Lecture 5: Asymptotics and Dynamics of the Volatility Skew

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12 Volatility Surface Asymptotics

In this section, we investigate the shape of the volatility surface for very generic models of the stochastic volatility with jumps type.

12.1 Short Expirations

We start by rewriting our original general stochastic volatility SDEs (1) and (2) in terms of the log-moneyness $x = \log(F/K)$ and under the risk neutral measure, specializing to the case where α and β do not depend on S or t.

$$\begin{aligned} dx &= -\frac{v}{2}dt + \sqrt{v}\,dZ_1 \\ dv &= \alpha(v)\,dt + \eta\sqrt{v}\beta(v)\,dZ_2 \end{aligned} \tag{56}$$

We may rewrite

$$dZ_2 = \rho dZ_1 + \varphi\, dZ_1^*$$

with $\varphi = \sqrt{1-\rho^2}$ and $\langle dZ_1^*, dZ_1\rangle = 0$. Eliminating $\sqrt{v}dZ_1$, we get

$$dv = \alpha(v,t)\,dt + \rho\eta\,\beta(v,t)\left\{dx + \frac{v}{2}dt\right\} + \varphi\eta\,\beta(v)\,\sqrt{v}\,dZ_1^*$$

Then,

$$\mathbf{E}\left[v + dv\,|dx\right] = v + \alpha(v)\,dt + \rho\eta\,\beta(v)\left\{dx + \frac{v}{2}dt\right\}$$

so for small times to expiration (relative to the variation of $\alpha(v)$ and $\beta(v)$), we have

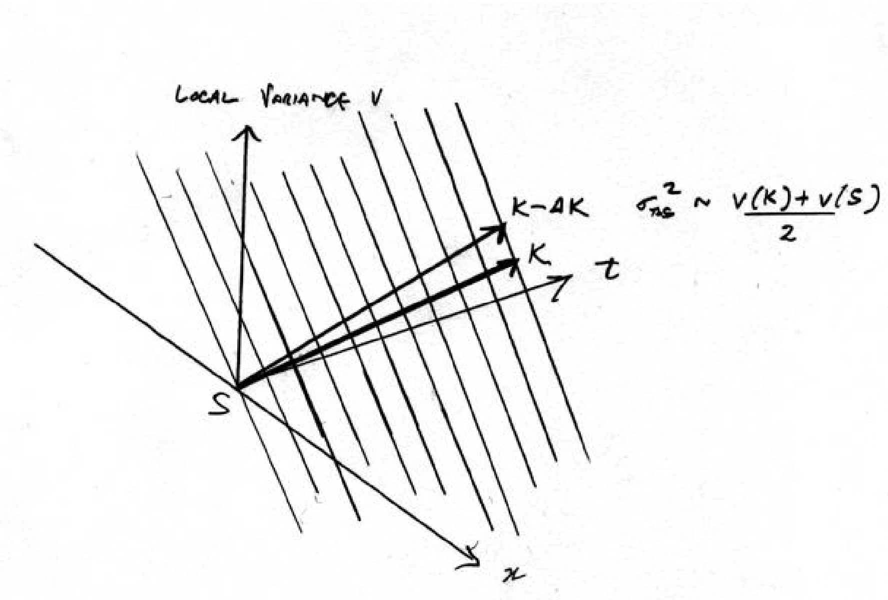
$$\begin{aligned} v_{loc}(x,t) &= \mathbb{E}\left[v_t\,|x_t = x\right] \\ &\approx v_0 + \left[\alpha(v_0) + \rho\eta\frac{v_0}{2}\beta(v_0)\right]t + \rho\eta\beta(v_0)\,x \end{aligned} \tag{57}$$

The coefficient of x (the slope of the skew) here agrees with that derived by Lee (2001) using a perturbation expansion approach.

To extend the result to implied volatility, we need the following lemma:

Lemma 12.1. *The local volatility skew is twice as steep as the implied volatility skew for short times to expiration.*

Figure 1: Integrating local variance to get implied variance



Proof. From Section 5.2, we know that BS implied total variance is the integral of local variance along the most probable path from the stock price on the valuation date to the strike price at expiration. This path is approximately a straight line (see Figure 1).

Also, from equation (57), we see that the slope of the local variance skew is a roughly constant $\beta(v_0)$ for short times. The BS implied variance skew, being the average of the local variance skews, is one half of the local variance skew. Formally,

$$\begin{aligned}
\sigma_{BS}(K,T)^2 &\approx \frac{1}{T}\int_0^T v_{loc}(\tilde{x}_t, t)\,dt \\
&\approx \text{const.} + \frac{1}{T}\int_0^T \rho\eta\beta(v_0)\tilde{x}_t\,dt \\
&\approx \text{const.} + \frac{1}{T}\int_0^T \rho\eta\beta(v_0)x_T\frac{t}{T}\,dt \\
&= \text{const.} + \frac{1}{2}\rho\eta\beta(v_0)\,x_T
\end{aligned}$$

where $\tilde{x}$ represents the "most probable" path from the stock price at time zero to the strike price at expiration. □

We conclude that for short times to expiration, the BS implied variance skew is given by

$$\frac{\partial}{\partial x}\sigma_{BS}(x,t)^2 \; = \; \frac{\rho\eta}{2}\beta(v_0) \tag{58}$$

□

Recall that in the Heston model, $\beta(v) = 1$; we see that equation (58) is consistent with the short-dated volatility skew behavior that we derived earlier in Section 5.2 for the Heston model.

