A Structural Model with Unobserved Default Boundary

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We consider a firm-value model similar to the one proposed by Black and Cox (1976) where additionally the firm value is allowed to jump and instead of assuming a constant and known default boundary, the default boundary is a random and unobserved process. This process has a Brownian component, reflecting the influence of uncertain effects on the precise timing of the default, and a jump component, which relates to abrupt changes in the policy of the company, exogenous events or changes in the debt structure. Interestingly, this setup admits a default intensity, so the reduced form methodology can be applied. We examine consequences for the relationship between equity and debt and consider the pricing of equity default swaps.

Keywords: structural model, equity default swaps, default boundary, jump-diffusion

1 Introduction

The seminal works Black and Scholes (1973) and Merton (1974) introduced the first structural models for describing the default risk[2] of companies. This paper belongs to the class of first-passage time models, pioneered by Black and Cox (1976), where default of a company is announced at the first time when the firm-value falls below a certain boundary. It has been shown by Leland and Toft (1996) that under certain assumptions this behaviour is optimal for the company owner. However, in these models it is a fundamental assumption that investors have complete information on the firm's value as well as on the default boundary. As in practice investors do not have complete information, there are several approaches which deal with this issue. All concentrate on first-passage time models. Duffie and Lando (2001) consider the case, where investors estimate the firm's value from noisy accounting reports. Coculescu, Geman, and Jeanblanc (2006) consider a model where investors observe a correlated index and Frey and Schmidt (2006) filter the firm-value from discretely observed news.

In contrast to these filtering approaches, there is a different branch of research where the investors have incomplete information of either firm's value or default barrier (or both), but no additional information. For example, Giesecke (2006) considers the case where the firm-value or default barrier (or both) may not be observed, while in Giesecke and Goldberg (2004) the firm value is observed, both in the case of a time-independent default barrier.

This paper extends to the case where the default barrier is allowed to be a stochastic process. Besides this, extending Zhou (2001), not only jumps in the firm-value but also jumps in the barrier are considered. Considering a constant debt level has serious drawbacks. If the firm value is observable, the default boundary necessarily must be smaller than the minimum of the firm value on the considered time interval. If the firm value is far above its minimum, this will lead to credit

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[2]For an introduction into credit risk we refer to the surveys by Giesecke (2004), Schmidt and Stute (2004) or one of the excellent textbooks Lando (2004), Schönbucher (2003), McNeil, Frey, and Embrechts (2005).

spreads which are unrealistically small. Considering a default barrier which is a stochastic process clearly remedies this. On the other side, structural models with a continuous firm value have difficulties in explaining short-term credit spreads. While some models almost overcome this, see for example Fouque, Sircar, and Sølna (2006), the model presented here is clearly able to solve this task as it has a default intensity and hence a positive credit spread for arbitrary small maturities.

The structure of the article is as follows. First, we formulate the problem in a quite general framework and consider several special cases thereafter. One is the case where firm value and debt level follow Brownian motions with linear drift. The analysis of the case where firm's value and debt level are geometric Brownian motions immediately follows from the Brownian case by taking logarithms. Thereafter, we consider a debt level which is allowed to jump. The jumps occur according to a Poisson process and the jump heights are random. We are also able to compute the default intensity in this framework. Further on, we consider the pricing of equity default swaps and compute the default intensity in several special cases of practical interest.

2 The general framework

Consider a structural model, where the firm value is denoted by the process $(V_t)_{t\geq 0}$. Following Black and Cox (1976), it is assumed that company owners declare bankruptcy, if the firm value falls below a certain boundary. This boundary is given by a stochastic process, denoted by $(D_t)_{t\geq 0}$ with $V_0 > D_0$. As default of the company is announced at the first time where V falls below D, the default time equals $\tau = \inf\{t \geq 0 : V_t \leq D_t\}$.

We always denote the inner filtration of a stochastic process, say V, by $\mathcal{F}^V$, i.e. $\mathcal{F}_t^V := \sigma(0 \leq s \leq t : V_s)$. If (V, D) is Markovian, the probability of $V - D$ not hitting zero in the interval $(t, T]$ given $\mathcal{F}_t^{V,D}$ can always be written as a function of t, T and $V_t - D_t$ and we set $H(V_t - D_t, t, T) := \mathbb{P}(\inf_{(t,T]}(V - D) > 0|\mathcal{F}_t^{V,D})$. Markovianity of (V, D) follows for example from Markovianity of V and D and independence. Another example is a two-dimensional Brownian motion with not necessarily independent components. First, we give results for general H and later on show how to compute H in several special cases.

The firm value is observable, but the default boundary is not. Investors also observe the default state of the company. The information available to investors is therefore represented by the filtration $\mathcal{G}_t := \sigma(s \in [0, t] : V_s, \mathbb{1}_{\{\tau > s\}})$.

Theorem 2.1. *Assume (V, D) is Markovian and denote the conditional distribution of D_t given $\mathcal{G}_t$ by $\mu_{D_t|\mathcal{G}_t} =: \mu_t^D$. Then,*

$$\mathbb{P}\big(\tau > T \mid \mathcal{G}_t\big) = \mathbb{1}_{\{\tau > t\}} \int_{-\infty}^{V_t} H(V_t - x, t, T)\, \mu_t^D(dx). \tag{1}$$

Proof. As $\mathbb{1}_{\{\tau > t\}}$ is measurable with respect to $\mathcal{G}_t$, it can be taken out. It remains to consider $\mathbb{1}_{\{\inf_{s\in(t,T]}(V_s - D_s) > 0\}}$. Set $A_t := \{\inf_{[0,t]}(V - D) > 0\}$. Then

$$\mathbb{P}\Big(\inf_{(0,T]}(V - D) > 0\big|\mathcal{F}_t^V, A_t\Big) = \mathbb{E}\Big(\mathbb{P}\Big(\inf_{s\in(t,T]}(V_s - D_s) > 0\big|\mathcal{F}_t^V \vee \mathcal{F}_t^D, A_t\Big)\big|\mathcal{F}_t^V, A_t\Big).$$

Because of the Markovian property the inner probability equals

$$\begin{aligned}\mathbb{P}\Big(\inf_{s\in(t,T]}(V_s - D_s) > 0\big|\mathcal{F}_t^V \vee \mathcal{F}_t^D, A_t\Big) &= \mathbb{P}\Big(\inf_{s\in(t,T]}(V_s - D_s) > 0\big|V_t, D_t\Big)\\ &= H(V_t - D_t, t, T).\end{aligned}$$

We therefore have

$$\mathbb{E}\Big(H(V_t - D_t, t, T)\big|\mathcal{F}_t^V, A_t\Big) = \int_{-\infty}^{V_t} H(V_t - x, t, T)\, \mu_t^D(dx),$$

as the integrand is zero on $\{D_t > V_t\}$. ■

For applications to credit risk it is an important question, if this model admits a default intensity.

Proposition 2.2. *Under the above assumptions and on $\{\tau > t\}$ the default intensity, if it exists, is given by*

$$\lambda_t = -\lim_{T\to t} \int_{-\infty}^{V_t} \frac{\partial}{\partial T} H(V_t - x, t, T)\, \mu_t^D(dx). \tag{2}$$

Proof. By results of Aven (1985) the default intensity equals, if it exists,

$$\lambda_t = -\frac{\partial}{\partial T}\Big|_{T=t} \ln \mathbb{P}(\tau > T \mid \mathcal{G}_t).$$

Considering the case where $\{\tau > t\}$ one finds that $\mathbb{P}(\tau > t|\mathcal{G}_t) = 1$, and therefore

$$\begin{aligned}
\mathbb{1}_{\{\tau>t\}}\lambda_t &= -\mathbb{1}_{\{\tau>t\}} \frac{\partial}{\partial T}\Big|_{T=t} \mathbb{P}\big(\tau > T \mid \mathcal{G}_t\big) \\
&= -\mathbb{1}_{\{\tau>t\}} \lim_{T\to t} \lim_{h\to 0} \int_{-\infty}^{V_t} \frac{H(V_t - x, t, T+h) - H(V_t - x, t, T)}{h}\, \mu_t^D(dx).
\end{aligned}$$

Recall, that $H(\cdot, t, T)$ is the probability of not hitting in the interval $(t, T]$ and therefore for $h > 0$ we have that $H(\cdot, t, t+h) \le H(\cdot, t, t)$. Using monotone convergence we conclude that

$$\mathbb{1}_{\{\tau>t\}}\lambda_t = -\mathbb{1}_{\{\tau>t\}} \lim_{T\to t} \int_{-\infty}^{V_t} \frac{\partial}{\partial T} H(V_t - x, t, T)\, \mu_t^D(dx).$$ ■

In the last equation it is typically not allowed to interchange limit and integration, as will be seen in the later examples. In the case where D is simply a constant random variable and V is a geometric Brownian motion an intensity does not exist, as shown in Giesecke (2006). Responsible for this is that default can only happen when V is at its running minimum. Therefore the compensator of $\mathbb{1}_{\{\tau>t\}}$ will not be absolutely continuous and hence its derivative, the default intensity, will not exist.

In the following case we consider the case where D and V are both jump-diffusions and as will be noted in Remark 3.5, then a default intensity always exists.

