Lecture 6: Volatility and Variance Swaps

Jim Gatheral, Merrill Lynch*

Case Studies in Financial Modelling Course Notes,
Courant Institute of Mathematical Sciences,
Fall Term, 2002

*I am indebted to Peter Friz for carefully reading these notes, providing corrections and suggesting useful improvements.

12 Spanning Generalized European Payoffs

As usual, we assume that European options with all possible strikes and expirations are traded. In the spirit of the paper by Carr and Madan (1998), we now show that any twice-differentiable payoff at time T may be statically hedged using a portfolio of European options expiring at time T.

From Breeden and Litzenberger (1978), we know that we may write the *pdf* of the stock price S_T at time T as

$$p(S_T, T; S_t, t) = \left. \frac{\partial^2 \tilde{C}(S_t, K, t, T)}{\partial K^2} \right|_{K=S_T} = \left. \frac{\partial^2 \tilde{P}(S_t, K, t, T)}{\partial K^2} \right|_{K=S_T}$$

where $\tilde{C}$ and $\tilde{P}$ represent undiscounted call and put prices respectively.

Then, the value of a claim with a generalized payoff $g(S_T)$ at time T is given by

$$\begin{aligned} \mathbf{E}\left[g(S_T) | \, S_t \right] &= \int_0^\infty dK \, p(K, T; S_t, t) \, g(K) \\ &= \int_0^F dK \, \frac{\partial^2 \tilde{P}}{\partial K^2} \, g(K) + \int_F^\infty dK \, \frac{\partial^2 \tilde{C}}{\partial K^2} \, g(K) \end{aligned}$$

where F represents the time-T forward price of the stock. Integrating by parts twice and using the put-call parity relation $\tilde{C}(K) - \tilde{P}(K) = F - K$ gives

$$\begin{aligned} \mathbf{E}\left[g(S_T) | \, S_t \right] &= \left. \frac{\partial \tilde{P}}{\partial K} \, g(K) \right|_0^F - \int_0^F dK \, \frac{\partial \tilde{P}}{\partial K} \, g'(K) + \left. \frac{\partial \tilde{C}}{\partial K} \, g(K) \right|_F^\infty - \int_F^\infty dK \, \frac{\partial \tilde{C}}{\partial K} \, g'(K) \\ &= g(F) - \int_0^F dK \, \frac{\partial \tilde{P}}{\partial K} \, g'(K) - \int_F^\infty dK \, \frac{\partial \tilde{C}}{\partial K} \, g'(K) \\ &= g(F) - \left. \tilde{P}(K) g'(K) \right|_0^F + \int_0^F dK \, \tilde{P}(K) \, g''(K) \\ &\quad - \left. \tilde{C}(K) g'(K) \right|_F^\infty + \int_F^\infty dK \, \tilde{C}(K) \, g''(K) \\ &= g(F) + \int_0^F dK \, \tilde{P}(K) \, g''(K) + \int_F^\infty dK \, \tilde{C}(K) \, g''(K) \end{aligned} \tag{59}$$

By letting $t \to T$ in equation (59), we see that any European-style twice-differentiable payoff may be replicated using a portfolio of European options with strikes from 0 to ∞ with the weight of each option equal to the second derivative of the payoff at the strike price of the option. This portfolio of European options is a static hedge because the weight of an option with a particular strike depends only on the strike price and the form of the payoff function and not on time or the level of the stock price. Note further that equation (59) is *completely model-independent*.

Example: European Options

In fact, using Dirac delta-functions, we can extend the above result to payoffs which are not twice-differentiable. Consider for example the portfolio of options required to hedge a single call option with payoff $(S_T - L)^+$. In this case $g''(K) = \delta(K - L)$ and equation (59) gives

$$\begin{aligned}
\mathbf{E}\left[(S_T - L)^+\right] &= (F - L)^+ + \int_0^F dK\,\tilde{P}(K)\,\delta(K - L) + \int_F^\infty dK\,\tilde{C}(K)\,\delta(K - L) \\
&= \begin{cases} (F - L) + \tilde{P}(L) & \text{if} \quad L < F \\ \tilde{C}(L) & \text{if} \quad L \geq F \end{cases} \\
&= \tilde{C}(L)
\end{aligned}$$

with the last step following from put-call parity as before. In other words, the replicating portfolio for a European option is just the option itself.

