Pricing and Hedging Options on Defaultable Assets

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Abstract

In general, contingent claims on assets which may default during the duration of the contract cannot be priced and hedged consistently. This is due to the fact that the possibility of a default event brings in an extra uncertain factor, and there are therefore too few assets to construct a hedge against all sources of uncertainty. In this paper we show that consistent pricing and hedging is still possible if we assume that (1) we can estimate the size of the loss in value (as a percentage) upon default and (2) default is the only non-systematic risk factor involved. Moreover, we show that the resulting formulas for prices and hedges do not depend on the intensity of the default process, but on a new riskfree intensity *which is an explicit function of other parameters in the model*, in contrast to most other models. We derive a simple tree method to implement the methodology that is proposed, and show how other pricing methods for claims on defaultable assets are linked to our method.

1 Introduction

Many assets in finance carry an inherent risk that they may *default* i.e. that they may experience a sudden loss in value at a time which cannot be predicted before the actual default event takes place. Such a default event creates a downward 'jump' in the value of the asset, and the usual methodology for pricing and hedging derivative contracts on assets no longer works. Indeed, it is well known that the

assumption of continuous sample paths for the value of the underlying asset is one of the essential ingredients for the classical Black-Scholes model for the pricing of derivative contracts. If the value of the underlying can change suddenly by a large amount, thus creating a discontinuity in time, the hedge proposed by Black and Scholes can no longer be used to eliminate all risk and therefore the option cannot be priced consistently using their methods. This important fact was already pointed out in (Merton 1976), and many methods have been proposed since to find consistent pricing methods for contingent claims on defaultable assets.

The most important differences in such methods arise in the modelling of the default event and we can broadly recognize two essentially different approaches in the existing literature. In the first approach, which is often referred to as the structural model approach, the default event for a firm is typically modelled as the first time that the value of the assets of a firm is smaller than the value of its liabilities. This means that we may not know in advance when default will occur, but that we can at least see the likelihood of default rise or fall if we assume that the value of assets and liabilities of the firm can be observed at any time. In the second approach, which is often called 'reduced-form modelling', default is seen as an exogenously given event, which only depends on the current state of the firm through its 'default intensity' which governs the probability that a jump will occur in the next small time interval. In mathematical terms, the first approach models the default time as a previsible stopping time, while the second one models it as a totally inaccessible stopping time.

The main problem with the first approach is the assumption that the value of the firm can be observed at all times, which is obviously quite problematic in practice. Therefore most models use the reduced-form formulation, and try to model the intensity of the default process using both market-wide as well as firm-specific factors. The default intensities cannot be observed directly either, but explicit pricing formulas, which are derived by imposing absence of arbitrage conditions, can be inverted to find estimates for them. Credit ratings, macro-economic factors and correlation between default events for different firms can all be incorporated in such models, but the estimation problem obviously becomes significantly harder when the model grows in complexity. See for example (Rogers and Hilberink 2000) for recent work on structural models, (Duffie and Singleton 1995; Lando 1994) for a general discussion of intensity-based models, and (Jarrow and Turnbull 1995) for the special case where a Markov chain of credit ratings is the only factor influencing the default intensities.

An important problem for those models of the default process which are based on many factors is the impossibility to hedge the derivative contracts. Prices for such contracts are written in the form of conditional expectations under a risk-neutral measure but no hedges are provided, often because replicating portfolios simply do not exist. A lot of contracts where the underlying asset can default therefore seem to take the form of *insurance* contracts, since a premium is paid to make sure that the expected profit over all such protective contracts will be positive in the long run.

In this paper we propose a model which will provide a consistent pricing method-

ology for derivatives on defaultable assets and explicit hedges against default risk. We are able to do so by two important assumptions which are essential for our model and its explicit solution:

- The decrease in value at the time of default is a known constant (given as a percentage of the pre-default value).
- The factor which causes default is the only factor in the dynamics of the asset price which is idiosyncratic.

Both assumptions are obviously quite restrictive and never met completely in practice, but they are necessary if we want to be able to hedge our contingent claims. If we do not know the decrease in value due to default (but assume instead that we know the probability distribution for this unknown quantity) then we cannot replicate the payoff of the derivative, see (Lamberton and Lapeyre 1996), Chapter 7. The second condition we impose is also necessary since we want to be able to diversify away all the risk which is not associated with default by using other assets in the market. In this sence, our approach is similar to the one reported in (Merton 1976) where the assumption that default represents risk which is uncorrelated with the market is used in a CAPM-type argument to derive a similar option pricing formula for this problem. The main difference between the approach we propose here and earlier models such as Merton's, or the models introduced in (Zhou 1997) and (Mason and Bhattacharya 1981), is the possibility to derive perfect hedges which replicate claims perfectly.

In (Beumee, Hilberink, Patel, and Walsh 1999) the problem of hedging derivative credit risk has been considered, and in this paper we show how to construct such hedges for a continuous-time model, but we will also show how hedges can be approximated on a tree in discrete time, thus highlighting the practical applicability of our models. We also consider the case where only one default is allowed to occur during the lifetime of the contract, which is not treated in the papers we mentioned, but which is of obvious practical interest.

As stated before, we will model simultaneously a defaultable asset S and an asset V that cannot default, which both depend on a common factor. In the continuous-time model that we will want to consider later, this common factor is modeled as a Brownian Motion process $\{(W_t,\, t \geq 0\}$, while the default event is modelled using a Poisson process $\{(N_t,\, t \geq 0\}$ with intensity λ:

$$dS_t = \mu_s\, S_{t-}\, dt + \sigma_s\, S_{t-}\, dW_t + (\alpha - 1)\, S_{t-}\, dN_t \tag{1.1}$$

$$dV_t = \mu_v\, V_t\, dt + \sigma_v\, V_t\, dW_t. \tag{1.2}$$

This means that the process S for the defaultable asset is a right-continuous diffusion process with jumps, which at the default times, i.e. the jump times of the process N, changes its current value S_{t-} to αS_{t-}.

This setup is typical for most reduced-form models for default events; see for example the papers (Lando 1994; Lando 1995; Zhou 1997; Davis and Mavroidis

1997; Dempster and Gotsis 1998). In our model, however, α is a deterministic and a priori given constant value in $]0,1[$, an assumption which we earlier specified as being essential for the possibility of hedging. The second fundamental assumption is the existence of the asset V which has a common factor with S but cannot default. We will show later on that if this assumption is not satisfied, we may alternatively assume the existence of an *insurer* for the default event and derive exactly the same results.

Consider a derivative contract C whose payout at the expiration time T depends on both S and V. Denote its value at time $t \in [0,T]$, depending on the current values $S_t = S$ and $V_t = V$, by $C(S,V,t)$. We will also assume the existence of a riskless bond (or bankaccount) the value B_t of which evolves as

$$dB_t \;=\; rB_t\,dt,$$

where r is a constant and known *riskless rate of return*.

We will show that within this framework we can price and hedge contingent claims C on the underlying assets S and V. Moreover, it turns out that we to do so we will only need market parameters with a sensible interpretation which can be estimated using market data.

The structure of this paper is as follows. In the next section we will consider a discrete time approximation for the processes under consideration, and construct a tree which can be used to price and hedge contingent claims. In section 3 we prove a convergence result, under rather mild conditions, for the case where we let the time step in the tree models go to zero. In section 4 we show how we can derive the partial differential equation for the pricing of the claims directly using the stochastic calculus for jump-diffusions. In the last section conclusions and suggestions for further research are formulated.

2 Discrete Time Models

In this section, we will not use the continuous-time processes defined by equations (1.1)-(1.2), but instead we first consider approximations of these processes with a discrete time parameter. To value contingent claims dependent on S and V, we approximate the diffusion processes on a tree with variable time grid h. The fundamental assumption underlying the construction of such a tree will be that we assume that at every time step $t = kh$, $(k \in 1,2,...,N(h))$ (with $hN(h) \leq T < h(N(h)+1)$) there are three possibilities:

1. the common factor W makes a (slight) upward movement, causing both assets to increase to $S_{(k+1)h} = u_s S_{kh}$ and $V_{(k+1)h} = u_v V_{kh}$ respectively, with u_s, $u_v > 1$

2. the factor W makes a (slight) downward movement, causing both assets to decrease to $S_{(k+1)h} = d_s S_{kh}$ and $V_{(k+1)h} = d_v V_{kh}$ respectively, with d_s, $d_v < 1$

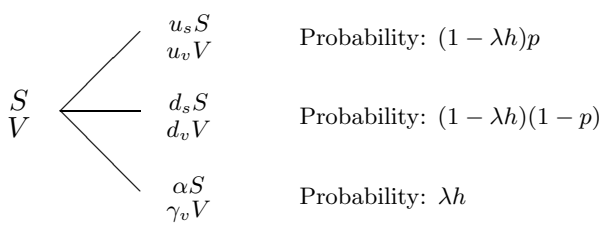
3. the asset S defaults to a value αS which is substantially smaller than its current value, while the asset V only changes slightly, so $S_{(k+1)h} = \alpha S_{kh}$ and $V_{(k+1)h} = \gamma_v V_{kh}$, with γ_v close to 1.

The free parameters u_s, d_s, u_v, d_v, γ_v and probabilities assigned to each of the three possibilities may depend on the time grid h, as long as we have that

- The default process is Poisson. This means that on the relevant time interval $[kh, (k+1)h]$ the probability of default happening in this interval is equal to the length of this interval h times the intensity parameter λ, and this default event is therefore independent of the number of defaults which have happened before,
- The first two conditional moments of the asset S, *conditioned on the event that no default occurs in this time interval*, should be the same as those of a lognormal process, with drift coefficient μ_s and diffusion coefficient σ_s respectively.
- The first two moments of the asset V should be the same as those of a lognormal process with drift coefficient μ_v and diffusion coefficient σ_v *in all cases* (i.e. both when a default occurs and if no default occurs).

Note that the factor W and the default events are not independent of each other in this model. We will also assume that a riskfree rate of interest r is a known constant during the entire time horizon that we wish to consider. This is for notational simplicity only: it can easily be seen in the sequel that our analysis can be generalized for an a priori known time-varying interest rates $\{r(kh),\ k = 1, 2, ..., N(h)\}$.

