappa S_{t-} \vee \bar{R}_t$: the effective recovery process,

$\xi^{cb} = \bar{N} \vee \kappa S_T + A_T$: the effective payoff at maturity,

$\bar{P} \leq \bar{C}$: the put and call nominal payments, respectively, such that $\bar{P} \leq \bar{N} \leq \bar{C}$,

$\delta \geq 0$: the length of the call notice period (see the detailed description below),

$t^\delta = (t+\delta) \wedge T$: the end date of the call notice period started at t.

Real-life convertible bonds typically include a positive *call notice period* δ so that if the issuer calls the bond at time τ_c, then the holder may either redeem the bond for $\bar{C}$ or convert the bond into κ shares of stock, at any time u at its convenience in $[\tau_c, \tau_c^\delta]$, where $\tau_c^\delta = (\tau_c + \delta) \wedge T$. Accounting for accrued interest, the effective call/conversion payment to the holder at time u is $\bar{C} \vee \kappa S_u + A_u$.

This clause makes CB with positive call notice period difficult to price directly. To handle this, we developed in [7] a recursive approach to value a CB with positive call notice period. In the first step, we value the CB upon call as a Reduced Convertible Bond (RB, see Definition 4.1 below). In the second step, we use this price as the payoff at call time of a CB with no call notice period.

Definition 4.1 (see [7]) A *reduced convertible bond* is a convertible security with recovery process R^{cb} and terminal payoffs L^{cb}, U^{cb}, ξ^{cb} such that

$$R^{cb}_{\tau_d} = (1-\eta)\kappa S_{\tau_d -} \vee \bar{R}_{\tau_d}, \quad L^{cb}_t = \bar{P} \vee \kappa S_t + A_t, \quad \xi^{cb} = \bar{N} \vee \kappa S_T + A_T \ ,$$

and

$$U^{cb}_t = \mathbb{1}_{\{t<\tau_d\}} \widetilde{U}^{cb}(t, S_t) + \mathbb{1}_{\{t \geq \tau_d\}} (\bar{C} \vee \kappa S_t + A_t), \quad t \in [0, T] \tag{18}$$

for a function $\widetilde{U}^{cb}(t, S)$ jointly continuous in time and space, except for negative left jumps of $-c^i$ at the T_is, and such that $\widetilde{U}^{cb}(t, S_t) \geq \bar{C} \vee \kappa S_t + A_t$ on the event $\{t < \tau_d\}$ (so $U^{cb}_t \geq \bar{C} \vee \kappa S_t + A_t$, $t \in [0, T]$).

So the discounted dividend process of an RB is given by, for $t \in [0, T]$:

$$\int_{[0,t]} \beta_u \, dD^{cb}_u := \int_{[0, t\wedge\tau_d]} \beta_u c_u du + \sum_{0 \leq T_i \leq t, \, T_i < \tau_d} \beta_{T_i} c^i + \mathbb{1}_{\{0 \leq \tau_d \leq t\}} \beta_{\tau_d} R^{cb}_{\tau_d} \ . \tag{19}$$

Clearly, a CB with no notice period ($\delta = 0$) is an RB, with

$$\widetilde{U}^{cb}(t, S) := \bar{C} \vee \kappa S + A(t).$$

More generally, the financial interpretation of the process U^{cb} in an RB is that U^{cb}_t represents the value of the RB upon a call at time t. In section 5, we shall actually prove that under mild regularity assumptions in the SMM, any CB (whether δ is positive or not) can be interpreted and priced as an RB.

4.2 Decomposition of a Reduced Convertible Bond

We consider an RB with dividend process D^{cb} given by (19), and an ES with the same coupon process as the RB, and with R^b and ξ^b as follows:

$$R^b_t = \bar{R}_t, \quad \xi^b = \bar{N} + A_T \tag{20}$$

so that

$$R^{cb}_t - R^b_t = (\kappa S_t - \bar{R}_t)^+ \geq 0, \quad \xi^{cb} - \xi^b = (\kappa S_T - \bar{N})^+ \geq 0.$$

So this ES corresponds to the defaultable bond with discounted cash flows given by the expression

$$\begin{aligned}\beta_t \phi_t &= \int_t^T \beta_u \, dD^{cb}_u + \mathbb{1}_{\{\tau_d > T\}} \, \beta_T \xi^b \\ &= \int_t^{T \wedge \tau_d} \beta_u c_u du + \sum_{t < T_i \leq T, T_i < \tau_d} \beta_{T_i} c^i + \mathbb{1}_{\{t < \tau_d \leq T\}} \, \beta_{\tau_d} R^b_{\tau_d} + \mathbb{1}_{\{\tau_d > T\}} \, \beta_T \xi^b\end{aligned}$$

and associated functions

$$f = \gamma R^b + G - \mu\Theta \ , \ g = \bar{N} \ .$$

This bond can be seen as the pure bond component of the RB (that is, the RB stripped of its optional clauses). Therefore, we shall call it the *bond embedded into the RB*, or simply the *embedded bond*.

In the sequel, in addition to the assumptions made so far, we work under the following regularity assumption.

Assumption 4.1 The functions $r(t)$, $q(t)$, $S\gamma(t, S)$, $S\sigma(t, S)$, $\gamma(t, S)\bar{R}(t, S)$ and c are continuously differentiable in time, and three times continuously differentiable in space, with bounded spatial partial derivatives of order one to three.

Remarks 4.1 **(i)** Note that under these assumptions, $S\sigma$, $S\gamma$, and $\gamma\bar{R}$ are Lipschitz in space. Also, under these assumptions, condition (9) is satified (see e.g. [38]).
(ii) These assumptions cover typical financial applications. In particular, they are satisfied when $\bar{R}$ is constant and for well-chosen parameterizations of σ and γ, which can be enforced at the time of the calibration of the model.

Lemma 4.1 *In the case of an RB, the elementary BSDE* $(\mathcal{E}')$ *associated to the embedded Bond admits a solution* $(\widehat{\Phi}, z)$ *Thus, denoting* $\widetilde{\Phi} = \widehat{\Phi} + A$, *the embedded bond admits a unique arbitrage price*

$$\Phi_t = \mathbb{1}_{t < \tau_d} \widetilde{\Phi}_t, \quad t \in [0, T],$$

by Theorem 3.2 (specified to the particular case of an ES). Moreover we have $\widehat{\Phi}_t = \widehat{\Phi}(t, \widetilde{S}_t)$, *where the function* $\widehat{\Phi}(t, S)$ *is jointly continuous in time and space and twice continuously differentiable in space, and the process* $\widehat{\Phi}(t, \widetilde{S}_t)$ *is an Itô process with strict martingale component, such that:*

$$\begin{aligned}d\widehat{\Phi}_t &= \widehat{u}_t \, dt + \widehat{v}_t \, dW_t \\ &:= (\mu_t \widehat{\Phi}_t - (\gamma_t R^b_t + G_t)) \, dt + \sigma(t, \widetilde{S}_t) \widetilde{S}_t \partial_S \widehat{\Phi}_t \, dW_t\end{aligned} \tag{21}$$

with $\widehat{v} \in \mathcal{H}^2(0, T)$.

