Premia 5

1. Standard European Option

1.1. Localization and Discretization. Let us consider a stock S described by the following stochastic equation:

$$\left\{\begin{array}{l} S_0 = y \\ \\ \frac{dS_t}{S_{t^-}} = \mu dt + \sigma dB_t + d(\sum_{j=1}^{N_t} U_j), \end{array}\right. \tag{1.1}$$

where y is the price spot at time 0, $(B_t)_{t\geq 0}$ is a Brownian motion, $(N_t)_{t\geq 0}$ is a Poisson process with deterministic jump intensity λ, $(U_t)_{j\geq 1}$ is a sequence of positive, independent stochastic variables with, at most, time-dependent density $\xi(.,t)$ and μ and σ are two constants, such that $\sigma > 0$. Then, we have

$$S_t = S_0\Big(\prod_{j=1}^{N_t}(U_j+1)\Big)e^{(\mu-\frac{\sigma^2}{2})t+\sigma B_t}. \tag{1.2}$$

We suppose that r is a deterministic risk-free interest rate such that

$$\mu = r - \lambda \mathbb{E} U_1. \tag{1.3}$$

We recall that the price of an European option in the Merton's model can be formulated in terms of the solution to a Partial Integro-Differential Equation (PIDE). After logarithmic transformation $X_t = \log(S_t)$, the price at time t of the option is $V_t = u(t, X_t)$ where u solves the integro-parabolic equation

$$\left\{\begin{array}{ll} \frac{\partial u}{\partial t} + \mathcal{A}u + \mathcal{B}u = 0 & [0,T)\times\mathbb{R} \\ \\ u(T,x) = \psi(x) & x \in \mathbb{R}, \end{array}\right. \tag{1.4}$$

where

$$\mathcal{A}u = \frac{b^2}{2}\frac{\partial^2 u}{\partial x^2} + a\frac{\partial u}{\partial x} - ru, \tag{1.5}$$

$$a = (r - \delta - \lambda\mathbb{E}U_1 - \tfrac{\sigma^2}{2}), \quad b = \sigma, \tag{1.6}$$

and

$$\mathcal{B}u = \lambda\int_{\mathbb{R}}(u(t,x+z) - u(t,x))\nu(dz). \tag{1.7}$$

The notations are

- x is the logarithm of the stock price
- σ is the volatility
- r is the interest rate
- δ is the instantaneous rate of dividend
- ψ is the pay-off
- T is the maturity
- $\mathbb{R}$ is the real line $(-\infty, +\infty)$

We start by limiting the domain in space: the problem will be solved in the finite interval $\Omega_l := [x-l, x+l]$. One chooses l so that

$$\P\,(\exists s \in [0,T],\ |X_s^x| \geq l) \leq \epsilon. \tag{1.8}$$

Once Ω_l is chosen, we solve the following local problem:

$$\left\{\begin{array}{ll} \frac{\partial u}{\partial t}+\mathcal{A}u+\mathcal{B}_l u=0 & (t,x)\in[0,T]\times\Omega_l \\ u(t,x)=\varphi(t,x) & (t,x)\in[0,T]\times\Omega_l^c \\ u(T,x)=\varphi(T,x)=\psi(x) & x\in\mathbb{R}, \end{array}\right. \tag{1.9}$$

where the differential operator $\mathcal{A}$ is defined in (1.5) and $\mathcal{B}_l$ is such that

$$\begin{array}{l} \mathcal{B}_l u=\lambda\int_{\Omega_{l,x}}(u(t,x+z)-u(t,x))\nu(dz)+ \\ +\lambda\int_{\Omega_{l,x}^c}(\varphi(t,x+z)-u(t,x))\nu(dz), \end{array} \tag{1.10}$$

with

$$\Omega_{l,x}=\{z\in\mathbb{R}\;:\;x+z\in\Omega_l\}.$$

We suppose now that the jump variable U_1 is log-normal distributed. Then the density function $\nu(dz)=\xi(z)dz$ is of the following form:

$$\xi(z)=\frac{1}{\sqrt{2\pi\gamma^2}}e^{-\frac{(z-\mu)^2}{2\gamma^2}}. \tag{1.11}$$

In this case, we can select the finite interval $[z_{\min},z_{\max}]$ such that

$$\forall\;\; z\in[z_{\min},z_{\max}]^c,\;\;|\,\xi(z)\,|\le\epsilon\quad\epsilon\ll 1. \tag{1.12}$$

