11.1.3 Fourier transform methods for option pricing

Contrary to the classical Black-Scholes case, in exponential-Lévy models there are no explicit formulae for call option prices, because the probability density of a Lévy process is typically not known in closed form. However, the characteristic function of this density can be expressed in terms of elementary functions for the majority of Lévy processes discussed in the literature. This has led to the development of Fourier-based option pricing methods for exponential-Lévy models. In these methods, one needs to evaluate one Fourier transform numerically but since they simultaneously give option prices for a range of strikes and the Fourier transform can be efficiently computed using the FFT algorithm, the overall complexity of the algorithm per option price is comparable to that of evaluating the Black-Scholes formula.

We will describe two Fourier-based methods for option pricing in exp-Lévy models. The first method, due to Carr and Madan [83] is somewhat easier to implement but has lower convergence rates. The second one, described by Lewis [262] converges faster but requires one intelligent decision which makes it more delicate to produce a robust automatic implementation. Recall the definition of the Fourier transform of a function f:

$$\mathbf{F}f(v) = \int_{-\infty}^{\infty} e^{ixv} f(x)dx$$

Usually v is real but it can also be taken to be a complex number. The inverse Fourier transform is given by:

$$\mathbf{F}^{-1}f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixv} f(v)dx$$

For $f \in L^2(\mathbb{R})$, $\mathbf{F}^{-1}\mathbf{F}f = f$, but this inversion formula holds in other cases as well. In what follows we denote by $k = \ln K$ the log strike and assume without loss of generality that $t = 0$.

Method of Carr and Madan [83] In this section we set $S_0 = 1$, i.e., at time 0 all prices are expressed in units of the underlying. An assumption necessary in this method is that the stock price have a moment of order $1+\alpha$ for some $\alpha > 0$:

(H1) $\qquad \exists\, \alpha > 0: \quad \int_{-\infty}^{\infty} \rho_T(s) e^{(1+\alpha)s} ds < \infty,$

where ρ_T is the risk-neutral density of X_T. In terms of the Lévy density it is equivalent to the condition

$$\exists\, \alpha > 0 \qquad \int_{|y|\geq 1} \nu(dy) e^{(1+\alpha)y} < \infty. \tag{11.15}$$

This hypothesis can be satisfied in all models discussed in Chapter 4 by putting a constraint on the exponential decay parameter for positive jumps (negative jumps do not affect it). In order to compute the price of a call option

$$C(k) = e^{-rT} E[(e^{rT+X_T} - e^k)^+]$$

we would like to express its Fourier transform in strike in terms of the characteristic function $\Phi_T(v)$ of X_T and then find the prices for a range of strikes by Fourier inversion. However we cannot do this directly because $C(k)$ is not integrable (it tends to a positive constant as $k \to -\infty$). The key idea of the method is to instead compute the Fourier transform of the (modified) time value of the option, that is, the function

$$z_T(k) = e^{-rT} E[(e^{rT+X_T} - e^k)^+] - (1 - e^{k-rT})^+. \tag{11.17}$$

Let $\zeta_T(v)$ denote the Fourier transform of the time value:

$$\zeta_T(v) = \mathbf{F} z_T(v) = \int_{-\infty}^{+\infty} e^{ivk} z_T(k) dk. \tag{11.18}$$

It can be expressed in terms of characteristic function of X_T in the following way. First, we note that since the discounted price process is a martingale, we can write

$$z_T(k) = e^{-rT} \int_{-\infty}^{\infty} \rho_T(x) dx (e^{rT+x} - e^k)(1_{k \le x+rT} - 1_{k \le rT}).$$

Condition (H1) enables us to compute $\zeta_T(v)$ by interchanging integrals:

$$\begin{aligned}
\zeta_T(v) &= e^{-rT} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dx e^{ivk} \rho_T(x)(e^{rT+x} - e^k)(1_{k \le x+rT} - 1_{k \le rT}) \\
&= e^{-rT} \int_{-\infty}^{\infty} \rho_T(x) dx \int_{x+rT}^{rT} e^{ivk}(e^k - e^{rT+x}) dk \\
&= \int_{-\infty}^{\infty} \rho_T(x) dx \left\{ \frac{e^{ivrT}(1 - e^x)}{iv + 1} - \frac{e^{x+ivrT}}{iv(iv+1)} + \frac{e^{(iv+1)x+ivrT}}{iv(iv+1)} \right\}
\end{aligned}$$

The first term in braces disappears due to martingale condition and, after computing the other two, we conclude that

$$\zeta_T(v) = e^{ivrT} \frac{\Phi_T(v - i) - 1}{iv(1 + iv)} \tag{11.19}$$

The martingale condition guarantees that the numerator is equal to zero for $v = 0$. Under the condition (H1), we see that the numerator becomes an analytic function and the fraction has a finite limit for $v \to 0$. Option prices can now be found by inverting the Fourier transform:

$$z_T(k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ivk} \zeta_T(v) dv \tag{11.20}$$

