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ap_call_carr

We assume that the underlying S_t evolves according to Merton's model:

$$S_t = S_0(e^{\sigma W_t + (\mu - \delta - \frac{\sigma^2}{2})t} \prod_{j=1}^{N_t} e^{m + \sqrt{v}g_j})$$

where

- W_t is a brownian motion.
- N_t is the counting function of an independent Poisson process with parameter λ .
- The $(g_j)_j$ are independent normal variables with mean 0 and variance 1, independent of both W and N .
- σ : volatility.
- μ : trend.
- δ : dividend rate.
- (m, v) : parameters of the jump law.

In other words, the spot evolves according to the Black-Scholes model between the jump times (τ_j) of the Poisson process, and jumps at times τ_j :

$$S_{\tau_j} = S_{\tau_j-}(e^{m + \sqrt{v}g_j})$$

We choose Merton's risk neutral probability measure in order to price the call option with maturity T and strike K . Therefore, we define the price of this option at time t by:

$$C_t = \mathbf{E}^*(e^{-r(T-t)}(S_T - K)_+ | \mathcal{F}_t)$$

where $\frac{d\mathbf{P}^*}{d\mathbf{P}} = e^{\int_0^T \frac{r - \lambda e^{\frac{m + \frac{v}{2} - \mu}}}{\sigma} dW_t - \frac{1}{2} \int_0^T (\frac{r - \lambda e^{\frac{m + \frac{v}{2} - \mu}}}{\sigma})^2 dt}$.

A method using the characteristic function

Let us consider the price of a European Call in this model, assuming that t is the pricing date and that T is the maturity date. One easily obtains $C_t = C(S, \theta)$, where $\theta = T - t$.

$$C(S, \theta) = \mathbf{E} \left(e^{-r\theta} (S e^{\sigma W_\theta + (r - \lambda e^{m+\frac{v}{2}} - \delta - \frac{\sigma^2}{2})\theta} \prod_{j=1}^{N_\theta} (e^{m+\sqrt{v}g_j}) - K)_+ \right).$$

We remind that N_t , W_t , and the $(g_j)_{j \geq 1}$ are independent.

We introduce $s_\theta = \ln(S) + \sigma W_\theta + (r - \lambda e^{m+\frac{v}{2}} - \delta - \frac{\sigma^2}{2})\theta + \sum_{k=1}^{N_\theta} (m + \sqrt{v}g_k)$ the logarithm of the underlying and

$$\phi_{\theta(u)} = \mathbf{E}(e^{ius_\theta}) = e^{\lambda\theta(e^{ium - u^2\frac{v}{2}} - 1) - \frac{u^2\sigma^2\theta^2}{2} + iu \left(s_0 + [r - \lambda e^{m+\frac{v}{2}} - \delta - \frac{\sigma^2}{2} + \lambda]\theta \right)} \quad (1)$$

its characteristic function[1].

We are going to express the call price $C(S, \theta)$ in terms of the characteristic function.

We write $C(S, \theta) = \mathbf{E} \left(e^{-r\theta} e^{s_\theta} \mathbf{1}_{\{s_\theta \geq \ln(K)\}} \right) - K e^{-r\theta} \mathbf{P}(s_\theta \geq \ln(K))$.

We are first going to express the cumulative distribution function $F_\theta(x) = \mathbf{P}(s_\theta \leq x)$ of s_θ in terms of ϕ_θ . We denote $p_\theta(x)$ the density of s_θ . $F_\theta(x) = \int_{-\infty}^x p_\theta(y) dy = H \star p_\theta(x)$, where H denotes the Heavyside function.

We deduce that the Fourier transform satisfies $\widehat{F}_\theta(u) = \widehat{H}(u)\phi_\theta(u)$. As $\widehat{H}(u) = \pi\delta_0 + i v p(\frac{1}{u})$ ($\langle v p(\frac{1}{u}), \varphi \rangle = \lim_{\epsilon \rightarrow 0} \int_{|u| > \epsilon} \frac{\varphi(u)}{u} du$), using the inverse

Fourier transform formula, we get

$$F_\theta(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-iux} \widehat{F}_\theta(u) du = \frac{1}{2} + \frac{1}{2\pi} \langle v p(\frac{1}{u}), i e^{-iux} \phi_\theta(u) \rangle.$$

