Modeling of Tradeable Securities with Dividends

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Abstract

We propose a generalized framework for the modeling of tradeable securities with dividends which are not necessarily cash dividends at fixed times or continuously paid dividends. In our setup the dividend processes are only required to be semi-martingales. We give a definition of self-financing replication which incorporates dividend processes, and we show how this allows us to translate standard results for the pricing and hedging of derivatives on assets without dividends to the case of assets with dividends. We then apply this framework to analyze and compare the different assumptions that have been made in earlier dividend models. We also study the case where we have uncertain dividend dates, and we look at securities which are not equity-based such as futures and credit default swaps, since our weaker assumptions on the dividend process allow us to consider these other applications as well.

Keywords: Financial modeling, Dividends, Futures, Credit Default Swaps.

1 Introduction

The pricing theory for derivatives on non-dividend paying stocks is well understood nowadays at a conceptual level, see for example Duffie (2001) and Musiela and Rutkowski (1997). In this paper we will clarify stock-price models *with* dividends, and show that our more general framework can also be used for securities other than stocks, such as futures and certain credit derivatives. By introducing the economic concept of a tradeable we are able to reduce models with dividends to models without dividends.

In the standard Black-Scholes model of option pricing (Black and Scholes (1973)) with one stock and one bank account with fixed interest rate $r > 0$ the stock and bank account are considered to be basic tradeables. For the stock it is clear that one can trade it and as our bank account is equivalent to a zero coupon bond on a fixed time interval $[0, T]$, it is also clear that we may view our bank account as a tradeable. It is assumed that there are no transaction costs, that shortselling is allowed and that the products are perfectly divisible. For the market participants we assume that they possess a perfect memory, an assumption that is reflected in the use of the concept of filtration. Given the assumptions above, every product that can be made from these two basic tradeables by a reasonable self-financing strategy (to be defined in a precise manner later on) is a tradeable as well.

Introducing dividends in such models means that the ex-dividend price process of the stock can no longer be thought of as a tradeable. Indeed it is clear that nobody would like to invest in such an asset without receiving the dividend stream. It is therefore useful to investigate how dividends can be incorporated into models for markets of tradeables in a consistent manner.

Quite often, certain assumptions have been made concerning the dividend payments which seem to have been specificly designed to simplify the computation of standard European option prices. The tractability of the Black-Scholes model is based upon the fact that the asset prices

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follow Geometric Brownian Motions, which leads to explicit closed-form formulas for standard options such as European calls and puts. Taking dividends proportional to the stock price on the dividend date retains this property, and it has therefore been a popular choice for academic models. But many practitioners, i.e. option market makers, actually prefer to model dividends as fixed amounts of cash, which are not dependent on the stock price just before the dividends are paid. If one continues to assume that in between dividend dates stock prices follow Geometric Brownian Motion, then the lognormal distribution no longer describes future values of the stock, and in general no closed-form pricing formulas can be derived anymore.

Therefore, different assumptions have been made for dividends in the literature which try to remedy this problem. In the *Escrowed Model* for dividends, for example, it is assumed that the asset price minus the present value of all dividends to be paid until the maturity of the option follows a Geometric Brownian Motion. In the *Forward Model* one assumes that the asset price plus the forward value of all dividends (from past dividend dates to today) follows a Geometric Brownian Motion. For both models one can use the original Black-Scholes pricing formulas for European-style calls and puts, when adjusted values for the strike or the current stock price are inserted in these formulas. Even more importantly, one can still use the powerful numerical method of binomial tree pricing to price options that can be exercised before maturity (American and Bermudean options). But the assumptions in these models may lead to inconsistencies, since they assume different dynamics for the underlying process when different options on the same underlying are considered simultaneously, and this may even lead to arbitrage opportunities in the market, see Beneder and Vorst (2002) and Frishling (2002). The reason for this is obvious: different asset price process dynamics are assumed for products up until the first dividend date. One can fix this by changing the definition to an assumption that the asset price minus the present value of all dividends to be paid in the future follows a Geometric Brownian Motion (an *Adjusted Escrowed Model*). But this would mean that the prices of options will depend on the dividends which are being paid after the options have expired. This is unsatisfactory as well, since this means that a trader would have to adjust the price of a two-year option once his view on the five-year dividend prediction changes. All this exemplifies the need for a consistent framework to model cash dividends.

In this paper we define such a consistent modeling framework to handle dividends. The dividend stream process and the ex-dividend stock price process can be freely specified and we then show how tradeable securities (i.e. the stock process which include dividends) can be generated. We note that in a very interesting recent paper by Korn and Rogers (Korn and Rogers (2004)) the same problem is being treated. Their solution is to define the stock price to be the net present value of all future dividends. They model the (discrete) dividend process directly and then derive the stock price from this. It turns out that under their assumptions, the dividends are proportional to the stock price on the dividend date, if it is assumed that dividends are announced before that date. We do not make that assumption in our model: ex-dividend stock prices and dividend values can be specified independently in our setup, since many market makers prefer models in which it is possible to specify in advance the exact discrete dividend amount that will be paid. The Escrowed, Forward, Korn-Rogers and other dividend models will be compared in examples given at the end of this paper.

This paper consists of three parts. In the first part we will briefly discuss continuous dividends of finite variation to get some intuition for the more general case, which is treated in the second part. As mentioned earlier, our emphasis on creating tradeable securities from the (not tradeable) ex-dividend process and the dividends can also be applied to the modelling problem for other securities, such as futures and credit default swaps. This will be investigated in the last part.

2 Continuous Dividends of Finite Variation

We assume given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ where the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ is the usual one associated with a given standard Brownian Motion $W : \Omega \times [0,T] \to \mathbb{R}$ with $T > 0$ a given fixed time-horizon. Throughout the paper, all filtrations we use are assumed to satisfy the

usual conditions. We use the notation $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$ and $\mathbb{R}_{++} = \{x \in \mathbb{R} \mid x > 0\}$.

We assume that the adapted càdlàg stochastic process $S : \Omega \times [0,T] \to \mathbb{R}_{++}$ describes the price of one unit of stock ex-dividend. The adapted stochastic process $\delta : \Omega \times [0,T] \to \mathbb{R}_+$ satisfying

$$\mathbb{E} \int_0^T \delta_u du \ < \ \infty$$

represents a continuous stream of dividend payments. The interpretation of this process is as follows. Assume you own x shares of stock during the time interval $]t, t+\epsilon]$ with ϵ infinitesimally small, then you will receive at time $t+\epsilon$ the amount of money $x\delta_t\epsilon$ as dividend. It goes without saying that in our context x may be negative or noninteger as well. We will not assume that S (the ex-dividend price process of the stock) is a tradeable in the market. In fact, we will need to construct the tradeable $\tilde{S}$ from S.

Informally we would reason as follows. Suppose we start at time zero with $x_0 = 1$ unit of stock. When we are at time $t \in [0,T]$ we have x_t stocks. Let $\epsilon > 0$ be small. At time $t+\epsilon$ we receive $\epsilon x_t \delta_t$ money units which we immediately invest in stock so

$$x_{t+\epsilon} \ = \ x_t + \frac{\epsilon x_t \delta_t}{S_{t+\epsilon}}$$

Now let us define $\tilde{S}_t = x_t S_t$. We are then inclined to view $\tilde{S}$ as the price process of a tradeable. We have

$$\begin{aligned} \tilde{S}_{t+\epsilon} &= x_{t+\epsilon} S_{t+\epsilon} \\ &= x_t S_{t+\epsilon} + x_t \delta_t \epsilon \\ &= x_t (S_t + S_{t+\epsilon} - S_t) + x_t \delta_t \epsilon \end{aligned}$$

or

$$\tilde{S}_{t+\epsilon} - \tilde{S}_t \ = \ x_t (S_{t+\epsilon} - S_t) + x_t \delta_t \epsilon$$

Formally we therefore proceed as follows. We are looking for a predictable adapted stochastic process $x : \Omega \times [0,T] \to \mathbb{R}$ with $x_0 = 1$ almost surely and an adapted stochastic process $S : \Omega \times [0,T] \to \mathbb{R}$ such that the following equations are satisfied simultaneously:

$$\tilde{S}_t \ = \ \tilde{S}_0 + \int_0^t x_u d(S_u + D_u) \tag{1}$$

$$\tilde{S}_t \ = \ x_t S_t \tag{2}$$

where

$$D_t \ = \ \int_0^t \delta_u du$$

defines the cumulative dividend process, and where we assume the above equations to be well-defined, i.e. $S + D$ is a semi-martingale.