Note that in this method we need the condition (H1) to derive the formulae but we do not need the exact value of α to do the computations, which makes the method easier to implement. The price to pay for this is a slower convergence of the algorithm: since typically $\Phi_T(z) \to 0$ as $\Re z \to \infty$, $\zeta_T(v)$ will behave like $|v|^{-2}$ at infinity which means that the truncation error in the numerical evaluation of integral (11.20) will be large. The reason of such a slow convergence is that the time value (11.17) is not smooth; therefore its Fourier transform does not decay sufficiently fast at infinity. For most models of Chapter 4 the convergence can be dramatically improved by replacing the time value with a smooth function of strike. Namely, instead of subtracting the intrinsic value of the option (which is non-differentiable) from its price, we suggest to subtract the Black-Scholes call price with suitable volatility (which is a smooth function). The resulting function will be both integrable and smooth. Denote

$$\tilde{z}_T(k) = e^{-rT}E[(e^{rT+X_T} - e^k)^+] - C^{\sigma}_{BS}(k),$$

where $C^{\sigma}_{BS}(k)$ is the Black-Scholes price of a call option with volatility σ and log-strike k for the same underlying value and the same interest rate. By a reasoning similar to the one used above, it can be shown that the Fourier transform of $\tilde{z}_T(k)$, denoted by $\tilde{\zeta}_T(v)$, satisfies

$$\tilde{\zeta}_T(v) = e^{ivrT}\frac{\Phi_T(v-i) - \Phi^{\sigma}_T(v-i)}{iv(1+iv)}, \tag{11.21}$$

where $\Phi^{\sigma}_T(v) = \exp(-\frac{\sigma^2 T}{2}(v^2+iv))$. Since for most models of Chapter 4 (more precisely, for all models except variance gamma) the characteristic function decays faster than every power of its argument at infinity, this means that the expression (11.21) will also decay faster than every power of v as $\Re v \to \infty$, and the integral in the inverse Fourier transform will converge very fast. This is true for *every* $\sigma > 0$ but some choices, or course, are better than others. Figure 11.4 shows the behavior of $|\tilde{\zeta}_T|$ for different values of σ compared to the behavior of $|\zeta_T|$ in the framework of Merton jump-diffusion model with volatility 0.2, jump intensity equal to 5 and jump parameters $\mu = -0.1$ and $\delta = 0.1$ for the time horizon $T = 0.5$. The convergence of $\tilde{\zeta}_T$ to zero is clearly very fast (faster than exponential) for all values of σ and it is particularly good for $\sigma = 0.3575$, the value of σ for which $\tilde{\zeta}(0) = 0$.

Method of Lewis [262] We present this method from a different angle than the previous one, in order to show how arbitrary payoff structures (and not just vanilla calls) can be priced. Since for an arbitrary payoff the notion of strike is not defined, we will show how to price options for a range of different initial values of the underlying. Let $s = \ln S_0$ denote the logarithm of current stock value and f be the payoff function of the option. The price of this option

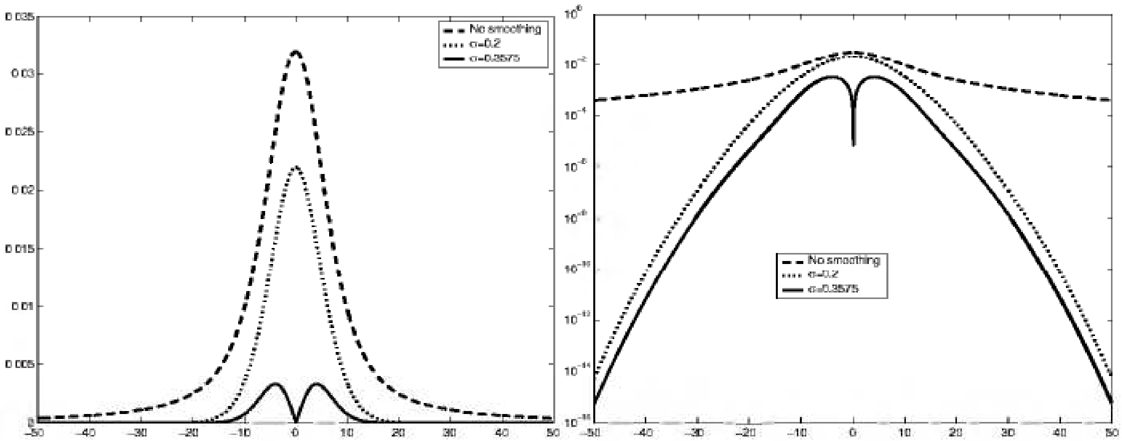

FIGURE 11.4: Convergence of Fourier transform of option's time value to zero in Merton model — see page 363. Left graph: linear scale; right graph: logarithmic scale.

