Premia 5

Cycle Postgrade des Grandes Ecoles
Ecole Polytechnique Fédérale de Lausanne – Ecole Polytechnique

Financial Mathematics

Pricing some derivatives

using

Laplace Transform

Clement AKEBOUE[†]

Academic supervisor Claude MARTINI [‡]
Professional supervisor Bernard Lapeyre [§]

September 1999

[†]Ecole Polytechnique ;akeboue@poly.polytechnique.fr
[‡] INRIA ; claude.martini@inria.fr
[§] Cermics ; bl@cermics.enpc.fr

Contents

1 Inverting the Laplace Transform

1.1 The Laplace Transform

Let $f(t)$ be complex function of the real variable t defined on the interval $0 \leq t < \infty$.

Let $f(t)$ be of bounded variation in the interval $0 \leq t < R$ for every positive R. This will be the case if and only if its real and imaginary parts have the same property. We assume that f is integrable.

Let z be a complex variable with real and imaginary parts a and b respectively,

$$z = a + ib$$

Its easy to understand from Stieltjes integral that the integral

$$\int_0^R e^{-zt} f(t) dt$$

exists for each positive R and for every complex s.

We now define the improper integral

$$\int_0^\infty e^{-zt} f(t)dt = \lim_{R\to\infty} \int_0^R e^{-zt} f(t)dt. \tag{1}$$

If its limit exist for a given value of z we say the integral $\int_0^R e^{-zt} f(t)dt$ converges (as $R \to \infty$) for that value of z. Using some well known (see [5]) theorems about the integral converging, if for some $z_0 = a_0 + ib_0$ we have

$$\limsup_{0\le x<\infty} |\int_0^x e^{-z_0 t} f(t)dt| = M < \infty \tag{2}$$

then (1.1) converges for every z for which $a > a_0$. For, this implies that the region of convergence of (1.1) is a half-plane. The divergence of (1.1) at a point z_0 implies its divergence at allpoints for which $a < a_0$. Hence three possibilities occure:

1. The integral converges for no point;
2. It converges for every point;
3. It converges for $a > a_c$ and diverges for $a < a_c$.

In case (3) we define the real number a_c as the *abscissa of convergence*, the lines $a = a_c$ the *axis of convergence*. In case (1) we write $a_c = +\infty$, in case (2), $a_c = -\infty$.

When the integral converges it defines a function os z which we denote by $F(z)$. This function is called the Laplace-Stieltjes transform, or simply the Laplace transform, of $f(t)$. If

$$F(z) = \int_0^\infty e^{-zt} f(t)dt \tag{3}$$

In either case $F(z)$ is called the *generating function* and $f(t)$ is referred to as the *determining function*.

1.2 Some properties of the Laplace Transform

We are not going to give all the properties of the Laplace Transform but only those which we will be helpfull for the understanding of the inversion and the determination of some Laplace transform ([5]).

First, it is easy to see that a Laplace Transform is an analytic function in its region of convergence.Moreover we have:

If the Laplace integral converges for $\infty > a > a_0$, then F(z) is analytic for $a > a_0$, and

$$F^{(k)}(z) = \int_0^\infty e^{-zt}(-t)^k f(t)dt \tag{4}$$

One other important property of the Laplace Transform is the uniqueness. More precisely, if $f(t)$ is normalised, its is uniquely determined by its Laplace Transform $F(z)$. For complete definitions and overview see [5] .

Now if we assume that$f(t)$ is a real valued function, we can see that

$$\begin{aligned} F(z) &= \int_0^\infty e^{-zt} f(t)dt \\ &= \int_0^\infty e^{-(a+ib)t} f(t)dt \ , \ (z = a + ib) \\ &= \int_0^\infty e^{-at} f(t)(\cos(bt) - i\sin(bt))dt \\ &= \int_0^\infty e^{-at} f(t)\cos(bt)dt - i\int_0^\infty e^{-at} f(t)\sin(bt)dt \\ & \quad \text{Which both exist as F(z) is defined} \end{aligned}$$

So we see that

$$Re(F(z)) = \int_0^\infty e^{-at} f(t)\cos(bt)dt$$

and

$$Im(F(z) = -\int_0^\infty e^{-at} f(t)\sin(bt)dt.$$

It is straightforward to see that, in the case of real valued determining function, $F(\overline{z}) = \overline{F(z)}$.

The last, and the key for the inversion, are the results from the familiar Dirichlet integral.

Theorem 1.1 *If $f(u)$ belongs to $L[-\infty, +\infty]$ and is of bounded variation in some two sided neighborhood of a point t, then*

$$\lim_{T\to\infty} \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{f(u)\sin(T(t-u))}{t-u} du = \frac{f(t+) + f(t-)}{2} \tag{5}$$

Theorem 1.2 *If $f(u)$ belongs to L in $[0, R]$ for every positive R and if the integral converges absolutely on the line $a = a_0$, then*

$$\lim_{T\to\infty} \frac{1}{2i\pi} \int_{c-iT}^{c+iT} F(z)e^{zt}dz = 0 \ (t < 0) \tag{6}$$

If in addition $f(u)$ is of bounded variation in some two sided neighborhood of a point $t \geq 0$, then

$$\begin{aligned} \lim_{T\to\infty} \tfrac{1}{2i\pi} \textstyle\int_{c-iT}^{c+iT} F(z)e^{zt}dz &= \tfrac{f(t+)+f(t-)}{2} \quad (t > 0) \\ &= \tfrac{f(0+)}{2} \quad (t = 0) \end{aligned}$$

For a complete proof of that one can consult [5] .

1.3 Numerical inversion of Laplace Transform of probability distibutions

The methods are convenient for calculating probability cumulative distribution function (cdf's) and other functions by numerically inverting Laplace Transforms. The methods work better when f is suitably smooth. But a different variant of the Fourier series method for generating functions should be used for cdf's of lattice distributions(see [6]). When f is otherwise not sufficiently smooth (continuous and differentiable), it may help to perfom convolution smoothing before doing the inversion.

