Pricing and Hedging of Portfolio Credit Derivatives with Interacting Default Intensities

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Abstract

We consider reduced-form models for portfolio credit risk with interacting default intensities. In this class of models default intensities are modelled as functions of time and of the default state of the entire portfolio, so that phenomena such as default contagion or counterparty risk can be modelled explicitly. In the present paper this class of models is analyzed by Markov process techniques. We study in detail the pricing and the hedging of portfolio-related credit derivatives such as basket default swaps and collaterized debt obligations (CDOs) and discuss the calibration to market data.

Keywords: Credit derivatives, CDOs, Hedging, Markov chains.

1 Introduction

With rapidly growing markets for portfolio credit derivatives such as collaterized debt obligations (CDOs) the development of suitable models for pricing these products has become an issue of high concern. The key element in any such model is the mechanism generating the dependence between default times. Here three major approaches falling within the broad category of reduced-form models can be distinguished: models with dependent default intensities but conditionally independent default times such as Duffie & Garleanu (2001); factor copula models such as Li (2001), Laurent & Gregory (2005), Hull & White (2004); models with direct interaction between default intensities such as Jarrow & Yu (2001), Davis & Lo (2001), Yu (2007), Bielecki & Vidozzi (2006) or the present paper. A detailed description of these model classes is given in McNeil, Frey & Embrechts (2005), Chapter 9.

At present factor-copula models are the market standard for pricing portfolio credit derivatives. In a nutshell, in these models one starts from assumptions on the risk-neutral distribution

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†A number of ideas in this paper go back to our earlier working paper Frey and Backhaus (2004); however, the focus of the present paper has shifted and the paper has been revised and extended substantially.

function of the default times under consideration. This distribution function can be decomposed into marginal distributions and copula (dependence structure) of the default times. The marginal distributions are determined by calibrating the model to defaultable-term-structure data, the copula is specified by the modeller. Usually the copula has a factor structure as in the case of the popular one-factor Gauss copula model, hence the name factor-copula models. The separation into marginal distribution and dependence structure facilitates the calibration of the model; this is the main reason for the popularity of this model class. Factor copula models are usually presented and used in a *static* fashion, i.e. with a focus on the distribution function of the default times. This makes it hard to to derive model-based hedging strategies and to gain intuition for dynamic aspects of the model.

In models with interacting intensities on the other hand one adopts the standard modelling practice in mathematical finance: start from assumptions on the dynamics of asset prices and state variables and derive distributional properties. Hence in this class of models default intensities are taken as model primitives; in particular, the modeller specifies explicitly the impact of the default of one firm on the default intensities of surviving firms. This approach allows for a very intuitive parameterization of default contagion and default dependence in general. Moreover, the dynamic formulation permits the derivation of model-based hedging strategies. On the downside, the calibration of the model to defaultable-term-structure data can be more involved, as marginal distributions are typically not available in closed form.

In the present paper we study the pricing and the hedging of credit derivatives in models with interacting intensities. In Section 2 we give a rigorous construction of the model as finite-state Markov chain on the set of all default configurations. Particular emphasis is put on the case where the portfolio consists of several homogenous groups. In Section 3 we study the pricing of basket default swaps and CDOs. In particular, we show that appropriately parameterized versions of our model are capable to explain the so-called implied correlation skew of synthetic CDO tranches in an intuitive way which is directly linked to the dynamics of the model. In Section 4 we derive dynamic hedging strategies for (basket) credit derivatives and analyze the impact of default contagion on the form of the hedging strategies.

2 The Model

2.1 General Setup

Notation. We consider a portfolio of m firms, indexed by $i \in \{1, \dots, m\}$. The evolution of the default state of the portfolio is described by a default indicator process $\mathbf{Y} = \big(Y_{t,1}, \dots, Y_{t,m}\big)_{t \geq 0}$, defined on some probability space $(\Omega, \mathcal{F}, P)$. We set $Y_{t,i} = 1$ if firm i has defaulted by time t and $Y_{t,i} = 0$ else, so that $\mathbf{Y}_t \in S^{\mathbf{Y}} := \{0,1\}^m$. The corresponding default times are denoted by $\tau_i := \inf\{t \geq 0\colon Y_{t,i} = 1\}$. Since we consider only models without simultaneous defaults, we can define the *ordered default times* $T_0 < T_1 < \ldots < T_m$ recursively by $T_0 = 0$ and $T_n = \min\{\tau_i : 1 \leq i \leq m, \tau_i > T_{n-1}\}$, $1 \leq n \leq m$. By $\xi_n \in \{1, \dots, m\}$ we denote the identity of the firm defaulting at time T_n, i.e. $\xi_n = i$ if $T_n = \tau_i$. The internal filtration of the process $\mathbf{Y}$ (the default history) is denoted by $(\mathcal{H}_t)$, i.e. $\mathcal{H}_t = \sigma(\mathbf{Y}_s : s \leq t)$. We use the following notation for flipping the ith coordinate of a default state: given $\boldsymbol{y} \in S^{\mathbf{Y}}$ we define $\boldsymbol{y}^i \in S^{\mathbf{Y}}$ by

$$y_i^i := 1 - y_i \text{ and } y_j^i := y_j,\ j \in \{1, \dots, m\} \setminus \{i\}\,. \tag{1}$$

Dynamics of Y. We assume that the default intensity of a non-defaulted firm i at time t is given by a function $\lambda_i(t, \mathbf{Y}_t)$ of time and of the current default state $\mathbf{Y}_t$.[1] Hence the default intensity of a firm may change if there is a change in the default state of other firms in the portfolio; in this way default contagion and counterparty risk can be modelled explicitly. We model the default indicator process by a time-inhomogeneous Markov chain with state space $S^{\mathbf{Y}}$. The next assumption summarizes the mathematical properties of $\mathbf{Y}$.

Assumption 2.1 (Markov family). Consider bounded and measurable functions $\lambda_i : [0,\infty) \times S^{\mathbf{Y}} \to \mathbb{R}_+$, $1 \le i \le m$. There is a family $P_{(t,\boldsymbol{y})}$, $(t, \boldsymbol{y}) \in [0,\infty) \times S$, of probability measures on $(\Omega, \mathcal{F}, (\mathcal{H}_t))$ such that $P_{(t,\boldsymbol{y})}(\mathbf{Y}_t = \boldsymbol{y}) = 1$ and such that $(\mathbf{Y}_s)_{s \ge t}$ is a finite-state Markov chain with state space $S^{\mathbf{Y}}$ and transition rates $\lambda(s, \boldsymbol{y}_1, \boldsymbol{y}_2)$ given by

$$\lambda(s, \boldsymbol{y}_1, \boldsymbol{y}_2) = \begin{cases} (1 - y_{1,i})\, \lambda_i(s, \boldsymbol{y}_1), & \text{if } \boldsymbol{y}_2 = \boldsymbol{y}_1^i \text{ for some } i \in \{1 \dots, m\}, \\ 0 & \text{else.} \end{cases} \tag{2}$$

Unless explicitly stated otherwise, $P = P_{(0,\mathbf{0})}$, i.e. we consider the chain $\mathbf{Y}$ starting at time 0 in the state $\mathbf{0} \in S^{\mathbf{Y}}$. Relation (2) has the following interpretation: In t the chain can jump only to the set of neighbors of the current state $\mathbf{Y}_t$ that differ from $\mathbf{Y}_t$ by exactly one default; in particular there are no joint defaults. The probability that firm i defaults in the small time interval $[t, t+h)$ thus corresponds to the probability that the chain jumps to the neighboring state $(\mathbf{Y}_t)^i$ in this time period. Since such a transition occurs with rate $\lambda_i(t, \mathbf{Y}_t)$, it should be intuitively obvious that under Assumption 2.1 $\lambda_i(t, \mathbf{Y}_t)$ is the default intensity of firm i at time t; a formal argument is given below. The *generator* of $\mathbf{Y}$ at time t equals

$$G_{[t]} f\,(\boldsymbol{y}) = \sum_{i=1}^{m} (1 - y_i) \lambda_i(t, \boldsymbol{y}) \big(f(\boldsymbol{y}^i) - f(\boldsymbol{y}) \big),\ \boldsymbol{y} \in S^{\mathbf{Y}}. \tag{3}$$

It is well-known that for any $f : S^{\mathbf{Y}} \to \mathbb{R}$ the process $f(\mathbf{Y}_t) - \int_0^t G_{[s]} f\,(\mathbf{Y}_s) ds$, $t \ge 0$, is a martingale. Let in particular $f_i(\boldsymbol{y}) := y_i$ and observe that $G_{[t]} f_i\,(\boldsymbol{y}) = (1 - y_i) \lambda_i(t, \boldsymbol{y})$. It follows that $Y_{t,i} - \int_0^{t \wedge \tau_i} \lambda_i(s, \mathbf{Y}_s) ds$ is a martingale, establishing formally that $\lambda_i(t, \mathbf{Y}_t)$ is the default intensity of firm i. The *transition probabilities* of the chain $\mathbf{Y}$ are given by

$$p(t, s, \boldsymbol{y}_1, \boldsymbol{y}_2) := P_{(t,\boldsymbol{y}_1)}(\mathbf{Y}_s = \boldsymbol{y}_2) = P(\mathbf{Y}_s = \boldsymbol{y}_2 \mid \mathbf{Y}_t = \boldsymbol{y}_1), \quad 0 \le t \le s < \infty. \tag{4}$$

The *numerical treatment* of the model can be based on Monte Carlo simulation or on the Kolmogorov forward and backward equation for the transition probabilities; see Appendix A for further information.

Note that the model outlined in Assumption 2.1 is similar to interacting particle systems studied in statistical mechanics. In particular the flip rate (default intensity) of a particle (firm) depends on the current configuration (default-state) $\mathbf{Y}_t$ of the system. The link between portfolio credit risk and interacting particle systems is explored further in (Frey 2003) (Giesecke & Weber 2006) or (Horst 2006), among others.

[1] It is possible to extend the model to stochastic default intensities of the form $\lambda_i(\Psi_t, \mathbf{Y}_t)$ where Ψ represents some economic factor process; see Frey & Backhaus (2004) for details.

