Variance optimal hedging for a given number of transactions

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1 Introduction

In the Black-Scholes setting for option pricing it is assumed that the market-maker designs a continuous-time hedge. This is not realistic from a practical point of view. In the presence of transaction costs for example the investor would like to hedge as little as possible. In fact even if there is no transaction costs nor liquidity restrictions, an investor can and will obviously in practice follow a discrete trading strategy at stopping times.

In this paper, we consider the problem of selecting the best hedging times and ratios given a maximum fixed number of trading times. As a criteria we take the variance of the replication error. Moreover we work under the martingale measure. Note that the optimal price of the option for this criteria is obviously the Black-Scholes price, so that we assume that the hedger of the option trades at this price.

2 Minimal variance hedging given n rebalancing

We consider the hedge ratios and hedging times as a control parameter. Let $(\tau_1^n, \tau_2^n, \ldots, \tau_n^n)$ denote the rebalancing (stopping) times and $\left(\delta_{\tau_1^n}, \delta_{\tau_2^n}, \ldots, \delta_{\tau_n^n}\right)$ the corresponding (adapted) hedging ratios chosen by the investor. In order to apply Dynamic Programming techniques, we shall consider an investor who initiates his strategy at time t, the value of the underlying being $S_t = x$, with the selling of the option and an initial hedge of an amount α of stocks. Let V_T denote the value of the portfolio at time T, $\varphi(x)$ the payoff of the option, $c(t,x)$ the Black-Scholes price, $F = (\mathcal{F}_t)_{0\leq t\leq T}$ the standard augmentation of the natural filtration of the Brownian motion, $\mathcal{T}_{t,T}$ the set of all stopping times of the filtration F which satisfy $t \leq \tau \leq T$ with probability one, E_t the conditional expectation with respect to F. Let also denote $\tilde{A}_t$ the quantity A_t discounted at time 0, that is $A_t e^{-rt}$. The associated variance of the tracking error is then:

$$\begin{aligned} I(t,x,\alpha,\mathcal{P}_n) &\equiv E\left[\{\tilde{V}_T - \tilde{\varphi}(S_T^{t,x})\}^2\right] \\ &= E\Big[\{\tilde{c}(t,S_t^{t,x}) + \alpha(\tilde{S}_{\tau_1^n}^{t,x} - \tilde{S}_t^{t,x}) \\ &\quad + \sum_{i=1}^{n} \delta_{\tau_i^n}(\tilde{S}_{\tau_{i+1}^n}^{t,x} - \tilde{S}_{\tau_i^n}^{t,x}) - \tilde{\varphi}(S_T^{t,x})\}^2\Big] \end{aligned}$$

where by convention $\tau_{n+1}^n = T$ and
$\mathcal{P}_n = \left(\tau_1^n, \tau_2^n, ..., \tau_n^n, \delta_{\tau_1^n}, \delta_{\tau_2^n}, ..., \delta_{\tau_n^n}\right)$.

In the expression above the process $(S_s^{t,x}, s \geq t)$ is the solution of the equation

$$\begin{cases} dS_s = rS_s ds + \sigma S_s dW_s \text{ for } s \geq t \\ S_t = x \text{ (and } S_u = x \text{ for } u \leq t) \end{cases}$$

The problem at hand is to characterize the optimal cost function v_n given the initial hedge:

$$v_n(t,x,\alpha) \equiv \inf_{\mathcal{P}_n} I(t,x,\alpha,\mathcal{P}_n) \tag{1}$$

and to find, if some, $\mathcal{P}_n^*$ minimizing $I(t,x,\alpha,\mathcal{P}_n)$ i.e

$$v_n(t,x,\alpha) = I(t,x,\alpha,\mathcal{P}_n^*)$$

and the optimal cost function

$$v_n^*(t,x) = \inf_{\alpha} v_n(t,x,\alpha)$$

The natural idea is to use Dynamic Programming to reduce our problem to a sequence of standard optimal stopping problems. Indeed since we work under the risk-neutral probability and with the square function, $I(t,x,\alpha,\mathcal{P}_n)$ splits naturally in the first local replication error until the next hedging time and the error starting afresh at that date:

Set

$$L = \tilde{c}\left(t, S_t^{t,x}\right) + \alpha\left(\tilde{S}_{\tau_1^{n+1}}^{t,x} - \tilde{S}_t^{t,x}\right) + \sum_{i=1}^{n+1} \delta_{\tau_i^{n+1}}\left(\tilde{S}_{\tau_{i+1}^{n+1}}^{t,x} - \tilde{S}_{\tau_i^{n+1}}^{t,x}\right) - \tilde{\varphi}\left(S_T^{t,x}\right)$$

Then

$$\begin{aligned} E\left[L^2\right] &= E\Big[\{\tilde{c}\left(t, S_t^{t,x}\right) + \alpha(\tilde{S}_{\tau_1^{n+1}}^{t,x} - \tilde{S}_t^{t,x}) - \tilde{c}\left(\tau_1^{n+1}, S_{\tau_1^{n+1}}^{t,x}\right)\}^2 \\ &+\{\tilde{c}\left(\tau_1^{n+1}, S_{\tau_1^{n+1}}^{t,x}\right) + \delta_{\tau_1^{n+1}}(\tilde{S}_{\tau_2^{n+1}}^{t,x} - \tilde{S}_{\tau_1^{n+1}}^{t,x}) \\ &+\sum_{i=2}^{n+1} \delta_{\tau_i^{n+1}}(\tilde{S}_{\tau_{i+1}^{n+1}}^{t,x} - \tilde{S}_{\tau_i^{n+1}}^{t,x}) - \tilde{\varphi}(S_T^{t,x})\}^2\Big] \end{aligned}$$

So it is natural to conjecture the following Dynamic Programming relation:

$$\begin{aligned} v_{n+1}(t,x,\alpha) &= \inf_{\tau \in \mathcal{T}_{t,T}} E\Big[\{\tilde{c}\left(t, S_t^{t,x}\right) + \alpha(\tilde{S}_\tau^{t,x} - \tilde{S}_t^{t,x}) - \tilde{c}\left(\tau, S_\tau^{t,x}\right)\}^2 \\ &+ v_n^*(\tau, S_\tau^{t,x})\Big] \quad (DP) \end{aligned}$$

or yet

$$\begin{aligned} v_{n+1}(t,x,\alpha) &= -\{\tilde{c}\left(t, S_t^{t,x}\right) - \alpha \widetilde{S}_t^{t,x}\}^2 \\ &+ \inf_{\tau \in \mathcal{T}_{t,T}} E\left[\{\tilde{c}\left(\tau, S_\tau^{t,x}\right) - \alpha \tilde{S}_\tau^{t,x}\}^2 + v_n^*(\tau, S_\tau^{t,x})\right] \end{aligned}$$

Observe now that v_n^* being given, we face a standard optimal stopping problem. The solution to such a stopping time problem is well known (cf [**?**]).

In [**?**], we give the proof of the Dynamic Programming equation. We also prove that the value function v_n is the unique viscosity solution of the following sequence of variational inequalities on $[0,T] \times \mathbb{R}^{+*} \times K$:

$$\left\{ \begin{array}{c} \frac{\partial v_n}{\partial t} + A v_n + \sigma^2 (x e^{-rt})^2 (\Delta(t,x) - \alpha)^2 \geq 0 \\ v_n \leq v_{n-1}^* \\ (v_n - v_{n-1}^*)(\frac{\partial v_n}{\partial t} + A v_n + \sigma^2 (x e^{-rt})^2 (\Delta(t,x) - \alpha)^2) = 0 \\ v_n(T,x,\alpha) = 0 \end{array} \right. \quad (2)$$

where A is the differential operator associated to S_t given by $Av\,(t,x) = rx\frac{\partial v}{\partial x}\,(t,x) + \frac{\sigma^2}{2} x^2 \frac{\partial^2 v}{\partial x^2}\,(t,x)$, $\Delta(t,x) = \frac{\partial c}{\partial x}(t,x)$ (i.e. the Black-Scholes delta) and K is a compact set where Δ takes its values (this is not a restriction as we shall see below). Notice than once $v_1, v_2, ..., v_n$ have been found, $\tau_1^n, \tau_2^n, ..., \tau_n^n$ can be constructed in the same way as done for optimal stopping problems.