Example: Amortizing Options

A useful variation on the payoff of the standard European option is given by the amortizing option with strike L with payoff

$$g(S_T) = \frac{(S_T - L)^+}{S_T}$$

Such options look particularly attractive when the volatility of the underlying stock is very high and the price of a standard European option is prohibitive . The payoff is effectively that of a European option whose notional amount declines as the option goes in-the-money. Then,

$$g''(K) = \left.\left\{-\frac{2L}{{S_T}^3}\theta(S_T - L) + \frac{\delta(S_T - L)}{S_T}\right\}\right|_{S_T = K}$$

Without loss of generality (but to make things easier), suppose $L > F$. Then substituting into equation (59) gives

$$\begin{aligned}\mathbf{E}\left[\frac{(S_T - L)^+}{S_T}\right] &= \int_F^\infty dK\, \tilde{C}(K)\, g''(K) \\ &= \frac{\tilde{C}(L)}{L} - 2L\int_L^\infty \frac{dK}{K^3}\, \tilde{C}(K)\end{aligned}$$

and we see that an Amortizing call option struck at L is equivalent to a European call option struck at L minus an infinite strip of European call options with strikes from L to ∞.

12.1 The Log Contract

Now consider a contract whose payoff at time T is $\log(\frac{S_T}{F})$. Then $g''(K) = -\left.\frac{1}{{S_T}^2}\right|_{S_T=K}$ and it follows from equation (59) that

$$\mathbf{E}\left[\log\left(\frac{S_T}{F}\right)\right] = -\int_0^F \frac{dK}{K^2}\, \tilde{P}(K) - \int_F^\infty \frac{dK}{K^2}\, \tilde{C}(K)$$

Rewriting this equation in terms of the log-strike variable $k \equiv \log\left(\frac{K}{F}\right)$, we get the promising-looking expression

$$\mathbf{E}\left[\log\left(\frac{S_T}{F}\right)\right] = -\int_{-\infty}^0 dk\, p(k) - \int_0^\infty dk\, c(k) \tag{60}$$

with $c(y) \equiv \frac{\tilde{C}(Fe^y)}{Fe^y}$ and $p(y) \equiv \frac{\tilde{P}(Fe^y)}{Fe^y}$ representing option prices expressed in terms of percentage of the strike price.

13 Variance and Volatility Swaps

We now revert to our usual assumption of zero interest rates and dividends. In this case, $F = S_0$ and applying Itô's Lemma, path-by-path

$$\begin{aligned}\log\left(\frac{S_T}{F}\right) &= \log\left(\frac{S_T}{S_0}\right) \\ &= \int_0^T d\log(S_t) \\ &= \int_0^T \frac{dS_t}{S_t} - \int_0^T \frac{{\sigma_{S_t}}^2}{2} dt\end{aligned} \tag{61}$$

The second term on the r.h.s of equation (61) is immediately recognizable as half the total variance W_T over the period $\{0, T\}$. The first term on the r.h.s represents the payoff of a hedging strategy which involves maintaining a constant dollar amount in stock (if the stock price increases, sell stock; if the stock price decreases, buy stock so as to maintain a constant dollar value of stock). Since the log payoff on the l.h.s can be hedged using a portfolio of European options as noted earlier, it follows that the total variance W_T may be replicated in a completely model-independent way so long as the stock price process is a diffusion. In particular, volatility may be stochastic or deterministic and equation (61) still applies.

Now taking the risk-neutral expectation of (61) and comparing with equation (60), we obtain

$$\mathbf{E}\left[\int_0^T \sigma_{S_t}^2 dt\right] = 2\,\mathbf{E}\left[\log\left(\frac{S_T}{F}\right)\right] = 2\left\{\int_{-\infty}^0 dk\, p(k) \;+\; \int_0^\infty dk\, c(k)\right\}$$

We see explicitly that the fair value of total variance is given by the value of an infinite strip of European options in a completely *model-independent* way so long as the underlying process is a diffusion.

13.1 Variance Swaps

Although variance and volatility swaps are relatively recent innovations, there is already a significant literature describing these contracts and the practicalities of hedging them including articles by Chriss and Morokoff (1999) and Demeterfi, Derman, Kamal, and Zou (1999).

In fact, a variance swap is not really a swap at all but a forward contract on the realized annualized variance. The payoff at time T is

$$N \times A \times \left\{\frac{1}{N}\sum_{i=1}^{N}\left\{\log\left(\frac{S_i}{S_{i-1}}\right)\right\}^2 \;-\; \left\{\frac{1}{N}\log\left(\frac{S_N}{S_0}\right)\right\}^2\right\} \;-\; N \times K_{var}$$

where N is the notional amount of the swap, A is the annualization factor and K_{var} is the strike price. Annualized variance may or may not be defined as mean-adjusted in practice so the corresponding drift term in the above payoff may or may not appear.