Our economically motivated intuition says that in an arbitrage-free market model there is precisely one predictable adapted process x such that the equations above are satisfied, and we will now show that this intuition is correct.

Theorem 2.1. *Assume that $S + D$ is a continuous semi-martingale. Then there exists a unique process x such that the equations (1)-(2) above are satisfied.*

Proof. We assumed that $S + D$ is a semi-martingale and as D is a semi-martingale as well it follows that S is also a semi-martingale. From the general theory of stochastic integration it follows that $\tilde{S}$ is a semi-martingale too. As S is assumed to be strictly positive and càdlàg it

follows from Ito's lemma and a localization argument (see for example Protter (2003)) that $1/S$ is a semi-martingale and hence x must be a semi-martingale too. We assumed that $S + D$ is continuous and since D is continuous, S has to be continuous too. But then

$$\begin{aligned}
[S, x]_t &= S_t x_t - \int_0^t S_u dx_u - \int_0^t x_u dS_u \\
&= S_0 x_0 - \int_0^t S_u dx_u + \int_0^t x_u dD_u
\end{aligned}$$

by equation (1)-(2) so this shows that $\int_0^t S_u dx_u$, and hence x, has finite variation. Applying Ito's rule and using the fact that x has finite variation we find

$$d\tilde{S}_t = S_t dx_t + x_t dS_t$$

and combining this with

$$d\tilde{S}_t = x_t dS_t + x_t dD_t$$

we find that

$$dx_t = \frac{x_t}{S_t} dD_t$$

As D is non-decreasing, so is x. From the general theory of stochastic differential equations it follows that

$$\begin{aligned}
dx_t &= \frac{x_t}{S_t} dD_t \\
x_0 &= 1
\end{aligned}$$

has a unique strong predictable solution on $[0, T]$ (see for example Protter (2003)). In fact

$$x_t = e^{\int_0^t \frac{\delta_u}{S_u} du}$$

which completes the proof. □

Note that a proof of existence for the process x_t could easily be settled using the last few equations, but it is the *uniqueness* which is of particular interest here. Also note that we assumed here that the semi-martingale $S + D$ is continuous and that the cumulative dividend process D is of finite variation. In the next section, where we discuss the more general case, we will show that we do not need to make these assumptions to show uniqueness and existence of tradeables $\tilde{S}$ in a more general setup.

3 General Dividend Processes

We would now like to be able to define on the same probability space, but with a filtration which need not necessarily be generated by a Brownian Motion, an asset process which may pay a discrete (i.e. cash) dividend equal to $\tilde{D}$ on time $t_D \in]0, T[$ where $\tilde{D} \in \mathcal{F}_{t_D}$ and such that

$$S_{t_D -} - \tilde{D} > 0 \qquad (P - \text{a.s.})$$

where S again describes the ex-dividend process. In fact, we would even like to consider cases where an asset pays both continuous and discrete dividends, or even more generally, where the cumulative dividend process is just assumed to be a semi-martingale.

Let V_t, S_t and B_t be adapted càdlàg ex-dividend price processes for assets V, S and B which are strictly positive and let D_t^V, D_t^S and D_t^B be the corresponding càdlàg adapted cumulative

dividend processes (which are not necessarily positive), such that $V_t + D_t^V$, $S_t + D_t^S$ and $B_t + D_t^B$ are all semi-martingales. We will assume that $D_0^S = D_0^B = D_0^V = 0$ throughout the paper. The asset B will often represent a bank account in this setup.

We would like to define the notion of *replicability* i.e. the idea that the price process of a certain asset V can be mimicked by trading in other assets.

Definition 3.1. *We say that an asset V can be replicated using assets S and B iff there exist adapted and predictable processes ϕ^S and ϕ^B such that for all $t \in [0, T]$*

$$V_{t-} = \phi_t^S S_{t-} + \phi_t^B B_{t-} \tag{3}$$

$$d(V_t + D_t^V) = \phi_t^S d(S_t + D_t^S) + \phi_t^B d(B_t + D_t^B) \tag{4}$$

where the first equation for $t = 0$ should be read as $V_0 = \phi_0^S S_0 + \phi_0^B B_0$ (i.e. without taking left-hand side limits).

Note that for continuous processes without dividends we find the classical definition of replication back in (3)-(4), but the lefthand side limits in the first equations are an important difference compared to the case without dividends. Indeed we can no longer say that

$$V_t = \phi_t^S S_t + \phi_t^B B_t$$

as in the usual formulations in the absence of dividends, but instead

$$V_t - \Delta V_t = \phi_t^S (S_t - \Delta S_t) + \phi_t^B (B_t - \Delta B_t)$$

which is of course a reformulation of (3) since $X_t - \Delta X_t = X_{t-}$.

If we define

$$\phi_t^S = \psi_t^S V_{t-}/S_{t-} \tag{5}$$

$$\phi_t^B = \psi_t^B V_{t-}/B_{t-} \tag{6}$$

then

$$\frac{dV_t + dD_t^V}{V_{t-}} = \psi_t^S \frac{dS_t + dD_t^S}{S_{t-}} + \psi_t^B \frac{dB_t + dD_t^B}{B_{t-}}$$

for certain predictable adapted processes ψ^S and ψ^B such that

$$\psi_t^S + \psi_t^B = 1$$

The interpretation is that the rate of return of V (which equals the difference in value based on changes in *both* the ex-dividend price *and* the dividends, divided by the price *before* any dividends have been paid out) is based on percentages invested in assets S and B. Working with percentages guarantees in an intuitive manner that we only consider strategies which do not necessitate cash withdrawal or injection, i.e. it is a convenient way to define self-financing strategies. However, our definition above is slightly more general in the sense that it allows the price processes becoming zero for certain times as well.

Throughout the paper we will assume $D^B = 0$ i.e. our bank account does not pay dividends (or coupons), only interest. Note that we have assumed that $S + D^S$ is a semi-martingale but we have not assumed it to be continuous, as we did in Theorem 2.1.

Theorem 3.1. *Let $S + D^S$ and B be semi-martingales satisfying the conditions stated above. Then there exists a unique asset price process $\tilde{V}$ with $D^{\tilde{V}} \equiv 0$ and $\tilde{V}_0 = S_0$ that can be replicated with $\phi^S \equiv 1$.*

To prove Theorem 3.1, we need the following result which is stated and proven in Jaschke (2003).

Theorem 3.2. *Let H be a semi-martingale and let Z be a semi-martingale with $Z_0 = 0$ and $\Delta Z_t \neq -1$, for all $t \in \mathbb{R}^+$. Then the solution of the equation*

$$X_t \;=\; H_t \;+\; \int_{0+}^{t} X_{s-} dZ_s$$

is given by

$$X_t \;=\; H_t - E_t \int_{0+}^{t} H_{s-} d(\tfrac{1}{E_s}) \tag{7}$$

$$E_t \;=\; e^{Z_t - \frac{1}{2}[Z,Z]_t^c} \prod_{0<s\leq t} (1+\Delta Z_s) e^{-\Delta Z_s} \tag{8}$$

and this solution is unique.

Proof of Theorem 3.1. Since we want $\phi^S \equiv 1$ we define $\phi_t^B = (\tilde{V}_{t-} - S_{t-})/B_{t-}$, so $\tilde{V}$ should satisfy

$$d\tilde{V}_t \;=\; d(S_t + D_t^S) \;+\; \frac{\tilde{V}_{t-} - S_{t-}}{B_{t-}} dB_t \tag{9}$$

We define $A_t = \tilde{V}_t - S_t$, then

$$dA_t \;=\; dD_t^S \;+\; A_{t-} \frac{dB_t}{B_{t-}}$$

so if we take $H_t = D_t^S$ and $Z_t = \int_0^t dB_u/B_{u-}$, we can apply Theorem 3.2 to prove the existence and uniqueness of the process A and hence of the process $\tilde{V} = A + S$. Indeed, substitution in (8) gives

$$\begin{aligned} E_t \;&=\; e^{\int_{0+}^t \frac{dB_u}{B_{u-}} - \frac{1}{2}[\int \frac{dB_u}{B_{u-}}, \int \frac{dB_u}{B_{u-}}]_t^c} \prod_{0<s\leq t} (1 + \Delta \int_{0+}^t \tfrac{dB_u}{B_{u-}}) e^{-\Delta \int_{0+}^t \frac{dB_u}{B_{u-}}} \\ &=\; e^{\int_{0+}^t d(\ln B_u)} \prod_{0<s\leq t} \frac{B_{s-}}{B_s} (1 + \tfrac{B_s - B_{s-}}{B_{s-}}) \;=\; B_t/B_0 \end{aligned}$$

so according to (7) the process

$$\begin{aligned} \tilde{V}_t \;&=\; S_t \;+\; A_t \\ &=\; S_t \;+\; D_t^S - B_t \int_{0+}^t D_{u-}^S d(\tfrac{1}{B_u}) \end{aligned} \tag{10}$$

satisfies our requirements. □

The interpretation of the result proven in the Theorem is of course that it should be possible to invest our dividend stream in the bank account and by doing so end up with a process which no longer pays any dividends. We will denote the process V constructed in the Theorem by S^B in the sequel.