2.1 Some properties of the solution

2.1.1 Optimal hedge ratios

If we know the trading dates it is easy to find the optimal deltas:

$$\begin{aligned}
E\left[\left\{\tilde{V}(T)-\tilde{\varphi}(S_T)\right\}^2\right] &= E\left[\left\{\sum_{i=0}^{n}\int_{\tau_i^*}^{\tau_{i+1}^*}\left(\Delta\left(t,S_t\right)-\delta_{\tau_i^*}\right)d\tilde{S}_t\right\}^2\right] \\
&= \sum_{i=0}^{n} E\left[\left\{\int_{\tau_i^*}^{\tau_{i+1}^*}\left(\Delta\left(t,S_t\right)-\delta_{\tau_i^*}\right)d\tilde{S}_t\right\}^2\right] \\
&= \sum_{i=0}^{n} E\left[\int_{\tau_i^*}^{\tau_{i+1}^*}\left(\Delta\left(t,S_t\right)-\delta_{\tau_i^*}\right)^2\sigma^2\tilde{S}_t^2\,dt\right] \\
&= \sum_{i=0}^{n} E\left[E_{\tau_i^*}\left(\int_{\tau_i^*}^{\tau_{i+1}^*}\Delta^2\left(t,S_t\right)\sigma^2\tilde{S}_t^2 dt\right)\right. \\
&\qquad -2\delta_{\tau_i^*}E_{\tau_i^*}\left(\int_{\tau_i^*}^{\tau_{i+1}^*}\Delta\left(t,S_t\right)\sigma^2\tilde{S}_t^2\,dt\right) \\
&\qquad \left.+\delta_{\tau_i^*}^2E_{\tau_i^*}\left(\int_{\tau_i^*}^{\tau_{i+1}^*}\sigma^2\tilde{S}_t^2\,dt\right)\right]
\end{aligned}$$

This entails

$$\delta_{\tau_i^*}=\frac{E_{\tau_i^*}\left[\int_{\tau_i^*}^{\tau_{i+1}^*}\Delta\left(t,S_t\right)\sigma^2\tilde{S}_t^2\,dt\right]}{E_{\tau_i^*}\left[\int_{\tau_i^*}^{\tau_{i+1}^*}\sigma^2\tilde{S}_t^2\,dt\right]}$$

which shows by the mean value theorem that $\delta_{\tau_i^*}=\Delta\left(u_i,S_{u_i}\right)$ for some random u_i between τ_i^* and τ_{i+1}^*.

Now if we assume that φ is a $k-$Lipschitz function, then $|\Delta\left(u_i,S_{u_i}\right)|\leq k$, so that we can restrict ourselves to $[-k,k]$, or better yet to the range of the function Δ. In the Call case, the interval $[0,1]$ is chosen, and $[-1,0]$ in a Put case.

2.1.2 Limit as $N\to\infty$

Obviously, the hedging error goes to zero as the number of hedges goes to infinity since it is true for the deterministic case $h=\frac{T-t}{N}$, $t_i=(i-1)h$.

Moreover in this case we know the convergence rate. Therefore

$$\begin{aligned} v_N(0,x,\alpha) &\leq R_N \\ &\equiv E\Big[\Big\{\tilde{c}\,(0,S_0)+\Delta(0,S_0)(\tilde{S}_{t_2}-\tilde{S}_0) \\ &\quad + \sum_{i=2}^{N}\Delta(t_i,S_{t_i})(\tilde{S}_{t_{i+1}}-\tilde{S}_{t_i})-\tilde{\varphi}(S_T)\Big\}^2\Big] \end{aligned}$$

with

$$\lim_{N\to\infty} NR_N = \frac{e^{-2rT}}{2}E\left[\int_0^T e^{-2rt}S_t^4\sigma^4(\frac{\partial^2 c}{\partial x^2}(t,S_t))^2\,dt\right]$$

where $\frac{\partial^2 c}{\partial x^2}$ is the gamma of the option. This result was shown by Zhang ([**?**]).

3 Finite difference method

This section deals with the numerical analysis of variational inequality 2.