From a theoretical perspective, the beauty of a variance swap is that it may be replicated perfectly assuming a diffusion process for the stock price as shown in the previous section. From a practical perspective, market operators may express views on volatility using variance swaps without having to delta hedge.

Variance swaps took off as a product in the aftermath of the LTCM meltdown in late 1998 when implied stock index volatility levels rose to unprecedented levels. Hedge funds took advantage of this by paying variance in swaps (selling the realized volatility at high implied levels). The key to their willingness to pay on a variance swap rather than sell options was that a variance swap is a pure play on realized volatility – no labor-intensive delta hedging or other path dependency is involved. Dealers were happy to buy vega at these high levels because they were structurally short vega (in the aggregate) through sales of guaranteed equity-linked investments to retail investors and were getting badly hurt by high implied volatility levels.

13.2 Variance Swaps in the Heston Model

Recall that in the Heston model, instantaneous variance v follows the process:

$$dv(t) = -\lambda(v(t) - \bar{v})dt + \eta\sqrt{v(t)}\,dZ$$

It follows that the expectation of the total variance W_T is given by

$$\begin{aligned}
\mathbf{E}[W_T] &= \mathbf{E}\left[\int_0^T v_t\,dt\right] \\
&= \int_0^T \hat{v}_t\,dt \\
&= \frac{1 - e^{-\lambda T}}{\lambda}(v - \bar{v}) + \bar{v}T
\end{aligned}$$

The expected annualized variance V_T is given by

$$V_T \equiv \frac{1}{T}\mathbf{E}[W_T] = \frac{1 - e^{-\lambda T}}{\lambda T}(v - \bar{v}) + \bar{v}$$

We see that the expected variance in the Heston model depends only on v, $\bar{v}$ and λ. It does not depend on the volatility of volatility η. Since the value of a variance swap depends only on the prices of European options, it follows that a variance swap would be priced identically by both Heston and our local volatility approximation to Heston.

13.3 Dependence on Skew and Curvature

We know that the implied volatility of an at-the-money forward option in the Heston model is lower than the square root of the expected variance (just think of the shape of the implied distribution of the final stock price in Heston). In practice, we start with a strip of European options of a given expiration and we would like to know how we should expect the price of a variance swap to relate to the at-the-money-forward implied volatility, the volatility skew and the volatility curvature (smile).

It turns out that there is a very elegant exact expression for the fair value of variance (which we will not prove here). Define

$$z(k) = d_2 = -\frac{k}{\sigma_{BS}(k)\sqrt{T}} + \frac{\sigma_{BS}(k)\sqrt{T}}{2}$$

Intuitively, z measures the log-moneyness of an option in implied standard deviations. Then,

$$\mathbf{E}[W_T] = \int_{-\infty}^{\infty} dz\, N'(z)\sigma_{BS}^2(z)T \tag{62}$$

To see this formula is plausible, it is obviously correct when there is no volatility skew.

Now consider the following simple parameterization of the BS implied variance skew:

$$\sigma_{BS}^2(z) = \sigma_0^2 + \alpha\, z + \beta\, z^2$$

Substituting into equation (62) and integrating, we obtain

$$\mathbf{E}[W_T] = \sigma_0^2 T + \beta T$$

We see that skew makes no contribution to this expression, only the curvature contributes. The intuition for this is simply that increasing the skew does not change the average level of volatility but increasing the curvature β increases the prices of puts and calls in equation (60) and always increases the fair value of variance.

13.4 The Effect of Jumps

Suppose we have some continuous process and add a jump of size ϵ that occurs between times t_i and t_{i+1}. Total variance is computed as a sum of terms of the form

$$\left(\log\left(\frac{S_{i+1}}{S_i}\right)\right)^2$$

The jump will change this particular term in the sum to

$$\left(\log\left(\frac{S_{i+1}e^{\epsilon}}{S_i}\right)\right)^2$$

The net impact of the jump on the total variance for this realization of the process is then given by

$$\epsilon^2 + 2\epsilon \,\log\left(\frac{S_{i+1}}{S_i}\right)$$

which in expectation is just $\mathbf{E}[\epsilon^2]$. So, assuming that jumps are independent of the underlying continuous process, to get the effect of jumps on the expected total variance, we need only figure out the expected number of jumps in the interval $(0,T]$ and their average squared magnitude.