Proof. Given that the related integrability conditions are satisfied in the SMM, then by direct application of the Brownian predictable representation theorem, the elementary BSDE $(\mathcal{E}')$ with data $(\gamma R^b + G - \mu\Theta, \bar{N})$ admits a solution $(\widehat{\Phi}, z)$. In particular, by $(\mathcal{E}')$:

$$\widehat{\Phi}_t = \mathbb{E}_{\mathbb{Q}}\Big(\int_t^T (\gamma_u R^b_u + G_u - \mu_u \widehat{\Phi}_u) du + (\xi^b - A_T) \,\Big|\, \mathcal{F}_t \Big) \ , \ t \in [0, T] \ ,$$

or equivalently (see [8])

$$\alpha_t \widehat{\Phi}_t = \mathbb{E}_{\mathbb{Q}}\Big(\int_t^T \alpha_u (\gamma_u R_u^b + G_u) du + \alpha_T \left(\xi^b - A_T\right) \Big| \mathcal{F}_t \Big) \;,\; t \in [0,T] \;,$$

and thus, using (16):

$$\alpha_t \widetilde{\Phi}_t = \mathbb{E}_{\mathbb{Q}}\Big(\int_t^T \alpha_u \big(\gamma_u R_u^b du + c_u\big) du + \sum_{t<T_i\le T} \alpha_{T_i} c^i + \alpha_T \xi^b \Big| \mathcal{F}_t \Big) \;,\; t \in [0,T] \;.$$

So, on each time interval $[T_{i-1}, T_i \wedge T)$, and also at T in the case where $i = K$, we have $\widetilde{\Phi}_t = \widehat{\Phi}_t^0 + \sum_{j=i}^N \widehat{\Phi}_t^j$, where we set

$$\alpha_t \widehat{\Phi}_t^0 = \mathbb{E}_{\mathbb{Q}}\Big(\int_t^T \alpha_u (\gamma_u R_u^b + c_u) du + \alpha_T \left(\bar{N} + A_T\right) \Big| \mathcal{F}_t \Big), \tag{22}$$

$$\alpha_t \widehat{\Phi}_t^i = \mathbb{E}_{\mathbb{Q}}\big(\alpha_{T_i} c^i \,\big|\, \mathcal{F}_t \big). \tag{23}$$

In particular, for any $\Theta = \widehat{\Phi}^0$ or $\widehat{\Phi}^i$, Θ is bounded and non negative. Let us denote generically T or T^i by $\mathcal{T}$, as appropriate according to the problem at hand. Given our regularity assumptions, we have $\Theta_t = \widehat{\Theta}(t, \widetilde{S}_t)$, where $\widehat{\Theta}$ is a $\mathcal{C}^{1,2}([0,\mathcal{T}) \times \mathbb{R}) \cap \mathcal{C}^0([0,\mathcal{T}] \times \mathbb{R})$-function [36, 24]. Therefore $\widehat{\Phi}_t = \widetilde{\Phi}_t - A_t$ is given by $\widehat{\Phi}(t, \widetilde{S}_t)$, for a function $\widehat{\Phi}(t,S)$ which is jointly continuous in time and space, and twice continuously differentiable in space. Moreover, given (22), (23) and the above $\mathcal{C}^{1,2}$ regularity results, we have on $(T_{i-1}, T_i \wedge T)$:

$$\begin{aligned} d\widehat{\Phi}_t^0 &= \Big(\mu_t \widehat{\Phi}_t^0 - (\gamma_t R_t^b + c_t)\Big) dt + \sigma(t, \widetilde{S}_t) \widetilde{S}_t \partial_S \widehat{\Phi}^0(t, \widetilde{S}_t) dW_t \\ d\widehat{\Phi}_t^i &= \mu_t \widehat{\Phi}_t^i dt + \sigma(t, \widetilde{S}_t) \widetilde{S}_t \partial_S \widehat{\Phi}^i(t, \widetilde{S}_t) dW_t, \quad i = 1, 2, \ldots, K, \\ dA(t) &= \rho_t dt \end{aligned}$$

Hence, by addition:

$$d\widehat{\Phi}(t, \widetilde{S}_t) = \Big(\mu_t \widetilde{\Phi}_t - (\gamma_t R_t^b + c_t + \rho)\Big)\, dt + \sigma(t, \widetilde{S}_t) \widetilde{S}_t \partial_S \widetilde{\Phi}(t, \widetilde{S}_t)\, dW_t = \widehat{u}_t\, dt + \widehat{v}_t\, dW_t \;.$$

Moreover, since $\widehat{\Phi}$ and $\widehat{u}$ are bounded in (21), we conclude that $\widehat{v} \in \mathcal{H}^2(0,T)$. □

We now define the *embedded Game Exchange Option* (GEO) as the RB with dividend process $D^{cb} - D^b$, payment at maturity $\xi^{cb} - \xi^b$, put payment $L_t^{cb} - \Phi_t$, call payment $U_t^{cb} - \Phi_t$ and call protection lifting time $\bar{\tau}$. So the embedded GEO is the zero-coupon CS with cash flows

$$\begin{aligned} \beta_t \psi(t; \tau_p, \tau_c) &= \mathbb{1}_{\{t<\tau_d\le\tau\}}\, \beta_{\tau_d} (R_{\tau_d}^{cb} - R_{\tau_d}^b) \\ &\quad + \mathbb{1}_{\{\tau_d>\tau\}}\, \beta_\tau \Big(\mathbb{1}_{\{\tau=\tau_p<T\}} \Big(L_{\tau_p}^{cb} - \Phi_{\tau_p}\Big) + \mathbb{1}_{\{\tau=\tau_c<\tau_p\}} \big(U_{\tau_c}^{cb} - \Phi_{\tau_c}\big) + \mathbb{1}_{\{\tau=T\}} (\xi^{cb} - \xi^b) \Big). \end{aligned} \tag{24}$$

Proposition 4.1 *Given an RB, the associated functions $f(t,S,\Theta)$, $g = g(S)$, $\ell = \ell(t,S)$ and $h = h(t,S)$, are:*
- $\gamma R^{cb} + G - \mu\Theta, \bar{N} \vee \kappa S, \bar{P} \vee \kappa S, \widetilde{U}^{cb} - A$, *for the RB;*
- $\gamma(R^{cb} - R^b) - \mu\Theta, (\kappa S - \bar{N})^+, \bar{P} \vee \kappa S - \widehat{\Phi}, \widetilde{U}^{cb} - A - \widehat{\Phi}$, *for the embedded GEO.* □

Remarks 4.2 (i) In the case of the RB, the function ℓ happens to be a function of S alone, $\ell(t,S) = \ell(S)$.
(ii) In the case of the Game Exchange Option, there are no coupons involved, so that we are always in the continuous coupon case (special case $C = C^c$).

4.3 Solution of the doubly reflected BSDEs

Theorem 4.1 *Under Assumption 4.1, the functions f, g, ℓ, h associated to an RB or to the embedded GEO (cf. Proposition 4.1), satisfy all the general assumptions in Theorem 3.3.*

Proof. Let us show the result for the RB. Then, in view of Proposition 4.1, the result for the GEO will follow from Lemma 4.1. The only non trivial issue is to verify assumption (iii) in Theorem 3.3. Note that $\ell(\widetilde{S}_t) = \bar{P} \vee \kappa \widetilde{S}_t$ is essentially equal to the payoff of a call option on $\kappa \widetilde{S}$. Therefore, a natural idea to approximate the function ℓ as it is required in this assumption, is to use known results on the Black–Scholes pricing function of a call option, where the corresponding fictitious volatility Σ^n (not to be confused with the volatility σ of the stock $\widetilde{S}$ in our SMM) will merely be used as an approximation parameter. So, introducing $\Sigma^n \searrow 0^+$ as $n \to \infty$, let us define

$$C^n(\kappa S) = \kappa S \mathcal{N}(d_+^n) - \bar{P}\mathcal{N}(d_-^n) \tag{25}$$

$$\ell^n(S) = \bar{P} + C^n(\kappa S) - C^n(\bar{P}) - \epsilon_n \tag{26}$$

where $\mathcal{N}$ is the standard Gaussian cumulative distribution function, $d_\pm^n := \frac{1}{\Sigma^n} \ln(\frac{\kappa S}{\bar{P}}) \pm \frac{1}{2}\Sigma^n$, and $\epsilon_n \to 0^+$ at infinity. Thus $C^n(\kappa S)$ is equal to the price of a one year call option struck at $\bar{P}$, on a Black–Scholes stock κS with volatility Σ^n and dividend and interest rates equal to 0. Note that we have, uniformly in S :

$$0 \le \ell^n(S) - \ell(S) = C^n(\kappa S) - (\kappa S - \bar{P})^+ - C^n(\bar{P}) - \epsilon_n ,$$

which tends to 0 as $n \to \infty$. So the functions ℓ^n satisfy all the requirements in assumption (iii) of Theorem 3.3.