Taking all these assumptions into account implies that a branch in our trinomial tree must have the following generic structure:



The free parameters in this model are u_s, d_s, u_v, d_v, γ_v and p, and all these parameters may depend on the timestep h. Because of the second assumption we must

have that for all k,

$$\begin{aligned}
\mathbb{E}[\,\ln S_{(k+1)h} - \ln S_{kh} \mid \text{no default}\ ] &= (\mu_s)h \\
\mathbb{E}[\,(\ln S_{(k+1)h} - \ln S_{kh})^2 \mid \text{no default}\ ] &= \sigma_s^2 h \ + \ (\mu_s)^2 h^2 \\
\mathbb{E}[\,\ln V_{(k+1)h} - \ln V_{kh} \mid \text{no default}\ ] &= (\mu_v)h \\
\mathbb{E}[\,(\ln V_{(k+1)h} - \ln V_{kh})^2 \mid \text{no default}\ ] &= \sigma_v^2 h \ + \ (\mu_v)^2 h^2,
\end{aligned}$$

implying that

$$p\,\ln u_s \ + \ (1-p)\ln d_s \ = \ (\mu_s)\,h \tag{2.3}$$

$$p\,(\ln u_s)^2 \ + \ (1-p)(\ln d_s)^2 \ = \ \sigma_s^2\,h \ + \ (\mu_s)^2 h^2 \tag{2.4}$$

$$(1-\lambda h)\ (p\,\ln u_v \ + \ (1-p)\ln d_v) \ + \ \lambda h(\ln \gamma_v) \ = \ (\mu_v)\,h \tag{2.5}$$

$$(1-\lambda h)\ \big(p\,\ln(u_v)^2 \ + \ (1-p)(\ln d_v)^2\big) \ + \ \lambda h(\ln \gamma_v)^2 \ = \ \sigma_v^2\,h \ + \ (\mu_v)^2 h^2 \tag{2.6}$$

This gives us 4 equations for the 6 unknown parameters u_s, d_s, u_v, d_v, γ_v and p. We will therefore have to show in the sequel that the results we obtain do not depend on our particular choices for the two degrees of freedom that we still have.

Suppose that at a certain time $t = kh$ we construct a portfolio Π containing an amount B of riskless bonds, and (possibly shorted) amounts ϵ and δ of the risky assets S and V respectively. The value of this portfolio at time $t = kh$ is then given by

$$\Pi_{kh} \ = \ \delta S_{kh} \ + \ \epsilon V_{kh} \ + \ B_{kh}.$$

The change in value of this portfolio Π and the derivative contract $C = C(S, V, t)$ over the time interval $[kh,\ (k+1)h]$ can be represented as follows:

$$\begin{array}{ccll}
 & & \delta S u_s + \epsilon V u_v + B e^{rh} & \\
 & & C_u = C(S u_s, V u_v, (k+1)h) & \text{Probability: } (1-\lambda h)p \\
\delta S + \epsilon V + B & & \delta S d_s + \epsilon V d_v + B e^{rh} & \\
C(S, V, kh) & \longrightarrow & C_d = C(S d_s, V d_v, (k+1)h) & \text{Probability: } (1-\lambda h)(1-p) \\
 & & \delta S \alpha + \epsilon V \gamma_v + B e^{rh} & \\
 & & C_\alpha = C(S\alpha, V\gamma_v, (k+1)h) & \text{Probability: } \lambda h.
\end{array}$$

If we want the value of the portfolio Π to be equal to the value of the derivative contract C on all three end points of the branch, we must have

$$\begin{aligned}
C_u &= \delta S u_s \ + \ \epsilon V u_v \ + \ B e^{rh} \\
C_d &= \delta S d_s \ + \ \epsilon V d_v \ + \ B e^{rh} \\
C_\alpha &= \delta S \alpha \ + \ \epsilon V \gamma_v \ + \ B e^{rh}.
\end{aligned}$$

These three equations can be solved to give us the three components of the portfolio Π, that is δS, ϵV and B, in terms of the other parameters in the model. A

standard arbitrage argument may then be used to conclude that the initial value of the derivative contract C at time kh should be equal to the initial portfolio value $\Pi = \delta S + \epsilon V + B$ which is the sum of these three components. Tedious but trivial calculations then give that

$$C \;=\; e^{-rh}\,(\,p_u C_u \;+\; p_d C_d \;+\; p_\alpha C_\alpha\,) \tag{2.7}$$

with

$$\begin{aligned}
p_u &= \frac{(e^{rh}-\alpha)(d_v-\gamma_v)+(\alpha-d_s)(e^{rh}-\gamma_v)}{(u_s-d_s)(d_v-\gamma_v)+(u_v-d_v)(\alpha-d_s)} \\
p_d &= \frac{(e^{rh}-\alpha)(\gamma_v-u_v)+(u_s-\alpha)(e^{rh}-\gamma_v)}{(u_s-d_s)(d_v-\gamma_v)+(u_v-d_v)(\alpha-d_s)} \\
p_\alpha &= \frac{(u_s-d_s)(d_v-e^{rh})+(u_v-d_v)(e^{rh}-d_s)}{(u_s-d_s)(d_v-\gamma_v)+(u_v-d_v)(\alpha-d_s)}
\end{aligned}$$

Appropriate conditions are to be imposed to ensure that all three of these quantities are in the interval $]0,1[$; these will indeed be shown to be satisfied in the sequel.

Once we know that (2.7) is correct, we can find the proper values of p_u, p_d and p_α by a simpler alternative method. Indeed, since $C(S,V,kh) = S$ and $C(S,V,kh) = V$ and $C(S,V,kh) = e^{rkh}$ should all satisfy this equation, we must have that

$$e^{rh}S \;=\; p_u S u_s \;+\; p_d S d_s \;+\; p_\alpha S\alpha \tag{2.8}$$

$$e^{rh}V \;=\; p_u V u_v \;+\; p_d V d_v \;+\; p_\alpha V\gamma_v. \tag{2.9}$$

$$1 \;=\; p_u \;+\; p_d \;+\; p_\alpha. \tag{2.10}$$

and the riskfree probabilities p_u, p_d and p_α follow immediately from these three equations.

3 The Partial Differential Equation

We now consider the case where we let the parameter h, which represents the time interval between nodes of our tree, go to zero. To do so, we will need the following asymptotic properties of the riskfree probability values.

Lemma 3.1. *Suppose that the h-dependent parameters u_s, u_v, d_s, d_v, $\gamma_v > 0$, $0 < p < 1$ satisfy for all $h > 0$ the equations (2.3)-(2.6), and also assume that*

$$\lim_{h\downarrow 0}\; p(h)(1-p(h)) \;\neq\; 0 \tag{3.1}$$

$$\lim_{h\downarrow 0}\; \gamma_v(h) \;=\; 1. \tag{3.2}$$

Then the solutions p_u, p_d and p_α of the equations (2.8)-(2.10) satisfy

$$p_u\,(u_s-1)^2 \;+\; p_d\,(d_s-1)^2 \;=\; \sigma_s^2 h \;+\; o(h) \tag{3.3}$$

$$p_u\,(u_v-1)^2 \;+\; p_d\,(d_v-1)^2 \;=\; \sigma_v^2 h \;+\; o(h) \tag{3.4}$$

$$p_u\,(u_s-1)(u_v-1) \;+\; p_d\,(d_s-1)(d_v-1) \;=\; \sigma_s\sigma_v h \;+\; o(h) \tag{3.5}$$

while $(u_v-1)^3$, $(d_v-1)^3$, $(u_s-1)^3$, $(d_s-1)^3$ are all of order $o(h)$ and

$$p_\alpha \;=\; \tilde\lambda h \;+\; o(h), \qquad\qquad \tilde\lambda \;=\; \frac{1}{1-\alpha}\left(\mu_s - r - \frac{\sigma_s}{\sigma_v}(\mu_v - r)\right). \tag{3.6}$$

A proof of this lemma can be found in the Appendix at the end of this paper. Remark that we can interpret the first equation as a condition that we do not destroy branches in the tree as we refine our grid, while the second one states that the expectation of the square of the increments of the V asset upon default vanishes when the time grid converges to zero. Using these conditions, we are now in a position to derive a partial differential equation for the value of a contingent claim on S and V.

Theorem 3.1. *Assume that we select the two free parameters $p = p(h)$ and $\gamma_v = \gamma_v(h)$ such that (3.3)-(3.4) are satisfied, and let the other parameters $u_s(h)$, $d_s(h)$, $u_v(h)$ and $d_v(h)$ satisfy (2.3)-(2.6). Then the solution of the riskfree probability pricing method on the tree given by (2.7) will converge to the solution of the following partial differential equation*

$$\begin{aligned} rC \;=\; & (r+\tilde\lambda(1-\alpha))\,S\frac{\partial C}{\partial S} \;+\; rV\frac{\partial C}{\partial V} \;+\; \tfrac{1}{2}\sigma_s^2 S^2\frac{\partial^2 C}{\partial S^2} \;+\; \tfrac{1}{2}\sigma_v^2 V^2\frac{\partial^2 C}{\partial V^2} \\ & +\; \sigma_s\sigma_v\,SV\frac{\partial^2 C}{\partial S\partial V} \;+\; \frac{\partial C}{\partial t} \;+\; \tilde\lambda\,[\,C(S\alpha,V,t) - C(S,V,t)\,]\,. \end{aligned} \tag{3.7}$$

when the time grid size h converges to zero, where

$$\tilde\lambda \;=\; \frac{1}{1-\alpha}\left(\mu_s - r - \frac{\sigma_s}{\sigma_v}(\mu_v - r)\right) \tag{3.8}$$

under the condition that this partial differential equation admits a unique solution $C(S,V,T)$ which is twice continuously differentiable in S and V and once differentiable with respect to t.

