Dynamic aspects of smile models

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Talk Outline

What would we like the model to accomplish?

A review of dynamic aspects of popular classes of models

- **How to price / hedge in incomplete markets?**
- **Stochastic volatility - Heston**
- **Jumps**
- **Levy + stochastic vol. extensions**

Conclusion - how could we improve on existing models?

What do we need models for ?

- The case of cliquets (even ATM)
- The case of path-dep cliquets
- What is the delta of a call?

What do we require from a model ?

- That it correctly captures the joint dynamics of spot / implied vols
  dynamics of ATM vols
  dynamics of skew
  spot / vol "correlation"
- That it fits today's implied vols reasonably well

Different approaches to generating implied vols dynamics

- Specifying dynamics on implied vols directly
- Specifying ad-hoc dynamics on the spot
- Other techniques (BGM-like spec. on forward variances, etc..)

Pricing in a general setting

Black-Scholes

- Delta is dP/dS. Delta strategy exactly generates payoff at maturity, with zero variance: there is just one price for an option
- Variance of final P&L is finite only because trading occurs at discrete dates (daily).

Other settings

- When there are jumps, or if volatility is stochastic final P&L has finite variance, even if trading occurs continuously. How do we price/ hedge an option?

  For a European option with payoff function f, the discounted P&L to the seller reads:

$$P\&L = -e^{-rT} f(S_T) + \int e^{-rt} \Delta(S_t, t, \ )\left(dS_t - (r-q)S_t dt\right)dt$$

  The pricing criterion used here is to minimize the variance of the final P&L. The function Δ is obtained as the solution of a stochastic control problem. It is a function of S, t, and may depend on other "hidden" variables.

  The price of the option is then set to: $P = -E[P\&L]$: "minimal risk" pricing.

Examples: two types of models

- Stochastic volatility
- Jumps / Levy processes

Pricing in a general setting

Stochastic volatility - Heston

The Heston dynamics reads:

$$
\begin{aligned}
dS &= \mu S dt + \sqrt{V} S dW \\
dV &= -k(V - V_0)dt + \sigma \sqrt{V} dZ
\end{aligned}
$$

Imagine we have sold a European option. Let $m_\Delta(t, S, V)$ and $W_\Delta(t, S, V)$ be the expectation and variance of the final P&L, discounted at time t. They are solutions of the following coupled equations:

$$
\begin{aligned}
\frac{\partial m}{\partial t} + \mathrm{L}m - rm &= -(\mu - r)S\Delta \\
\frac{\partial W}{\partial t} + \mathrm{L}W - 2rW &= -VS^2\left(\Delta + \frac{\partial m}{\partial S} + \frac{\rho\sigma}{S}\frac{\partial m}{\partial V}\right)^2 - (1-\rho^2)\sigma^2 V\left(\frac{\partial m}{\partial V}\right)^2
\end{aligned}
$$

with

$$
\begin{aligned}
m(T, S, V) &= -\text{Payoff}\,(S_T) \\
W(T, S, V) &= 0
\end{aligned}
$$

$$
\mathrm{L} = \mu S \frac{\partial}{\partial S} - k(V - V_0)\frac{\partial}{\partial V} + \frac{1}{2}VS^2\frac{\partial^2}{\partial S^2} + \frac{1}{2}\sigma^2 V\frac{\partial^2}{\partial V^2} + \rho\sigma S V\frac{\partial^2}{\partial S \partial V}
$$

Pricing in a general setting

Stochastic volatility - Heston

By (variationally) differentiating w.r.t. Δ, we get (1) the optimal delta and (2) the pricing equation:

$$\Delta = \frac{\partial P}{\partial S} + \frac{\rho\sigma}{S}\frac{\partial P}{\partial V}$$

$$\frac{\partial P}{\partial t} + (r-q)S\frac{\partial P}{\partial S} - k(V-\overline{V_0})\frac{\partial P}{\partial V} + \frac{1}{2}VS^2\frac{\partial^2 P}{\partial S^2} + \frac{1}{2}\sigma^2 V\frac{\partial^2 P}{\partial V^2} + \rho\sigma SV\frac{\partial^2 P}{\partial S\partial V} = rP$$

where $\overline{V_0} = V_0 - \frac{(\mu-r)\rho\sigma}{k}$ and $P = -m$

- Drift for spot is still financing cost - as in B.S.
- V_o is renormalized but - luckily - pricing equation keeps usual form. Volatility degrees of freedom are partially hedged with the stock ⇨ impacts the drift for V.
- Variance of final P&L is now finite