Besides, we have by the Itô–Tanaka formula, for $t \in [0, T]$:

$$\ell_t = \bar{P} \vee \kappa \widetilde{S}_t = \bar{P} \vee \kappa S_0 + \int_0^t \mathbb{1}_{\kappa \widetilde{S}_u \ge \bar{P}}((r(u) - q(u) + \eta\gamma(u, \widetilde{S}_u))\kappa \widetilde{S}_u du + \int_0^t \mathbb{1}_{\kappa \widetilde{S}_u \ge \bar{P}} \sigma(u, \widetilde{S}_u)\kappa \widetilde{S}_u dW_u + \Lambda_T \tag{27}$$

where Λ_t is the local time spent by the process κS at level $\bar{P}$ between 0 and t. Since

$$\bar{P} \vee \kappa \widetilde{S}_T \in L^2(T), \ \int_{t=0}^T \widetilde{S}_t dt \in L^2(T)$$

and (9) is verified, so the process ℓ_t is a semimartingale with strict martingale component. In addition, the finite variation component of ℓ_t admits the following decomposition:

$$dV_t^+ = \mathbb{1}_{\kappa \widetilde{S}_t \ge \bar{P}}((r(t) + \eta\gamma(t, \widetilde{S}_t))\kappa \widetilde{S}_t dt \ + \ d\Lambda_t$$
$$dV_t^- = \mathbb{1}_{\kappa \widetilde{S}_t \ge \bar{P}} q(t)\kappa \widetilde{S}_t dt$$

with $V^\pm \in \mathcal{A}^2(0, T)$. We have thus proven that the process ℓ_t is a continuous semimartingale with strict martingale component and finite variation components in $\mathcal{A}^2(0, T)$, such that

$$\ell^n(\widetilde{S}_t) < \ell_t \le h_t \ , \ 0 \le t \le T \ ,$$

so that the RB satisfies the general assumptions in Theorem 3.3.

□

Theorem 4.2 *Given an RB under Assumption 4.1, denote by $\widehat{\Pi}$ and $\widehat{\Psi}$, the state-processes of the solutions to the related R2BSDEs, by Theorems 3.3 and 4.1.*
(i) *Ψ_t defined as $\mathbb{1}_{t<\tau_d}\widehat{\Psi}_t$ is the unique arbitrage price of the embedded GEO, and $(\Psi_t, \zeta^*, \tau_c^*)$ (resp. $(\Psi_t, -\zeta^*, \tau_p^*)$) as defined in Theorem 3.2 is an issuer (resp. holder) hedge with initial value Ψ_t at time t for the embedded GEO;*
(ii) *Π_t defined as $\mathbb{1}_{t<\tau_d}\widetilde{\Pi}_t$, with $\widetilde{\Pi} := \widehat{\Pi} + A$, is the unique arbitrage price of the RB, and $(\Pi_t, \zeta^*, \tau_c^*)$ (resp. $(\Pi_t, -\zeta^*, \tau_p^*)$) as defined in Theorem 3.2 is an issuer (resp. holder) hedge with initial value Π_t at time t for the RB.*
(iii) *$\widehat{\Phi}$ and Φ being defined as in Lemma 4.1, we have $\Pi = \Phi + \Psi$, $\widehat{\Pi} = \widehat{\Phi} + \widehat{\Psi}$.*

Note in particular that the same strategies provide bilateral hedges for the RB and for the embedded GEO.

Proof of Theorem 4.2 (i) and (ii) follow by application of Theorems 3.3 and 4.1. (iii) then follows from the general results in [7, 8]. □

We can now give analytical characterizations of the so-called pre-default *clean* prices $\widehat{\Theta}$ (pre-default prices $\widetilde{\Theta}$ less accrued interest, see [8]) as viscosity solutions to the associated variational inequalities (see also [1, 2, 21, 22]). To get the corresponding pre-default prices, it suffices to add to the clean price process $\widehat{\Theta}$ the related accrued interest process. Note that in the case of the Game Exchange Option, there are no discrete coupons involved, therefore the Pre-Default Clean Price and the Pre-Default Price coincide.

4.4 Variational Inequalities for the No-Protection Clean Prices

We first assume that $\bar{\tau} = 0$ (no call protection). By application of Theorems 4.1 and 3.3.1, we have the following result.

Theorem 4.3 (No-Protection Clean Prices) *In the case where $\bar{\tau} = 0$ (no call protection), we define the following analytical problems $(\mathcal{P})$ on $\mathcal{D} = [0, T] \times \mathbb{R}$:*

1. *Defaultable Bond*

$$\begin{gathered} -\mathcal{L}\widehat{\Phi} + \mu\widehat{\Phi} - (\gamma R^b + G) = 0,\, t < T \\ \widehat{\Phi}(T, S) = \bar{N} \end{gathered} \tag{28}$$

2. *GEO*

$$\begin{gathered} \max\left(\min\left(-\mathcal{L}\widehat{\Psi} + \mu\widehat{\Psi} - \gamma(R^{cb} - R^b), \widehat{\Psi} - \left(\bar{P} \vee \kappa S - \widehat{\Phi}\right)\right), \widehat{\Psi} - \left(\widetilde{U}^{cb} - A - \widehat{\Phi}\right)\right) = 0,\ t < T \\ \widehat{\Psi}(T, S) = (\kappa S - \bar{N})^+ \end{gathered} \tag{29}$$

3. *RB*

$$\begin{gathered} \max\left(\min\left(-\mathcal{L}\widehat{\Pi} + \mu\widehat{\Pi} - (\gamma R^{cb} + G), \widehat{\Pi} - \bar{P} \vee \kappa S\right), \widehat{\Pi} - (\widetilde{U}^{cb} - A)\right) = 0,\, t < T \\ \widehat{\Pi}(T, S) = \bar{N} \vee \kappa S \end{gathered} \tag{30}$$

Then for any of the problems $(\mathcal{P})$ above, the corresponding No Protection (Pre-default) Clean Price $\widehat{\Theta}^0_t$ can be written as $\widehat{\Theta}^0(t, \widetilde{S}_t)$, where the function $\widehat{\Theta}^0$ is a $\mathcal{S}$-solution of $(\mathcal{P})$ on $\mathcal{D}$. Moreover, in the continuous coupon cases (hence always, in the case of the GEO), we have uniqueness of the $\mathcal{S}$-solution, and any stable, monotonous and consistent approximation scheme for $\widehat{\Theta}^0$ converges locally uniformly to $\widehat{\Theta}^0$ on $\mathcal{D}$ as $h \to 0^+$, provided it converges to $\widehat{\Theta}^0$ at T. □

Corollary 4.1 *A pair of No Protection Pre-default optimal stopping times (τ_p^*, τ_c^*) (see Theorem 3.2), both in the case of the Game Exchange Option embedded in the RB, and of the RB itself, is given by*

$$\tau_p^* = \inf\{u \in [t, T]\, ;\, \widetilde{S}_u \in \mathcal{E}x_p\} \wedge T,\ \tau_c^* = \inf\{u \in [t, T]\, ;\, \widetilde{S}_u \in \mathcal{E}x_c\} \wedge T$$

where

$$\mathcal{E}x_p := \{(t, S) \in [0, T]\, ;\, \widehat{\Pi}(t, S) = \bar{P} \vee \kappa S\},\ \mathcal{E}x_c := \{(t, S) \in [0, T]\, ;\, \widehat{\Pi}(t, S) = \widetilde{U}^{cb}(t, S) - A_t\}$$

are the No Protection Pre-default Put or Conversion Zone and the No Protection Pre-default Call Zone.

