Modelling without tears

Dariusz Gatarek demonstrates that using a BGM model to manage an interest rate options portfolio need not be as complicated as many might believe. He presents a simple building-block approach to pricing a large class of instruments using forward Libor volatilities and the yield curve

Monsieur Jourdain, hero of Molière's *Le Bourgeois Gentil-Homme*, discovers that he has "... been speaking in prose without knowing it!". Something similar happened to many traders and risk managers: the popular Black (1976) model, when used for interest rate derivatives, became the Brace-Gatarek-Musiela (BGM, 1997) model.

The Black model and the BGM model coincide for a large class of options, but there is a big difference in the way they are perceived. The Black model is considered "simple", while the BGM model is seen as somehow more "difficult" to use. The aim of this article is to justify the likely approach that Monsieur Jourdain would have taken to risk management: use of the Black formula alone is sensible for a very large class of interest rate derivatives. A simple "building blocks" approach (designed especially for management of an interest rate options portfolio) is presented. Volatilities of forward Libor rates and correlation parameters constitute the basic "building blocks" and are determined by the volatilities of the most liquid instruments (caps, floors and swaptions). A prescription for how to price a large class of instruments using forward Libor volatilities and the yield curve is presented. Instantaneous volatilities are not used for model calibration and so the procedure is quite straightforward. In addition, only implied volatilities are considered.

This paper builds on work detailed in several previous articles. The lognormal approximation of swaption prices used here was first introduced in Andersen & Andreasen (1997) and developed in Hull & White (1999). Single-factor approximation was introduced in Brace & Musiela (1994) and developed in Brace, Dun & Barton (1999). Correlation of risk factors was extensively studied in Rebonato (1999).

Let δ be the accrual period for both interest rates and swaps. For simplicity, assume that δ is a constant. Define consecutive swap points as those satisfying the condition $T_{i+1} = T_i + \delta$, for a specified initial condition $T = T_0$ where T_0 is greater than the exercise time t of the swap option. Forward Libor rates are defined as:

$$L_n(t) = \frac{1}{\delta}\left(\frac{B(t,T_{n-1})}{B(t,T_n)} - 1\right) \tag{1}$$

and forward swap rates as:

$$S_N(t) = \frac{\sum_{i=1}^{N} B(t,T_i)L_i(t)}{\sum_{i=1}^{N} B(t,T_i)} = \frac{B(t,T_0) - B(t,T_N)}{\delta\sum_{i=1}^{N} B(t,T_i)} \tag{2}$$

By the arbitrage pricing theory the European-style swaption price is equal:

$$\text{Swaption}_N(t) = \delta\sum_{i=1}^{N} B(t,T_i)E_N\left\{c\left(S_N(T_0)\right)\right\} \tag{3}$$

where E_N is the martingale measure associated with the swap, with respect to which the process $S_N(T)$ is a martingale and c is the intrinsic value (of a call, put or other option). In the BGM model of interest rate dynamics, it is assumed that the Libor rates satisfy the following stochastic equations:

$$dL_n(t) = a_n(t)dt + \gamma_n(t)L_n(t)dW(t) \tag{4}$$

where $a_n(t)$ is the drift coefficient, $\gamma_n(t)$ the instantaneous volatility and $W(t)$ a Wiener process. Determining $\gamma_n(t)$ such that market prices coincide with those given by the model is called calibration. We will not go so far. We will determine some parameters with more intuitive interpretation that are easier to calculate, which will help us price quite a large family of less liquid instruments. The model will be calibrated to the most liquid products: caps, floors and swaptions, options that are quoted in terms of Black volatility.