Remark 2.3. Models where D is a constant have difficulties if the firm's value decreases and thereafter rises substantially, such that V is far above its running minimum. In this case credit spreads for small maturities are too small, as may be seen in Figure 2 in Giesecke (2006). Letting D be a stochastic process, and in particular one admitting upward jumps, clearly helps to overcome this. In particular, this is also reflected in the existence of a default intensity of the latter models.

3 The Brownian case

In this section we derive the conditional distribution of D given V and $\mathbb{1}_{\{\tau>t\}}$. Then we draw an interesting connection to Doob's h-Transform[3] and the survival measure.

We assume that V and D are two independent Brownian motions. Denote the probability that a Brownian motion stays below a given function $v : [0,t] \mapsto \mathbb{R}$ in the interval $[s,t]$ by $N(v,s,t)$. We will show later on how to derive this probability approximately. Denote by δ_t the dirac-function in t, i.e. $\delta_t(s) = \mathbb{1}_{\{s=t\}}$ and recall that $A_t = \{\inf_{[0,t]}(V-D) > 0\}$. Because of the independence of V and D we have

$$\mathbb{P}\Big(D_t \le x \mid \mathcal{F}_t^V, A_t\Big) = \frac{\mathbb{P}\big(D_t \le x, A_t|\mathcal{F}_t^V\big)}{\mathbb{P}\big(A_t|\mathcal{F}_t^V\big)} = \begin{cases} 1 & \text{for } x > V_t \\ \frac{N(V-(V_t-x)\delta_t,0,t)}{N(V,0,t)} & \text{otherwise.} \end{cases}$$

The above probability is similar to a change to the survival measure which has been discussed in Collin-Dufresne, Goldstein, and Hugonnier (2004) and Schönbucher (2004). More precisely we show that this is the case conditional on $V = v$, and under this survival measure D is a Brownian motion with drift. Starting point is

$$\mathbb{P}\Big(D_t \le x \mid \mathcal{F}_t^V, A_t\Big) = \frac{\mathbb{P}\big(D_t \le x, A_t|\mathcal{F}_t^V\big)}{\mathbb{P}\big(A_t|\mathcal{F}_t^V\big)} = \mathbb{E}\Big(\frac{\mathbb{1}_{A_t}}{\mathbb{P}(A_t|\mathcal{F}_t^V)}\mathbb{1}_{\{D_t\le x\}} \mid \mathcal{F}_t^V\Big)$$

In the following we aim at computing the factorization, ie

$$\mathbb{E}\Big(\frac{\mathbb{1}_{\{\inf_{[0,t]}(v-D)>0\}}}{\mathbb{P}(A_t|V=v)}\mathbb{1}_{\{D_t\le x\}} \mid V = v\Big) = \mathbb{E}^*\Big(\mathbb{1}_{\{D_t\le x\}} \mid V = v\Big), \tag{3}$$

where the last expectation $\mathbb{E}^*$ is computed under the probability measure $\mathbb{P}^*$, to be defined as follows: set $A_t^v := \{\inf_{[0,t]}(v-D) > 0\}$. Then we define $\mathbb{P}^*$ by

$$d\mathbb{P}^* = \frac{\mathbb{1}_{A_t^v}}{\mathbb{P}(A_t^v)}\, d\mathbb{P}.$$

Note that this is not an equivalent change of measure. Under $\mathbb{P}^*$, the event that a default occurs prior to t has the probability zero (conditionally on $V = v$). The likelihood process of the measure transform is given by

$$L_s = \mathbb{E}\Big(\frac{\mathbb{1}_{A_t^v}}{\mathbb{P}(A_t^v)} \mid \mathcal{F}_s^D\Big) = \mathbb{1}_{A_s}\frac{1}{\mathbb{P}(A_t^v)}\,\mathbb{E}\Big(\inf_{[s,t]}(v-D) > 0 \mid \mathcal{F}_s^D\Big).$$

As the Brownian motion has independent increments, the likelihood process can be written as

$$dL_s = \mathbb{1}_{A_s}\frac{1}{N(v,0,t)}\,N\big(v-D_s,s,t\big) =: \mathbb{1}_{A_s}h(s,D_s).$$

Here $v - D_s$ is short for the function given by $u \mapsto v_u - D_s$. The function $h(s,x)$ denotes the h-transform. By Equation (39.4) in Rogers and Willams (1994) the Brownian motion D under $\mathbb{P}^*$ has the drift

$$\frac{\partial}{\partial x}\ln h(t,x) = \frac{1}{N(v-D_s,s,t)}\frac{\partial}{\partial x}N\big(v-D_s,s,t\big).$$

[3] See Rogers and Willams (1994).

3.1 Approximation in the Brownian case

In the structural model considered by Black and Cox (1976) the firm value follows a geometric Brownian motion. Taking logarithms, one arrives at a Brownian motion with drift where default refers to hitting an affine barrier. To generalize this model to a random barrier we therefore need to consider a Brownian motion with drift hitting a second Brownian motion with possibly different drift.

Denote by B^V a Brownian motion and assume $V = v_0 + \sigma_V B^V$ with a volatility $\sigma_V > 0$ and $v_0 > 0$. The default boundary is assumed to be a Brownian motion with drift mt, so that $D_t = \sigma_D B_t + mt$ for $\sigma_D > 0$ and a Brownian motion $(B_t)_{t\geq 0}$ which is assumed to be independent of V and w.l.o.g. starting at zero. Recall, that a default happens at the first time V hits D, i.e. $\tau = \inf\{s \geq 0 : V_s = D_s\}$. As previously, the information of the investor is $\mathcal{G}_t = \sigma\big(s \in [0,t] : V_s, \mathbb{1}_{\{\tau > s\}}\big)$. We are interested in the default probability

$$\mathbb{P}\big(\tau > T \mid \mathcal{G}_t\big) = \mathbb{1}_{\{\tau>t\}}\mathbb{P}\big(\tau > T \mid \mathcal{F}_t^V, \inf_{[0,t]}(V-D) > 0\big).$$

Because a Brownian motion has independent increments and its boundary crossing probabilities are well known for affine barriers, one easily reaches the following result.

Proposition 3.1. *Denote the distribution of D_t conditioned on $\mathcal{G}_t$ by μ_t^D. Then, with $\sigma := \sqrt{\sigma_V^2 + \sigma_D^2}$,*

$$\mathbb{P}\big(\tau > T \mid \mathcal{G}_t\big) = \mathbb{1}_{\{\tau>t\}} \int_{-\infty}^{V_t} H(V_t - x, T-t, \sigma)\, \mu_t^D(dx),$$

where H is given by

$$H(y,t,\sigma) := \Phi\Big(\frac{y-mt}{\sigma\sqrt{t}}\Big) - e^{\frac{2my}{\sigma^2}}\Phi\Big(-\frac{y+mt}{\sigma\sqrt{t}}\Big). \tag{4}$$

Proof. According to Theorem 2.1 we have to derive the conditional probability of $V-D$ not hitting zero in the interval $[t,T]$. We therefore consider

$$\begin{aligned}\big\{\inf_{(t,T]}(V-D) > 0\big\} &= \big\{\inf_{s\in(t,T]}\big(V_s - V_t - (D_s - D_t)\big) > -(V_t - D_s)\big\}\\ &= \big\{\inf_{s\in(t,T]}\big(\sigma_V(B_s^V - B_t^V) - m(s-t) - \sigma_D(B_s - B_t)\big) > -(V_t - D_s)\big\}\end{aligned}$$

Note, that $\sigma_V B^V - \sigma_D B$ is again a Gaussian process, which is equivalent (in distribution) to σB. As Brownian motions have independent and stationary increments, the conditional probability of this event can be derived using Corollary A.2. We obtain

$$\mathbb{P}\Big(\inf_{(t,T]}(V-D) > 0 \Big| V_t, D_t\Big) = \Phi\Big(\frac{-mt + (V_t - D_t)}{\sqrt{\sigma(T-t)}}\Big) - e^{\frac{2m(V_t - D_t)}{\sigma^2}}\Phi\Big(\frac{-mt - (V_t - D_t)}{\sigma\sqrt{(T-t)}}\Big)$$

and the conclusion follows. ∎

Hence, to compute the default probability it is sufficient to compute the conditional distribution of $V_t - D_t$, i.e. μ^D.

3.1.1 The conditional distribution

Next, we derive an approximation of the conditional distribution of D. Note that the structure of the observation is quite complicated. Essentially we observe a realization of V, which will be denoted by $v : [0,t] \mapsto \mathbb{R}$ and that D stayed below v in the whole interval $[0,t]$.

