# Hedging default risks of CDOs in Markovian contagion models

**J.-P. Laurent, A. Cousin, J-D. Fermanian**[1]

**First version: 10 May 2007**
**This version: 26 September 2007**

### Abstract

We describe a hedging strategy of CDO tranches based upon dynamic trading of the corresponding credit default swap index. We rely upon a homogeneous Markovian contagion framework, where only single defaults occur. In our framework, a CDO tranche can be perfectly replicated by dynamically trading the credit default swap index and a risk-free asset. Default intensities of the names only depend upon the number of defaults and are calibrated onto an input loss surface. Numerical implementation can be carried out fairly easily thanks to a recombining tree describing the dynamics of the aggregate loss. Both continuous time market and its discrete approximation are complete. The computed credit deltas can be seen as a credit default hedge and may also be used as a benchmark to be compared with the market credit deltas. Though the model is quite simple, it provides some meaningful results which are discussed in detail. We study the robustness of the hedging strategies with respect to recovery rate and examine how input loss distributions drive the credit deltas. Using market inputs, we find that the deltas of the equity tranche are lower than those computed in the standard base correlation framework and relate this to the dynamics of dependence between defaults.

*Keywords*: CDOs, hedging, complete markets, contagion model, Markov chain, recombining tree.

## Introduction

When dealing with CDO tranches, the market approach to the derivation of credit default swap deltas consists in bumping the credit curves of the names and computing the ratios of changes in present value of the CDO tranches and the hedging credit default swaps. This

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[1] Jean-Paul Laurent is professor at ISFA Actuarial School, Université Lyon 1 and a scientific consultant for BNP Paribas (laurent.jeanpaul@free.fr or laurent.jeanpaul@univ-lyon1.fr, http://laurent.jeanpaul.free.fr), 50 avenue Tony Garnier, 69007, LYON, FRANCE.
Areski Cousin (areski.cousin@univ-lyon1.fr) is a PhD candidate at ISFA Actuarial School, Université Lyon 1, 50 avenue Tony Garnier, 69007, LYON, FRANCE.
Jean-David Fermanian (jean-david.fermanian@uk.bnpparibas.com) is a senior quantitative analyst within FIRST, Quantitative Credit Derivatives Research at BNP-Paribas, 10 Harewood Avenue, LONDON NW1 6AA.
The authors thank Matthias Arnsdorf, Fahd Belfatmi, Xavier Burtschell, Rama Cont, Michel Crouhy, Rüdiger Frey, Kay Giesecke, Michael Gordy, Jon Gregory, Alexander Herbertsson, Steven Hutt, Monique Jeanblanc, Vivek Kapoor, Pierre Miralles, Marek Musiela, Marek Rutkowski, Antoine Savine, Olivier Vigneron and the participants at the Global Derivatives Trading and Risk Management conference in Paris, the Credit Risk Summit in London, the 4th WBS fixed income conference and at the doctoral seminars of the University of Dijon and "séminaire Bachelier" for useful discussions and comments. We also thank Fahd Belfatmi, Marouen Dimassi and Pierre Miralles for very useful help regarding implementation and calibration issues. All remaining errors are ours. This paper has an academic purpose and may not be related to the way BNP Paribas hedges its credit derivatives books.

involves a pricing engine for CDO tranches, usually some mixture of copula and base correlation approaches, leading to so-called sticky deltas. The only rationale of this modus operandi is local hedging with respect to credit spread risks, provided that the trading books are marked-to-market with the same pricing engine. Even when dealing with small changes in credit spreads, there is no guarantee that this would lead to appropriate credit deltas. For instance one can think of changes in base correlation correlated with changes in credit spreads. Moreover, the standard approach is not associated with a replicating theory, thus inducing the possibility of unexplained drifts and time decay effects in the present value of hedged portfolios (see Petrelli *et al.* (2006)).

Unfortunately, the trading desks cannot rely on a sound theory to determine replicating prices of CDO tranches. This is partly due to the dimensionality issue, partly to the stacking of credit spread and default risks. Laurent (2006) considers the case of multivariate intensities in a conditionally independent framework and shows that for large portfolios where default risks are well diversified, one can concentrate on the hedging of credit spread risks and control the hedging errors. In this approach, the key assumption is the absence of contagion effects which implies that credit spreads of survival names do not jump at default times, or equivalently that defaults are not informative. Whether one should rely on this assumption is to be considered with caution as discussed in Das *et al*. (2007). Anecdotal evidence such as the failures of Delphi, Enron, Parmalat and WorldCom also show mixed results.

