Lecture 4: More on Jumps

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10 Merton's Model of Default

As we have come to expect, Wilmott (2000) gives an excellent introduction to the modelling of default risk. There are two broad types of default-risk model used by practitioners: so-called *structural* models and so-called *reduced form* models. I found the following useful description by JabairuStork on Wilmott.com:

"A *structural model* (of firm default) postulates that default occurs when some economic variable (like firm value) crosses some barrier (like debt value), typically using a contingent claims model to support this assertion and to find the probability of default. Both H-W and Creditgrades[1] are models of this form."

"A *reduced form* model models default as a random occurrence - there is no observable or latent variable which triggers the default event, it just happens. The Duffie-Singleton model (Duffie and Singleton 1999) is a reduced form model. These models are easy to calibrate, but because they lack any ability to explain why default happens, I think they make most people nervous. Basically, you estimate an intensity for the arrival of default (possibly as a function of time, possibly as a stochastic process, possibly as a function of other things.)"

Merton's model (Merton 1974) is the simplest possible example of a reduced form model. It supposes that there some probability $\lambda(t)$ per unit time of the stock price jumping to zero (the *hazard rate*) whereupon default occurs. Jumps are independent of the stock price process. Then, contingent claims must satisfy the jump-diffusion valuation equation (44) with $\mathbb{E}[J] = 0$. It is particulary straightforward to value a call option because for a call, $V(SJ, t) = 0$. Substitution into equation (44) gives:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV - \lambda(t)\left\{V - S\,\frac{\partial V}{\partial S}\right\} = 0 \tag{54}$$

We immediately recognize equation (54) as the Black-Scholes equation with a shifted interest rate $r + \lambda$. Its solution is of course the Black-Scholes formula with this shifted rate.

The meaning of this shifted rate is particularly clear if we assume no recovery (in the case of default) on the issuer's bonds so that $B(JS, t) = 0$. Then, the risky bond price $B(t, T)$ must also satisfy equation (54) with the

[1]See Finkelstein (2002) and Lardy (2002)

solution

$$B(t,T) = e^{-\int_t^T (r(s)+\lambda(s))ds}$$

We identify the shifted rate $r + \lambda$ with the yield (risk-free rate plus credit spread) of a risky bond. The situation is a little more complicated (but not too much more) if we allow some recovery R on default.

Intuition

It may at first seem surprising that the Black-Scholes formula could be a solution of an equation that has a jump to zero (the so-called *jump to ruin*) in it. There is an economic reason for this however.

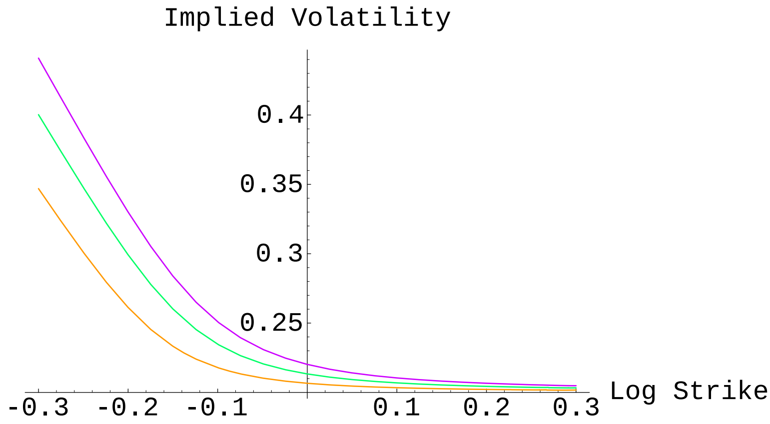
Recall that the derivation of the Black-Scholes formula involves the construction of a replicating portfolio for a call option involving just stock and risk-free bonds. Suppose instead, we were to construct this portfolio using stock and risky bonds. So long as there is no jump to ruin, the derivation goes through as before and the portfolio is self-financing. If there is a jump to ruin, assuming no recovery on the bond, both the bond and the stock jump to zero – the portfolio is still self-financing!