In the Merton and SVJ models, jumps are generated by a Poisson process with parameter λ and the size of the jump is lognormally distributed with mean α and standard deviation δ. In this case, the expected number of jumps is just λT and the average squared magnitude is given by $\alpha^2+\delta^2$. The expected total variance $\mathbf{E}[W_T]$ is then given by

$$\mathbf{E}[W_T] = \mathbf{E}[W_T]^c + \lambda T\left(\alpha^2+\delta^2\right)$$

where $\mathbf{E}[W_T^c]$ denotes the expected variance of the same process without jumps. This formula tells us in absolute terms how the value of a variance swap would change but not in terms of the European option hedge whose value obviously must also change when we add jumps to the underlying process.

To see how the value of the variance swap relates to the value of the log-contract (the infinite strip of European options), we note that from the definition of the characteristic function

$$\mathbf{E}\left[\log\left(\frac{S_T}{F}\right)\right] = -i\,\left.\frac{\partial}{\partial u}\phi_T(u)\right|_{u=0}$$

Also, recall that if jumps are independent of the continuous process as they are in both the Merton and SVJ models, the characteristic function may be written as the product of a continuous part and a jump part

$$\phi_T(u) = \phi_T^c(u)\,\phi_T^j(u)$$

Then

$$\mathbf{E}\left[\log\left(\frac{S_T}{F}\right)\right] = -i\ \frac{\partial}{\partial u}\phi_T^c(u)\bigg|_{u=0} - i\ \frac{\partial}{\partial u}\phi_T^j(u)\bigg|_{u=0}$$

From equation (61), the first term on the r.h.s is just $-\frac{1}{2}\mathbf{E}[W_T^c]$. To get the second term, we use the explicit form of $\phi_T^j(u)$ given by

$$\phi_T^j(u) = \exp\left\{-iu\lambda T\left(e^{\alpha+\delta^2/2}-1\right) + \lambda T\left(e^{iu\alpha-u^2\delta^2/2}-1\right)\right\}$$

Then

$$-i\ \frac{\partial}{\partial u}\phi_T^j(u)\bigg|_{u=0} = \lambda T\left(1+\alpha-e^{\alpha+\delta^2/2}\right)$$

So

$$\mathbf{E}\left[\log\left(\frac{S_T}{F}\right)\right] = -\frac{1}{2}\mathbf{E}[W_T^c] + \lambda T\left(1+\alpha-e^{\alpha+\delta^2/2}\right)$$

and the difference in value between the log-contract hedge (2 log contracts) and the variance swap is given by

$$\begin{aligned}\mathbf{E}[W_T] + 2\mathbf{E}\left[\log\left(\frac{S_T}{F}\right)\right] &= \lambda T\left(\alpha^2+\delta^2\right) + 2\lambda T\left(1+\alpha-e^{\alpha+\delta^2/2}\right)\\ &= -\frac{1}{3}\lambda T\alpha\left(\alpha^2+3\delta^2\right) + \text{higher order terms}\end{aligned}$$

This shows that two log contracts is a good average hedge for a variance swap even with jumps. Putting $\lambda = 0.6$, $\alpha = -0.15$ and $\delta = 0.1$, we get an error of only 0.0016 on a one-year variance swap which corresponds to around 0.25 vol points if the volatility level is 30%.

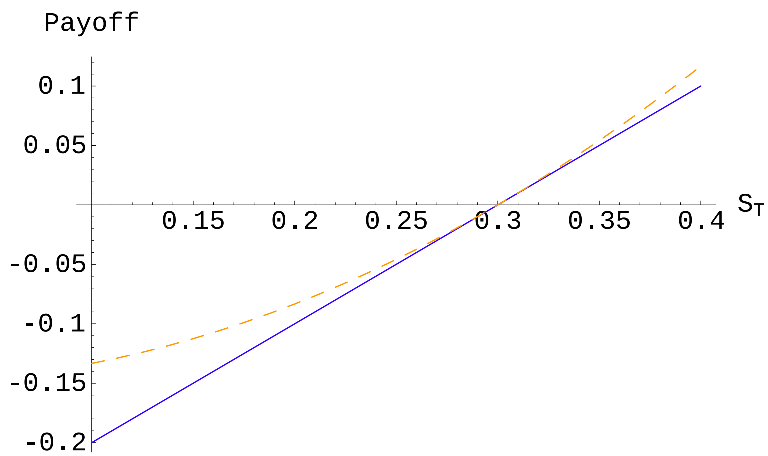
13.5 Volatility Swaps

Realized volatility Σ_T is the square root of realized variance V_T and we know that the expectation of the square root of a random variable is less than (or equal to) the square root of the expectation. The difference between $\sqrt{V_T}$ and Σ_T is known as the *convexity adjustment*.