Proof. We have, from (2.7), that

$$\begin{aligned} e^{rh}C(S,V,kh) \;=\; & p_u\,C(Su_s,Vu_v,(k+1)h) \;+\; p_d\,C(Sd_s,Vd_v,(k+1)h) \\ & +\; p_\alpha\,C(S\alpha,V\gamma_v,(k+1)h). \end{aligned}$$

We perform a Taylor series expansion in h around $h=0$ on both sides. Since we have shown in the previous Lemma that terms like $(u_s-1)^3$ are in fact of small

order $o(h)$, we only need to consider a second order expansion for an expansion up to order h. Such a Taylor series expansion is indeed possible since we assumed that the solution C to the PDE satisfies the appropriate differentiability requirements. We find

$$
\begin{aligned}
&(1+rh)C(S,V,kh) \\
&= \; p_u \left[C(S,V,kh) + \tfrac{\partial C}{\partial S}(S,V,kh)\cdot S(u_s-1) \;+\; \tfrac{1}{2}\tfrac{\partial^2 C}{\partial S^2}(S,V,kh)\cdot S^2(u_s-1)^2 \right] \;+ \\
&\quad p_u \left[\tfrac{\partial^2 C}{\partial S \partial V}(S,V,kh)\cdot S(u_s-1)V(u_v-1) \;+\; \tfrac{\partial C}{\partial V}(S,V,kh)\cdot V(u_v-1) \right] \;+ \\
&\quad p_u \left[\tfrac{1}{2}\tfrac{\partial^2 C}{\partial V^2}(S,V,kh)\cdot V^2(u_v-1)^2 \;+\; \tfrac{\partial C}{\partial t}(S,V,kh)\cdot h \right] \;+ \\
&\quad p_d \left[C(S,V,kh) + \tfrac{\partial C}{\partial S}(S,V,kh)\cdot S(d_s-1) \;+\; \tfrac{1}{2}\tfrac{\partial^2 C}{\partial S^2}(S,V,kh)\cdot S^2(d_s-1)^2 \right] \;+ \\
&\quad p_d \left[\tfrac{\partial^2 C}{\partial S \partial V}(S,V,kh)\cdot S(d_s-1)V(d_v-1) \;+\; \tfrac{\partial C}{\partial V}(S,V,kh)\cdot V(d_v-1) \right] \;+ \\
&\quad p_d \left[\tfrac{1}{2}\tfrac{\partial^2 C}{\partial V^2}(S,V,kh)\cdot V^2(d_v-1)^2 \;+\; \tfrac{\partial C}{\partial t}(S,V,kh)\cdot h \right] \;+ \\
&\quad p_\alpha \; C(S\alpha, V\gamma_v, (k+1)h) \;+\; o(h).
\end{aligned}
$$

Collecting and rearranging terms gives

$$
\begin{aligned}
&rh\,C(S,V,kh) \qquad\qquad (3.9) \\
&= \; \tfrac{\partial C}{\partial S}(S,V,kh)\cdot S\left[p_u u_s + p_d d_s - (p_u+p_d)\right] \;+ \\
&\quad \tfrac{\partial C}{\partial V}(S,V,kh)\cdot V\left[p_u u_v + p_d d_v - (p_u+p_d)\right] \;+ \\
&\quad \tfrac{1}{2}\tfrac{\partial^2 C}{\partial S^2}(S,V,kh)\cdot S^2\left[p_u(u_s-1)^2 + p_d(d_s-1)^2\right] \;+ \\
&\quad \tfrac{1}{2}\tfrac{\partial^2 C}{\partial V^2}(S,V,kh)\cdot V^2\left[p_u(u_v-1)^2 + p_d(d_v-1)^2\right] \;+ \\
&\quad \tfrac{\partial^2 C}{\partial S \partial V}(S,V,kh)\cdot S\cdot V\left[p_u(u_s-1)(u_v-1) + p_d(d_s-1)(d_v-1)\right] \;+ \\
&\quad p_\alpha \left[-C(S,V,kh) + C(S\alpha,V,kh) + \tfrac{\partial C}{\partial t}(S\alpha,V,kh)\cdot h + \tfrac{\partial C}{\partial V}(S\alpha,V,kh)\cdot V(\gamma_v-1) \right] \\
&\qquad\qquad + \tfrac{1}{2}\tfrac{\partial^2 C}{\partial V^2}(S\alpha,V,kh)\cdot V^2(\gamma_v-1)^2 \Big] \;+ \\
&\quad (1-p_\alpha)\,\tfrac{\partial C}{\partial t}(S,V,kh)\cdot h.
\end{aligned}
$$

From (2.8) and (2.9) we may immediately conclude that

$$
\begin{aligned}
e^{rh}-1 &= p_u(u_s-1) \;+\; p_d(d_s-1) \;+\; p_\alpha(\alpha-1) \\
e^{rh}-1 &= p_u(u_v-1) \;+\; p_d(d_v-1) \;+\; p_\alpha(\gamma_v-1)
\end{aligned}
$$

so we have that

$$
\begin{aligned}
p_u u_s + p_d d_s - (p_u+p_d) &= rh + (1-\alpha)p_\alpha + O(h^2) \\
p_u u_v + p_d d_v - (p_u+p_d) &= rh + (1-\gamma_v)p_\alpha + O(h^2).
\end{aligned}
$$

Substituting these equations, and the results from Lemma 3.1 into (3.9) and dividing by h then gives the result if we let h tend to zero. ■

We can now easily see that if we wish to price a contingent claim which only depends on the defaultable asset S, and not on V, then the price of such an asset also depends on S alone. However, from the proof we can see that the *hedge* for such a claim *does* include the asset V. But the price of V never enters our calculations; only the *market price of risk* of V, $\frac{\mu_v - r}{\sigma_v}$ appears in the pricing formula, thus showing that our results hold for a generic choice of the asset V.

4 Analysis in Continuous Time

Having found the limiting solution for trees with time steps h converging to zero, we now turn to the full continuous time problem, which will enable us to derive the partial differential equation for the option price C directly. We want to model the defaultable asset S as a diffusion process with jumps which are governed by a Poisson process. To do so, we start with a probability space $(\Omega, \mathcal{F}, P)$ and a filtration $\{\mathcal{F}_t,\, t \geq 0\}$ satisfying all the usual conditions (i.e. it is right-continuous and $\mathcal{F}_0$ contains all P-negligible sets). We assume the space to be rich enough to be able to define on it a Brownian Motion process $\{(W_t, \mathcal{F}_t),\, 0 \leq t \leq T\}$ and an infinite countable number of stochastic variables $\{Z_k,\, k \in \mathbb{N}\}$ which have an exponential distribution with parameter λ. Define $T_0 = 0$ and for all $k \in \mathbb{N}$ construct the random variables $T_{k+1} = T_k + Z_k$ so

$$\mathbb{P}(T_{k+1} - T_k \leq s) \;=\; \left(1 - e^{-\lambda s}\right) \mathbf{1}_{\{s \geq 0\}}.$$

Then we define the asset process $\{S_t,\, 0 \leq t \leq T\}$ to be of the following form:

$$S_t \;=\; S_0\, e^{(\mu_s - \frac{1}{2}\sigma_s^2)t + \sigma_s W_t} \sum_{k=0}^{\infty} \alpha^k \mathbf{1}_{\{\, t \in [T_k,\, T_{k+1}[\,\}}.$$

Remark that we have not, so far, specified anything about the relation between the random times between default $\{Z_k,\, k \in \mathbb{N}\}$ and the Brownian motion process $\{(W_t, \mathcal{F}_t),\, 0 \leq t \leq T\}$, and in particular we have not assumed them to be independent.

The process S is defined to be right-continuous with left-hand limits: it belongs to the class of *corlol* processes in the usual terminology. We define the left hand limit process $\{S_{t-},\, 0 < t \leq T\}$ associated with S by $S_{t-} = \lim_{u \uparrow t} S_u$, so

$$S_{t-} \;=\; S_0\, e^{(\mu_s - \frac{1}{2}\sigma_s^2)t + \sigma_s W_t}\, \mathbf{1}_{\{\, t \in]0,\, T_1]\,\}} \;+\; S_0\, e^{(\mu_s - \frac{1}{2}\sigma_s^2)t + \sigma_s W_t} \sum_{k=1}^{\infty} \alpha^k \mathbf{1}_{\{\, t \in]T_k,\, T_{k+1}]\,\}},$$

and we introduce the corresponding jump process

$$\Delta S_t \;=\; S_t \;-\; S_{t-}.$$

This jump process is obviously equal to zero at all times except the default times $t = T_k$ for $k \in \mathbb{N} \setminus \{0\}$, where one can easily show that

$$\Delta S_{T_k} \;=\; S_0\, e^{(\mu_s - \frac{1}{2}\sigma_s^2)T_k + \sigma_s W_{T_k}}\,(\,\alpha^k - \alpha^{k-1}\,) \;=\; (\alpha - 1)\, S_{T_k -}.$$

We will shortly start analyzing portfolios with value

$$\Pi_t \;=\; \rho_t B_t \;+\; \delta_t S_t \;+\; \epsilon_t V_t.$$

with $B_t = e^{rt}$ and we will require the hedging processes $\{\rho_t,\, 0 \le t \le T\}$, $\{\delta_t,\, 0 \le t \le T\}$ and $\{\epsilon_t,\, 0 \le t \le T\}$ to be left-continuous processes:

$$\begin{aligned}
\rho_t &= \sum_{k=0}^{\infty} \rho_t^k\, \mathbf{1}_{\{\, t \in]T_k,\, T_{k+1}]\,\}} \\
\delta_t &= \sum_{k=0}^{\infty} \delta_t^k\, \mathbf{1}_{\{\, t \in]T_k,\, T_{k+1}]\,\}} \\
\epsilon_t &= \sum_{k=0}^{\infty} \epsilon_t^k\, \mathbf{1}_{\{\, t \in]T_k,\, T_{k+1}]\,\}}.
\end{aligned}$$

This means that we do not want the hedging strategies to anticipate the default event, i.e. the hedging position at a certain time t should be based on all the information about S_u at times $u < t$ but not on S_t itself. Remark that the definitions imply that $\rho_t = \rho_{t-}$, $\delta_t = \delta_{t-}$ and $\epsilon_t = \epsilon_{t-}$ so $\Delta\rho_t = \Delta\delta_t = \Delta\epsilon_t = 0$ for all t in $[0, T]$.