Conditional expectations and densities. In the next proposition we derive analytical expressions for the density of the default time τ_{i_0} of a given firm and compute certain related conditional expectations. These will come in handy in the pricing of (basket) default swaps later in the paper.

Proposition 2.2. *For $i_0 \in \{1, \dots, m\}$, $0 \le s \le t$ and $\bar{\boldsymbol{y}} \in S^{\mathbf{Y}}$ with $\bar{y}_{i_0} = 0$, the density of τ_{i_0} with respect to $P_{(s,\bar{\boldsymbol{y}})}$ equals*

$$P_{(s,\bar{\boldsymbol{y}})}(\tau_{i_0} \in dt) = \sum_{\{\boldsymbol{y} \in S^{\mathbf{Y}}:\ y_{i_0}=0\}} \lambda_{i_0}(t, \boldsymbol{y})\, p(s, t, \bar{\boldsymbol{y}}, \boldsymbol{y})\,. \tag{5}$$

Moreover, we have for $\boldsymbol{y} \in S^{\mathbf{Y}}$

$$P_{(s,\bar{\boldsymbol{y}})}\big(\mathbf{Y}_t = \boldsymbol{y} \mid \tau_{i_0} = t\big) = y_{i_0} P_{(s,\bar{\boldsymbol{y}})}(\tau_{i_0} \in dt)^{-1} \lambda_{i_0}(t, \boldsymbol{y}^{i_0})\, p(s, t, \bar{\boldsymbol{y}}, \boldsymbol{y}^{i_0}). \tag{6}$$

Proof. It suffices to consider the case $(s, \bar{\boldsymbol{y}}) = (0, \mathbf{0})$. We first show that for $\boldsymbol{y} \in S^{\mathbf{Y}}$ with $y_{i_0} = 1$ we have

$$\lim_{\epsilon \to 0+} \frac{1}{\epsilon}\, P\big(\mathbf{Y}_t = \boldsymbol{y},\ \tau_{i_0} \in (t - \epsilon, t]\big) = \lambda_{i_0}(t, \boldsymbol{y}^{i_0}) P\big(\mathbf{Y}_t = \boldsymbol{y}^{i_0}\big). \tag{7}$$

To verify (7) we argue as follows. The probability to have more than one default in $(t - \epsilon, t]$ is of order $o(\epsilon)$. Thus we have $P\big(\mathbf{Y}_t = \boldsymbol{y},\ \tau_{i_0} \in (t - \epsilon, t]\big) = P\big(\mathbf{Y}_{t-\epsilon} = \boldsymbol{y}^{i_0},\ \tau_{i_0} \in (t - \epsilon, t]\big) + o(\epsilon)$. Now we get, using the Markov property of $\mathbf{Y}$,

$$\begin{aligned} P\left(\mathbf{Y}_{t-\epsilon} = \boldsymbol{y}^{i_0}, \tau_{i_0} \in (t - \epsilon, t]\right) &= E\Big(E\big(\mathbf{1}_{\{\mathbf{Y}_{t-\epsilon}=\boldsymbol{y}^{i_0}\}} \mathbf{1}_{\{\tau_{i_0} \in (t-\epsilon,t]\}} \mid \mathcal{H}_{t-\epsilon}\big)\Big) \\ &= E\Big(\mathbf{1}_{\{\mathbf{Y}_{t-\epsilon}=\boldsymbol{y}^{i_0}\}} \mathbf{1}_{\{\tau_{i_0} > t-\epsilon\}} P_{(t-\epsilon, \mathbf{Y}_{t-\epsilon})}(\tau_{i_0} \le \epsilon)\Big). \end{aligned} \tag{8}$$

Moreover, $P_{(t-\epsilon,\boldsymbol{y}^{i_0})}(\tau_{i_0} \le \epsilon) = \epsilon \lambda_{i_0}(t - \epsilon, \boldsymbol{y}^{i_0}) + o(\epsilon)$, and $\tau_{i_0} > t - \epsilon$ on $\{\mathbf{Y}_{t-\epsilon} = \boldsymbol{y}^{i_0}\}$. Hence (8) equals $\epsilon E\Big(\mathbf{1}_{\{\mathbf{Y}_{t-\epsilon}=\boldsymbol{y}^{i_0}\}} \lambda_{i_0}(t - \epsilon, \boldsymbol{y}^{i_0})\Big) + o(\epsilon)$, and (7) follows.

The proof of the proposition is now straightforward. Relation (5) follows from (7) and the fact that $P\big(\tau_{i_0} \in (t-\epsilon, t]\big) = \sum_{\{\boldsymbol{y} \in S^{\mathbf{Y}}:\ y_{i_0}=1\}} P\big(\mathbf{Y}_t = \boldsymbol{y},\ \tau_{i_0} \in (t-\epsilon, t]\big)$; relation (6) follows from (7), the definition of the elementary conditional expectation and a standard limit argument. □

2.2 Models with Homogeneous-Group Structure

If the portfolio size m is large it is natural to assume that the portfolio has a homogeneous-group structure. This assumption gives rise to intuitive parameterizations for the default intensities; moreover, the homogeneous-group structure leads to a substantial reduction in the size of the state space of the model.

Homogeneous-group structure. Assume that we can divide our portfolio of m firms into k groups (typically $k \ll m$) within which risks are exchangeable. A group might correspond to firms with identical credit rating or to firms from the same industrial sector. Let $\kappa(i) \in \{1, \dots, k\}$ give the group membership of firm i and denote by $m_\kappa := \sum_{i=1}^m 1_{\{\kappa(i)=\kappa\}}$ the number of firms in group κ. Define functions M, $M_\kappa : S^{\mathbf{Y}} \to \{1, \dots, m\}$ by $M(\boldsymbol{y}) := \sum_{i=1}^m y_i$ respectively $M_\kappa(\boldsymbol{y}) := \sum_{i=1}^m 1_{\{\kappa(i)=\kappa\}} y_i$, and put $M_t := M(\mathbf{Y}_t)$ respectively $M_{t,\kappa} := M_\kappa(\mathbf{Y}_t)$, so that M_t and $M_{t,\kappa}$ give the number of firms which have defaulted by time t in the entire portfolio respectively in group κ. Define the vector process $\boldsymbol{M} = (M_{t,1}, \dots, M_{t,k})_{t \ge 0}$. Note that $\boldsymbol{M}$ takes values in the set $S^{\boldsymbol{M}} := \big\{\boldsymbol{l} = (l_1, \dots, l_k) : l_\kappa \in \{0, \dots, m_\kappa\}, 1 \le \kappa \le k\big\}\,.$

Assumption 2.3 (Homogeneous-group structure). The default intensities of firms belonging to the same group are identical and of the form $\lambda_i\left(t, \mathbf{Y}_t\right) = h_{\kappa(i)}\big(t, \boldsymbol{M}_t\big)$ for bounded and measurable functions $h_\kappa : [0, \infty) \times S^{\boldsymbol{M}} \to [0, \infty)$, $1 \leq \kappa \leq k$.

Assumption 2.3 implies that for all κ the default indicator processes $\{(Y_{t,i})_{t \geq 0} : 1 \leq i \leq m, \, \kappa(i) = \kappa\}$ of firms belonging to the same group are *exchangeable*. Conversely, consider an arbitrary portfolio of m firms with default indicators satisfying Assumption 2.1, and suppose that the portfolio can be split in $k < m$ homogeneous groups. Homogeneity implies that a) the default intensities are invariant under permutations π of $\{1, \ldots, m\}$ leaving the group structure invariant ($\lambda_i(t, \boldsymbol{y}) = \lambda_i(t, \pi(\boldsymbol{y}))$ for all i and all permutations π with $\kappa(\pi(j)) = \kappa(j)$ for all j), and b) that default intensities of different firms from the same group are identical. Condition a) immediately yields that $\lambda_i(t, \boldsymbol{y}) = h_i(t, M_1(\boldsymbol{y}), \ldots, M_k(\boldsymbol{y}))$ for functions $h_i : [0, \infty) \times S^{\boldsymbol{M}} \to [0, \infty)$; together with b) this implies that the default intensities necessarily satisfy Assumption 2.3.

Examples. For later use we introduce several parameterizations for default intensities satisfying Assumption 2.3. We begin with exchangeable models where the entire portfolio forms a single homogeneous group; these form a very useful benchmark case. As just discussed, in exchangeable models default intensities are necessarily of the form $\lambda_i(t, \mathbf{Y}_t) = h(t, M_t)$ for some $h : [0, \infty) \times \{0, \ldots, m\} \to \mathbb{R}_+$, and the process M is a standard pure death process. Note that the assumption that default intensities depend on $\mathbf{Y}_t$ via the overall number of defaulted firms $M(\mathbf{Y}_t)$ makes sense also from an economic viewpoint, as unusually many defaults might have a negative impact on the liquidity of credit markets or on the business climate in general.

The simplest exchangeable model is the *linear counterparty-risk* model. Here

$$h(t, l) = \lambda_0 + \lambda_1 l \,, \quad \lambda_0 > 0, \lambda_1 \geq 0. \tag{9}$$

The interpretation of (9) is straightforward: upon default of some firm the default intensity of the surviving firms increases by the constant amount λ_1 so that default dependence increases with λ_1; for $\lambda_1 = 0$ defaults are independent. Model (9) is the homogeneous version of the so-called looping-defaults model of Jarrow & Yu (2001).

The next model generalizes the linear counterparty-risk model in two important ways: first, we introduce time-dependence and assume that a default event at time t increases the default intensity of surviving firms only if M_t exceeds some deterministic threshold $\mu(t)$ measuring the expected number of defaulted firms up to time t; second, we assume that on $\{l > \mu(t)\}$ the function $h(t, \cdot)$ is strictly *convex*. Convexity of h implies that large values of M_t lead to very high values of default intensities, thus triggering a cascade of further defaults. This will be important in explaining properties of observed CDO prices in Section 3.3. The following specific model with the above features will be particularly useful:

$$h(t, l) = \lambda_0 + \frac{\lambda_1}{\lambda_2} \left(\exp\left(\lambda_2 \frac{(l - \mu(t))^+}{m} \right) - 1 \right) , \quad \lambda_0 > 0, \lambda_1 \geq 0, \lambda_2 \geq 0 \,; \tag{10}$$

in the sequel we call (10) *convex counterparty-risk* model. In (10) λ_0 is a level parameter that mainly influences credit quality. λ_1 gives the slope of $h(t, l)$ at $\mu(t)$; intuitively this parameter models the strength of default interaction for "normal" realisations of M_t. The parameter λ_2

controls the degree of convexity of h and hence the tendency of the model to generate default cascades; note that for $\lambda_2 \to 0$ (and $\mu(t) \equiv 0$) (10) reduces to the linear model (9).

