Pricing Complex Barrier Options with General Features Using Sharp Large Deviation Estimates

Paolo Baldi* Lucia Caramellino† Maria Gabriella Iovino‡

February 5, 2003

Premia 5

Abstract

In this paper we adapt the simulation procedures, already developed in a previous paper, in order to evaluate single and double barrier options with cash rebates and Parisian barrier options. Our method is based on Sharp Large Deviation estimates, which allow one to improve the usual Monte Carlo procedure. Numerical results are provided and show the validity of the proposed simulation algorithm.

1 Introduction

Barrier options have recently become increasingly popular in the financial markets since they are less expensive than conventional options and are a valuable tool for risk management purposes. In fact, they are defined in such a way that investors do not have to pay for states they believe are unlikely to occur. More precisely, barrier options differ from the well known European conventional options by means of the introduction of one or two boundaries, deterministic and possibly time-dependent, which are contractually specified,

*Dipartimento di Matematica, University of Rome - Tor Vergata, Via della Ricerca Scientifica, 00133 Roma, Italy. e-mail baldi@mat.uniroma2.it

†Dipartimento di Matematica, University of Rome - Roma Tre, Largo San L. Murialdo 1, 00146 Roma, Italy. e-mail lucia@mat.uniroma3.it

‡Istituto di Matematica Finanziaria, University of Perugia, Via A. Pascoli 1, 06100 Perugia, Italy. e-mail iovino@unipg.it

and which may nullify the value of the option if breached by the underlying asset price. For instance, a *knock-and-out* barrier call is equivalent to the corresponding standard call, provided that the underlying asset price does not hit either barrier, otherwise its payoff is set equal to zero. Similarly, one may define the *knock-and-in* call by setting to zero its value if the asset price does not cross either barrier. More generally, it is possible to include a pre-specified cash rebate which is paid out if a knock-and-out option is knocked out during its life. On the other hand, knock-and-in barrier options are contractually specified to pay the cash rebate at maturity if the option expires without knocking in a barrier. Therefore, the knock-and-out and the knock-and-in features lower the price of the option with respect to the conventional one.

Although the introduction of cash rebates guarantees that the option holder is not left uncovered when the option is nullified, these instruments still carry significant risk. In fact, the spot price needs only hit the barrier once for the option to be terminated. In addition, as it has been pointed out by several authors ([1]), "when large positions of knock-and-out options with the same barrier are accumulated in the market, traders can drive the price of the underlying asset to the barrier, thus creating massive losses by triggering the barriers". To this purpose, recent literature has proposed the introduction of new types of barrier options that mean to overcome these problems. In particular, Parisian barrier options are delayed options such that they can be nullified (or activated) only if the underlying spot price is observed to have spent uninterruptedly a pre-specified length of time beyond a barrier (occupation time).

Barrier options with cash rebates and Parisian barrier options belong to the wide class of *path dependent* options, where closed form solutions are available only under particular frameworks. For barrier options with single or double barrier and zero rebates, closed form solutions have been provided by in [5]the Black and Scholes model framework when the boundaries are of the exponential-type. When cash rebates are included, the price of the option is known in closed form in the single barrier case [3] Finally, [4], developed Laplace transform techniques in order to price Parisian barrier options when the barrier is single and constant.

Unfortunately Monte Carlo simulations, which usually provide a flexible and easy approach, do not perform well in the context of barrier options with cash rebates and delayed barrier options. The reason is that the underlying asset price is checked at discrete instants through simulations. If the option contractually specified the discrete monitoring of the security price, as it happens in several cases (see for instance [8]), Monte Carlo methods would provide an unbiased estimator for the price. In our case, since the contract is supposed to be continuously monitored, standard Monte Carlo simulations always give an over-estimation of the hitting time and therefore misprice the option. In fact, the barrier might have been hit without being detected.

The problem of improving the performance of Monte Carlo methods has already been considered in literature, see e.g. [7] and [6]. Also, in a paper by [2] , making use of Sharp Large Deviation techniques, simple formulas are derived in order to obtain precise estimates of the probability that the underlying asset hits a barrier during each step of the simulation. Estimates for the exit probability of the Brownian bridge are provided in situations in which its exact value is not known (such as double and time-dependent barrier options with underlying asset price driven by a general diffusion process).

In this paper those results are extended to include barrier options with cash rebates and Parisian barrier options, both with single and double barriers. The formulas obtained, suitable to be implemented in a simulation program, allow one to improve the standard Monte Carlo estimates, as shown by the numerical experiments described in Section 4.

The paper is organized as follows. In Section 2 we give the asymptotic formulas for the exit probabilities when cash rebates are included and the underlying security price is supposed to evolve as a geometric Brownian motion. Moreover, the simulation procedure for the pricing of Parisian barrier options is described.

In Section 3, we show how these estimates can be adapted to the case when the underlying security price evolves as a general diffusion process. In particular, the CEV (Constant Elasticity of Variance) model will be considered. In such a framework, numerical results are produced in Section 4 and compared with those obtained in [9].

In Section 4, numerical results are shown. Comparisons with results obtained by means of alternative analytical and numerical methods known in literature show good improvements of our procedure with respect to the standard Monte Carlo simulation.