Since $X_t Y_t = \int X_{t-} dY_t + \int Y_{t-} dX_t + [X,Y]_t$ for all semi-martingales X and Y, we can rewrite the formula (10) derived in the Theorem as follows:

$$\begin{aligned} S_t^B \;&=\; S_t \;+\; D_t^S - B_t \int_{0+}^t D_{u-}^S d(\tfrac{1}{B_u}) \\ &=\; S_t \;+\; B_t \int_{0+}^t \frac{dD_u^S}{B_{u-}} \;+\; B_t\, [D^S,\, B^{-1}]_t \end{aligned} \tag{11}$$

The necessity of the last bracket term to compensate for the fact that paid out cashflows and the bank account may have nonzero covariation was already noted in Norberg and Steffensen (2005).

If we assume that B is continuous and of finite variation then we simply find

$$S_t^B \;=\; S_t \;+\; B_t \int_{0+}^{t} \frac{dD_u^S}{B_u}.$$

In the special case for just one cash dividend $D_t^S = \tilde{D}\mathbf{1}_{[t_D,T]}(t)$, we can reduce this to

$$S_t^B \;=\; S_t + \mathbf{1}_{[t_D,T]}(t)\tilde{D}\frac{B_t}{B_{t_D}}$$

Note that if we work on a Brownian filtration, then $S+D^S$ and $V+D^V$ are continuous processes, so $\Delta D^S = -\Delta S$ and the times at which D^S and S are discontinuous thus have to coincide. In general we would have for adapted processes X on this filtration that $X_{t-} = X_t - \Delta X_t = X_t + \Delta D_t^X$ and the first replication equation (3) would then boil down to

$$V_t + \Delta D_t^V \;=\; \phi_t^S(S_t + \Delta D_t^S) \;+\; \phi_t^B(B_t + \Delta D_t^B)$$

which is the classical notion of a *gains process* to model dividend, and has been introduced earlier in the literature, see for example Duffie (2001). This may seem a natural alternative choice for the first equation in our definition of replication, but it will not generalize in a nice way when we use other filtrations than those generated by Brownian Motion, since we will see later that on filtrations which are not left-continuous we may not always have that $\Delta D^S = -\Delta S$. On such filtrations our definition is therefore different from the one in Duffie (2001).

Theorem 3.3. *Let $S + D^S$ and B be semi-martingales satisfying the conditions stated above. Then there exists a unique asset price process V with $D^V \equiv 0$ and $V_0 = S_0$ such that V can be replicated using S only, i.e. such that $\phi^B \equiv 0$. This asset price process $V = \tilde{S}$ can, together with B, replicate S^B.*

Proof. We are looking for a process V such that $D^V \equiv 0$, with $\phi^B \equiv 0$. But this last assumption implies that $\phi_t^S = V_{t-}/S_{t-}$ so we need to prove that there exists a unique process V such that

$$dV_t \;=\; \frac{V_{t-}}{S_{t-}} d(S_t + D_t^S) \tag{12}$$

with $V_0 = S_0$, so

$$V_t \;=\; S_0 + \int_{0+}^{t} \frac{V_{u-}}{S_{u-}} d(S_u + D_u^S)$$

We can thus apply Theorem 3.2 again with $Z_t = \int_0^t \frac{d(S_u+D_u^S)}{S_{u-}}$ and $H \equiv S_0$ to prove existence and uniqueness. Finally, the asset $\tilde{V} = S^B$ can be replicated using $V = \tilde{S}$ and B since

$$\begin{aligned} \tilde{V}_{t-} &= \phi_t^{\tilde{S}} \tilde{S}_{t-} \;+\; \phi_t^B B_{t-} \\ d\tilde{V}_t &= \phi_t^{\tilde{S}} d\tilde{S}_t \;+\; \phi_t^B dB_t \end{aligned}$$

if we take $\phi^{\tilde{S}} = S_{t-}/\tilde{S}_{t-}$ and $\phi_t^B = (\tilde{V}_{t-} - S_{t-})/B_{t-}$, as can be seen from (12) and (9). □

Note that in the special case of a single discrete dividend $\tilde{D}$ at one particular time t_D, i.e. when $D_t^S = \tilde{D}\mathbf{1}_{[t_D,T]}(t)$, we can simplify the expression for the process V based on formula (7)

considerably. In fact, we then find that $V = \tilde{S}$ with

$$\begin{aligned}
\tilde{S}_t &= S_0 e^{\int_{0+}^t \frac{d(S_u+D_u^S)}{S_{u-}} - \frac{1}{2}[\int_{0+}^t \frac{d(S_u+D_u^S)}{S_{u-}}, \int_{0+}^t \frac{d(S_u+D_u^S)}{S_{u-}}]_t^c} (1 + \frac{\tilde{D}\mathbf{1}_{[t_D,T]}(t)}{S_{t_D-}}) e^{-\frac{\tilde{D}\mathbf{1}_{[t_D,T]}(t)}{S_{t_D-}}} \\
&= S_0 e^{\int_{0+}^t d(\ln S_u) - \mathbf{1}_{[t_D,T]}(t)\Delta(\ln S_t)} (1 + \frac{\tilde{D}\mathbf{1}_{[t_D,T]}(t)}{S_{t_D-}}) \\
&= S_t + \mathbf{1}_{[t_D,T]}(t) \frac{\tilde{D}}{S(t_D)} S_t
\end{aligned}$$

and this process indeed satisfies the requirements. Note that this expression represents our economic intuition of what happens when we reinvest dividend proceeds in the underlying asset S.

To check that this expression models the correct behavior, first notice that

$$t < t_D \ : \qquad \tilde{S}_t = S_t \qquad \tilde{S}_{t-} = S_{t-} \tag{13}$$

$$t = t_D \ : \qquad \tilde{S}_{t_D} = S_{t_D} + \tilde{D}\frac{S_{t_D}}{S_{t_D-}} \qquad \tilde{S}_{t_D-} = S_{t_D-} \tag{14}$$

$$t > t_D \ : \qquad \tilde{S}_t = S_t(1 + \frac{\tilde{D}}{S_{t_D-}}) \qquad \tilde{S}_{t-} = S_{t-}(1 + \frac{\tilde{D}}{S_{t_D-}}) \tag{15}$$

We can now show directly that $V = \tilde{S}$ satisfies (12). Indeed, we have for $t < t_D$ that

$$d\tilde{S}_t = dS_t = dS_t + dD_t^S = \frac{\tilde{S}_{t-}}{S_{t-}} d(S_t + D_t^S)$$

as required, where we have used (13) and the definition of D_t^S. For $t = t_D$ we find, using (14)

$$d\tilde{S}_t = dS_t + dD_t^S = \frac{\tilde{S}_{t-}}{S_{t-}} d(S_t + D_t^S)$$

and finally, for $t > t_D$

$$\begin{aligned}
d\tilde{S}_t &= dS_t + \frac{\tilde{D}}{S_{t_D}} dS_t = (1 + \frac{\tilde{D}}{S_{t_D}}) d(S_t + D_t^S) \\
&= \frac{\tilde{S}_{t-}}{S_{t-}} d(S_t + D_t^S)
\end{aligned}$$

so we are done.

The approach taken in the proof of Theorem 3.3 formalizes the idea that we could reinvest dividend payouts in the asset which pays the dividends, instead of the approach taken in the previous Theorem, where the dividend proceeds were invested in the bank account.