In order to illustrate the modelling possibilities under Assumption 2.3, we finally introduce a model which might be suitable for firms from k different industry groups. It is well-known that defaults of firms from the same industry exhibit stronger dependence than defaults of firms from different industries. To mimic this effect the default intensity of firms from group κ is modelled by the function

$$h_\kappa(t, \boldsymbol{l}) = \lambda_{\kappa,0} + \frac{\lambda_1}{\lambda_2}\Big\{ \exp\Big(\lambda_2 \gamma \big(l_\kappa - \mu_\kappa(t)\big)^+ + \lambda_2(1-\gamma)\Big(\sum_{i=1}^k l_i - \mu(t)\Big)^+\Big) - 1\Big\}, \tag{11}$$

with parameters $\lambda_{\kappa,0} > 0$, $\lambda_1 \geq 0$, $\lambda_2 \geq 0$ and $\gamma \in [0,1]$. The first term in the argument of the exponential function in (11) reflects the interaction between firms from the same industry group; the second term captures the global interaction between defaults in the entire portfolio. The relative strength of these effects is governed by the parameter γ.

Now we turn to certain mathematical properties of the homogeneous-group model.

Lemma 2.4 (Markov property of $\boldsymbol{M}$). *Assume that the default intensities satisfy Assumption 2.3. Then under $P_{(t,\boldsymbol{y})}$ the process $(\boldsymbol{M}_s)_{s\geq t}$ follows a time-inhomogeneous Markov chain with state space $S^{\boldsymbol{M}}$, initial value $\boldsymbol{M}_t = (M_1(\boldsymbol{y}), \ldots, M_k(\boldsymbol{y}))$, and generator*

$$G^M_{[s]} f(\boldsymbol{l}) = \sum_{\kappa=1}^k (m_\kappa - l_\kappa) h_\kappa(s, \boldsymbol{l})\big(f(\boldsymbol{l} + \boldsymbol{e}_\kappa) - f(\boldsymbol{l})\big), \ \boldsymbol{e}_\kappa \text{ the } \kappa\text{th unit vector in } \mathbb{R}^k. \tag{12}$$

Proof. Suppose that $\boldsymbol{M}_s = (l_1, \ldots, l_k)$. Since there are no joint defaults, $\boldsymbol{M}_s$ can only jump to the points $\{\boldsymbol{M}_s + \boldsymbol{e}_\kappa \colon 1 \leq \kappa \leq k,\ M_{s,\kappa} < m_\kappa\}$. Now $\boldsymbol{M}_s$ jumps to $\boldsymbol{M}_s + \boldsymbol{e}_\kappa$ if and only if the next defaulting firm belongs to group κ. Hence the transition rate from $\boldsymbol{M}_s$ to $\boldsymbol{M}_s + \boldsymbol{e}_\kappa$ equals

$$\sum_{i=1}^m 1_{\{\kappa(i)=\kappa\}}\,(1 - Y_{s,i})\ \lambda_i(s, \mathbf{Y}_s) = h_\kappa(s, \boldsymbol{M}_s)\big(m_\kappa - M_{s,\kappa}\big).$$

The Markovianity of $\boldsymbol{M}$ follows, as this transition-rate depends on $\mathbf{Y}_s$ only via $\boldsymbol{M}_s$. The form of the generator $G^{\boldsymbol{M}}_{[s]}$ is obvious. □

The law of the chain $(\boldsymbol{M}_s)_{s\geq t}$ starting at time t in the point $\boldsymbol{l} \in S^{\boldsymbol{M}}$ will be denoted by $P_{(t,\boldsymbol{l})}$. Lemma 2.4 is useful for the numerical analysis of the model: note that the cardinality of the state space of $\boldsymbol{M}$ is $|S^{\boldsymbol{M}}| = (m_1+1)\cdots(m_k+1)$, so that for k fixed $|S^{\boldsymbol{M}}|$ grows in m at most at rate $(m/k)^k$ (polynomial growth) whereas $\big|S^{\mathbf{Y}}\big| = 2^m$ grows exponentially. Hence for k small the distribution of $\boldsymbol{M}_t$ can be inferred via the Kolmogorov equations for $\boldsymbol{M}$ even for m relatively large. Moreover, under Assumption 2.3 the random variables $\{Y_{T,i} : \kappa(i) = \kappa\}$ are exchangeable, so that the probability function of $\mathbf{Y}_T$ can be expressed in terms of the probability function of $\boldsymbol{M}_T$. Consider $\boldsymbol{y} \in S^{\mathbf{Y}}$ and put $\boldsymbol{l} := (M_1(\boldsymbol{y}), \ldots M_k(\boldsymbol{y}))$. We have

$$P(\mathbf{Y}_T = \boldsymbol{y}) = \frac{P(\boldsymbol{M}_T = \boldsymbol{l})}{|\{\boldsymbol{x} \in S^{\mathbf{Y}} \colon M_1(\boldsymbol{x}) = l_1, \ldots, M_k(\boldsymbol{x}) = l_k\}|} = \frac{P(\boldsymbol{M}_T = \boldsymbol{l})}{\prod_{\kappa=1}^k \binom{m_\kappa}{l_\kappa}}. \tag{13}$$

Default probabilities can therefore be inferred from the probability function of $\boldsymbol{M}_T$. We get $P\left(Y_{T,i} = 1 \mid M_{T,\kappa(i)}\right) = M_{T,\kappa(i)}/m_{\kappa(i)}$, and hence

$$P\left(Y_{T,i} = 1\right) = E\left(P(Y_{T,i} = 1 \mid M_{T,\kappa(i)})\right) = m^{-1}_{\kappa(i)} E(M_{T,\kappa(i)}).$$

In the homogeneous-group case Proposition 2.2 can be refined as well:

Corollary 2.5. *Consider a model that satisfies Assumption 2.3. We have for* $1 \le i \le m$

$$P(\tau_i \in dt) = \sum_{\{\boldsymbol{l}: l_{\kappa(i)}>0\}} \frac{m_{\kappa(i)} - l_{\kappa(i)} + 1}{m_{\kappa(i)}} \, h_{\kappa(i)}(t, \boldsymbol{l} - \boldsymbol{e}_{\kappa(i)}) \, P(\boldsymbol{M}_t = \boldsymbol{l} - \boldsymbol{e}_{\kappa(i)}) \tag{14}$$

and for $\bar{\boldsymbol{l}} \in S^{\boldsymbol{M}}$ *fixed,*

$$P\big(\boldsymbol{M}_t = \bar{\boldsymbol{l}} \mid \tau_i = t\big) = \frac{(m_{\kappa(i)} - \bar{l}_{\kappa(i)} + 1)\, h_{\kappa(i)}(t, \bar{\boldsymbol{l}} - \boldsymbol{e}_{\kappa(i)}) \, P(\boldsymbol{M}_t = \bar{\boldsymbol{l}} - \boldsymbol{e}_{\kappa(i)})}{\sum_{\{\boldsymbol{l}: l_{\kappa(i)}>0\}} \big\{ (m_{\kappa(i)} - l_{\kappa(i)} + 1)\, h_{\kappa(i)}(t, \boldsymbol{l} - \boldsymbol{e}_{\kappa(i)}) \, P(\boldsymbol{M}_t = \boldsymbol{l} - \boldsymbol{e}_{\kappa(i)}) \big\}} \, . \tag{15}$$

The result follows from Proposition 2.2 via combinatoric arguments; we omit the details.

3 Pricing of Credit Derivatives

3.1 Our Setup

We consider a fixed portfolio of m firms with default indicator process $\mathbf{Y}$. The following assumptions on the structure of the market will be in force in the remainder of the paper.

Assumption 3.1 (Market structure).

(i) The investor-information at time t is given by the default history $\mathcal{H}_t = \sigma(\mathbf{Y}_s : s \le t)$.

(ii) The default-free interest rate is deterministic and equal to $r(t) \ge 0$; $p_0(t,T) = e^{-\int_t^T r(s)ds}$ denotes the default-free zero-coupon bond with maturity $T \ge t$.

(iii) (Martingale modelling.) There is a risk neutral pricing measure, denoted P, such that the price in t of any $\mathcal{H}_T$-measurable claim H is given by $H_t := p_0(t,T)E(H \mid \mathcal{H}_t)$. Moreover, $\mathbf{Y}$ satisfies Assumption 2.1 under P.

Comments. (i) The choice of $(\mathcal{H}_t)$ as underlying filtration is natural in view of the structure of our model (see Assumption 2.1); it is moreover in line with the literature on the dynamic structure of factor copula models such as Schönbucher (2004).
(ii) The assumption of deterministic interest rates is routinely made in the literature on portfolio credit derivatives, as the additional complexity of stochastic interest rates is not warranted given the huge degree of uncertainty in the modelling of the dependence structure of default times.
(iii) Martingale modelling is standard practice in the literature, because credit derivatives are usually priced relative to traded credit products such as corporate bonds or single-name credit default swaps (CDSs). Note however that from a methodological point of view, the choice of P as pricing measure is unambiguously justified only if there are enough traded credit products so that the market for credit derivatives can be considered complete. We come back to this issue in Section 4.