We first set $X_t = ln(S_t)$ and if we define u_n^α by the relation

$$v_n^\alpha(t,x) = u_n^\alpha(t,\ln(x))$$

then u_n^α solves in $[0,T]\times\mathbb{R}$ the following parabolic inequality:

$$\left\{\begin{array}{c} \frac{\partial u_n^\alpha}{\partial t}+\tilde{A}u_n^\alpha+\sigma^2(e^x e^{-rt})^2(\Delta(t,e^x)-\alpha)^2\geq 0 \\ u_n^\alpha \leq \inf_\delta u_{n-1}^\delta \\ (u_n^\alpha-\inf_\delta u_{n-1}^\delta)(\frac{\partial u_n^\alpha}{\partial t}+\tilde{A}u_n^\alpha+\sigma^2(e^x e^{-rt})^2(\Delta(t,e^x)-\alpha)^2)=0 \\ u_n^\alpha(T,x)=0 \end{array}\right. \tag{3}$$

Since this variational inequality is defined on the whole real line $\mathbb{R}$, we need first to localize it in an interval $[-l,+l]$ where l is a constant. The localized problem is then solved by the finite different method.

3.1 Localization

To localize the variational inequality, we introduce

$$T_l^{t,x} = \inf\{s>t; |X_s^{t,x}|>l\}$$

and we denote

$$\begin{aligned}
u^{\alpha}_{n,l}(t,x) &= -\{\tilde{c}(t,e^x) - \alpha e^x e^{-rt}\}^2 \\
&+ \inf_{\tau \in \mathcal{T}_{t,T}} E\left[\{\tilde{c}((\tau \wedge T^{t,x}_l), e^{X^{t,x}_{(\tau\wedge T^{t,x}_l)}}) - \alpha e^{-r(\tau\wedge T^{t,x}_l)} e^{X^{t,x}_{(\tau\wedge T^{t,x}_l)}}\}^2\right. \\
&+ \left. u^*_{n-1}((\tau \wedge T^{t,x}_l), e^{X^{t,x}_{(\tau\wedge T^{t,x}_l)}})\right] \\
&= -\{\tilde{c}(t,e^x) - \alpha e^x e^{-rt}\}^2 \\
&+ \inf_{\tau \in \mathcal{T}_{t,T}} E\left[\Phi(\tau, X^{t,x}_{\tau\wedge T^{t,x}_l})\right]
\end{aligned}$$

where $\Phi(t,x) = \{\tilde{c}(t,e^x) - \alpha e^{-rt} e^x\}^2 + u^*_{n-1}(t,e^x)$.

The following proposition (see [?]) implies that to calculate u^{α}_n, it suffices to compute the value of the function $u^{\alpha}_{n,l}$.

Proposition 1 *$u^{\alpha}_{n,l}$ converges uniformly to u^{α}_n on any compact as l goes to infinity i.e.:*

$$\forall R > 0, \lim_{l\to\infty} \sup_{(t,x)\in[0,T]\times[-R,R]} |u^{\alpha}_n(t,x) - u^{\alpha}_{n,l}(t,x)| = 0$$

Remark 1 *More precisely, we have the following estimate:*

$$|u^{\alpha}_n(t,x) - u^{\alpha}_{n,l}(t,x)| \le C e^{-\frac{(l-R-|r-\frac{\sigma^2}{2}|T)^2}{\sigma^2 T}}$$

where C is a constant independent of l. In practice, for a given precision ϵ, a suitable value for l can be found by using the above estimate.

3.2 Discretization

We discretize now the problem using the finite difference method. We introduce a grid of mesh points $(t_j, x_i) = (j * k, -l + ih)$ where $k = \frac{T}{M}$ and $h = \frac{2l}{N+1}$. We note $u_h(t) = (u_h(t,i))_{1\le i\le N}$ where $u_h(t,i)$ is an approximation of $u(t,x_i)$.

