Gatheral's homework 4: *solution by Peter Friz*

The numerical simulation shows that Lockback options are systematically cheaper under stochastic vol than under the corresponding local vol. Why? We give two different explanations:

I. First observe

$$(x-K)^+ \approx \sum_{i\geq 1} \Delta K \times \mathbf{1}_{[K+i\Delta K,\infty)}(x)$$

One simple implication is that a call option can be decomposed into a strip of binary call options with increasing strikes $K+i\Delta K$, just replaced x by $S(T)$.
More interestingly, replace x by $S^*(T)$ the (path−wise) maximum of $S(t)$ up to time T .
This yields a decomposition of a lookback option into a strip of one−touch options.
We saw in the course (Fig 4, p46) that a one−touch has a higher value under local vol than under stochastic vol which therefore explains the result.

Let's give explanations why one−touches behave this way. Suppose we have put together a static hedge for the one−touch option which consists of a strip of European binary option struck at the barrier level with all expirations from today to time T . The value of this hedge today does not depend on modeling assumptions. When the barrier is hit, the hedge must be liquidated. Because the forward vol skew is flat in a local vol model but is as steep as today's skew in a stochastic vol model, we can sell the outstanding binary calls (which are now at−the−money by definition) for more money in the stochastic vol case than in the local vol case (see p38 for a discussion of skew&binary calls).
We conclude that the upfront premium required to break even on the hedge is less in the stochastic vol case than in the local vol case and the valuation of a one−touch should be correspondingly lower in the stochastic vol case.
An independent explanation a la Taleb goes as follows: stochastic vol has "fat tails" which means that in some std. deviation interval around the mean "fat tail"−distributions have, say, 78% of the mass as opposed to, say, a normal law which only has 67% of mass in this interval. On the other hand "fat tails"−densities do not go to zero as fast as, say, normal laws which explains the name "fat tail". Nevertheless, since most mass is around the mean the probability of going far out is smaller under a "fat−tail"−distribution than under some "non−fat−tail"−distribution as the one of a local vol model. Clearly, the fair value of a one−touch option is just the probability of hitting the barrier. But as explained before, this probability is smaller under a ("fat−tail"−)stochastic vol model than under local vol. Again, we find that the valuation of a one−touch should be lower in the stochastic vol case.

II. Use the lookback−hedge discussed in section 11.4 of the notes. There we saw that whenever the stock price does reach K and increases by some small increment ΔK the value of the lookcack option must increase by ΔK. The two−call hedge discussed there generates a profit by rebalancing

$$2C(K+\Delta K,K) - 2C(K+\Delta K,K+\Delta K) \approx -2\partial C/\partial K \text{ (at } S=K) \, \Delta K$$

which equals ΔK in a BS−world with log−drift zero. Bringing the skew into the game we see that the r.h.s. of the above expression is larger under stochastic vol. Indeed, recall from sectionthat $-\partial C/\partial K$ is larger when there is skew (= future situations under stochastic vol) thanwithout (= future situation under local vol). Hence, the rebalancing profit is larger under stochastic vol and correspondingly the fair price for the lookback option should be smaller under stochastic vol than under local vol. Exactly, as expected.