Instead of considering arbitrary continuous v we will approximate the observation by a piecewise constant function, $\bar{v}$. To simplify notation, we assume that the grid on which $\bar{v}$ is constant is equally spaced and therefore divides the interval $[0,t]$ in n intervals of length Δ. Then $\bar{v}$ can be written as follows:

$$\bar{v}(t) = \sum_{i=1}^{n} a_{i-1} \mathbb{1}_{\{t \in ((i-1)\Delta, i\Delta]\}}. \tag{5}$$

For example, we could set $a_i := v(i\Delta)$. We first derive the probability of D staying below the approximating boundary, $\bar{v}$, and thereafter give a result which shows that for suitably approximated boundaries the limit will yield the desired probability. Note that we assumed that V starts in v_0, such that we have $a_0 = v_0$. To determine the conditional distribution we look at the cumulative distribution function, and for $x < V_t$

$$\mathbb{P}\Big(D_t \le x \mid \mathcal{F}_t^V, \tau > t\Big) = \frac{\mathbb{P}\Big(D_t \le x, \inf_{[0,t]}(V-D) > 0 \mid \mathcal{F}_t^V\Big)}{\mathbb{P}\Big(\inf_{[0,t]}(V-D) > 0 \mid \mathcal{F}_t^V\Big)}. \tag{6}$$

First, we consider the numerator and then easily conclude for the denominator. We need some notation. Denote by $N(\bar{v}, t)$ the probability that $(\sigma_D B_s + ms)_{s \ge 0}$ stays below the barrier $\bar{v}$ in the whole interval $[0,t]$. Define $c := 2/\sqrt{\Delta} \cdot \exp(-m^2\Delta/2\sigma_D^2)$, set $S_n := \sum_{i=1}^n y_i$ and

$$g(a,b) := \phi\Big(\frac{a}{\sigma_D\sqrt{\Delta}}\Big) - \phi\Big(\frac{2b-a}{\sigma_D\sqrt{\Delta}}\Big).$$

Furthermore, we write $\mathbf{y}$ for the vector $(y_1, \dots, y_{n-1})$ and set

$$I(\mathbf{y}, \bar{v}) := \mathbb{1}_{\{y_1 < a_1, \dots, S_{n-1} < a_{n-1}\}} c^{n-1} e^{\frac{m S_{n-1}}{\sigma_D^2}} \cdot g\big(y_{n-1}, S_{n-2} - a_{n-2}\big) \cdots g\big(y_1, -a_0\big).$$

Theorem 3.2. *For a piecewise constant default barrier $\bar{v}$ with representation* (5) *we have that*

$$\begin{aligned} N(\bar{v}, t) = &\int_{-a_0}^{\infty} \cdots \int_{S_{n-2}-a_{n-2}}^{\infty} I(\mathbf{y}, \bar{v}) \\ &\Big[\Phi\Big(\frac{a_{n-1} - S_{n-1} - m\Delta}{\sigma_D\sqrt{\Delta}}\Big) - e^{2m(a_{n-1}-S_{n-1})/\sigma_D^2} \Phi\Big(\frac{S_{n-1} - a_{n-1} - m\Delta}{\sigma_D\sqrt{\Delta}}\Big)\Big] \, d\mathbf{y}, \end{aligned} \tag{7}$$

where $d\mathbf{y}$ is short for $dy_1 \dots dy_{n-1}$. Furthermore, μ_t^D admits the density $f_{D_t|\bar{v}}(x)$, given by

$$\begin{aligned} f_{D_t|\bar{v}}(x) = &\frac{1}{N(\bar{v}, t)} \int_{-a_0}^{\infty} \cdots \int_{S_{n-2}-a_{n-2}}^{\infty} I(\mathbf{y}, \bar{v}) \\ &\frac{1}{\sigma_D\sqrt{\Delta}} \Big[\phi\Big(\frac{x - S_{n-1} - m\Delta}{\sigma_D\sqrt{\Delta}}\Big) - e^{2m(a_{n-1}-S_{n-1})/\sigma_D^2} \phi\Big(\frac{x + S_{n-1} - 2a_{n-1} - m\Delta}{\sigma_D\sqrt{\Delta}}\Big)\Big] \, d\mathbf{y}. \end{aligned} \tag{8}$$

By the way, for $\sigma_D = 1$ and $m = 0$, N is the probability that a Brownian motion stays below the piecewise constant barrier written as $\bar{v} = \sum a_{i-1} \mathbb{1}_{(t_{i-1}, t_i]}$ on the whole interval $[0,t]$.

The above results immediately leads to the following approximation of the survival probability:

$$P\big(\tau > T \mid A_t, V = \bar{v} \text{ on } [0,t]\big) = \int_{-\infty}^{V_t} H(V_t - x, T - t, \sigma_D) f_{D_t|\bar{v}}(x)\, dx. \tag{9}$$

Proof. As V and B are independent, the numerator in (6) is a measurable function of $(V)_{[0,t]}$, which we will denote by $F_x(V)$. For a $\bar{v}$ with representation (5) we have that

$$F_x(\bar{v}) = \mathbb{P}\Big(D_t \le x, \inf_{[0,t]}(\bar{v} - D) > 0\Big).$$

If $x > a_{n-1}$ the set $\{D_t \le x\}$ is included in $\tilde{A}_t := \{\inf_{[0,t]}(\bar{v} - D) > 0\}$. With a slight abuse of notation we write $\tilde{A}_k$ for $\tilde{A}_{k\Delta}$ (and similarly $\mathcal{F}_k$ and D_k). The above probability can be computed recursively. If D stays below the boundary $\bar{v}$ on $[0, k\Delta]$ it does stay below $\bar{v}$ on $[0,(k-1)\Delta]$ and on $[(k-1)\Delta, k\Delta]$ (and vice versa). Therefore

$$\tilde{A}_k = \tilde{A}_{k-1} \cap \Big\{ \inf_{((k-1)\Delta, k\Delta]} -D > -a_{k-1} \Big\}.$$

We start with

$$F_x(\bar{v}) = \mathbb{E}\Big(\mathbb{1}_{\tilde{A}_{n-1}} \mathbb{P}\Big[D_t \le x, \inf_{((n-1)\Delta, n\Delta]} -D > -a_{n-1} \;\Big|\; \mathcal{F}^D_{n-1} \Big]\Big).$$

Fist, consider the inner probability. Its factorization (where we replace D_{n-1} by d) is equal to

$$\mathbb{P}\Big[D_\Delta \le x - d, \inf_{s \in (0,\Delta]} -D_s > -(a_{n-1} - d) \Big] =: P(d, x).$$

Notice, that if $d > a_{n-1}$ this probability equals zero.

We will apply the formulas on boundary crossing probabilities from the appendix to calculate $P(d,x)$. Write $\tilde{m} := m/\sigma_D^2$. With Corollary A.2 we obtain for $d < a_{n-1}$

$$\begin{aligned} P(d,x) &= \mathbb{P}\Big[-D_\Delta > -(x-d), \inf_{(0,\Delta]} -D > -(a_{n-1} - d) \Big] \\ &= \Phi\left(\frac{-m\Delta + (x \wedge a_{n-1}) - d}{\sigma_D \sqrt{\Delta}} \right) - e^{2\tilde{m}(a_{n-1} - d)} \Phi\left(\frac{(x \wedge a_{n-1}) + d - 2a_{n-1} - m\Delta}{\sigma_D \sqrt{\Delta}} \right). \end{aligned} \tag{10}$$

Coming back to F, we found

$$F(\bar{v}) = \mathbb{E}\Big(\mathbb{1}_{\tilde{A}_{n-1}} P(D_{n-1}, x) \Big) = \mathbb{E}\Big[\mathbb{1}_{\tilde{A}_{n-2}} \mathbb{E}\Big(\mathbb{1}_{\{\inf_{((n-2)\Delta,(n-1)\Delta]} D > -a_{n-2}\}} P(D_{n-1}, x) \mid \mathcal{F}^D_{n-2} \Big)\Big].$$

Set $E_{n-1}(d,x) := \mathbb{1}_{\{d < a_{n-1}\}} P(d,x)$ and for $k \in \{0, \dots, n-2\}$

$$E_k(d,x) := \mathbb{E}\Big(\mathbb{1}_{\{\inf_{(0,\Delta]} D > -(a_k - d)\}} E_{k+1}(d + D_\Delta, x) \Big).$$

Then

$$\begin{aligned} F(\bar{v}) &= \mathbb{E}\Big[\mathbb{1}_{\tilde{A}_{n-2}} E_{n-2}(D_{n-2}, x) \Big] = \mathbb{E}\Big[\mathbb{1}_{\tilde{A}_{n-3}} E_{n-3}(D_{n-3}, x) \Big] \\ &= \mathbb{E}\Big[\mathbb{1}_{\tilde{A}_1} E_1(D_1, x) \Big] = E_0(0, x). \end{aligned}$$

The common density of a Brownian motion and its minimum will lead to a formula for E_k. As we primarily consider the probability of D being below a step function over the time interval $[0,t]$ which is divided into n subintervals, it is clear that we have to compute the probability of an

n-dimensional Gaussian vector with dependent coordinates. In Corollary A.3 we state this density and therewith obtain the following formula

$$\begin{aligned}
E_k(d,x) &= \int_{d-a_k}^{0} \int_{z}^{\infty} E_{k+1}(d+y,x) f_{M,B}(z,y)\,dy\,dz \\
&= \int_{d-a_k}^{0} \int_{z}^{\infty} E_{k+1}(d+y,x) \frac{2(y-2z)}{(\sigma_D\sqrt{\Delta})^3}\, e^{2z\tilde{m}}\, \phi\left(\frac{m\Delta+2z-y}{\sigma_D\sqrt{\Delta}}\right) dy\,dz \\
&= \int_{d-a_k}^{\infty} E_{k+1}(d+y,x) \int_{d-a_k}^{y\wedge 0} \frac{2(y-2z)}{(\sigma_D\sqrt{\Delta})^3}\, e^{2z\tilde{m}}\, \phi\left(\frac{m\Delta+2z-y}{\sigma_D\sqrt{\Delta}}\right) dz\,dy.
\end{aligned}$$