2 The simulation procedure

Let S_t denote the underlying stock price which we assume here to evolve as a geometric Brownian motion in the time-interval $[0, T]$, that is

$$dS_t = \mu S_t dt + \sigma S_t dB_t \tag{1}$$

where μ and σ are constant and B is a one-dimensional Brownian motion. Let $\tilde{U}$ and $\tilde{L}$ be two real functions, $\tilde{U}, \tilde{L} : [0, +\infty) \to [0, +\infty)$, such that $\tilde{L}(t) < \tilde{U}(t)$ for every t. $\tilde{U}$ and $\tilde{L}$ are the *upper* and *lower* barrier respectively.

All the path-dependent options we are going to study have pricing formulas expressed as the conditional expectation (under the risk neutral probability measure) of a functional which depends on the first time, τ, the underlying asset price hits a barrier, i.e.

$$\tau = \inf\{t > 0 \,:\, S_t \leq \tilde{L}(t) \text{ or } S_t \geq \tilde{U}(t)\}. \tag{2}$$

In order to numerically estimate τ, let us consider a partition $t_0 \equiv 0 < t_1 < \cdots < t_n \equiv T$ of the time interval $[0, T]$ with $t_{i+1} - t_i = \varepsilon$ for $i = 0, \ldots, n-1$. Since Equation (1) admits an explicit solution, at each step the asset price $S_{t_{i+1}}$ is simulated by means of its value at time t_i, i.e. under the risk neutral probability measure one has,

$$S_{t_{i+1}} = S_{t_i} \mathrm{e}^{(r - \frac{\sigma^2}{2})\varepsilon + \sigma(B_{t_{i+1}} - B_{t_i})},$$

where r denotes the (constant) spot rate. We need to know the conditional probability that S_t hits the barriers during the time-interval (t_i, t_{i+1}), given the observations S_{t_i} and $S_{t_{i+1}}$. This is equivalent to determining the probability that the process $W_t = \log S_t$ hits the barriers $L(t) = \log \tilde{L}(t)$ or $U(t) = \log \tilde{U}(t)$ in the time interval (t_i, t_{i+1}), given the observations $W_{t_i} = \log S_{t_i} = \zeta$ and $W_{t_{i+1}} = \log S_{t_{i+1}} = y$. In other words, we look for the probability p_i^ε that a Brownian bridge starting at ζ at time t_i and conditional to be in y at time t_{i+1} breaches the barriers. The exact value of p_i^ε is known in the case of single or double barrier, if it is constant. For double constant barriers an exact expression is known in the form of an infinite series: Kunitomo and Ikeda (1992) and Revuz and Yor (1994) p.105–106. No exact formulas are available for general (single or double) time-dependent barriers. However as we shall point out later, an exact formula holds for a single linear barrier, a fact which is not widely known. In this section, using Sharp Large Deviation arguments, we provide an approximation of p_i^ε by studying its asymptotic behaviour as $\varepsilon \to 0$.

More precisely, let us denote such a probability with $p^\varepsilon_{U,L}(t_i, \zeta, y)$. Suppose that U and L are continuous with Lipschitz continuous derivatives. Then for every $\zeta, y \in (L(t_i), U(t_i))$ for every $i = 0, \ldots, n-1$ one has

$$p^\varepsilon_{U,L}(t_i, \zeta, y) = \exp\left\{ -\frac{Q_{U,L}(t_i, \zeta, y)}{\varepsilon} - R_{U,L}(t_i, \zeta, y) \right\} (1 + O(\varepsilon)) \tag{3}$$

where

$$Q_{U,L}(t_i, \zeta, y) = \begin{cases} Q_U(t_i, \zeta, y) & \textit{if} \quad \zeta + y > U(t_i) + L(t_i) \\ Q_L(t_i, \zeta, y) & \textit{if} \quad \zeta + y < U(t_i) + L(t_i) \end{cases}$$

and

$$R_{U,L}(t_i,\zeta,y) = \begin{cases} R_U(t_i,\zeta,y) & if \quad \zeta + y > U(t_i) + L(t_i) \\ R_L(t_i,\zeta,y) & if \quad \zeta + y < U(t_i) + L(t_i) \end{cases}$$

Q_U, R_U, Q_L, R_L being

$$Q_U(t_i,\zeta,y) = \frac{2}{\sigma^2}\left(U(t_i) - \zeta\right)\left(U(t_i) - y\right),$$

$$R_U(t_i,\zeta,y) = \frac{2}{\sigma^2}\left(U(t_i) - \zeta\right)U'(t_i),$$

$$Q_L(t_i,\zeta,y) = \frac{2}{\sigma^2}\left(\zeta - L(t_i)\right)\left(y - L(t_i)\right),$$

and

$$R_L(t_i,\zeta,y) = -\frac{2}{\sigma^2}\left(\zeta - L(t_i)\right)L'(t_i).$$

Choosing respectively $L(t) = -\infty$ or $U(t) = +\infty$ (i.e. $\tilde{L}(t) = 0$ or $\tilde{U}(t) = +\infty$), we easily obtain the exit probability $p_U^\varepsilon(t_i,\zeta,y)$ or $p_L^\varepsilon(t_i,\zeta,y)$ from a single (upper or lower) barrier:

Under the assumptions of Theorem 2, for every $\zeta,\, y \in (L(t_i),\, U(t_i))$

$$p_U^\varepsilon(t_i,\zeta,y) = \exp\left\{-\frac{Q_U(t_i,\zeta,y)}{\varepsilon} - R_U(t_i,\zeta,y)\right\}(1 + O(\varepsilon))$$

$$p_L^\varepsilon(t_i,\zeta,y) = \exp\left\{-\frac{Q_L(t_i,\zeta,y)}{\varepsilon} - R_L(t_i,\zeta,y)\right\}(1 + O(\varepsilon)).$$

It is worthwhile to point out that if we consider the case of a single and linear barrier (constant barrier in particular), the approximation is exact, i.e. $O(\varepsilon) = 0$. Concerning the double constant barrier case, our results turn out to be very precise: the quantity $O(\varepsilon)$ has to be replaced with $o(\varepsilon^k)$, for any k. For details,[2], where also the proofs of Theorem 2 and Corollary 2 can be found.

The approximation of the exit probabilities given above is now used to numerically price single and double barrier options with cash rebates. The option pricing formula of a knock-and-out call with cash rebate Γ is:

$$C(0) = \mathbb{E}_0\left[\mathrm{e}^{-rT}\max(S_T - K, 0)_{\tau > T} + \mathrm{e}^{-r\tau}\Gamma_{\tau \le T}\right] \tag{4}$$

where K stands for the exercise price and the rebate is paid out at time τ, i.e. when the barrier is hit. For a knock-and-in call, the events $\{\tau > T\}$ and $\{\tau \le T\}$ have to be switched:

$$C(0) = \mathbb{E}_0\left[\mathrm{e}^{-rT}\max(S_T - K, 0)_{\tau \le T} + \mathrm{e}^{-rT}\Gamma_{\tau > T}\right] \tag{5}$$

and the rebate is paid if the option has not been activated, i.e. at expiration.

The numerical procedure for the pricing of a knock-and-out call can be implemented as follows. At each partition instant t_i, by neglecting the terms of order ε in Equation (3), we approximate the hitting probability p_i^ε with:

$$\exp\left\{-\frac{Q_{U,L}(t_i, \log S_{t_i}, \log S_{t_{i+1}})}{\varepsilon} - R_{U,L}(t_i, \log S_{t_i}, \log S_{t_{i+1}})\right\}, \tag{6}$$

where $Q_{U,L}$ and $R_{U,L}$ are defined in Theorem 2 and Corollary 2, depending on the case of interest. Now, with probability equal to p_i^ε we stop the simulation and set t_i as the hitting time τ^ε; with probability $1 - p_i^\varepsilon$ we carry on the simulation. The approximation of the knock-and-out call option price is thus

$$C^\varepsilon(0) = \mathbb{E}_0\left[\mathrm{e}^{-rT}\max(S_T - K, 0)_{\tau^\varepsilon > T} + \mathrm{e}^{-r\tau^\varepsilon}\Gamma_{\tau^\varepsilon \le T}\right],$$

which, as usual, is numerically evaluated by simulating a large number of independent paths.

In the double barrier framework, it is interesting to study the case of two different cash rebates, denoted by Γ_U and Γ_L, which depend on whether the underlying asset price hits on the upper or on the lower barrier respectively. Setting τ_U and τ_L the exit times from the upper and lower boundary respectively, obviously τ, defined in (2), coincides with $\tau_U \wedge \tau_L$. The value of a knock-and-out call with double cash rebates is:

$$\begin{aligned} C(0) = \mathbb{E}_0 \quad &\left[\mathrm{e}^{-rT}\max(S_T - K, 0)_{\tau > T}\right. \\ &\left. + \mathrm{e}^{-r\tau_U} R_{U\tau=\tau_U \le T} + \mathrm{e}^{-r\tau_L} R_{L\tau=\tau_L \le T}\right]. \end{aligned} \tag{7}$$

Of course, the introduction of two different rebates does not make sense in a knock-and-in framework.

The algorithm described above cannot be immediately implemented in order to compute numerically (7) because Theorem 2 does not give an approximation of the probabilities of the events $\{\tau = \tau_U\}$ and $\{\tau = \tau_L\}$.

To this purpose, we set p_{1U}^ε and p_{1L}^ε as the conditional probabilities of $\{\tau = \tau_U\}$ and $\{\tau = \tau_L\}$, respectively, during the time-interval $[t_i, t_{i+1}]$, given the observations. It is not hard to state an approximation for such probabilities; these turn out to be asymptotically equal to p_U^ε and p_L^ε, respectively. Under the assumptions of Theorem 2, for every ζ, $y \in (L(t_i), U(t_i))$, one has:

$$p_{1U}^\varepsilon(t_i, \zeta, y) = \exp\left\{-\frac{Q_U(t_i, \zeta, y)}{\varepsilon} - R_U(t_i, \zeta, y)\right\}(1 + O(\varepsilon))$$

$$p_{1L}^\varepsilon(t_i, \zeta, y) = \exp\left\{-\frac{Q_L(t_i, \zeta, y)}{\varepsilon} - R_L(t_i, \zeta, y)\right\}(1 + O(\varepsilon)).$$

Proof. We come back to the notations introduced at the beginning of this section and we refer to Baldi, Caramellino and Iovino (1998), Section 5, for the Large Deviation arguments we are going to use.