Single-name credit derivatives. Throughout, we will calibrate the model to observed prices of single-name credit derivatives (defaultable bonds or single-name CDSs). In the spirit of Lando (1998), we reduce the pricing of these claims to the analysis of two building blocks, survival claims and default payments. A *survival claim* (zero-recovery defaultable zero-coupon bond) with maturity T on firm i has payoff $1 - Y_{T,i}$. By the Markov-property of $\mathbf{Y}$, the price of this claim in $t < T$ is given by

$$p_i(t,T) = p_0(t,T)E_{(t,\mathbf{Y}_t)}\big((1 - Y_{T,i})\big) =: v^i(t,\mathbf{Y}_t)\,. \tag{16}$$

The function $v^i(t,\mathbf{Y}_t)$ can be computed using the Kolmogorov backward equation for $\mathbf{Y}$, or, under Assumption 2.3, the backward equation for $\boldsymbol{M}$; see the discussion following (13). A *default payment* with deterministic payoff δ and maturity T on firm i is a claim which pays δ at the default time τ_i if $\tau_i \le T$; otherwise there is no payment. On $\{\tau_i > t\}$ the price equals

$$H_t = \delta E\Big(\exp\big(-\int_t^{\tau_i} r(s)\,ds\big)\,Y_{T,i} \mid \mathcal{H}_t\Big) = \delta\int_t^T p_0(t,s)P_{(t,\mathbf{Y}_t)}(\tau_i \in ds)\,ds\,; \tag{17}$$

this expression can be evaluated numerically using (5) or, in the homogeneous-group model, (14). Of course, (16) or (17) can alternatively be computed via Monte Carlo simulation; see Appendix A. Assuming a deterministic loss given default δ, the price of a defaultable zero-coupon bonds under various recovery models as well as the fair swap spread of a plain vanilla single-name CDS is easily computed from the pricing formulas for survival claims and default payments; see for instance Section 9.4 of McNeil et al. (2005) for details.

3.2 Pricing of kth-to-default swaps

Payoff description. We consider a portfolio of m names with nominal N_i and deterministic loss given default $(LGD)_i = \delta_i N_i$, $1 \le i \le m$, $\delta_i \in (0,1)$. We start with a description of the *default payment leg* of the structure. If the kth default time T_k is smaller than the maturity T of the swap, the protection buyer receives at time T_k the loss of the portfolio incurred at the kth default given by $(LGD)_{\xi_k}$. Note that the size of this payment is typically unknown at time 0, as it depends on the identity ξ_k of the kth defaulting firm. The *premium payment leg* consists of regular premium payments of size $x^{k\text{th}}(t_n - t_{n-1})$ at fixed dates $t_1, t_2, \ldots, t_N = T$, provided $t_n \le T_k$; after T_k the regular premium payments stop. The quantity $x^{k\text{th}}$ is the *swap spread* of the kth-to-default swap. Moreover, if $T_k \in (t_{n-1}, t_n)$, the protection seller is entitled to an *accrued premium payment* at T_k of size $x^{k\text{th}}(T_k - t_{n-1})$. This structure can be priced using Proposition 2.2:

Default payment leg. Under the above assumptions the value of the default payment leg at $t = 0$ can be written as

$$V^{\text{def}} := \sum_{j=1}^m (LGD)_j\, E\Big(\exp\big(-\int_0^{\tau_j} r(s)\,ds\big)\,1_{\{\tau_j \le T\}}1_{\{M_{\tau_j}=k\}}\Big).$$

Now we obtain by conditioning on τ_j

$$E\Big(\exp\big(-\int_0^{\tau_j} r(s)\,ds\big)\,1_{\{\tau_j \le T\}}1_{\{M_{\tau_j}=k\}}\Big) = \int_0^T p_0(0,t)P\big(M_t = k \big| \tau_j = t\big)P(\tau_j \in dt)\,dt. \tag{18}$$

Defining for $l, j \in \{1, \dots, m\}$ the set $A_1(l, j) := \{\boldsymbol{y} \in S^{\mathbf{Y}} : M(\boldsymbol{y}) = l, y_j = 1\}$, we have $P\big(M_t = k \mid \tau_j = t\big) = \sum_{\boldsymbol{y} \in A_1(k,j)} P(\mathbf{Y}_t = \boldsymbol{y} \mid \tau_j = t)$. Hence we get from (6) and (18)

$$V^{\text{def}} = \sum_{j=1}^{m} (LGD)_j \sum_{\boldsymbol{y} \in A_1(k,j)} \int_0^T p_0(0,t) \lambda_j(t, \boldsymbol{y}^j) P(\mathbf{Y}_t = \boldsymbol{y}^j)\, dt\,; \tag{19}$$

for m small this expression can be computed using the forward equation for $\mathbf{Y}$ and numerical integration. Under Assumption 2.3, Corollary 2.5 and the forward equation for $\boldsymbol{M}$ can be used to evaluate (18). For instance, we get in an exchangeable model with only one group the relation

$$V^{\text{def}} = \int_0^T p_0(0,t)\, \frac{m-k+1}{m}\, h(t, k-1) P(M_t = k-1)\, dt \Big(\sum_{j=1}^{m} (LGD)_j \Big). \tag{20}$$

Premium payment leg. The premium payment leg consists of the sum of the value of the regular premium payments and the accrued premium payment; since $\{T_k \le t\} = \{M_t \ge k\}$, given a generic swap spread x, its value in $t = 0$ can be written as

$$V^{\text{prem}}(x) = x \sum_{n=1}^{N} \Big\{ (t_n - t_{n-1}) p_0(0, t_n) P\big(M_{t_n} < k\big) + E\Big(e^{-\int_0^{T_k} r(s) ds} (T_k - t_{n-1}) 1_{\{t_{n-1} < T_k \le t_n\}} \Big) \Big\}. \tag{21}$$

Using partial integration we get for the second term

$$\begin{aligned} E\Big(e^{-\int_0^{T_k} r(s) ds} (T_k - t_{n-1}) 1_{\{t_{n-1} < T_k \le t_n\}} \Big) &= \int_{t_{n-1}}^{t_n} p_0(0,t)(t - t_{n-1}) P(T_k \in dt)\, dt \\ &= p_0(0, t_n)(t_n - t_{n-1}) P(M_{t_n} \ge k) - \int_{t_{n-1}}^{t_n} p_0(0,t)\big(1 - r(t)(t - t_{n-1})\big) P(M_t \ge k) dt\,. \end{aligned}$$

Hence we get from (21)

$$V^{\text{prem}}(x) = x \sum_{n=1}^{N} \Big\{ p_0(0, t_n)(t_n - t_{n-1}) - \int_{t_{n-1}}^{t_n} p_0(0,t)\big(1 - r(t)(t - t_{n-1})\big) P(M_t \ge k)\, dt \Big\}, \tag{22}$$

which is easily computed given the distribution of M. The fair spread $x^{k\text{th}}$ is finally obtained by solving the equation $V^{\text{def}} = V^{\text{prem}}(x^{k\text{th}})$.

Example 3.2. We consider a portfolio of $m = 5$ firms with corresponding 5-year CDS spreads of 80bp, 90bp, 100bp, 110bp and 120bp (1bp =0.01%). Exposure N and loss given default δ are identical across firms and given by $N \equiv 1$ and $\delta = 0.6$. For simplicity we set $r(t) \equiv 0$. Default intensities are given by a variant of the convex counterparty-risk model:

$$\lambda_i(t, \mathbf{Y}_t) = \lambda_{0,i} + \frac{\lambda_1}{\lambda_2} \left(e^{\lambda_2 \left(\frac{(M(\mathbf{Y}_t) - \mu(t))^+}{m} \wedge 0.37 \right)} - 1 \right)\,; \tag{23}$$

the cap at 0.37 has been introduced in order to avoid an "explosion" of the intensity for high values of λ_2. Since the average CDS spread of the portfolio is 0.01 (100bp) we take $\mu(t) := 5\big(1 - \exp(-\frac{0.01}{0.6} t)\big)$ as approximation for the expected number of defaulted firms. It is interesting to compare the results of the Markov model with the industry standard one-factor

kth-to-default spreads in bp	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
Gauss copula with $\rho = 10\%$	469.73	81.18	11.32	1.07	0.05
Markov model with $\lambda_2 = 0$	469.73	80.34	11.62	0.94	0.03
Markov model with $\lambda_2 = 5$	469.73	77.47	13.76	1.51	0.07
Markov model with $\lambda_2 = 10$	469.73	73.11	16.63	2.74	0.19

Table 1: Fair spread of kth-to-default swaps in Gauss copula model and Markov model for varying convexity parameter λ_2. Details are given in the text.

Gauss copula model, as this reveals the impact of the convexity parameter λ_2 on swap spreads. For this we first computed the fair kth-to default spreads in the one-dimensional Gauss copula model. Then the basket swap was priced in the Markov model for varying values of λ_2; $\lambda_{0,i}$ and λ_1 were calibrated to the single-name CDS spreads and the first-to-default spread $x^{1\text{st}}$ obtained in the copula model. The results are contained in Table 1: as we increase λ_2 the higher order spreads $x^{3\text{rd}}, \dots, x^{5\text{th}}$ increase even for $x^{1\text{st}}$ fixed. This shows that higher values of λ_2 lead to a fatter right tail of the portfolio loss distribution even if the left tail and the mean of the distribution are kept fixed.

3.3 Synthetic CDO-Tranches

In this section we turn to an analysis of synthetic CDO tranches; in particular, we are interested in modelling the well-known implied correlation skew.

Payoff description. A synthetic CDO tranche is based on a portfolio of m single-name CDSs on m different reference entities. Let N_i denote the nominal exposure of the ith swap, δ_i the percentage loss given default of company i and $N := \sum_{i=1}^m N_i$ the overall exposure. The *cumulative loss* of the portfolio up to time t due to default events is then given by $L_t = \sum_{i=1}^m \delta_i N_i Y_{t,i}$. A *synthetic CDO tranche* is characterized by a maturity date T and fixed percentage lower and upper attachment points $0 \le l \le u \le 1$. The tranche consists of a default payment leg and a premium payment leg. In order to describe the corresponding payments we define the cumulative loss of the tranche by

$$L_t^{[l,u]} := v^{[l,u]}(L_t) := (L_t - lN)^+ - (L_t - uN)^+; \tag{24}$$

note that $L_t^{[l,u]}$ gives the part of the cumulative loss falling in the layer $[lN, uN]$. The *notional* of the tranche at time t is defined as $n^{[l,u]}(L_t) := (u-l)N - v^{[l,u]}(L_t)$. At a default time $T_k \le T$ there is a default payment of size $\Delta L_{T_k}^{[l,u]} := L_{T_k}^{[l,u]} - L_{T_k-}^{[l,u]}$.

