Asian options pricing with finite elements

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In [1] Ben Hameur, Breton and Lecuyer are interested in the pricing of fixed strike Amerasian options based on arithmetic average. They approximate the Amerasian option by a Bermudean Asian option with n possible exercise dates. The pricing of Bermudean style options is naturally formulated as a stochastic dynamic programming (DP) problem. At each exercise date the price of the option is obtained as the maximum of the exercise value of the option and the holding value. Ben Hameur, Breton and Lecuyer propose a numerical approach based on piecewise bilinear interpolation over rectangular finite elements to solve the (DP) equations under the Black and Scholes model. Their method can be adapted to the pricing of European Asian option by keeping the computation of the holding value but removing the maximisation with the exercise value at each exercise date. So from now on, we focus on the American case.

Notations

We consider a single primitive asset whose price $(S_t)_{0\leq t\leq T}$ satisfies the Black and Scholes equation under the risk-neutral probability measure :

$$\begin{aligned} &S_0 > 0 \\ &dS_t = S_t((r-d)dt + \sigma dW_t) \quad 0 \leq t \leq T \end{aligned} \tag{1}$$

where

- T is the maturity date
- r is the risk free rate

- d is the dividend rate
- σ is the volatility parameter
- $(W_t)_{0\leq t\leq T}$ is a standart brownian motion
- $\mu = r - d - \sigma^2/2$

Let $h = T/n$ and $t_m = mh$ be the observation dates for $0 \leq m \leq n$; for $1 \leq m \leq n$, t_m is a possible exercise date and the corresponding payoff is

- for the American call : $(\bar{S}_m - K)^+$
- for the American put : $(K - \bar{S}_m)^+$

where $(x)^+ = \max(x, 0)$ and $\bar{S}_m = \frac{S_{t_1}+\cdots+S_{t_m}}{m}$. We note at time $t_m : S_{t_m} = s$ et $\bar{S}_m = \bar{s}$, and we focus only on the Amerasian call.

The DP equation

The value function

For $m = 0, \ldots, n$ the price $v_m(s, \bar{s})$ of the Bermudean Asian option at time t_m when spot is s and average $\bar{s}$ is obtained as the maximum of *the exercice value* :

$$v_m^e(\bar{s}) = (\bar{s} - K)^+ \tag{2}$$

and the *the holding value* (which is the value of the option if it is not exercised at time t_m but is exercised optimally in the future) :

$$v_m^h(s, \bar{s}) = \left\{ \begin{array}{ll} \rho E\left[v_1\left(S_{t_1}, S_{t_1}\right) | S_0 = s\right] & \text{if} \quad m = 0 \\ \rho E\left[v_{m+1}\left(S_{t_{m+1}}, \frac{m\bar{s} + S_{t_{m+1}}}{m+1}\right) | S_{t_m} = s\right] & \text{if} \quad 1 \leq m \leq n-1 \end{array} \right. \tag{3}$$

where $\rho = e^{-rh}$.
Hence :

$$v_m(s, \bar{s}) = \left\{ \begin{array}{ll} \max\left\{v_m^h(s, \bar{s}), v_m^e(\bar{s})\right\} & \text{if} \quad 0 \leq m \leq n-1 \\ v_m^e(\bar{s}) & \text{if} \quad m = n \end{array} \right. \tag{4}$$

The change of variable $\bar{s}' = \frac{m\bar{s} - s}{m-1}$ if $m > 1$ and 0 if not simplifies the dynamic programming equations. If we set $w_m^h(s, \bar{s}') = v_m^h\left(s, \frac{(m-1)\bar{s}' + s}{m}\right)$ and $w_m(s, \bar{s}') = v_m\left(s, \frac{(m-1)\bar{s}' + s}{m}\right)$ we have for $1 \leq m$,

$$w_m^h(s, \bar{s}') = \rho E\left[w_{m+1}\left(s\, e^{\sigma(W_{t_{m+1}} - W_{t_m}) + \mu(t_{m+1} - t_m)}, \frac{(m-1)\bar{s}' + s}{m}\right)\right] \tag{5}$$

and

$$w_m(s, \bar{s}') = \max\left\{\left(\frac{(m-1)\bar{s}' + s}{m} - K\right)^+, w_m^h(s, \bar{s}')\right\} \tag{6}$$

For $m = 0$,

$$w_0(S_0) = \rho E\left[w_1\left(S_0\, e^{\sigma W_{t_1} + \mu t_1}, 0\right)\right] \tag{7}$$

The advantage of the change of variable is that the second variable of the function w_{m+1} in the expectation giving w_m^h becomes deterministic.