Figure 1 shows how the payoff of a variance swap compares with the payoff of a volatility swap.

Intuitively, the magnitude of the convexity adjustment must depend on the volatility of realized volatility. Note that volatility does not have to be stochastic for realized volatility to be volatile; realized volatility Σ_T varies according to the path of the stock price even in a local volatility model.

Figure 1: Payoff of a variance swap (dashed line) and volatility swap (solid line) as a function of realized volatility. Both swaps are stuck at 30% volatility. Σ_T



In fact, there is no replicating portfolio for a volatility swap and the magnitude of the convexity adjustment is highly model-dependent. As a consequence, market makers' prices for volatility swaps are both wide (in terms of bid-offer) and widely dispersed. As in the case of live-out options, price takers such as hedge funds may occasionally have the luxury of being able to cross the bid-offer – that is, buy on one dealer's offer and sell on the other dealer's bid.

Assuming no jumps however (Matytsin (1999) discusses the impact of jumps), the convexity adjustment is not so model dependent. We will now compute it for the Heston model.

13.6 Convexity Adjustment in the Heston Model

To compute the expectation of volatility in the Heston model we use the following trick:

$$\mathbf{E}\left[\sqrt{W_T}\right] = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{1 - \mathbf{E}\left[e^{-\psi W_T}\right]}{\psi^{3/2}} \, d\psi \tag{63}$$

From Cox, Ingersoll, and Ross (1985), the Laplace transform of the total variance $W_T = \int_0^T v_t\,dt$ is given by

$$\mathbf{E}\left[e^{-\psi W_T}\right] = A\,e^{-\psi v B}$$

where

$$\begin{aligned} A &= \left\{\frac{2\phi\,e^{(\phi+\lambda)T/2}}{(\phi+\lambda)(e^{\phi T}-1)+2\phi}\right\}^{2\lambda\bar{v}/\eta^2} \\ B &= \frac{2\,(e^{\phi T}-1)}{(\phi+\lambda)(e^{\phi T}-1)+2\phi} \end{aligned}$$

with $\phi = \sqrt{\lambda^2 + 2\psi\eta^2}$.

With some tedious algebra, we may verify that

$$\begin{aligned} \mathbf{E}\left[W_T\right] &= -\frac{\partial}{\partial\psi}\mathbf{E}\left[e^{-\psi W_T}\right]\Big|_{\psi=0} \\ &= \frac{1-e^{-\lambda T}}{\lambda}\,(v-\bar{v}) + \bar{v}T \end{aligned}$$

as we found earlier in Section 13.2.

Computing the integral in equation (63) numerically using the usual parameters from Homework 2 ($v = 0.04, \bar{v} = 0.04, \lambda = 10.0, \eta = 1.0$), we get the graph of the convexity adjustment as a function of time to expiration shown in Figure 2.

Using Bakshi, Cao and Chen parameters ($v = 0.04, \bar{v} = 0.04, \lambda = 1.15, \eta = 0.39$), we get the graph of the convexity adjustment as a function of time to expiration shown in Figure 3.

To get intuition for what is going on here, compute the limit of the variance of V_T as $T \to \infty$ with $v = \bar{v}$ using

$$\begin{aligned} \mathrm{var}\left[\mathrm{W_T}\right] &= \mathbf{E}\left[{W_T}^2\right] - \left\{\mathbf{E}\left[W_T\right]\right\}^2 \\ &= \frac{\partial^2}{\partial\psi^2}\mathbf{E}\left[e^{-\psi W_T}\right]\Big|_{\psi=0} - \left\{\frac{\partial}{\partial\psi}\mathbf{E}\left[e^{-\psi W_T}\right]\Big|_{\psi=0}\right\}^2 \\ &= \bar{v}T\frac{\eta^2}{\lambda^2} + O(T^0) \end{aligned}$$

Then, as $T \to \infty$, the standard deviation of *annualized* variance has the leading order behavior $\sqrt{\frac{\bar{v}}{T}}\frac{\eta}{\lambda}$. The convexity adjustment should be of the

Figure 2: Annualized Heston convexity adjustment as a function of T with parameters from Homework 2.