More in general, we can take $[z_{\min},z_{\max}]$, such that

$$\int_{z_{\min}}^{z_{\max}}\xi(z)dz\approx\int_{-\infty}^{\infty}\xi(z)dz-\epsilon=1-\epsilon,\quad\epsilon\ll 1.$$

From (1.12), we get

$$\left\{\begin{array}{l} z_{\min}=\mu-\sqrt{2\gamma^2\ln\frac{1}{\epsilon\sqrt{2\pi\gamma^2}}} \\ z_{\max}=\mu+\sqrt{2\gamma^2\ln\frac{1}{\epsilon\sqrt{2\pi\gamma^2}}}, \end{array}\right. \tag{1.13}$$

Then, we can solve the numerical problem (1.9), in the following interval:

$$[x_{\min},x_{\max}]=[\bar{x}-l+\min(0,z_{\min}),\bar{x}+l+\max(0,z_{\max})],$$

where $\bar{x}$ is an input stock value, $\bar{x}=\ln\bar{s}$.
We define,

$$h=\frac{x_{\max}-x_{\min}}{N},\quad x_j=x_{\min}+jh\quad j=0,...,N$$

We write $u(t_n,x_j)=u_j^n$ and we set

$$\Omega_h=\{x_j\in\mathbb{R}\mid x_j\in\Omega_l\},$$

$$j_l=\Big[\frac{|\min(0,z_{\min})\,|}{h}\Big],$$

$$j_u=N-\Big[\frac{|\max(0,z_{\max})\,|}{h}\Big].$$

1.2. $\mathcal{B}_l$ **approximation.** We set $\nu(dz) = \xi(t,z)dz$ then the $\mathcal{B}_l$ operator is

$$\mathcal{B}_l u = \Big[\lambda \int_{\Omega_{l,x}} u(t,x+z)\xi(t,z)dz - \lambda u(t,x)\Big] +$$

$$+\Big[\lambda \int_{\Omega_{l,x}^c} \varphi(t,x+z)\xi(t,z)dz\Big] = \bar{\mathcal{B}}_l u(t,x) + \phi(t,x).$$

- *Standard approximation*
  For $j = 0, ..., N$,

$$\bar{\mathcal{B}}_l u(.,x_j) \approx \bar{\mathcal{B}}_h u(.,x_j) = \lambda h \sum_{i=j_l-j}^{j_u-j} u(.,x_{j+i})\xi(.,x_i) - \lambda u(.,x_j),$$

$$\phi(.,x_j) \approx \phi_h(.,x_j) = \lambda h \sum_{\substack{i+j<j_l \\ i+j>j_u}} \varphi(.,x_{j+i})\xi(.,x_i).$$

- *FT approximation*
  The discrete correlation of two functions g_i, h_i, each periodic with period N, is defined by

$$Corr(g,h)_j = \sum_{i=0}^{N-1} g_{j+i}h_i.$$

  The *discrete correlation theorem* says that the discrete Fourier transform of the discrete correlation of two real function g and h is such that

$$< Corr(g,h)_j > = < g_i >< h_i^* > \tag{1.14}$$

  where $< . >$ is the discrete Fourier transforms operator, and the asterisk denotes complex conjugation. Then, we can compute correlation using the FFT as follows: FFT the two data sets, multiply one resulting transform by the complex conjugate of the other and inverse transform the product. The result will formally be a complex vector of length N. However, it will turn out to have all its imaginary parts zero since the original data sets were both real.
  We can apply this procedure to the $\bar{\mathcal{B}}_h$ and ϕ_h operator.

1.3. **Finite Differences.** For the differential operator $\mathcal{A}$, we write:

$$\mathcal{A}_h u(.,x_j) = [\frac{b^2}{2}\delta_{xx}^2 + a\delta_x - r]u(.,x_j),$$

where

$$\delta_{xx}^2 u(.,x_j) = \tfrac{u(.,x_{j+1}) - 2u(.,x_j) + u(.,x_{j-1})}{h^2},$$

$$\delta_x u(.,x_j) = \tfrac{u(.,x_j) - u(.,x_{j-1})}{h} + \alpha \tfrac{u(.,x_{j+1}) - 2u(.,x_j) + u(.,x_{j-1})}{h},$$

where α is chosen such that:

$$\begin{cases} ha \leq \frac{b^2}{2} & \alpha = \frac{1}{2} \\ ha > \frac{b^2}{2} & \begin{cases} \alpha = 0 & a > 0 \\ \alpha = 1 & a < 0 \end{cases} \end{cases}$$