1.3.1 The EULER method

The first method is called EULER because it employs Euler summation. It is based on the Bromwich contour inversion integral, which can be expressed as the integral of a real valued function of a real variable by choosing a specific contour. The contour is choosed to be any vertical line $z = a$ such that $F(z)$ has no singularities on or to the right of it, we then obtain

$$
\begin{aligned}
f(t) &= \tfrac{1}{2\pi i}\int_{a-\infty}^{a+\infty} e^{zt}F(z)dz \\
&= \tfrac{1}{2\pi}\int_{-\infty}^{+\infty} e^{(a+iu)t}F(a+iu)du \\
&= \tfrac{e^{at}}{2\pi}\int_{-\infty}^{+\infty}(\cos ut + i\sin ut)F(a+iu)du \\
&= \tfrac{e^{at}}{2\pi}\int_{-\infty}^{+\infty}[Re(F(a+iu))\cos ut - Im(F(a+iu))\sin ut]du \\
&\qquad \text{(as } f(t) \text{ is real)} \\
&= \tfrac{2e^{at}}{\pi}\int_{0}^{+\infty} Re(F(a+iu))\cos utdu
\end{aligned}
$$

The last line is true if we have

$$\int_{-\infty}^{+\infty} Re(F(a+iu))\cos utdu = -\int_{-\infty}^{+\infty} Im(F(a+iu))\sin utdu$$

That is

$$
\begin{aligned}
\textstyle\int_{-\infty}^{+\infty} Re(F(a+iu))\cos utdu + \int_{-\infty}^{+\infty} Im(F(a+iu))\sin utdu &= 0 \\
\tfrac{1}{e^{at}}Re(\textstyle\int_{-\infty}^{+\infty} F(a+iu)e^{(a+iu)t}du) &= 0 \\
\tfrac{1}{ie^{at}}Re(\textstyle\int_{a-\infty}^{a+\infty} F(z)e^{-zt}dz) &= 0 \\
\tfrac{1}{ie^{at}}Re(\textstyle\int_{a-\infty}^{a+\infty} F(z)e^{z(-t)}dz) &= 0
\end{aligned}
$$

The last is true ([5]) as $t > 0$ *and*

$$\int_{a-\infty}^{a+\infty} F(z)e^{z(t')}dz = 0 \text{ with } t' < 0$$

Then we have,

$$
\begin{aligned}
f(t) &= \tfrac{e^{at}}{\pi}\int_{-\infty}^{+\infty} Re(F(a+iu))\cos utdu \\
&= \tfrac{e^{at}}{\pi}\int_{0}^{+\infty} Re(F(a+iu))\cos utdu + \tfrac{e^{at}}{\pi}\int_{-infty}^{0} Re(F(a+iu))\cos utdu \\
&= \tfrac{2e^{at}}{\pi}\int_{0}^{+\infty} Re(F(a+iu))\cos utdu
\end{aligned}
$$

With an appropriate change of variable in the second integral and using the fact that cos is odd.

We the numerically evaluate the integral by means of the trapezoidal rule. Using a step size h, we get.

$$f(t) \approx f_h(t) \equiv \frac{he^{at}}{\pi} Re(F(a)) + \frac{2he^{at}}{\pi} \sum_{k=1}^{\infty} Re(F(a+ikh)) \cos hkt \tag{7}$$

Now letting $h = \frac{\pi}{2t}$ and $a = \frac{A}{2t}$, we obtain the series

$$f_h(t) = \frac{e^{\frac{A}{2}}}{2t} Re(F(\frac{A}{2t})) + \frac{e^{\frac{A}{2}}}{t} \sum_{k=1}^{\infty} (-1)^k Re(F(\frac{A+2ik\pi}{2t})). \tag{8}$$

We now use the Poisson fummation formula to identify the discretization error. The idea is to replace the damped function $g(t) = e^{-bt} f(t)$ for $b > 0$ by the *periodic function*:

$$g_p(t) = \sum_{k=-\infty}^{\infty} g(t + \frac{2k\pi}{h}) \tag{9}$$

of period $\frac{2\pi}{h}$. We then represent the periodic function $g_p(t)$ by its complex Fourier series

$$g_p(t) = \sum_{k=-\infty}^{\infty} c_k e^{ikht} \tag{10}$$

where c_k is the kth Fourier coefficient of $g_p(t)$,

$$\begin{aligned} c_k &= \tfrac{h}{2\pi} \textstyle\int_{-\frac{\pi}{h}}^{+\frac{\pi}{h}} g_p(t) e^{-ikht} dt \\ &= \tfrac{h}{2\pi} \textstyle\int_{-\frac{\pi}{h}}^{+\frac{\pi}{h}} \sum_{k=-\infty}^{\infty} g(t + \frac{2k\pi}{h}) e^{-kht} dt \\ &= \tfrac{h}{2\pi} \textstyle\int_{-\infty}^{\infty} g(t) e^{-ikht} dt \\ &= \tfrac{h}{2\pi} \textstyle\int_{0}^{\infty} e^{-bt} f(t) e^{-ikht} dt \\ &= \tfrac{h}{2\pi} F(b+ikh) \end{aligned}$$

Combining the value of c_k and (1.9) we obtain a version of the Poisson summation formula,

$$\begin{aligned} g_p(t) &= \textstyle\sum_{k=-\infty}^{\infty} g(t + \frac{2k\pi}{h}) \\ &= \textstyle\sum_{k=-\infty}^{\infty} f(t + \frac{2k\pi}{h}) e^{-b(t+\frac{2k\pi}{h})} \\ &= \tfrac{h}{2\pi} \textstyle\sum_{k=-\infty}^{\infty} F(b+ikh) e^{ikht} \end{aligned}$$

Letting $h = \frac{\pi}{t}$ and $b = \frac{A}{2t}$ we get,

$$f(t) = \frac{e^{\frac{A}{2}}}{2t} \sum_{k=-\infty}^{\infty} (-1)^k Re(F(\frac{A + 2ik\pi}{2t})) - \sum_{k=1}^{\infty} e^{-kA} f((2k+1)t) \tag{11}$$

As the first term on the right of (1.11) coincides with the trapezoidal-rule approximation in (1.8) , so that the second term is the discretization error associated with the trapezoidal rule, i.e,

$$e_d = \sum_{k=1}^{\infty} e^{-kA} f((2k+1)t) \tag{12}$$

With probability applications, $|f(t)| \leq 1$ for all t, then the error is bounded by

$$|e_d| \leq \frac{e^{-A}}{1 - e^{-A}} \tag{13}$$

, which is nearly equal to e^{-A}. Hence, to have at most $10^{-\beta}$ discretisation error, we let $A = \beta \ln(10)$.