12.2 The Medvedev-Scaillet Result

It turns out that we can do much better than the heuristic discussion of Section 12.1. In a recent working paper, Medvedev and Scaillet (2004)[1] develop a perturbation expansion for small times to expiration τ and fixed normalized log-strike z defined as

$$z := \frac{k}{\sigma_{BS}(k,\tau)}$$

First specializing their result to the case where the underlying process is a diffusion of the form

$$\begin{aligned} \frac{dS_t}{S_t} &= \sigma_t\, dZ_1 \\ d\sigma_t &= a\,(\sigma_t)\, dt \; + \; b\,(\sigma_t)\; dZ_2 \end{aligned} \tag{59}$$

we find that the implied volatility I has the following short-term asymptotics

$$I(z,\tau;\sigma) = \sigma + I_1(z;\sigma)\,\sqrt{\tau} + I_2(z;\sigma)\,\tau + O(\tau\,\sqrt{\tau})$$

where I_1 and I_2 are functions of the z and the instantaneous volatility $\sigma =$

[1]Thanks to Peter Friz for drawing this paper to my attention

$\sqrt{v}$ only:

$$
\begin{aligned}
I_1(z;\sigma) &= \frac{\rho\, b(\sigma)\, z}{2} \\
I_2(z;\sigma) &= \frac{1}{6}\left\{\frac{b(\sigma)^2\,(1-\rho^2)}{\sigma} + \rho^2\, b(\sigma)\,\partial_\sigma b(\sigma)\right\} z^2 \\
&\quad + \frac{a(\sigma)}{2} + \frac{\rho\,\sigma\, b(\sigma)}{4} + \frac{1}{24}\,\frac{\rho^2\, b(\sigma)^2}{\sigma} + \frac{1}{12}\,\frac{b(\sigma)^2}{\sigma} - \frac{1}{6}\,\rho^2\, b(\sigma)\,\partial_\sigma b(\sigma)
\end{aligned}
\tag{60}
$$

We note that the limit of implied volatility as the log-strike $k \to 0$ and the time to expiration $\tau \to 0$ is just the instantaneous volatility σ. So although critics of stochastic volatility models love to point out that instantaneous volatility is not an observable and that this is a deficiency of such models, we see that this deficiency is not a major limitation in practice; in liquid option markets, the implied volatility surface is typically very smooth and we can extrapolate to the zero expiration, at-the-money strike limit with little uncertainty.

To compute the short-dated volatility skew, we substitute

$$z = \frac{k}{\sigma\,\sqrt{\tau}}$$

into (60) and take the limit $\tau \to 0$, to obtain

$$\left.\frac{\partial I}{\partial k}\right|_{k=0} \to \frac{\rho\, b(\sigma)}{2\,\sigma} \tag{61}$$

which is exactly consistent our with earlier result (58) derived using heuristic methods.

We note that the short-dated volatility skew is not explicitly time-dependent; it depends only on the form of the SDE for volatility.

In contrast, as we shall see, local volatility models imply short-dated skews that decay rapidly as time advances. So even if we find a stochastic volatility model and a local volatility model that price all European options identically today, forward-starting options (that is options whose strikes are to be set some time in the future) cannot possibly be priced identically by these two models. Both models fit the options market today but the volatility surface dynamics implied by the two models are quite different.

Equations (58) and (61) suggest a wild generalization: perhaps all stochastic volatility models, whether analytically tractable or not, have similar implications for the BS implied volatility skew up to a factor of $\beta(v)$. In Section 12.4, we will investigate the behavior of the volatility skew at long expirations and present further evidence that makes this claim more plausible.

12.3 Including Jumps

Medvedev and Scaillet's main result is a more complicated expression for models that include jumps in the stock price. The authors note that adding jumps in volatility would make the model more realistic but as we also noted earlier in Section 9.3, there is no contribution to the shape of the volatility surface from the jump in volatility for very short expirations.