In this paper, we take an alternative route, concentrating on contagion effects and default risks and neglecting specific credit spread dynamics. Contagion models were introduced to the credit field by Davis and Lo (2001), Jarrow and Yu (2001) and further studied by Yu (2007). Schönbucher and Schubert (2001) show that copula models exhibit some contagion effects and relate jumps of credit spreads at default times to the partial derivatives of the copula. This is also the framework used by Bielecki *et al*. (2007) to address the hedging issue. A similar but somehow more tractable approach has been considered by Frey and Backhaus (2007a), since the latter paper considers some Markovian models of contagion. In a copula model, the contagion effects are computed from the dependence structure of default times, while in contagion models the intensity dynamics are the inputs from which the dependence structure of default times is derived. In both approaches, credit spreads shifts occur only at default times. Thanks to this quite simplistic assumption, and provided that no simultaneous defaults occurs, it can be shown that the CDO market is complete, i.e. CDO tranche cash-flows can be fully replicated by dynamically trading individual credit spread swaps or, in some cases, by trading the credit default swap index.

Lately, Frey and Backhaus (2007b) have considered the hedging of CDO tranches in a Markov chain credit risk model allowing for spread and contagion risk. In this framework, when the hedging instruments are credit default swaps with a given maturity, the market is incomplete. In order to derive dynamic hedging strategies, Frey and Backhaus (2007b) use risk minimization techniques. In a multivariate Poisson model, Elouerkhaoui (2006) also addresses the hedging problem thanks to the risk minimization approach. As can be seen from the previous papers, practical implementation can be cumbersome, especially when dealing the hedging ratios at different points in time and different states.

As far as applications are concerned, calibration of the credit dynamics to market inputs is critical. Calibration of Markov chain models similar to ours have recently been considered by a number of authors including van der Voort (2006), Schönbucher (2006), Arnsdorf and Halperin (2007), de Koch and Kraft (2007), Epple *et al*. (2007), Lopatin and Misirpashaev

(2007), Herbertsson (2007a, 2007b), Cont and Savescu (2007). The aim of the previous papers is to construct arbitrage-free, consistent with some market inputs, Markovian models of aggregate losses, possibly in incomplete markets, without detailing the feasibility and implementation of replication strategies. Regarding the hedging issues, a nice feature of our specification is that the market inputs completely determine the credit dynamics, thanks to the forward Kolmogorov equations. This parallels the approach of Dupire (1994) in the equity derivatives context. Thanks to this feature and the completeness of the market, one can unambiguously derive dynamic hedging strategies of CDO tranches. This can be seen as a benchmark for the study of more sophisticated, model or criteria dependent, hedging strategies.

For the paper to be self-contained, we recall in Section 1 the mathematics behind the perfect replicating strategy. The main tool there is a martingale representation theorem for multivariate point processes. In Section 2, we restrict ourselves to the case of homogeneous portfolios with Markovian intensities which results in a dramatic dimensionality reduction for the (risk-neutral) valuation of CDO tranches and the hedging of such tranches as well. We find out that the aggregate loss is associated with a pure birth process, which is now well documented in the credit literature. In line with several new papers, Section 3 provides some calibration procedures of such contagion models based on the marginal distributions of the number of defaults. Section 4 details the computation of replicating strategies of CDO tranches with respect to the credit default swap index, through a recombining tree on the aggregate loss. We look for the dependency of the hedging strategy upon the chosen recovery rate. We eventually discuss how hedging strategies are related to dependence assumptions in Gaussian copula and base correlation frameworks.

# 1 Theoretical framework

## 1.1 Default times

Throughout the paper, we will consider $n$ obligors and a random vector of default times $(\tau_1,\dots,\tau_n)$ defined on a probability space $(\Omega, A, P)$. We denote by $N_1(t)=1_{\{\tau_1\le t\}},\dots,$ $N_n(t)=1_{\{\tau_n\le t\}}$ the default indicator processes and by $H_{i,t}=\sigma\left(N_i(s), s\le t\right)$, $i=1,\dots,n$, $H_t=\overset{n}{\underset{i=1}{\vee}} H_{i,t}$. $(H_t)_{t\in\mathbb{R}^+}$ is the natural filtration associated with the default times.