Pricing in a general setting

Jumps

Now imagine that vol is not stochastic, but there is an additional jump process. Let J be the (random) magnitude of the jumps and λ their intensity. The equations for m and W read:

$$\frac{\partial m}{\partial t} + \mathrm{L}m - rm = -(\mu + \lambda \overline{J} - r)S\Delta$$

$$\frac{\partial W}{\partial t} + \mathrm{L}W - 2rW = -\Delta^2 S^2(\sigma^2 + \lambda \overline{J^2}) - 2\Delta S\left(\lambda \overline{\delta m J} + \sigma^2 S \frac{\partial m}{\partial S}\right) - \sigma^2 S^2 \left(\frac{\partial m}{\partial S}\right)^2 - \lambda \overline{\delta m^2}$$

with

$$\delta m = m(S(1+J)) - m(S)$$

$$m(T,S,V) = -\text{Payoff }(S_T)$$

$$W(T,S,V) = 0$$

$$\mathrm{L}f = \mu S \frac{\partial f}{\partial S} + \lambda \overline{\delta f} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}$$

Jumps

We get the delta $\Delta = \dfrac{\sigma^2 \dfrac{\partial P}{\partial S} + \lambda \dfrac{1}{S}\overline{\delta P J}}{\sigma^2 + \lambda \overline{J^2}}$

For small jumps: $\Delta = \dfrac{\partial P}{\partial S} + \lambda \overline{J^3} S \dfrac{\partial^2 P}{\partial S^2} + \ldots$

Plug the expression for delta in the equation for m ➪ we get a pricing equation in which the historical drift of the spot appears. This is to be expected as delta is different than dP/dS.

Is it reasonable to take a position on the stock and bet on a value of the historical drift? - the “optimal” delta may not be optimal, since jumps are probably ill-specified with respect to the historical behavior of stock prices.

Let us then decide that the delta is dP/dS. The historical drift of the spot disappears from the pricing equation which now reads:

$$\frac{\partial P}{\partial t} + (r - q - \lambda \overline{J}) S \frac{\partial P}{\partial S} + \lambda \overline{\delta P} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} = rP$$

This holds for Levy processes as well.

Static / dynamic behavior of models

Heston - static

- Parameters: $V,\quad k,\rho,\sigma,V_0$

- Model is homogeneous: $\hat{\sigma} = f\left(\frac{K}{F}, V\right)$

$$dS = \mu S dt + \sqrt{V} S dW$$
$$dV = -k(V - V_0)dt + \sigma\sqrt{V} dZ$$

- Expansion in powers of vol of vol at order 1 yields:

$$\text{Short term}: \quad K\left.\frac{\partial\hat{\sigma}}{\partial K}\right|_F = \frac{\rho\sigma}{4\sqrt{V}}$$

$$\text{Long term}: \quad K\left.\frac{\partial\hat{\sigma}}{\partial K}\right|_F = \frac{\rho\sigma}{2kT\sqrt{V_0}}$$

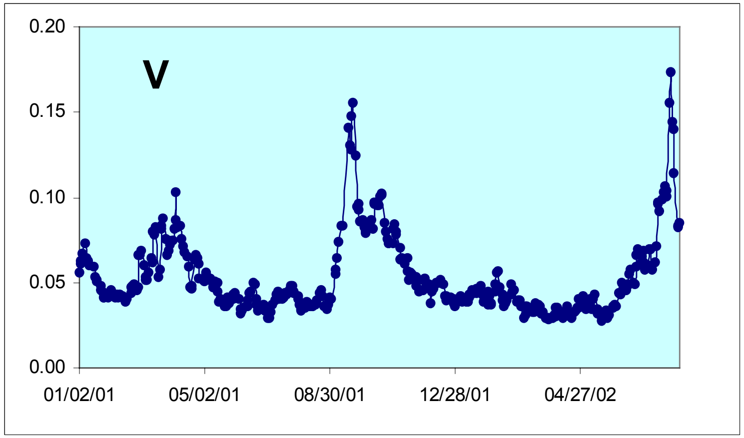
Crossover is set by mean-reversion time $\tau = 1/k$. For maturities longer than τ, increments of $\ln(S)$ become stationary and independent ➪ skew decays as $1/T$. For short-term, no explosion of the skew.