The closed form for w_{n-1}

Setting $\bar{K} = nK - ((n-2)\bar{s}' + s)$ one obtains :

$$\begin{aligned} w^h_{n-1}(s,\bar{s}') &= \rho E\left[\left(\frac{(n-2)\bar{s}' + s(1+e^{\sigma(W_{t_n}-W_{t_{n-1}})+\mu(t_n-t_{n-1})})}{n} - K\right)^+\right] \\ &= \frac{\rho}{n}E\left[\left(s\,e^{\sigma(W_{t_n}-W_{t_{n-1}})+\mu(t_n-t_{n-1})} - \bar{K}\right)^+\right] \end{aligned}$$

Hence :

$$w^h_{n-1}(s,\bar{s}') = \begin{cases} \frac{s}{n}e^{-dh} - \rho\frac{\bar{K}}{n} & \text{if} \quad \bar{K} \leq 0 \\ \frac{1}{n}(N(d_1)se^{-dh} - \rho\bar{K}N(d_1 - \sigma\sqrt{h})) & \text{if} \quad \bar{K} > 0 \end{cases} \tag{8}$$

where $d_1 = \frac{\ln(s/\bar{K})+(r-d+\sigma^2/2)h}{\sigma\sqrt{h}}$ and N denotes the standart normal cumulative distribution function. One can finally express w_{n-1} :

$$w_{n-1}(s,\bar{s}') = \max\left\{w^h_{n-1}(s,\bar{s}'), \left(\frac{(n-2)\bar{s}'+s}{n-1} - K\right)^+\right\} \tag{9}$$

Numerical Solution

Notations

For $0 \leq m \leq n-2$, the computation of w_m from w_{m+1} is based on backward recursion using finite elements over $(s,\bar{s}') \in [0,\infty[^2$. Ben Hameur, Breton and Lecuyer propose to take the grid defined by the points (a_i, b_j) with $i = 0,\ldots,p+1$ and $j = 0,\ldots,q+1$ where :

- $a_0 = 0$
- $a_1 = S_0 e^{\mu t_{n-1} - 3\sigma\sqrt{t_{n-1}}}$
- $a_i =$ the quantile of order $\frac{i-1}{p-2}$ of the lognormal distribution with parameters μt_{n-1} and $\sigma\sqrt{t_{n-1}}$ for $i = 2,\ldots,p-2$
- $a_{p-1} = S_0 e^{\mu t_{n-1} + 3\sigma\sqrt{t_{n-1}}}$
- $a_p = S_0 e^{\mu t_{n-1} + 4\sigma\sqrt{t_{n-1}}}$
- $a_{p+1} = +\infty$

and

- $b_0 = 0$
- $b_1 = S_0 e^{\mu t_{n-1} - 2\sigma\sqrt{t_{n-1}}}$
- $b_{q/4} = K\frac{(n-1)\rho-1}{n-2}$
- $b_{3q/4} = \frac{nK}{n-2}$
- $b_q = S_0 e^{\mu t_{n-1} + 3.9\sigma\sqrt{t_{n-1}}}$
- $b_{q+1} = +\infty$
- the other b_j are equally spaced.

The function w_m is approximated by a bilinear function over each rectangle:

$$\tilde{w}_m(s, \bar{s}') = \sum_{i=0}^{p} \sum_{j=0}^{q} I_{s,\bar{s}'}^{i,j} (\alpha_{ij}^m + \beta_{ij}^m s + \gamma_{ij}^m \bar{s}' + \delta_{ij}^m s\bar{s}') \tag{10}$$

where $I_{s,\bar{s}'}^{i,j} = 1$ if $a_i \le s < a_{i+1}$ and $b_j \le \bar{s}' < b_{j+1}$ and 0 if not.

How to proceed

The backward recursion is initialized by choosing the coefficient α_{ij}^{n-1}, β_{ij}^{n-1}, γ_{ij}^{n-1} and δ_{ij}^{n-1} so that the values of $\tilde{w}_{n-1}$ coincides with the values given by the closed form expression of w_{n-1} at the points of the grid.