Swaptions

In the BGM model, the price of a European-style caplet with strike K and option maturity T_{n-1} at time zero is given by the Black formula:

$$\text{Caplet}_n(0) = \delta B(0,T_n)\left(L_n(0)N(d_1) - KN(d_2)\right) \tag{5}$$

where:

$$d_1 = \frac{\ln\left(L_n(0)/K\right) + T_{n-1}\sigma_n^2/2}{\sigma_n\sqrt{T_{n-1}}} \tag{6}$$

$$d_2 = d_1 - \sigma_n\sqrt{T_{n-1}} \tag{7}$$

$$T_{n-1}\sigma_n^2 = \int_0^{T_{n-1}} \gamma_n^2(t)dt \tag{8}$$

In practice, swap options are also priced by the Black formula, ie, a call option with strike = K and option maturity = T.

$$\text{Swaption}_N(0) = \delta\sum_{i=1}^{N} B(0,T_i)\left(S_N(0)N(d_1) - KN(d_2)\right) \tag{9}$$

where:

$$d_1 = \frac{\ln\left(S_N(0)/K\right) + T\sigma^2/2}{\sigma\sqrt{T}} \tag{10}$$

$$d_2 = d_1 - \sigma\sqrt{T} \tag{11}$$

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Two questions may be posed:

- May we price both Libor and swap options with the Black formula?
- If yes, what is the relation between cap (floor) volatilities $\{\sigma_n\}_{n=1,2,..,N}$, and swaption volatility σ?

A purely academic answer to the first question would be: "No, because swap and Libor cannot be lognormal simultaneously." A practitioner, however, would answer: "Yes, it is exactly what we do." An argument in support of the assertion that "swap rates are close to lognormality" may be developed as follows: by equation (2), $S_N(t)$ is an average of lognormal variables. Notice the following:

- $L_n(t)$ are strongly correlated.
- Volatility of the weight function:

$$q_i = \frac{B(t,T_i)}{\sum_{i=1}^{N} B(t,T_i)}$$

is small when compared with the volatility of $L_n(t)$.

If the correlation of $L_n(t)$ was complete and q_i were deterministic, $S_N(t)$ would be lognormal. Since this is only close to the truth, $S_N(t)$ is only "almost" lognormal.

Before we try to answer the second question, let us state the dynamics of the forward swap rate. Since $S_N(t)$ is a positive martingale with respect to the measure E_N:

$$dS_N(t) = \gamma(t) S_N(t) dW(t) \tag{12}$$

where $\gamma(t)$ is the stochastic instantaneous volatility of $S_N(t)$. On the other hand:

$$dS_N(t) = \sum_{n=1}^{N} \frac{\partial S_N(t)}{\partial L_n(t)} L_n(t) \gamma_n(t) dW(t) \tag{13}$$

Calculate:

$$\frac{1}{S_N(t)} \frac{\partial S_N(t)}{\partial L_n(t)} = \frac{\delta}{1+\delta L_n(t)} \tag{14}$$

$$\frac{B(t,T_0)\sum_{k=n}^{N} B(t,T_k) + B(t,T_N)\sum_{k=1}^{n-1} B(t,T_k)}{\left(B(t,T_0) - B(t,T_N)\right)\sum_{k=1}^{N} B(t,T_k)} \tag{14}$$

Denote $\Gamma(t) = [\gamma_1(t), \gamma_2(t), \ldots, \gamma_N(t)]$:

$$R_n^N(t) = \frac{\delta L_n(t)}{1+\delta L_n(t)} \tag{15}$$

$$\frac{B(t,T_0)\sum_{k=n}^{N} B(t,T_k) + B(t,T_N)\sum_{k=1}^{n-1} B(t,T_k)}{\left(B(t,T_0) - B(t,T_N)\right)\sum_{k=1}^{N} B(t,T_k)}$$

and $R^N(t) = [R_1^N(t), R_2^N, \ldots, R_N^N(t)]$. Then:

$$\gamma(t) = R^N(t)\,\Gamma(t) = \sum_{k=1}^{N} R_k^N(t)\,\gamma_k(t) \tag{16}$$

Assuming that most of the interest rate movements are parallel shifts, then the approximation $R^N(t) \cong R^N(0)$ may be used. If one accepts that this is a good approximation of $\gamma(t)$ (at least on average), the following expression is valid:

$$\sigma = \sqrt{\left(R^N(0)\right)^T \Phi R^N(0)} \tag{17}$$

where:

$$\Phi = \begin{bmatrix} \varphi_{11} & \varphi_{12} & \cdots & \varphi_{1N} \\ \varphi_{21} & \varphi_{22} & \cdots & \varphi_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ \varphi_{N1} & \varphi_{N2} & \cdots & \varphi_{NN} \end{bmatrix} \tag{18}$$