We approximate the derivatives by

$$\frac{\partial u}{\partial x}(t,x_i) \sim \frac{u_h(t,i+1) - u_h(t,i-1)}{2h}$$

and

$$\frac{\partial^2 u}{\partial x^2}(t,x_i) \sim \frac{u_h(t,i+1) - 2u_h(t,i) + u_h(t,i-1)}{h^2}.$$

We get

$$(\tilde{A}_h u_h(t))_i = (-\frac{r-\frac{\sigma^2}{2}}{2h}+\frac{\sigma^2}{2h^2})u_h(t,i-1)-\frac{\sigma^2}{h^2}u_h(t,i)+(\frac{r-\frac{\sigma^2}{2}}{2h}+\frac{\sigma^2}{2h^2})u_h(t,i+1)$$

One then seeks the vector $(u_h(t,x_i), 0 \leq i \leq N)$ such that, there holds

- in the case of natural Dirichlet boundary conditions:

$$\begin{cases} \frac{du_h}{dt}(t,i) + \tilde{A}u_h(t,i) + f_h(t,i) \geq 0, \ \forall 0 \leq t \leq T, \ \forall 1 \leq i \leq N \\ u_h(t,i) \leq \phi_h(t,i) \ \forall 0 \leq t \leq T, \ \forall 1 \leq i \leq N \\ (\frac{du_h}{dt}(t,i) + \tilde{A}u_h(t,i) + f_h(t,i))(u_h(t,i) - \phi_h(t,i)) = 0 \\ u_h(T,i) = 0, \ \forall 1 \leq i \leq N \\ u_h(t,0) = u_h(t,N+1) = 0, \ \forall 0 \leq t \leq T \end{cases}$$

- in the case of Neumann boundary conditions:

$$\begin{cases} \frac{du_h}{dt}(t,i) + \tilde{A}u_h(t,i) + f_h(t,i) \geq 0, \ \forall 0 \leq t \leq T, \ \forall 1 \leq i \leq N \\ u_h(t,i) \leq \phi_h(t,i) \ \forall 0 \leq t \leq T, \ \forall 1 \leq i \leq N \\ (\frac{du_h}{dt}(t,i) + \tilde{A}u_h(t,i) + f_h(t,i))(u_h(t,i) - \phi_h(t,i)) = 0 \\ u_h(T,i) = 0, \ \forall 1 \leq i \leq N \\ u_h(t,1) = u_h(t,0), \ \forall 0 \leq t \leq T \\ u_h(t,N) = u_h(t,N+1), \ \forall 0 \leq t \leq T \end{cases}$$

where $f_h(t) = (f_h(t,i))_{1 \leq i \leq N}$ are $\phi_h(t) = (\phi_h(t,i))_{1 \leq i \leq N}$ are some vectors defined by:

$$f_h(t,i) = \sigma^2 (e^{x_i} e^{-rt})^2 (\Delta(t, e^{x_i}) - \alpha)^2$$

and

$$\phi_h(t,i) = \inf_{\delta} u_{n-1}^{\delta}(t,x_i)$$

We set:

$$\begin{cases} \alpha = (-\frac{r-\frac{\sigma^2}{2}}{2h} + \frac{\sigma^2}{2h^2}) \\ \beta = -\frac{\sigma^2}{h^2} \\ \gamma = (\frac{r-\frac{\sigma^2}{2}}{2h} + \frac{\sigma^2}{2h^2}) \end{cases}$$

The operator $\tilde{A}_h$ is then defined by:

- in the case of natural Dirichlet boundary conditions:

$$\tilde{A}^h = \begin{bmatrix} \beta & \gamma & 0 & \cdots & 0 & 0 \\ \alpha & \beta & \gamma & 0 & \cdots & 0 \\ 0 & \alpha & \beta & \gamma & \cdots & 0 \\ 0 & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha & \beta & \gamma \\ 0 & 0 & 0 & \cdots & \alpha & \beta \end{bmatrix}$$

- in the case of Neumann boundary conditions:

$$\tilde{A}^h = \begin{bmatrix} \beta+\alpha & \gamma & 0 & \cdots & 0 & 0 \\ \alpha & \beta & \gamma & 0 & \cdots & 0 \\ 0 & \alpha & \beta & \gamma & \cdots & 0 \\ 0 & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha & \beta & \gamma \\ 0 & 0 & 0 & \cdots & \alpha & \beta+\gamma \end{bmatrix}$$