The first step to obtain $\tilde{w}_m$ from $\tilde{w}_{m+1}$ consists in computing $\tilde{w}_m$ at the points of the grid.

$$\begin{aligned} \tilde{w}_m^h(a_k, b_l) &= \rho \textstyle\sum_{i=0}^{p} \left([\alpha_{i\xi}^{m+1} + \gamma_{i\xi}^{m+1} c_{kl}] P_{ik} + [\beta_{i\xi}^{m+1} + \delta_{i\xi}^{m+1} c_{kl}] a_k Q_{ik} \right) \\ \tilde{w}_m(a_k, b_l) &= \max\{\tilde{w}_m^h(a_k, b_l), (c_{kl} - K)^+\} \end{aligned} \tag{11}$$

where[1]

- τ is lognormal with parameters μh and $\sigma\sqrt{h}$
- I is the indicator function
- $P_{ik} = E[I_{\{a_i \le a_k \tau < a_{i+1}\}}] = N\left(\frac{\ln(a_{i+1}/a_k) - h\mu}{\sigma\sqrt{h}}\right) - N\left(\frac{\ln(a_i/a_k) - h\mu}{\sigma\sqrt{h}}\right)$
- $Q_{ik} = E[\tau I_{\{a_i \le a_k \tau < a_{i+1}\}}] = e^{h\sigma^2/2 + \mu h}\left[N\left(\frac{\ln(a_{i+1}/a_k) - h\mu}{\sigma\sqrt{h}}\right) - N\left(\frac{\ln(a_i/a_k) - h\mu}{\sigma\sqrt{h}}\right)\right]$

[1] with the conventions $N(-\infty) = 0$ and $N(+\infty) = 1$

- $c_{kl} = \frac{(m-1)b_l + a_k}{m} \quad k = 0, \ldots, p \quad l = 0, \ldots, q$
- ξ is such that $c_{kl} \in [b_\xi, b_{\xi+1}[$

The second step is devoted to the computation of α_{ij}^m, β_{ij}^m, γ_{ij}^m and δ_{ij}^m. Setting for $i = 0, \ldots, p-1$ and $j = 0, \ldots, q-1$: $w_1 = \tilde{w}_m(a_i, b_j)$, $w_2 = \tilde{w}_m(a_{i+1}, b_j)$, $w_3 = \tilde{w}_m(a_i, b_{j+1})$ and $w_4 = \tilde{w}_m(a_{i+1}, b_{j+1})$. We obtain by an easy computation:

- $\delta_{ij}^m = \frac{w_4 - w_3 - w_2 + w_1}{(b_{j+1} - b_j)(a_{i+1} - a_i)}$
- $\beta_{ij}^m = \frac{w_2 - w_1}{a_{i+1} - a_i} - \delta_{ij}^m b_j$
- $\gamma_{ij}^m = \frac{w_3 - w_1}{b_{j+1} - b_j} - \delta_{ij}^m a_i$
- $\alpha_{ij}^m = w_1 - \beta_{ij}^m a_i - \gamma_{ij}^m b_j - \delta_{ij}^m a_i b_j$

By extrapolation α_{p+1j}^m, β_{p+1j}^m, γ_{p+1j}^m, δ_{p+1j}^m (resp. α_{iq+1}^m, β_{iq+1}^m, γ_{iq+1}^m, δ_{iq+1}^m) are chosen equal to α_{pj}^m, β_{pj}^m, γ_{pj}^m, δ_{pj}^m (resp. α_{iq}^m, β_{iq}^m, γ_{iq}^m, δ_{iq}^m).

At time t_1 the variable $\bar{s}'$ is identically equal to 0. The approximation $\tilde{w}_1(s, 0)$ of $w_1(s, 0)$ is chosen affine on each interval $[a_i, a_{i+1}]$ and obtained after computation of $\tilde{w}_1(a_i, 0)$ and $\tilde{w}_1(a_{i+1}, 0)$ as above. Let η be such that the initial spot S_0 belongs to $[a_\eta, a_{\eta+1}[$. Then $\tilde{w}_0(a_\eta)$ and $\tilde{w}_0(a_{\eta+1})$ are computed by :

$$\begin{array}{l} \tilde{w}_0(a_\eta) = \rho \sum_{i=0}^{p} (\alpha_{i0}^1 P_{i\eta} + \beta_{i0}^1 a_\eta Q_{i\eta}) \\ \tilde{w}_0(a_{\eta+1}) = \rho \sum_{i=0}^{p} (\alpha_{i0}^1 P_{i\eta+1} + \beta_{i0}^1 a_{\eta+1} Q_{i\eta+1}) \end{array} \tag{12}$$

The approximated price and delta of the option returned by the method are respectively $\tilde{w}_0(a_\eta) + \frac{\tilde{w}_0(a_{\eta+1}) - \tilde{w}_0(a_\eta)}{a_{\eta+1} - a_\eta}(S_0 - a_\eta)$ and $\frac{\tilde{w}_0(a_{\eta+1}) - \tilde{w}_0(a_\eta)}{a_{\eta+1} - a_\eta}$.

References

[1] H.Ben Hameur M.Breton P. L'Ecuyer. A numerical procedure for pricing american-style asian option. *preprint*, 1999. 1