The left-continuity will now enable us to use the stochastic integration theory for general processes. The integral

$$I_t \;=\; \int_0^t \delta_{u-}\, dS_u$$

is defined to be equal to

$$I_t \;=\; \sum_{k=0}^{\infty} I_t^k\, \mathbf{1}_{\{\, t \in [T_k,\, T_{k+1}[\,\}},$$

with

$$I_t^k \;=\; I_{T_k}^{k-1} \;+\; \delta_{T_k}^{k-1}\, \Delta S_{T_k} \;+\; \int_{T_k}^t \delta_u^k\, dS_u.$$

The key element here is that we incorporate the value of $\delta_{T_k}^{k-1}$ at the jump time T_k, and not the value of $\delta_{T_k}^k$, since we should not anticipate on beforehand that a jump is about to happen. Note that I will be in the class of corlol processes as well, i.e. it is right-continuous and has left-hand limits. Moreover, with these definitions we can now use the following two key results from the general theory of stochastic processes, proofs of which, and details concerning their interpretation, can be found for example in the books (Elliott 1982; Rogers and Williams 1987) and (Shiryayev 1984).

Lemma 4.2. (Partial Integration Formula for Semimartingales)
Using the definitions given above, we have that for all $t \in [0,T]$

$$\delta_t S_t \;-\; \delta_0 S_0 \;=\; \int_0^t S_{u-}\, d\delta_u \;+\; \int_0^t \delta_{u-}\, dS_u \;+\; \langle \delta^{\mathrm{c}},\, S^{\mathrm{c}} \rangle_t \;+\; \sum_{0 \le u \le t} \Delta S_u\, \Delta \delta_u$$

where $\langle \delta^{\mathrm{c}},\, S^{\mathrm{c}} \rangle$ is the joint quadratic variation of the continuous parts of the S and δ processes.

Lemma 4.3. (Change of Variable Formula for Semimartingales)
For every twice differentiable function F with bounded first and second order derivatives we have that for all $t \in [0,T]$

$$\begin{aligned} F(S_t) \;-\; F(S_0) \;=\; & \int_0^t F'(S_{u-})\, dS_u \;+\; \tfrac{1}{2} \int_0^t F''(X_{u-})\, d\,\langle S^{\mathrm{c}},\, S^{\mathrm{c}} \rangle_u \\ & + \sum_{0 < u \le t} (\, F(S_u) - F(S_{u-}) - F'(S_{u-}) \Delta S_u \,), \end{aligned}$$

where $\langle S^{\mathrm{c}},\, S^{\mathrm{c}} \rangle$ is the quadratic variation of the continuous part of the S process.

We now define the Poisson process N associated with the jump times $\{T_k,\, k \in N\}$ as

$$N_t \;=\; \sum_{k=0}^{\infty} k\, \mathbf{1}_{\{\, t \in [T_k,\, T_{k+1}[\,\}}.$$

This Poisson process is obviously in the class of corlol processes, and its intensity is λ. We can then easily show that our original process S solves the stochastic differential equation

$$S_t \;-\; S_0 \;=\; \mu_s \int_0^t S_{u-}\, du \;+\; \sigma_s \int_0^t S_{u-}\, dW_u \;+\; (\alpha - 1) \int_0^t S_{u-}\, dN_u, \quad (4.1)$$

since application of the Change of Variable Formula for $F(x) = \ln x$ gives that

$$\begin{aligned} \ln S_t - \ln S_0 \;=\; & \int_0^t \frac{\mu_s\, S_{u-}\, du \;+\; \sigma_s S_{u-}\, dW_u \;+\; (\alpha - 1) S_{u-}\, dN_u}{S_{u-}} \;+\; \tfrac{1}{2} \int_0^t \frac{-1}{S_{u-}^2} \sigma_s^2 S_{u-}^2\, du \\ & + \sum_{0 < u \le t} (\, \ln \alpha S_{u-} - \ln S_{u-} - \frac{1}{S_{u-}} \cdot (\alpha - 1) S_{u-}\,) \mathbf{1}_{\{S_u \ne S_{u-}\}} \\ =\; & (\mu_s - \tfrac{1}{2}\sigma_s^2) t \;+\; \sigma_s W_t \;+\; (\alpha - 1) N_t \;+\; N_t \cdot (\, \ln \alpha - (\alpha - 1)\,) \end{aligned}$$

so

$$\begin{aligned} S_t \;=\; & S_0\, e^{(\mu_s - \frac{1}{2}\sigma_s^2)\, t \;+\; \sigma_s\, W_t \;+\; (\ln \alpha)\, N_t} \\ =\; & S_0\, e^{(\mu_s - \frac{1}{2}\sigma_s^2) t + \sigma_s W_t} \sum_{k=0}^{\infty} \alpha^k \mathbf{1}_{\{\, t \in [T_k,\, T_{k+1}[\,\}}, \end{aligned}$$

which is exactly the definition of the S process we introduced earlier. But substituting (4.1) in the Change of Variable Formula then gives that for our process S we have that

$$\begin{aligned}
F(S_t) \;-\; F(S_0) & \\
= \;& \int_0^t F'(S_{u-})\,[\,\mu_s S_{u-}\,du \;+\; \sigma_s S_{u-}\,dW_u \;+\; (\alpha-1)S_{u-}\,dN_u\,] \\
& + \int_0^t \tfrac{1}{2}F''(S_{u-})\,\sigma_s^2(S_{u-})^2\,du \\
& + \sum_{0<u\le t} \big[\,(\,F(\alpha S_{u-}) - F(S_{u-})\,)\cdot \mathbf{1}_{\{S_u\neq S_{u-}\}} \;-\; F'(S_{u-})(\alpha-1)S_{u-}\cdot \mathbf{1}_{\{S_u\neq S_{u-}\}}\,\big] \\
= \;& \int_0^t [\,F'(S_{u-})\,\mu_s S_{u-} \;+\; \tfrac{1}{2}F''(S_{u-})\,\sigma_s^2(S_{u-})^2\,]\,du \;+\; \int_0^t F'(S_{u-})\sigma_s S_{u-}\,dW_u \\
& + \int_0^t [\,F(\alpha S_{u-}) - F(S_{u-})\,]\,dN_u. \qquad (4.2)
\end{aligned}$$

We can use this equation to construct a (dynamic) portfolio Π_t which will satisfy the following conditions:

- the value of this portfolio will at all times during the life time of the contract equal the value of the derivative contract C,
- the portfolio only consists of prescribed amounts δ_t of the risky assets S, prescribed amounts ϵ_t of the riskless asset V, and a prescribed amount ρ_t of cash,
- the invested amounts ϵ_t, δ_t and ρ_t are left-continuous, so they do not anticipate the default event, and they are self-financing, so no money is injected into, or withdrawn from, the portfolio during the life time of the derivative contract.

The last requirement can be formulated as follows: we require the portfolio

$$\Pi_t \;=\; \rho_t B_t \;+\; \delta_t S_t \;+\; \epsilon_t V_t \qquad (4.3)$$

to satisfy

$$\Pi_t - \Pi_0 \;=\; \int_0^t \rho_u\,dB_u \;+\; \int_0^t \delta_u\,dS_u \;+\; \int_0^t \epsilon_u\,dV_u, \qquad (4.4)$$

at all times $0 \le t \le T$, which means that the changes in value of the portfolio are only a result of price movements in the market and not due to our selling or buying of assets. A self-financing portfolio which satisfies all conditions mentioned above is called a replicating portfolio, and we equate its value at all times with the value of the derivative C at that time, using a standard arbitrage argument. This will then enable us to prove the following.

Theorem 4.2. *Let C be a contingent claim paying $G(S_T, V_T)$ at the expiration date T, and let the two assets S and V evolve in continuous time according to (1.1)-(1.2). Assume that the diffusion factor and default factor have zero crossvariation at all times: $d\langle W, N\rangle_t = 0$ for all $t \in [0,T]$. Then C must satisfy the partial differential equation*

$$\begin{aligned} rC &= (r + \tilde{\lambda}(1-\alpha))\, S\frac{\partial C}{\partial S} \;+\; rV\frac{\partial C}{\partial V} \;+\; \tfrac{1}{2}\sigma_s^2 S^2 \frac{\partial^2 C}{\partial S^2} \;+\; \tfrac{1}{2}\sigma_v^2 V^2 \frac{\partial^2 C}{\partial V^2} \\ &\quad + \sigma_s\sigma_v\, SV\frac{\partial^2 C}{\partial S \partial V} \;+\; \frac{\partial C}{\partial t} \;+\; \tilde{\lambda}\,[\,C(S\alpha, V, t) - C(S, V, t)\,], \end{aligned} \tag{4.5}$$

with boundary condition

$$C(S, V, T) \;=\; G(S, V)$$

if this PDE has a unique solution on $t \in [0,T]$ for this boundary condition, which is twice continuously differentiable in S and V and continuously differentiable in t.