The premium payment leg consists of regular premium payments, to be made at fixed dates $0 = t_0 < t_1 < \cdots < t_N = T$, and of accrued premium payments, to be made at the default times T_k with $T_k \le T$. The regular premium payment at date t_n is given by $x^{[l,u]}(t_n - t_{n-1}) n^{[l,u]}(L_{t_n})$, where $x^{[l,u]}$ is the fair annualized *tranche spread*. At a default time $T_k \in (t_n, t_{n+1}]$ there is an accrued payment of size $x^{[l,u]}(T_k - t_n)\Delta L_{T_k}^{[l,u]}$. There are no initial payments so that the fair tranche spread $x^{[l,u]}$ is computed by equating the value of default- and premium payment leg of the structure.

Computation of tranche spreads. Given a generic tranche spread x, the value of the regular payments equals

$$V_{[l,u]}^{\text{prem},1}(x) := x \sum_{n=1}^{N} p_0(0,t_n)(t_n - t_{n-1}) E\left(n^{[l,u]}(L_{t_n}) \right). \tag{25}$$

The value of the default payments in $t = 0$ is given by

$$V_{[l,u]}^{\text{def}} := E\left(\int_0^T \exp\big(-\int_0^t r(s)ds\big)\, dL_t^{[l,u]} \right). \tag{26}$$

Theses expressions can be computed via standard Monte Carlo using Algorithm A.1. For most synthetic CDO tranches it is assumed that $N_i \equiv N$ and $\delta_i \equiv \delta$ for some $\delta \in (0,1]$. If we moreover assume that default intensities have the homogeneous group structure of Assumption 2.3, we can alternatively use a semi-analytic pricing approach based on the Kolmogorov forward equation for $\boldsymbol{M}$: first, we transform the Stieltjes integral in (26) using partial integration; since $L_0^{[l,u]} = 0$ and since $L_t^{[l,u]} = v^{[l,u]}(L_t)$ we get

$$V_{[l,u]}^{\text{def}} = p_0(0,T) E\left(v^{[l,u]}(L_T) \right) + \int_0^T r(t) p_0(0,t) E\left(v^{[l,u]}(L_t) \right) dt. \tag{27}$$

The value of the accrued premium payment $V_{[l,u]}^{\text{prem},2}(x)$ can be transformed via partial integration using a similar argument as in (27); we omit the details. Under our homogeneity assumptions, $L_t = \delta N M_t = \delta N \sum_{\kappa=1}^{K} M_{t,\kappa}$; moreover, the distribution of $\boldsymbol{M}_t$ can be determined via the Kolmogorov forward equation (38) for $\boldsymbol{M}$.[2]

Implied correlation skews. By now there exists a liquid market for synthetic CDO tranches on major CDS indices, and the properties of the corresponding tranche-spreads serve as reference point for many academic studies. In these markets spreads of synthetic CDO tranches are usually expressed in terms of *implied correlation* in an exchangeable Gauss copula model with $\text{Exp}(\gamma)$-distributed margins and exchangeable Gauss copula with correlation parameter ρ. The parameter γ is calibrated to the CDS-index level, and the correlation parameter ρ is calibrated to the observed tranche spread. Here to different notions of implied correlation are being used: implied *tranche correlation* is the value of ρ that leads to the observed tranche spread; implied *base correlation* is the implied tranche correlation corresponding to a hypothetical set of tranche spreads for equity tranches $[0, u_k]$; see for instance McGinty, Beinstein, Ahluwalia & Watts (2004) for details.

Implied correlations show a typical pattern known as the *implied correlation skew*. As a benchmark case we present the tranche spreads on the DJ iTraxx Europe (the major European CDS index) on August 4, 2004. Standardized CDO-tranches on this index have a maturity of 3, 5, 7 or 10 years with 5 year tranches being particularly liquid, quaterly premium payments and attachment points equal to 0%, 3%, 6%, 9%, 12%, 22%. Table 2 gives the CDO-spreads observed on the market that day and the corresponding implied correlations. We observe that implied tranche correlations are first decreasing and then increasing, whereas implied base correlations are strictly increasing. Moreover, spreads for senior tranches (tranches with high upper attachment point) are comparatively high. This behavior is typical for CDO markets.[3]

[2] Of course, this is computationally feasible only if the number of groups k is relatively small.

[3] While tranche spreads have changed a lot since 2004, the qualitative properties of tranche spreads are unchanged; see also Table 4 in the appendix.

	Index	[0,3]	[3,6]	[6,9]	[9,12]	[12,22]
market spread	42bp	27.6%	168bp	70bp	43bp	20bp
tranche correlation		22.4%	5.0%	15.3%	22.6%	30.6%
base correlation		22.4%	32.1%	38.8%	43.8%	57.1%

Table 2: Market quotes and implied correlations for the underlying index and synthetic CDO-tranche spreads and corresponding implied correlations for the DJ iTraxx Europe. Note that the value of 27.6% for the equity tranche corresponds to an upfront payment of 27.6% of the notional; the running spread is set to 5% by market convention. The intensity parameter was calibrated to $\gamma = 0.007$. Spread data are from Hull and White (2004).

Explaining Correlation Skews Implied correlation skews reflect deficiencies of the exchangeable Gauss copula model. In particular, the high implied correlations for the senior tranches show that market participants expect large clusters of defaults to occur more frequently than is consistent with a Gauss copula model. Most attempts to explain correlation skews start from this observation. For instance Hull & White (2004), Kalemanova, Schmid & Werner (2005), Guegan & Houdain (2005) or Elouerkhaoui (2006) consider models based on alternative copulas leading to more frequently occurring default clusters. Graziano & Rogers (2006) use a model with conditionally independent defaults driven by a Markov-chain; jumps of the chain may moreover cause simultaneous defaults of several names.

We show next that is possible to generate correlation skews in the convex counterparty risk model (10) respectively (23). The basic idea is simple: by increasing λ_2 we can generate occasional large clusters of defaults without affecting the left tail of the distribution of L_t too much; in this way we can reproduce the high spread of the CDO tranches in a way which is consistent with the observed spread of the equity tranche. To confirm this intuition we consider the market data introduced in Table 2. In Table 3 we give the CDO spreads if the convexity parameter λ_2 is varied; λ_0 and λ_1 were calibrated to the index level and the observed market quote of the equity tranche. The results show that for appropriate values of λ_2 the model can reproduce the qualitative behavior of the observed tranche spreads in a very satisfactory way. This observation is interesting as it provides an explanation of correlation skews of CDOs in terms of the *dynamics* of the default indicator process.[4] Similarly as in Andersen & Sidenius (2004), the model fit can be improved further by considering a state-dependent loss given default of the form $\delta_t = \delta_0 + \delta_1 M_t$ for δ_0 and $\delta_1 > 0$; see again Table 3 for details.

Implied correlations for CDO tranches on the iTraxx Europe have changed substantially since August 2004. More importantly, the analysis presented in Table 3 presents only a "snapshot" of the CDO market at a single day. For these reasons we recalibrated the convex counterparty risk model 23 to 6 months of observed 5-year tranche spreads on the iTraxx Europe in the period 23.9.2005–03.03.2006. In order to assess the issue of parameter-stability over time we compared two different calibration approaches: first we did a *full calibration* where at a given day $\lambda_0, \lambda_1 and \lambda_2$ were calibrated to the index level and the tranche spread of equity and junior mezzanine tranche observed at that day; second, we did a *partial calibration* where at a given day λ_0 was calibrated to the index level observed at that day whereas for λ_1 and λ_2 we used the values obtained by full calibration on Sept, 23 2005 (the first day of the sample). The

[4] Qualitatively similar results were recently obtained in the related paper Herbertsson (2007).

tranches	[0,3]	[3,6]	[6,9]	[9,12]	[12,22]	
market spreads	27.6%	168.0bp	70.0bp	43.0bp	20.0bp	
model spreads						$\sum$ abs. err.
$\lambda_2 = 0$	27.60%	223.1bp	114.5bp	61.1bp	16.9bp	120.8bp
$\lambda_2 = 5$	27.60%	194.2bp	95.7bp	54.9bp	23.3bp	67.1bp
$\lambda_2 = 8$	27.60%	172.1bp	80.0bp	46.7bp	23.7bp	21.5bp
$\lambda_2 = 8.54$	27.60%	168.0bp	77.1bp	45.1bp	23.5bp	12.7bp
$\lambda_2 = 10$	27.60%	156.9bp	69.4bp	40.7bp	22.7bp	16.7bp
state-dependent LGD $\delta_0 = 0.5$; $\delta_1 = 7.5$	27.60%	168.0bp	71.2bp	39.3bp	19.6bp	5.3bp

Table 3: CDO-spreads in the convex counterparty risk model (23) for varying λ_2. λ_0 and λ_1 were calibrated to the index level of 42bp and the market quote for the equity tranche, assuming $\delta = 0.6$. For $\lambda_2 \in [8, 10]$ the qualitative properties of the model-generated CDO-spreads resemble closely the behaviour of the market spreads; with state-dependent LGD the fit is almost perfect.

motivation for this distinction is as follows: λ_0 is a level parameter which is mainly influenced by the randomly fluctuating index spread so that one expects a lot of variability in this parameter; λ_1 and λ_2 on the other hand are structure parameters which should be reasonably stable over time The results are contained in Table 4 in the appendix. A comparison of the tranche spreads obtained via full and partial calibration shows that partial calibration performs quite good, indicating that the model does indeed give a reasonable description of the dynamics of CDO markets.

4 Hedging Credit Derivatives

In this section we study the hedging of credit derivatives under Assumption 3.1. In particular, we give conditions ensuring that every $\mathcal{H}_T$-measurable claim can be replicated by dynamic trading in a portfolio of defaultable zero-coupon bonds and cash. Not surprisingly, it turns out that our results depend strongly on the dynamic structure of the model, in particular on the choice of $(\mathcal{H}_t)$ as underlying filtration. This highlights a point made already in the introduction: starting directly with assumptions on the joint distribution of $\tau_1, \ldots, \tau_m$ and neglecting the dynamic aspects of the model - as it is done in most of the literature on factor copula models - might be bad modelling practice. The hedging of credit risky securities is also studied in (Bielecki, Jeanblanc & Rutkowski 2004) and (Elouerkhaoui 2006).