The unique processes $\tilde{S}$ and S^B that we have created and which do *not* contain any dividends, can now be used for replication purposes, so the original ex-dividend process S and its dividend process D^S have become superfluous in this sense:

Corollary 3.1. *If an asset V can be replicated using the assets S and B, then it can be replicated using the assets $\tilde{S}$ and B.*
If an asset V can be replicated using the assets S and B, then it can be replicated using the assets S^B and B.

Proof. If an asset V is replicated using S and B we may write

$$V_{t-} = \phi_t^S S_{t-} + \phi_t^B B_{t-} \tag{16}$$

$$d(V_t + D_t^V) = \phi_t^S d(S_t + D_t^S) + \phi_t^B dB_t \tag{17}$$

but using (12) we can rewrite this as

$$\begin{aligned} V_{t-} &= \phi_t^S \frac{S_{t-}}{\tilde{S}_{t-}} \tilde{S}_{t-} \;+\; \phi_t^B B_{t-} \\ d(V_t + D_t^V) &= \phi_t^S \frac{S_{t-}}{\tilde{S}_{t-}} d\tilde{S}_t \;+\; \phi_t^B dB_t \end{aligned}$$

so taking $\phi_t^{\tilde{S}} = \phi_t^S S_{t-}/\tilde{S}_{t-}$ shows the first result. The second result follows when we use (12) to rewrite (16)-(17) in the form

$$\begin{aligned} V_{t-} &= \phi_t^S S_t^B \;+\; (\phi_t^B - \phi_t^S \frac{S_{t-}^B - S_{t-}}{B_{t-}}) B_t \\ d(V_t + D_t^V) &= \phi_t^S dS_t^B \;+\; (\phi_t^B - \phi_t^S \frac{S_{t-}^B - S_{t-}}{B_{t-}}) dB_t \end{aligned}$$

so replication is possible in this case as well. □

We now consider an arbitrage-free market with the assets $(\tilde{S}, B)$ in it. We know that there exists a measure $\mathbb{Q}$, equivalent to our original measure P, such that $\tilde{S}/B$ is a martingale under $\mathbb{Q}$.

Definition 3.2. *We say that V is the price process of a tradeable asset iff*

1. *It can be replicated using $\tilde{S}$ and B*
2. *The process $\mathcal{D}(V)$ is a martingale under $\mathbb{Q}$, where*[1]

$$\mathcal{D}(V) \;=\; \frac{V + D^V}{B} - D_-^V \bullet B^{-1}$$

Due to the corollary proven above, we might as well have required that V can be replicated using S^B and B.

We noted before in (11) that we may rewrite $\mathcal{D}(V)_t$ as

$$\mathcal{D}(V) \;=\; \frac{V}{B} \;+\; B_-^{-1} \bullet D^V \;+\; [D^V, B^{-1}]$$

but we prefer the notation used in the definition since it does not involve a bracket. The main point of the definition given above is that we would like $\mathcal{D}$ to be a martingale, and not just a *local* martingale. That it is a local martingale is already guaranteed by the first part of the definition, as the following result shows. This representation theorem is the main result of the paper, which shows how the usual martingale representation theory for assets without dividends carries over to our more general case.

Theorem 3.4. *If an asset price process V can be replicated using S and B then there exists an adapted predictable process ϕ such that*

$$d\mathcal{D}(V)_t \;=\; \phi_t \; d\left(\frac{\tilde{S}_t}{B_t}\right)$$

Proof. We apply Ito's rule for (not necessarily continuous) semi-martingales which states that for twice continuously differentiable functions $f : \mathbb{R}^n \to \mathbb{R}$ and semi-martingales X on $\mathbb{R}^n$ we have

$$\begin{aligned} f(X_t) - f(X_0) &= \sum_{i=1}^n \int_{0+}^t \tfrac{\partial f}{\partial x_i}(X_{s-}) dX_s^i \;+\; \tfrac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \int_{0+}^t \tfrac{\partial^2 f}{\partial x_i \partial x_j}(X_{s-}) d[X^i, X^j]_s^c \\ &\quad + \sum_{0<s\le t} [\, f(X_s) - f(X_{s-}) - \sum_{i=1}^n \tfrac{\partial f}{\partial x_i}(X_{s-}) \Delta X_s^i \,] \end{aligned}$$

[1] We use the common notation $Z = X_- \bullet Y$ for a process Z satisfying $dZ_t = X_{t-} dY_t$.

In particular, for $f(x,y) = x/y$ we find that

$$d\frac{X_t}{Y_t} = \frac{dX_t}{Y_{t-}} - \frac{X_{t-}}{Y_{t-}}\frac{dY_t}{Y_{t-}} - \frac{d[X,Y]^c_t}{Y^2_{t-}} + \frac{X_{t-}d[Y,Y]^c_t}{Y^3_{t-}} + \left(\frac{\Delta Y_t}{Y_{t-}}\right)\left(\frac{X_{t-}}{Y_t}\frac{\Delta Y_t}{Y_{t-}} - \frac{\Delta X_t}{Y_t}\right)$$

If V can be replicated using S and B, it can be replicated using $\tilde{S}$ and B by the previous corollary, so there exist $\phi^{\tilde{S}}$ and ϕ^B such that

$$V_{t-} = \phi^{\tilde{S}}_t \tilde{S}_{t-} + \phi^B_t B_{t-} \tag{18}$$

$$d(V_t + D^V_t) = \phi^{\tilde{S}}_t d\tilde{S}_t + \phi^B_t dB_t \tag{19}$$

where we have used the fact that $D^{\tilde{S}} = D^B \equiv 0$. But then

$$\begin{aligned}
d\frac{V_t + D^V_t}{B_t} &= \frac{d(V_t + D^V_t)}{B_{t-}} - \frac{V_{t-} + D^V_{t-}}{B_{t-}}\frac{dB_t}{B_{t-}} - \frac{d[V + D^V, B]^c_t}{B^2_{t-}} + \frac{V_{t-} + D^V_{t-}}{B^3_{t-}}d[B,B]^c_t \\
&\quad + \left(\frac{\Delta B_t}{B_{t-}}\right)\left(\frac{(V_{t-} + D^V_{t-})}{B_t}\frac{\Delta B_t}{B_{t-}} - \frac{\Delta(V_t + D^V_t)}{B_t}\right) \\
-D^V_{t-}d\frac{1}{B_t} &= -D^V_{t-}\left[\frac{-dB_t}{B^2_{t-}} + \frac{d[B,B]^c_t}{B^3_{t-}} + \left(\frac{\Delta B_t}{B_{t-}}\right)\left(\frac{\Delta B_t}{B_t B_{t-}}\right)\right] \\
-\phi^{\tilde{S}}_t d\frac{\tilde{S}_t}{B_t} &= -\phi^{\tilde{S}}_t\left[\frac{d\tilde{S}_t}{B_{t-}} - \frac{\tilde{S}_{t-}}{B_{t-}}\frac{dB_t}{B_{t-}} - \frac{d[\tilde{S},B]^c_t}{B^2_{t-}} + \frac{\tilde{S}_{t-}}{B^3_{t-}}d[B,B]^c_t\right] \\
&\quad - \phi^{\tilde{S}}_t\left(\frac{\Delta B_t}{B_{t-}}\right)\left(\frac{\tilde{S}_{t-}}{B_t}\frac{\Delta B_t}{B_{t-}} - \frac{\Delta\tilde{S}_t}{B_t}\right)
\end{aligned}$$