Proof. Let $C(S, V, t)$ be the unique solution for the PDE with the given boundary condition. Define a self-financing portfolio Π with value Π_t at time t as in (4.3), with hedging processes

$$\begin{aligned} \delta_t &= \tfrac{C(\alpha S_{t-}, V_t, t) - C(S_{t-}, V_t, t)}{(\alpha-1)S_{t-}} \\ \epsilon_t &= \tfrac{\partial C}{\partial V}(S_{t-}, V_t, t) \;+ \\ &\qquad \tfrac{\sigma_s}{\sigma_v(1-\alpha)V_t}\,\big[\,C(\alpha S_{t-}, V_t, t) - C(S_{t-}, V_t, t) - S_{t-}(\alpha-1)\tfrac{\partial C}{\partial S}(S_{t-}, V_t, t)\,\big]\,, \\ \rho_t &= \frac{1}{B_t}\,[\,C(S_{t-}, V_t, t) \;-\; \delta_t S_{t-} \;-\; \epsilon_t V_t\,]\,. \end{aligned}$$

Remark that all these hedging strategy processes are left-continuous and that our choice implies that $\Pi_0 = C(S_0, V_0, 0)$ as can be easily seen in the equation for ρ_t given above. We now find that

$$\begin{aligned} &d\Pi_t \\ &= \rho_t\, dB_t \;+\; \delta_t\, dS_t \;+\; \epsilon_t\, dV_t \;=\; r\rho_t B_t\, dt \;+\; \delta_t\, dS_t \;+\; \epsilon_t\, dV_t \\ &= rC(s_{t-}, V_t, t)\, dt \;+\; \delta_t(dS_t - rS_{t-}dt) + \epsilon_t(dV_t - rV_t dt) \\ &\overset{(4.5)}{=} \Big((r + \tilde{\lambda}(1-\alpha))\tfrac{\partial C}{\partial S}S_{t-} + r\,\tfrac{\partial C}{\partial V}V_t + \tfrac{1}{2}\tfrac{\partial^2 C}{\partial S^2}\,\sigma_s^2 S_{t-}^2 + \tfrac{1}{2}\tfrac{\partial^2 C}{\partial V^2}\,\sigma_v^2 V_t^2 \\ &\qquad + \tfrac{\partial^2 C}{\partial S \partial V}\,\sigma_s\sigma_v S_{t-}V_t + \tfrac{\partial C}{\partial t} \;+\; \tilde{\lambda}\,[\,C(\alpha S_{t-}, V_t, t) \;-\; C(S_{t-}, V_t, t)\,]\Big)\, dt \\ &\qquad + \tfrac{C(\alpha S_{t-}, V_t, t) \;-\; C(S_{t-}, V_t, t)}{\alpha-1}\,(\mu_s\, dt \;+\; \sigma_s\, dW_t \;+\; (\alpha-1)\, dN_t \;-\; r\, dt) \\ &\qquad + \Big(\tfrac{\partial C}{\partial V}\,V_t + \tfrac{\sigma_s}{\sigma_v(1-\alpha)}(C(\alpha S_{t-}, V_t, t) - C(S_{t-}, V_t, t) - (\alpha-1)\tfrac{\partial C}{\partial S}S_{t-})\Big)\cdot \\ &\qquad\qquad \cdot\,(\mu_v\, dt \;+\; \sigma_v\, dW_t \;-\; r\, dt) \end{aligned}$$

We collect terms to obtain

$$
\begin{aligned}
& d\Pi_t \\
= \; & \left(\tfrac{1}{2}\tfrac{\partial^2 C}{\partial S^2}\,\sigma_s^2 S_{t-}^2 + \tfrac{1}{2}\tfrac{\partial^2 C}{\partial V^2}\,\sigma_v^2 V_t^2 + \tfrac{\partial^2 C}{\partial S \partial V}\,\sigma_s \sigma_v S_{t-} V_t + \tfrac{\partial C}{\partial t} \right) dt \\
& + (\, C(\alpha S_{t-}, V_t, t) \;-\; C(S_{t-}, V_t, t)\,) \cdot \\
& \quad \cdot \left((\tilde{\lambda} + \tfrac{\mu_s - r - (\mu_v - r)\frac{\sigma_s}{\sigma_v}}{\alpha - 1})dt \;+\; (\tfrac{\sigma_s}{\alpha-1} + \tfrac{-\frac{\sigma_s}{\sigma_v}\sigma_v}{\alpha-1})\, dW_t \;+\; dN_t \right) \\
& + \tfrac{\partial C}{\partial S}\, S_{t-} \left((r + \tilde{\lambda}(1-\alpha) + \tfrac{\sigma_s(1-\alpha)(\mu_v - r)}{\sigma_v(1-\alpha)})dt \;-\; \tfrac{\sigma_s(\alpha-1)}{\sigma_v(1-\alpha)}\sigma_v\, dW_t \right) \\
& + \tfrac{\partial C}{\partial V}\, V_t\, (\, r\, dt \;+\; \mu_v\, dt \;+\; \sigma_v\, dW_t \;-\; r\, dt\,) \\
= \; & \left(\tfrac{1}{2}\tfrac{\partial^2 C}{\partial S^2}\,\sigma_s^2 S_{t-}^2 + \tfrac{1}{2}\tfrac{\partial^2 C}{\partial V^2}\,\sigma_v^2 V_t^2 + \tfrac{\partial^2 C}{\partial S \partial V}\,\sigma_s \sigma_v S_{t-} V_t + \tfrac{\partial C}{\partial t} \right) dt \\
& + \;(\, C(\alpha S_{t-}, V_t, t) \;-\; C(S_{t-}, V_t, t)\,)\; dN_t \\
& \quad + \tfrac{\partial C}{\partial S}\, S_{t-} \left((r + (\mu_s - r) - \tfrac{\sigma_s(\mu_s - r)}{\sigma_v} + \tfrac{\sigma_s(\mu_s - r)}{\sigma_v})\, dt \;+\; \sigma_s dW_t \right) \\
& \quad + \tfrac{\partial C}{\partial V}\, V_t\, (\, \mu_v\, dt \;+\; \sigma_v dW_t\,) \\
= \; & dC(S_{t-}, V_t, t).
\end{aligned}
$$

But since $\Pi_T = C(S_{T-}, V_T, T) = G(S_{T-}, V_T)$ we have thus shown that the portfolio Π is a perfect hedge for the contingent claim G, and the value of this claim at time zero is therefore equal to $\Pi_0 = C(S_0, V_0, 0)$. ■

5 Solution of the Partial Differential Equation

We may solve the PDE that we have now derived using Fourier Transforms, but once the form of the solution has been found, it may be derived in a much simpler way.

Let $F(S,t)$ denote the solution of the ordinary Black-Scholes equation with adjusted interest rate $r + \tilde{\lambda}(1-\alpha)$ for a contingent claim G:

$$
\begin{aligned}
0 &= \tfrac{1}{2}\sigma_s^2 S^2 \frac{\partial^2 F}{\partial S^2} \;+\; (r + \tilde{\lambda}(1-\alpha))\, S \frac{\partial F}{\partial S} \;+\; \frac{\partial F}{\partial t} \;-\; (r + \tilde{\lambda}(1-\alpha))F \\
G(S) &= F(S,T), \qquad (5.1)
\end{aligned}
$$

and look for a solution for our PDE of the form

$$
C(S,t) \;=\; \sum_{n=0}^{\infty} a_n(t)\, F(\alpha^n S, t). \qquad (5.2)
$$

We immediately find the boundary conditions

$$
a_0(T) = 1, \qquad\qquad a_n(T) = 0, \quad n \in \mathbb{N} \setminus \{0\}. \qquad (5.3)
$$

Taking our PDE (3.7)

$$\begin{aligned} 0 \;=\; & \tfrac{1}{2}\sigma_s^2 S^2 \frac{\partial^2 C}{\partial S^2} \;+\; (r+\tilde{\lambda}(1-\alpha))\, S\frac{\partial C}{\partial S} \;+\; \frac{\partial C}{\partial t} \\ & -\; (r+\tilde{\lambda}(1-\alpha))C \;+\; \tilde{\lambda}(1-\alpha)C \;+\; \tilde{\lambda}\,(\,C(S\alpha,t)-C(S,t)\,) \end{aligned}$$

and substituting (5.2) gives

$$\begin{aligned} 0 \;=\; & \sum_{n=0}^{\infty} a_n(t)\Big[\,\tfrac{1}{2}\sigma_s^2 S^2 \frac{\partial^2}{\partial S^2}\alpha^{2n}\,F(\alpha^n S,t) \;+\; (r+\tilde{\lambda}(1-\alpha))\alpha^n\, S\frac{\partial}{\partial S}F(\alpha^n S,t) \\ & \qquad\qquad + \frac{\partial}{\partial t}F(\alpha^n S,t) \;-\; (r+\tilde{\lambda}(1-\alpha))F(\alpha^n S,t)\Big] \\ & +\; \sum_{n=0}^{\infty}\Big[\,F(\alpha^n S,t)\tfrac{d}{dt}a_n(t) \;+\; \tilde{\lambda}(1-\alpha)a_n(t)F(\alpha^n S,t) \\ & \qquad\qquad +\tilde{\lambda}a_n(t)\,\big(\,F(\alpha^{n+1}S,t)-F(\alpha^n S,t)\,\big)\Big] \end{aligned}$$

and using (5.1) for the value $S \to S\alpha^n$ then gives

$$0 \;=\; 0 \;+\; \sum_{n=0}^{\infty}\Big[\,\tfrac{d}{dt}a_n(t) \;+\; \tilde{\lambda}a_{n-1}(t) \;-\; \alpha\tilde{\lambda}a_n(t)\Big]\,F(\alpha^n S,t)$$

if we take $a_{-1}(t)=0$ for all t, and therefore

$$\begin{aligned} \tfrac{d}{dt}a_n(t) \;&=\; \tilde{\lambda}\,(\alpha\, a_n(t)-a_{n-1}(t)), \qquad\qquad n\in\mathbb{N}\setminus\{0\}, \\ \tfrac{d}{dt}a_0(t) \;&=\; \tilde{\lambda}\alpha\, a_0(t). \end{aligned}$$

Solving for the functions a_n satisfying these equations and the boundary conditions (5.3) gives

$$a_n(t) \;=\; \frac{[\tilde{\lambda}(T-t)]^n}{n!}e^{-\tilde{\lambda}\alpha(T-t)},$$

so if we define

$$\tilde{r} \;=\; r \;-\; \tfrac{1}{2}\sigma_s^2 \;+\; \tilde{\lambda}(1-\alpha)$$

a solution is given by

$$\begin{aligned} & C(S,t) \\ & = \sum_{n=0}^{\infty}\frac{[\tilde{\lambda}(T-t)]^n}{n!}e^{-\tilde{\lambda}\alpha(T-t)}\,\cdot \\ & \qquad \mathbb{E}\Big[\,e^{-(r+\tilde{\lambda}(1-\alpha))(T-t)}G(S_T)\;\mid\;\ln S_T\sim N(\ln S\alpha^n+\tilde{r}(T-t),\,\sigma_s\sqrt{T-t})\Big] \\ & = \sum_{n=0}^{\infty}\frac{[\tilde{\lambda}(T-t)]^n e^{-\tilde{\lambda}(T-t)}}{n!}\,\cdot \qquad\qquad (5.4) \\ & \qquad \mathbb{E}\Big[\,e^{-r(T-t)}G(S_T)\;\mid\;\ln S_T\sim N(\ln S\alpha^n+\tilde{r}(T-t),\,\sigma_s\sqrt{T-t})\Big]\,. \end{aligned}$$

Let $P(\tilde{\lambda})$ denote a Poisson process with parameter $\tilde{\lambda}$. We have then proven

Theorem 5.3. *The solution to the partial differential equation (3.7) can be written in the form of the following conditional expectation:*

$$C(S,t) \;=\; \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}\,G(S_T\cdot\alpha^{N_{T-t}})\right]$$

for all $t\in[0,T]$, where $\mathbb{Q}$ is a measure under which the stochastic variables N and S_T are assumed to be independent with marginal distributions

$$\begin{aligned}\ln S_T &\sim N(\ln S+\tilde{r}(T-t),\,\sigma_s\sqrt{T-t})\\ N &\sim P(\tilde{\lambda})\end{aligned}$$

and where

$$\tilde{r} \;=\; r \;-\; \tfrac{1}{2}\sigma_s^2 \;+\; \tilde{\lambda}(1-\alpha)$$

Note that this means that calculation of the price of the derivative contract may be interpreted as calculating an expectation by conditioning on n jumps having occurred, where the *jump intensity has changed in the riskfree world to a new value* $\tilde{\lambda}$, instead of the real world jump intensity λ.