3.3 The "θ-scheme"

We give $\theta \in [0,1]$. We construct an approximation

$$u_{h,k}(t,x) = \sum_{n=0}^{N} u_h^n(x)\mathbf{1}_{[nk,(n+1)k[}(t)$$

where $u_h^0, \dots, u_h^N$ satisfy

$$\begin{cases} \frac{u_h^{n+1}(i)-u_h^n(i)}{k} + \theta\tilde{A}_h u_h^n(i) + (1-\theta)\tilde{A}_h u_h^{n+1}(i) + f_i^n \geq 0 \\ u_h^n(i) \leq \phi_i^n \\ (\frac{u_h^{n+1}(i)-u_h^n(i)}{k} + \theta\tilde{A}_h u_h^n(i) + (1-\theta)\tilde{A}_h u_h^{n+1}(i) + f_i^n)(u_h^n(i) - \phi_i^n) = 0 \\ u_h^M(i) = 0 \end{cases}$$

Besides, one must add the appropriate boundary conditions.
For Dirichlet boundary conditions we have:

$$\begin{cases} u_h^n(0) & = & 0 \\ u_h^n(N+1) & = & 0 \end{cases}$$

for Neumann boundary conditions:

$$\begin{cases} u_h^n(0) & = & u_h^n(1) \\ u_h^n(N+1) & = & u_h^n(N) \end{cases}$$

3.4 Explicit Method

First, let us discuss the case $\theta = 0$. The approximating scheme is reduced

$$\begin{cases} \frac{u_h^{n+1}(i)-u_h^n(i)}{k} + \tilde{A}_h u_h^{n+1}(i) + f_i^n \geq 0 \\ u_h^n(i) \leq \phi_i^n \\ (\frac{u_h^{n+1}(i)-u_h^n(i)}{k} + \tilde{A}_h u_h^{n+1}(i) + f_i^n)(u_h^n(i) - \phi_i^n) = 0 \\ u_h^M(i) = 0 \end{cases}$$

The solution of the system is given by

$$u_h^n(i) = \min(g_i^n, \phi_i^n)$$

where $g_i^n = u_h^{n+1}(i) + k\tilde{A}_h u_h^{n+1}(i) + kf_i^n$.

3.5 Implicit Methods

When we choose $1 \geq \theta > 0$, we have to solve at each time step, a linear system of the type

$$\begin{cases} TX \geq G \\ X \leq F \\ (TX - G, X - F) = 0 \end{cases}$$

where

$$\begin{cases} T = \theta k \tilde{A}_h - I \\ G = -kf^n - u^{n+1} - (1-\theta)k\tilde{A}_h u^{n+1} \\ F = \phi^n \\ X = u^n \end{cases}$$

T is a tridiagonal matrix:

$$T = \begin{bmatrix} b+a & c & 0 & \cdots & 0 & 0 \\ a & b & c & 0 & \cdots & 0 \\ 0 & a & b & c & \cdots & 0 \\ 0 & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & a & b & c \\ 0 & 0 & 0 & \cdots & a & b+c \end{bmatrix}$$

with

$$\begin{cases} a = \theta k(-(r - \frac{\sigma^2}{2})\frac{1}{2h} + \frac{\sigma^2}{2h^2}) \\ b = -\theta k \frac{\sigma^2}{h^2} - 1 \\ c = \theta k((r - \frac{\sigma^2}{2})\frac{1}{2h} + \frac{\sigma^2}{2h^2}) \end{cases}$$

3.6 The algorithm of Howard

At each time step, we have to solve a variational inequality in some domain D:

$$\min(Au - \theta, u - \phi) = 0 \tag{4}$$

Let $\epsilon > 0$ given. The algorithm of Howard ([?]) reads as follows: the solution is approximated by a sequence (u^k) and the stopping criteria is

$$\|u^{k+1} - u^k\|_\infty < \varepsilon. \tag{5}$$

- Let u^k be given, we compute a partition $D_1^k \cup D_2^k$ de D such that:

  $$Au^k - \theta \geq u^k - \phi \text{ in } D_1^k$$

  $$Au^k - \theta < u^k - \phi \text{ in } D_2^k$$

- at the next step, we solve a linear system:

  $$Au - \theta = 0 \text{ in } D_1^k$$

  with $u = \phi$ in D_2^k. This, again a well posed problem which leads to u^{k+1}.

We thus obtain by this method the solution of the discrete variational inequality and the boundary of the continuation set.

4 Formulation of the problem in the binomial model

Among the N possible dates of trading, the hedger will decide to hedge only n $(< N)$ times. So he can not any longer duplicate the payoff by constructing a self-financing strategy. His goal is to minimize the variance of the tracking error under the risk neutral probability.