The asymptotic behavior of p^ε_{1U} can be investigated by means of the Large Deviation Principle satisfied, as $\varepsilon \to 0$, by the conditional law of $(W_{t_i+t})_{t\in(0,t_{i+1}-t_i)}$ given the observation $\{W_{t_i}=\zeta, W_{t_{i+1}}=y\}$. In particular, two special paths exist, say γ_U and γ_{1U}, which minimise the rate function over the set of the paths reaching the upper barrier and the set of the paths reaching the upper barrier before the lower one, respectively. Such paths are extremely important because they give the asymptotic behavior of p^ε_U and p^ε_{1U}. In fact, by setting $B_\delta(\gamma)=\{h\,:\,\sup_{t\in[0,t_{i+1}-t_i]}|h(t)-\gamma(t)|<\delta\}$, with $\delta>0$, by a typical Large Deviation argument, the asymptotics of p^ε_U and p^ε_{1U} are the same as

$$P\Big(\{\tau_U<t_{i+1}\}\cap\{W_{t_i+\cdot}\in B_\delta(\gamma_U)\}|W_{t_i}=\zeta,W_{t_{i+1}}=y\Big) \tag{8}$$

and

$$P\Big(\{\tau=\tau_U<t_{i+1}\}\cap\{W_{t_i+\cdot}\in B_\delta(\gamma_{1U})\}|W_{t_i}=\zeta,W_{t_{i+1}}=y\Big) \tag{9}$$

respectively. Since γ_U reaches the upper but not the lower barrier, if δ is small enough all paths in $B_\delta(\gamma_U)$ behave in the same way, so that (8) is equal to

$$P\Big(\{\tau=\tau_U<t_{i+1}\}\cap\{W_{t_i+\cdot}\in B_\delta(\gamma_U)\}|W_{t_i}=\zeta,W_{t_{i+1}}=y\Big) \tag{10}$$

It is easy to check that $\gamma_U=\gamma_{1U}$, so that, by comparing (10) and (9) it follows that the asymptotics of p^ε_U and p^ε_{1U} do coincide, i.e. $p^\varepsilon_{1U}=p^\varepsilon_U(1+o(\varepsilon^k))$, for any k. By Corollary 2, the statement holds. The same proof holds for the case of the lower barrier. ♠

Since $p^\varepsilon_{U,L}=p^\varepsilon_{1U}+p^\varepsilon_{1L}$, one could think that Lemma 2 is in contrast with Theorem 2. However, this is not the case. In fact, the condition $\zeta+y>U(t_i)+L(t_i)$ is equivalent to $Q_U<Q_L$ (see the proof of Theorem 2.1 in Baldi, Caramellino and Iovino (1998)), so that

$$\begin{aligned}p^\varepsilon_{U,L}=p^\varepsilon_{1U}+p^\varepsilon_{1L}=p^\varepsilon_{1U}\Big(1+\frac{p^\varepsilon_{1L}}{p^\varepsilon_{1U}}\Big)\\ =p^\varepsilon_{1U}\Big(1+\exp\Big(-\frac{Q_L-Q_U}{\varepsilon}-R_L+R_U\Big)\Big)=p^\varepsilon_{1U}(1+o(\varepsilon^k)).\end{aligned}$$

The same arguments hold if $\zeta+y<U(t_i)+L(t_i)$. Finally, we state the result that is used in the simulation algorithm for the pricing of double barrier options with cash rebates: Under the assumptions of Theorem 2, for

every ζ, $y \in (L(t_i), U(t_i))$, for every $k > 0$ one has:

if $\zeta + y > U(t_i) + L(t_i)$ then

$$P\Big(\tau = \tau_U | W_{t_i} = \zeta, W_{t_{i+1}} = y, \tau \leq t_{i+1}\Big) = 1 + o(\varepsilon^k);$$

if $\zeta + y < U(t_i) + L(t_i)$ then

$$P\Big(\tau = \tau_L | W_{t_i} = \zeta, W_{t_{i+1}} = y, \tau \leq t_{i+1}\Big) = 1 + o(\varepsilon^k)$$

Proof. By Remark 2 and Proposition 2, the statement follows immediately: if $\zeta + y > U(t_i) + L(t_i)$ then

$$P\Big(\tau = \tau_U | W_{t_i} = \zeta, W_{t_{i+1}} = y, \tau \leq t_{i+1} \leq t_{i+1}\Big) = \frac{p^\varepsilon_{1U}}{p^\varepsilon_{U,L}} = 1 + o(\varepsilon^k).$$

Similarly, if $\zeta + y < U(t_i) + L(t_i)$ then

$$P\Big(\tau = \tau_L | W_{t_i} = \zeta, W_{t_{i+1}} = y, \tau \leq t_{i+1} \leq t_{i+1}\Big) = \frac{p^\varepsilon_{1L}}{p^\varepsilon_{U,L}} = 1 + o(\varepsilon^k).$$

♠

Now, at each step, we compute the approximation p^ε_i of the exit probability by means of Formula (6). With probability $1 - p^\varepsilon_i$ the simulation is carried on. Otherwise, i.e. with probability p^ε_i, one states that the path has reached the boundary, so that, by Proposition 2, we put:

- if $\log S_{t_i} + \log S_{t_{i+1}} > U(t_i) + L(t_i)$ then $\tau^\varepsilon_U = t_i$;
- if $\log S_{t_i} + \log S_{t_{i+1}} < U(t_i) + L(t_i)$ then $\tau^\varepsilon_L = t_i$.