Proof. This follows immediately of Theorems 4.3 and 4.2. □

So assuming that there is no call protection, and that the CB is still alive:
(i) an optimal call time for the issuer of the CB is given by the first hitting time of $\mathcal{E}x_c$ (if any before $T \wedge \tau_d$) by S;
(ii) an optimal put/conversion time for the holder of the CB consists in putting or converting, whichever is best, at the first hitting time of $\mathcal{E}x_p$ (if any before $T \wedge \tau_d$) by S.

4.5 Variational Inequalities for the Post-protection Prices

For any $\bar{\tau} \in \mathcal{F}^0_T$, the associated Pre-default Price coincides on $[\bar{\tau}, T]$ with the Pre-default Price corresponding to a lifting time of call protection that would be given by $\bar{\tau}^0 := 0$. This follows from the general results in [9], using also the fact that the BSDEs related to the problems with lifting times of call protection $\bar{\tau}$ and $\bar{\tau}^0$ both have solutions, by the previous results.

Thus No Protection Prices (pre-default prices for lifting time of call protection $:= \bar{\tau}^0 = 0$) can be also be interpreted as Post-protection (Pre-default) Prices for arbitrary $\bar{\tau} \in \mathcal{G}^0_T$. Therefore the results of Section 4.4 also apply to Post-Protection Clean Prices. So,

Theorem 4.4 (Post-protection Clean Prices) *Let $\bar{\tau}$ be arbitrarily fixed in $\mathcal{F}^0_T$. Then for any of the pricing problems considered in Theorem 4.3, the corresponding Pre-default Clean Price $\widehat{\Theta}_t$ coincides on $[\bar{\tau}, T]$ with $\widehat{\Theta}^0(t, \widetilde{S}_t)$, where $\widehat{\Theta}^0$ is the related function in Theorem 4.3.* □

Corollary 4.2 *The pair of No-Protection Pre-default optimal stopping times (τ^*_p, τ^*_c), and the associated No-Protection Pre-default Call and Put Zones $\mathcal{E}x_c$ and $\mathcal{E}x_p$ (see Corollary 4.1), can also be interpreted as a pair of Post-protection optimal stopping times and Post-protection Call and Put Zones, respectively.*

So assuming that call protection have already been lifted (namely, for $t \geq \bar{\tau}$), and that the CB is still alive:
(i) an optimal call time for the issuer of the CB is given by the first hitting time of $\mathcal{E}x_c$ (if any before $T \wedge \tau_d$) by S;
(ii) an optimal put/conversion time for the holder of the CB consists in putting or converting, whichever is best, at the first hitting time of $\mathcal{E}x_p$ (if any before $T \wedge \tau_d$) by S.

4.6 Variational Inequalities for the Protection Prices

We finally consider Protection Clean Prices $\bar{\Theta}$, namely, by definition, Pre-Default Clean Prices stopped at $\bar{\tau}$. As usual, to get the corresponding Protection (Pre-default) Prices, it suffices to add the related accrued interest process (if there are any discrete coupons involved). Let $\widehat{\Phi}$, $\widehat{\Psi}$, $\widehat{\Pi}$ denote the Post Protection Clean Prices Functions (or No Protection Clean Prices Functions) defined in Theorems 4.3–4.4.

4.6.1 Hard Call Protections

In the case of hard call protection $\bar{\tau} = \bar{T}$ for some $\bar{T} \leq T$, the protectionclean prices functions $\bar{\Theta}$ are solutions of analytical problems as in Theorem 4.3 (special case with one obstacle, $h = +\infty$) with $\bar{T}$ instead of T, and terminal conditions equal to the corresponding Post Protection Clean Prices Functions $\widehat{\Theta}$ (or No Protection Clean Prices Functions $\widehat{\Theta}^0$, cf Theorem 4.4) at $\bar{T}$. So,

Theorem 4.5 (Hard Protection Clean Prices) *In the case where $\bar{\tau} = \bar{T}$ for some $\bar{T} \leq T$, we define the following analytical problems $(\mathcal{P})$ on $\mathcal{D} := \bar{\mathcal{D}} = [0, \bar{T}] \times \mathbb{R}$,*

1. *GEO*

$$\min\left(-\mathcal{L}\bar{\Psi} + \mu\bar{\Psi} - \gamma(R^{cb} - R^b), \widehat{\Psi} - \left(\bar{P} \vee \kappa S - \widehat{\Phi}\right)\right) = 0,\ t < T$$
$$\bar{\Psi}(\bar{T}, S) = \widehat{\Psi}(\bar{T}, S) \tag{31}$$

2. ***CB***

$$\min\left(-\mathcal{L}\bar\Pi+\mu\bar\Pi-(\gamma R^{cb}+G),\bar\Pi-\bar P\vee\kappa S\right)=0,\ t<T$$
$$\bar\Pi(\bar T,S)=\widehat\Pi(\bar T,S) \tag{32}$$

Then for either problem, the corresponding Hard Protection Clean Price $\bar\Theta$ can be written as $\bar\Theta(t,\widetilde S_t)$, where the function $\bar\Theta$ is a $\mathcal{S}$-solution of ($\mathcal{P}$) on $\bar{\mathcal{D}}$. Moreover, in the continuous coupon cases (hence always, in the case of the GEO), we have uniqueness of the $\mathcal{S}$-solution, and any stable, monotonous and consistent approximation scheme for $\bar\Theta$ converges locally uniformly to $\bar\Theta$ on $\mathcal{D}$ as $h\to 0^+$, provided it converges to $\bar\Theta(=\widehat\Theta)$ at $\bar T$.

Proof of Theorem 4.5 We know by Theorem 4.3 that the terminal conditions $\widehat\Theta(\bar T,\cdot)$ of all problems ($\mathcal{P}$) considered in Theorem 4.5 are continuous with polynomial growth in S. Therefore Theorem 4.5 can be proven in the same way as Theorem 4.3, by application of Theorem 3.3.1. □

Corollary 4.3 *A Hard Protection Pre-default optimal stopping time τ_p^* for the Game Exchange Option problem, and for the RB problem as well, is given by*

$$\tau_p^*=\inf\left\{u\in[t,\bar T]\,;\,\widetilde S_u\in\mathcal{E}x^h\right\}\wedge T$$

where

$$\mathcal{E}x^h:=\left\{(t,S)\in[0,\bar T]\,;\,\bar\Pi(u,\widetilde S_t)=\bar P\vee\kappa\widetilde S_t\right\}\wedge T$$

is the Hard Protection Pre-default Put or Conversion Zone.