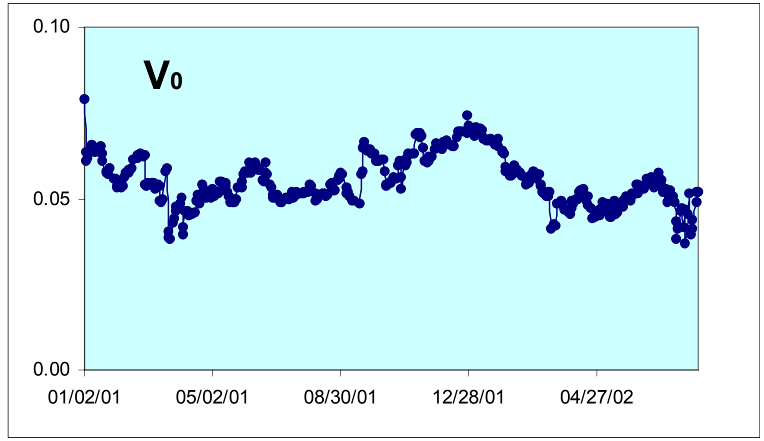
- Variance Swap vol equals Log Swap vol: $\hat{\sigma}_{VS}^2 = V_0 + (V - V_0)\frac{1-e^{-kT}}{kT}$

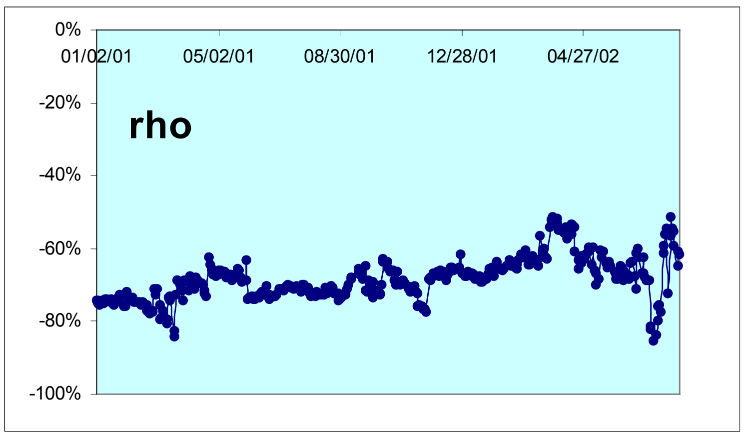
Static / dynamic behavior of models

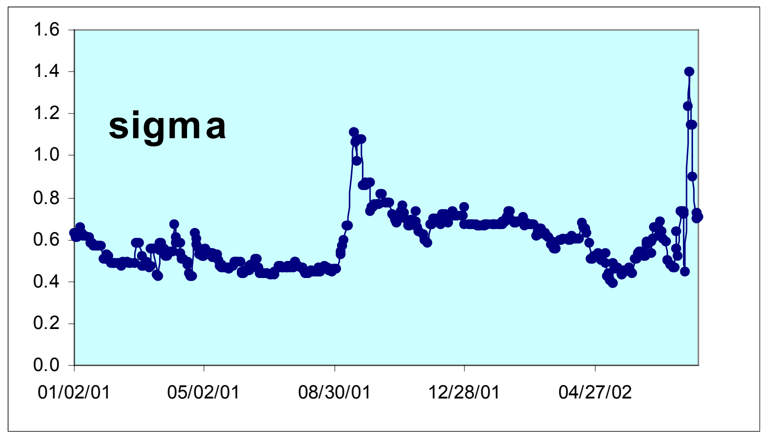
Heston - dynamic

$V, V_0, \rho, \sigma,$ are calibrated on market smiles. Daily fit of the S&P500 smile up to 1 year maturity -1/k is set equal to 6 months.