The hedging problem. We use the setup introduced in Assumption 3.1. The set of hedging instruments consists of defaultable zero-coupon bonds issued by the m firms in the portfolio; for simplicity the bonds are assumed to have a zero recovery rate and fixed common maturity T. Recall that the price of the bond issued by firm j is $v^j(t, \mathbf{Y}_t) := p_0(t,T) E_{(t,\mathbf{Y}_t)}\big((1 - Y_{T,j})\big)$. In the sequel we will work with discounted quantities using the default-free zero coupon bond $p_0(\cdot, T)$ as numéraire; the discounted price of the bond issued by firm j is then

$$\tilde{v}^j(t, \mathbf{Y}_t) = E_{(t,\mathbf{Y}_t)}\big((1 - Y_{T,j})\big) \,. \tag{28}$$

We are aware that this choice of hedging instruments is not in line with market practice - most practitioners regard single-name CDSs as natural hedging instruments for portfolio credit derivatives - but it is a useful first step. In principle, our arguments apply also to the problem of hedging with CDSs. However, the gains-from-trade process of these instruments takes on a quite cumbersome form, thus complicating the analysis considerably.

We consider the problem of hedging a claim with maturity T, $\mathcal{H}_T$-measurable payoff H and discounted price process $\tilde{H}_t = E(H \mid \mathcal{H}_t)$; since $\mathbf{Y}$ is Markov, in most cases of interest $\tilde{H}_t$ is in fact of the form $\tilde{v}^H(t, \mathbf{Y}_t)$ for some function $\tilde{v}^H : [0,T] \times S^{\mathbf{Y}} \to \mathbb{R}$. We are looking for a selffinancing portfolio strategy $\boldsymbol{\theta}^H = (\theta^H_{t,0}, \ldots, \theta^H_{t,m})_{0 \le t \le T}$ in the savings account and in the defaultable zero-coupon bonds $p_j(\cdot, T)$, $1 \le j \le m$, that replicates the claim H. By standard results on numéraire-invariance, this is equivalent to finding a representation of the martingale $\tilde{H}$ of the form

$$\tilde{H}_t = \tilde{H}_0 + \sum_{j=1}^{m} \int_0^t \theta^H_{s,j} \, d\tilde{p}_j(s,T) \,, \quad 0 \le t \le T \,. \tag{29}$$

We will approach this problem in two steps. First, we derive a martingale representation of $\tilde{H}$ in terms of the compensated default indicator processes $N_{t,i} := Y_{t,i} - \int_0^{t \wedge \tau_i} \lambda_i(s, \mathbf{Y}_s) ds$, $1 \le i \le m$; in Step 2 we use this representation to give conditions for the existence of a martingale representation of $\tilde{H}$ in terms of the discounted bond price processes $\tilde{p}_j(\cdot, T)$.

Step 1. Since there are no joint defaults in our model, the mark space of the marked point process $(T_n, \xi_n)_{1 \le n \le m}$, is given by the set $\{1, \ldots, m\}$. By standard results from stochastic calculus - see for instance Jacod (1975) - every $(\mathcal{H}_t)$-martingale can therefore be represented as stochastic integral with respect to the m martingales $N_{t,i}, \ldots, N_{t,m}$, i.e. there are predicable processes $\phi^H_{t,1}, \ldots, \phi^H_{t,m}$ such that

$$\tilde{H}_t = \tilde{H}_0 + \sum_{i=1}^{m} \int_0^t \phi^H_{s,i} dN_{s,i}. \tag{30}$$

The process $\boldsymbol{\phi}^H$ is in fact easily determined: denote by $A_t := \{1 \le i \le m \colon Y_{t,i} = 0\}$ the set of surviving firms at time t. Since $\Delta \tilde{H}_t = \sum_{i=1}^m \phi^H_{t,i} \Delta Y_{t,i}$ we get that

$$\phi^H_{t,i_0} = \sum_{n=0}^{m-1} 1_{]\!]T_n \, T_{n+1}]\!]}(t) 1_{A_{T_n}}(i_0) \Big(E\big(\tilde{H}_T \mid \mathcal{H}_{T_n}, T_{n+1} = t, \xi_{n+1} = i_0\big) - \tilde{H}_t \Big) \,; \tag{31}$$

if $\tilde{H}_t = \tilde{v}^H(t, \mathbf{Y}_t)$, (31) reduces to $\phi^H_{t,i_0} = 1_{A_t}(i_0) \big(\tilde{v}^H(t, \mathbf{Y}^{i_0}_t) - \tilde{v}^H(t, \mathbf{Y}_t) \big)$. Note that ϕ^H is by definition left-continuous.

Step 2. In analogy with (30), there are predictable vector-valued processes $\boldsymbol{\phi}^j$ such that

$$\tilde{p}_j(t,T) = \tilde{p}_j(0,T) + \sum_{i=1}^{m} \int_0^t \phi^j_{s,i} dN_{s,i}, \quad 1 \le j \le m \,. \tag{32}$$

Hence the desired representation (29) can be written in the form

$$\tilde{H}_t = \tilde{H}_0 + \sum_{j=1}^{m} \int_0^t \theta^H_{s,j} \, d\Big(\sum_{i=1}^{m} \int_0^s \phi^j_{u,i} \, dN_{u,i} \Big) = \tilde{H}_0 + \sum_{i=1}^{m} \int_0^t \Big(\sum_{j=1}^{m} \theta^H_{s,j} \phi^j_{s,i} \Big) \, dN_{s,i} \,. \tag{33}$$

Comparing (30) and (33), we obtain the following equations for the random variables $\theta^H_{t,j}$, $j \in A_t$,

$$\sum_{j \in A_t} \phi^j_{t,i}\, \theta^H_{t,j} = \phi^H_{t,i}, \quad i \in A_t\,, 0 \le t \le T\,; \tag{34}$$

for $j \notin A_t$ we let $\theta^H_{t,j} = 0$. Note that (34) is a linear system of $|A_t|$ equations for $|A_t|$ unknowns with coefficient matrix $\mathbf{\Phi}_t := \big(\phi^j_{t,i}\big)_{i,j \in A_t}$. Summing up, we therefore have

Proposition 4.1. *Suppose that almost surely the matrix $\mathbf{\Phi}_t$ has full rank for all $t \in [0,T]$. Then every $\mathcal{H}_T$-measurable claim can be replicated by dynamic trading in the savings account and the defaultable zero-coupon bonds $p_j(\cdot,T)$, $1 \le j \le m$. The trading strategy $(\theta^H_{t,1},\dots,\theta^H_{t,m})$ is given as solution to the linear system* (34) *with coefficients $\phi^j_{t,i}$ determined in* (32) *and right hand side $\phi^H_{t,i}$ determined in* (30)*; θ^H_0 is determined by the selffinancing-condition.*

Comments. 1. The system (34) can be simplified in the homogeneous-group case. In that case $\phi^j_{t,i} = \eta^{\kappa(j)}_{t,\kappa(i)}$ and $\phi^H_{t,i} = \eta^H_{\kappa(i)}$ for stochastic processes $\eta^\kappa_{t,\nu}$, $\eta^H_{t,\nu}$, $1 \le \nu, \kappa \le k$. Hence also $\theta^H_{t,j} = \psi^H_{t,\kappa(j)}$ and the stochastic processes $\psi^H_{t,\kappa}$, $1 \le \kappa \le k$ are determined by the following system of k equations:

$$\sum_{\kappa=1}^{k} (m_\kappa - M_{t,\kappa}) \eta^\kappa_{t,\nu} \psi^H_{t,\kappa} = \eta^H_{t,\nu}\,, \quad 1 \le \nu \le k. \tag{35}$$

2. The same argument applies to other hedging instruments such as single-name CDSs; the only thing that changes is the form of the integrands in the martingale representation of the corresponding discounted gains-from-trade process with respect to $N_{t,1},\dots,N_{t,m}$.
3. The assumption that asset price processes are $(\mathcal{H}_t)$-adapted is crucial for our analysis; without this assumption the martingale representations (30) and (32) break down. This assumption is not as innocent as it may seem: it implies that prices or spreads evolve deterministically between defaults and react in a predictable way to default events in the portfolio. In market-language, in our model there is *event risk* but between default times there is no *spread- or market risk*. This is admittedly unrealistic, so that our analysis constitutes only a first step in the development of a systematic theory for hedging credit derivatives. In particular, we expect that in more realistic models markets will typically be *incomplete*. On the other hand, at least from a conceptual point of view, the model-based hedging theory derived in Proposition 4.1 is clearly an advance over the ad hoc hedging approaches that are currently being used in practice; see for instance Neugebauer (2006).

The full-rank condition on $\mathbf{\Phi}_t$. For a given model and a given set of hedging instruments the full rank condition is straightforward to verify. In the case of zero-recovery zero coupon bonds as hedging instruments we can expect the full-rank condition to hold if the interaction between defaults is not too strong or if the time-to maturity is not too large, as we now show. By a similar reasoning as in (31), we obtain for $i, j \in A_t$

$$\phi^j_{t,i} = \tilde{v}^j(t, \mathbf{Y}^i_t) - \tilde{v}^j(t, \mathbf{Y}_t)\,. \tag{36}$$

Note that for $j \in A_t$ the element $\phi^j_{t,j} = -\tilde{v}^j(t, \mathbf{Y}_t)$ corresponds to the change in the value of the bond due to default of the issuing firm j, whereas the off-diagonal elements $\phi^j_{t,i}$, $i \neq j$, reflect the change in the value of the bond caused by the impact of the default of firm i on the conditional survival probability of the issuer j. In particular, with independent defaults $\phi_{t,i}^j = 0$ for $i \neq j$ and $\phi_{t,j}^j < 0$ for all $j \in A_t$. Hence $\mathbf{\Phi}_t$ is a diagonal matrix with non-vanishing diagonal elements, and the full-rank condition is obviously satisfied. Now recall the following simple result from linear algebra: a generic matrix $(a_{ij})_{1 \le i,j \le n}$ is non-singular if it has a *dominant diagonal*, i.e. if $|a_{ii}| > \sum_{i \neq j} |a_{ij}|$ for all $i \in \{1, \dots, n\}$. Since with independent defaults $\mathbf{\Phi}_t$ is diagonal, we expect that $\mathbf{\Phi}_t$ has a dominant diagonal and therefore full rank, if the interaction between default intensities is not to strong. A similar qualitative statement applies in the case of a small time to maturity: Since $\tilde{v}^j(T, \boldsymbol{y}) = p_0(0,T)(1 - y_j)$, we get that

$$\mathbf{\Phi}_T = \operatorname{diag}\big(-p_0(0,T), \dots, -p_0(0,T)\big).$$

Hence $\mathbf{\Phi}_t$ has a dominant diagonal for t close to T (note that $\tilde{v}$ is continuous in t).

