Multigrid methods for options pricing

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1 Introduction

The characteristic feature of the multigrid methods is a rapid rate of convergence. The convergence speed does not deteriorate when the discretisation is refined whereas classical iterative methods slow down for decreasing grid size. This unfavourable property of basic iterative methods is due to the inability of the methods to quickly dampen the low frequency components of the error. However, the high frequency components tend to be quickly damped and this smoothing property of classical iterative methods is exploited by the multigrid algorithm.

The basic idea of the multigrid algorithms is to compute on a sequence of a nested grids. The principle is to approximate the low frequency components of the error on coarser grids. The high frequency components are reduced withing a small number of iterations with a basic method on a finer grid. Therefore, the computation proceeds on a grid until the error becomes smooth and the rate of convergence slows, at which point the computation is transferred to a coarser grid. When the error has been reduced on the coarser grid, the solution on the finer grid is corrected using interpolated values from the coarser grid. This two grid method is applied recursively to obtain a multigrid method.

1.1 Multigrid method for stationary problems

The iteration consists of a "smoothing step" and a "coarse grid correction" involving a sequence of coarser grids (cf [2]). We assume that there are $l > 1$ grids with grid spacing $h,\ 2h,,\ 4h,,\ldots,,\ Lh = 2^{l-1}h$ and denote the linear discrete problem by

$$L^h u^h = f^h. \tag{1}$$

The multigrid iteration (V-cycle) at level l for solving 1 is defined by the following recursive procedure:

$$v_h \leftarrow V^h(v^h, f^h)$$

1. Relax m_1 times on $L^h u^h = f^h$ with a given initial guess v^h.
2. If Ω^h is the coarsest grid then go to step 4.

   Else
   $$\begin{aligned} &f^{2h} \leftarrow I_h^{2h}(f^h - L^h v^h). \\ &v^{2h} \leftarrow 0. \\ &v^{2h} \leftarrow V^{2h}(v^{2h}, f^{2h}) \end{aligned}$$
3. Correct $v^h \leftarrow v^h + I_{2h}^h v^{2h}$.
4. Relax m_2 times on $L^h u^h = f^h$ with initial guess v^h.

where I_{2h}^h is the linear interpolation operator and I_h^{2h} the restriction operator.

The V-cycle is just one of a family of multigrid cycling schemes. The entire family is called the μ-cycle method and is defined recursively by the following:

$$v_h \leftarrow M\mu^h(v^h, f^h)$$

1. Relax m_1 times on $L^h u^h = f^h$ with a given initial guess v^h.

2. If Ω^h is the coarsest grid then go to step 4.

   Else
$$\begin{aligned}&f^{2h} \leftarrow I_h^{2h}(f^h - L^h v^h).\\&v^{2h} \leftarrow 0.\\&v^{2h} \leftarrow V^{2h}(v^{2h}, f^{2h})\ \mu \text{ times}\end{aligned}$$

3. Correct $v^h \leftarrow v^h + I_{2h}^h v^{2h}$.

4. Relax m_2 times on $L^h u^h = f^h$ with initial guess v^h.

In practice, only $\mu = 1$ (which gives the V-cycle) and $\mu = 2$ are used.

Multigrid basic cycling schemes start with relaxations on the fine grid, then proceed to coarser levels to reduce smooth error components. Yet, it is perhaps better to start on coarser levels and proceed to finer grids only when sufficiently good approximation to the solution of the original equation have been achieved. In effect, we are thus led to using nested iteration again, but this time on the multigrid scheme itself. This yields the so-called full multigrid (FMG) cycling scheme. The algorithm has the following form:

$$v_h \leftarrow FMG^h(f^h)$$

1. If Ω^h = coarsest grid, set $v^h \leftarrow 0$ and go to step 3.

   Else
$$\begin{aligned}&f^{2h} \leftarrow I_h^{2h}(f^h),\\&v^{2h} \leftarrow FMG^{2h}(f^{2h}).\end{aligned}$$

2. Correct $v^h \leftarrow I_{2h}^h v^{2h}$.

3. $v^h \leftarrow V^h(v^h, f^h)$ m_0 times.

The basic aim of FMG is to guarantee a good initial guess for level h before any processing is done there.

1.2 Multigrid method for parabolic equations

A linear parabolic problem is given by

$$\frac{\partial u}{\partial t} + Lu = f$$

together with initial and boundary conditions. Assume that different levels of discretizations are given characterized by a spatial grid size h. In addition there are time steps δt. A simple discretization is the implicit formula

$$\frac{u^h(t) - u^h(t - \delta t)}{\delta t} + L^h(t)u^h(t) = f^h(t). \tag{2}$$

The conventional approach is to solve the equation 2 time step by time step: $u^h(t)$ is computed from $u^h(t+\delta t)$, then $u^h(t+\delta t)$ from $u^h(t)$ etc.