Our procedure can now be adjusted in order to evaluate Parisian barrier options. The payoff of a knock-and-out call with double barrier is defined as the payoff of a standard call provided that during its life the underlying stock price stays above the barrier $\tilde{U}$ or below the barrier $\tilde{L}$ uninterruptedly for longer than a pre-specified time length D. In formulas, the price of a, say, knock-and-out call is

$$C(0) = \mathbb{E}_0 \left[\max(S_T - K, 0)_{H^{U,L}_D > T} \right] \tag{11}$$

where

$$H^{U,L}_D = \inf\{t > 0 \, : \, (t - g^{U,L}_t)_{\{S_t \geq \tilde{U} \text{ or } S_t \leq \tilde{L}\}} \geq D\}$$

and

$$g_t^{U,L} = \sup\{u \le t \,:\, S_u = \tilde{U} \text{ or } S_u = \tilde{L}\}.$$

Similarly, one can define a Parisian option with the knock-and-in feature, simply by replacing $\{H_D^{U,L} > T\}$ with $\{H_D^{U,L} \le T\}$.

If a standard Monte Carlo procedure is implemented, one sets initially $\hat{g}_0^{U,L} = 0$ (being $S_0 \in (\tilde{L}, \tilde{U})$ typically). As long as the price is below $\tilde{U}$ and above $\tilde{L}$, $\hat{g}_{t_i}^{U,L}$ is set equal to the current time t_i. We define $\hat{H}_D^{U,L} = t_i - \hat{g}_{t_i}^{U,L}$. So, as long as the process belongs to $(\tilde{L}, \tilde{U})$, $\hat{H}_D^{U,L} = 0$.

The variable $\hat{g}_{t_i}^{U,L}$ is updated at the first instant t_i such that $S_{t_i} \ge \tilde{U}$ or $S_{t_i} \le \tilde{L}$ and set to

$$\hat{g}_{t_i}^{U,L} = \begin{cases} t_{i-1} + \dfrac{U - \log S_{t_{i-1}}}{\log S_{t_i} - \log S_{t_{i-1}}}\varepsilon, & \text{if } S_{t_i} > \tilde{U} \\[2ex] t_{i-1} + \dfrac{\log S_{t_{i-1}} - L}{\log S_{t_{i-1}} - \log S_{t_i}}\varepsilon, & \text{if } S_{t_i} < \tilde{L} \end{cases}$$

where $U = \log \tilde{U}$ and $L = \log \tilde{L}$ (recall that ε stands for the step-size). This is implied from Brownian bridge arguments. In fact, suppose for instance that $S_{t_i} > \tilde{U}$: we would need to know the last time $u \in (t_{i-1}, t_i)$ such that $S_u = \tilde{U}$, conditional on the observations $S_{t_{i-1}} < \tilde{U}$ and $S_{t_i} \ge \tilde{U}$. Now, the conditional law of $W_t = \log S_t$ is equal to the law induced by a Brownian bridge on $[t_{i-1}, t_i]$ with endpoints $\zeta = \log S_{t_{i-1}}$ and $y = \log S_{t_i}$, i.e.

$$\zeta + \frac{y - \zeta}{\varepsilon}(t - t_i) + V_t^\varepsilon$$

where $V_t^\varepsilon = \sigma(B_t - B_{t_{i-1}} - (t - t_{i-1})(B_{t_i} - B_{t_{i-1}})/\varepsilon)$. Thus the expression above for $\hat{g}_{t_i}^{U,L}$ amounts to neglect the (small) term V^ε.

The value of $\hat{g}^{U,L}$ is not changed for all the instants t_j, with $j > i$, when the underlying asset price stays beyond the barriers (i.e. above $\tilde{U}$ if $S_{t_i} > \tilde{U}$ and below $\tilde{L}$ if $S_{t_i} < \tilde{L}$). Only during these steps the value of $\hat{H}_D^{U,L}$ increases. The algorithm checks the steps at which $\hat{H}_D^{U,L}$ increases: as soon as $\hat{H}_D^{U,L} > D$ the simulation path is stopped, otherwise it is carried on. If in a future instant the process comes back into the interval $(\tilde{L}, \tilde{U})$ before $\hat{H}_D^U$ has reached D, then again $\hat{g}^{U,L}$ is set equal to the current time and so on.

