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## mc\_fixedasian\_glassermann

### Input parameters

- Number of iterations  $N$
- Generator type
- Increment  $inc$
- Confidence Value

### Output parameters

- Price  $P$
- Error price  $\sigma_P$
- Delta  $\delta$
- Error delta  $\sigma_{delta}$
- Price Confidence Interval:  $ICp$  [Inf Price, Sup Price]
- Delta Confidence Interval:  $ICp$  [Inf Delta, Sup Delta]

## Description

Computation of the price of a asian option when the underlying asset follows the Black and Scholes model.

*/\*The model\*/*

Under the standard Black and Scholes assumptions the price of the underlying asset is driven by the SDE

$$dS_t = S_t((r - q)dt + \sigma dW_t), \quad S_{T_0} = x, \quad (1)$$

with  $r$  the risk-free, continuously compounded interest rate,  $\sigma(t, y)$  the asset volatility,  $W$  a Brownian motion, and  $x$  fixed.

The solution to this equation can be simulated without discretization error on a discrete grid of points  $T_0 < T_1 < \dots < T_m = T$ , by setting

$$S_{T_i} = S_{T_{i-1}} \exp((r - \frac{1}{2}\sigma^2)\delta t + \sigma\sqrt{\delta t}Z_i), \quad i = 1, \dots, m,$$

where  $Z = (Z_1, \dots, Z_m) \sim \mathcal{N}(0, I_m)$  and  $I_m$  is the identity matrix of  $\mathbb{R}^m$ .

/\*The option real and approximate prices\*/

For arbitrage reasons, the price of an option with payoff  $\psi(S_t, t \leq T)$  is given by

$$V_0 = \mathbb{E}[e^{-r(T-T_0)}\psi(S_t, t \leq T)].$$

For a call option we have  $\psi(S_t, t \leq T) = \left(\frac{1}{T-T_0} \int_{T_0}^T S_t dt - K\right)^+$ . which we rewrite

$$G(Z) = e^{-r(T-T_0)} \left(\hat{A}(T_0, T, Z) - K\right)^+,$$

where  $Z$  is a random gaussian vector,  $\hat{A}(T_0, T, Z)$  is the discretized mean and  $G$  is a function we can compute by using the discretization of the mean  $A(T_0, T) = \frac{1}{T-T_0} \int_{T_0}^T S_t dt$  and the payoff function. Thus the approximate price of the option is given by

$$\hat{V}_0 = \mathbb{E}[G(Z)].$$

## Importance sampling

We change the law of  $Z = (Z_1, \dots, Z_m)$  by adding a drift vector  $\mu = (\mu_1, \dots, \mu_m)$ . An elementary version of Girsanov theorem leads to the following representation of  $\hat{V}_0$ :

$$\hat{V}_0 = \mathbb{E}[g(\mu, Z)],$$

with

$$g(\mu, Z) = G(Z + \mu) e^{-\mu \cdot Z - \frac{1}{2}\|\mu\|^2}, \quad (2)$$

where  $\|x\|$  denotes the Euclidean norm of a vector  $x \in \mathbb{R}^m$  and  $x \cdot y$  is the inner product of two vectors  $x, y \in \mathbb{R}^m$ . In (2) the optimal  $\mu$  solves the problem

$$\min_{\mu} \mathbb{E}[G(Z)^2 e^{-\mu \cdot Z + \frac{1}{2}\|\mu\|^2}].$$

Note that even if the optimal  $\mu$  can be found, it will not in general provide a zero-variance estimator. In practice, finding the optimal  $\mu$  exactly is infeasible and some approximation is required. In their paper the authors of the method have shown that this optimal  $\mu$  maximizes the function  $F(z) - \frac{1}{2}z \cdot z$  with  $F(z) = \log(G(z))$ . That is equivalent to finding the solution of the fixed point problem

$$\nabla F(z) = z.$$

It is proved that the solution to this problem is (asymptotically) optimal in some sens.

## Asian option

In the sequel we will restrict our attention to the case of a *Riemannian* (or *Euler*) discretization of the mean  $A(T_0, T) = \frac{1}{T-T_0} \int_{T_0}^T S_t dt$ . Due to the structure of the asian options, we can find particularly efficient solution of this optimization problem.

Consider the discretized payoff  $G(z) = (\hat{A}(T_0, T, Z) - K)^+$ , it clearly suffices to consider the points  $z$  at which  $G(z) \neq 0$  and thus  $G$  and  $F$  are differentiable.

/\*The algorithm\*/

The first-order conditions for optimality become

$$z_j = \frac{\sigma \sqrt{\Delta t} \sum_{i=j}^m S_i}{mG(z)}, \quad j = 1, \dots, m,$$

where we  $S_i$  for  $S_{i\Delta t}$ . This implies that

$$z_1 = \frac{\sigma \sqrt{\Delta t} [G(z) + K]}{G(z)}, \quad z_{j+1} = z_j - \frac{\sigma \sqrt{\Delta t} S_j}{mG(z)}, \quad j = 1, \dots, m-1. \quad (3)$$

Given a value  $G(z) \equiv y$ , equation (3) determines  $z$  together with

$$S_j = S_{j-1} e^{(r-q-\frac{1}{2}\sigma^2)\Delta t + \sigma \sqrt{\Delta t} z_j}, \quad j = 1, \dots, m. \quad (4)$$

Subject to the first-order conditions, we may therefore view the  $S_i$  as functions of the scalar  $y$  rather than the vector  $z$ . The optimization problem thus reduces to finding the  $y$  that indeed produced a payoff of  $y$  at  $S_1(y), \dots, S_m(y)$ ; that is, finding the root of the equation

$$g(y) \equiv \frac{1}{m} \sum_{j=1}^m S_j(y) - K - y = 0.$$

There is no proof that this equation has a unique root, but numerically this appears to be the case. Bisections find the root very quickly, and given this scalar  $y$ , equations (3) and (4) recover  $z$  efficiently. We denote this vector by  $\mu^*$ .

/\*The MC price computation\*/

If  $(Z^n)_{1 \leq n \leq N}$  is an *i.i.d.* sample from the gaussian law  $\mathcal{N}(0, I_m)$  then the MC price of the option is given by

$$\hat{V}_0 \sim \frac{1}{N} \sum_{n=1}^N G(Z^n + \mu^*) e^{-\mu^* \cdot Z^n - \frac{1}{2} \|\mu^*\|^2}.$$

See [1] for more details.

## References

- [1] P.GLASSERMAN P.HEIDELBERGER P.SHAHABUDDIN. Asymptotically optimal importance sampling and stratification for pricing path-dependent options. *Mathematical Finance*, 2, April:117–152, 1999. 4