Its clear that (1.13) can be used to get discretisation error bounds under other assumptions about f.

1.3.2 The POST-WIDDER method

This method is based on the Post-Widder Theorem, which express $f(t)$ as the pointwise limit, as $n \to \infty$ of

$$f_n(t) = \frac{(-1)^n}{n!} (\frac{n+1}{t})^{n+1} F^{(n)}(\frac{n+1}{t}),$$

where $F^{(n)}(z)$ is the nth derivative of the Laplace Transform at z. Feller shows that the Post-Widder formula is easy to understand probabilistically. By differentiating the transform, it is easy to see that $f_n(t) = E[f(X_{n,t})]$, where $X_{n,t}$ is a random variable with gamma distribution on $[0, \infty[$ with mean t and variance $\frac{t}{n+1}$. Then $X_{n,t}$ converges in probability as $n \to \infty$ to the random variable X_t with $P(X_t = t) = 1$, and $f_n(t) \to f(t)$ as $n \to \infty$ for all bounded real-valued f that are continuous at t, and for other f as well.

$$G(z) \equiv \sum_{n=0}^{\infty} a_n(t) z^n = \frac{n+1}{t} F(\frac{n+1}{t}(1 - z)) \tag{14}$$

, whose nth coefficient is $f_n(t)$, ie, $a_n(t) = f_n(t)$. Using the Cauchy contour integral, we have

$$f_n(t) = \frac{1}{2i\pi} \int_{C_r} \frac{G(z)}{z^{n+1}} dz \tag{15}$$

, where C_r is a circle of radius r. From this, if the analytical expression of F is knonw, when can get an analytical expression of $f_n(t)$ by using the residus theorem and go throw the numericals. If it is not, then (when f is a real function),

Doing the change of variables $z = re^{iu}$, we get

$$\begin{aligned} f_n(t) &= \tfrac{1}{2\pi r^n} \textstyle\int_0^{2\pi} G(re^{iu}) e^{-inu} \\ &= \tfrac{n+1}{t} \tfrac{1}{2\pi r^n} \textstyle\int_0^{2\pi} F(\tfrac{n+1}{t}(1 - re^{iu})) e^{-inu} du \end{aligned}$$

then applying Poisson summation formula again,

$$\begin{aligned} f_n(t) &= \tfrac{n+1}{2tnr^n} \textstyle\sum_{k=1}^{2n} (-1)^k Re(F(\tfrac{n+1}{t}(1 - re^{\frac{1\pi k}{n}}))) - e_d \\ &= \tfrac{n+1}{2tnr^n} * [F(\tfrac{(n+1)(1-r)}{t}) + (-1)^n F(\tfrac{(n+1)(1+r)}{t}) + 2 \textstyle\sum_{k=1}^{n-1} (-1)^k Re(F(\tfrac{n+1}{t}(1 - re^{\frac{1\pi k}{n}})))] - e_d \end{aligned}$$

where

$$e_d = \sum_{j=1}^{\infty} f_{n+jm}(t + \frac{2tjm}{n+1}) r^{2jn}$$

Assuming that $|f(t)| \leq 1$ for all t , we have $|f_n(t)| \leq 1$ for all n and t , so that,

$$|e_d| \leq \frac{r^{2n}}{1 - r^{2n}} \cong r^{2n} \tag{16}$$

In order to enhance the accuracy, we use a linear combination of the terms, ie,

$$f_{j,m}(t) = \sum_{k=1}^{m} w(k,m) f_{jk}(t),$$

and

$$w(k,m) = (-1)^{m-k} \frac{k^m}{k!(m-k)!},$$

The proof are similar to the Euler method, see [5] .

2 Pricing Exotic Options

We take as given a complete probability space $(\Omega, \mathcal{F}, P)$ with a filtration $(\mathcal{F}_t)_{0\leq t\leq\infty}$, which is rigth-continuous and such that $\mathcal{F}_0$ contains all the P - null sets of $\mathcal{F}$. We assume the existence of a "risk-neutral" probability measure $\mathcal{Q}$ (equivalent to P) under which the underlying stock price dynamics are driven by

$$dS(t) = yS(t)dt + \sigma S(t)dW(t) \tag{17}$$

2.1 Continuous time Asian Options_1

Suppose we have call option maturing at time T, on a random variable A, determined at time T, that pays at T, the excess of A over a prespecified strike of K. In the case of the continuous time asian option A is the integral of the stock price from time 0 to time T divided by T (the mean until time T). We suppose that risk free investment at the constant interest rate of r per unit time is available over any time subinterval and let the rik neutral measure Q, be the one associated with discounting by the money market accumulation factor e^{rt}.

2.1.1 The Laplace Transform and the strike Transform of the call price

Let the risk neutral density of A at t be $f_{t,T}(a)$, with Laplace Transfom:

$$\psi_{t,T}(\lambda) = \int_0^\infty e^{-\lambda a} f_{t,T}(a)da \tag{18}$$

Further, let $c_{t,T}(K)$ be the price at t of a call option on A maturing at T, with strike K. It follows from martingale theory pricing principles

$$c_{t,T}(K) = e^{-r(T-t)} \int_K^\infty (a-K) f_{t,T}(a)da \tag{19}$$

Consider the Laplace Transform in K of the call option price defined as

$$\phi_{t,T}(\lambda) = \int_0^\infty e^{-\lambda K} c_{t,T}(K)dK \tag{20}$$

Then the pricing stategy is based on the relationship between $\psi_{t,T}(\lambda)$ and $\phi_{t,T}(\lambda)$. We then develop an expression for $\psi_{t,T}(\lambda)$ and obtain option prices by inverting the implied transform $\phi_{t,T}(\lambda)$.