Observe that $E_k(d,x)=0$ for $d-a_k>0$. For the inner integral,

$$\begin{aligned}
&\int_{d-a_k}^{y\wedge 0} \frac{2(y-2z)}{(\sigma_D\sqrt{\Delta})^3}\, e^{2z\tilde{m}}\, \phi\left(\frac{m\Delta+2z-y}{\sigma_D\sqrt{\Delta}}\right) dz \\
&\qquad = \frac{1}{(\sigma_D\sqrt{\Delta})^3}\, e^{2z\tilde{m}}\, \phi\left(\frac{m\Delta+2z-y}{\sigma_D\sqrt{\Delta}}\right)\Big|_{d-a_k}^{y\wedge 0} \\
&\qquad = \frac{2}{\sigma_D\sqrt{\Delta}} e^{-\frac{m^2\Delta}{2\sigma_D^2}+y\tilde{m}}\, \phi\Big(\frac{2z-y}{\sigma\sqrt{\Delta}}\Big)\Big|_{d-a_k}^{y\wedge 0}. \qquad (11)
\end{aligned}$$

Because of the symmetry of ϕ we obtain

$$\phi\left(2y\wedge 0-y\right)=\phi\left(y\right),$$

and thus have

$$(11) = \frac{2}{\sigma_D\sqrt{\Delta}} e^{-\frac{m^2\Delta}{2\sigma_D^2}+y\tilde{m}} \left[\phi\left(\frac{y}{\sigma_D\sqrt{\Delta}}\right) - \phi\left(\frac{2(d-a_k)-y}{\sigma_D\sqrt{\Delta}}\right)\right] = c\,e^{y\tilde{m}}\, g(y, v-a_k),$$

while for E_k this yields

$$E_k(d,x) = \mathbb{1}_{\{d<a_k\}} \int_{d-a_k}^{\infty} E_{k+1}(d+y,x)\, c\, e^{y\tilde{m}}\, g(y,v-a_k)\,dy.$$

Recursively applied, we obtain

$$\begin{aligned}
E_0(0,x) &= \int_{-a_0}^{\infty} E_1(y_1,x) c e^{y_1\tilde{m}}\, g(y_1,-a_0)\,dy_1 \\
&= \int_{-a_0}^{\infty} \int_{y_1-a_1}^{\infty} \mathbb{1}_{\{y_1<a_1\}} c^2 e^{(y_2+y_2)\tilde{m}} E_2(y_1+y_2,x) g(y_2,y_1-a_1) g(y_1,-a_0)\,dy_2\,dy_1 \\
&= \int_{-a_0}^{\infty} \cdots \int_{S_{n-2}-a_{n-2}}^{\infty} \mathbb{1}_{\{y_1<a_1,\ldots,S_{n-2}<a_{n-2}\}} \\
&\qquad\qquad \cdot c^{n-1} e^{\tilde{m}S_{n-1}} E_{n-1}(S_{n-1},x)\, g(y_{n-1}, S_{n-2}-a_{n-2}) \cdots g(y_1,-a_0)\,d\mathbf{y}.
\end{aligned}$$

As we have $E_{n-1}(d,x) = P(d,x)$, inserting (10) and using the definition of I leads to

$$E_0(0,x) = \int_{-a_0}^{\infty} \cdots \int_{S_{n-2}-a_{n-2}}^{\infty} I(\mathbf{y}_1, \bar{v}) \mathbb{1}_{\{S_{n-1}<a_{n-1}\}}$$
$$\Big[\Phi\Big(\frac{-m\Delta - S_{n-1} + x \wedge a_{n-1}}{\sigma_D\sqrt{\Delta}}\Big) - e^{2\tilde{m}(a_{n-1}-S_{n-1})}\Phi\Big(\frac{S_{n-1} - 2a_{n-1} - m\Delta + x \wedge a_{n-1}}{\sigma_D\sqrt{\Delta}}\Big)\Big] d\mathbf{y}.$$

To put everything together, we have to look back at Equation (6). For the denominator we can proceed as previously. Note that the only difference occurs in the formulation of P, which is for the denominator

$$\tilde{P}(d) = \mathbb{1}_{\{d \le a_{n-1}\}} \left[\Phi\Big(\frac{-m\Delta + a_{n-1} - d}{\sigma_D\sqrt{\Delta}}\Big) - e^{2\tilde{m}(a_{n-1}-d)}\Phi\Big(\frac{-m\Delta - a_{n-1} + d}{\sigma_D\sqrt{\Delta}}\Big)\right].$$

Therefore we arrive at the desired expression for $N(\bar{v}, t)$.

For the approximation of the conditional density of D we differentiate (6) w.r.t. x and obtain zero for $x > a_{n-1}$ and on $\{x \le a_{n-1}\}$ the given expression for f. ■

It remains to be shown, that the approximating probability $N(\bar{v}, t)$ is close to the target probability, $N(v, t)$. Surprisingly, a quite crude approximation already shows that for $\| \bar{v} - v \| := \sup_{s\in[0,t]}(\bar{v}(s) - v(s)) \to 0$, the considered probabilities converge.

Proposition 3.3. *Consider a piecewise linear function $\bar{v}$ with representation* (5) *and assume that $\| v - \bar{v} \| \le \varepsilon$ for a $\varepsilon > 0$. Then*

$$|N(v,t) - N(\bar{v},t)| \le 1 - 2\Phi\Big(\frac{-2\varepsilon}{\sigma_D\sqrt{t}}\Big).$$

Note that the r.h.s. converges to zero with $\varepsilon \to 0$.

Proof. According to the representation of $\bar{v}$, (5), the interval $[0,t]$ is divided into n intervals of the length Δ. As $\bar{v}$ is a piecewise constant function and $\| v - \bar{v} \| \le \varepsilon$, we have that

$$\sup_{((i-1)\Delta, i\Delta]} v - \inf_{((i-1)\Delta, i\Delta]} v \le 2\varepsilon, \qquad i = 1, \dots, n.$$

For $s \in ((i-1)\Delta, i\Delta]$ set $g^*(s) := \sup_{((i-1)\Delta, i\Delta]} v + \epsilon$ and $g^*(s) := \inf_{((i-1)\Delta, i\Delta]} v - \epsilon$, respectively. Then any piecewise linear function $\tilde{v}$ with $\| \tilde{v} - v \| \le 2\epsilon$ lies in between g_* and g^*, where g_* and g^* are again piecewise linear and $\sup_{s\in((i-1)\Delta, i\Delta), i=1,2,\dots,n}(g^*(s) - g_*(s)) \le 4\epsilon$. Furthermore,

$$|N(v,t) - N(\bar{v},t)| \le |N(g^*,t) - N(g_*,t)|. \tag{12}$$

The probability on the right hand side refers to the probability of $(D_s)_{s\in[0,t]} = (ms + \sigma_D B_s)_{s\in[0,t]}$ staying in a sequence of pipes with width 4ϵ which are shifted at each grid point. This probability is smaller than the probability of $(\sigma_D B_s)_{s\in[0,t]}$ staying in $\{(x,y) : x \in [0,t], |y| \le 2\epsilon\}$. Hence,

$$(12) \le \mathbb{P}\Big(\sup_{[0,t]}(\sigma_D B) \le 2\varepsilon, \inf_{[0,t]}(\sigma_D B) \ge -2\varepsilon\Big) \le \mathbb{P}\Big(\inf_{[0,t]}(\sigma_D B) \ge -2\varepsilon\Big) = 1 - 2\Phi\Big(\frac{-2\varepsilon}{\sigma_D\sqrt{t}}\Big),$$

using Corollary A.2 for the last equality. This shows the result. ■

Finally, the question is if for any $\varepsilon > 0$ and a realization v of the Brownian motion a piecewise linear approximation $\bar{v}$ with $\| v - \bar{v} \|$ may be found. This is indeed the case. By Lévy 's construction of Brownian motion it is known that v can be uniformly approximated by piecewise linear functions. Thus, there exists a piecewise linear $\tilde{v}$, s.t. $\| v - \tilde{v} \| < \varepsilon/2$. Moreover, it is clear that for a piecewise linear function on $[0,t]$ with a finite number of kinks there exists a piecewise constant function $\bar{v}$, s.t. $\| \tilde{v} - \bar{v} \| < \varepsilon/2$ and so for this $\bar{v}$, $\| v - \bar{v} \| < \varepsilon$.

3.2 The default intensity

In this section we examine the question if a non-zero default intensity exists in our framework. As already pointed out, it is sufficient to differentiate the survival probability. Not surprisingly, we obtain a nonzero intensity in this setup. After obtaining the intensity, pricing and hedging of securities can be done using the standard framework for intensity based models, as may be found e.g. in Bielecki and Rutkowski (2002). Recall the definitions prior to Theorem 3.2.