If $\alpha=1$ or $\tilde{\lambda}=0$ we find back the usual Black-Scholes equation, as was to be expected. In particular, note that the intensity of the default process N_t does not enter the partial differential equation. If we hedge perfectly, it should not make any difference how *often* we have to hedge, since we have to be ready for the possibility of a jump occurring at all times during the lifetime of the contract.

Equation (5.4) shows how we may easily compute the value of a call option on a defaultable asset S. Indeed, it is simply a linear combination of terms indexed by n, with weight

$$\frac{[\tilde{\lambda}(T-t)]^n e^{-\tilde{\lambda}(T-t)}}{n!}$$

while the main term

$$\mathbb{E}\left[e^{-r(T-t)}G(S_T)\;\mid\;\ln S_T\sim N(\ln S\alpha^n+\tilde{r}(T-t),\,\sigma_s\sqrt{T-t})\right].$$

can easily be shown to be the value of a vanilla Black-Scholes call option with strike price K, time to maturity $T-t$, volatility σ and riskfree interest rate r, if we substitute $S\alpha^n e^{-\tilde{\lambda}(1-\alpha)(T-t)}$ for the stock price into the Black-Scholes formula, instead of S. The term $e^{-\tilde{\lambda}(1-\alpha)(T-t)}$ may be interpreted as a negative dividend on the risky asset S.

If the derivative asset C only depends on the defaultable asset S, while the other asset V is priced in a Black-Scholes complete market with market price of risk equal to φ, the equation simplifies to

$$rC \;=\; (\,\mu_s - \sigma_s\varphi\,)\,S\frac{\partial C}{\partial S} \;+\; \tfrac{1}{2}\sigma_s^2 S^2\frac{\partial^2 C}{\partial S^2} \;+\; \frac{\partial C}{\partial t} \;-\; (\,\mu_s - \sigma_s\varphi - r\,)\frac{C(S\alpha,V,t)-C(S,V,t)}{\alpha-1}.$$

Note that if we define an analogous *market price of default risk* φ_{def} by

$$\frac{\mu_s - r}{\sigma_s} \;=\; \varphi \;+\; \varphi_{\text{def}}$$

then we may write

$$rC \;=\; r\,S\frac{\partial C}{\partial S} \;+\; \tfrac{1}{2}\sigma_s^2 S^2\frac{\partial^2 C}{\partial S^2} \;+\; \frac{\partial C}{\partial t} \;-\; \sigma_s\varphi_{\text{def}}\frac{C(S\alpha,V,t)-C(S,V,t)-\frac{\partial C}{\partial S}(S\alpha-S)}{\alpha-1}.$$

which shows clearly that we arrive yet again at the ordinary Black-Scholes formula if there is no default risk (i.e. if $\varphi_{\text{def}} = 0$). As we noted before, the price of V itself never enters our pricing formula, and any non-defaultable asset V with this market price of risk can therefore be used to hedge our position.

We can give a nice and simple interpretation of the new riskfree default intensity in terms of an *insurance contract* for default risk. Suppose we construct a portfolio consisting of $\sigma_s S_{t-}$ assets V and $-\sigma_v V_t$ (i.e. shorted) assets S. The value of this portfolio will then be

$$\Pi_t \;=\; \sigma_s S_{t-} V_t - \sigma_v V_t S_t.$$

If we use $\tilde{\lambda}$ to denote the intensity which makes $\tilde{N}_t = N_t - \tilde{\lambda}t$ a martingale under the risk-neutral measure $\mathbb{Q}$, so

$$\mathbb{E}^{\mathbb{Q}}[N_t - N_s \mid N_s] \;=\; \tilde{\lambda}(t-s)$$

then we have, under the assumption that the portfolio is self-financing,

$$\begin{aligned} d\Pi_t \;&=\; \sigma_s S_{t-}(\mu_v dt + \sigma_v dW_t)V_t \\ &\qquad - \sigma_v V_t(\,(\mu_s + \tilde{\lambda}(\alpha-1))dt + \sigma_s dW_t + (\alpha-1)(dN_t - \tilde{\lambda}dt)\,)S_{t-} \\ &=\; (\sigma_s\mu_v - \sigma_v\mu_s + \sigma_v\tilde{\lambda}(1-\alpha))S_{t-}V_t\,dt \;+\; \sigma_v(1-\alpha)\tilde{\lambda}d\tilde{N}_t \end{aligned}$$

but since $\tilde{N}_t$ is a martingale under the risk-neutral measure, the first term must equal $r\Pi_{t-}\,dt$ and since

$$r\Pi_{t-} \;=\; r(\sigma_s - \sigma_v)S_{t-}V_t$$

we thus have

$$\sigma_s\mu_v - \sigma_v\mu_s + \sigma_v\tilde{\lambda}(1-\alpha) \;=\; r(\sigma_s - \sigma_v)$$

or

$$\tilde{\lambda} \;=\; \frac{1}{1-\alpha}\left(\mu_s - r - \frac{\sigma_s}{\sigma_v}(\mu_v - r)\right)$$

This result helps to give an appropriate interpretation of the new market parameter $\tilde{\lambda}$ that we have introduced. The decomposition

$$dN_t \;=\; \tilde{\lambda}\,dt \;+\; d\tilde{N}_t \tag{5.5}$$

under the risk-neutral measure can be interpreted as the decomposition of an insurance contract N_t (which pays \$ 1 at time T for every default that has happened during the time interval $[0, T]$, i.e. the lifetime of the contract) into a martingale part $\tilde{N}_t$ and a risk premium $\tilde{\lambda}$ per time unit. The derivation above therefore shows that our original assumption, that all risk due to the diffusion term dW_t in the asset dynamics can be diversified away using the asset V, is in fact equivalent with the existence of an insurance contract with this premium. The assumption that an asset V exists which shares the diffiusion term with S but cannot itself default is therefore no longer necessary if one can find someone who is willing to *'insure jumps'*. This statement should actually be of help in risk management problems which involve default or credit derivatives. The statement that risk which cannot be diversified away in defaultable assets should be interpreted as insurance seems logical, and the derivation above may constitute a first step to the explicit quantification of such a statement.

6 Exclusion of Multiple Default Events.

For some markets it is more realistic to model the default process in such a way that either zero or one default can occur, but not more than one. It is relatively easy to describe this process in stochastic calculus: one may simply stop the Poisson process N_t at its first jump time τ to give a new process $\tilde{N}_t = N_{\min(t,\tau)}$. It can be shown (Davis 1993) that the compensator for $\tilde{N}_t$ is equal to $\lambda(1 - \tilde{N}_{t-})$ at all times $t \geq 0$. by the optional stopping theorem. By a slight abuse of notation, we will again use the notation N_t for the stopped process, instead of $\tilde{N}_t$. We thus have for this case

$$dS_t \;=\; \mu_s S_{t-}\,dt \;+\; \sigma_s S_{t-}\,dW_t \;+\; (\alpha - 1)S_{t-}(1 - N_{t-})\,dN_t \tag{6.1}$$

Having maximally one jump means that the information structure of the problem changes. When we considered a Poisson process N_t, the value of the derivative contract $C(S, V, t)$ did not depend on the current value N_t, the number of defaults so far. This is because the conditional distribution of the future of a Poisson process does not depend on the number of jumps which have occurred in the past. This is no longer true for the stopped Poisson process. Once a jump has occurred no jump can occur in the future, so the conditional distribution of the future changes at the time

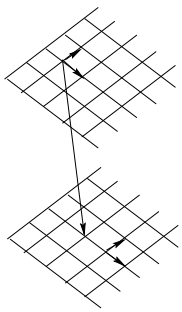
of the jump (to a trivial one, in fact). This means that the contingent claim will now depend on the jump having or not having occurred: $C = C(S_t, V_t, N_t, t)$, and we need to consider the cases $C(S, V, 0, t)$ and $C(S, V, 1, t)$ separately. We will from now on only consider contingent claims which do not depend on V, so $C(S, V, N, t) = C(S, N, t)$. The case where the claim does depend on V can be treated using an obvious extension of the analysis presented here.

It is not hard to see that if the contingent claim at time T is of the form $G(S)$, i.e. it does not depend on N_T, then $C(S, 1, t)$ will evolve according to the ordinary Black-Scholes equation:

$$C(S,1,t) \quad = \quad \mathbb{E}[\, e^{-r(T-t)} G(S_T) \mid \ln S_T \sim N(\ln S + (r - \tfrac{1}{2}\sigma^2)(T-t),\, \sigma\sqrt{T-t})\,]$$

since no jumps are to be expected after the jump has occurred, and we will simply be able to hedge our position using the standard hedge, as prescribed by Black and Scholes. The price of the contract from then on must then be equal to the price of constructing these hedges, which is the solution to the ordinary Black-Scholes equation.

But from a simple discrete tree argument we can then find the partial differential equation for $C(S, 0, t)$. In fact, we should now combine *two* trees, one for the case where a default has already occurred, and one for the case where a default has not (yet) occurred.