Theorem 2.1 *The Laplace Transform of the call option price in the strike price $\phi_{t,T}(\lambda)$ is related to the Laplace Transform of the risk neutral density $\psi_{t,T}(\lambda)$ by,*

$$\phi_{t,T}(\lambda) = e^{-r(T-t)} \frac{E_{t,T}[A]\lambda + \psi_{t,T}(\lambda) - 1}{\lambda^2} \tag{21}$$

where $E_{t,T}[A]$ is the mean of the density $f_{t,T}(a)$

The proof is easy to perform by a simple rewritting of (2.4), one can see also [4]

2.1.2 The Laplace Transform of integral of geometric Brownian motion

The objective of this section is to develop analytical expression for $\psi_{t,T}(\lambda)$.

Let $S(t)$ be the stock price at time t and $W(t)$ be standard brownian motion. We suppose that $S(t)$ satisfies the stochastic differential equation

$$dS(t) = rS(t)dt + \sigma S(t)dW(t) \tag{22}$$

with solution given by

$$S(t) = S(0)e^{rt+\sigma W(t)-\frac{\sigma^2 t}{2}} \tag{23}$$

Let the mean of the stock price at time t be $a(t) = \frac{\int_0^t S(u)du}{t}$. Then the payoff of the call option at maturity T is given by,(we call it $w(t)$ at time t)

$$w(T) = (a(T) - K)^+ \tag{24}$$

The Laplace Transform of $a(T)$ is,

$$\psi_{t,T}(\lambda) = E_t^Q[e^{-\lambda a(T)}] \tag{25}$$

Then with $A(T) = \int_0^T S(u)du$

$$\begin{aligned}
\psi_{t,T}(\lambda) &= E_t^Q[e^{-\lambda a(T)}] \\
&= E_t^Q[e^{-\frac{\lambda}{T}A(T)}] \\
&= E_t^Q[e^{-\frac{\lambda}{T}A(t)-\frac{\lambda}{T}(A(T)-A(t))}] \\
&= e^{-\frac{\lambda t}{T}a(t)} E_t^Q[e^{-\lambda \int_t^T S(u)du}]
\end{aligned}$$

Define the Laplace transform of the remaining uncertainty by

$$\Phi(t,\lambda,T) = E_t^Q[e^{-\lambda\int_t^T S(u)du}] \tag{26}$$

we may write

$$\psi_{t,T}(\lambda) = e^{-\frac{\lambda t}{T}a(t)}\Phi(t,\frac{\lambda}{T},T) \tag{27}$$

It folows that

$$\Phi(t,\lambda,T) = \Psi(t,S,T,\lambda) \tag{28}$$

where $S(t) = S$ and Ψ satisfies the partial differential equation

$$\Psi_t + rS\Psi_S + \frac{1}{2}\sigma^2 S^2 \Psi_{SS} = \lambda S\Psi \tag{29}$$

subject to the boundary condition

$$\Psi(t,S,t,\lambda) = 1.$$

Then the final theorem.

Theorem 2.2 *The solution to the partial differential equation subject to the boundary condition is given by*

$$\Psi(t,S,T,\lambda) = U(\ln(S) - (r - \frac{\sigma^2}{2})(T-t), T-t, \lambda) \tag{30}$$

Where the transform of $U(x,\tau,\lambda)$ in τ is given by

$$\int_0^\infty e^{-\nu\tau}U(x,\tau,\lambda)d\tau = \frac{1}{\nu}F_{1,2}[1; 1+\frac{\sqrt{2\nu}}{\sigma}; 1-\frac{\sqrt{2\nu}}{\sigma}, \frac{2\lambda}{\sigma^2}e^x] \tag{31}$$

and $F_{1,2}$ is the generalised hypergeometric function.

The proof is made by resolving the partial differential equation (2.13) after the change of variable (2.14). For that, see [4].

2.2 Continuous time Asian Options_2

Here again, for simplicity, we assume that r and σ are both constant over $[t,T]$. Considering the process $(A(x), x \leq 0)$, with

$$A(x) = \frac{1}{x-t}\int_t^x S(u)du \tag{32}$$

the payoff of the asian option at maturity is

$$\max[(A(T)-k),0]=(A(T)-k)^{+}$$

where k is the fixed-strike price of the option. By arbitrage arguments and because the interest rate and the dividend are constant over the lifetime of the option, the value at time t of the Asian call option is

$$C_{t,T}(k)=e^{-r(T-t)}E_{\mathcal{Q}}[(A(T)-k)^{+}|\mathcal{F}_t]$$

Then with simplifying steps,

$$C_{t,T}(k)=\frac{e^{-r(T-t)}}{T-t}(\frac{4S(t)}{\sigma^2})C^{\nu}(h,q) \tag{33}$$

where

$$\nu=\frac{2y}{\sigma^2}-1\ ,\ h=\frac{\sigma^2}{4}(T-t)\ ,\ q=\frac{\sigma^2}{4S(t)}\{k(T-t)\}$$

Then from the use of bessel process we have:

- When $q \leq 0$. The calculation is easy ([2]) and gives

$$C^{\nu}(h,q)=\{\frac{1}{2(\nu+1)}[\exp(2(\nu+1)h)-1]\}-q$$

  from this we have a closed formula for the Asian call option price,

$$C_{t,T}(k)=S(t)(\frac{1-e^{-r(T-t)}}{r(T-t)})-e^{-r(T-t)}k \tag{34}$$

- For $q>0$ we do not have such a simple reduction of $C^{\nu}(h,q)$, but providing an expression of its Laplace transform with respect to the variable h, we obtain a “closed form” in terms of inverting a Laplace transform.