Specifically, consider the stochastic volatility with jump model

$$\begin{aligned} \frac{dS_t}{S_t} &= \sigma_t\, dZ_1 + J(\sigma_t)\, dq_t \\ d\sigma_t &= a\,(\sigma_t)\, dt \,+\, b\,(\sigma_t)\; dZ_2 \end{aligned} \tag{62}$$

The jump term dq is a standard Poisson process with intensity $\lambda_J(\sigma_t)$ and $J(\sigma_t)$ is a $(-1,\infty)$-valued random variable with density f sampled at each jump. As before, the jump compensator μ_J is given by

$$\mu_J = \lambda_J \int_{-1}^{+\infty} f(x)\, dx$$

In this model, short-dated implied volatilities are given by

$$I(z,\tau;\sigma) = \sigma + \tilde{I}_1(z;\sigma)\,\sqrt{\tau} + \tilde{I}_2(z;\sigma)\,\tau + O(\tau\,\sqrt{\tau})$$

where $\tilde{I}_1$ and $\tilde{I}_2$ are given by

$$\begin{aligned} \tilde{I}_1(z;\sigma) &= I_1(z;\sigma) - \mu_J\, g(z) + \eta_J\, h(z) \\ \tilde{I}_2(z;\sigma) &= I_2(z;\sigma) + \frac{1}{2\,\sigma}\,\left(\mu_J\, g(z) - \eta_J\, h(z)\right)^2\, z^2 \\ &\quad - \left\{ -\frac{\mu_J\,\sigma}{2} - \sigma\,\lambda_J + \frac{\mu_J^2}{\sigma} + \frac{\mu_J\, b(\sigma)\,\rho}{2\,\sigma} \right\}\, g(z)\, z \\ &\quad - \left\{ \frac{\eta_J\,\sigma}{2} + \sigma\,\chi_J - \frac{\mu_J\,\eta_J}{\sigma} - \frac{\eta_J\, b(\sigma)\,\rho}{2\,\sigma} \right\}\, h(z)\, z \\ &\quad + \frac{\rho\, b(\sigma)\,\mu_J}{2\,\sigma} - \frac{\rho\,\partial_\sigma b(\sigma)\,\mu_J}{2} + \frac{\mu_J^2}{2\,\sigma} - \frac{\sigma\,\mu_J}{2} - \lambda_J\,\sigma \end{aligned} \tag{63}$$

where $\eta_J = \lambda_J \int_0^\infty x\, f(x)\, dx$, $\chi = \lambda_J \int_0^\infty f(x)\, dx$ are respectively the positive part of the jump compensator and the probability of an upwards jump and

$$g(z) = \frac{N(-z)}{N'(z)};\; h(z) = \frac{1}{N'(z)}$$

As expected, all jump-related terms (with subscript J) vanish if there are no jumps.

Corollary 12.2. *In a jump diffusion model (with volatility deterministic) the limit of the implied volatility skew as $\tau \to 0$ is given by*

$$\left.\frac{\partial I}{\partial k}\right|_{k=0} \to -\frac{\mu_J}{\sigma}$$

To get this result, note that $g'(0) = 1$ and $h'(0) = 0$. The result is exactly consistent with our earlier heuristic derivation in Section 8.6.2.

Corollary 12.3. *In the SVJ model, the limit of the implied volatility skew as $\tau \to 0$ is given by*

$$\left.\frac{\partial I}{\partial k}\right|_{k=0} \to \frac{\rho\, b(\sigma)}{2\,\sigma} - \frac{\mu_J}{\sigma}$$

This is consistent with our observation in Section 9.1 that the jump and stochastic volatility effects on the at-the-money variance skew are approximately additive. In fact we have

$$\left.\frac{\partial v_{BS}}{\partial k}\right|_{k=0} \to \rho\, b(\sigma) - 2\,\mu_J \text{ as } \tau \to 0$$

so they are exactly additive at $\tau = 0$!

12.4 Long Expirations: Fouque, Papanicolaou and Sircar

Fouque, Papanicolaou, and Sircar (1999) and Fouque, Papanicolaou, and Sircar (2000) show using a perturbation expansion approach that in any stochastic volatility model where volatility is mean-reverting, Black-Scholes implied volatility can be well approximated by a simple function of log-moneyness and time to expiration for long-dated options. In particular, they study a model where the log-volatility is a Orenstein-Uhlenbeck process (log-OU for short). That is:

$$\begin{aligned} dx &= -\frac{\sigma^2}{2}\,dt + \sigma\,dZ_1 \\ d\log(\sigma) &= -\lambda[\,\log(\sigma) - \overline{\log(\sigma)}\,]dt + \xi\,dZ_2 \end{aligned}$$

They find that the slope of the BS implied volatility skew is given (for large λT) by

$$\frac{\partial}{\partial x}\,\sigma_{BS}(x,T) \approx \frac{\rho\,\xi}{\lambda\,T} \tag{64}$$

To recast this in terms of v to be consistent with the form of the generic process we wrote down in equation (56), we note that (considering random terms only), $dv \sim 2\,\sigma\,d\sigma$ and in the log-OU model,

$$d\sigma \sim \xi\,\sigma dZ_2$$

So

$$dv \sim 2\,\xi\,v\,dZ_2$$

Then $\beta(v)$ as defined in equation (56) is given by

$$\eta\,\beta(v) = 2\xi\,\sqrt{v}$$

and, from equation (64), the BS implied variance skew is given by

$$\frac{\partial}{\partial x}\,\sigma_{BS}(x,T)^2 \approx \frac{2\,\rho\,\xi\,\sqrt{v}}{\lambda\,T} = \frac{\rho\,\eta\,\beta(v)}{\lambda\,T}$$

Looking back at section 5.2 again, we see that the Heston skew (where $\beta(v) = 1$) has the same behavior for large $\lambda\,T$. We now have enough evidence to make our generalization more plausible: it seems that both for long and short expirations, the skew behavior may be identical for all stochastic volatility models up to a factor of $\beta(v)$. Supposing this claim were true, what would be the natural way to interpolate the asymptotic skew behaviors between long and short expirations?