As we remarked above, on the lower tree the contract simply evolves according to the Black Scholes equation, while on the upper tree we have to use the riskneutral probabilities p_u, p_d and p_α which we introduced before. We can then go through exactly the same calculations. Since $C(S, 1, t)$ follows the Black-Scholes equation we use the notation $C(S, t) = C(S, 0, t)$. Calculations eventually lead to the partial differential equation

$$rC \quad = \quad (r + \tilde{\lambda}(1-\alpha))\, S\frac{\partial C}{\partial S} \;+\; \tfrac{1}{2}\sigma_s^2 S^2 \frac{\partial^2 C}{\partial S^2} \;+\; \frac{\partial C}{\partial t} \;+\; \tilde{\lambda}\,[\,C(S\alpha, 1, t) - C(S, t)\,]\,,$$

which may be solved explicitly

Theorem 6.4. *Let $C(S_t, N_t, t)$ be a contingent claim paying $G_{N_T}(S_T)$ at the expiration date T, and let the asset S evolve in continuous time according to (6.1). Assume that the diffusion factor and default factor have zero crossvariation at all times: $d\langle W, N\rangle_t = 0$ for all $t \in [0,T]$. Then C must satisfy*

$$\begin{aligned} & C(S,0,t) \\ &= e^{-r(T-t)}\,\mathbb{E}^{\mathbb{Q}}\left[G_0(S_T e^{\tilde{\lambda}(1-\alpha)(T-t)})\,\mathbf{1}_{\{\tau\geq T\}} + G_1(\alpha S_T e^{\tilde{\lambda}(1-\alpha)(\tau-t)})\mathbf{1}_{\{\tau\leq T\}}\right] \\ & C(S,1,t) \\ &= e^{-r(T-t)}\,\mathbb{E}^{\mathbb{Q}}\left[\, G_1\left(S_T\right)\,\right] \end{aligned}$$

where $\mathbb{Q}$ is a measure such that S_T and τ are independent with

$$\ln S_T \;\sim\; N(\ln S + (r - \tfrac{1}{2}\sigma_s^2)(T-t),\, \sigma\sqrt{T-t})$$

while τ has an exponential distribution with parameter $\tilde{\lambda}$, as in (3.8) conditioned on the event that $\tau \geq t$.

Proof. We solve the PDE, with boundary conditions

$$\begin{aligned} C(S,0,T) &= G_0(S) \\ C(S,1,T) &= G_1(S) \end{aligned}$$

by Fourier analysis. First we use the coordinate transformation $S = e^x$ with

$$\begin{aligned} C(S,0,t) &= C(e^x,0,t) = \phi_0(x,t), \\ C(S,1,t) &= C(e^x,1,t) = \phi_1(x,t), \end{aligned}$$

which gives that

$$\begin{aligned} \frac{\partial \phi_0}{\partial x} &= S\,\frac{\partial C}{\partial S}, \\ \frac{\partial^2 \phi_0}{\partial x^2} &= S^2\,\frac{\partial C}{\partial S} + S\,\frac{\partial C}{\partial S}, \end{aligned}$$

so the transformed PDE becomes

$$r\phi_0 = (r + \tilde{\lambda}(1-\alpha) - \tfrac{1}{2}\sigma_s^2)\,\frac{\partial \phi_0}{\partial x} + \tfrac{1}{2}\sigma_s^2 \frac{\partial^2 \phi_0}{\partial x^2} + \frac{\partial \phi_0}{\partial t} + \tilde{\lambda}\,[\,\phi_1(x + \ln\alpha, t) - \phi_0(x,t)\,]\,.$$

Denoting the Fourier transforms[1] as

$$\Phi_k(\omega,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x}\phi_k(x,t)\,dx$$

[1] We will not go into details concerning the existence and uniqueness of the Fourier transforms here. The Fourier Transform of $G(e^x)$ may not be well defined but one can show that such problems can be circumvented when we incorporate the boundary conditions $G(e^x)$ into the PDE itself by solving the PDE's for $C(e^x,i,t) - G(e^x)$ instead of the PDE's for $C(e^x,i,t)$. It then turns out that the solutions do exist for vanilla products such as ordinary calls or puts. We do not introduce this (rather cumbersome) method here but proceed formally, since we may always check that the solution we find does indeed satisfy the PDE.

(for $k = 0, 1$) we find that

$$\left\{ \begin{array}{rcl} \frac{\partial \Phi_0}{\partial t} & = & \Phi_0(\omega, t)\,[\, r + \tilde{\lambda} - i\omega(r + \tilde{\lambda}(1-\alpha) - \frac{1}{2}\sigma_s^2) + \frac{1}{2}\sigma_s^2\omega^2\,] \\ & & - \tilde{\lambda}\Phi_1(\omega, t)e^{i\omega \ln \alpha} \\ \Phi_0(\omega, T) & = & \Phi^{G_0}(\omega) \end{array} \right. \tag{6.2}$$

where $\Phi^{G_k}(\omega)$ denotes the Fourier Transform of $G_k(e^x)$. We can solve this to find

$$\begin{aligned} &\Phi_0(\omega, t) \\ &= \Phi^{G_0}(\omega)e^{[\,R(\omega)+\tilde{\lambda}(1-i\omega(1-\alpha))\,](t-T)} \\ &\qquad - \tilde{\lambda}e^{i\omega \ln \alpha}\int_T^t e^{[\,\tilde{\lambda}(1-i\omega(1-\alpha))+R(\omega)\,](t-s)}\Phi_1(\omega, s)ds \end{aligned} \tag{6.3}$$

with

$$R(\omega) \;=\; r - i\omega(r - \tfrac{1}{2}\sigma_s^2) + \tfrac{1}{2}\sigma_s^2\omega^2.$$

For $\tilde{\lambda} = 0$ we must have that the resulting expression is the Fourier transform of the solution of the Black Scholes equation. But $\Phi_1(\omega, s)$ *is* the Fourier transform of the solution of the Black Scholes equation, which must therefore equal

$$\begin{aligned} &\Phi_1(\omega, s) = \Phi^{G_1}(\omega)e^{R(\omega)\cdot(s-T)} \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\omega x}\,\mathbb{E}\left[\, e^{-r(T-s)}G_1(S_T) \;\mid\; \ln S_T \sim N(x + (r - \tfrac{1}{2}\sigma_s^2)(T-s),\, \sigma_s\sqrt{T-s})\,\right] dx \end{aligned} \tag{6.4}$$

We can then substitute this result in (6.3) to find

$$\begin{aligned} &\Phi_0(\omega, t) \\ &= e^{\tilde{\lambda}(t-T)}e^{i\omega\tilde{\lambda}(1-\alpha)(T-t)} \cdot \left[\Phi^{G_0}(\omega)e^{R(\omega)(t-T)}\right] \\ &\qquad + \tilde{\lambda}e^{i\omega \ln \alpha}\int_t^T e^{\tilde{\lambda}(1-i\omega(1-\alpha))(t-s)}\left[e^{R(\omega)\cdot(t-s)}e^{R(\omega)\cdot(s-T)}\Phi^{G_1}(\omega)\right] ds \end{aligned}$$

and substituting (6.4) in the terms in brackets gives

$$\begin{aligned} &\Phi_0(\omega, t) \\ &= e^{-\tilde{\lambda}(T-t)}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\omega(x+\tilde{\lambda}(1-\alpha)(T-t))}\,\cdot \\ &\qquad\qquad \mathbb{E}\left[\, e^{-r(T-t)}G_0(S_T) \;\mid\; \ln S_T \sim N(x + (r - \tfrac{1}{2}\sigma_s^2)(T-t),\, \sigma_s\sqrt{T-t})\,\right] dx \\ &\quad + \frac{1}{\sqrt{2\pi}}\int_t^T \tilde{\lambda}e^{-\tilde{\lambda}(s-t)}\int_{-\infty}^{\infty} e^{i\omega(x+\tilde{\lambda}(1-\alpha)(s-t)+\ln \alpha)}\,\cdot \\ &\qquad\qquad \mathbb{E}\left[\, e^{-r(T-t)}G_1(S_T) \;\mid\; \ln S_T \sim N(x + (r - \tfrac{1}{2}\sigma_s^2)(T-t),\, \sigma_s\sqrt{T-t})\,\right] dx \end{aligned}$$

and applying the inverse Fourier transform then gives the solution

$$\begin{aligned}
&\phi_0(x,t) \\
&= \left(\int_T^\infty \tilde{\lambda} e^{-\tilde{\lambda}(s-t)}\, ds\right) \mathbb{E}[e^{-r(T-t)} G_0(S_T) \mid \ln S_T \sim N(x + a(T), \sigma\sqrt{T-t})] \\
&\quad + \int_t^T \tilde{\lambda} e^{-\tilde{\lambda}(s-t)} \mathbb{E}[e^{-r(T-t)} G_1(S_T) \mid \ln S_T \sim N(x + \ln\alpha + a(s), \sigma\sqrt{T-t})]\, ds
\end{aligned}$$

with

$$a(s) = (r - \tfrac{1}{2}\sigma_s^2)(T-t) + \tilde{\lambda}(1-\alpha)(s-t)$$

which proves the result. ■

Notice that in this new formula we can no longer calculate the value of a vanilla call option as a countable infinite sum of solutions of the usual Black-Scholes formula, since the stochastic time of default τ enters our calculations explicitly. This is due to the changed nature of the problem: the distribution of the future of the default process does not depend on time when we allow an infinite number of defaults, but if only one default can maximally occur, this is no longer the case. In our earlier result we could substitute $S \to S\alpha^n e^{\tilde{\lambda}(1-\alpha)(T-t)}$ in the ordinary Black-Scholes call formula, but we see that in the present case we must substitute $S \to S\alpha^0 e^{\tilde{\lambda}(1-\alpha)(T-t)}$ for the first term and $S \to S\alpha^1 e^{\tilde{\lambda}(1-\alpha)(\tau-t)}$ for the second term. Taking the expectation of the second term then involves an integration over the whole (riskfree) conditional distribution of τ, thus making the pricing formula more complex.