1.3 Multigrid algorithm (FMGH) for solving Hamilton-Jacobi-Bellman equations

In this part, we describe a multigrid algorithm based on the "Howard algorithm" (policy iteration) [3] and the multigrid method [4] to solve Hamilton-Jacobi-Bellman equations

$$\min_{u\in U} (A^u v + C^u) = 0$$

where U is the set of admissible policies.

The domain $\Omega \subset \mathbb{R}^n$ is approximated by a sequence of grids

$$\Omega_0 \subset \ldots \subset \Omega_l \ldots$$

with corresponding grid sizes

$$h_0 > \ldots > h_l \ldots$$

such that $h_l = \frac{h_{l-1}}{2}$.

Then, on Ω^h the difference equation approximating takes the form

$$\min_{u\in U} (A^u_k v + C^u_k) = 0$$

FMGH algorithm:

Let v^0_0 and u^0_0 be given, we define the sequences v^n_k and u^n_k for $0 \leq n \leq \bar{n}$ and $k \in \mathbb{N}$ by:

For $0 \leq n \leq \bar{n}$,

- $u^{n+1}_k = \mathcal{N}^{\mu_{n+1}}_k(v^n_k)$ is an approximation of u^n_k of $Argmin(A^u_k v^n_k + c^u_k)$.
- $v^{n+1}_k = \mathcal{M}^{m_{n+1}}_k(u^{n+1}_k)v^n_k$ approximated solution of $A^{u^{n+1}_k}_k v + c^{u^{n+1}_k}_k = 0$

Then, we get $v^{\bar{n}}_k$ and $u^{\bar{n}}_k$ and

- $u^0_{k+1} = \mathcal{IU}^{k+1}_k u^{\bar{n}}_k$
- $v^0_{k+1} = \mathcal{I}^{k+1}_k v^{\bar{n}}_k$

where $\mathcal{IU}^{k+1}_k : \mathcal{U}_k \to \mathcal{U}_{k+1}$ is an interpolation operator for control.

For a better understanding, we refer to Akian [1] for a detailed presentation.

2 Multigrid for options pricing

2.1 Standard European option

I recall that the price of a European option in the Black and Scholes model can be formulated in terms of the solution to a parabolic Partial Differential Equation (cf. fd_doc.pdf).

To solve the PDE by multigrid method, we start by limiting the integration domain in space: the problem will be solved in a finite interval. The domain of the problem is partitioned in space and in time, and the differential equation is replaced by a second-order finite difference approximation. Then, the finite difference discretization requires the solution of a sequence of discrete linear problem

$$M_h u_h^n = v_h^n.$$

2.1.1 Multigrid for European options

At each time step, we have to solve a linear discrete problem.

To define completely the multigrid method, we have to chose the smoother, the restriction operator and the the prolongation operator.

The smoother is the Gauss-Seidel method.

For the one-dimensional problem, the prolongation operator is denoted I_{2h}^h. It takes coarse-grid vectors and produces fine-grid vectors according to the rule $I_{2h}^h v^{2h} = v^h$ where

$$\begin{aligned} &v_{2j}^h = v_j^{2h} \\ &v_{2j+1}^h = \tfrac{1}{2}(v_j^{2h} + v_{j+1}^{2h}),\ 0 \le j \le \tfrac{n}{2} - 1. \end{aligned}$$

The restriction operator is denoted I_h^{2h}. The most obvious operator is injection. It is defined by $I_h^{2h} v^h = v^{2h}$, where

$$v_j^{2h} = v_{2j}^h.$$

For the two-dimensional problem, the prolongation operator may be defined in a similar way. If we let $I_{2h}^h v^{2h} = v^h$, then the components of v^h are given by

$$v^h_{2i,2j} = v^{2h}_{ij},$$

$$v^h_{2i+1,2j+1} = \tfrac{1}{2}(v^{2h}_{ij} + v^{2h}_{i+1,j}),$$

$$v^h_{2i,2j+1} = \tfrac{1}{2}(v^{2h}_{ij} + v^h_{i,j+1}),$$

$$v^h_{2i+1,2j+1} = \tfrac{1}{4}(v^{2h}_{ij} + v^{2h}_{i+1,j} + v^{2h}_{i,j+1} + v^{2h}_{i+1,j+1}),$$

$$0 \leq i,j \leq \tfrac{n}{2} - 1.$$

2.2 Standard American option

It is well known that the price of an american option can be expressed after discretisation as a linear complementary problem:

at each time step n, we have to solve

$$\begin{cases} M_h X_h^n \geq G_h^n \\ X_h^n \geq \Phi_h^n \\ (M_h X_h^n - G_h^n, X_h^n - \Phi_h^n) = 0 \end{cases}$$

We solve this equation using the multigrid-Howard algorithm presented in 1.3.

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