Clearly, the most plausible interpolation function between short expiration and long expiration volatility skews is the one we already derived for the Heston model in Section 5.2 and

$$\frac{\partial}{\partial x}\,\sigma_{BS}(x,T)^2 \approx \frac{\rho\,\eta\,\beta(v)}{\lambda'\,T}\left\{1 - \frac{\left(1-e^{-\lambda' T}\right)}{\lambda'\,T}\right\} \tag{65}$$

with $\lambda' = \lambda - \frac{1}{2}\,\rho\,\eta\,\beta(v)$.

12.5 Small Volatility of Volatility: Lewis

Lewis (2000) performs perturbation expansions of implied volatility with respect to the volatility of volatility parameter (assumed small) in any stochastic volatility model of the form (56) for general choices of $\beta(v)$.

According to equation (3.14) on page 143, we have

$$v_{BS}(k,t) = \beta_0(v,t) + \beta_1(v,t)\,k + \beta_2(v,t)\,k^2 + O(\eta)^3$$

where

$$\begin{aligned}
\beta_0(v,t) &= v + \frac{1}{2}\frac{\eta}{t}\,J^{(1)} + \\
&\quad \eta^2\left[\frac{J^{(2)}}{t} - \frac{1}{2}\frac{J^{(3)}}{v\,t^2}\left(1+\frac{1}{4}\,v\,t\right) - \frac{J^{(4)}}{v\,t^2}\left(1-\frac{1}{4}\,v\,t\right) + \frac{(J^{(1)})^2}{v^2\,t^3}\left(\frac{3}{4}+\frac{1}{16}\,v\,t\right)\right] \\
\beta_1(v,t) &= \frac{\eta}{vt^2}\,J^{(1)} + \eta^2\left[-\frac{J^{(4)}}{v\,t^2} - \frac{(J^{(1)})^2}{v^2\,t^3}\right] \\
\beta_2(v,t) &= \eta^2\left[\frac{1}{2}\frac{J^{(3)}}{v^2\,t^3} + \frac{J^{(4)}}{v^2\,t^3} - \frac{5}{4}\frac{(J^{(1)})^2}{v^3\,t^4}\right]
\end{aligned} \qquad (66)$$

Example 4 on page 144 deals with the case of most interest to us:

$$dv = -\lambda\,(v-\bar{v})\,dt + \eta\;v^{\phi}\,dZ$$

For this volatility process, in the special case $v=\bar{v}$, we have

$$\begin{aligned}
J^{(1)} &= \bar{v}^{1/2+\phi}t\frac{\rho}{\lambda}\left\{1-\frac{1-e^{-\lambda t}}{\lambda t}\right\} \\
J^{(3)} &= \bar{v}^{2\phi}\frac{\rho}{2\lambda^3}\left\{\frac{3}{2}+\lambda t + 2e^{-\lambda t} - \frac{1}{2}e^{-2\lambda t}\right\} \\
J^{(4)} &= \bar{v}^{2\phi}\frac{\rho^2}{\lambda^3}\left(\frac{1}{2}+\phi\right)\left\{-2+\lambda t + (2+\lambda t)\,e^{-\lambda t}\right\}
\end{aligned}$$

Substituting back into equation (66) gives

$$\left.\frac{\partial v_{BS}}{\partial k}\right|_{k=0} = \frac{\rho\,\eta\,v^{\phi-1/2}}{\lambda\,t}\left\{1-\frac{1-e^{-\lambda t}}{\lambda t}\right\}$$

and we see that (65) is not only plausible but is exactly correct to first order in the volatility of volatility η.

12.6 Extreme Strikes: Roger Lee

In a beautiful paper, Roger Lee (Lee 2004) shows that implied variance is linear in the log-strike $k = \ln(K/F)$ as $|k| \to \infty$. Moreover, he shows how to relate the gradients of the wings of the implied variance skew to the maximal finite moments of the underlying process.

Specifically, let $q^* := \sup \left\{ q : \mathbb{E}\, S_T^{-q} < \infty \right\}$ and

$$\beta^* := \limsup_{k \to -\infty} \frac{\sigma_{\mathrm{BS}}^2(k,T)\,T}{\mathrm{k}}$$

Then $\beta^* \in [0,2]$ and

$$q^* = \frac{1}{2}\left(\frac{1}{\sqrt{\beta^*}} - \frac{\sqrt{\beta^*}}{2} \right)^2$$

Also let $p^* := \sup \left\{ p : \mathbb{E}\, S_T^{1+p} < \infty \right\}$ and

$$\alpha^* := \limsup_{k \to +\infty} \frac{\sigma_{\mathrm{BS}}^2(k,T)\,T}{k}$$

Then $\alpha^* \in [0,2]$ and

$$p^* = \frac{1}{2}\left(\frac{1}{\sqrt{\alpha^*}} - \frac{\sqrt{\alpha^*}}{2} \right)^2$$

Lee's derivation assumes only the existence of a Martingale measure: it makes no assumptions on the distribution of underlying returns. His result is completely model-independent.