  The Laplace transform of $C^{\nu}(h,q)$ with respect to h can be written as (see [2])

$$\Theta_q(\lambda)=\int_0^{\infty}e^{-\lambda h}C^{\nu}(h,q)dh=\frac{\int_0^{\frac{1}{2q}}e^{-x}x^{\frac{\mu-\nu}{2}-2}(1-2qx)^{\frac{\mu+\nu}{2}+1}dx}{\lambda(\lambda-2-2\nu)\Gamma(\frac{\mu-\nu}{2}-1)} \tag{35}$$

then we have,

$$C_{t,T}(k) = \frac{e^{-r(T-t)}}{T-t}(\frac{4S(t)}{\sigma^2})\mathcal{L}^{-1}(\Theta_q(\lambda))(h) \tag{36}$$

2.3 Pricing double-barrier options

A double knock option is characterized by two barriers, L (lower barrier) and U (upper barrier);the option knocks out if either barrier is touched. Otherwise, the option gives at maturity T the standard Black and Scholes ([1]) payoff $\max(0, S(T) - k)$, where k, the strike price of the option, satisfies $L < k < U$.

In our setting the uncertainty in the economy is represented by a filtered probability space $(\Omega, \Im, \Im_t, P)$, where $\Im_t$ is the information available at time t and P is the objective probability. From the no-arbitrage assumption, we have a probability measure Q equivalent to P such that the discounted prices of the basic securities are Q-martingales. Under Q, the dynamics of the price $S(t)$ of the underlying asset are driven by the stochastic differential equation

$$\frac{dS_t}{S_t} = ydt + \sigma d\tilde{W}_t \tag{37}$$

where $\tilde{W}(t)$ is a Q-Brownian motion, σ is constant, and the risk-neutral drift is assumed to be a general constant. So we can incorporate the case wherethe underlying instrument is a dividend-paying stock, currency, or a commdity. Equation (2.21) gives,

$$S(t) = S(0)e^{(y-\frac{\sigma^2}{2})t+\sigma\tilde{W}(t)}. \tag{38}$$

The call price is then

$$C_{L,U}(t) = e^{-r(T-t)}E_Q[(S(T)-k)^+\mathbf{1}_{(\sum_{L,U}>T)}/\Im_t], \tag{39}$$

where $\sum_{L,U}$ is the first exit time of the process $(S_t)_{t\geq 0}$ out of the interval $[L, U]$.

Then we can write, if the double-barriercall has not been knocked out prior to time t, its price attime t is equal to

$$C_{L,U}(t) = e^{-r(T-t)}S(t)E_Q[(S_1(\tau)-h)^+\mathbf{1}_{(\hat{\sum}_{m,M}>\tau)}], \tag{40}$$

where $\tau = T - t$, $h = \frac{k}{S(t)}$, $m = \frac{L}{S(t)}$, $M = \frac{U}{S(t)}$, and $\hat{\sum}_{\alpha,\beta}$ is the first exit time of the process $(S_1(s))_{s\geq 0}$ out of the interval $[\alpha, \beta]$ where $S_1(s) = \exp\{(y - \frac{\sigma^2}{2})s + \sigma \hat{W}_s\}$, $(\hat{W}(s) = \tilde{W}(t+s) - \tilde{W}(t), s \geq 0)$ is a new Brownian motion under Q and finally for all $u \geq t \geq 0$,

$$S(u) = S(t) S_1(u - t). \tag{41}$$

¿From this and after some long calculuous (see [3]) we have

$$C_{L,U}(t) = S(t)\{BS(0, 1, \sigma, \tau, h) - e^{-r\tau} \mathcal{L}^{-1}\psi\}(\tau)\} \tag{42}$$

With the following notations,

- $BS(0, 1, \sigma, \tau, h)$ is the Black and Scholes price of a standard call with maturity $\tau = T - t$, strike price $h = \frac{k}{S(t)}$, assumed to be written on an underlying asset S such that $S(0) = 1$.

- We use the formula

$$\psi(\lambda) = \frac{1}{\sigma^2}\Phi(\frac{\lambda}{\sigma^2}) \tag{43}$$

  and denote $\mathcal{L}^{-1}$ to the inverse of the Laplace transform operator. So we find

$$\{\mathcal{L}^{-1}\psi\}(\tau) = \{\mathcal{L}^{-1}\Phi\}(\sigma^2\tau). \tag{44}$$

- With

$$\Phi(\theta) = \frac{\sinh(\mu b)}{\sinh[\mu(a+b)]} g_1(e^{-a}) + \frac{\sinh(\mu a)}{\sinh[\mu(a+b)]} g_1(e^b) \tag{45}$$

  and $\mu = \sqrt{2\theta + \nu^2}$, $m = e^{-a}$, $M = e^b$, $\nu = (\frac{1}{\sigma^2})(y - \frac{\sigma^2}{2})$.

- Finally

$$g_1(e^{-a}) = \frac{h^{\nu+1-\mu} e^{-\mu a}}{\mu(\mu - \nu)(\mu - \nu - 1)} \tag{46}$$

  and

$$g_1(e^b) = 2\{\frac{e^{b(\nu+1)}}{\mu^2 - (\nu+1)^2} - \frac{h e^{b\nu}}{\mu^2 - \nu^2}\} + \frac{e^{-\mu b} h^{\nu+1+\mu}}{\mu(\mu+\nu)(\mu+\nu+1)} \tag{47}$$

2.4 Computation

2.4.1 Asians options method_1

We did not use this method for pricing the asian option as there is a double inversion and most because the inversion is made from a complex function to another complex function. With [6] , using the Euler method, we only compute the inversion from a complex function to a real function.