7 Conclusions

In this paper we have presented a hedging strategy for contingent claims on defaultable assets, under the important assumption that we know the size of the (relative) loss in value upon default, and the assumption that market-wide risk in the asset can be hedged away using other assets in the market. The usefulness of the model therefore critically depends on whether these two assumptions are a reasonable approximation for a particular case that one wants to consider. However, more important is the general conclusion that if one manages to hedge away (part of) the market risk in a defaultable security, hedging the default risk should in principle be possible if the underlying asset can be shorted. We have shown that the resulting equations may be interpreted as the introduction of an *insurance contract* N_t against default risk, which pays one dollar at the end of the contract for every default that has occurred. The parameter $\tilde{\lambda}$, which was shown in (5.5) to determine the risk premium on such an insurance contract, is a market parameter which is not directly observable but it could be estimated using statistical estimation techniques. Note that we do not need to estimate $\tilde{\lambda}$ directly since we have shown that $\tilde{\lambda}$ follows from the drift and volatility parameters of S and *any* asset V which cannot default.

This is the important difference with earlier models such as (Merton 1976) in which it is shown that an option pricing formula based on a hedge with just the two assets S and B cannot be constructed, but that one may still find an option pricing formula if one assumes that the defaults represent 'non-systematic' risk which will be uncorrelated with the market, having a β of zero. The option formula which is derived there is very similar to our result when the number of defaults is not limited to one, but in that model the parameter λ which is used represents the *original* intensity of the default process.

The difference with our model is that we find a new *riskneutral* intensity $\tilde{\lambda}$, due to our consideration for the rest of the market, modelled by the asset V. Indeed, in our model the original intensity of the default process does not play a role in the pricing or hedging formulas. And if our position is properly hedged *all the time* the average number of default that may happen should indeed not enter our calculations. The fact that the intensity of the jump process should be changed to a new riskneutral value in the pricing formula has been remarked upon earlier in (Aase 1988). In that paper a more general setup is used and it is shown that claims can be replicated, but the equivalent martingale measure (and hence prices) are not unique. Also the new intensity $\tilde{\lambda}$ under the martingale measure is not linked to other default-free assets V in the market. The same is true for the paper (Lando 1995) in which a special case of our setup (with $\alpha = 0$ and no restriction on the number of defaults that may happen) has been solved.

Note that the only characteristic of V which enters our equations is its market price of risk (which should be the same for *all* default-free assets in our model) and the *price* of V at any given moment is only used to calculate the hedge for the option; it does not play a role in the option price when the payoff at maturity does not involve V. This suggests that our framework may also be applicable in more complicated models for defaultable assets, and these are now the topic for ongoing research.

Appendix

Proof of Lemma 3.1. Subtracting the square of equation (2.3) from (2.4) gives

$$p(1-p)\ (\ln u_s - \ln d_s\ )^2 \ = \ \sigma_s^2 h,$$

and so we have (since we assumed that $u_s > d_s$)

$$\ln u_s \ - \ \ln d_s \ = \ \frac{\sigma_s}{\sqrt{p(1-p)}} \cdot \sqrt{h}. \tag{7.1}$$

If we substitute this back into (2.3) we find that

$$\ln d_s \ = \ (\mu_s)h - p(\ln u_s - \ln d_s) \ = \ -\sigma_s \sqrt{\tfrac{p}{(1-p)}} \cdot \sqrt{h} \ + \ (\mu_s)h \tag{7.2}$$

and using (7.1) this means that

$$\ln u_s \;=\; \sigma_s\sqrt{\tfrac{1-p}{p}}\,\sqrt{h} \;+\; (\mu_s)h. \tag{7.3}$$

Similar calculations for (2.5) and (2.6) show that

$$\begin{aligned}
\ln u_v &= \sqrt{\tfrac{1-p}{p}}\cdot\sqrt{\sigma_v^2-2\lambda\ln\gamma_v^2}\cdot\sqrt{h} \;+\; (\mu_v-\lambda\ln\gamma_v)\cdot h \;+\; O(h\sqrt{h})\\
\ln d_v &= -\sqrt{\tfrac{p}{1-p}}\cdot\sqrt{\sigma_v^2-2\lambda\ln\gamma_v^2}\cdot\sqrt{h} \;+\; (\mu_v-\lambda\ln\gamma_v)\cdot h \;+\; O(h\sqrt{h})
\end{aligned}$$

We conclude that under the conditions stated in (3.1)-(3.2), $\ln u_s$, $\ln d_s$, $\ln u_v$, $\ln d_v$ are all $O(\sqrt{h})$.

To simplify future notation, define

$$\bar{p} \;=\; \sqrt{\frac{1-p}{p}} \;+\; \sqrt{\frac{p}{1-p}} \;=\; \frac{1}{\sqrt{p(1-p)}}.$$

Using the asymptotic expansions in h of the parameters as derived previously, we find that

$$u_s \;=\; e^{\ln(u_s)} \;=\; 1 \;+\; \ln(u_s) \;+\; o(h)$$

so $(u_s-1)^3 = O(h\sqrt{h})$ and analogous expressions can be derived for d_s, u_v and d_v. The equations (3.3)-(3.4) of the Lemma now follow immediately from (2.4)-(2.6), while

$$\begin{aligned}
u_v-d_v &= (u_v-1)-(d_v-1) \;=\; \ln(u_v)-\ln(d_v) \;+\; o(\sqrt{h})\\
&= \bar{p}\sqrt{\sigma_v^2-\lambda\ln\gamma_v^2}\sqrt{h}+o(\sqrt{h}) && (7.4)\\
u_s-d_s &= (u_s-1)-(d_s-1) \;=\; \ln(u_s)-\ln(d_s) \;+\; o(\sqrt{h}) && (7.5)\\
&= \bar{p}\sigma_s\sqrt{h}+o(\sqrt{h}) && (7.6)\\
d_s-e^{rh} &= (d_s-1)-(e^{rh}-1)\\
&= -\sigma_s\sqrt{\tfrac{p}{1-p}}\sqrt{h}+(\mu_s-r)h+o(h) && (7.7)\\
d_v-e^{rh} &= (d_v-1)-(e^{rh}-1)\\
&= -\sqrt{\sigma_v^2-\lambda\ln\gamma_v^2}\sqrt{\tfrac{p}{1-p}}\cdot\sqrt{h}+(\mu_v-r-\lambda\ln\gamma_v)h+o(h). && (7.8)
\end{aligned}$$

Since

$$p_\alpha \;=\; \frac{(u_s-d_s)(d_v-e^{rh})+(u_v-d_v)(e^{rh}-d_s)}{(u_s-d_s)(d_v-\gamma_v)+(u_v-d_v)(\alpha-d_s)}$$

we calculate for the numerator

$$
\begin{aligned}
&(u_s - d_s)(d_v - e^{rh}) \\
&\quad = \quad (\bar{p}\sigma_s\sqrt{h} + o(\sqrt{h}))(-\sqrt{\sigma_v^2 - \lambda\ln(\gamma_v)^2}\sqrt{\tfrac{p}{1-p}}\sqrt{h} + (\mu_v - r - \lambda\ln(\gamma_v))h + o(h)) \\
&(u_v - d_v)(e^{rh} - d_s) \\
&\quad = \quad (\bar{p}\sigma_v\sqrt{h} + o(\sqrt{h}))(\sigma_s\sqrt{\tfrac{p}{1-p}}\sqrt{h} - (\mu_s - r)h + o(h)) \\
&(u_s - d_s)(d_v - e^{rh}) + (u_v - d_v)(e^{rh} - d_s) \\
&\quad = \quad ((\mu_v - r)\sigma_s - (\mu_s - r)\sigma_v)\, h\sqrt{h} \;+\; o(h\sqrt{h})
\end{aligned}
$$

and for the denominator

$$
\begin{aligned}
&(u_s - d_s)(d_v - \gamma_v) \\
&\quad = \quad (\bar{p}\sigma_s\sqrt{h} + o(\sqrt{h}))(1 - \gamma_v - \sqrt{\sigma_v^2 - \lambda\ln\gamma_v^2}\sqrt{\tfrac{p}{1-p}}\cdot\sqrt{h} + (\mu_v - r - \lambda\ln\gamma_v)h + o(h)) \\
&(u_v - d_v)(\alpha - d_s) \\
&\quad = \quad (\bar{p}\sqrt{\sigma_v^2 - \lambda\ln\gamma_v^2}\sqrt{h} + o(\sqrt{h}))(\alpha - 1 + \sigma_s\sqrt{\tfrac{p}{1-p}}\sqrt{h} - (\mu_s - r)h + o(h)) \\
&(u_s - d_s)(d_v - \gamma_v) + (u_v - d_v)(\alpha - d_s) \\
&\quad = \quad (\sigma_s(1 - \gamma_v) - \sigma_v(1 - \alpha))\,\sqrt{h} \;+\; o(\sqrt{h})
\end{aligned}
$$

and since $\lim_{h\downarrow 0}\gamma_v = 1$ we have that

$$
p_\alpha \quad = \quad \frac{(\mu_v - r)\sigma_s - (\mu_s - r)\sigma_v}{\sigma_v(\alpha - 1)}\, h \;+\; o(h)
$$

which proves (3.6). To prove (3.5) we note that

$$
\begin{aligned}
&p_u(u_s - 1)(u_v - 1) + p_d(d_s - 1)(d_v - 1) \\
&\quad = \quad p_u\left(\sigma_s\sqrt{\tfrac{1-p}{p}}\sqrt{h}\right)\left(\sigma_v\sqrt{\tfrac{1-p}{p}}\sqrt{h}\right) \\
&\qquad\qquad + p_d\left(-\sigma_s\sqrt{\tfrac{p}{1-p}}\sqrt{h}\right)\left(-\sigma_v\sqrt{\tfrac{p}{1-p}}\sqrt{h}\right) \\
&\quad = \quad \sigma_s\sigma_v h\left(\tfrac{1-p}{p}p_u + \tfrac{p}{1-p}p_d\right).
\end{aligned}
$$

Using an argument which is similar to the one given above one may show that

$$
\begin{aligned}
p_u \quad &= \quad p \;+\; o(\sqrt{h}) \\
p_d \quad &= \quad 1 - p \;+\; o(\sqrt{h})
\end{aligned}
$$

and (3.5) follows. ■

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