1.4. **The "θ-scheme".** We define the time grid size $k > 0$ such that

$$T_k = \{t_n \mid t_n = kn, \ \ n = 0, ..., M-1\}.$$

Then we consider the following discrete operator,

$$\begin{aligned} &\tfrac{u_j^{n+1}-u_j^n}{k} + \mathcal{A}_h[\theta_A u_j^n + (1-\theta_A)u_j^{n+1}] + \\ &+\theta_B[\bar{\mathcal{B}}_h u_j^n + \phi_{h,j}^n] + (1-\theta_B)[\bar{\mathcal{B}}_h u_j^{n+1} + \phi_{h,j}^{n+1}], \end{aligned} \tag{1.15}$$

where $\theta_A, \ \theta_B \in [0, 1]$. We set the two following choices:

1.5. *Explicit scheme.* $\theta_A = \theta_B = 0$, computationally feasible but potentially unstable and suffer from the drawback that their convergence in the time domain are only of $O(k)$.
For every $n = M-1, ..., 0$ we solve,
(1.16)

$$\begin{cases} u_j^n = \varphi_j^n, \quad j = 0, ..., j_l - 1 \ \ and \ \ j = j_u + 1, ..., N \\ u_j^n = p_1 u_{j-1}^{n+1} + p_2 u_j^{n+1} + p_3 u_{j+1}^{n+1} + k\lambda(h \sum_{i=i_l}^{i_u} \xi_i u_{j+i}^{n+1} - u_j^{n+1}), \quad j = j_l, ..., j_u, \end{cases}$$

where

$$p_1 = k\Big(\frac{b^2}{2h^2} + \frac{a}{h}(1-\alpha)\Big), \ \ p_2 = 1 - k\Big(\frac{b^2}{h^2} + \frac{a}{h}(1-2\alpha) + r\Big), \ \ p_3 = k\Big(\frac{b^2}{2h^2} - \frac{a}{h}\alpha\Big).$$

1.6. *"Asymmetric" scheme.* $\theta_A = 1/2$, $\theta_B = 0$, stable and efficient but accuracy is lost due to the asymmetric treatment of the continuous and jump part.
We have to solve, for $n = M-1, ..., 0$, the following linear system

$$\mathbf{A}u^n = \mathbf{B}^{n+1}, \tag{1.17}$$

where $u^n = (u_0^n, ..., u_{j_l}^n, ..., u_{j_u}^n, ..., u_N^n)^T$,

$$\mathbf{A} = \begin{pmatrix} I_l & & 0 \\ & \tilde{A} & \\ 0 & & I_u \end{pmatrix}, \tag{1.18}$$

I_l and I_u are two identity matrix, $(j_l - 1) \times (j_l - 1)$ and $(N - (j_u + 1)) \times (N - (j_u + 1))$ respectively, and $\tilde{A}$ is a $(j_u - j_l) \times (j_u - j_l)$ tridiagonal matrix such that

$$\tilde{A} = \begin{pmatrix} a_1 & a_2 & 0 & \cdot & \cdot & 0 \\ a_0 & a_1 & a_2 & 0 & \cdot & 0 \\ 0 & a_0 & a_1 & a_2 & 0 & \cdot \\ \cdot & 0 & & & & 0 \\ \cdot & \cdot & 0 & a_0 & a_1 & a_2 \\ 0 & \cdot & \cdot & 0 & a_0 & a_1 \end{pmatrix} \qquad \begin{aligned} a_0 &= -\tfrac{k}{2}\Big(\tfrac{b^2}{2h^2} + \tfrac{a}{h}(1-\alpha)\Big) \\ a_1 &= 1 + \tfrac{k}{2}\Big(\tfrac{b^2}{h^2} + \tfrac{a}{h}(1-2\alpha) + r\Big) \\ a_2 &= -\tfrac{k}{2}\Big(\tfrac{b^2}{2h^2} - \tfrac{a}{h}\alpha\Big) \end{aligned}$$

and

$$\mathbf{B}^{n+1} = (\varphi_0^{n+1}, ..., \varphi_{j_l-1}^{n+1}, f_{j_l}^{n+1}, ..., f_j^{n+1}, ..., f_{j_u}^{n+1}, \varphi_{j_u+1}^{n+1}, ..., \varphi_N^{n+1})^T, \tag{1.19}$$

where, for $j = j_l, ..., j_u$,

$$f_j^{n+1} = -a_0 u_{j-1}^{n+1} + (2-a_1)u_j^{n+1} - a_2 u_{j+1}^{n+1} + k\lambda\Big(h\sum_{i=i_l}^{i_u} \xi_i u_{j+i}^{n+1} - u_j^{n+1}\Big).$$