So assuming that the CB is still alive at some time $t<\bar T$, an optimal strategy for the holder of the CB consists in putting or converting, whichever is best, at the first hitting time of $\mathcal{E}x^h$ (if any before $T\wedge\tau_d\wedge\bar T$) by S.

4.6.2 Soft Call Protections

Let us now treat the case of (standard) soft call protections. By application of Theorems 4.1 and 3.3.2, we have the following result.

Theorem 4.6 (Soft Protection Clean Prices) *Assuming that $\bar\tau=\inf\{t>0\,;\,\widetilde S_t\geq\bar S\}\wedge T$ for some $\bar S\in\mathbb{R}_+^\star$, we define the following analytical problems ($\mathcal{P}$) on $\bar{\mathcal{D}}:=[0,T]\times(-\infty,\bar S]$: 4.4.*

1. ***GEO***

$$\min\left(-\mathcal{L}\bar\Psi+\mu\bar\Psi-\gamma(R^{cb}-R^b),\bar\Psi-\left(\bar P\vee\kappa S-\widehat\Phi\right)\right)=0\,;\ t<T\,,S<\bar S$$
$$\bar\Psi(t,\bar S)=\widehat\Psi(t,\bar S)\,,t\leq T \tag{33}$$
$$\bar\Psi(T,S)=(\kappa S-\bar N)^+\,,S\leq\bar S$$

2. ***CB***

$$\min\left(-\mathcal{L}\bar\Pi+\mu\bar\Pi-(\gamma R^{cb}+G),\bar\Pi-\bar P\vee\kappa S\right)=0,\ t<T$$
$$\bar\Pi(t,\bar S)=\widehat\Pi(t,\bar S),\ t\leq T$$
$$\bar\Pi(T,S)=\bar N\vee\kappa S\,,S\leq\bar S \tag{34}$$

Then for either problem, the corresponding Soft Protection Clean Price $\bar\Theta$ can be written as $\bar\Theta(t,\widetilde S_t)$, where the function $\bar\Theta$ is a $\mathcal{S}$-solution of ($\mathcal{P}$) on $\bar{\mathcal{D}}$. Moreover, in the continuous coupon cases (hence always, in the case of the GEO), we have uniqueness of the $\mathcal{S}$-solution, and any stable, monotonous and consistent approximation scheme for $\bar\Theta$ converges locally uniformly to $\bar\Theta$ on $\bar{\mathcal{D}}$ as $h\to 0^+$, provided it converges to $\bar\Theta(=\widehat\Theta)$ at T and at $\bar S$. □

Corollary 4.4 *A Soft Protection Pre-default optimal stopping time τ_p^* for the Game $\mathbb{Q}$-Exchange Option problem, and for the CB problem as well, is given by*

$$\tau_p^* = \inf\Big\{ u \in [t,T]\,;\, \bar{S}_u \in Ex_s \Big\} \wedge T = \inf\Big\{ u \in [t,\bar{\tau}]\,;\, \widetilde{S}_u \in Ex_s \Big\} \wedge T$$

where

$$Ex_s = \{(t,S) \in [0,T]\,;\, \bar{\Pi}(t,S) = \bar{P} \vee \kappa S\Big\}$$

is the Soft Protection Pre-default Put or Conversion Zone.

So assuming that the stock has not reached the level $\bar{S}$ yet, and that the CB is still alive, an optimal strategy for the holder of the CB consists in putting or converting, whichever is best, at the first hitting time of Ex_s (if any before $T \wedge \tau_d \wedge \bar{\tau}$) by $\widetilde{S}$.

5 Convertible Bonds with positive Call Notice Period

We now consider the case of a Convertible Bond with positive Call Notice Period.

5.1 Reduced Convertible Bond interpretation

Note that between the call time t and the end of the notice period t^δ, a CB actually becomes a PB, that is, a CB with no call clause (formally, we set $\bar{\tau} = T$), called the t-PB, with effective put payment equal to the effective call payment C_u of the original CB, and effective payment at maturity C_{t^δ}. (see [7]).

Proposition 5.1 *In the case of the t-PB ($t \in [0,T]$), the associated functions $f(t,S,\Theta)$, $g = g(S)$ and $\ell = \ell(t,S)$ are ($h = +\infty$):*
• *$\gamma R^b + G - \mu\Theta, \bar{C}, -\infty$, for the embedded Bond (the* t-Bond, *in the sequel);*
• *$\gamma(R^{cb} - R^b) - \mu\Theta, \bar{C} \vee \kappa S - \widehat{\Phi}^t(t^\delta, S), \bar{C} \vee \kappa S - \widehat{\Phi}^t$, where $\widehat{\Phi}^t$ is the No Protection Clean Price Function of the t-Bond (obtained by application of Theorem 4.3), for the embedded GEO (the* t-GEO, *in the sequel);*
• *$\gamma R^{cb} + G, \bar{C} \vee \kappa S, \bar{C} \vee \kappa S$, for the t-PB itself.*

Theorem 5.1 (Variational Inequalities for the embedded PBs) *Given $t \in [0,T]$, we define the following problems ($\mathcal{P}$) on $\mathcal{D}^t := [t,t^\delta] \times \mathbb{R}$:*
1. *t-Bond*

$$\begin{gathered} -\mathcal{L}\widehat{\Phi}^t + \mu\widehat{\Phi}^t - \big(\gamma R^b + G\big) = 0,\ u < t^\delta \\ \widehat{\Phi}^t(t^\delta, S) = \bar{C} \end{gathered} \tag{35}$$

2. *t-GEO*

$$\begin{gathered} \min\Big(-\mathcal{L}\widehat{\Psi}^t + \mu\widehat{\Psi}^t - \gamma(R^{cb} - R^b), \widehat{\Psi}^t - \Big(\bar{C} \vee \kappa S - \widehat{\Phi}^t\Big)\Big) = 0,\ u < t^\delta \\ \widehat{\Psi}^t(t^\delta, S) = \bar{C} \vee \kappa S - \widehat{\Phi}^t(t^\delta, S) \end{gathered} \tag{36}$$

3. *t-PB*

$$\begin{gathered} \min\Big(-\mathcal{L}\widehat{\Pi}^t + \mu\widehat{\Pi}^t - (\gamma R^{cb} + G), \widehat{\Pi}^t - \bar{C} \vee \kappa S\Big) = 0,\ u < t^\delta \\ \widehat{\Pi}^t(t^\delta, S) = \bar{C} \vee \kappa S. \end{gathered} \tag{37}$$

Then for any of the problems ($\mathcal{P}$) above, the corresponding Pre-default Clean t-Price $\widehat{\Theta}^t$ can be written as $\widehat{\Theta}^t(t, \widetilde{S}_t)$, where the function $\widehat{\Theta}^t$ is a $\mathcal{S}$-solution of ($\mathcal{P}$) on $\mathcal{D}^t$. Moreover, in the continuous *coupon cases (hence always, in the case of the GEO), we have uniqueness of the $\mathcal{S}$-solution, and any stable, monotonous and consistent approximation scheme for $\widehat{\Theta}^t$ converges locally uniformly to $\widehat{\Theta}^t$ on $\mathcal{D}$ as $h \to 0^+$, provided it converges to $\widehat{\Theta}^t$ at t^δ.*

In addition, the function $\widehat{\Pi}^t(t,S)$ is jointly continuous in time and space (in any case, whether we are in the continuous coupon case or not), hence the function $\widetilde{\Pi}^t(t,S) = \widehat{\Pi}^t(t,\widetilde{S}_t) + A(t)$ is also continuous with respect to (t,S), except for left jumps of $-c^i$ at the T_i.