$$T\varphi_{lk} = \int_0^T \gamma_l(t)\,\gamma_k(t)\,dt \tag{19}$$

Notice that $\varphi_{11} = \sigma_1^2$ does not necessarily imply that $\varphi_{kk} = \sigma_k^2$ for any $k > 1$. This is because σ_k is the volatility of a Libor option and $\varphi_k = \sqrt{\varphi_{kk}}$ is the volatility of a forward Libor option, ie:

$$\sigma_n^2 = T_{n-1}^{-1} \int_0^{T_{n-1}} \gamma_n^2(t)\,dt \tag{20}$$

$$\varphi_n^2 = \varphi_{nn} = T^{-1}\int_0^T \gamma_n^2(t)\,dt \tag{21}$$

Calibration

Calibration is usually taken to mean determining the set of instantaneous volatilities $\Gamma(t) = [\gamma_1(t), \gamma_2(t), \ldots, \gamma_N(t)]$. If we do it, we are able to price everything – at least in theory. In practice, there is insufficient data to determine instantaneous volatilities and further assumptions are necessitated. Evaluation of the matrix Φ alone is much easier and, in addition, fewer assumptions are required.

Two of many possible parameterisations of the matrix Φ will be studied: single factor and leading factor. Their suitability depends on the number and liquidity of interest rate options in the market.

- **Single-factor approximation.** The assumption that the covariance matrix Φ is single factor implies that $\varphi_{nk} = \varphi_n\varphi_k$. In other words, the Libor rates are fully correlated. Under this assumption, the following approximations are valid:

$$\sigma^N = \sum_{i=1}^{N} R_i^N(0)\,\varphi_i \tag{22}$$

Single-factor approximation provides a direct way of calibrating the model to one caplet (or floorlet) and $(N-1)$ swaptions $S_2, S_3, \ldots, S_N$. The required results are now obtained in terms of the solution to a set of N linear equations:

$$\varphi_1 = \sigma_1 \tag{23}$$

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$$\varphi_n = \frac{\sigma^n - \sum_{i=1}^{n-1} R_i^n(0)\varphi_i}{R_n^n(0)} \quad (24)$$

☐ **Leading-factor approximation.** If it is assumed that a direct dependence between φ_k and σ_k ($\varphi_k = \sigma_k$, for instance) exists, then a leading-factor approximation may be used. In this case, the model is calibrated to $(N - 1)$ swaptions and N caps (floors). The general concept comes from the portfolio theory. We assume that there exists a leading factor in all volatilities of $L_n(t)$, ie:

$$\int_0^T \gamma_k(t)dW(t) = \varphi_k\left(p_k\xi + \xi_k\sqrt{1-p_k^2}\right) \quad (25)$$

where ξ, $\xi_{k,\ k=1,\ldots,N}$ are independent normally distributed random variables and $0 \le p_k \le 1$ are weights. Then $\varphi_{nk} = \varphi_n\varphi_k p_k p_n$ for any $k \ne n$ and:

$$\tilde{\sigma}^N = \sqrt{\left(\sum_{k=1}^N p_k\varphi_k R_k^N(0)\right)^2 + \sum_{k=1}^N \left(1-p_k^2\right)\varphi_{kk}R_k^N(0)^2} \quad (26)$$

Notice that the leading-factor approximation is not a generalisation of the single-factor approximation, since for single-factor approximation we do not assume any dependence between φ_k and σ_k. All the other approximations (two factors, three factors, etc) are also tractable within the model.

We use the same technique if the swap accrual period is larger than the interest rate accrual period, for instance twice, as in the dollar case. Let δ be the accrual period for swaps and $\delta/2$ the accrual period for interest rates. Denote:

$$F_{2n}(t) = \frac{2}{\delta}\left(\frac{B(t,T_n-\delta/2)}{B(t,T_n)} - 1\right) \quad (27)$$

$$F_{2n+1}(t) = \frac{2}{\delta}\left(\frac{B(t,T_n)}{B(t,T_n+\delta/2)} - 1\right) \quad (28)$$