Static / dynamic behavior of models

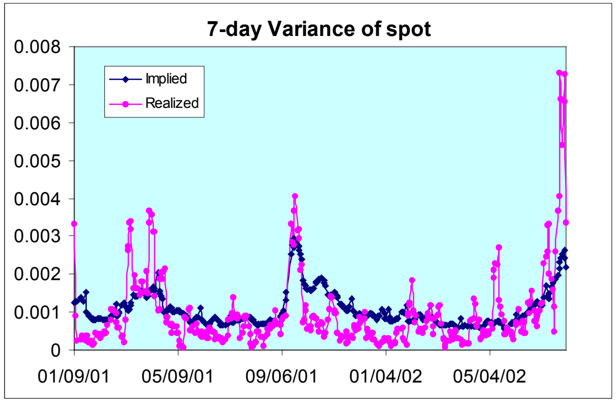
Heston - dynamic

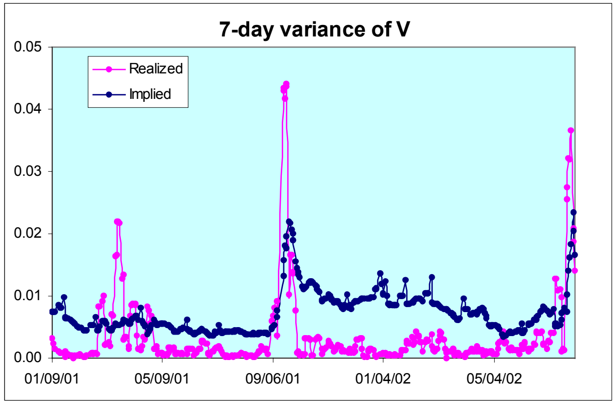
- Parameters are determined by fitting implied vols every day. Only k is kept constant. Are their values in agreement with their dynamics? Look at following averages:

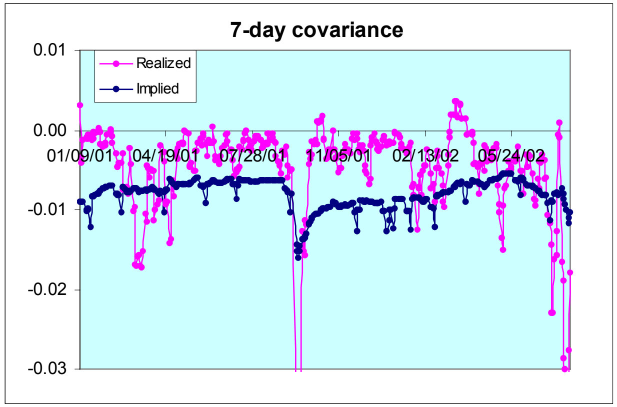
$$\frac{\overline{\delta S^2}}{S^2 V \delta t} = 1.02 \qquad \frac{\overline{\delta V^2}}{\sigma^2 V \delta t} = 0.52 \qquad \frac{\overline{\delta S \delta V}}{S V \rho \sigma \delta t} = 0.65$$

suggesting that $\frac{\sigma_{\text{implied}}}{\sigma_{\text{realized}}} \approx 1.4 \qquad \frac{\rho_{\text{implied}}}{\rho_{\text{realized}}} \approx 1.1$

- However look at graphs:







➪ Dynamics of (short) implied vols is not in agreement with model's anticipation

We may be asking too much from the vol of vol
- create a skew
- drive dynamics of implied vols

Static / dynamic behavior of models

Heston - dynamic

- Dynamics of implied vols: look at variance swap variance

$$V_{VS} = V_0 + (V - V_0)\frac{1-e^{-kT}}{kT}$$

$$dV_{VS} \propto \frac{1-e^{-kT}}{kT}\sigma\sqrt{V}\,dZ$$

Term structure of variance of implied vols is controlled by k. For maturities $T >> 1/k$, vols do not move.

In stationary regime

$$E\left[(V_t - V_0)(V_{t'} - V_0)\right] = \frac{\sigma^2 V_0}{2k} e^{-k|t-t'|}$$

Variance of variance swap vol is:

decays like 1/T for long maturities

$$\mathrm{V}ar\left(\frac{1}{T}\int_t^{t+T} V_u du\right) = \frac{\sigma^2 V_0}{2k}\frac{2}{(kT)^2}\left[kT - 1 + e^{-kT}\right]$$

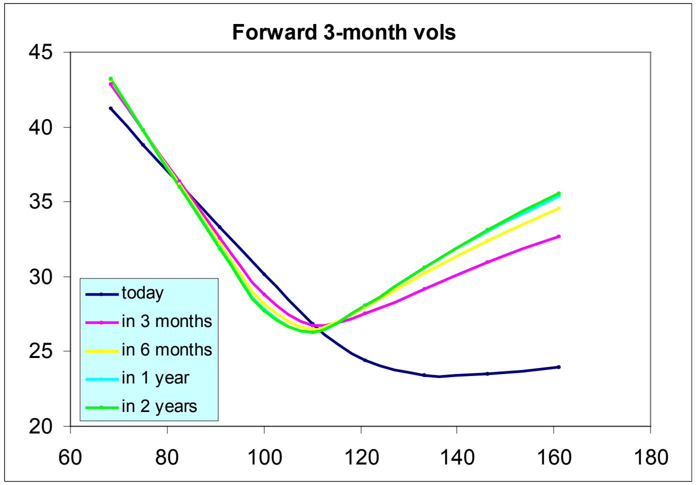
We may need more than 1 factor on vol to control the term structure of the variance of implied vols.