Since the conjugate of $\frac{i e^{-iux} \phi_\theta(u)}{u}$ is $\frac{i e^{iux} \phi_\theta(-u)}{-u}$, one has:

$$\langle v p(\frac{1}{u}), i e^{-iux} \phi_\theta(u) \rangle = 2 \lim_{\epsilon \rightarrow 0} \int_{\epsilon}^{+\infty} \operatorname{Re} \left[\frac{i e^{-iux} \phi_\theta(u)}{u} \right] du = 2 \int_0^{+\infty} \operatorname{Re} \left[\frac{i e^{-iux} \phi_\theta(u)}{u} \right] du.$$

Hence

$$\Pi_2 = \mathbf{P}\left(s_\theta \geq \ln(K)\right) = \frac{1}{2} + \frac{1}{\pi} \int_0^{+\infty} \operatorname{Re} \left[\frac{e^{-iuln(K)} \phi_\theta(u)}{iu} \right] du. \quad (2)$$

We introduce the probability measure $\tilde{\mathbf{P}}$ equivalent to \mathbf{P} defined by:

$$\frac{d\tilde{\mathbf{P}}}{d\mathbf{P}} = \frac{e^{s_\theta}}{\mathbf{E}(e^{s_\theta})}.$$

$$\mathbf{E}(e^{-r\theta} e^{s_\theta} \mathbf{1}_{\{s_\theta \geq \ln(K)\}}) = \mathbf{E}(e^{-r\theta} e^{s_\theta}) \tilde{\mathbf{P}}(s_\theta \geq \ln(K)) = e^{-\delta\theta} S \tilde{\mathbf{P}}(s_\theta \geq \ln(K)).$$

The characteristic function of s_θ under $\tilde{\mathbf{P}}$ is given by:

$$\tilde{\mathbf{E}}(e^{ius_\theta}) = \frac{\mathbf{E}(e^{s_\theta} e^{ius_\theta})}{\mathbf{E}(e^{s_\theta})} = \frac{\phi_\theta(u-i)}{\phi_\theta(-i)}.$$

Hence, similarly to (2),

$$\Pi_1 = \tilde{\mathbf{P}}\left(s_\theta \geq \ln(K)\right) = \frac{1}{2} + \frac{1}{\pi} \int_0^{+\infty} \operatorname{Re} \left[\frac{e^{-iuln(K)} \phi_\theta(u-i)}{iu\phi_\theta(-i)} \right] du. \quad (3)$$

We conclude that

$$C(S, \theta) = S e^{-\delta\theta} \Pi_1 - K e^{-r\theta} \Pi_2. \quad (4)$$

Furthermore, the correspondent delta is given by:

$$\Delta = \frac{\partial C(S, \theta)}{\partial S} = e^{-\delta\theta} \Pi_1.$$

Both Π_1 and Π_2 can be computed by inserting (1) in their above expressions and numerically discretizing the corresponding integrals.

The aim is to compute the Call price: $C(\theta, S) = S e^{-\delta\theta} \Pi_1 - K e^{-r\theta} \Pi_2$ and the delta $\Delta = \frac{\partial C(\theta, S)}{\partial S} = e^{-\delta\theta} \Pi_1$, with

$$\Pi_1 = \frac{1}{2} + \frac{1}{\pi} \int_0^{+\infty} \operatorname{Re} \left[\frac{e^{-iuln(K)} \phi_\theta(u-i)}{iu\phi_\theta(-i)} \right] du,$$

$$\Pi_2 = \frac{1}{2} + \frac{1}{\pi} \int_0^{+\infty} \operatorname{Re} \left[\frac{e^{-iuln(K)} \phi_\theta(u)}{iu} \right] du,$$

and

$$\phi_{\theta(u)} = \mathbf{E}(e^{ius_\theta}) = e^{\lambda\theta(e^{i\theta m - u^2 \frac{v}{2}} - 1) - \frac{u^2 \sigma^2 \theta^2}{2} + iu \left(s_0 + [r - \lambda e^{m + \frac{v}{2}} - \delta - \frac{\sigma^2}{2} + \lambda] \theta \right)}.$$

/*Carr*/

Computes Π_1 and Π_2 by discretizing the above integrals by the trapezoid method. More precisely, we choose a step h ($h = 0.01$) and

$$0 = x_0 < x_1 < \dots < x_{\frac{M}{h}} = M$$

($M = 100$) such that $\forall k, x_{k+1} - x_k = h$. If we denote by f one of the two integrands, we approximate $\int_0^M f(u) du$ by $\frac{h}{2} \left(f(x_0) + f(x_{\frac{M}{h}}) + 2 \sum_{j=1}^{\frac{M}{h}-1} f(x_j) \right)$.

References

- [1] P. CARR D.B.MADAN. Option valuation using the fast fourier transform. Technical report, 1999. 2