Proposition 3.4. *In the case of a piecewise constant observation boundary $\bar{v}$ the default intensity is given by*

$$\lambda_t = \lambda_t(\bar{v}) = \int_{-a_0}^{\infty} \cdots \int_{S_{n-2}-a_{n-2}}^{\infty} \frac{I(\mathbf{y},\bar{v})}{2\Delta^{3/2}N(\bar{v},t)} \Big[\Big(S_{n-1} - V_t - 2m\Delta\Big)\phi\Big(\frac{V_t - S_{n-1} + m\Delta}{\sigma\sqrt{\Delta}}\Big) + \Big(V_t + S_{n-1} - 2a_{n-1}\Big) e^{2m\frac{a_{n-1}-S_{n-1}}{\sigma\sqrt{\Delta}}} \phi\Big(\frac{V_t + S_{n-1} - 2a_{n-1} - m\Delta}{\sigma\sqrt{\Delta}}\Big)\Big] \, d\mathbf{y}$$

Proof. As we want to apply Proposition 2.2 we need to derive H. We obtain

$$\begin{aligned} \frac{\partial}{\partial T} H(x,T,\sigma) &= -\frac{x+mT}{2\sigma T^{3/2}}\phi\Big(\frac{x-mT}{\sigma\sqrt{T}}\Big) + e^{\frac{2mx}{\sigma^2}} \frac{-x+mT}{2\sigma T^{3/2}}\phi\Big(\frac{-x-mT}{\sigma\sqrt{T}}\Big) \\ &= -\frac{x}{\sigma T^{3/2}}\,\phi\Big(\frac{x-mT}{\sigma\sqrt{T}}\Big). \end{aligned}$$

For the intensity, we obtain

$$\begin{aligned} \lambda_t &= -\lim_{T\to t} \int_{-\infty}^{V_t} \frac{\partial}{\partial T} H(V_t - x, T-t, \sigma) f_{D_t|\bar{v}}(x)\, dx \\ &= \lim_{T\to t} \frac{1}{\sigma(T-t)^{3/2}} \int_0^{\infty} x\,\phi\Big(\frac{x - m(T-t)}{\sigma\sqrt{T-t}}\Big) f_{D_t|\bar{v}}(V_t - x)\, dx. \end{aligned} \tag{13}$$

At this point Lemma A.5 enables us to directly conclude for the several limits. Inserting the form for the conditional density (8) and consequently applying Lemma A.5 yields the desired form of the intensity. ■

The default intensity allows to apply the highly developed machinery available for reduced form models to our setting. However, the above formulation of the default intensity is quite complicated, such that computations directly in the structural model will be preferable, compare for example the computation of the survival probability in (9).

Remark 3.5. From the above Proposition it immediately follows, that as long as $\sigma_D > 0$, a default intensity exists. Indeed, setting $v^* := \sup_{[0,t]} V$ and $v_* := \inf_{[0,t]} V$ one immediately obtains upper and lower bounds in the case $v_* > 0$ for $\lambda_t(V)$ by $\lambda_t(v^*)$ and $\lambda_t(v_*)$. If $v_* \leq 0$, there exists an $\varepsilon > 0$, s.t. $v_\varepsilon := \inf_{[0,\varepsilon)} V > 0$ and one obtains a lower boundary using $v_{**}(s) := \mathbb{1}_{\{0\leq s<\varepsilon\}} v_\varepsilon + \mathbb{1}_{\{s\geq\varepsilon\}} v_*$. It is clear, that if an default intensity exists in the Brownian case, it also exists after introducing jumps, compare Section 5.2.2.

4 Including Jumps

Up to now, the unobserved default boundary was not allowed to jump. For the applications to credit risk this is a serious restriction. In the following we relax the conditions and allow the underlying

process to have jumps. The jump times are obtained as the jump times of a Poisson process and the jump sizes are assumed to be independent and identically distributed. It is straightforward to allow for non identically distributed jump sizes, but at a certain notational burden, so we do not include this here. The same applies for a time inhomogeneous intensity or for considering a Cox process instead of the Poisson process. Consider first the case, where D is a compound Poisson process M.

Assumption 4.1. *Consider a Poisson process $\tilde{N}$ with intensity l and jump times $(\tau_i)_{i\geq 1}$. The jump sizes $(J_i)_{i\geq 1}$ are independent, identically distributed (i.i.d.) random variables with cumulative distribution function F_J and have finite expectations. The considered compound Poisson process is*

$$D_t = M_t := \sum_{\tau_i \leq t} J_i, \qquad t \geq 0.$$

4.1 First hitting time distribution

We derive the probability of a Brownian motion B hitting M in the time interval $[0, T]$.

Denote the distribution given through $F_J(dj_1) \cdots F_J(dj_k)\, du_1 \cdots du_k$ by $\mu^k_{J,u}$. This is the distribution of the jumps and the jump times, conditioned on having k jumps and therefore operates on $\mathbb{R}^k \times [0,1]^k$. We write short $d\mu^k_{J,u}(m^k)$ for $\mu^k_{J,u}(dj1, \ldots, dj_k, du_1, \ldots, du_k)$.

Proposition 4.2. *Under assumption 4.1, the following holds for any $x < B_0$:*

$$\mathbb{P}\big(\inf_{[0,t]}\{B - M\} \geq x\big) = \sum_{k=0}^{\infty} e^{-lt}\frac{(lt)^k}{k!} \int\limits_{[0,1]^k \times \mathbb{R}^k} N\big(-\sum_{i=1}^{k} j_i \mathbb{1}_{\{tu_i \leq \cdot\}} - x\big)\, d\mu^k_{J,u}(m^k).$$

Proof. We condition on the number of jumps in the interval,

$$\begin{aligned}\mathbb{P}\big(\inf_{[0,t]}\{B - M\} \geq x\big) &= \mathbb{P}\big(\inf_{[0,t]}\{B - M - x\} \geq 0\big) \\ &= \sum_{k=0}^{\infty} \mathbb{P}\big(\tilde{N}_t = k, \inf_{[0,t]}\{B - M - x\} \geq 0\big). \qquad (14)\end{aligned}$$

The conditional distribution of the τ_i's can be replaced by an unconditional one[4], because for $t < T$

$$\mathcal{L}\big(\tau_{\tilde{N}_t+1}, \ldots, \tau_{\tilde{N}_T} \big| \tilde{N}_T - \tilde{N}_t = k\big) = \mathcal{L}\big(\eta_{1:k}, \ldots, \eta_{k:k}\big),$$

where the η_i are i.i.d. $U[0, T]$. Hence,

$$(14) = \sum_{k=0}^{\infty} e^{-lt}\frac{(lt)^k}{k!}\mathbb{P}\Big(\inf_{s\in[0,t]}(B_s - \sum_{i=1}^{k} J_i \mathbb{1}_{\{\eta_{i:k} \leq s\}} - x) \geq 0\Big).$$

As the jump sizes are interchangeable, the probability does not depend on the order of the jump times and $\eta_{i:k}$ can be replaced by η_i. Now we condition on the jump times and the jump sizes. Denote the obtained path of M by

$$m^k(s) = \sum_{i=1}^{k} j_i \mathbb{1}_{\{tu_i \leq s\}}, \quad 0 \leq s \leq t$$

[4]See Rolski, Schmidli, Schmidt, and Teugels (1999), p.502. The $\eta_{i:k}$ denote the order statistics of η_i, that is the η_i are ordered, such that $\eta_{1:k} \leq \eta_{2:k} \leq \cdots \leq \eta_{k:k}$.

where we think of u_i being the realizations of i.i.d. $U[0,1]$ random variables. The conditioned probability is the probability that B stays above $m^k + x$ in the whole interval. In Theorem 2.1 we derived the probability of staying below such a barrier, denoted by $N(m^k)$. Using the equivalence of B and $-B$, we get

$$\begin{aligned}
&\mathbb{P}\big(\inf_{s\in[0,t]}(B_s - \sum_{i=1}^k J_i \mathbb{1}_{\{\eta_i \le s\}} - x) \ge 0\big) \\
&\qquad = \mathbb{P}\big(\inf_{s\in[0,t]}(-\sum_{i=1}^k J_i \mathbb{1}_{\{\eta_i \le s\}} - x - B_s) \ge 0\big) \\
&\qquad = \int_{[0,1]^k\times\mathbb{R}^k} N\big(-\sum_{i=1}^k j_i \mathbb{1}_{\{tu_i \le \cdot\}} - x\big)\, F_J(dj_1)\cdots F_J(dj_k)\, du_1 \cdots du_k. \qquad \blacksquare
\end{aligned}$$

It is easy to generalize the result to the case where the boundary process is a Brownian motion plus jump component and arbitrary volatilities are allowed. Assume that $\tilde{B}$ is a Brownian motion independent from B. Then, observe that in $\inf_{[0,T]}(\sigma_V B - \sigma_D \tilde{B} - M) \ge x$ the difference $B - \tilde{B}$ is again a Brownian motion, and in distribution equal to σB, with $\sigma = \sqrt{\sigma_V^2 + \sigma_D^2}$. So the previous results apply.

Corollary 4.3. *Assume, that Assumption 4.1 holds. Consider a Brownian motion $\tilde{B}$, independent from all other components and $x < B_0$. Then*

$$\begin{aligned}
&\mathbb{P}\big(\inf_{s\in[0,t]}\{\sigma_V B_s - ms - (\sigma_D \tilde{B}_s + M_s)\} \ge x\big) \\
&\qquad = \sum_{k=0}^\infty e^{-lt}\frac{(lt)^k}{k!} \int_{[0,1]^k\times\mathbb{R}^k} N\Big(-\frac{1}{\sigma}\Big(\sum_{i=1}^k j_i \mathbb{1}_{\{tu_i \le \cdot\}} + x\Big), \frac{m}{\sigma}, \sigma\Big)\, d\mu_{J,u}^k(m^k),
\end{aligned}$$

where $N(\cdot, m, \sigma)$ is the probability that $(\sigma B_s + ms)_{s\in[0,t]}$ stays below a piecewise linear barrier.

For later references we denote the above probability by $M^*(x, m, t)$.