We sum the three expressions to calculate $d\mathcal{D}(V)_t$ and collect terms:

$$\begin{aligned}
&d\mathcal{D}(V)_t - \phi^{\tilde{S}}_t d\frac{\tilde{S}_t}{B_t} \\
&= d\frac{V_t + D^V_t}{B_t} - D^V_{t-}d\frac{1}{B_t} - \phi^{\tilde{S}}_t d\frac{\tilde{S}_t}{B_t} \\
&= \frac{d(V_t + D^V_t)}{B_{t-}} - \frac{V_{t-} + D^V_{t-}}{B_{t-}}\frac{dB_t}{B_{t-}} - D^V_{t-}\frac{-dB_t}{B^2_{t-}} - \phi^{\tilde{S}}_t\left[\frac{d\tilde{S}_t}{B_{t-}} - \frac{\tilde{S}_{t-}}{B_{t-}}\frac{dB_t}{B_{t-}}\right] \\
&\quad - \frac{d[V + D^V, B]^c_t}{B^2_{t-}} + \frac{V_{t-} + D^V_{t-}}{B^3_{t-}}d[B,B]^c_t - D^V_{t-}\frac{d[B,B]^c_t}{B^3_{t-}} - \phi^{\tilde{S}}_t\left[-\frac{d[\tilde{S},B]^c_t}{B^2_{t-}} + \frac{\tilde{S}_{t-}}{B^3_{t-}}d[B,B]^c_t\right] \\
&\quad \left(\frac{\Delta B_t}{B_{t-}}\right)\left[\frac{(V_{t-} + D^V_{t-})}{B_t}\frac{\Delta B_t}{B_{t-}} - \frac{\Delta(V_t + D^V_t)}{B_t} - D^V_{t-}\frac{\Delta B_t}{B_t B_{t-}} - \phi^{\tilde{S}}_t\frac{\tilde{S}_{t-}}{B_t}\frac{\Delta B_t}{B_{t-}} + \phi^{\tilde{S}}_t\frac{\Delta\tilde{S}_t}{B_t}\right]
\end{aligned}$$

We substitute (18)-(19) and get

$$\begin{aligned}
&= \frac{1}{B^2_{t-}}\left[B_{t-}(\phi^{\tilde{S}}_t d\tilde{S}_t + \phi^B_t dB_t) - (V_{t-} + D^V_{t-})dB_t + D^V_{t-}dB_t - \phi^{\tilde{S}}_t(B_{t-}d\tilde{S}_t - \tilde{S}_{t-}dB_t)\right] \\
&\quad + \frac{1}{B^3_{t-}}\left[-B_{t-}d[V + D^V, B]^c_t + V_{t-}d[B,B]^c_t + \phi^{\tilde{S}}_t B_{t-}d[\tilde{S},B]^c_t - \phi^{\tilde{S}}_t\tilde{S}_{t-}d[B,B]^c_t\right] \\
&\quad \left(\frac{\Delta B_t}{B_{t-}}\right)\left[\frac{V_{t-}\Delta B_t - B_{t-}\Delta(V_t + D^V_t) - \phi^{\tilde{S}}_{t-}\tilde{S}_{t-}\Delta B_t + B_{t-}\phi^{\tilde{S}}_{t-}\Delta\tilde{S}_t}{B_t B_{t-}}\right] \\
&= \frac{1}{B^2_{t-}}\left[B_{t-}\phi^B_t - V_{t-} + \phi^{\tilde{S}}_t\tilde{S}_{t-}\right]dB_t \\
&\quad + \frac{B_{t-}}{B^3_{t-}}\left[-d[V + D^V, B]^c_t + \phi^{\tilde{S}}_t d[\tilde{S},B]^c_t + \phi^B_t d[B,B]^c_t\right] \\
&\quad + \left(\frac{\Delta B_t}{B_{t-}}\right)\left(\frac{1}{B_t B_{t-}}\right)\left[(V_{t-} - \phi^{\tilde{S}}_t\tilde{S}_t)\Delta B_t - B_{t-}\Delta(V_t + D^V_t) + B_{t-}\phi^{\tilde{S}}_t\Delta\tilde{S}_t\right]
\end{aligned}$$

and we see this is zero by using (18)-(19) again, and using the fact that (19) implies that

$$\Delta(V_t + D_t^V) \quad = \quad \phi_t^{\tilde{S}} \Delta \tilde{S}_t \; + \; \phi_t^B \Delta B_t$$

This completes the proof. □

We have thus proven that asset price processes V that can be constructed in a self-financing manner using stock and the bank account, inherit the local martingale property from the underlying assets: if the discounted version of $\tilde{S}$ is a local martingale under $\mathbb{Q}$, then so is $\mathcal{D}(V)$, the properly discounted version of V and its dividend process D^V. This will allow us to apply the usual theory for option pricing in arbitrage-free markets without dividends.

Note that we allow tradeables here to have dividend processes. Alternatively we could say that V is a tradeable whenever $D^V \equiv 0$ and $\frac{V}{B}$ is a $\mathbb{Q}$-martingale, but we will see in the applications of the next section that this would be too restrictive for many financial applications.

Since $\mathcal{D}(V)_t$ is a $\mathbb{Q}$-martingale we have that $\mathbb{E}^{\mathbb{Q}}[\mathcal{D}(V)_t \mid \mathcal{F}_s] = \mathcal{D}(V)_s$ and taking limits $s \uparrow t$ we find that $\mathbb{E}^{\mathbb{Q}}[\Delta\mathcal{D}(V)_t \mid \mathcal{F}_{t-}] = 0$. So when B is continuous and of finite variation we must have that

$$\mathbb{E}^{\mathbb{Q}}[\Delta V_t + \Delta D_t^V \mid \mathcal{F}_{t-}] \quad = \quad 0$$

This expression immediately shows that on left-continuous filtrations (such as those generated by Brownian Motion) where $\mathcal{F}_{t-} = \mathcal{F}_t$, we must have that $\Delta V = -\Delta D^V$ since both V and D^V are adapted. But if the underlying filtration is not left-continuous this is no longer necessary, even if cash dividend payments are announced in advance (i.e. when ΔD^V is $\mathcal{F}_{t-}$-measurable). We then only know that

$$\mathbb{E}^{\mathbb{Q}}[\Delta V_t \mid \mathcal{F}_{t-}] = -\Delta D_t^V$$

so the jump in the ex-dividend process of a tradeable does not necessarily cancel the jump due to a dividend payment. This was already noted in Heath and Jarrow (1988) and Battauz (2002). In the last paper an asset price model is formulated in which $D_t^V = D\mathbf{1}_{t \geq t_D}$ with D and t_D deterministic, and $\Delta V_{t_D} = -D + Y(V_{t_D-} - D)$ for a stochastic variable Y with support $]-1, 1[$ and such that $\mathbb{E}^{\mathbb{Q}}[Y \mid \mathcal{F}_{t_D-}] = 0$. This provides a nice example of a tractable dividend model where $\Delta V \neq -\Delta D^V$.

4 Examples

We will now show how the framework developed so far can be applied to different types of securities. In all the different products we consider the key notion that we will use is the fact that if an asset V is tradeable in an arbitrage-free and complete market on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in \mathbb{R}_+})$, then there exists a unique equivalent martingale measure $\mathbb{Q}$ such that the process $\mathcal{D}(V)$ is a martingale under $\mathbb{Q}$. Throughout this section the processes B will be of finite variation and continuous, so $[D^V, B^{-1}] \equiv 0$ and $\mathcal{D}(V)$ being a $\mathbb{Q}$-martingale then leads to

$$V_t \quad = \quad B_t \, \mathbb{E}^{\mathbb{Q}} \left[\frac{V_T}{B_T} + \int_t^T \frac{dD_u^V}{B_u} \middle| \mathcal{F}_t \right] \tag{20}$$

Note that this expression has a nice interpretation: the current price of a tradeable can be seen as the price of a derivative which represents the sum of the ex-dividend price at a later date and all the cashflows paid out by the tradeable until that date, after all these have been properly discounted.

4.1 Equity Dividend Models: Deterministic processes for Dividends

In the first section of this paper we mentioned some different approaches to handle the incorporation of dividends in equity price processes. As we explained there, the Escrowed Model for dividends assumes that the (cumulative) dividend process is deterministic and that asset price minus the present value of all dividends to be paid until the maturity of the option follows a Geometric Brownian Motion. This means that

$$V_t = S_t - \int_t^T \frac{B_t}{B_u} dD_u^S$$

is a Geometric Brownian Motion, and if it is also a tradeable, it must be a $\mathbb{Q}$-martingale after discounting, so

$$\frac{S_t}{B_t} - \int_t^T \frac{1}{B_u} dD_u^S = \left(\frac{S_0}{B_0} - \int_0^T \frac{1}{B_u} dD_u^S \right) e^{\int_0^t \sigma_u dW_u^{\mathbb{Q}} - \frac{1}{2}\int_0^t \sigma_u^2 du}$$

for some deterministic process $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ and $W^{\mathbb{Q}}$ a Brownian Motion under $\mathbb{Q}$. The standard European Call option has a payoff $(S_T - K)^+$ which under $\mathbb{Q}$ can be written as

$$\frac{B_T}{B_t} \left(V_t e^{\int_t^T \sigma_u dW_u^{\mathbb{Q}} - \frac{1}{2}\int_t^T \sigma_u^2 du} - K \right)^+$$

This shows that the original Black-Scholes formula can be used to calculate the Call Option price $\frac{B_t}{B_T}\mathbb{E}^{\mathbb{Q}}[(S_T - K)^+ \mid \mathcal{F}_t]$, if one inserts a different starting value for the asset price process: instead of the Black-Scholes formula with current asset price S_t we now use a Black-Scholes formula with current asset price V_t.

