1 Generalized Black–Scholes model

In market finance, a European *call* (respectively *put*) option with maturity date T and exercise price K, on an underlying asset S, denotes a right to buy (respectively sell), at price K, a unit of S at time T. Let us then consider a theoretical financial market, with two traded assets: cash, with constant interest-rate r, and a risky stock, with diffusion price process

$$dS_t = S_t(\rho(t,S_t)dt + \sigma(t,S_t)dW_t)\ , \quad t > t_0\ ; \quad S_{t_0} = S_0\ .$$

Here W means a standard Brownian motion. Moreover, the stock is assumed to yield a continuously compounded dividend at constant rate q. Suppose finally, that the market is liquid, non arbitrable, and perfect. These assumptions mean, respectively, that first, there are always buyers and sellers, second, there can be no opportunity that a riskless investment can earn more than the interest-rate of the economy r, and third, there are no restrictions of any kind on the sales, neither transaction costs. Under these assumptions the market is complete. This means that any option can be duplicated by a portfolio of cash and stock. Moreover, a European call/put on S has a theoretical fair price within the model, that we will denote by $\Pi^{+/-}_{T,K}(t_0,S_0;r,q,a)$, where $a \equiv \sigma^2/2$, and

$$\Pi^{+/-}_{T,K}(t_0,S_0;r,q,a) = e^{-r(T-t_0)}\mathrm{E}_P^{t_0,S_0}(S_T - K)^{+/-}\ . \tag{1}$$

Here P denotes the so-called *risk-neutral* probability, under which

$$dS_t = S_t((r-q)dt + \sigma(t,S_t)dW_t)\ , \quad t > t_0\ ; \quad S_{t_0} = S_0\ . \tag{2}$$

Alternatively to the probabilistic representation (1), the prices $\Pi^{+/-}$ can be given as the solution to a differential equation. One can use either the *Black–Scholes* backward parabolic equation, in the variables (t_0,S_0), which is

$$\begin{cases} -\partial_t\Pi - (r-q)S\partial_S\Pi - a(t,S)S^2\partial^2_{S^2}\Pi + r\Pi \ = \ 0, \quad t < T \\ \Pi|_T \ \equiv \ (S-K)^{+/-}\ , \end{cases} \tag{3}$$

or the *Dupire* forward parabolic equation, in the variables (T,K), given by

$$\begin{cases} \partial_T\Pi - (q-r)K\partial_K\Pi - a(T,K)K^2\partial^2_{K^2}\Pi + q\Pi \ = \ 0, \quad T > t_0 \\ \Pi|_{t_0} \ \equiv \ (S_0-K)^{+/-}\ . \end{cases} \tag{4}$$

We will show in lemma **??** and theorem **??** that equations (1) or (3)–(4) hold for an arbitrary measurable, positively bounded local volatility function a. However, let us give a less formal insight by recalling the Black–Scholes seminal analysis [1], valid in the special case where the volatility depends on time alone. We consider a self-financing portfolio, short one option and long $\partial_S\Pi$ shares of the underlying stock. The value V of the risky component of the portfolio then evolves as

$$\begin{aligned} dV_t &= -d\Pi(t,S_t) + \partial_S\Pi(dS_t + qS_tdt) \\ &= -(\partial_t\Pi - qS\partial_S\Pi + aS^2\partial^2_{S^2}\Pi)dt\ , \end{aligned}$$