In [?], we have shown that the value function v_n solves an optimal stopping problem:

$$v_n(t,x,\alpha) = -(\tilde{c}_t - \alpha x e^{-rt})^2 + \underset{\tau \in \mathcal{T}_{t,T}}{inf} \; E[\{\tilde{c}\left(\tau, S_\tau^{t,x}\right) - \alpha \tilde{S}_\tau^{t,x}\}^2 + v_{n-1}^*(\tau, S_\tau^{t,x})]$$

where $\mathcal{T}_{t,T}$ is the set of all stopping time which satisfy $t \leq \tau \leq T$. Recall that the optimal stopping time which achieves the minimum is given by

$$\tau_t^* = \inf\{u \in [t,T]/v_{n-1}^*(u, S_u^{t,x}) \leq v_n(u, S_u^{t,x}, \alpha)\}$$

If n is fixed, an application of the Bellman principle reduces the multiple-times optimal stopping problem to a sequence of traditional optimal stopping problems. In other words, we have to solve at step $n+1$ an American-type option pricing problem in which the payoff function is given by the value function at step n.

This procedure may be implemented in a straightforward manner in the corresponding problem in the binomial setting. We can show the following corollary to the discrete-time version of (DP) which allows us to find the value function by an explicit recursive procedure:

Corollary 1 *Let $V_p^{n,\alpha}, p \leq N$ be the nonnegative adapted process defined recursively by*

$$V_p^{0,\alpha} = -(\tilde{c}_p - \alpha\widetilde{S}_p)^2 + E\left[\{\tilde{c}\left(p+1, S_{p+1}^{p,S_p}\right) - \alpha\tilde{S}_{p+1}^{p,S_p}\}^2 + V_{p+1}^{0,\alpha} \mid \mathcal{F}_p\right]$$

and for $n \geq 1$ and $p \leq N-1$

$$\begin{aligned} V_p^{n,\alpha} &= min\{E\left[V_{p+1}^{n,\alpha} \mid \mathcal{F}_p\right] + E\left[\{\tilde{c}\left(p+1, S_{p+1}^{p,S_p}\right) - \alpha\tilde{S}_{p+1}^{p,S_p}\}^2 \mid \mathcal{F}_p\right] \\ &\quad -(\tilde{c}_p - \alpha\widetilde{S}_p)^2, V_p^{n-1,*}\} \end{aligned}$$

where $V_N^{n,\alpha} = 0$, $V_p^{n-1,*} = \min\limits_{\alpha} V_p^{n-1,\alpha}$, and $\mathcal{F}_p = \sigma(S_0, S_1.., S_p)$,

then the optimal variance v_n is given at time p by:

$$v_n(p, S_p, \alpha) = V_p^{n,\alpha}$$

Moreover, the first optimal date of trading (after p) is given by:

$$\tau_p^* = \min\{u \in \{p, ..., N\}/V_u^{n-1,*} \leq V_u^{n,\alpha}\}$$

The above result gives the following algorithm: we first calculate the variance of the tracking error for $n = 0$ at every node at every time step working backwards throughout the tree. Using $V_N^{1,\alpha} = 0$, we can compute the error at every node at time N. We can then apply the Dynamic Programming equation to compute the error at time $N-1$ which is the minimum of the immediate trading value and the present value of continuing without trading. We can then reapply this procedure at every node at every time step as the step n.

The scheme at hand is of complexity $n * N^3$: we solve for each resolution of (DP) a family of optimal stopping problem (complexity N^2) for every level α (which is discretized with N levels). There are n such steps.