In the Forward Model, the the asset price plus the forward value of all dividends (from past dividend dates to today) is assumed to follow a Geometric Brownian Motion, so

$$V_t = S_t + \int_0^t \frac{B_t}{B_u} dD_u^S$$

is a Geometric Brownian Motion, and since it has to be a tradeable as well we find that

$$\frac{S_t}{B_t} + \int_0^t \frac{1}{B_u} dD_u^S = \frac{S_0}{B_0} e^{\int_0^t \sigma_u dW_u^{\mathbb{Q}} - \frac{1}{2}\int_0^t \sigma_u^2 du}$$

and the European Call payoff can be written as

$$\frac{B_T}{B_t} \left(V_t e^{\int_t^T \sigma_u dW_u^{\mathbb{Q}} - \frac{1}{2}\int_t^T \sigma_u^2 du} - [\int_0^T \frac{B_t}{B_u} dD_u^S + K] \right)^+$$

so we see that this time we can use the original Black-Scholes formula with a different strike: instead of the strike K we need to insert the strike $K + \int_0^T \frac{B_t}{B_u} dD_u^S$ into the Black-Scholes formula for European Calls, and insert V_t instead of S_t for the current asset price.

4.2 Korn-Rogers Model: Bounded Variation processes for Dividends

In the model of Korn and Rogers, stochastic dividends are paid at dividend times which are known a priori while the ex-dividend asset price process S equals the conditional expectation, under the equivalent martingale measure, of the sum of all (discounted) future dividends, so

$$S_t = B_t \, \mathbb{E}^{\mathbb{Q}} \left[\int_t^\infty \frac{dD_u^S}{B_u} \middle| \mathcal{F}_t \right]$$

In the case treated by Korn and Rogers, the filtration is generated by a Lévy process. We define

$$V_t = S_t + \int_0^t \frac{B_t}{B_u} dD_u^S$$

which implies that

$$\frac{V_t}{B_t} = \mathbb{E}^{\mathbb{Q}}\left[\int_t^\infty \frac{dD_u^S}{B_u}\middle| \mathcal{F}_t\right] + \int_0^t \frac{dD_u^S}{B_u} = \mathbb{E}^{\mathbb{Q}}\left[D^\infty | \mathcal{F}_t\right]$$

where

$$D^\infty = \int_0^\infty \frac{dD_u^S}{B_u}.$$

is assumed to be a well-defined finite stochastic variable, which is integrable with respect to $\mathbb{Q}$. It is thus immediately clear that in this model S is automatically a tradeable. Korn and Rogers let the process D^S have the specific form $D_t^S = \sum_{i=1}^\infty \mathbf{1}_{t \geq t_i} X_{t_i}$ with X an exponentiated Lévy process and the times t_i deterministic. Obviously, D^S is of bounded variation in that case.

4.3 Futures: Îto-processes for Dividends

A futures contract is an exchange-traded standardized contract which gives the holder the obligation to buy or sell a certain commodity (or another financial contract) at a certain date in the future, the delivery date, for a price specified on that day, the settlement price. It should be contrasted with a forward contract, which gives the holder the obligation to buy or sell at a date in the future for a price specified today but paid or received at the future date (today's forward price for the commodity or underlying contract). Forwards are conceptually easier but more complicated in practice, since it assumes that a buyer and a seller agree on cash being paid today and delivery taking place at a future date.

If one wants to buy the commodity on a specific delivery date in the future one can obtain a future contract, at zero costs today. Today's futures price for that delivery date tells you for what price you will obtain the commodity at that time, but instead of paying that amount right now (which you would do if you had taken out a forward contract) you pay nothing now. Instead, you open a bank account on the exchange, the so-called margin account. From now until the delivery date (or until the first date before that date on which you get rid of the future) you will receive every day, after the new futures price for your commodity and your delivery date has been specified, the difference between the new futures price and the previous day's futures price, if this difference is positive. When this difference is negative, the corresponding amount it taken from your account. The net effect of this is that you end up paying the futures price at which you obtained your contract in the market: you pay the futures price on the delivery date (which must equal the price of the commodity on that date, of course) but you have been compensated on a daily basis if that price is higher than the futures price at which you got in. On the other hand, if the futures price on the delivery date is lower, you have actually paid that difference by the daily adjustments before that date.

The futures contract has therefore three essential elements:

- Going long or short any number of futures contracts is free at all times
- With every future contract we enter, we can associate a margin account in which the differences between the current and previous futures price is being paid (if we are long one contract) or withdrawn (if we are short one contract).
- This margin account earns interest.

We will use these three elements as the basis of a definition of a futures price.

Definition 4.1. *We call $m : \Omega \times [0,T] \to \mathbb{R}$ the futures price process associated with delivery of asset S at time T if the following holds:*

- *m is a semi-martingale and $m_T = S_T$*
- *For* **all** *bounded previsible processes ψ the following process M is a tradeable:*

$$\left\{ \begin{array}{rcl} dM_t & = & M_{t-}\frac{dB_t}{B_{t-}} \; + \; \psi_t dm_t \\ M_0 & = & 0 \end{array} \right. \tag{21}$$

Notice that delivery involves the ex-dividend price, and not the price of the tradeable.

We will use the notation M^ψ for the process M to remind ourselves that it depends on the process ψ. Note that the process ψ in the definition above has the interpretation of a futures trading strategy: ψ_t represents the number of futures contracts in our position at time t. Our definition reflects the fact that we may enter the futures market at any time at zero costs. What we do is to 'invest' the proceeds of the futures strategy ψ into the margin-account M which earns the riskfree rate.

This approach is different from the usual one (see for example Bjork (2004)) where margin accounts are never taken into account explicitly. The only exception we know of is the work of Duffie and Stanton (1992) in which the margin account is mentioned directly. Our treatment here is inspired by the paper by Pozdnyakov and Steele on the martingale framework for futures pricing, Pozdnyakov and Steele (2004), but our definition differs from theirs. We only impose that m is such that M^ψ/B is a $\mathbb{Q}$-martingale on $[0,T]$ (i.e. that M^ψ is a tradeable in economic parlance) and we do *not* need to impose any regularity conditions on m from the start. Another difference with the approach in Pozdnyakov and Steele (2004) is that we introduce a whole *collection* of tradeables from the very beginning and this is completely in line with the fact that one may enter a futures contract at any time in real life.

The following two results are then immediate:

Theorem 4.1. *The margin account process can be replicated using a zero ex-dividend process with pays continuous dividends equal to the futures price.*

Proof. Taking $\phi_t^{\hat{S}} = \psi_t,\ \phi_t^B = M_t/B_t$ and $\hat{S}_t = 0,\ D_t^{\hat{S}} = m_t$ replicates $V_t = M_t$ with $D_t^V = 0$, see equations (3)-(4). □

Theorem 4.2. *We have for all $t \in [0,T]$ that m satisfies*

$$m_t \;=\; \mathbb{E}^{\mathbb{Q}}\left[S_T - \int_t^T \frac{d[\ \mathbb{E}^{\mathbb{Q}}[S_T \mid \mathcal{F}_t],\ B\ ]_u}{B_u} \middle| \mathcal{F}_t \right]$$

Proof. From the proof of Theorem 3.1. we see that we can solve (21) for M. In fact we have that

$$d\left(\frac{M_t}{B_t}\right) \;=\; \psi_t \left(\frac{dm_t}{B_{t-}} \;+\; d[m, B^{-1}]_t \right) \;=\; \frac{\psi_t}{B_{t-}} \left(dm_t \;-\; \frac{d[m,B]_t}{B_t} \right)$$

By the definition of tradeable, $\mathcal{D}(M)$ must be a $\mathbb{Q}$-martingale, but since M pays no dividends, this means that M/B must be a $\mathbb{Q}$-martingale. If we take $\psi_t = B_{t-}$ for all t, we thus have that

$$\mathbb{E}^{\mathbb{Q}}[m_T - m_t - \int_t^T \frac{d[m,B]_u}{B_u} \mid \mathcal{F}_t] \;=\; 0$$

and since $m_T = S_T$ we thus find the result

$$m_t \;=\; \mathbb{E}^{\mathbb{Q}}[S_T - \int_t^T \frac{d[m,B]_u}{B_u} \mid \mathcal{F}_t]$$

Write $m_t = \mathbb{E}^{\mathbb{Q}}[S_T \mid \mathcal{F}_t] - A_t$ then A_t has finite variation, so $[m,B] = [\mathbb{E}^{\mathbb{Q}}[S_T \mid \mathcal{F}_t], B]$ and the result follows. □

In many models it is assumed that B is continuous and of finite variation, and in this case we get the well-known result that $m_t = \mathbb{E}^{\mathbb{Q}}[S_T \mid \mathcal{F}_t]$.