1.7. **The ADI-FFT scheme (Andersen and Andreasen).** In the article [1], Andersen and Andreasen propose an FFT-ADI (Fast Fourier Transform - Alternating Implicit Direction) scheme to avoid instability problem of the explicit methods. The FFT technique is applied to the correlation term and coupled with an ADI method, where each time step is split into two half-steps.

step 1) $t_{n+1} \longrightarrow t_{n+1/2}$
We set in (1.15) $\theta_A = 1$ and $\theta_B = 0$, which gives us

$$[\frac{2}{k} - \mathcal{A}_h]u^{n+\frac{1}{2}} = [\frac{2}{k} - \lambda + \lambda Corr(\xi, .)]u^{n+1}, \tag{1.20}$$

that can be solved by first computing the correlation $Corr(\xi, u^{n+1})$ in the discrete Fourier space, where by (1.14),

$$< Corr(\xi, u^{n+1}) >=< \xi >< u^{n+1} > .$$

If we observe that $< \xi >$ only needs to be compute once, the computational cost associated with the correlation part is one FFT and one inverse FFT, i.e. $O(N \log_2 N)$. Second, by solving the tridiagonal system.

step 2) $t_{n+1/2} \longrightarrow t_n$
We set in (1.15) $\theta_A = 0$ and $\theta_B = 1$, whereby

$$[\frac{2}{k} + \lambda - \lambda Corr(\xi, .)]u^n = [\frac{2}{k} + \mathcal{A}_h]u^{n+\frac{1}{2}}. \tag{1.21}$$

If we let

$$y = [\frac{2}{k} + \mathcal{A}_h]u^{n+\frac{1}{2}}$$

we can write the Fourier transform of (1.21) to arrive at

$$(\frac{2}{k} + \lambda) < u^n > -\lambda < \xi >< u^n >=< y > \Rightarrow$$

$$\Rightarrow < u^n >= \frac{< y >}{(\frac{2}{k} + \lambda - \lambda < \xi >)}.$$

We can now transform the equation back to obtain u^n. All in all this requires one tridiagonal matrix multiplication, one FFT and one inverse FFT, i.e. a procedure with a computational burden of $O(N \log_2 N)$.

1.8. **Boundary conditions for the PIDE arising in the jump-diffusion models.** The question is: how to choose the function $\varphi(t, x)$ in the PIDE equation (1.9)?
A good choice is, of coarse, to get $\varphi(t, x)$ as the payoff function $\psi(x)$. More in general (see [1]), we could assume $\varphi(t, x)$ to be linear in e^x and we could write

$$\varphi(t, x) \approx [g_l(t)e^x + h_l(t)]\mathbf{1}_{\{x<\bar{x}-l\}} + [g_u(t)e^x + h_u(t)]\mathbf{1}_{\{x>\bar{x}+l\}}$$

where g_l, h_l, g_u, h_u are deterministic functions. Those technique is only directly possible when we explicitly know the asymptotic behavior of u, i.e. the coefficients g_l, h_l, g_u, h_u. However, if we assume asymptotic linearity of u in e^x, we have that

$$| u(t, x) - f(t, x) | \longrightarrow 0 \quad when \quad | x | \to \infty,$$

where f solves the Balck-Scholes PDE

$$\frac{\partial f}{\partial t} + (a + \lambda \mathbb{E}U_1)\frac{\partial f}{\partial x} + \frac{b^2}{2} - rf = 0, \tag{1.22}$$

subject to the same boundary as (1.9). Hence we can make the approximation

$$\varphi \approx f\mathbf{1}_{\{x\in[\bar{x}-l,\bar{x}+l]^c\}}. \tag{1.23}$$

Using this we can solve (1.22) on a wide and possibly coarse Crank-Nicolson grid on some interval $[x_1, x_2] \supset [\bar{x} - l, \bar{x} + l] = [x_1^*, x_2^*]$ to obtain the discrete solutions for f, and then use these solutions to find the coefficients g_l, h_l, g_u, h_u as the solutions to

$$f(t, x_1) = g_l(t)e^{x_1} + h_l(t); \quad f(t, x_2) = g_l(t)e^{x_2} + h_l(t)$$

$$f(t, x_1^*) = g_l(t)e^{x_1^*} + h_l(t); \quad f(t, x_2^*) = g_l(t)e^{x_2^*} + h_l(t).$$

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