What would happen if we were to hedge a short call option position using stock and *risk-free* bonds following the standard Black-Scholes hedging recipe (as most practitioners actually do)? We would be long stock and short risk-free bonds and in the case of default, the call would end up worthless, the stock would be worthless and we would get full recovery on our risk-free bonds. In other words, on default, we would have a windfall gain. On the other hand, relative to hedging with risky bonds, we would forego the higher carry (or yield).

Implications for the Volatility Skew

All issuers of stock have some probability of defaulting. There is a very active credit derivative market (see DefaultRisk.com for background) which prices default-risk. Black-Scholes implied volatilities are computed by inserting the risk-free rate into the Black-Scholes formula. However, as we just showed, in Merton's model, call option prices are correctly obtained by substituting the risky rate into the Black-Scholes formula. This induces a skew which can become extremely steep for short-dated options on stocks whose issuers have high credit spreads.

Figure 1: 3 month implied volatilities from the Merton model assuming a stock volatility of 20% and credit spreads of 100bp (orange), 200bp (green) and 300bp (magenta).



In Figure 1, we graph the implied volatility for various issuer credit spreads assuming that options are correctly priced using the Merton model. We see that the downside skew that the model generates can be extreme.

10.1 Capital Structure Arbitrage

Capital structure arbitrage is the term used to describe the current fashion for arbitraging equity claims against fixed income and convertible claims. At its most sophisticated, practitioners build elaborate models of the capital structure of a company to determine the relative values of the various claims - in particular, stock, bonds and convertible bonds. At its simplest, the trader looks to see if equity puts are cheaper than credit derivatives and if so buys the one and sells the other. To understand this, we review put-call parity.

Put-Call Parity

We saw above that in the Merton model, the value of an equity call option is given by the Black-Scholes formula for a call with the risk-free rate replaced by the risky-rate. What about put options? To make the arguments above work, the put option would need to be worthless after the jump to ruin occurs. That would be the case if the put in question were to be written by the issuer of the stock. In that case, when default occurs, assuming zero recovery, the put options would also be worth nothing. So the Black-Scholes formula for a put with the risk-free rate replaced by the risky-rate does value put options written by the issuer.

What about put options written by some default-free counterparty (for example an exchange)? When default occurs, this put option should be worth the strike price. We already know how to value a call written by a default-free counterparty; by definition, the issuer of a stock cannot default on a call on his own stock so the value of a call written by the issuer of the stock equals the value of a call written by a default-free counterparty. We obtain the value of a put by put-call parity: using risk-free bonds in the case of the default-free counterparty and risky bonds in the case of the risky counterparty.

Denoting the value of a risk-free put, call and bond by P_0,C_0 and B_0 and the value of risky claims on the issuer of the stock by P_I, C_I and B_I (I for issuer), we obtain

$$\begin{aligned} P_0 &= C_0 + KB_0 - S \text{ (from put-call parity with risk-free bonds)} \\ &= C_I + KB_0 - S \text{ (risk-free and issuer-written calls have the same value)} \\ &= P_I + S - KB_I + KB_0 - S \text{ (from put-call parity with risky bonds)} \\ &= P_I + K(B_0 - B_I) \end{aligned}$$

As we would expect, the risk-free put is worth more than the risky put. The excess value is equal to the difference in risky and risk-free bond prices (times the strike price). With maturity-independent rates and credit spreads for clarity and setting $t = 0$, we obtain

$$B_0 - B_I = e^{-rT}\left(1 - e^{-\lambda T}\right)$$

which is just the discounted probability of default in the Merton model. In words, the extra value is the strike price times the (pseudo-) probability that default occurs. This payoff is also more or less exactly the payoff of a default put in the credit derivatives market.