is

$$C(s) = e^{-rT} \int_{-\infty}^{\infty} f(e^{s+x+rT}) \rho_T(x) dx.$$

In this method, instead of subtracting something from the call price to obtain an integrable function, one computes the Fourier transform for complex values of the argument: the Fourier transform is defined, as usual, by

$$\mathbf{F}g(z) = \int_{-\infty}^{\infty} e^{iuz} g(u) du$$

but z may now be a complex number.

For $a, b \in \mathbb{R}$ we say that $g(u)$ is Fourier integrable in a strip (a, b) if $\int_{-\infty}^{\infty} e^{-au}|g(u)|du < \infty$ and $\int_{-\infty}^{\infty} e^{-bu}|g(u)|du < \infty$. In this case $\mathbf{F}g(z)$ exists and is analytic for all z such that $a < \Im z < b$. Moreover, within this strip the generalized Fourier transform may be inverted by integrating along a straight line parallel to the real axis (see [262]):

$$g(x) = \frac{1}{2\pi} \int_{iw-\infty}^{iw+\infty} e^{-izx} \mathbf{F}g(z) dz \qquad (11.22)$$

with $a < w < b$. To proceed, we need an additional hypothesis ($\bar{S}$ denotes the complex conjugate set of S):

(H2) $\rho_T(x)$ is Fourier integrable in some strip S_1,
$f^*(x) \equiv f(e^{x+rT})$ is Fourier integrable in some strip S_2 and the intersection of $\bar{S}_1$ with S_2 is nonempty: $S = \bar{S}_1 \cap S_2 \neq \emptyset$.

Using this hypothesis we can now compute the generalized Fourier transform of $C(s)$ by interchanging integrals: for every $z \in S$

$$\int_{-\infty}^{\infty} e^{izs} C(s) ds = e^{-rT} \int_{-\infty}^{\infty} \rho_T(x) dx \int_{-\infty}^{\infty} e^{izs} f(e^{s+x+rT}) ds$$
$$= e^{-rT} \int_{-\infty}^{\infty} e^{-izx} \rho_T(x) dx \int_{-\infty}^{\infty} f(e^{y+rT}) e^{izy} dy.$$

Finally we obtain

$$\mathbf{F}C(z) = e^{-rT} \Phi_T(-z) \mathbf{F} f^*(z) \quad \forall z \in S.$$

Option prices can now be computed from (11.22) for a range of initial values using the FFT algorithm (see below).

Application to call options The payoff function of a European call option is Fourier integrable in the region $\Im z > 1$, where its generalized Fourier transform can be computed explicitly:

$$\mathbf{F} f^*(z) = \int_{-\infty}^{\infty} e^{iyz} (e^{y+rT} - e^k)^+ dy = \frac{e^{k+iz(k-rT)}}{iz(iz+1)}.$$

The hypothesis (H2) now requires that $\rho_T(x)$ be integrable in a strip (a, b) with $a < -1$. Since $\rho_T(x)$ is a probability density, 0 belongs to its strip of integrability which means that $b \geq 0$. Therefore, in this setting the hypothesis (H2) is equivalent to (H1).

Finally, the generalized Fourier transform of call option price takes the form

$$\mathbf{F}C(z) = \frac{\Phi_T(-z) e^{(1+iz)(k-rT)}}{iz(iz+1)}, \quad 1 < \Im z < 1 + \alpha.$$

The option price can be computed using the inversion formula (11.22), which simplifies to

$$C(x) = \frac{\exp(wx + (1-w)(k-rT))}{2\pi} \int_{-\infty}^{\infty} \frac{e^{iu(k-rT-x)} \Phi_T(-iw-u) du}{(iu-w)(1+iu-w)}$$

for some $w \in (1, 1+\alpha)$. The integral in this formula is much easier to approximate at infinity than the one in (11.20) because the integrand decays exponentially (due to the presence of characteristic function). However, the price to pay for this is having to choose w. This choice is a delicate issue because choosing big w leads to slower decay rates at infinity and bigger truncation errors and when w is close to one, the denominator diverges and the discretization error becomes large. For models with exponentially decaying tails of Lévy measure, w cannot be chosen a priori and must be adjusted depending on the model parameters.