So, not only is the implied variance surface linear in k for $|k|$ large but the slope of the wings may be computed explicitly in terms of model parameters if the moments can be explicitly computed. This is certainly the case when the characteristic function is known in closed form as in the Heston model for example.

12.7 Asymptotics in Summary

It's quite clear from the results presented here that the general shape of the volatility surface doesn't depend very much on the specific choice of model. Any stochastic volatility with jump model should generate a similar shape of volatility surface with appropriate numerical choices of the parameters.

13 Dynamics of the Volatility Surface

13.1 Dynamics of the Volatility Skew under Stochastic Volatility

At first it might seem that a result that says that all stochastic volatility models have essentially the same implications for the shape of the volatility surface would it make it hard to differentiate between models. That would certainly be the case if we were to confine our attention to the shape of the volatility surface today. However, if instead we were to study the dynamics of the volatility skew – in particular, how the observed volatility skew depends on the overall level of volatility, we would be able to differentiate between models.

Empirical studies of the dynamics of the volatility skew show that $\frac{\partial}{\partial k}\sigma(k,t)$ is approximately independent of volatility level over time. Translating this into a statement about the implied variance skew, we get

$$\frac{\partial}{\partial k}\,\sigma_{BS}(k,t)^2 \;=\; 2\,\sigma_{BS}(k,t)\,\frac{\partial}{\partial k}\,\sigma_{BS}(k,t) \;\sim\; \sqrt{v(k,t)}.$$

This in turn implies that $\beta(v) \sim \sqrt{v}$ and that v is approximately lognormal in contrast to the square root process assumed by Heston. This makes intuitive sense given that we would expect volatility to be more volatile if the volatility level is high than if the volatility level itself is low.

Does it matter whether we model variance as a square root process or as lognormal? In certain cases it does. After all, we are using our model to hedge and the hedge should approximately generate the correct payoff at the boundary. If the payoff that we are hedging depends (directly or indirectly) on the volatility skew, and our assumption is that the variance skew is independent of the volatility level, we could end up losing a lot of money if that's not how the market actually behaves.

Is any stochastic volatility model better than none at all? The answer here has to be yes because whereas having the wrong stochastic volatility model will cause the hedger to generate a payoff corresponding to a skew that may perhaps be off by a factor of 1.5 if volatility doubles, having only a local volatility model will cause the hedger to generate a payoff that corresponds to almost no forward skew at all. We will now show this.

13.2 Dynamics of the Volatility Skew under Local Volatility

Empirically, the slope of the volatility skew decreases with time to expiration. From the above, in the case of mean-reverting stochastic volatility, the term structure of the BS implied variance skew will look something like equation (65). In particular, the slope of the volatility skew will decay over time according to the time behavior of the coefficient

$$\frac{1}{\lambda' T}\left\{1-\frac{\left(1-e^{-\lambda' T}\right)}{\lambda' T}\right\}.$$

Recall from Section 2.3 the formula for local volatility in terms of implied volatility:

$$v_{loc} = \frac{\frac{\partial w}{\partial T}}{1-\frac{k}{w}\frac{\partial w}{\partial k}+\frac{1}{4}\left(-\frac{1}{4}-\frac{1}{w}+\frac{k^2}{w^2}\right)\left(\frac{\partial w}{\partial k}\right)^2+\frac{1}{2}\frac{\partial^2 w}{\partial k^2}}$$

Differentiating with respect to x and considering only the leading term in $\frac{\partial w}{\partial k}$ (which is small for large T) , we find

$$\frac{\partial v_{loc}}{\partial k} \approx \frac{\partial}{\partial T}\frac{\partial w}{\partial k} + \frac{1}{w}\frac{\partial w}{\partial T}\frac{\partial w}{\partial k}$$

That is, the local variance skew $\frac{\partial v_{loc}}{\partial k}$ decays with the BS implied total variance skew $\frac{\partial w}{\partial k}$.

To get the forward volatility surface from the local volatility surface in a local volatility model, we integrate over the local volatilities from the (forward) valuation date to the expiration of the option along the most probable path joining the current stock price to the strike price using the trick presented in Section 5.2. It is obvious that the forward implied volatility surface will be substantially flatter than today's because the forward local volatility skews are all flatter.

Contrast this with a stochastic volatility model where implied volatility skews are approximately time-homogeneous. In other words, local volatility models imply that future BS implied volatility surfaces will be flat (relative to today's) and stochastic volatility models imply that future BS implied volatility surfaces will look like today's.