Proof. Except for the last part, this follows by application of Theorems 4.1 and 3.3 to the t-PB, in view of Proposition 5.1. Thus it only remains to show that the function $\widehat{\Pi}^t(t,S)$ is jointly continuous in time and space. Let $(t_n,S_n) \to (t,S)$ as $n \to \infty$. We decompose

$$\widehat{\Pi}^{t_n}(t_n,S_n) = \widehat{\Pi}^t(t_n,S_n) + (\widehat{\Pi}^{t_n}(t_n,S_n) - \widehat{\Pi}^t(t_n,S_n)) \, .$$

By Theorem 5.1 (first part already proven at this point), $\widehat{\Pi}^t(t_n,S_n) \to \widehat{\Pi}^t(t,S)$ as $n \to \infty$. Moreover, denoting $\widehat{C}_t = \bar{C} \vee \widetilde{S}_t$, we have:

$$\alpha_u \widehat{\Pi}^t_u \quad = \quad \mathrm{esssup}_{\tau_p \in \mathcal{F}^u_{t^\delta}} \mathbb{E}_{\mathbb{Q}}\Big(\int_u^{\tau_p} \alpha_v F_v dv + \alpha_{\tau_p} \widehat{C}_{\tau_p} \,\Big|\, \mathcal{F}_u \Big)$$

So, assuming t_n sufficiently close to the left of t

$$\begin{aligned}
\alpha_{t_n} \widehat{\Pi}^{t_n}(t_n,S_n) &= \mathrm{esssup}_{\tau_p \in \mathcal{F}^{t_n}_{t_n^\delta}} \mathbb{E}_{\mathbb{Q}}\Big(\int_{t_n}^{\tau_p} \alpha_v F_v dv + \alpha_{\tau_p} \widehat{C}_{\tau_p} \,\Big|\, \mathcal{F}_{t_n} \Big) \\
&\le \mathrm{esssup}_{\tau_p \in \mathcal{F}^{t_n}_{t^\delta}} \mathbb{E}_{\mathbb{Q}}\Big(\int_{t_n}^{\tau_p} \alpha_v F_v dv + \alpha_{\tau_p} \widehat{C}_{\tau_p} \,\Big|\, \mathcal{F}_{t_n} \Big) = \alpha_{t_n} \widehat{\Pi}^t(t_n,S_n)
\end{aligned}$$

Conversely, for any $\tau_p \in \mathcal{F}^{t_n}_{t^\delta}$, we have $\tau_p{}' := \tau_p \wedge t_n^\delta \in \mathcal{F}^{t_n}_{t_n^\delta}$, $0 \le \tau_p - \tau_p{}' \le t - t_n$ and

$$\begin{aligned}
&\Big| \int_{t_n}^{\tau_p} \alpha_v F_v dv + \alpha_{\tau_p} \widehat{C}_{\tau_p} - \int_{t_n}^{\tau_p{}'} \alpha_v F_v dv - \alpha_{\tau_p{}'} \widehat{C}_{\tau_p{}'} \Big| \\
&\quad \le \int_{\tau_p{}'}^{\tau_p} \alpha_v F_v dv + |\alpha_{\tau_p} \widehat{C}_{\tau_p} - \alpha_{\tau_p{}'} \widehat{C}_{\tau_p{}'}|
\end{aligned}$$

Therefore

$$\begin{aligned}
&\Big|\mathbb{E}_{\mathbb{Q}}\Big(\int_{t_n}^{\tau_p} \alpha_v F_v dv + \alpha_{\tau_p} \widehat{C}_{\tau_p} \,\Big|\, \mathcal{F}_{t_n} \Big) - \mathbb{E}_{\mathbb{Q}}\Big(\int_{t_n}^{\tau_p{}'} \alpha_v F_v dv + \alpha_{\tau_p{}'} \widehat{C}_{\tau_p{}'} \,\Big|\, \mathcal{F}_{t_n} \Big)\Big| \\
&\quad \le \mathbb{E}_{\mathbb{Q}}\Big(\int_{\tau_p{}'}^{\tau_p} \alpha_v F_v dv \,\Big|\, \mathcal{F}_{t_n} \Big) + \mathbb{E}_{\mathbb{Q}}\Big(|\alpha_{\tau_p} \widehat{C}_{\tau_p} - \alpha_{\tau_p{}'} \widehat{C}_{\tau_p{}'}| \,\Big|\, \mathcal{F}_{t_n} \Big) \\
&\quad \le \sqrt{t - t_n} \|F\|_{\mathcal{H}^2(0,T)} + \mathbb{E}_{\mathbb{Q}}\Big(|\alpha_{\tau_p} \widehat{C}_{\tau_p} - \alpha_{\tau_p{}'} \widehat{C}_{\tau_p{}'}| \,\Big|\, \mathcal{F}_{t_n} \Big).
\end{aligned}$$

We conclude that $\widehat{\Pi}^{t_n}(t_n,S_n) - \widehat{\Pi}^t(t_n,S_n) \to 0$ as $t_n \to t^-$. But this is also true as $t_n \to t^+$ (same proof), hence $\widehat{\Pi}^{t_n}(t_n,S_n) - \widehat{\Pi}^t(t_n,S_n) \to 0$ as $t_n \to t$. Finally $\widehat{\Pi}^{t_n}(t_n,S_n) \to \widehat{\Pi}^t(t,S)$ as $t_n \to t$, as desired. □

Theorem 5.2 *Assuming No Arbitrage (so that in particular the embedded t-PBs price processes Π^t are arbitrage-free, $0 \le t \le T$), then a CB with positive notice period $\delta > 0$ can be interpreted as an RB, with $\widetilde{U}^{cb}(t,S) = \widetilde{\Pi}^t(t,S)$, so that*

$$U^{cb}_t = \bar{U}^{cb}_t := \mathbb{1}_{\{\tau_d > t\}} \widetilde{\Pi}^t_t + \mathbb{1}_{\{\tau_d \le t\}} (\bar{C} \vee \kappa S_t + A_t) \, . \tag{38}$$

Proof. First, the t-PB related reflected BSDE has a solution, by Theorems 4.1 and 3.3.1 applied to the t-PB. Thus the t-PB has a unique arbitrage price process $\Pi^t_u = \mathbb{1}_{\{u < \tau_d\}} \widetilde{\Pi}^t_u$ with $\widetilde{\Pi}^t_u = \widehat{\Pi}^t_u + A_u$, by Theorem 3.2.3. So the arbitrage price of the CB upon call time t (assuming the CB still alive at time t) is well defined, as Π_t^t, which is also equal to $\widetilde{\Pi}^t(t, \widetilde{S}_t)$, by Theorem 5.1.

Moreover, by Theorem 5.1 (last part), the function $\widetilde{\Pi}^t(t, S)$ is jointly continuous in time and space, except for negative left jumps of $-c^i$ at the T_is, and $\Pi^t(t, S_t) = \Pi_t^t \geq \bar{C} \vee \kappa S_t + A_t$ on the event $\{\tau_d > t\}$, by the general results in [7]. So $\bar{U}^{cb}$ satisfies all the requirements in (18). □

Therefore all the results of section 4 are applicable to a CB, interpreted as an RB under the assumptions 4.1(i).