Computing Fourier transforms In order to implement the algorithms above, one needs to numerically compute Fourier transform in an efficient manner. This can be done using the discrete Fourier transform

$$F_n = \sum_{k=0}^{N-1} f_k e^{-2\pi i n k/N}, \quad n = 0 \ldots N-1.$$

To compute $F_0, \ldots, F_{N-1}$, one needs a priori N^2 operations, but when N is a power of 2, an algorithm due to Cooley and Tukey and known as the *fast Fourier transform* (FFT) reduces the computational complexity to $O(N \ln N)$ operations, see [321]. Subroutines implementing the FFT algorithm are available in most high-level scientific computation environments. A C language library called FFTW can be downloaded (under GPL license) from www.fftw.org.

Suppose that we would like to approximate the inverse Fourier transform of a function $f(x)$ with a discrete Fourier transform. The integral must then be truncated and discretized as follows:

$$\int_{-\infty}^{\infty} e^{-iux} f(x)dx \approx \int_{-A/2}^{A/2} e^{-iux} f(x)dx \approx \frac{A}{N} \sum_{k=0}^{N-1} w_k f(x_k) e^{-iux_k},$$

where $x_k = -A/2 + k\Delta$, $\Delta = A/(N-1)$ is the discretization step and w_k are weights corresponding to the chosen integration rule (for instance, for the trapezoidal rule $w_0 = w_{N-1} = 1/2$ and all other weights are equal to 1). Now, setting $u_n = \frac{2\pi n}{N\Delta}$ we see that the sum in the last term becomes a discrete Fourier transform:

$$\mathbf{F}f(u_n) \approx \frac{A}{N} e^{iuA/2} \sum_{k=0}^{N-1} w_k f(x_k) e^{-2\pi i n k/N}$$

Therefore, the FFT algorithm allows to compute $\mathbf{F}f(u)$ at the points $u_n = \frac{2\pi n}{N\Delta}$. Notice that the grid step d in the Fourier space is related to the initial grid step Δ:

$$d\Delta = \frac{2\pi}{N}$$

This means that if we want to compute option prices on a fine grid of strikes, and at the same time keep the discretization error low, we must use a large number of points. Another limitation of the FFT method is that the grid must always be uniform and the grid size a power of 2. The functions that one has to integrate are typically irregular at the money and smooth elsewhere but increasing the resolution close to the money without doing so in other regions is not possible. These remarks show that the use of FFT is only justified when one needs to price a large number of options with the same maturity (let us say, more than 10) — for pricing a single option adaptive variable-step integration algorithms perform much better.

When one only needs to price a single option, Boyarchenko and Levendorskiĭ [71] suggest to transform the contour of integration in the complex plane (using Cauchy theorem) to achieve a better convergence. Their method, called "integration along cut" is applicable to tempered stable model with $\alpha \leq 1$ and in many other cases and performs especially well for options close to maturity. However, when one only needs to price a single option, the speed is not really an issue since on modern computers *all methods* can do this computation quickly. Speed becomes an issue when one repeatedly needs to price a large number of options (as in calibration or in scenario simulation for portfolios). If one is interested in pricing several options with the same maturity, FFT is hard to beat.

11.2 Forward start options

A forward start (or delayed) option is an option with some contractual feature, typically the strike price, that will be determined at some future date before expiration, called the fixing date. A typical example of forward start option is a stock option. When an employee begins to work, the company may promise that he or she will receive call options on the company's stock at some future date. The strike of these options will be such that the options are at the money or, for instance, 10% out of the money at the fixing date of the strike. Due to the homogeneous nature of exponential-Lévy models, forward start options are easy to price in this framework.

The simplest type of a forward start option is a forward start call, starting at T_1 and expiring at T_2, with strike equal to a fixed proportion m of the stock price at T_1 (often $m = 1$: the option is at the money on the date of issue). The payoff of such an option at T_2 is therefore

$$H = (S_{T_2} - mS_{T_1})^+. \tag{11.23}$$

At date T_1 the value of this option is equal to that of a European call with maturity T_2 and strike mS_{T_1}. Denoting the price of a forward start option by P_t we obtain

$$P_{T_1} = S_{T_1} e^{-r(T_2-T_1)} E\{(e^{r(T_2-T_1)+X_{T_2-T_1}} - m)^+\}.$$

Notice that the expectation is not conditional: the result is deterministic. Therefore the value of option at T_1 is equal to S_{T_1} times a constant. This option is then simply equivalent to a certain number of forward contracts and its price at t is given by

$$\begin{aligned} P_t &= e^{-r(T_1-t)} E\{S_{T_1}|\mathcal{F}_t\} e^{-r(T_2-T_1)} E\{(e^{r(T_2-T_1)+X_{T_2-T_1}} - m)^+\} \\ &= S_t e^{-r(T_2-T_1)} E\{(e^{r(T_2-T_1)+X_{T_2-T_1}} - m)^+\}. \end{aligned} \tag{11.24}$$