13.3 Stochastic Implied Volatility Models

Many authors including Brace, Goldys, Klebaner, and Womersley (2001), Ledoit, Santa-Clara, and Yan (2002) and Schönbucher (1999) have looked at models that allow the entire implied volatility surface to diffuse. It turns out that the statics and dynamics of the implied volatility surface in these models are highly constrained.

In particular, non-discounted option prices are risk-neutral expectations of future cashflows and as such must be martingales. Changes in the call price reflect changes in the underlying and changes in implied volatility. Imposing the martingale constraint

$$\mathbb{E}[dC_t] = 0$$

gives a tight relationship between the various sensitivities and many results such as equation (58) follow immediately from this.

14 Digital Options and Digital Cliquets

Applying our insights to the valuation of actual derivative contracts, we choose to study digital options because their valuation involves the volatility skew directly.

14.1 Valuing Digital Options

A digital (call) option $D(K,T)$ pays 1 if the stock price S_T at expiration T is greater than the strike price K and zero otherwise. It may be valued as the limit of a call spread as the spread between the strikes is reduced to zero.

$$D(K,T) = -\frac{\partial C(K,T)}{\partial K} \tag{67}$$

where $C(K,T)$ represents the price of a European call option with strike K expiring at time T.

To see that its price is very sensitive to the volatility skew, we rewrite the European call price in equation (67) in terms of its Black-Scholes implied volatility $\sigma_{BS}(K,T)$.

$$\begin{aligned} D(K,T) &= -\frac{\partial}{\partial K} C_{BS}\left(K,T,\sigma_{BS}(K,T)\right) \\ &= -\frac{\partial C_{BS}}{\partial K} - \frac{\partial C_{BS}}{\partial \sigma_{BS}} \frac{\partial \sigma_{BS}}{\partial K} \end{aligned}$$

To get an idea of the impact of the skew in practice, consider our usual idealized market with zero interest rate and dividends and a one year digital option struck at-the-money. Suppose further that at-the-money volatility is 25% and the volatility skew (typical of SPX for example) is 3% per 10% change in strike. Its value is given by:

$$\begin{aligned} D(1,1) &= -\frac{\partial C_{BS}}{\partial K} - \frac{\partial C_{BS}}{\partial \sigma_{BS}} \frac{\partial \sigma_{BS}}{\partial K} \\ &= N\left(-\frac{\sigma}{2}\right) - \text{vega} \times \text{skew} \\ &= N\left(-\frac{\sigma}{2}\right) + \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \times 0.3 \\ &\approx N\left(-\frac{\sigma}{2}\right) + 0.4 \times 0.3 \end{aligned}$$

If we had ignored the skew contribution, we would have got the price of the digital option wrong by 12% of notional!

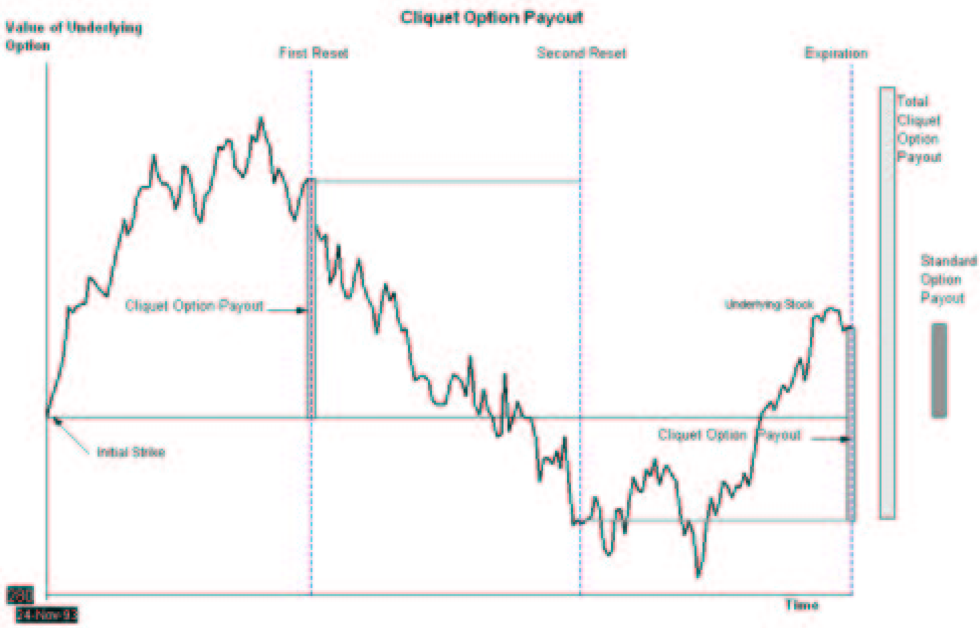
14.2 Digital Cliquets

For an example of an actual digital cliquet contract, see Appendix A. Here is a description of the Cliquet from the Dictionary of Financial Risk Management at http://www.amex.com:

"The French like the sound of 'cliquet' and seem prepared to apply the term to any remotely appropriate option structure. (1) Originally a periodic reset option with multiple payouts or a ratchet option (from vilbrequin à cliquet – ratchet brace). Also called Ratchet Option. See Multi-period Strike Reset Option (MSRO), Stock Market Annual Reset Term (SMART) Note. See also Coupon Indexed Note. (2) See Ladder Option or Note (diagram). Also called Lock-Step Option. See also Stock Upside Note Security (SUNS). (3) Less commonly, a rolling spread with strike price resets, usually at regular intervals. (4) An exploding or knockout option such as CAPS (from cliqueter – to knock)."

