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ap_put_carr

We assume that the underlying S_t evolves according to Merton's model:

$$S_t = S_0(e^{\sigma W_t + (\mu - \delta - \frac{\sigma^2}{2})t} \prod_{j=1}^{N_t} e^{m + \sqrt{v} g_j})$$

where
$\cdot W_t$ is a brownian motion.
$\cdot N_t$ is the counting function of an independent Poisson process with parameter λ.
$\cdot$The $(g_j)_j$ are independent normal variables with mean 0 and variance 1, independent of both W and N.
$\cdot\sigma$: volatility.
$\cdot\mu$: trend.
$\cdot\delta$: dividend rate.
$\cdot(m, v)$: parameters of the jump law.

In other words, the spot evolves according to the Black-Scholes model between the jump times (τ_j) of the Poisson process, and jumps at times τ_j:

$$S_{\tau_j} = S_{\tau_{j-}}(e^{m + \sqrt{v} g_j})$$

We choose Merton's risk neutral probability measure in order to price the call option with maturity T and strike K. Therefore, we define the price of this option at time t by:

$$C_t = \mathbf{E}^*(e^{-r(T-t)}(K - S_T)_+ | \mathcal{F}_t)$$

where $\frac{d\mathbf{P}^*}{d\mathbf{P}} = e^{\int_0^T \frac{r - \lambda e^{m + \frac{v}{2}} - \mu}{\sigma} \, dW_t - \frac{1}{2} \int_0^T (\frac{r - \lambda e^{m + \frac{v}{2}} - \mu}{\sigma})^2 \, dt}$.

A method using the characteristic function

Let us consider the price of a European Call in this model, assuming that t is the pricing date and that T is the maturity date. One easily obtains $C_t = C(S,\theta)$, where $\theta = T - t$.

$$C(S,\theta) = \mathbf{E}\Big(e^{-r\theta}(K - Se^{\sigma W_\theta + (r-\lambda e^{m+\frac{v}{2}}-\delta-\frac{\sigma^2}{2})\theta}\prod_{j=1}^{N_\theta} e^{m+\sqrt{v}g_j})_+\Big).$$

We remind that N_t, W_t, and the $(g_j)_{j\geq 1}$ are independent.

We introduce $s_\theta = ln(S) + \sigma W_\theta + (r - \lambda e^{m+\frac{v}{2}} - \delta - \frac{\sigma^2}{2})\theta + \sum\limits_{k=1}^{N_\theta}(m + \sqrt{v}g_k)$ the logarithm of the underlying and

$$\phi_{\theta(u)} = \mathbf{E}(e^{ius_\theta}) = e^{\lambda\theta(e^{ium-u^2\frac{v}{2}}-1)-\frac{u^2\sigma^2\theta^2}{2}+iu\Big(s_0+[r-\lambda e^{m+\frac{v}{2}}-\delta-\frac{\sigma^2}{2}+\lambda]\theta\Big)} \tag{1}$$

its characteristic function[1].
We are going to express the call price $C(S,\theta)$ in terms of the characteristic function.

We write $C(S,\theta) = Ke^{-r\theta}\mathbf{P}(s_\theta \leq ln(K) - \mathbf{E}\Big(e^{-r\theta}e^{s_\theta}\mathbf{1}_{\{s_\theta\leq ln(K)\}}\Big)$.

We are first going to express the cumulative distribution function $F_\theta(x) = \mathbf{P}(s_\theta \leq x)$ of s_θ in terms of ϕ_θ. We denote $p_\theta(x)$ the density of s_θ.
$F_\theta(x) = \int\limits_{-\infty}^{x} p_\theta(y)\,dy = H \star p_\theta(x)$, where H denotes the Heavyside function.
We deduce that the Fourier transform satisfies $\widehat{F}_\theta(u) = \widehat{H}(u)\phi_\theta(u)$. As $\widehat{H}(u) = \pi\delta_0 + ivp(\frac{1}{u})$ $(< vp(\frac{1}{u}), \varphi >= \lim_{\epsilon\to 0}\int\limits_{|u|>\epsilon}\frac{\varphi(u)}{u}\,du)$, using the inverse Fourier transform formula, we get

$$F_\theta(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-iux}\widehat{F}_\theta(u)\,du = \frac{1}{2} + \frac{1}{2\pi} < vp(\frac{1}{u}), ie^{-iux}\phi_\theta(u) > .$$