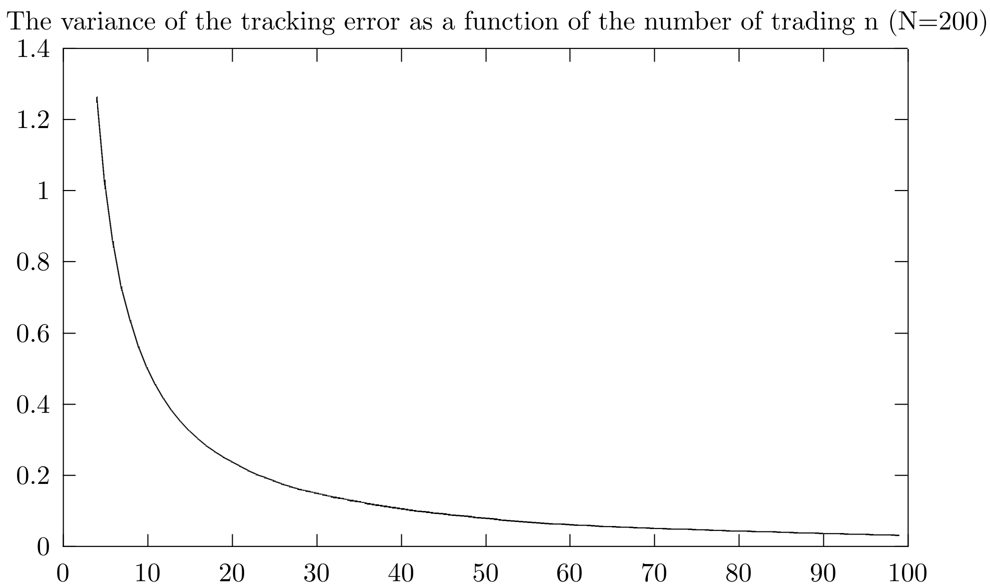
5 Numerical results

The option considered here is a European Call expiring at date $T = 0.333$ with strike price K. The value of the parameters are the following:

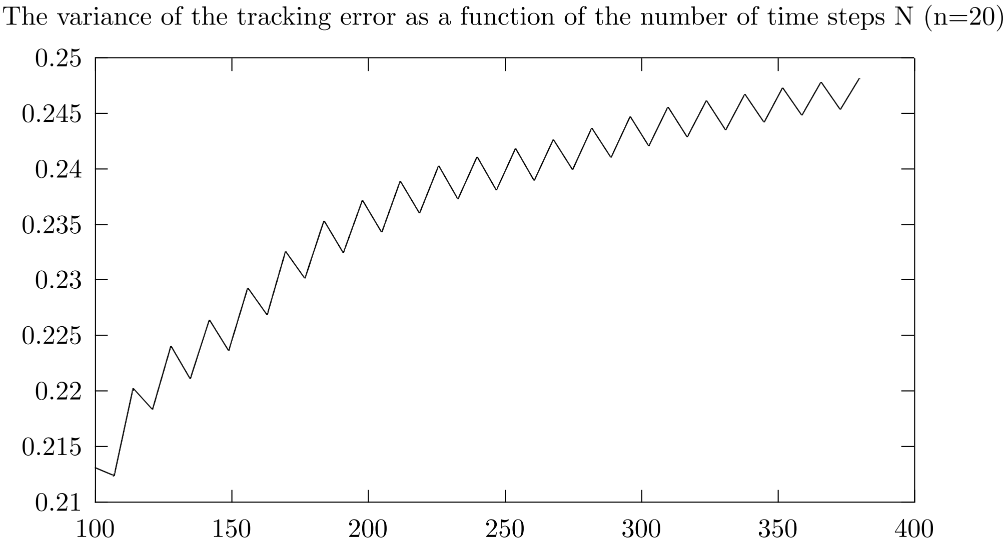
$$S_0 = 100, K = 100, r = 0, \sigma = 0.2$$

5.1 Tracking error in the binomial scheme

The following figure plots the variance of the tracking error as a function of the number of rebalancing. This figure illustrates the rapid decrease of the tracking error as the number of trading increases.



In the next plot, we fix the number of hedging, and we draw the error as a function of the number of time steps. Note that it displays the typical oscillatory behaviour of a binomial method.



5.2 Comparison with deterministic strategies

In the following table, we compare the variance of the tracking error in our model with the error obtained in the deterministic case from Zhang ([**?**]).

Number of hedging times (n)	Error for optimal stopping times	Error for deterministic times
10	0.500	1.662
20	0.236	0.831
30	0.149	0.554
40	0.105	0.415
50	0.078	0.332
60	0.061	0.277
70	0.051	0.237
80	0.043	0.207
90	0.036	0.184
100	0.031	0.166

Thus it shows that the variance of the tracking error for optimal stopping times can be reduced significantly compared to the deterministic hedging case.