A cliquet payoff diagram is shown in Figure 2.

Figure 2: Illustration of a Cliquet Payoff.



For our purposes, a cliquet is just a series of options whose strikes are set on a sequence of futures dates. In particular, a digital cliquet is a sequence of digital options whose strikes will be set (usually) at the prevailing stock price on the relevant reset date. Denoting the set of reset dates by $\{t_1, t_2, ..., t_n\}$, the digital cliquet pays Coupon $\times\, \theta\big(\mathrm{S}_{\mathrm{t_i}} - \mathrm{S}_{\mathrm{t_{i-1}}}\big)$ at t_i where $\theta(.)$ represents the Heaviside function.

One can see immediately that the package consisting of a zero coupon bond together with a digital cliquet makes a very natural product for a risk-averse retail investor – he typically gets an above market coupon if the underlying stock index is up for the period (usually a year) and a below market coupon (usually zero) if the underlying stock index is down. Not surprisingly, this product was and is very popular and as a result, many equity derivatives dealers have digital cliquets on their books.

From the foregoing, the price of a digital cliquet may vary very substantially depending on the modelling assumptions made by the seller. Those sellers using local volatility models will certainly value a digital cliquet at a lower price than sellers using a stochastic volatility (or more practically, those guessing that the forward skew should look like today's). Perversely then, those sellers using an inadequate model will almost certainly win the deal and end up short a portfolio of misvalued forward-starting digital options. Or even worse, a dealer could have an appropriate valuation approach but be pushed internally by the salespeople to match (mistaken) competitor's lower prices. The homework assignment deals with exactly this set of circumstances.

How wrong could the price of the digital cliquet be? Taking the example of the deal documented in the addendum, neglecting the first coupon (because we suppose that all dealers can price a digital which sets today), the error could be up to 12% of the sum of the remaining coupons (52%) or 6.24% of Notional. In the actual deal, the digital are struck out-of-themoney and interest rates and dividends are not zero. Nevertheless, a pricing error of this magnitude is a big multiple of the typical margin on such a trade and would cause the dealer a substantial loss.

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A An Example of a Cliquet Contract

Merrill Lynch

Confirmation of OTC Swap Transaction

Dated : February 23, 2000 **ML Ref : Based on a Real Document**

To :	**Banca Sbagliata** **Via Dolorosa** **Roma, Italia**
Attention :	**Dottore Michele Angelo** Telephone: **(39) 69-69-69**
From :	**Merrill Lynch International (“ML” and “MLI”)** tel: (212) 123-4567 fax: (212) 123-4567

Dear Sir / Madam,

The purpose of this letter agreement (this “Confirmation”) is to confirm the terms and conditions of the above referenced transaction entered into between Counterparty and ML on the Trade Date specified below (the “Transaction”). This Confirmation constitutes a “Confirmation” as referred to in the ISDA Master Agreement specified below.

The definitions and provisions contained in the 1991 ISDA Definitions (the “Swap Definitions”) and the 1996 ISDA Equity Derivatives Definitions (the “Equity Definitions” and together with the Swap Definitions, the “Definitions”) in each case as published by the International Swaps and Derivatives Association, Inc., are incorporated into this Confirmation. In the event of any inconsistency between the Swap Definitions and the Equity Definitions, the Equity Definitions will govern and in the event of any inconsistency between either the Swap Definitions, the Equity Definitions and this Confirmation, this Confirmation will govern.

This Confirmation evidences a complete binding agreement between you and us as to the terms of the Transaction to which this Confirmation relates. In addition, you and we agree to use all reasonable efforts promptly to negotiate, execute and deliver an agreement in the form of the ISDA Master Agreement (Multicurrency-Cross Border) (the “ISDA Form”), with such modifications as you and we will in good faith agree. Upon the execution by you and us of such an agreement, this Confirmation will supplement, form a part of, and be subject to that agreement and Merrill Lynch & Co., Inc. will deliver a guarantee of ML’s obligations thereunder. All provisions contained or incorporated by reference in that agreement upon its execution will govern this Confirmation except as expressly modified below. Until we execute and deliver that agreement, this Confirmation together with all other documents referring to the ISDA Form (each a “Confirmation”) confirming transactions (each a “Transaction”) entered into between us (notwithstanding anything to the contrary in a Confirmation), shall supplement, form a part of, and be subject to an agreement in the form of the ISDA Form as if we had executed an agreement in such form (but without any Schedule) on the Trade Date of the first such Transaction between us. In the event of any inconsistency between the provisions of that agreement and this Confirmation, this Confirmation will prevail for the purpose of this Transaction.