More interesting is the case where the bank account B has quadratic variation. Let S and B be driven by Brownian Motions V and W with correlation coefficient ρ in a market that is completed by additional assets, i.e. under $\mathbb{Q}$ (the martingale measure for numeraire B) we have

$$\begin{aligned} dS_t/S_t &= rdt + \sigma^S dV_t \\ dB_t/B_t &= rdt + \sigma^B dW_t \end{aligned}$$

for known constants r, σ^S and σ^B. Then the futures price m equals

$$\begin{aligned} m_t &= \mathbb{E}^{\mathbb{Q}}[S_T - \int_t^T \frac{d[\mathbb{E}^{\mathbb{Q}}[S_T \mid \mathcal{F}_s], B]_s}{B_s} \mid \mathcal{F}_t] \\ &= S_t e^{r(T-t)} - \int_t^T \frac{d[S_s e^{r(T-s)}, B_s]_s}{B_s} \\ &= S_t e^{r(T-t)} - \int_t^T \frac{e^{r(T-s)}\sigma^S S_s \sigma^B B_s \rho \, ds}{B_s} \\ &= S_t e^{r(T-t)} - \rho\sigma^S\sigma^B e^{rT} \int_t^T \mathbb{E}^{\mathbb{Q}}\left[e^{-rs}S_s \middle| \mathcal{F}_t\right] ds \\ &= S_t e^{r(T-t)} - \rho\sigma^S\sigma^B e^{rT} \int_t^T e^{-rs} S_t e^{r(s-t)} ds \\ &= S_t e^{r(T-t)} \left[1 - \rho\sigma^S\sigma^B (T-t)\right] \end{aligned}$$

Note that the futures price may thus become negative for positively correlated S and B processes when the time to maturity is large! In practice, of course, the interest rate earned on a futures margin account is usually fixed or certainly of finite bounded variation. In that case, negative future prices will not occur for positive asset price processes S.

4.4 Credit Default Swaps: Stopped Jump Process for Dividends

We now consider credit default swaps, as an example of a filtration for the dividend process which is not a Brownian filtration. In particular, we would like to derive a trading strategy which allows us to hedge a position in credit default swaps using defaultable coupon bonds.

We will use the same setup as Bielecki, Jeanblanc, and Rutkowski (2005). Define on a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ a strictly positive stochastic variable τ and

$$\begin{aligned} E_t &= \mathbf{1}_{t \geq \tau} \\ p(t) &= \mathbb{E}^{\mathbb{Q}}(1 - E_t) = \mathbb{Q}(\tau > t) \end{aligned}$$

and let $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ be the filtration generated by the process E, then τ is obviously a stopping time with respect to this filtration. We assume that p is a continuous function on $\mathbb{R}_+$. In a market with a *deterministic* bond process B which is continuous and of finite variation, we define a credit default swap with maturity T for the default event τ as an asset S such that

$$S_T = 0, \qquad D_t^S = -A(t \wedge \tau \wedge T) + I(\tau)E_t$$

where A and I are deterministic continuous functions which represent a (cumulative) amount paid as long as there is no default, and an amount received upon default respectively. We assume that A is differentiable as well. The process

$$M_t = E_t + \int_0^t (1 - E_{u-}) \frac{dp(u)}{p(u)} \tag{22}$$

is a $\mathbb{Q}$-martingale on $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$, since a direct computation verifies that $\mathbb{E}^{\mathbb{Q}}[M_t - M_s|\mathcal{F}_s] = 0$ for all $0 \le s < t$:

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[E_t - E_s \mid \mathcal{F}_s] &= \mathbb{E}^{\mathbb{Q}}[\mathbf{1}_{s<\tau\le t} \mid \mathcal{F}_s] \;=\; \mathbf{1}_{s<\tau}\mathbb{E}^{\mathbb{Q}}[\mathbf{1}_{s<\tau\le t}] \;=\; \mathbf{1}_{s<\tau}(p(s)-p(t)) \\
\mathbb{E}^{\mathbb{Q}}[\int_s^t (1-E_{u-})\tfrac{dp(u)}{p(u)} \mid \mathcal{F}_s] &= \mathbf{1}_{s<\tau}\mathbb{E}^{\mathbb{Q}}[\mathbf{1}_{s<\tau}\int_s^t (1-E_{u-})\tfrac{dp(u)}{p(u)}] \\
&= \mathbf{1}_{s<\tau}\mathbb{E}^{\mathbb{Q}}[\mathbf{1}_{s<\tau}\mathbf{1}_{\tau\le t}\int_s^\tau \tfrac{dp(u)}{p(u)} + \mathbf{1}_{s<\tau}\mathbf{1}_{\tau>t}\int_s^t \tfrac{dp(u)}{p(u)}] \\
&= \mathbf{1}_{s<\tau}\int_s^t\int_s^v \tfrac{-dp(u)}{p(u)}dp(v) + p(t)\int_s^t \tfrac{dp(u)}{p(u)}] \;=\; \mathbf{1}_{s<\tau}(p(t)-p(s)).
\end{aligned}$$

If we want S to be a tradeable, equation (20) then gives for the ex-dividend price:

$$S_t \;=\; B_t\,\mathbb{E}^{\mathbb{Q}}\left[\frac{S_T}{B_T} + \int_t^T \frac{dD_u^S}{B_u}\middle|\mathcal{F}_t\right] \;=\; \mathbb{E}^{\mathbb{Q}}\left[\int_t^T \frac{-(1-E_{u-})dA(u) + I(u)dE_u}{B_u}\middle|\mathcal{F}_t\right]$$

Using standard results this can be rewritten as

$$\begin{aligned}
S_t &= B_t\,\frac{\mathbf{1}_{\tau>t}}{\mathbb{E}^{\mathbb{Q}}\mathbf{1}_{\tau>t}}\mathbb{E}^{\mathbb{Q}}\left[-\mathbf{1}_{\tau>t}\int_t^{\tau\wedge T}\frac{-dA(u)}{B_u} + \frac{I(\tau)}{B_\tau}\mathbf{1}_{t<\tau\le T}\right] \\
&= B_t L_t K(t) \qquad (23)
\end{aligned}$$

with $L_t = (1-E_t)/p(t)$ and

$$K(t) \;=\; -\int_t^T p(u)\frac{-dA(u)}{B_u} - \int_t^T I(u)\frac{dp(u)}{B_u}$$

a deterministic continuous process. The $\mathbb{Q}$-martingale $S/B + B^{-1}\bullet D^S$ can be represented in terms of M. We have

$$dL_t \;=\; \frac{d(1-E_t)}{p(t)} - \frac{(1-E_{t-})dp(t)}{p(t)^2} \;=\; \frac{-dM_t}{p(t)} \;=\; \frac{-dM_t(1-E_{t-})}{p(t)} \;=\; -L_t\,dM_t$$

and we find for $t \le \tau$, since $S_t = B_t L_t K(t)$,

$$\begin{aligned}
d\frac{S_t}{B_t} + \frac{dD_t^S}{B_t} &= d(L_tK(t)) + \frac{-(1-E_{t-})dA(t) + I(t)dE_t}{B_t} \\
&= K(t)dL_t + L_{t-}dK(t) + \frac{-(1-E_{t-})dA(t) + I(t)dE_t}{B_t} \\
&= -K(t)L_{t-}dM_t + \frac{I(t)}{B_t}(dE_t + (1-E_{t-})\frac{dp(t)}{p(t)}) \\
&= \frac{1}{B_t}(I(t) - S_{t-})dM_t \qquad (24)
\end{aligned}$$