Note that N is given in Theorem 3.2, where at this place we write $N(\bar{v}, m, \sigma_D)$ to state the explicit dependance on m and σ_D. For practical purposes, it is important to note that for small l the series converges very fast, so possibly just few summands must be considered.

4.2 The conditional probability

Consider the piecewise constant default boundary from (5). Now, V is a Brownian motion and $D_t = B_t + M_t$, i.e. the sum of a Brownian motion and a marked point process. Similar to the case without jump we compute

$$\mathbb{P}\Big(D_t + M_t \le x, \inf_{[0,t]}(V - D - M) > 0 \;\Big|\; \mathcal{F}_t^V\Big) \tag{15}$$

considering piecewise constant paths for V. This can be achieved by conditioning on M.

Theorem 4.4. *Under Assumption 4.1 the probability of default is*

$$\mathbb{P}\big(\tau > T \;\big|\; \mathcal{G}_t\big) = \int_{-\infty}^V M^*(V_t - x, m, T - t)\, f_{D_t|\bar{v}}(x)\, dx,$$

where the density $f_{D_t|\bar{v}}(x)$ is given in (16).

Proof. Consider a realization of M with k jumps, denoted by m^k or $\big(\sum_{i=1}^k j_i \mathbb{1}_{\{u_i \le s\}}\big)_{s\in[0,t]}$, respectively. Then the two piecewise constant functions $\bar{v}$ and m^k add up to a piecewise linear function $\tilde{v} := \bar{v} - m^k$. Now the previous results can be applied.

After conditioning on the number of jumps we need to consider

$$\mathbb{P}\Big(D_t \le x - m^k(t), \inf_{[0,t]}(\tilde{v} - D) > 0 \;\Big|\; V = \bar{v}, M = m^k\Big) = E_0(0, x, \tilde{b}),$$

where $E_0(0, x, \bar{v})$ was calculated in the proof of Theorem 3.2. We just state the dependence on $\bar{v}$ explicitly, which now is replaced by $\tilde{b}$.

Also recall, that $N(\tilde{b})$ denotes the probability for D staying below $\tilde{b}$ in the interval $[0, t]$. Therefore, we obtain

$$\mathbb{P}\Big(D_t \le x \;\Big|\; \mathcal{F}_t^V, A_t\Big) = \sum_{k=0}^{\infty} e^{-lt}\frac{(lt)^k}{k!} \int\limits_{[0,1]^k\times\mathbb{R}^k} \frac{E_0(0, x, \bar{v} - m^k)}{N(\bar{v} - m^k)}\, d\mu_{J,u}^k(m^k).$$

Coming to the conditional density we have to derive w.r.t. x. Here, the conditional density comes into play and we find that in our case the conditional density of D is

$$f_{\tilde{B}_t + M_t|\bar{v}}(x) = \sum_{k=0}^{\infty} e^{-lt}\frac{(lt)^k}{k!} \int\limits_{[0,1]^k\times\mathbb{R}^k} f_{\tilde{B}_t|\bar{v}-m^k}(x)\, d\mu_{J,u}^k(m^k). \tag{16}$$

The conditional density $f_{\tilde{B}_t|\cdot}(x)$ may be found in (8). Applying the results of Section 4.1 immediately yields the conclusion. ■

4.3 Joint distribution of a jump-diffusion and its minimum

For considering barrier options on jump-diffusions it is necessary to determine the joint distribution of the underlying and its minimum. The following result will also be used to derive the equity value of the considered company in the next chapter.

Theorem 4.5. *Let $\bar{d}$ be a piecewise constant function of the form* (5)*. Under Assumption 4.1 the joint distribution of V and its minimum is given by*

$$\begin{aligned} F_{V,\min}(V_0, T, x, \bar{d}) &:= \mathbb{P}(V_T \le x, \inf_{[0,T]}(V - \bar{d}) > 0) \\ &= \sum_{k=0}^{\infty} e^{-lt}\frac{(lt)^k}{k!} \int\limits_{[0,1]^k\times\mathbb{R}^k} E_0(0, x, m^k - \bar{d}) d\mu_{J,u}^k(m^k). \end{aligned} \tag{17}$$

Proof. Recall that a realization with k jumps was denoted by m^k or $m^k(s) = \sum_{i=1}^k j_i \mathbb{1}_{\{u_i \le s\}}$, respectively. Proceeding as in Theorem 4.4 immediately gives the result. ■

5 Applications

This section will present several applications of the considered frameworks. As the model integrates equity as well as credit risky securities, pricing of securities which are related to both of this quantity is of interest. We show how to proceed for pricing equity default swaps.

5.1 Application to equity default swaps

Besides integrating nicely into the reduced form models, our approach offers a direct connection between equity and default as in the structural models. In this section we exploit this connection to show how the obtained results can be used for pricing so-called equity default swaps (EDS).

Equity default swaps are similar to credit default swaps: a fixed premium is exchanged with a payment at "default", just that in this case the payment is done, when equity falls to a very low level, typically 30% from the equity value at initiation of the contract. Also the payment is fixed, so that there is no recovery risk. For ease of exposition we assume a constant interest rate r.

Denote by τ^* the first time where the equity, denoted by $(S_t)_{t\geq 0}$, hits the pre-specified boundary S^* which is typically 30% of S_0. The fee leg of an EDS pays the premium κ regularly at time $T_1, \ldots, T_N$ provided $T_i < \tau^*$. If $\tau^* \leq T$ then the insurance payment, normalized to 1, is paid at τ^*. The κ is chosen such that the initial value of the EDS is zero, ie

$$\kappa = \frac{\mathbb{E}\Big(e^{-r\tau^*}\mathbb{1}_{\{\tau^*\leq T_N\}}\Big)}{\sum_{i=1}^N \mathbb{E}\Big(e^{-rT_i}\mathbb{1}_{\{\tau^*>T_i\}}\Big)}.$$

Assume for a moment that we have $F_{\tau^*}(x) := Q(\tau^* \leq x)$ at hand. Then

$$\mathbb{E}\Big(e^{-r\tau^*}\mathbb{1}_{\{\tau^*\leq T_N\}}\Big) = \int_0^{T_N} e^{-rx} F_{\tau^*}(dx),$$

thus for pricing the EDS it is sufficient to compute F.

In Black and Scholes (1973) equity was modelled as a call on the firm value with strike equal to the debt level. In first-passage time models, equity equals a down-and-out call on the firm's value. In the considered framework the debt level obviously can be different from the default boundary; however, we concentrate on the case where these two boundaries equal. We assume that the debt level is a known constant K, while noting that the previously developed theory also allows to price the EDS when the debt level is a jump-diffusion process. However, not to obstruct ideas with technical difficulties we concentrate on the easier case. Recall that the firm value V is given by a jump-diffusion. Then, Theorem 4.5 allows to compute the price of this down-and-out call.

Corollary 5.1. *The price of a down-and-out call on V at time t with barrier K and maturity T is given by*

$$C_{do}(V_t, T) = \mathbb{1}_{\{V_t>K\}} \int_K^\infty (v-K)\, dF_{V,\min}(V_t, T-t, v, K),$$

where the joint distribution of V and its minimum is given in (17).

Writing short $C_{do}(V_{t-})$ for $C_{do}(V_{t-}, T-t)$, we obtain by Itô 's formula the following dynamics for the firm's equity:

$$dE_t = \frac{\partial}{\partial t} C_{do}(V_{t-})\, dt + \frac{\partial}{\partial x} C_{do}(V_{t-}) dV_t^c + \frac{1}{2}\frac{\partial^2}{\partial x^2} C_{do}(V_{t-}) d[V,V]_t^c + \Delta C_{do}(V_t)$$

which again is a jump-diffusion. The first hitting time of equity with the debt level K can be computed with Proposition 4.2 which in turn leads to the price of the EDS.

5.2 Special cases

This section considers several special cases which illustrate the applicability of the obtained results.

5.2.1 Brownian case with simplified observation

It the firm's value did not change very much in the past, the past observation may be well approximated by a constant (also, upper and lower bounds can be obtained this way). In this paragraph we therefore consider the case where $\bar{v}_s = K \in \mathbb{R}$ for all $s \in [0,t]$. The previously obtained results immediately allow to derive the default intensity. Observe that the default intensity depends on the barrier K, as well as on volatility σ_D and drift m of D. Setting

$$N := \Phi\left(\frac{K - mt}{\sigma_D\sqrt{t}}\right) - \exp\left(\frac{2Km}{\sigma_D^2}\right)\Phi\left(-\frac{K + mt}{\sigma_D\sqrt{t}}\right)$$

we have the following result.