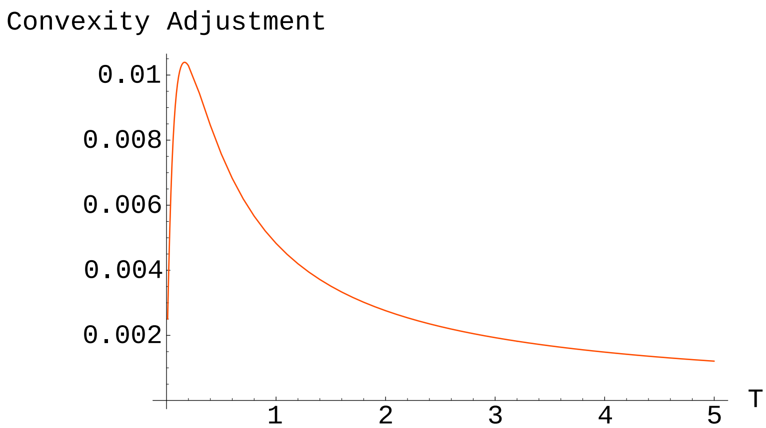
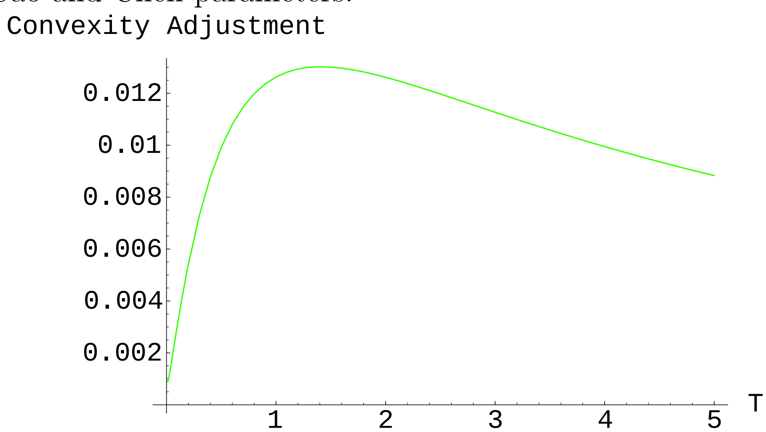


Figure 3: Annualized Heston convexity adjustment as a function of T with Bakshi, Cao and Chen parameters.



order of the standard deviation of annualized volatility over the life of the contract. From the last result, we expect this to scale as $\frac{\eta}{\lambda}$. Comparing Bakshi, Cao and Chen (BCC) parameters with Homework 2 parameters, we deduce that the convexity adjustment should be roughly 3.39 times greater for BCC parameters and that's what we see in the graphs.

14 Epilog

I hope that this series of lectures has given students an insight into how financial mathematics is used in the derivatives industry. It should be apparent that modelling is an art in the true sense of the word – not a science, although when it comes to implementing the chosen solution approach, science becomes necessary. We have seen several examples of claims which may be priced differently under different modelling assumptions even though the models generate identical prices for European options. The importance of lateral thinking outside the framework of a given model cannot be overemphasized. For example, what is the impact of jumps? of stochastic volatility? of skew? and so on. Finally, intuition together with the ability to express this intuition clearly is ultimately what counts; without this intuition, financial mathematics is useless in practice given that the ultimate users of models are overwhelmingly non-mathematicians.

References

Breeden, D., and R. Litzenberger, 1978, Prices of state-contingent claims implicit in option prices, *Journal of Business* 51, 621–651.

Carr, Peter, and Dilip Madan, 1998, Towards a theory of volatility trading, in Robert A. Jarrow, ed.: *Volatility: New Estimation Techniques for Pricing Derivatives* . chap. 29, pp. 417–427 (Risk Books: London).

Chriss, Neil, and William Morokoff, 1999, Market risk for variance swaps, *Risk* 12, 55–59.

Cox, John C., Jonathan E. Ingersoll, and Steven A. Ross, 1985, A theory of the term structure of interest rates, *Econometrica* 53, 385–407.

Demeterfi, Kresimir, Emanuel Derman, Michael Kamal, and Joseph Zou, 1999, A guide to volatility and variance swaps, *Journal of Derivatives* 6, 9–32.

Matytsin, Andrew, 1999, Modeling volatility and volatility derivatives, Columbia Practitioners Conference on the Mathematics of Finance.