Since the conjugate of $\frac{ie^{-iux}\phi_\theta(u)}{u}$ is $\frac{ie^{iux}\phi_\theta(-u)}{-u}$, one has:

$$< vp(\frac{1}{u}), ie^{-iux}\phi_\theta(u) >= 2\lim_{\epsilon\to 0}\int_{\epsilon}^{+\infty} Re\Big[\frac{ie^{-iux}\phi_\theta(u)}{u}\Big]\,du = 2\int_{0}^{+\infty} Re\Big[\frac{ie^{-iux}\phi_\theta(u)}{u}\Big]\,du.$$

Hence

$$\Pi_2 = \mathbf{P}\Big(s_\theta \leq ln(K)\Big) = \frac{1}{2} - \frac{1}{\pi}\int\limits_0^{+\infty} Re\Big[\frac{e^{-iuln(K)}\phi_\theta(u)}{iu}\Big]\, du. \tag{2}$$

We introduce the probability measure $\widetilde{\mathbf{P}}$ equivalent to $\mathbf{P}$ defined by:

$$\frac{d\widetilde{\mathbf{P}}}{d\mathbf{P}} = \frac{e^{s_\theta}}{\mathbf{E}(e^{s_\theta})}.$$

$$\mathbf{E}(e^{-r\theta}e^{s_\theta}\mathbf{1}_{\{s_\theta \geq ln(K)\}}) = \mathbf{E}(e^{-r\theta}e^{s_\theta})\widetilde{\mathbf{P}}(s_\theta \geq ln(K)) = e^{-\delta\theta}S\widetilde{\mathbf{P}}(s_\theta \geq ln(K)).$$

The characteristic function of s_θ under $\widetilde{\mathbf{P}}$ is given by:

$$\widetilde{\mathbf{E}}(e^{ius_\theta}) = \frac{\mathbf{E}(e^{s_\theta}e^{ius_\theta})}{\mathbf{E}(e^{s_\theta})} = \frac{\phi_\theta(u-i)}{\phi_\theta(-i)}.$$

Hence, similarly to (2),

$$\Pi_1 = \widetilde{\mathbf{P}}\Big(s_\theta \leq ln(K)\Big) = \frac{1}{2} - \frac{1}{\pi}\int\limits_0^{+\infty} Re\Big[\frac{e^{-iuln(K)}\phi_\theta(u-i)}{iu\phi_\theta(-i)}\Big]\, du. \tag{3}$$

We conclude that

$$C(S,\theta) = Ke^{-r\theta}\Pi_2 - Se^{-\delta\theta}\Pi_1. \tag{4}$$

Furthermore, the correspondent delta is given by:

$$\Delta = \frac{\partial C(S,\theta)}{\partial S} = -e^{-\delta\theta}\Pi_1.$$

Both Π_1 and Π_2 can be computed by inserting (1) in their above expressions and numerically discretizing the corresponding integrals.

The aim is to compute the Put price: $C(\theta, S) = Ke^{-r\theta}\Pi_2 - Se^{-\delta\theta}\Pi_1$ and the delta $\Delta = \frac{\partial C(\theta,S)}{\partial S} = -e^{-\delta\theta}\Pi_1$, with

$$\Pi_1 = \frac{1}{2} - \frac{1}{\pi}\int\limits_0^{+\infty} Re\Big[\frac{e^{-iuln(K)}\phi_\theta(u-i)}{iu\phi_\theta(-i)}\Big]\, du,$$

$$\Pi_2 = \frac{1}{2} - \frac{1}{\pi}\int\limits_0^{+\infty} Re\Big[\frac{e^{-iuln(K)}\phi_\theta(u)}{iu}\Big]\, du,$$

and

$$\phi_{\theta(u)} = \mathbf{E}(e^{ius_\theta}) = e^{\lambda\theta(e^{ium-u^2\frac{v}{2}}-1)-\frac{u^2\sigma^2\theta^2}{2}+iu\left(s_0+[r-\lambda e^{m+\frac{v}{2}}-\delta-\frac{\sigma^2}{2}+\lambda]\theta\right)}.$$

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Computes Π_1 and Π_2 by descretizing the above integrals by the trapezoid method. More precisely, we choose a step h ($h = 0.01$) and

$$0 = x_0 < x_1 < ... < x_{\frac{M}{h}} = M$$

($M = 100$) such that $\forall k,\ x_{k+1} - x_k = h$. If we denote by f one of the two integrands, we approximate $\int\limits_0^M f(u)\,du$ by $\frac{h}{2}\Big(f(x_0) + f(x_{\frac{M}{h}}) + 2\sum\limits_{j=1}^{\frac{M}{h}-1} f(x_j)\Big)$.

References

[1] P. CARR D.B.MADAN. Option valuation using the fast fourier transform. Technical report, 1999. 2