Proposition 5.2. *Consider the constant $K > 0$. Conditional on $\inf_{[0,t]}(K - D) > 0$, the default intensity equals*

$$\lambda_t = \frac{K}{2N\sigma_D t^{3/2}}\phi\left(\frac{mt - K}{\sigma_D\sqrt{t}}\right)\left(1 + e^{\frac{2Km}{\sigma_D^2}}\right)$$

Proof. We proceed analogously to the proof of Proposition 3.4, in particular we use Equation (13). First, we need to compute the conditional density of D using (6). Observe that for $x < K$

$$\begin{aligned}\mathbb{P}\Big(D_t \le x, \inf_{[0,t]}(V - D) > 0 | \mathcal{F}_t^V\Big) &= \mathbb{P}\Big(\sigma_D B_t + mt \le x, \inf_{s\in[0,t]}(K - \sigma_D B_s - ms) > 0\Big)\\ &= \mathbb{P}\Big(\sigma_D B_t^* - mt \ge -x, \inf_{s\in[0,t]}(\sigma_D B_s^* - ms) > -K\Big), \qquad (18)\end{aligned}$$

where $B^* := -B$ is also a standard Brownian motion. The last probability is given in Corollary A.2; as $-x > -K$ we obtain

$$(18) = \Phi\left(\frac{x - mt}{\sigma_D\sqrt{t}}\right) - \exp\left(\frac{2Km}{\sigma_D^2}\right)\Phi\left(\frac{x - mt - 2K}{\sigma_D\sqrt{t}}\right)$$

and

$$\begin{aligned}\mathbb{P}\Big(\inf_{[0,t]}(V - D) > 0 | \mathcal{F}_t^V\Big) &= \mathbb{P}\Big(\inf_{s\in[0,t]}(\sigma_D B_s^* - ms) > -K\Big)\\ &= \Phi\left(\frac{K - mt}{\sigma_D\sqrt{t}}\right) - \exp\left(\frac{2Km}{\sigma_D^2}\right)\Phi\left(-\frac{K + mt}{\sigma_D\sqrt{t}}\right) = N.\end{aligned}$$

According to (6) the conditional distribution is given by the fraction of these two expressions and we immediately obtain the conditional density

$$f_{D_t|K} = \mathbb{1}_{\{x\le K\}}\frac{1}{N}\left(\frac{1}{\sigma_D\sqrt{t}}\phi\left(\frac{x - mt}{\sigma_D\sqrt{t}}\right) - \exp\left(\frac{2Km}{\sigma_D^2}\right)\frac{1}{\sigma_D\sqrt{t}}\phi\left(\frac{x - mt - 2K}{\sigma_D\sqrt{t}}\right)\right). \qquad (19)$$

Inserting this into (13) we obtain, letting $\tau := T - t$,

$$\lambda_t = \frac{1}{N\sigma_D\sqrt{t}}\lim_{\tau\to 0}\frac{1}{\sigma_D\tau^{3/2}}\int_0^\infty x\,\phi\Big(\frac{x - m\tau}{\sigma_D\sqrt{\tau}}\Big)\left(\phi\left(\frac{V_t - x - mt}{\sigma_D\sqrt{t}}\right) - e^{\frac{2Km}{\sigma_D^2}}\phi\left(\frac{V_t - x - mt - 2K}{\sigma_D\sqrt{t}}\right)\right)dx.$$

To compute the limit we make use of Lemma A.5. Hence,

$$\begin{aligned}\lambda_t = -\frac{\sigma_D}{N\sigma_D^2\sqrt{t}}\Bigg[&\frac{(mt - V_t)\sigma_D^2 - m\sigma_D^2 t}{2\sigma_D^2 t}\phi\left(\frac{mt - V_t}{\sigma_D\sqrt{t}}\right)\\ &- e^{\frac{2Km}{\sigma_D^2}}\frac{(mt + 2K - V_t)\sigma_D^2 - m\sigma_D^2 t}{2\sigma_D^2 t}\phi\left(\frac{mt + 2K - V_t}{\sigma_D\sqrt{t}}\right)\Bigg]\\ = \frac{1}{2N\sigma_D t^{3/2}}\phi&\left(\frac{mt - K}{\sigma_D\sqrt{t}}\right)\left[K + e^{\frac{2Km}{\sigma_D^2}}K\right],\end{aligned}$$

where we used $V_t = K$ for the last step. ∎

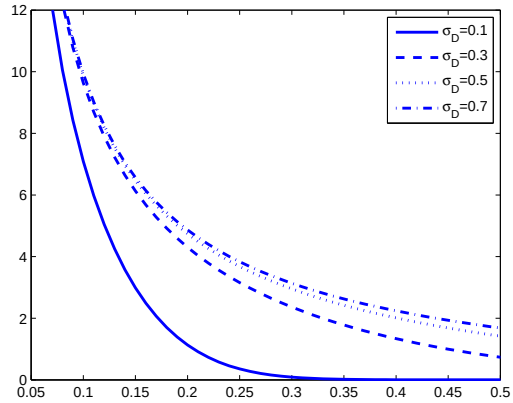

Figure 1: The default intensity λ_1 in the Brownian case for $(\bar{v}_s)_{s\in[0,1]} \equiv K$ for different volatilities σ_D and zero drift m.

In Figure 1 we show the default intensity in dependence on K. It is seen from the graph that with decreasing K the default intensity increases. In particular, for very small K the default boundary D is quite likely to be close to K, so the default intensity rises sharply. Also, for increasing volatility σ_D the default intensity increases. Further simulations (not shown) also show that with decreasing drift m the default intensity increases, as was to be expected.

5.2.2 Debt with jumps

In practice, debt levels will typically jump. The default intensity when the debt is a jump-diffusion is a sum of the default intensity of the diffusive part and the default intensity of the pure-jump part. We therefore analyse the default intensity of the pure-jump case in this section.

First, we assume that the debt level at time t is known. Since jumps occur with intensity l and we have that $\mathbb{P}(V_{\tau_1} \leq D_{\tau_1} | F^{V,D}_{\tau_1-}) = 1 - F_J(V_{\tau_1}/D_{\tau_1-})$. If we assume that V is continuous, we obtain

$$\lambda_t = l \cdot \Big(1 - F_J\Big(\frac{V_t}{D_{t-}}\Big)\Big).$$

If V also admits jumps, one can proceed analogously.

In the case where the debt level is not known at time t, one has to use the results from Section 4 to obtain the conditional distribution of D. This is more interesting in the case where D has also a diffusion part. It may be reasonable to assume that the jumps in the history are known; and this jumps are incorporated in the default boundary. Hence, from the perspective of our model, D will not have jumps in the past. Then the approximation of the conditional distribution becomes much easier, for example the results in the previous section could be used. The default intensity is the sum of the intensity for the diffusion part and of the intensity for the jump part. Assuming no jumps in V and, as before, that the past default boundary is simply K we obtain

$$\lambda_t = \frac{K}{2N\sigma_D t^{3/2}} \phi\left(\frac{mt - K}{\sigma_D\sqrt{t}}\right)\left(1 + e^{\frac{2Km}{\sigma_D^2}}\right) + \int_{-\infty}^{K} l\big(1 - F_J(K/y)\big) f_{D_t|K}(y)dy,$$

where $f_{D_t|K}$ is given in (19).

6 Conclusion

This article propose a generalization of several structural models with incomplete information. Default is triggered by the firm value crossing a random barrier, which itself is allowed to be a stochastic process. Both, firm value and default boundary may have jumps. While the firm value is observed, the default boundary is not. It is shown that under this assumption, generally a default intensity exists and how to compute it. This makes use of boundary crossing probabilities for jump-diffusions. A main criticism of incomplete information models with constant default boundary is remedied, namely that for firm values far above its running minimum credit spreads are to small.

A Appendix

We need some results on boundary crossing probabilities of Brownian motions. Results A.1 - A.3 are well-known and we state them here in for convenience of the reader. Results A.4 - A.5 compute certain integrals or limits used in the text. We start with the joint distribution of a Brownian motion with drift and its minimum in the following lemma.

Lemma A.1. *For $c < 0$ and a standard Brownian motion B it holds for $c < d$*

$$\mathbb{P}\big(\inf_{[0,t]}(B_s + ms) \leq c, B_t + mt > d\big) = e^{2cm}\,\Phi\left(\frac{mt + 2c - d}{\sqrt{t}}\right) \tag{20}$$

while, for $c \geq d$, this probability equals

$$\Phi\left(\frac{d - mt}{\sqrt{t}}\right) - \Phi\left(\frac{c - mt}{\sqrt{t}}\right) + e^{2cm}\Phi\left(\frac{c + mt}{\sqrt{t}}\right).$$

Proof. By the Girsanov theorem we can change to a measure P^*, where $B_t^* := B_t + mt$ is a Brownian motion. Define

$$d\mathbb{P}^* := \exp\Big(-mB_t - \frac{m^2 t}{2}\Big)\,d\mathbb{P} = \exp\Big(-mB_t^* + \frac{m^2 t}{2}\Big)\,d\mathbb{P}.$$

Then

$$\begin{aligned}(20) &= \int \mathbb{1}_{\{\inf_{[0,t]}(B_s+ms)\leq c, B_t+mt>d\}} e^{mB_t^* - \frac{m^2 t}{2}}\,d\mathbb{P}^* \\ &= \mathbb{E}^*\left[e^{mB_t^* - \frac{m^2 t}{2}}\mathbb{P}^*\Big(\inf_{[0,t]} B^* \leq c, B^* > d \;\Big|\; B_t^*\Big)\right].\end{aligned}$$

First, consider the inner probability. We have that

$$\mathbb{P}^*\Big(\inf_{[0,t]} B^* \leq c, B^* > d \;\Big|\; B_t^*\Big) = \mathbb{1}_{\{B_t^* > d\}}\mathbb{P}^*\Big(\inf_{[0,t]} B^* \leq c \;\Big|\; B_t^*\Big).$$

The conditional probability on the right side is easily computed (see Schmidt (2003, p. 142)) and equals on $\{B_t^* > c\}$

$$\mathbb{P}^*\Big(\inf_{[0,t]} B^* \leq c \;\Big|\; B_t^*\Big) = \exp\Big(\frac{2c}{t}(B_t^* - c)\Big).$$