We can use this to calculate how the Credit Default Swap can be hedged with a defaultable bond. We define the defaulable bond P with maturity T and known coupon payments C_i at known times $t_i \le T$ for $i = 1...n$ as a tradeable with

$$P_T = 0, \qquad D_t^P = \sum_{i=1}^n C_i\mathbf{1}_{t\ge t_i}\mathbf{1}_{\tau>t_i}.$$

Similar calculations as above then give the ex-dividend price as

$$P_t \;=\; B_t\,\mathbb{E}^{\mathbb{Q}}\left[\frac{P_T}{B_T} + \int_t^T \frac{dD_u^P}{B_u}\middle|\mathcal{F}_t\right] \;=\; B_t\,\frac{1-E_t}{p(t)}\sum_{i=1}^n C_i\frac{p(t_i)}{B_{t_i}}\mathbf{1}_{t<t_i}$$

and this allows us to write

$$P_t = B_t L_t \sum_{i=1}^{n} v_i (1 - R_t^i), \qquad D_t^P = \sum_{i=1}^{n} p(t_i) L_{t_i} R_t^i$$

with

$$v_i = C_i \, p(t_i)/B_{t_i}, \qquad R_t^i = \mathbf{1}_{t \geq t_i}$$

But then

$$d\frac{P_t}{B_t} + \frac{dD_t^P}{B_t} = \sum_{i=1}^{n} v_i d\left[(1 - R_t^i)L_t + L_{t_i} R_t^i\right]$$

and

$$\begin{aligned}
d\left[(1 - R_t^i)L_t + L_{t_i} R_t^i\right] &= dL_t + d[(L_{t_i} - L_t)R_t^i] = dL_t + L_{t_i}\Delta R_t^i - d(L_t R_t^i) \\
&= dL_t + L_{t_i}\Delta R_t^i - [L_{t-} dR_t^i + R_{t-}^i dL_t + L_t R_t^i - L_{t-} R_{t-}^i - L_{t-}\Delta R_t^i - R_{t-}^i \Delta L_t] \\
&= dL_t + L_{t_i}\Delta R_t^i - R_{t-}^i dL_t - L_t \Delta R_t^i \\
&= (1 - R_{t-})dL_t + (L_{t_i} - L_t)\Delta R_t^i = -(1 - R_{t-})L_{t-} dM_t + 0 \\
&= \frac{-\mathbf{1}_{t \leq t_i}\mathbf{1}_{t \leq \tau}}{p(t)} dM_t
\end{aligned}$$

so

$$d\frac{P_t}{B_t} + \frac{dD_t^P}{B_t} = \sum_{i=1}^{n} \frac{C_i}{B_{t_i}} \frac{p(t_i)}{p(t)} \mathbf{1}_{t \leq t_i} \mathbf{1}_{t \leq \tau} dM_t \tag{25}$$

and equations (24)-(25) thus show that if we want to replicate the Credit Default Swap using defaultable coupon bonds, the amount of bonds per swap to hold in our portfolio equals for $t \leq \tau$

$$\phi_t = \frac{I(t) - S_{t-}}{B_t} \left[\frac{1}{p(t)} \sum_{\{i : t_i \geq t\}} \frac{C_i p(t_i)}{B_{t_i}} \right]^{-1}.$$

4.5 Uncertain Dividend Dates: Jump-diffusion Processes

The results of the previous subsection concerning dividend processes generated by jumps at random times can also be used to model uncertainty in dividend dates. Suppose we have an ex-dividend stock price process S and a dividend of known magnitude $\tilde{D}$ which will be paid out at the unknown dividend date τ which has a known distribution p under an equivalent martingale measure $\mathbb{Q}$, i.e.

$$D_t^S = \tilde{D} E_t, \qquad E_t = \mathbf{1}_{t \geq \tau}, \qquad \mathbb{Q}(\tau > t) = p(t)$$

As before, we also have a bank account given by $B_t = B_0 e^{rt}$. Since we would like our model to have some tractability, we would like the process S to be adapted to $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$, the filtration generated by the process E and a standard Brownian Motion W. We define the process

$$V_t = S_t + B_t \int_0^t \frac{dD_u^S}{B_u}$$

and this should be a $\mathbb{Q}$-martingale after discounting if we want the stock to be a tradeable. Notice that we use a model here which is similar to the Forward model we mentioned in subsection 4.1 because the alternative, the Escrowed model which uses $V_t = S_t - \int_t^T dD_u^S/B_u$, is no longer adapted when the cumulative dividend process D^S is stochastic. If V/B is a $\mathbb{Q}$-martingale, then by

predictable representation theorems (see for example Protter (2003)) there must exist predictable processes A_t and J_t such that $d(V_t/B_t) = A_t dW_t + J_t dM_t$ where M_t has been defined in the previous subsection. Since $\Delta(V_t/B_t) = J_t \Delta M_t$ it is clear that $J_t = \tilde{D}/B_t$ so we find

$$\begin{aligned} d\frac{S_t}{B_t} &= d\frac{V_t}{B_t} - \frac{dD_t^s}{B_t} = A_t dW_t + \frac{\tilde{D}}{B_t} dM_t - \frac{\tilde{D}}{B_t} dE_t \\ &= A_t dW_t + \frac{\tilde{D}}{B_t}(1 - E_{t-})\frac{dp(t)}{p(t)} \end{aligned}$$

or

$$d(\frac{S_t}{B_t} - \tilde{D}\int_0^{t\wedge\tau} \frac{1}{B_u}\frac{dp(u)}{p(u)}) = A_t dW_t$$

To arrive at a tractable formula we *choose* A_t now in such a way that we can solve this SDE, which is certainly the case if the lefthand side becomes the increment of a lognormal process, i.e. we take

$$\frac{S_t}{B_t} - \tilde{D}\int_0^{t\wedge\tau} \frac{1}{B_u}\frac{dp(u)}{p(u)} = (\frac{S_0}{B_0} - 0)e^{\sigma W_t - \frac{1}{2}\sigma^2 t}$$

so the ex-dividend price then becomes

$$S_t = S_0 e^{(r-\frac{1}{2}\sigma^2)t + \sigma W_t} + \tilde{D}\int_0^{t\wedge\tau} \frac{B_t}{B_u}\frac{dp(u)}{p(u)}$$

If the martingale measure $\mathbb{Q}$ is unique, a vanilla call with payoff $(S_T - K)^+$ based on the ex-dividend price must then have the price

$$\begin{aligned} &\frac{B_t}{B_T}\mathbb{E}^{\mathbb{Q}}[(S_T - K)^+ \mid \mathcal{F}_t] \\ &= \frac{B_t}{B_T}\mathbb{E}^{\mathbb{Q}}[\left((S_t - \tilde{D}\int_0^{t\wedge\tau}\frac{B_t}{B_u}\frac{dp(u)}{p(u)})e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(W_T-W_t)} + \tilde{D}\int_0^{T\wedge\tau}\frac{B_T}{B_u}\frac{dp(u)}{p(u)} - K\right)^+ \mid \mathcal{F}_t] \end{aligned}$$

If one assumes that the processes W and E are independent, this leads to a European option pricing formula that can be written in terms of integrals of the Black-Scholes call option formula over different strike and stock values.

5 Conclusions

We have shown how dividends can be modeled consistently in arbitrage-free markets by the introduction of tradeable securities without dividends that can be replicated using underlying assets *with* dividends. We believe that our definition of what replication should mean in the presence of dividends provides a natural concept for the modelling of dividends, as witnessed by the many different examples given in the previous section.

The last example given there (where the dividend dates are uncertain) shows that we need to be careful when defining a model for the ex-dividend process if we want the combination of ex-dividend and dividend processes to be a tradeable in an abitrage-free market: it is obvious that when dividends are present, the ex-dividend process cannot be a martingale under an equivalent martingale measure after discounting. But once tradeables have been defined in a proper manner by reinvesting dividend proceeds, Theorem 3.4 shows that pricing and hedging problems can be addressed using the well-known tools of martingale representation theorems in stochastic calculus.

We therefore believe that our results may be of some interest when designing hedging strategies for financial products which include dividends or when designing hedging strategies that use securities that have dividend payoffs themselves in the hedge.

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