Our correction procedure is activated only if at the current and previous step the process has been outside $(\tilde{L}, \tilde{U})$: if both S_{t_i} and $S_{t_{i-1}}$ are above $\tilde{U}$ (below $\tilde{L}$), we check if it has crossed in between by using our approximation of the probability of exit from $\tilde{U}$ ($\tilde{L}$), which corresponds to a lower (upper) boundary. If it did, we set $\hat{g}_{t_i}^{U,L} = t_i$ (the current time) so that $\hat{H}_D^U = 0$.

In contrast with the algorithms previuosly introduced, in this procedure the standard Monte Carlo kills more trajectories than the corrected Monte Carlo one and the price is lower in the first case.

The simulation methods proposed here are straightforward to implement since the correction probability has a simple form. Moreover, the complexity of the algorithm does not change by introducing the correction.

3 Generalization of the procedure

We now investigate how these procedures can be adapted when the underlying asset price S evolves, with respect to the risk neutral probability measure, as a diffusion process with general coefficients, including the case of a volatility depending on the state of the process:

$$dS_t = \mu(S_t,t)dt + \sigma(S_t,t)dB_t \tag{12}$$

where the coefficients μ and σ satisfy the usual assumptions which guarantee the existence and the uniqueness of the solution.

Let $t_0 \equiv 0 < t_1 < \cdots < t_n \equiv T$ be a partition of the time interval $[0,T]$ with $t_{i+1}-t_i = \varepsilon$, $i = 0,1,\ldots,n-1$. S_t may be estimated by using the Euler approximation scheme: $(S_t)_{t\in(t_i,t_{i+1})}$ is approximated by the process

$$S^\varepsilon_t = S^\varepsilon_{t_i} + \mu(S^\varepsilon_{t_i},t_i)(t-t_i) + \sigma(S^\varepsilon_{t_i},t_i)(B_t - B_{t_i}).$$

As in the previous section, we would need to know the conditional probability that S hits the barriers during the time-interval (t_i,t_{i+1}), given the observations $S_{t_i} = \zeta$ and $S_{t_{i+1}} = y$. Since this is not feasible in general, a reasonable approximation of the exit probability may be obtained by replacing the conditional distribution of S on $[t_i,t_{i+1}]$ with the distribution of the diffusion S^ε, whose (constant) coefficients are $\mu_0 = \mu(\zeta,t_i), \sigma_0 = \sigma(\zeta,t_i)$, with $\zeta = S^\varepsilon_{t_i}$. The exit probability (or at least its asymptotic behaviour as $\varepsilon \to 0$) of this conditioned diffusion can easily be computed, since it reduces to the computations already made for the Brownian bridge.

More precisely the proposed estimate of the exit probability is

$$p^\varepsilon_{U,L}(t_i,\zeta,y) = \exp\left\{-\frac{Q_{U,L}(t_i,\zeta,y)}{\varepsilon} - R_{U,L}(t_i,\zeta,y)\right\}$$

where

$$Q_{U,L}(t_i,\zeta,y) = \begin{cases} Q_U(t_i,\zeta,y) & if \quad \zeta + y > U(t_i) + L(t_i) \\ Q_L(t_i,\zeta,y) & if \quad \zeta + y < U(t_i) + L(t_i) \end{cases}$$

and

$$R_{U,L}(t_i,\zeta,y)=\begin{cases} R_U(t_i,\zeta,y) & if \quad \zeta+y>U(t_i)+L(t_i)\\ R_L(t_i,\zeta,y) & if \quad \zeta+y<U(t_i)+L(t_i)\end{cases}$$

Q_U, R_U, Q_L, R_L being

$$Q_U(t_i,\zeta,y)=\frac{2}{\sigma^2(\zeta,t_i)}(U(t_i)-\zeta)(U(t_i)-y),$$

$$R_U(t_i,\zeta,y)=\frac{2}{\sigma^2(\zeta,t_i)}(U(t_i)-\zeta)U'(t_i)),$$

$$Q_L(t_i,\zeta,y)=\frac{2}{\sigma^2(\zeta,t_i)}(\zeta-L(t_i))(y-L(t_i)),$$

and

$$R_L(t_i,\zeta,y)=-\frac{2}{\sigma^2(\zeta,t_i)}(\zeta-L(t_i))L'(t_i).$$

In the case of a single barrier the approximations of the exit probabilities are

$$p_U^\varepsilon(t_i,\zeta,y)=\exp\left\{-\frac{Q_U(t_i,\zeta,y)}{\varepsilon}-R_U(t_i,\zeta,y)\right\}$$
$$p_L^\varepsilon(t_i,\zeta,y)=\exp\left\{-\frac{Q_L(t_i,\zeta,y)}{\varepsilon}-R_L(t_i,\zeta,y)\right\}$$

We point out that, for a single constant barrier these are the approximate values for the exit probability introduced by Beaglehole, Dybvig and Zhou (1997).

In the next section, we apply the setting above to the context of the CEV model, first studied by Cox (1975). Equation (12) is then

$$dS_t=rS_t du+\sigma {S_t}^{\frac{\alpha}{2}} dB_t, \tag{13}$$

where r and σ are constant and the elasticity factor α is such that $0<\alpha\leq 2$. When α is equal to 2, S is a geometric Brownian motion, which was handled in the previous section. Comparisons with the outcomes obtained by [9] (who determine the price of double constant barrier options under the CEV process by means of a numerical procedure) are provided in the following section.

