mc_variancereduction

1 Computation of the price by Monte Carlo methods

We want to compute the price at time t $\in [0,T]$ of a "down and out" call option:

$$\Pi_{i=1}^{j} 1\left(S_{t_i} > L\right) \exp\left(-r(T-t)\right) E^{*}\left[\Pi_{i=j+1}^{m} 1\left(S_{t_i} > L\right)\left(S_T - K\right)_{+} | F_t\right]$$

where $\mathrm{E}^{*}[.]$ denote the expectation under the risk neutral probability and $\mathrm{t}_j \leq t \leq t_{j+1}$ In order to simplify the notations, let us place at time 0. Our aim is then to approximate the quantity $\mathrm{P}{=}\mathrm{E}^{*}\left[\Pi_{i=1}^{m} 1\left(S_{t_i} > L\right)\left(S_T - K\right)_{+} | F_t\right]$ by using Monte Carlo methods, while attaching ourselve to reduce the variance of the estimator.

2 Variance Reduction

First, for an integrable random variable Y and an event A we can write

$$\begin{aligned} E\left[Y 1_A\right] &= P(A) E\left[Y \frac{1_A}{P(A)}\right] \\ &= P(A) E^{Q}\left[Y\right] \end{aligned}$$

where

$$\frac{dQ}{dP} = \frac{1_A}{P(A)}$$

So, to approximate

$\mathrm{E}[Y 1_A]$ we can either simulate the natural estimator $\mathrm{Y}1_A$ under the real probability or simulate $\mathrm{P}(A)\, Y$ under the probability Q, that is equivalent to simulate Y ($P\left(A\right)$ is a constant) conditionaly to the event A. Let

Y_1 be a random variable such that $\pounds(Y_1) \equiv \pounds(Y|A)$ It's quite straightforward that: $\sigma^2(P(A)Y_1) \leq \sigma^2(Y1_A)$ indeed we have $\sigma^2(P(A)Y_1) = (P(A))^2\left(E[Y^2|A] - (E[Y|A])^2\right)$ and $\sigma^2(Y1_A) = P(A)E[Y^2|A] - (P(A))^2(E[Y|A])^2$ but

$$P(A) > (P(A))^2$$

Let us now consider the price P of a "down and out" call option with barrier L and 2 monitoring instant t_1, t_2 :

$$\begin{aligned} P &= E^*\left[1(S_{t_1} > L)1(S_{t_2} > L)(S_T - K)_+\right] \\ &= E^*\left[1(S_{t_1} > L)1(S_{t_2} > L)(S_T - K)_+ \,|\, (S_{t_1} > L)\right]P(S_{t_1} > L) \\ &= E^*\left[1(S_{t_1} > L)1\left(\frac{S_{t_2}}{S_{t_1}}S_{t_1} > L\right)\left(\frac{S_T}{S_{t_2}}\frac{S_{t_2}}{S_{t_1}}S_{t_1} - K\right)_+ |(S_{t_1} > L)\right]P(S_{t_1} > L) \end{aligned}$$

with $\mathrm{X}_1 = S_{t_1}, X_2 = \frac{S_{t_2}}{S_{t_1}}, X_3 = \frac{S_T}{S_{t_2}}$ P=E$^*\left[1(X_2X_1 > L)(X_3X_2X_1 - K)_+ \,|\, (X_1 > L)\right]P(X_1 > L)$ Denote Y_1 a random variable independant of X_2 and X_3 such that $\pounds(\mathrm{Y}_1) \equiv \pounds(X_1|X_1 > L)$ then we can write

$$\begin{aligned} P &= E^*\left[1(X_2Y_1 > L)(X_3X_2Y_1 - K)_+\right]P(X_1 > L) \\ &= E^*\left[E^*\left[(X_3X_2Y_1 - K)_+ \,|\, \left(X_2 > \frac{L}{Y_1}\right)\right]P\left(X_2 > \frac{L}{Y_1}\right)\right]P(X_1 > L) \end{aligned}$$

Denote Y_2 a random variable independant of X_3 such that $\pounds(\mathrm{Y}_2) \equiv \pounds\left(X_2|X_2 > \frac{L}{Y_1}\right)$ then we can write P=E$^*\left[(X_3Y_2Y_1 - K)_+ P\left(X_2 > \frac{L}{Y_1}\right)\right]P(X_1 > L)$ this leads to the new estimator $(X_3Y_2Y_1 - K)_+ P\left(X_2 > \frac{L}{Y_1}\right)P(X_1 > L)$ We generalize this result to the case with n monitoring instant $t_1, ..., t_n$ and obtain the general form of the new estimator that we denote N N=$(\mathrm{X}_{n+1}Y_nY_{n-1}...Y_1)_+P\left(X_{n+1} > \frac{L}{Y_nY_{n-1}...Y_1}\right)...P$ where

$$\begin{aligned} X_1 &= S_{t_1} \\ X_i &= \frac{S_{t_i}}{S_{t_{i-1}}} 2 \leq i \leq n \\ X_{n+1} &= \frac{S_T}{S_{t_n}} \\ \pounds(Y_1) &= \pounds(X_1|X_1 > L) \\ \pounds(Y_i) &= \pounds\left(X_i|X_i > \frac{L}{Y_{i-1}...Y_1}\right) 2 \leq i \leq n \end{aligned}$$

3 Simulation

3.1 Simulation of N

To simulate N we proceed as follows : (1) We simulate Y_1 (2) Knowing the value of $\frac{L}{\mathrm{Y}_1}$ we can simulate Y_2 ... (i) Knowing the value of $\frac{L}{Y_{i-1}...Y_1}$ we can simulate Y_i for $2 \leq i \leq n$ ... (n+1) X_{n+1} being independant of $Y_1, ..., Y_n$ we can simulate it (n+2) Knowing the values of $Y_1, ..., Y_n$ we can compute $P\left(X_{n+1} > \frac{L}{Y_n Y_{n-1}...Y_1}\right) ...P\left(X_2 > \frac{L}{\mathrm{Y}_1}\right) P\left(X_1 > L\right)$

3.2 Simulation of $Y_n, Y_{n-1}, ..., Y_1$

In our example, the simulation of the random variable $Y_n, Y_{n-1}, ..., Y_1$ amounts to the simulation of random variables $(\mathrm{g}_i)_{i=1...n}$ normaly distributed with mean 0 and variance 1 conditionaly to the events $(g_i \in [a_i, +\infty])_{i=1...n}$ To do this, we use the following method: Let Z be a random variable such that $\pounds(Z) \equiv \pounds\left(g_i | g_i \in [a_i, +\infty]\right)$ and u$\geq a_i$ we can write

$$\begin{aligned} P(Z \leq u) &= P(X \leq u | X \in [a_i, +\infty]) \\ \Leftrightarrow F_Z(u) &= \frac{P(a_i \leq X \leq u)}{P(a_i \leq X)} \\ \Leftrightarrow F_Z(u) &= \frac{F_{g_i}(u) - F_{g_i}(a_i)}{1 - F_{g_i}(a_i)} \end{aligned}$$

it follows then $\mathrm{F}_Z^{-1}(y) = F_{g_i}^{-1}\left(F_{g_i}(a_i) + y\left(1 - F_{g_i}(a_i)\right)\right)$ But we know that if the random variable U is uniformly distributed on $[0, 1]$ then $\pounds\left(F_Z^{-1}(U)\right) \equiv \pounds\left(Z\right)$ has the same law than Z. Since we can easily simulate U and compute $F_{g_i}^{-1}\left(U\right)$ the simulation of Z doesn't raise any technical problem.

References