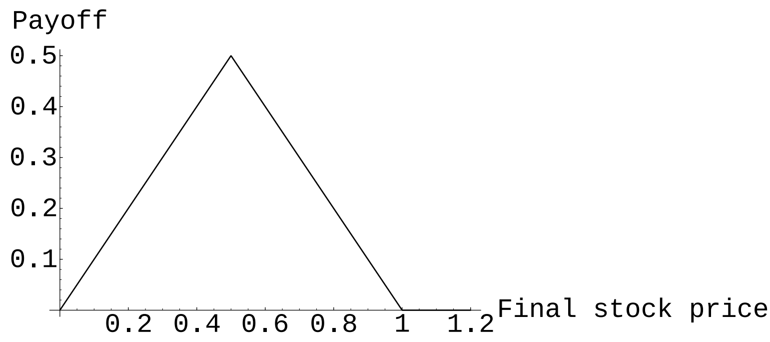
The Arbitrage

Referring back to Figure 1, we see that the downside implied volatility skew can be extreme for stocks whose issuers have high credit spreads. Equity option market makers (until recently at least) made do with heuristic rules to determine whether a skew looked reasonable or not; implied volatility skews of the magnitude shown in Figure 1 seemed just too extreme to be considered reasonable. Taking advantage of the market maker's lack of understanding, the trader buys an equity option on the exchange at a "very high" (but of course insufficiently high) implied volatility and sells a default put on the same stock in the credit derivatives market locking in a risk-free return.

This actually happened and hedge funds were able to lock in risk-free gains for a period of time. During this period of course, market makers saw what were to them extremely steep volatility skews get even steeper and they lost money.

Ultimately, skews got so steep that the hedge funds made money risk-free the other way round – through put spreads. A popular trade for a hedge fund was to buy one at-the-money put and sell two puts struck at half the current stock price. As we can see from Figure 2, this strategy has only positive payoffs so if this can be traded flat or for a net credit, it is a pure arbitrage.

Figure 2: Payoff of the 1x2 put spread combination: buy one put with strike 1.0 and sell two puts with strike 0.5



The life of a market maker is not a happy one – it seems that at least a Masters in Finance is needed to avoid getting arbitraged!

We see that there is a lower bound to the price of a put given by the credit default swap market and an upper bound given by spread arbitrage. To make this more concrete, consider the upper and lower bounds for a 0.5

strike one-year option with credit spreads of $100, 200$ and 300bp displayed in Table 1.

Table 1: Upper and lower arbitrage bounds for one year 0.5 strike options for various credit spreads (at-the-money volatility is 20%)

Credit spread (bp)	**Lower bound**	**Upper bound**
250	0.0123	0.0398
500	0.0244	0.0398
750	0.0361	0.0398

Assuming 20% at-the-money volatility, the upper bound is computed as half the value of an at-the-money option which is 0.0398 in each case. On the other hand, the lower bounds are just the present value[2] of the strike price times the probability of default. We see that the lower bound increases steadily towards the upper bound as the credit spread increases *for fixed at-the-money implied volatility.* It's easy to see how a market maker could have exceeded the upper bound given the steady increase in skews.

10.2 Local and implied volatility in the jump-to-ruin model

As noted above, the value of a call option is given by the Black-Scholes formula with the interest rate shifted by the hazard rate.

We recall the formula (5) for local volatility from Lecture 1:

$$\sigma^2_{loc}(K,T,S) = \frac{\frac{\partial C}{\partial T}}{\frac{1}{2}\,K^2\,\frac{\partial^2 C}{\partial K^2}} \tag{55}$$

Because the Black-Scholes formula C for a call option is linearly homogenous in the stock price S and the strike price K, we have the relation

$$C = S\,\frac{\partial C}{\partial S} + K\,\frac{\partial C}{\partial K}$$

It follows that

$$K^2\,\frac{\partial^2 C}{\partial K^2} = S^2\,\frac{\partial^2 C}{\partial S^2}$$

[2]We're still assuming zero rates so the PV factor is always one.