Example 4.2 (Hedging of a survival claim and a first-to-default swap). We consider a portfolio of $m = 5$ firms with intensity function $\lambda_i(t, \mathbf{Y}_t) = \lambda_{0,i} + \lambda_1\big(\frac{M(\mathbf{Y}_t)}{m} \wedge 0.37\big)$. Model parameters are calibrated to the spread data of Example 3.2; for simplicity we take $r(t) \equiv 0$. Throughout we use zero-coupon bonds with zero recovery rate and a maturity of $T = 5$ years as hedging instruments.

We start with the replication of a zero-coupon bond issued by Firm 1 with a maturity of $T_0 = 0.5$. With independent defaults ($\lambda_1 = 0$), the hedge portfolio is constant and given by $\theta_{t,1} \equiv p_1(0,T_0)/p_1(0,T)$; the cash position and the position in the bonds issued by Firm 2–5 (the firms not underlying the transaction) are identically zero. For $\lambda_1 > 0$ the situation changes. In that case the default of, say, Firm 2 leads to an increase in the default intensity of Firm 1, reducing the value of zero coupon bonds issued by Firm 1. This effect becomes stronger with increasing time to maturity, so that $|\Delta p_1(\tau_2, T_0)| < |\Delta p_1(\tau_2, T)|$. To make up for the ensuing loss, the hedger has to take a short position in $p_2(\cdot, T)$. A similar argument applies to Firm 3, 4 and 5. Numerical values for the position in the risky zero-coupon bonds are plotted in Figure 1. The example shows that in the presence of default contagion the replicating portfolio of a single-name credit derivative may contain defaultable securities issued by firms not directly underlying the transaction.

Next we consider the default payment of a first-to-default swap. The corresponding hedging strategy is illustrated numerically in Figure 2. The portfolio consists of a short position in all the defaultable bonds underlying the transaction; in this way the portfolio produces a gain at T_1 which compensates the default payment. For $\lambda_1 > 0$ the absolute size of this short-position is reduced compared to the case of independent defaults. Intuitively, this is due to the fact that with default contagion $\Delta p_i(T_1, T) < 0$ also for the surviving firms $i \in A_{T_1}$. Since $\theta_{T_1,i} < 0$ this leads to a an additional increase in the value of the hedge portfolio at T_1 which contributes to financing the payoff of the claim.

A Numerical tools for the Markov model

We present standard approaches for the numerical treatment of (time-inhomogeneous) finite-state markov chains; see for instance Norris (1997) for the theoretical foundations.

A.1 Monte Carlo

For the convenience of the reader we recall the standard algorithm for generating trajectories of $\boldsymbol{Y}$ (or equivalently realisations of the sequence $(T_n, \xi_n)_{1 \le n \le m}$).

Algorithm A.1. 1. Generate independent random variables $Z_1, \dots, Z_m$, $U_1, \dots, U_m$ with $Z_i \sim \text{Exp}(1)$, $U_i \sim \text{U}(0,1)$. Put $T_0 = 0$, $\boldsymbol{y}^{(0)} = \boldsymbol{0}$, $n = 1$, and define $\bar{\lambda}_t^{(1)} := \sum_{i=1}^m (1 - y_i^{(0)}) \lambda_i(t, \boldsymbol{y}^{(0)})$.

2. Given T_{n-1}, $\boldsymbol{y}^{(n-1)}$, $(\bar{\lambda}_t^{(n)})_{t \ge T_{n-1}}$, let $T_n := \inf \left\{ t \ge T_{n-1} : \int_{T_{n-1}}^t \bar{\lambda}_s^{(n)} ds \ge Z_n \right\}$ and let $\xi_n := i$ if $\sum_{j=1}^{i-1} (1 - \boldsymbol{y}_j^{(n-1)}) \lambda_j(T_n, \boldsymbol{y}^{(n-1)}) \le \bar{\lambda}_{T_n}^{(n)} U_n < \sum_{j=1}^{i} (1 - \boldsymbol{y}_j^{(n-1)}) \lambda_j(T_n, \boldsymbol{y}^{(n-1)})$.

3. If $n = m$ stop. Else, set $\boldsymbol{y}^{(n)} := \left(y_1^{(n-1)}, \dots, 1 - y_{\xi_n}^{(n-1)}, \dots, y_m^{(n-1)} \right)$, define for $t \ge T_n$ $\bar{\lambda}_t^{(n+1)} := \sum_{i=1}^m (1 - y_i^{(n)}) \lambda_i(t, \boldsymbol{y}^{(n)})$, replace n with $n+1$ and continue with Step 2.

A.2 Kolmogorov equations

The backward equation is an ODE system for the function $(t, \boldsymbol{y}_1) \to p(t, s, \boldsymbol{y}_1, \boldsymbol{y}_2)$, $0 \le t \le s$; s and $\boldsymbol{y}_2$ are considered as parameters. The equation has the form

$$\frac{\partial}{\partial t} p(t, s, \boldsymbol{y}_1, \boldsymbol{y}_2) + G_{[t]} p(t, s, \boldsymbol{y}_1, \boldsymbol{y}_2) = 0 \text{ for } 0 \le t < s, \quad p(s, s, \boldsymbol{y}_1, \boldsymbol{y}_2) = 1_{\{\boldsymbol{y}_2\}}(\boldsymbol{y}_1). \tag{37}$$

The forward-equation is an ODE-System for the function $(s, \boldsymbol{y}_2) \to p(t, s, \boldsymbol{y}_1, \boldsymbol{y}_2)$, $s \ge t$, which is governed by the adjoint operator $G_{[s]}^*$. In its general form the forward equation reads $\frac{\partial}{\partial s} p(t, s, \boldsymbol{y}_1, \boldsymbol{y}_2) = G_{[s]}^* p(t, s, \boldsymbol{y}_1, \boldsymbol{y}_2)$ with initial condition $p(t, t, \boldsymbol{y}_1, \boldsymbol{y}_2) = 1_{\{\boldsymbol{y}_1\}}(\boldsymbol{y}_2)$. This leads to the following system of ODEs:

$$\frac{\partial p(t, s, \boldsymbol{y}_1, \boldsymbol{y}_2)}{\partial s} = \sum_{\{j \colon y_{1,j} = 1\}} \lambda_j(s, \boldsymbol{y}_2^j) p(t, s, \boldsymbol{y}_1, \boldsymbol{y}_2^j) - \sum_{\{j \colon y_{2,j} = 0\}} \lambda_j(s, \boldsymbol{y}_2) p(t, s, \boldsymbol{y}_1, \boldsymbol{y}_2); \tag{38}$$

for a formal proof see Appendix A.2 of Frey & Backhaus (2004). Note that the first term on the right in (38) gives the instantaneous increase in the probability $p(t, s, \boldsymbol{y}_1, \boldsymbol{y}_2)$ due to jumps from neighboring states $\boldsymbol{y}_2^j$ into the state $\boldsymbol{y}_2$; the second term gives the instantaneous decrease due to jumps from $\boldsymbol{y}_2$ to the neighboring states $\boldsymbol{y}_2^j$, $1 \le j \le m$.

Of course, it is also possible to derive the Kolmogorov equations for the transition probabilities of $\boldsymbol{M}$. The exact form of the ODE-system for the backward equation is obvious. For the forward equation we obtain the following ODE-system

$$\begin{aligned} \frac{\partial p^{\boldsymbol{M}}(t, s, \boldsymbol{l}^1, \boldsymbol{l}^2)}{\partial s} = & \sum_{\kappa=1}^k 1_{\{l_\kappa^2 > 0\}} \big(m_\kappa - l_\kappa^2) + 1 \big) h_\kappa \big(s, \boldsymbol{l}^2 - \boldsymbol{e}_\kappa \big) p^{\boldsymbol{M}} \big(t, s, \boldsymbol{l}^1, \boldsymbol{l}^2 - \boldsymbol{e}_\kappa \big) \\ & - \sum_{\kappa=1}^k m_\kappa - l_\kappa^2 h_\kappa(s, \boldsymbol{l}^2) p^{\boldsymbol{M}}(t, s, \boldsymbol{l}^1, \boldsymbol{l}^2) \end{aligned}$$

with initial condition $p^{\boldsymbol{M}}(t, t, \boldsymbol{l}^1, \boldsymbol{l}^2) = 1_{\{\boldsymbol{l}^1\}}(\boldsymbol{l}^2)$.