This Confirmation supplements, forms part of, and is subject to, the ISDA Master Agreement dated as of 01 January 2000, as amended and supplemented from time to time (the “Agreement”), between you and us, with the obligations of ML under the Agreement guaranteed by Merrill Lynch & Co., Inc. All provisions contained in the Agreement govern this Confirmation except as expressly modified below.

The terms of the particular Transaction to which this Confirmation relates are as follows:

General Terms:

Trade Date:	**February 23, 2000**
Effective Date:	**February 25, 2000**
Termination Date:	Two Currency Business Days following the final Valuation Date
Index:	New World Index
Notional Amout:	EUR 100,000,000
Exchange(s):	Brussels
Related Exchange(s):	Any exchange(s) on which futures and/or options contracts related to the Index is principally traded.
Business Day Convention:	Following

Amounts payable by: Merrill Lynch

Upfront Fee: EUR 2,000,000 being 2% of the Notional Amount

Upfront Fee Payment Date: February 25, 2000

Maturity Payment: MLI shall pay to Banca Sbagliata on the Termination Date an amount as determined by the Calculation Agent in accordance with the following:

Notional Amount multiplied by the sum of the Percentage Levels from the table below for each time that the level of the Index as at the Valuation Time on the relevant Valuation Date, with the exception of the first Valuation Date, is greater than or equal to 100% of the level of the Index as at the Valuation Time on the immediately preceding Valuation Date.

Number	Percentage Level
First	5%
Second	7%
Third	8%
Fourth	10%
Fifth	9%

Initial Index Level: 1,000.00

Initial Strike Price: 1,000.00 being 100% of the Initial Index Level.

Business Days: Rome, Brussels

Valuation Terms:

Valuation Time: The close of trading on the Exchange

Valuation Dates: **February 23 in each year commencing 2000 and ending 2005** and the Termination Date subject to adjustment in accordance with the Following Business Day Convention.

Amounts Payable by Banca Sbagliata:

Payment Date(s): The second Currency Business Day following each Valuation Date.

Fixed Rate: **5.35%** (exclusive of spread) being the EUR 5 year mid-rate as displayed on Reuters page ICAPEURO on February 23, 2000 plus Spread.

Spread: **minus 100** basis points

Fixed Rate Day Count: 30/360

Business Days: Rome, Brussels

Calculation Agent: Merrill Lynch

Account Details:

Account for payments to Counterparty: Please advise

Account for payment to ML: First National Bank, Sioux City
ABA# 031200029
FAO: ML Equity Derivatives
A/C: 02232000

Non-Reliance: Each party represents to the other party that it is acting for its own account, and has made its own independent decisions to enter into this Transaction and as to whether this Transaction is appropriate or proper for it based on its own judgment and upon advice from such advisors as it has deemed necessary. It is not relying on any communication (written or oral) of the other party as investment advice or as a recommendation to enter into this Transaction, it being understood that information and explanations related to the terms and conditions of this Transaction shall not be considered investment advice or a recommendation to enter into this Transaction. No communication (written or oral) received from the other party shall be deemed to be an assurance or guarantee as to the expected results of this Transaction.

Transfer: Neither party may transfer its rights or obligations under this Transaction except in accordance with Section 7 of the Master Agreement; *provided however* that ML may assign its rights and delegate its obligations hereunder, in whole or in part, to any affiliate (an "Assignee") of Merrill Lynch & Co., Inc. ("ML&Co."), effective (the "Transfer Effective Date") upon delivery to Counterparty of (a) an executed acceptance and assumption by the Assignee (an "Assumption") of the transferred obligations of ML under this Transaction (the "Transferred Obligations"); (b) and an executed guarantee (the "Guarantee") of ML&Co. of the Transferred Obligations. On the Transfer Effective Date, (a) ML shall be released from all obligations and liabilities arising under the Transferred Obligations; and (b) the Transferred Obligations shall cease to be a Transaction(s) under the Agreement and shall be deemed to be a Transaction(s) under the ISDA Master Agreement between Assignee and Counterparty, provided that, if at such time Assignee and Counterparty have not entered into a ISDA Master Agreement, Assignee and Counterparty shall be deemed to have entered into an ISDA form of Master Agreement (Multicurrency-Cross Border) without any Schedule attached thereto.

Governing law: Unless otherwise provided in the Agreement (in which case the law so specified shall govern), this Confirmation shall be governed by and construed in accordance with the laws of the State of New York.

ML is regulated by The Securities and Futures Authority Limited and has entered into this Transaction as principal.

Please confirm that the foregoing correctly sets forth the terms of our agreement by executing the copy of this Confirmation enclosed for that purpose and returning it to us by facsimile transmission to the Attention of: Jim Gatheral (Telecopier No. 212 123-4567).

Very truly yours,

Merrill Lynch International

By:

____

Name:
Title:

Confirmed as of the date first above written:

Banca Sbagliata

By: ______________________________
Name:
Title:

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