Also, in the jump-to-ruin case with zero interest rates and dividends, we have

$$\frac{\partial C}{\partial T} = \frac{1}{2}\,\sigma^2\,S^2\,\frac{\partial^2 C}{\partial S^2} + \lambda\,S\,\frac{\partial C}{\partial S} - \lambda\,C$$

where σ is the volatility (diffusion coefficient) and λ is the hazard rate. Rewriting this in terms of derivatives with respect to K gives

$$\frac{\partial C}{\partial T} = \frac{1}{2}\,\sigma^2\,K^2\,\frac{\partial^2 C}{\partial K^2} - \lambda\,K\,\frac{\partial C}{\partial K}$$

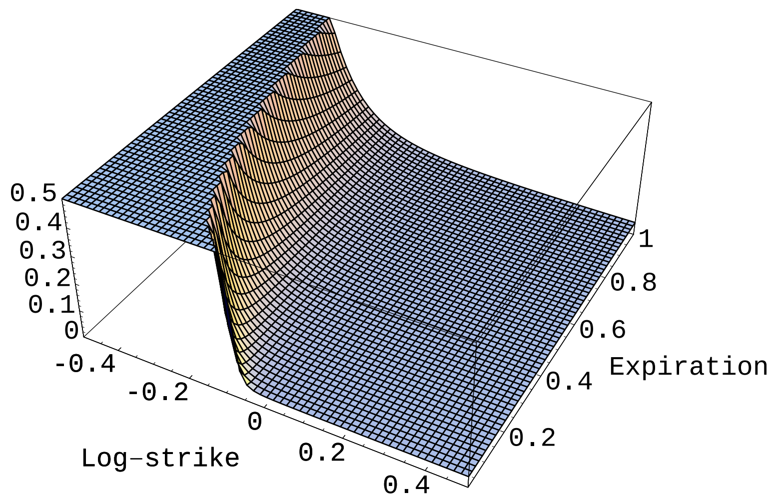
Substituting into equation (55) gives

$$\begin{aligned}\sigma^2_{loc}(K,T,S) &= \sigma^2 - \lambda\,\frac{K\,\frac{\partial C}{\partial K}}{\frac{1}{2}\,K^2\,\frac{\partial^2 C}{\partial K^2}} \\ &= \sigma^2 + 2\,\lambda\,\sigma\,\sqrt{T}\,\frac{N(d_2)}{N'(d_2)}\end{aligned}$$

with

$$d_2 = \frac{\ln S/K + \lambda\,T}{\sigma\,\sqrt{T}} - \frac{\sigma\,\sqrt{T}}{2}$$

Figure 3: Local variance plot with $\lambda = 0.05$ and $\sigma = 0.2$



For very low strikes $K/S \ll 1$, we have $d_2 \gg 0$ and

$$\begin{aligned} N(d_2) &\approx 1 \\ N'(d_2) &= \frac{1}{\sqrt{2\pi}}\, e^{-d_2^2/2} \end{aligned}$$

Then, for very low strikes,

$$\sigma_{loc}^2(K,T,S) \approx \sigma^2 + 2\,\lambda\,\sigma\,\sqrt{T}\,\sqrt{2\pi}\,e^{+d_2^2/2}$$

Figure 3 shows a typical jump-to-ruin local variance surface. From Lecture 2, we know that implied variance (volatility squared) is a gamma-weighted average of local variances. It follows that implied volatility in the jump-to-ruin model gets exponentially high for small strikes and tends to the constant σ for high strikes – exactly consistent with Figure 1 and quite different from the stochastic volatility case.

10.3 The effect of default risk on option prices

To make the foregoing a little more concrete, consider the implied volatilities[3] of January-05 options on GT (Goodyear Tire and Rubber) as of October 20, 2004.

We noted earlier that the price of a European call option in the Merton jump-to-ruin model is given by the Black-Scholes formula with a shifted interest rate. Our experiment is to find the shifted rate and constant volatility that generate prices closest to the European option prices computed using these implied volatilities (*i.e.* the best fit parameters of the Merton model).

We find the best fit parameters:

$$\lambda = 0.01934; \sigma = 0.3946$$

With these parameters, we may compute call option prices and compute the standard (risk-free) Black-Scholes implied volatilities. The results are shown in Table 2.