We assume that the dynamics of $F_n(t)$ are lognormal. Obviously:

$$\frac{\partial S(t)}{\partial F_{2n}(t)} = \frac{\partial S(t)}{\partial L_n(t)}\frac{\partial L_n(t)}{\partial F_{2n}(t)} = \frac{1}{2}\frac{\partial S(t)}{\partial L_n(t)}\left(\delta F_{2n-1} + 2\right) \quad (29)$$

$$\frac{\partial S(t)}{\partial F_{2n-1}(t)} = \frac{\partial S(t)}{\partial L_n(t)}\frac{\partial L_n(t)}{\partial F_{2n-1}(t)} = \frac{1}{2}\frac{\partial S(t)}{\partial L_n(t)}\left(\delta F_{2n} + 2\right) \quad (30)$$

The volatility σ can be approximated in the manner discussed previously. Before considering pricing with this procedure, it may be useful to summarise some key facts. The volatilities of caplets (floorlets) $\{\sigma_n\}_{n=1,2,\ldots,N}$ and swaptions $\{\sigma^n\}_{n=1,2,\ldots,N}$ were used as the input data. The output data obtained were the volatilities of forward Libor rates $\{\varphi_n\}_{n=1,2,\ldots,N}$ (for the single-factor approximation), or correlation parameters $\{p_n\}_{n=1,2,\ldots,N}$ (for the leading-factor approximation).

Pricing

The volatilities of forward Libor rates $\{\varphi_n\}_{n=1,2,\ldots,N}$ and correlation parameters $\{p_n\}_{n=1,2,\ldots,N}$ are the "building blocks" from which a large class of instruments can be priced. Let FI be a financial instrument based on interest rates, eg, a swap with a sinking fund, etc. Let $FI(t)$ denote the price at time t. Let c be the payout function of a European-style option set at time $T > t$ and settled according to a certain cashflow CF (of a call, put, binary or other option). By the arbitrage pricing theory the price is equal:

$$\text{Option}(t) = DF(t)E_{CF}\left\{c\left(FI(T)\right)\right\} \quad (31)$$

where E_{CF} is the martingale measure associated with the instrument FI, with respect to which the process $FI(t)$ is a martingale and $DF(t)$ is the discount factor of the cashflow CF. Define:

$$R_n = \frac{L_n(0)}{FI(0)}\frac{\partial FI(0)}{\partial L_n(0)} \quad (32)$$

In the single-factor approximation:

$$\sigma = \sum_{k=1}^N \varphi_k R_k \quad (33)$$

and for the leading-factor approximation:

$$\sigma = \sqrt{\left(\sum_{k=1}^N p_k\varphi_k R_k\right)^2 + \sum_{k=1}^N\left(1-p_k^2\right)\varphi_{kk}R_k^2} \quad (34)$$

The price of a call option with strike K and option maturity T is therefore equal:

$$\text{Option}(0) = DF(0)\left(FI(0)N(d_1) - KN(d_2)\right) \quad (35)$$

where:

$$d_1 = \frac{\ln\left(S_N(0)/K\right) + T\sigma^2/2}{\sigma\sqrt{T}} \quad (36)$$

$$d_2 = d_1 - \sigma\sqrt{T} \quad (37)$$

This method can also be used for Bermudan options. The other (perhaps more important) use of this method is to manage a portfolio of interest rate options. The volatility of all caps, floors, swaptions, etc, can be decomposed into "building blocks" (ie, the volatilities of forward Libor rates). Calculating Greeks (with respect to forward Libor rates) of such a portfolio is straightforward. A naive (Black) approach does not allow such a possibility because when one uses models in the traditional way, exact calibration of model parameters is required.

One may ask what happens if there are some path-dependent options in the portfolio that require a more sophisticated treatment. In this situation, one is compelled to determine the instantaneous volatility. The simplified calibration procedure should be considered as a first step. Having already determined the matrix Φ, one can proceed in the usual way (see Pedersen, 1999, for example). The smile effect can be treated following Andersen & Andreasen, 1997, and Hull & White, 1999, by substituting equation (4) with:

$$dL_n(t) = a_n(t)\,dt + \gamma_n(t)c\left(L_n(t)\right)dW(t) \quad (38)$$

where c is a deterministic function (usually a power). ■

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