All the simulation procedures described in the previous section, which allow one to price cash rebates and Parisian barrier options, may be adapted to this more general framework, making use of the exit probabilities given above.

The procedures outlined in this and in the previous section can also be implemented with only piecewise continuous barriers. To do this, just choose the partition time-intervals t_i so that in any sub-interval (t_i, t_{i+1}) there are no discontinuities.

A number of detailed remarks about the correction algorithm, may be found in Baldi, Caramellino and Iovino (1998), Remark 3.1.

4 Numerical results

The procedure outlined in Sections 2 and 3 (from here on *corrected Monte Carlo*) are now applied to compute cash rebates and Parisian double-barrier option prices. We have evaluated, both for the standard and corrected Monte Carlo algorithms, 100 numerical evaluations of the option price, each obtained by simulating 10000 paths of the underlying asset process. The step-size is set equal to 1/365 unless otherwise specified. The standard deviation is displayed in brackets below the evaluated prices.

In order to compare the results where closed-form solutions are available, our method is firstly used in the Black-Scholes framework. The price of a double-barrier option with zero cash rebate is computed assuming that the barriers are constant or of the exponential type. Comparisons with Kunitomo and Ikeda (K-I), who allow the barriers to evolve exponentially with time as well as to be constant, are provided in Table 1, where the prices are computed for upper and lower barriers defined as $\tilde{U}(t) = B\exp(\delta_1 t)$ and $\tilde{L}(t) = A\exp(\delta_2 t)$ respectively and for several values of the parameters. We have set $A = 1.5$, $B = 2.5$ and (δ_1, δ_2) takes the values $(-0.1, 0.1)$, $(0, 0)$, and $(0.1, -0.1)$, in order to show the effects of the upward-downward behaviour of the two barriers on the option price.

In Table 1, the time-to-maturity is set equal to 1 year and the initial underlying asset price S_0 is set equal to 2. For the volatility parameter and for the spot interest rate we consider two levels: 20% per annum for the first and 2% for the second.

In Table 2 the effects of the time-to-maturity on the option prices are investigated. As expected, since Monte Carlo procedures are heavily affected by the chosen step-size, this phenomenon becomes more noticeable as the time-to-maturity decreases. The corrected procedure shows very good approximations of the option prices even when the step-size is not too small. Indeed, in the particular case of a single constant barrier, the estimator provided by the corrected algorithm is unbiased even with step-size $\varepsilon = T$ (being the exit probability exact in this particular case). Table 2 provides the results of the simulations when the time-to-maturity of the option is 1 month

Table 1: Double Knock-and-Out Call with time to maturity 1 year under the Black and Scholes model ($S_0 = 2$, $\sigma = 0.2$, $r = 0.02$, $k = 2$, $A = 1.5$, $B = 2.5$)

(δ_1, δ_2)	$(-0.1, 0.1)$	$(0, 0)$	$(0.1, -0.1)$
K-I	0.00916	0.04109	0.08544
cor.M-C	0.00910	0.04104	0.08568
	(0.0003)	(0.0010)	(0.0015)
st.M-C	0.01060	0.04413	0.08929
	(0.0004)	(0.0010)	(0.0016)

Table 2: Double Knock-and-out Call with time to maturity 1 month under the Black and Scholes model ($S_0 = 2.4$, $\sigma = 0.2$, $r = 0.02$, $k = 2$, $A = 1.5$, $B = 2.5$)

	step-size $\varepsilon = \frac{1}{365}$			step-size $\varepsilon = \frac{1}{4\times365}$		
(δ_1, δ_2)	$(-0.1, 0.1)$	$(0, 0)$	$(0.1, -0.1)$	$(-0.1, 0.1)$	$(0, 0)$	$(0.1, -0.1)$
K-I	0.14269	0.16282	0.18336	0.14269	0.16282	0.18336
cor.M-C	0.14080	0.16092	0.18161	0.14236	0.16251	0.18318
	(0.0016)	(0.0017)	(0.0016)	(0.0016)	(0.0017)	(0.0017)
st.M-C	0.16394	0.18513	0.20667	0.15434	0.17505	0.19611
	(0.0018)	(0.0017)	(0.0016)	(0.0016)	(0.0018)	(0.0018)

and the step-size are $\frac{1}{365}$ and $\frac{1}{4\times365}$ respectively.

In Table 3, we compare the prices obtained in Table 1 when cash rebates are introduced. More precisely, we have allowed the pair of rebates (Γ_L, Γ_U) to vary among values which cover the following cases: zero rebates, single lower and single upper rebate, double equal rebates and double different rebates.

The outcomes show that standard Monte Carlo procedures systematically overprice the knock-and-out call option.

In Table 4, we make use of the algorithm described in Section 2 in order to determine the price of Parisian options. The values taken by the time length D range between 0 and 0.15, in order to show the occupation time effects on the price. Remark that the standard Monte Carlo procedure kills many more paths than the corrected one, therefore lowering the price.