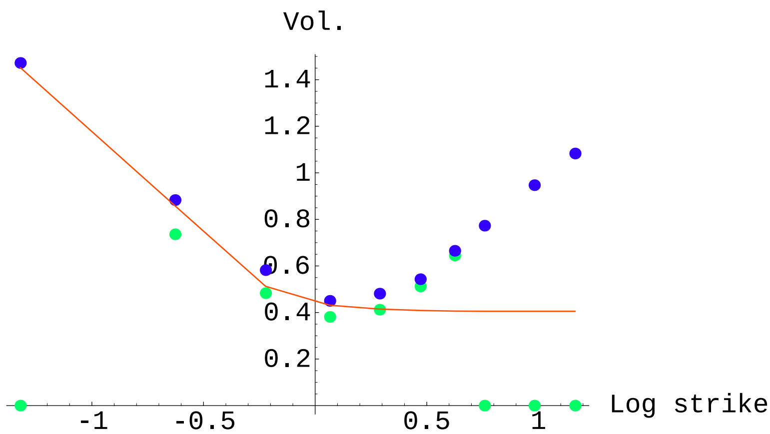
In fact, because GT credit spreads are very high, the Merton model fits the left wing of the volatility skew very well as shown in Figure 4.

[3]Note that traded options are American; we are making an assumption here that American implied volatilities are the same as European implied volatilities

Table 2: Implied volatilities for Jan-2005 options on GT as of 20-Oct-2004 (GT was trading at 9.40). Merton vols are volatilities generated from the Merton model with fitted parameters.

Strike	**Bid vol**	**Ask vol**	**Merton vol**
2.50		147.2%	145.2%
5.00	73.6%	88.3%	85.8%
7.50	48.3%	58.2%	51.2%
10.00	38.1%	45.0%	43.1%
12.50	41.2%	48.1%	41.5%
15.00	51.2%	54.3%	40.9%
17.50	64.5%	66.5%	40.6%
20.00		77.3%	40.0%
25.00		94.7%	40.0%
30.00		108.3%	40.0%

Figure 4: The blue dots are ask vols, the green dots are bid vols and the solid red line is the Merton model fit.



However, the Merton model produces a skew that is a little too steep for low strikes and as predicted generates no right wing (high strike structure) at all and that's just not consistent with the data.

Finally, we might ask whether or not the fitted parameters are realistic. The volatility estimate $\sigma = 39.46\%$ is clearly realistic from inspection of the implied volatilities. To see that the hazard rate estimate of $\lambda = 0.01934$ is also realistic, we note that the fair price of a zero coupon bond of GT (assuming zero rates) should be given by

$$P_t = e^{-\lambda t} R + (1 - e^{-\lambda t})$$

where R is the recovery rate. With Bloomberg's standard assumed recovery rate of 0.4, the credit spread of this bond would be found by solving

$$P_t = e^{-ct} = e^{-\lambda t} R + (1 - e^{-\lambda t})$$

With the above choices of λ and R, we obtain

$$c = 4.58\%$$

This compares with the 5-year credit default swap (CDS) rate of over 5% for GT; the derived credit spread is almost certainly too high for 3 month paper. However, the main point remains: most of the volatility skew for stocks with high credit spreads can be ascribed to default risk.

11 Extreme Jump Dependence: The Baseball Trade

The baseball trade takes its name from the familiar "three strikes and you're out" rule. It is one of the most model-dependent structures ever traded. Its terms are as follows:

- We establish an initial range for the stock price (95 to 105 say).
- If the stock price has exited the range at any reset point, we reset the barriers to be equidistant from the *new* stock price.
- We repeat this procedure until the third time the stock price exits a range whereupon the trade expires worthless.
- If the stock price is still within one of the specified ranges at maturity, the trade pays \$1.

In fact, this structure was popular for a while in the FX markets where jumps are uncommon. In equity markets however, where jumps are common, the extreme model-dependence is poisonous.

To see this, suppose we were to value it using a jump-diffusion model with Andersen-Andreasen parameters as in Table 5 of Section 9.2 (8.5% probability per year of a −55% jump in asset prices). The diffusion component of the process would have rather low volatility and the probability of having more than one jump in one year say is negligible. If and when a jump occurs, the new range is set around the point reached *not* around the old range. If the original range is sufficiently wide the probability of this claim paying $1 at the end is very high.

On the other hand, if we value the same trade using a local volatility model where the local volatilities are calibrated to return jump-diffusion model European option prices, the volatility will be that much higher and the probability of not getting $1 that much greater.

References

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