B Figures and additional tables

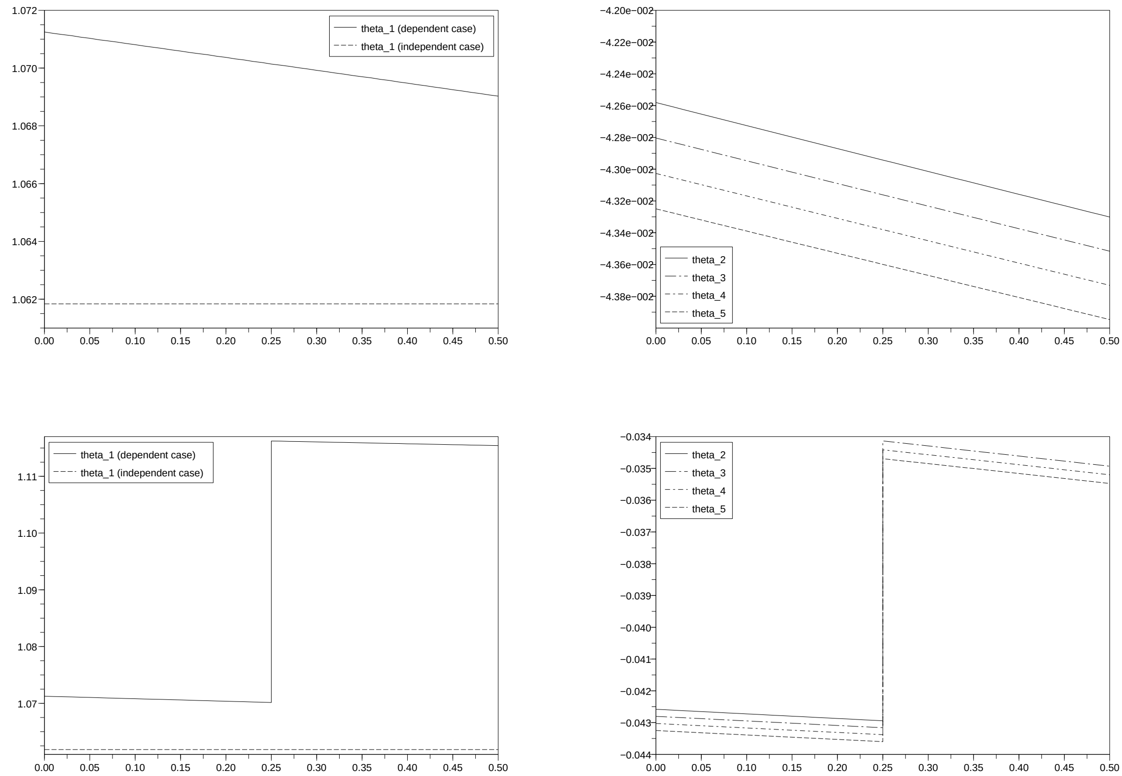

Figure 1: Evolution over time of the hedge portfolio of a survival claim with maturity date $T = 0.5$ (only positions in the defaultable zero-coupon bonds). Upper row: hedge portfolio if $T_1 > 0.5$; lower row: hedge portfolio if $T_1 = 0.25$. Details are given in the text

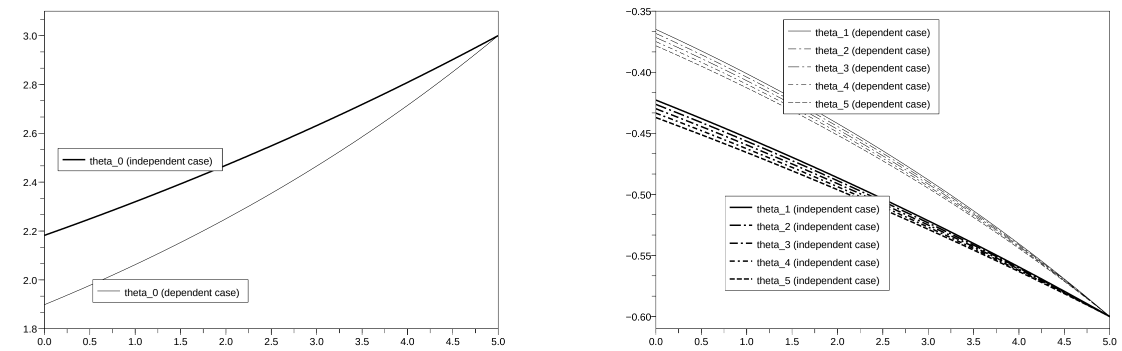

Figure 2: Evolution over time of the hedge portfolio for the default payment of a first-to-default swap if $t < T_1$. Left picture: cash-position $\theta_{t,0}$; right picture: position $\theta_{t,1}, \ldots, \theta_{t,5}$ in the defaultable zero-coupon bonds.

		[0,3] tranche			[3,6] tranche			[6,9] tranche			[9,12] tranche			[12,22] tranche			
		Spread Tr. Corr.			Spread Tr. Corr.			Spread Tr. Corr.			Spread Tr. Corr.			Spread Tr. Corr.			
Date	Index spread	market	full	partial	market	full	partial	market	full	partial	market	full	partial	market	full	partial	λ_0
23.09.2005	39	31	31	31	103	103	103	36	29	29	16	17	17	9	12	12	0.004668
		15.7	15.7	15.7	2.2	2.2	2.2	10.4	9.1	9.1	14.6	14.8	14.8	23.4	26.2	26.2	
30.09.2005	36.3	28	28	27	92	92	92	27	25	26	12	14	15	7	10	11	0.004318
		15.9	15.9	16.3	3.0	3.0	3.0	10.0	9.7	9.7	14.2	15.3	15.4	22.9	26.4	26.6	
07.10.2005	37.5	30	30	29	98	98	97	28	26	27	13	14	16	6	10	12	0.004481
		14.9	14.9	16.3	2.8	2.8	2.7	9.6	9.3	9.5	14.1	14.8	15.2	21.2	25.6	26.6	
04.11.2005	37	29	29	28	93	93	94	23	25	26	12	14	15	6	10	11	0.004445
		14.9	14.9	16.2	3.0	3.0	3.0	9.0	9.4	9.7	14.1	14.9	15.4	21.6	25.9	26.6	
02.12.2005	35	26	26	26	75	75	85	22	20	24	11	12	13	6	10	10	0.004218
		15.9	15.9	16.6	2.0	2.0	3.8	9.9	9.4	10.3	14.7	15.2	15.9	22.3	26.8	27.0	
06.01.2006	36	26	26	27	83	83	89	28	23	24	13	14	13	6	11	10	0.004388
		16.9	16.9	15.8	3.3	3.3	3.7	11.0	9.9	10.1	15.4	15.8	15.6	22.5	22.8	26.4	
03.02.2006	36	27	27	27	76	76	88	26	19	23	11	11	13	5	9	9	0.004423
		15.1	15.1	15.4	3.0	3.0	3.9	10.8	9.1	10.1	14.7	14.9	15.5	21.5	26.3	26.1	
03.03.2006	35.5	27	27	26	67	67	85	22	16	22	12	10	12	5	8	9	0.004391
		13.9	13.9	15.2	2.8	2.8	4.2	10.3	8.6	10.2	15.6	14.4	15.5	21.9	25.8	26.0	

Table 4: Calibration of the model (23) to iTraxx Europe index and tranche spreads using full and partial calibration. The parameters used for partial calibration are $\lambda_1 = 0.1921$ and $\lambda_2 = 20.73$. A comparison of the tranche spreads obtained via full and partial calibration shows that partial calibration performs quite good; details are given in the text.

References

Andersen, L. & Sidenius, J. (2004), 'Extensions to the Gaussian copula: Random recovery and random factor loadings', *Journal of Credit Risk* **1**, 29–70.

Bielecki, T., Jeanblanc, M. & Rutkowski, M. (2004), Hedging of defaultable claims, *in* 'Paris-Princeton Lectures on Mathematical Finance', Vol. 1847 of *Springer Lecture Notes in Mathematics*, Springer.

Bielecki, T. Vidozzi, A. & Vidozzi, L. (2006), 'An efficient approach to valuation of credit basket products and rating-triggered step-up bonds', working paper, Illinois Institute of Technology.

Davis, M. & Lo, V. (2001), 'Infectious defaults', *Quant. Finance* **1**, 382–387.

Duffie, D. & Garleanu, N. (2001), 'Risk and valuation of collateralized debt obligations', *Financial Analyst's Journal* **57**(1), 41–59.

Elouerkhaoui, Y. (2006), Etude des Problèmes de Corrélation et d'Incomplétude dans les Marchés de Crédit, PhD thesis, Université Paris IX Dauphine (in English).

Frey, R. (2003), 'A mean-field model for interacting defaults and counterparty risk', Bulletin of the International Statistical Institute.

Frey, R. & Backhaus, J. (2004), 'Portfolio credit risk models with interacting default intensities: a Markovian approach', Preprint, Universität Leipzig. available from `www.math.uni-leipzig.de/~frey`.

Giesecke, K. & Weber, S. (2006), 'Credit contagion and aggregate losses', *J. Econom. Dynam. Control* **30**, 741–767.

Graziano, G. & Rogers, C. (2006), 'A dynamic approach to the modelling of correlation credit derivatives using Markov chains', working paper, Statistical Laboratory, University of Cambridge.

Guegan, D. & Houdain, J. (2005), 'Collateralized debt obligations pricing and factor models: A new methodology using normal inverse gaussian distributions', working paper, ENS, Cachan.

Herbertsson, A. (2007), 'Pricing synthetic CDO tranches in a model with default contagion using the matrix-analytic approach', working paper, Göteborg University.

Horst, U. (2006), 'Stochastic cascades, credit contagion and large portfolio losses', *Journal of Economic Behavior and Organization* . to appear.

Hull, J. & White, A. (2004), 'Valuation of a CDO and a nth to default CDS without Monte Carlo simulation', *Journal of Derivatives* **12**, 8–23.

Jacod, J. (1975), 'Multivariate point processes: predictable projection, Radon-Nikodym derivatives, representation of martingales', *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **31**, 235–253.

Jarrow, R. & Yu, F. (2001), 'Counterparty risk and the pricing of defaultable securities', *J. Finance* **53**, 2225–2243.

Kalemanova, A., Schmid, B. & Werner, R. (2005), 'The normal inverse gaussian distribution for synthetic CDO pricing', working paper, Risklab Germany.

Lando, D. (1998), 'Cox processes and credit risky securities', *Rev. Derivatives Res.* **2**, 99–120.

Laurent, J. & Gregory, J. (2005), 'Basket default swaps, CDOs and factor copulas', *Journal of Risk* **7**, 103–122.

Li, D. (2001), 'On default correlation: a copula function approach', *J. of Fixed Income* **9**, 43–54.

McGinty, L., Beinstein, E., Ahluwalia, R. & Watts, M. (2004), 'Introducing base correlation', Preprint, JP Morgan.

McNeil, A., Frey, R. & Embrechts, P. (2005), *Quantitative Risk Management: Concepts, Techniques and Tools*, Princeton University Press, Princeton, New Jersey.

Neugebauer, M. (2006), 'Understanding and hedging risks in synthetic CDO tranchse', Fitch Special Report.

Norris, J. (1997), *Markov Chains*, Cambridge University Press.

Schönbucher, P. (2004), 'Information-driven default contagion', Preprint, Department of Mathematics, ETH Zürich.

Yu, F. (2007), 'Correlated defaults in intensity-based models', *Math. Finance* **17**, 155–173.