We therefore have

$$\begin{aligned}(20) &= \mathbb{E}^*\left(\mathbb{1}_{\{B_t^* > d\}}\exp\Big(mB_t^* - \frac{m^2 t}{2} + \frac{2c}{t}(B_t^* - c)\Big)\right) \\ &= \exp\Big(-\frac{m^2 t}{2} - \frac{2c^2}{t}\Big)\mathbb{E}^*\left(\mathbb{1}_{\{B_t^* > d\}}\exp\Big((m + \frac{2c}{t})B_t^*\Big)\right).\end{aligned}$$

For the expectation it is easy to see that

$$\mathbb{E}\Big(\mathbb{1}_{\{B_t>d\}}e^{\alpha B_t}\Big) = \int_d^\infty \frac{1}{\sqrt{2\pi t}} e^{\alpha x - \frac{x^2}{2t}}\, dx = e^{\frac{\alpha^2 t}{2}}\, \Phi\left(\frac{\alpha t - d}{\sqrt{t}}\right)$$

and alltogether we get that

$$(20) = \exp\Big(-\frac{m^2 t}{2} - \frac{2c^2}{t} + \frac{\big(m+2c/t\big)^2 t}{2}\Big)\Phi\left(\frac{mt+2c-d}{\sqrt{t}}\right) = e^{2cm}\,\Phi\left(\frac{mt+2c-d}{\sqrt{t}}\right).$$

Proceeding similarly, one also obtains the second equation. ■

On the other side, if the case where the minimum is greater than a certain level is of interest, we may use the following:

$$\begin{aligned}\mathbb{P}\big(B_t + mt > d\big) &= \mathbb{P}\big(\inf_{[0,t]}(B_s+ms) \le c, B_t + mt > d\big)\\ &\quad + \mathbb{P}\big(\inf_{[0,t]}(B_s+ms) > c, B_t + mt > d\big),\end{aligned}$$

such that we immediately get for $c < d$ and $c < 0$

$$\mathbb{P}\big(\inf_{[0,t]}(B_s+ms) > c, B_t + mt > d\big) = \Phi\left(\frac{mt-d}{\sqrt{t}}\right) - e^{2cm}\,\Phi\left(\frac{mt+2c-d}{\sqrt{t}}\right).$$

For the case $c \ge d$, we have that[5]

$$\begin{aligned}\mathbb{P}\big(\inf_{[0,t]}(B_s+ms) > c, B_t + mt > d\big) &= \mathbb{P}\big(\inf_{[0,t]}(B_s+ms) > c\big)\\ &= \Phi\left(-\frac{c-mt}{\sqrt{t}}\right) - e^{2cm}\,\Phi\left(\frac{c+mt}{\sqrt{t}}\right). \qquad (21)\end{aligned}$$

Obviously the last formula equals the previous formula with $c = d$. We thus have the following:

Corollary A.2. *Denote $d \vee c := \max\{d, c\}$. For constants $c < 0$, $\sigma > 0$ and a standard Brownian motion B it holds that*

$$\mathbb{P}\big(\inf_{[0,t]}(\sigma B_s + ms) > c,\ \sigma B_t + mt > d\big) = \Phi\left(\frac{mt - d\vee c}{\sigma\sqrt{t}}\right) - e^{\frac{2cm}{\sigma^2}}\,\Phi\left(\frac{mt+2c-d\vee c}{\sigma\sqrt{t}}\right)$$

and

$$\mathbb{P}\big(\inf_{[0,t]}(\sigma B_s + ms) > c\big) = \Phi\left(\frac{mt-c}{\sigma\sqrt{t}}\right) - e^{2\frac{cm}{\sigma^2}}\,\Phi\left(\frac{c+mt}{\sigma\sqrt{t}}\right).$$

These results enable us to compute the common density of a Brownian motion and its minimum.

Corollary A.3. *For $c < 0$, a Brownian motion with drift mt and its minimum have the common density*

$$f_{M,B}(c,d) = \mathbb{1}_{\{c<d\}}\frac{2(d-2c)}{t^{3/2}}\, e^{2cm}\,\phi\left(\frac{mt+2c-d}{\sqrt{t}}\right).$$

[5] See, e.g., Pechtl (1996) for a proof of $\mathbb{P}\big(\inf_{[0,t]}(B_s+ms) \le c\big) = \Phi\left(\frac{c-mt}{\sqrt{t}}\right) + e^{2cm}\,\Phi\left(\frac{c+mt}{\sqrt{t}}\right)$.

Proof. The density is obtained by partially deriving the distribution function. As we want to apply (20), we have to take care the second part is $\{\cdots > d\}$, which leads us to the negative sign in the following:

$$\begin{aligned} f_{M,B}(c,d) &= -\frac{\partial^2}{\partial c \partial d} e^{2cm} \Phi\left(\frac{mt+2c-d}{\sqrt{t}}\right) \\ &= \frac{2m}{\sqrt{t}} e^{2cm} \phi\left(\frac{mt+2c-d}{\sqrt{t}}\right) - \frac{1}{\sqrt{2\pi t}} \frac{2(mt+2c-d)}{t} e^{2cm - \frac{(mt+2c-d)^2}{2t}} \\ &= \frac{2(d-2c)}{t^{3/2}} e^{2cm} \phi\left(\frac{mt+2c-d}{\sqrt{t}}\right). \end{aligned}$$

■

Finally, we state two auxiliary results.

Lemma A.4.

$$\begin{aligned} &\int_0^\infty x\phi\left(\frac{x+a}{\sqrt{b}}\right) \phi\left(\frac{x+d}{\sqrt{c}}\right) dx \\ &\quad = \frac{bc}{b+c}\phi\left(\frac{a-d}{\sqrt{b+c}}\right) \phi\left(\frac{ac+db}{\sqrt{bc(b+c)}}\right) - \sqrt{bc}\frac{ac+db}{(b+c)^{\frac{3}{2}}} \phi\left(\frac{a-d}{\sqrt{b+c}}\right) \Phi\left(-\frac{ac+db}{\sqrt{bc(b+c)}}\right). \end{aligned}$$

Proof. We have

$$\begin{aligned} \int_{-\infty}^0 x\, e^{-\frac{(x-a)^2}{2b}} dx &= b \int_{-\infty}^0 \frac{x-a}{b} e^{-\frac{(x-a)^2}{2b}} dx + a \int_{-\infty}^0 e^{-\frac{(x-a)^2}{2b}} dx \\ &= -b\exp(-\frac{a^2}{2b}) + a\sqrt{2\pi b}\frac{1}{\sqrt{2\pi b}} \int_{-\infty}^0 e^{-\frac{(x-a)^2}{2b}} dx = -b\exp(-\frac{a^2}{2b}) + a\sqrt{2\pi b}\ \Phi\left(-\frac{a}{\sqrt{b}}\right). \end{aligned} \tag{22}$$

Therefore,

$$\begin{aligned} &\int_{-\infty}^0 x\exp\left(-\frac{(x-a)^2}{2b} - \frac{(x-d)^2}{2c}\right) dx \\ &\quad = \int_{-\infty}^0 x\exp\left(-\frac{\left(x-\frac{ac+db}{b+c}\right)^2}{2bc/(b+c)} - \frac{(a-d)^2}{2(b+c)}\right) dx \\ &\quad \overset{(22)}{=} \exp\left(-\frac{(a-d)^2}{2(b+c)}\right)\left[-\frac{bc}{b+c}\exp\left(-\frac{(ac+db)^2}{2bc(b+c)}\right) + \sqrt{2\pi bc}\frac{ac+db}{(b+c)^{\frac{3}{2}}}\Phi\left(-\frac{ac+db}{\sqrt{bc(b+c)}}\right)\right]. \end{aligned}$$

Substituting $-x$ for x we end up with the desired result. ■

Lemma A.5. *Consider $\tau := T - t$ and $\sigma > 0$. For the limit $\tau \to 0$ we obtain*

$$\frac{1}{\tau^{3/2}} \int_0^\infty x\,\phi\left(\frac{x+a}{\sqrt{b}}\right) \phi\left(\frac{x+m\tau}{\sigma\sqrt{\tau}}\right) dx \to -\sigma\frac{\sigma^2 a + mb}{2b}\phi\left(\frac{a}{\sqrt{b}}\right).$$

Proof. By Lemma A.4 the integral is a sum of two terms. For the first term the leading factors are

$$\frac{1}{\tau^{3/2}} \frac{b\,\sigma^2\tau}{b+\sigma^2\tau}$$

such that the first term converges to zero. The second term equals

$$-\frac{\sqrt{b\,\sigma^2\tau}}{\tau^{3/2}}\,\frac{\sigma^2 a\tau + m\tau b}{(b+\sigma^2\tau)^{3/2}}\phi\left(\frac{a-m\tau}{\sqrt{b+\sigma^2\tau}}\right)\,\Phi\left(-\frac{\sigma^2 a\tau + m\tau b}{(b+\sigma^2\tau)^{3/2}}\right).$$

Note that $\Phi(\cdot) \to \frac{1}{2}$ and so we conclude by observing that the second term converges to

$$-\,\sigma\frac{\sigma^2 a + mb}{2b}\phi\left(\frac{a}{\sqrt{b}}\right). \qquad \blacksquare$$

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