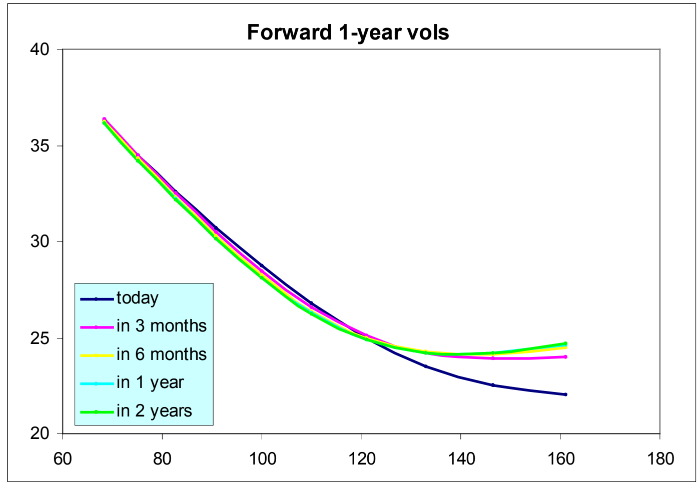
Static / dynamic behavior of models

Heston - dynamic - forward smile

- Look at forward smile

Set $V = V_0 = 0.1, \quad \sigma = 0.7, \quad \rho = -0.7, \quad k = 2$

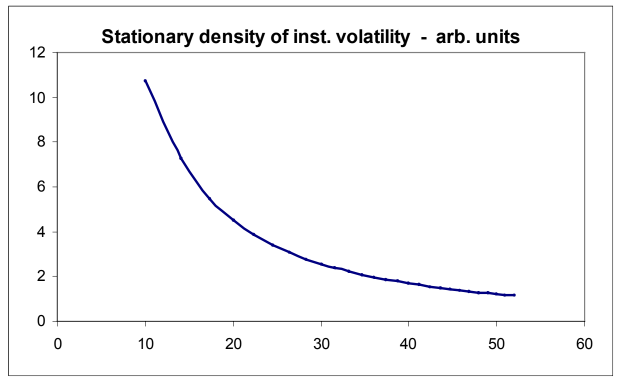




- Shape of forward smile is generated by:

  - density of V at forward date

  - dependence of smile on value of inst. variance



Static / dynamic behavior of models

Jumps

- Skew decays like *1/T*
- Smile is static - implied vols (as a function of moneyness) are frozen.

  Forward smiles are the same as today's smile.

- Jumps / Levy processes are a neat trick for generating a skew without extra degrees of freedom
- However, be careful about prices of very path-dependent options

  Ex. variance swaps: should we use
  - the Variance Swap vol?
  - the Log Swap vol?

Static / dynamic behavior of models

Levy processes and stochastic vol extensions (D. Madan, P. Carr et al.)

- Pick your favorite Levy process
- Replace physical time with the integral of some positive process:

$$t \quad \rightarrow \quad \int_0^t \lambda_u du \qquad \overline{\lambda_u} = 1$$

$$dt \quad = \quad \lambda_u \, du$$

- What kind of dynamics does this generate for implied vols?

1st order perturbation in skewness of ditribution of $\ln(S_T)$ yields:

$$\hat{\sigma}_{ATM} \approx \sqrt{\frac{\text{Variance}}{T}}$$

$$K \left.\frac{\partial \hat{\sigma}}{\partial K}\right|_{ATM} \approx \frac{\text{Skewness}}{6\sqrt{T}}$$

For short maturities $\text{Variance} \propto \lambda, \qquad \text{Skewness} \propto \frac{1}{\sqrt{\lambda}}$

i.e. $K \left.\frac{\partial \hat{\sigma}}{\partial K}\right|_{ATM} \propto \frac{1}{\hat{\sigma}_F}$

➪ Structural constraint on the dynamics of the smile

Static / dynamic behavior of models

Conclusion - how do we improve on existing models ?

- Set priorities:
  - behavior of skew w.r.t. ATM vol - correlation with spot
  - term stucture of variance of implied vols
  - impact of misspecification of process for the spot

- Need some kind of stoc. vol.
  - probably more than 1 factor needed

- Modelling choices
  - directly modelling spot process has advantages
    - no arbitrage in forward smiles
    - process for the spot is under control
    - delta is an explicit output of pricing model

- **Ultimately reliable prices and hedges !!**