2.4.2 Asians options method_2

$C_{t,T}(k) = \frac{\exp(-r(T-t))}{T-t}\left(\frac{4S(t)}{\sigma^2}\right)\pounds^{-1}\left(\Theta_q(\lambda)\right)(h)$
and

$$\begin{aligned}
\Theta_q(\lambda) &= \frac{\int_0^{\frac{1}{2q}}\exp(-x)x^{\frac{\mu-\nu}{2}-2}(1-2qx)^{\frac{\mu+\nu}{2}+1}dx}{\lambda(\lambda-2-2\nu)\Gamma\left(\frac{\mu-\nu}{2}-1\right)} \\
&= \frac{1}{\lambda(\lambda-2-2\nu)\Gamma\left(\frac{\mu-\nu}{2}-1\right)(2q)^{\frac{\mu-\nu}{2}-1}}\int_0^1\exp\left(-\frac{u}{2q}\right)\left(\frac{u}{2q}\right)^{\frac{\mu-\nu}{2}-2}(1-u)^{\frac{\mu+\nu}{2}+1}du \\
&= \frac{1}{(2q)^{\frac{\mu-\nu}{2}-1}\lambda(\lambda-2-2\nu)\Gamma\left(\frac{\mu-\nu}{2}-1\right)}\sum_{p=0}^{\infty}\frac{(-1)^p}{p!}\frac{1}{(2q)^p}\int_0^1 u^{\frac{\mu-\nu}{2}-2+p}(1-u)^{\frac{\mu+\nu}{2}+1}du \\
&= \frac{1}{(2q)^{\frac{\mu-\nu}{2}-1}\lambda(\lambda-2-2\nu)\Gamma\left(\frac{\mu-\nu}{2}-1\right)}\sum_{p=0}^{\infty}\frac{(-1)^p}{p!}\frac{1}{(2q)^p}\beta\left(\frac{\mu+\nu}{2}+2,\frac{\mu-\nu}{2}-1+p\right)
\end{aligned}$$

The convergence of this sum is simple to proove as the integral is finite.
Then we have,

$\beta\left(\frac{\mu+\nu}{2}+2,\frac{\mu-\nu}{2}-1+p\right) = \frac{\Gamma\left(\frac{\mu+\nu}{2}+2\right)\Gamma\left(\frac{\mu-\nu}{2}-1+p\right)}{\Gamma(\mu+p+1)}$
so

$\Theta_q(\lambda) = \frac{1}{(2q)^{\frac{\mu-\nu}{2}-1}\lambda(\lambda-2-2\nu)\Gamma\left(\frac{\mu-\nu}{2}-1\right)}\sum_{p=0}^{\infty}\frac{(-1)^p}{p!}\frac{1}{(2q)^p}\frac{\Gamma\left(\frac{\mu+\nu}{2}+2\right)\Gamma\left(\frac{\mu-\nu}{2}-1+p\right)}{\Gamma(\mu+p+1)}$

Computation

We deal with the inversion of the laplace transform $\Theta_q(\lambda)$. Because the values given by the computation are finites, we need some boundness for the variable q and we also need a finite estimation for the infinite sum.

Recalling that the termes μ, λ are connected by $\mu = \sqrt{2\lambda+\nu^2}$, $\nu = \frac{2r}{\sigma^2}-1$, and the crutial variable q (for the computation) is connected with the volatility σ by $q = \frac{\sigma^2}{4S(t)} \{k(T-t)\}$(in our simple case).

For $t = 0$, it is always possible to get into this case), $q = \frac{\sigma^2}{4S(0)} kT \simeq \frac{T\sigma^2}{4}$ (when $k \simeq S(0)$).

We see that the small values of σ or T will be the cause of computation mistake (as $\frac{1}{(2q)^p} = NaN$, because it goes throw the computeur capacity).

Recall that for the computation that we compute

$s_{n+j} = \frac{e^{\left(\frac{A}{2}\right)}}{2T} Re(F)\left(\frac{A}{2T}\right) + \frac{e^{\left(\frac{A}{2}\right)}}{T} \sum_{m=1}^{n+j} (-1)^m a_m(T) \quad (\tau = T),\ \ a_m(T) = Re(F)\left(\frac{A}{2T} + i\frac{\pi m}{T}\right)$

with $A = 19.1, \quad n = 15, \quad j = 0..11, \quad m = 1..26, \quad \lambda = \frac{A}{2T} + i\frac{\pi m}{T},$ $\nu = \frac{2r}{\sigma^2} - 1$

then $\mu = \sqrt{2\lambda + \nu^2} = \sqrt{\frac{A}{T} + 2i\frac{\pi m}{T} + \nu^2}$

for example, if we take $m = 26,\ T = 0.05, \quad r = 0.05, \quad$ and $\sigma = 0.1$ $(with S(0) = k)$

- we find : $q = \frac{T\sigma^2}{4} = 0.000125 \Rightarrow\ 2q = 0.00025 \Rightarrow\ \frac{1}{2q} = 4.10^3$
- $\mu = \sqrt{\frac{A}{T} + 2i\frac{\pi m}{T} + \nu^2} \simeq \sqrt{382 + 3267.26i + 81} \simeq \sqrt{463 + 3267.26i} \simeq 57.45\sqrt{\cos(\varphi) + i\sin(\varphi)}$ φ such that $\tan(\varphi) \simeq 7.06 \Rightarrow \tan\left(\frac{\varphi}{2}\right) \simeq 0.93$
- $\frac{\mu-\nu}{2} - 1 \simeq 15.53 + 19.56i$ then $\left(\frac{1}{2q}\right)^{\frac{\mu-\nu}{2}-1} = (4.10^3)^{\frac{\mu-\nu}{2}-1}$
- If $m = 15$ (minimum value) $\frac{1}{m!(2q)^m} = \frac{4^{15}.10^{45}}{15!} \simeq 8.2\ 10^{41}$

from the we can see that the maximum value of the "double" in C is 10^{37}. The lower bound for q is absolutly necessary.