5.2 Numerical Solution of the related Variational Inequalities

Under Assumption 4.1 and assuming that $\bar{\tau} = 0$ (no call protection), then, once we have specified all the parameters for one of the problems (28), (29) or (30), including, in the case of (29) or (30), the function $\widetilde{U}^{cb}$, one can solve the problem numerically [2, 31], and we know that under mild conditions, standard finite differences or elements numerical schemes will converge towards the $\mathcal{S}$-solution of the problem as the discretization step goes to 0. This is at least the case in the continuous coupon cases, e.g. for the embedded GEO. Solving the embedded Bond-related PDEs is of course standard, we shall not comment on this issue here. Finally, in the case of the CB itself, and if there are discrete coupons involved, it is still possible to use the results related to the GEO, via the relation $\Pi = \Pi + \Psi$.

To have a fully endogenous specification of the problem, one can then take $\widetilde{U}^{cb}(t, S) := \widetilde{\Pi}^t(t, S)$ in (29) or (30), by Theorem 5.2, and again, at least in the continuous coupon cases, standard finite differences or elements numerical schemes will typically converge towards the $\mathcal{S}$-solutions of the t-problems (37) as the discretization step goes to 0, by Theorem 5.1. Here is a practical algorithm for solving (30) with $\widetilde{U}^{cb}(t, S) := \widetilde{\Pi}^t(t, S)$, using for example a fully implicit finite difference scheme (see for instance [35]) to discretize $\mathcal{L}$:

1. Localize problems (37) for the embedded t-PBs and problem (30) for the CB. In the case of the PBs, we know in advance that the price is equal to κS for κS larger than $\bar{C}$. So a natural choice, obviously in the special case of the PBs, but also for the CB, is to localize the problems on the spatial domain $(-\infty, \bar{C}/\kappa]$;

2. Discretize the domain $\mathcal{D} = [0, T] \times (-\infty, \bar{C}/\kappa]$, using, say, one time step per day market day between 0 and T;

3. Discretize problems (37) for the embedded PBs on $\mathcal{D}^t = \mathcal{D} \cap [t, t^\delta]$, for t in the time grid (one problem per value of t in the time grid);

4. Solve for $\widehat{\Pi}^t$ the discretized problems (37) corresponding to the embedded PBs for t in the time grid (one problem per value of t in the time grid);

5. Discretize problem (30) for the CB on $\mathcal{D}$ and solve the discretized problem, using the numerical approximation of $\widetilde{\Pi}^t(t, S) := \widehat{\Pi}^t(t, S) + A(t)$ as an input for $\widetilde{U}^{cb}(t, S)$.

Since the problem for the t-PB only has to be solved on the subdomain $\mathcal{D}^t$ of $\mathcal{D}$, the overall computational cost for solving a CB problem (30) with positive call notice period is roughly the same as that required for solving one CB problem without call notice period, plus n PB problems that would be defined on the whole grid, where n is the number of days in the notice period — typically one month, that is $n = 20$ market days.

Finally if a call protection is in force then we proceed along essentially the same lines following the indications in Section 3.3.

An alternative to numerical solution for the VIs would be to use simulation methods adapted from simulated methods for American options [34, 33, 39], that allow one to handle early exercise by simulation. Note that these methods are not very much used in the industry at this stage. Beyond the fact that they are computationally more intensive than standard Monte carlo simulations for European options, another reason is that they do not give a confidence interval. When the space dimension of a model is three or less, determnistic numerical schemes are recommended. They give the CB price and Greeks at the same time and for a whole range of values of the spot, and they can easily deal with all the early exercice features. Otherwise or if one wants to take into account non standard soft call protection clauses and be able to deal with any kinds of path-dependency, simulation methods may be considered.

Still another alternative would be to use numerical methods for BSDEs [13, 14], resorting to a notion of weak solutions in weighted Sobolev spaces for the variational inequalities rewritten in divergence form [5, 3]. Given the solution $(\Theta, z, \mathbf{k})$ of a R2BSDE with respect to $\mathbb{F} := \mathbb{F}^W$ in a Markovian set-up, the interest of the Sobolev solutions approach is to provide PDE representations not only for the state-process Θ (the price of the CS) as $\Theta_t := \Theta(t, S_t)$, but also for z (the 'delta' of the CS) and $\mathbf{k}$. So, typically, in our notation: $z_t = \sigma(t, S_t) S_t \partial_S \Theta(t, S_t)$. From the computational point of view this also gives numerical schemes to approximate Θ (and maybe also z). Note however that the associated convergences are to be understood in probability only. This direction of investigation is left for future research.

Another numerical issue is the calibration of the model, which consists in letting undetermined some specific parameters of the model, such as the local volatility σ and the local intensity γ in our Markovian model for $\mathbb{Q}$ above, and trying to fix these parameters in order to mark model $\mathbb{Q}$ to the market. Various input instruments can be used in this calibration process, such as vanilla options on the underlying equity and/or CDS traded on bonds of the issuer [1].

A Viscosity solutions of double obstacle variational inequalities

For the convenience of the reader, we recall in this Appendix the definition of viscosity solutions [17] in the case of variational inequalities, as well as the notions of stable, monotonous and consistent approximation schemes [6].

Definition A.1 Given $\bar{S} \leq \infty$, let $\mathcal{D} := [0, T] \times \mathrm{Adh}(-\infty, \bar{S})$, and let

$$\overset{\circ}{\mathcal{D}}_p = [0, T) \times (-\infty, \bar{S}) \quad \text{and} \quad \partial_p \mathcal{D} := \mathcal{D} \setminus \overset{\circ}{\mathcal{D}}_p$$

stand for the *parabolic interior* and the *parabolic boundary* of $\mathcal{D}$, respectively. We denote by $(\mathcal{P})$ the following variational inequality with double obstacle on $\mathcal{D}$:

$$\max\left(\min\left(-\mathcal{L}\Theta(t, S) - f(t, S, \Theta(t, S)), \Theta(t, S) - \ell(t, S)\right), \Theta(t, S) - h(t, S)\right) = 0 \,.$$

(i) A locally bounded upper semicontinuous function Θ on $\mathcal{D}$ is called a *viscosity subsolution* of $(\mathcal{P})$ on $\overset{\circ}{\mathcal{D}}_p$ if and only if for any $(t, S) \in \overset{\circ}{\mathcal{D}}_p$ and any $\varphi \in \mathcal{C}^{1,2}(\mathcal{D})$ such that $\Theta - \varphi$ reaches its maximum on $\mathcal{D}$ at (t, S), then

$$\max\left(\min\left(-\mathcal{L}\varphi(t, S) - f(t, S, \Theta(t, S)), \Theta(t, S) - \ell(t, S)\right), \Theta(t, S) - h(t, S)\right) \leq 0 \tag{39}$$

Equivalently, Θ is a viscosity subsolution of $(\mathcal{P})$ on $\overset{\circ}{\mathcal{D}}_p$ if and only if $\Theta \leq u$, and $\Theta(t, S) > \ell(t, S)$ implies

$$-\mathcal{L}\varphi(t, S) - f(t, S, \Theta(t, S)) \leq 0 \,, \tag{40}$$

for any $(t, S) \in \overset{\circ}{\mathcal{D}}_p$ and $\varphi \in \mathcal{C}^{1,2}(\mathcal{D})$ such that $\Theta - \varphi$ reaches its maximum on $\mathcal{D}$ at (t, S).