Finally, by means of the approximations introduced in Section 3 we have implemented our procedure and made comparisons with the numerical results

Table 3: Double Knock-and-out Call with cash rebates, time to maturity 1 year, under the Black and Scholes model ($S_0 = 2$, $\sigma = 0.5$, $r = 0.05$, $k = 2$, $A = 1.5$, $B = 3$)

(Γ_L, Γ_U)	$(0, 0)$	$(0.01, 0)$	$(0, 0.01)$	$(0.01, 0.01)$	$(0.015, 0.01)$	$(0.01, 0.015)$
cor.M-C	0.04104	0.04260	0.04375	0.04516	0.04580	0.04645
	(0.0010)	(0.0009)	(0.0010)	(0.0008)	(0.0009)	(0.0009)
st.M-C	0.04413	0.00456	0.04679	0.04800	0.04886	0.04916
	(0.0010)	(0.0010)	(0.0010)	(0.0008)	(0.0010)	(0.0009)

Table 4: Double Knock-and-out Parisian Call with time to maturity 1 year under the Black and Scholes model ($S_0 = 2$, $\sigma = 0.2$, $r = 0.02$, $k = 2$, $A = 1.5$, $B = 2.5$)

D	0	0.01	0.05	0.1	0.15
cor.M-C	0.04104	0.05701	0.07636	0.09214	0.10509
	(0.0010)	(0.0012)	(0.0013)	(0.0017)	(0.0017)
st.M-C	0.04413	0.05492	0.07337	0.08920	0.10228
	(0.0010)	(0.0012)	(0.0013)	(0.0016)	(0.0017)

Table 5: Double Knock-and-out Call with time to maturity 6 months under the CEV model ($K = 105$, $U = 120$, $L = 90$)

model	$\alpha = 0.5$	$\alpha = 1$	$\alpha = 1.5$	$\alpha = 2$
B-T	0.5510	0.5115	0.4746	0.4404
cor.M-C	0.5518	0.5105	0.4760	0.4414
	(0.0176)	(0.0208)	(0.0178)	(0.0171)
st.M-C	0.6931	0.6469	0.6061	0.5676
	(0.0203)	(0.0225)	(0.0210)	(0.0195)

of Boyle and Tian (1997), in the context of the CEV process with elasticity factor $\alpha \in (0, 2])$. We test our procedure giving to α the values 0.5, 1, 1.5, 2.

Table 5 exhibits knock-and-out call option prices when the time to maturity is 6 months, the risk-free rate r is set equal to 0.1 per annum, the strike price $K = 105$ and the initial underlying asset price is 100. The upper and lower (constant) barriers are set equal to 90 and 120, respectively. In accordance with Boyle and Tian, we have set $\sigma = \sigma_{BS} S_0^{1-\frac{\alpha}{2}}$, where σ_{BS} is the instantaneous volatility of the Black and Scholes model ($\alpha = 2$). In the tables below σ_{BS} has been set equal to 0.25.

In all cases, the prices determined by means of the corrected procedure show high accordance with Boyle and Tian (B-T) prices. As expected, the standard Monte Carlo procedure strongly overprices the option.

In conclusion, the corrected Monte Carlo procedure improves the results obtained by the standard Monte Carlo algorithm and also the complexity of the latter is not changed by the introduction of the correction. Of course, the introduction of the correction increases the computing time of the standard algorithm. However, in order for the standard Monte Carlo procedure to achieve the same accuracy a radically smaller step size would be needed, since empirical evidence has shown that the bias of the standard Monte Carlo estimator decreases very slowly. In such a case, the computation time would be much longer than by applying the correction procedure with the bigger step size.

Acknowledgements. The authors are grateful to Emmanuel Buffet and Claude Martini for useful comments and discussions.

References

[1] V. LINETSKY. Step options (the feynman-kac approach to occupation time derivatives). Technical Report 96-18, University of Michigan, IOE Technical Report, 1996. 2

[2] P.BALDI L.CARAMELLINO M.G.IOVINO. Pricing general barrier options: a numerical approach using sharp large deviations. *To appear in Mathematical Finance (1999)*, 1999. 3, 5

[3] E.REINER M.RUBINSTEIN. Breaking down the barriers. *Risk*, 4:28–35, 191. 2

[4] M. CHESNEY M.JEANBLANC-PIQUÉ M.YOR. Brownian excursion and parisian barrier options. *Advances in Applied Probability*, 29:165–184, 1997. 2

[5] N.KUNIMOTO N.IKEDA. Pricing options with curved boundaries. *Mathematical finance*, 2:275–298, 1992. 2

[6] D.BEAGKEHOLE D.DYBVIG P.ZHOU. Going to extremes:correcting simulation bias in exotic option valuation. *Financial Analyst Journal*, pages 62–68, Jan-Fev 1997. 3

[7] L.ANDERSON R.BROTHERTON-RATCLIFFE. Exact exotics. *Risk*, (9):85–89, Oct 1996. 3

[8] M.BROADIE P.GLASSERMANN S.KOU. A continuity correction for discrete barrier options. *Mathematical Finance*, 7, 1997. 2

[9] P.BOYLE Y.TIAN. Pricing path–dependent options under the cev process. *Working Paper*, 1997. 3, 11