To end these notes we must deal with the Γ values. In fact they are not so difficult to compute as $(p = 1..\infty)$:

$\frac{\Gamma\left(\frac{\mu-\nu}{2}-1+p\right)}{\Gamma\left(\frac{\mu-\nu}{2}-1\right)\Gamma(\mu+p+1)} = \frac{\Gamma\left(\frac{\mu-\nu}{2}+p-2\right)}{\Gamma\left(\frac{\mu-\nu}{2}-1\right)\Gamma(\mu+p)} \frac{\frac{\mu-\nu}{2}+p-2}{\mu+p}$

If we define the next suite $(u_p, p = 0, ...\infty)$

$u_0 = 1; u_{p+1} = u_p \frac{1}{p+1} \frac{\frac{\mu-\nu}{2}+p-1}{\mu+p+1} \Rightarrow \Theta_q(\lambda) = \frac{\Gamma\left(\frac{\mu+\nu}{2}+2\right)}{(2q)^{\frac{\mu-\nu}{2}-1}\lambda(\lambda-2-2\nu)\Gamma(\mu+1)} \sum_{p=0}^{\infty} \left(\frac{-1}{2q}\right)^p u_p$

we can get more if we put $v_p = \left(\frac{-1}{2q}\right)^p u_p$ then with $v_{p+1} = \left(\frac{-1}{2q}\right) \frac{1}{p+1} \frac{\frac{\mu-\nu}{2}+p-1}{\mu+p+1} v_p$, $v_0 = 1$

$$\boxed{\Theta_q(\lambda) = \frac{\Gamma\left(\frac{\mu+\nu}{2}+2\right)}{(2q)^{\frac{\mu-\nu}{2}-1}\lambda(\lambda-2-2\nu)\Gamma(\mu+1)} \sum_{p=0}^{\infty} v_p}$$

the main problem is the computation of $\frac{1}{(2q)^{\frac{\mu-\nu}{2}-1}}$. To solve this, we fix the lower bound of $2q$ to 10^{-2}

if $2q$ is less than this value, we'll use a linear interpolation.

denoting $\lambda = \lambda.r + i\lambda.i$ we have $Re\left(\mu\right) = \sqrt{\frac{\lambda.i^2+(\lambda.r+\nu^2)^2+(\lambda.r+\nu^2)}{2}}$

then

$Re\left(\frac{\mu-\nu}{2}-1\right) = \sqrt{\frac{\lambda.i^2+(\lambda.r+\nu^2)^2+(\lambda.r+\nu^2)}{8}} - \frac{\nu}{2} - 1 = \sqrt{\frac{\left(2\frac{\pi m}{T}\right)^2+\left(\frac{A}{T}+\left(\frac{2r}{\sigma^2}-1\right)^2\right)^2+\left(\frac{A}{T}+\left(\frac{2r}{\sigma^2}-1\right)^2\right)}{8}} - \frac{\frac{2r}{\sigma^2}-1}{2} - 1$

We decide not to go throw 10^{15} then we need $\frac{1}{(2q)^{Re\left(\frac{\mu-\nu}{2}-1\right)}} \leq 10^{15} \Leftrightarrow -Re\left(\frac{\mu-\nu}{2}-1\right)\ln\left(2q\right) \leq 15\ln\left(10\right)$

$\Leftrightarrow Re\left(\frac{\mu-\nu}{2}-1\right) \leq 7.5$

so we have to verify the next inegality, with the known value of

$$\boxed{\sqrt{\frac{\left(2\frac{\pi m}{T}\right)^2+\left(\frac{A}{T}+\left(\frac{2r}{\sigma^2}-1\right)^2\right)^2+\left(\frac{A}{T}+\left(\frac{2r}{\sigma^2}-1\right)^2\right)}{8}} - \frac{\frac{2r}{\sigma^2}-1}{2} \leq 8.5 \quad \forall\ m = 0..26}$$

to end this computation we must approxim the infinite sum ; $\sum_{p=0}^{\infty} v_p\left(2q\right)$ as it is increasing with $2q$, we must get a good approximation for the maximum of the value of $2q$.

Recalling that $v_{p+1}\left(2q\right) = \left(\frac{-1}{2q}\right)\frac{1}{p+1}\frac{\frac{\mu-\nu}{2}+p-1}{\mu+p+1}v_p\left(2q\right)$, we find the minimum number of our sum to have ;

$\left|\frac{v_{p+1}}{v_p}\right| \leq 10^{-3} \Leftarrow \left(\frac{1}{2q}\right)\frac{1}{p+1} \leq 10^{-3} \Leftarrow p \geq 10^5$

To end we are going to compute the value

$\Theta_q\left(\lambda\right) = \frac{\Gamma\left(\frac{\mu+\nu}{2}+2\right)}{(2q)^{\frac{\mu-\nu}{2}-1}\lambda(\lambda-2-2\nu)\Gamma(\mu+1)}\sum_{p=0}^{10^5} v_p$ with the condition on the exposant of $\left(\frac{1}{2q}\right)$

One other method, which is that we use, is to compute directly the integral, as a Riemman sum;

$$\Theta_q\left(\lambda\right) = \frac{\int_0^{\frac{1}{2q}} \exp\left(-x\right) x^{\frac{\mu-\nu}{2}-2}\left(1-2qx\right)^{\frac{\mu+\nu}{2}+1} dx}{\lambda\left(\lambda-2-2\nu\right)\Gamma\left(\frac{\mu-\nu}{2}-1\right)}$$

From the previous calculuous we know that this computation problems will occur. According to that, we decide to go not throw a certain value of q

depending for most of the values σ , $(T-t)$ and the quantity $\frac{K}{S(t)}$ (not only because in the computation, the parameter r is used (by $\nu = \frac{r}{\sigma^2} - 0.5$) , then for some of their values the maximun value of q can be lower or higher)

The limit value of q is 0.0009 when the ration $\frac{K}{S(t)}$ is equal to 1 .

then to get a good computation, the user should verifie this inegality ;

$$\boxed{q = \frac{T\sigma^2 k}{4S(t)} \geq 0.0009}$$

2.4.3 Double_barrier options

The price of the Double_barrier call option is given by

$C_{L,U}(t) = S(t)\left\{BS(0,1,\sigma,\tau,h) - e^{-r\tau}\pounds^{-1}\Psi(\tau)\right\}$

with the following notations ;

$BS(0,1,\sigma,\tau,h)$ is the Black and Scholes price of a standard call with maturity τ , strike price $h = \frac{k}{S(t)}$, assumed to be written on an underlying asset S such that $S(0) = 1$.