(ii) A locally bounded lower semicontinuous function Θ on $\mathcal{D}$ is called a *viscosity supersolution* of $(\mathcal{P})$ on $\overset{\circ}{\mathcal{D}}_p$ if and only if for any $(t, S) \in \overset{\circ}{\mathcal{D}}_p$ and $\varphi \in \mathcal{C}^{1,2}(\mathcal{D})$ such that $\Theta - \varphi$ reaches its minimum on $\mathcal{D}$ at (t, S), then

$$\max\left(\min\left(-\mathcal{L}\varphi(t, S) - f(t, S, \Theta(t, S)), \Theta(t, S) - \ell(t, S)\right), \Theta(t, S) - h(t, S)\right) \geq 0 \tag{41}$$

Equivalently, Θ is a viscosity supersolution of $(\mathcal{P})$ on $\overset{\circ}{\mathcal{D}}_p$ if and only if $\Theta \geq l$, and $\Theta(t, S) < h(t, S)$ implies

$$-\mathcal{L}\varphi(t, S) \; - \; f(t, S, \Theta(t, S)) \geq 0 \,, \tag{42}$$

for any $(t,S) \in \overset{\circ}{\mathcal{D}}_p$ and $\varphi \in \mathcal{C}^{1,2}(\mathcal{D})$ such that $\Theta - \varphi$ reaches its minimum on $\mathcal{D}$ at (t,S).

(iii) Θ is called a *viscosity solution* of $(\mathcal{P})$ on $\overset{\circ}{\mathcal{D}}_p$ if and only if it is both a viscosity subsolution and a viscosity supersolution of $(\mathcal{P})$ on $\overset{\circ}{\mathcal{D}}_p$ —in which case, Θ is a continuous function.

Remarks A.1 (i) Given the potential discontinuities in f, we should a priori use the general definition of viscosity solutions for discontinuous operators. However, due to the very simple nature of the discontinuities involved, the standard definition is sufficient for our purposes.
(ii) A classic solution of $(\mathcal{P})$ (if any) is necessarily a viscosity solution of $(\mathcal{P})$.
(iii) A viscosity subsolution (resp. supersolution) Θ of $(\mathcal{P})$ does not need to verify $\Theta \geq \ell$ (resp. $\Theta \leq h$). A viscosity solution (in particular, a classic solution, if any) Θ of $(\mathcal{P})$ necessarily satisfies $\ell \leq \Theta \leq h$.
(iv) So problems $(\mathcal{P})$ are posed over spatial domains going to $-\infty$, though only the positive part of the domain has a financial interpretation (cf. Remark 2.1(i)). If we decided to pose problems $(\mathcal{P})$ over bounded spatial domains, then we would also have to impose some appropriate non-trivial boundary condition at the lower space boundary, in order to get a well-posed problem.
(iv) By a classic trick for viscosity solutions of parabolic problems ("two-sided sided implies one-sided", see for instance [32]), and due to the very simple nature of the discontinuities involved in our operators, a continuous function which is a viscosity solution of our problem on the right of T_i, is also a viscosity solution at T_i.

Let us now define approximation meshes and suitable notions of limits on these meshes.

Lemma A.1 *Let $(\mathcal{D}^h)_{h>0}$ denote a family of subsets of $\mathcal{D}$, such that for any $(t,S) \in \mathcal{D}$, there exists sequences h_n, t^{h_n}, S^{h_n} verifying*

$$h_n \to 0^+ \text{ and } \mathcal{D}^{h_n} \ni (t^{h_n}, S^{h_n}) \to (t,S) \text{ as } n \to \infty.$$

Let $(\Theta^h)_{h>0}$ be a family of uniformly locally bounded real functions defined on the sets $(\mathcal{D}^h)_{h>0}$.
(i) *For any $(t,S) \in \mathcal{D}$, the set of limits of the following kind:*

$$\lim_{n\to+\infty} \Theta^{h_n}(t^{h_n}, S^{h_n}) \in \mathbb{R} \tag{43}$$

with $h_n \to 0^+$ and $\mathcal{D}^{h_n} \ni (t^{h_n}, S^{h_n}) \to (t,S)$ as $n \to \infty$, is non empty and compact in $\mathbb{R}$. It admits as such a smallest and a greatest element: $\underline{\Theta}(t,S) \leq \overline{\Theta}(t,S)$ in $\mathbb{R}$.
(ii) *The function $\underline{\Theta}$, respectively $\overline{\Theta}$, defined in this way, is locally bounded and lower semi-continuous on $\mathcal{D}$, respectively locally bounded and upper semi-continuous on $\mathcal{D}$. We shall call it the lower limit, respectively upper limit, of Θ^h at (t,S) as $h \to 0^+$. We shall say that Θ^h converges to l at $(t,S) \in \mathcal{D}$ as $h \to 0^+$, and we shall denote :*

$$\lim_{\substack{h\to 0^+ \\ \mathcal{D}^h \ni (t^h, S^h) \to (t,S)}} \Theta^h(t^h, S^h) \ = \ l \ ,$$

if and only if $\underline{\Theta}(t,S) = \overline{\Theta}(t,S) = l$, or, equivalently:

$$\lim_{n\to+\infty} \Theta^{h_n}(t^{h_n}, S^{h_n}) \ = \ l$$

for any $h_n \to 0^+$ et $\mathcal{D}^{h_n} \ni (t^{h_n}, S^{h_n}) \to (t,S)$.
(iii) *If Θ^h converges pointwise everywhere to a continuous function Θ on $\mathcal{D}$, then this convergence is locally uniform:*

$$\max_{\mathcal{D}^h \cap \mathcal{C}} |\Theta^h - \Theta| \ \to 0$$

as $h \to 0^+$, for any compact subset $\mathcal{C}$ of $\mathcal{D}$. □

Definition A.2 Let us be given a family $(\delta_h\mathcal{L})_{h>0}$ of operators such that $\delta_h\mathcal{L} := \delta_h\mathcal{L}(t,S,\Theta(t,S),\Theta)$, acting on locally bounded functions Θ on $\mathcal{D}^h$. We shall say that $(\delta_h\mathcal{L})_{h>0}$ is *monotonous*, if and only if

$$\delta_h\mathcal{L}(t,S,\Theta(t,S),\Theta) \le \delta_h\mathcal{L}(t,S,\Theta'(t,S),\Theta') \text{ for any } \Theta(t,S) = \Theta'(t,S) \text{ and } \Theta \ge \Theta' \text{ on } \mathcal{D}^h,$$

and consistent with $\mathcal{L}$, if and only if for any regular function φ

$$h^{-1}\delta_h\mathcal{L}(t,S,\varphi(t,S)+\xi^h\,\varphi+\xi^h) \to \mathcal{L}\varphi(t,S) \text{ as } h\to 0^+\ ,\ \mathcal{D}^h \ni (t^h,S^h) \to (t,S) \in \overset{\circ}{\mathcal{D}}_p \ \text{ and } \mathbb{R} \ni \xi^h \to 0\ .$$

Remarks A.2 In the continuous coupon case, standard finite differences or finite elements numerical schemes furnish monotonous and consistent approximation schemes for $\mathcal{L}$.

Lemma A.2 *Let $(\Theta_h)_{h>0}$ be uniformly polynomially bounded (Θ_h bounded by $C(1+S^p)$ for some C and p independent of h) and satisfy*

$$\delta_h\mathcal{L}(t,S,\Theta_h(t,S),\Theta_h) + hf(t,S,\Theta_h(t,S)) = 0$$

on $\mathcal{D}^h$ for any $h>0$, where $(\delta_h\mathcal{L})_{h>0}$ is monotonous and consistent with $\mathcal{L}$. Then the upper and lower limits $\overline{\Theta}$ and $\underline{\Theta}$ of Θ_h as $h\to 0^+$, are respectively viscosity subsolutions and supersolutions of $(\mathcal{P})$ on $\overset{\circ}{\mathcal{D}}_p$.

Proof. This follows by application of [6] (see also [18]). □

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