We use $\Psi(\lambda) = \frac{1}{\sigma^2}\Phi\left(\frac{\lambda}{\sigma^2}\right)$ with

$\Phi(\theta) = \frac{\sinh(\mu b)}{\sinh(\mu(a+b))} g_1\left(e^{-a}\right) + \frac{\sinh(\mu a)}{\sinh(\mu(a+b))} g_1\left(e^{b}\right)$

$\mu = \sqrt{2\theta + \nu^2}$, $\frac{L}{S(t)} = m = e^{-a}$, $\frac{U}{S(t)} = M = e^{b}$, $\nu = \left(\frac{1}{\sigma^2}\right)\left(y - \frac{\sigma^2}{2}\right)$, $y = r - \delta$

$g_1\left(e^{-a}\right) = \frac{h^{\nu+1-\mu}e^{-\mu a}}{\mu(\mu-\nu)(\mu-\nu-1)}$

$g_1\left(e^{b}\right) = 2\left\{\frac{e^{b(\nu+1)}}{\mu^2-(\nu+1)^2} - \frac{he^{b\nu}}{\mu^2-\nu^2}\right\} + \frac{h^{\nu+1+\mu}e^{-\mu b}}{\mu(\mu+\nu)(\mu+\nu+1)}$

We here deal with the Laplace transform of $\Psi(\lambda) = \frac{1}{\sigma^2}\Phi\left(\frac{\lambda}{\sigma^2}\right)$. From the computation method it can be seen that the definition of all the parameter give no problems unless σ and τ throw ν and μ as $\nu = \left(\frac{1}{\sigma^2}\right)\left(y - \frac{\sigma^2}{2}\right) = \frac{r-\delta}{\sigma^2} - 0.5$ and $\mu = \sqrt{2\theta + \nu^2}$.

Let then viewing all the terms of $\frac{1}{\sigma^2}\Phi\left(\frac{\lambda}{\sigma^2}\right) = \frac{1}{\sigma^2}\left\{\frac{\sinh(\mu b)}{\sinh(\mu(a+b))} g_1\left(e^{-a}\right) + \frac{\sinh(\mu a)}{\sinh(\mu(a+b))} g_1\left(e^{b}\right)\right\}$ with $\mu = \frac{1}{\sigma}\sqrt{2\lambda + \frac{1}{\sigma^2}\left(r - \delta - \frac{\sigma^2}{2}\right)^2}$

$$\frac{1}{\sigma^2}\Phi\left(\frac{\lambda}{\sigma^2}\right) =$$

$$\frac{1}{\sigma^2}\left\{\frac{m^{\mu}\left(M^{2\mu}-1\right)}{M^{2\mu}-m^{2\mu}}g_1\left(m\right)+\frac{M^{\mu}\left(1-m^{2\mu}\right)}{M^{2\mu}-m^{2\mu}}g_1\left(M\right)\right\} = \frac{1}{\sigma^2}\left\{\frac{m^{\mu}\left(1-M^{-2\mu}\right)}{1-\left(\frac{m}{M}\right)^{2\mu}}g_1\left(m\right)+\frac{M^{-\mu}\left(1-m^{2\mu}\right)}{1-\left(\frac{m}{M}\right)^{2\mu}}g_1\left(M\right)\right\}$$

where

$$m^{\mu}g_1\left(m\right) = \frac{m^{\mu}h^{\nu+1}}{\mu(\mu-\nu)(\mu-\nu-1)}\left(\frac{m}{h}\right)^{\mu}$$

$$M^{-\mu}g_1\left(M\right) = 2\left\{\frac{M^{\nu+1-\mu}}{\mu^2-(\nu+1)^2}-\frac{hM^{\nu-\mu}}{\mu^2-\nu^2}\right\}+\frac{M^{-\mu}h^{\nu+1}}{\mu(\mu+\nu)(\mu+\nu+1)}\left(\frac{h}{M}\right)^{\mu}$$

$m^{\mu}g_1\left(m\right)$ and $M^{-\mu}g_1\left(M\right)$ bounded for reasonable values of σ ($\sigma \geq 0.001$) (not forget the definitions of the parameters).

So are the values $\frac{\left(1-M^{-2\mu}\right)}{1-\left(\frac{m}{M}\right)^{2\mu}}$ and $\frac{\left(1-m^{2\mu}\right)}{1-\left(\frac{m}{M}\right)^{2\mu}}$ (go to 1 as μ takes infinite values).

But if we compute the numbers separatly, without any attention, some of them may be "infinite" (regarding from the computer, as $h^{\nu+1}$ when σ goes to 0 and $h \succ 1$), so a deep control of the values calculated must occur in order to avoid going throw the computer capacity (in the example we compute $M^{-\mu}h^{\nu+1}$ instead of $h^{\nu+1}$ and $M^{-\mu}$ separately).

References

[1] F.BLACK M.SCHOLES. The pricing of Options and Corporate Liabilities. *Journal of Political Economy*, 81:635–654, 1973. 13

[2] H.GEMAN M.YOR. Besssel processes, asian options, and perpetuities. *Mathematical finance*, 3:349–375, 1993. 12

[3] H.GEMAN M.YOR. Pricing and hedging double barrier options: a probabilistic approach. *Mathematical finance*, 6:365–378, 1996. 14

[4] FU MADAN WANG. Pricing continuous time asian options: A comparaison of analytical and monte carlo methods. *College of Business and Management*, 1996. 10, 11

[5] DAVID VERNON WIDDER. *The Laplace Transform.* Princeton University press, 1941. 2, 3, 4, 5, 8

[6] J.ABATE W.WHITT. Numerical inversion of laplace transform of probability distribution. *ORSA Journal of Computing*, 7(1), Winter 1995. 4, 15