We denote by $\tau^1,\dots,\tau^n$ the ordered default times and assume that no simultaneous defaults can occur, i.e. $\tau^1<\dots<\tau^n$, $P-a.s$. This assumption is important with respect to the completeness of the market. As shown below, it allows to dynamically hedge basket default swaps and CDOs with $n$ credit default swaps[2].

[2] In the general case where multiple defaults could occur, we have to consider possibly $2^n$ states, and we would require non standard credit default swaps with default payments conditionally on all sets of multiple defaults to hedge CDO tranches.

We moreover assume that there exists some $\left(P, H_t\right)$ intensities for the counting processes $N_i(t)$, $i = 1, \dots, n$, i.e. there exists some (non negative) $H_t -$ predictable processes $\alpha_1^P, \dots, \alpha_n^P$, such that $t \to N_i(t) - \int_0^t \alpha_i^P(s)ds$ are $\left(P, H_t\right)$ martingales.

### 1.2 Market assumptions

For the sake of simplicity, let us assume for a while that instantaneous digital default swaps are traded on the names. An instantaneous digital credit default swap on name $i$ traded at $t$, provides a payoff equal to $dN_i(t) - \alpha_i^Q(t)dt$ at $t + dt$. $dN_i(t)$ is the payment on the default leg and $\alpha_i^Q(t)$ is the (short term) premium on the default swap. Note that considering such instantaneous digital default swaps rather than actually traded credit default swaps is not a limitation of our purpose. This can rather be seen as a convenient choice of basis from a theoretical point of view. Of course, we will compute credit deltas with respect to traded credit default swaps in the applications below.

Since we deal with the filtration generated by default times, the credit default swap premiums are deterministic between two default events. Therefore, we restrain ourselves to a market where only default risks occurs and credit spreads themselves are driven by the occurrence of defaults. In our simple setting, there is no specific credit spread risk. This corresponds to the framework of Bielecki *et al*. (2007).

For simplicity, we further assume that (continuously compounded) default-free interest rates are constant and equal to $r$. Given some initial investment $V_0$ and some $H_t -$ predictable processes $\delta_1(\bullet), \dots, \delta_n(\bullet)$ associated with some self-financed trading strategy in instantaneous digital credit default swaps, we attain at time $T$ the payoff $V_0 e^{rT} + \sum_{i=1}^{n} \int_0^T \delta_i(s) e^{r(T-s)} \left(dN_i(s) - \alpha_i^Q(s)ds\right)$. $\delta_i(s)$ is the nominal amount of instantaneous digital credit default swap on name $i$ held at time $s$. This induces a net cash-flow of $\delta_i(s) \times \left(dN_i(s) - \alpha_i^Q(s)ds\right)$ at time $s + ds$, which has to be invested in the default-free savings account up to time $T$.

### 1.3 Hedging and martingale representation theorem

From the absence of arbitrage opportunities, $\alpha_1^Q, \dots, \alpha_n^Q$ are non negative $H_t -$ predictable processes. From the same reason, $\left\{\alpha_i^Q(t) > 0\right\} \overset{P-a.s.}{=} \left\{\alpha_i^P(t) > 0\right\}$. Under mild regularity assumptions, there thus exists a probability $Q$ equivalent to $P$ such that, $\alpha_1^Q, \dots, \alpha_n^Q$ are the $\left(Q, H_t\right)$ intensities associated with the default times (see Brémaud, chapter VI)[3].

Let us consider some $H_T$ – measurable $Q$ – integrable payoff $M$. Since $M$ depends upon the default indicators of the names up to time $T$, this encompasses the cases of CDO tranches and basket default swaps, provided that recovery rates are deterministic. Thanks to the

[3] Let us remark that the assumption of no simultaneous defaults also holds for $Q$.

integral representation theorem of point process martingales (see Brémaud, chapter III), there exists some $H_t$ - predictable processes $\theta_1,\ldots,\theta_n$ such that:

$$M = E^Q\left[M\right] + \sum_{i=1}^{n}\int_0^T \theta_i(s)\left(dN_i(s) - \alpha_i^Q(s)ds\right).$$

As a consequence, we can replicate $M$ with the initial investment $E^Q\left[Me^{-rT}\right]$ and the trading strategy based on instantaneous digital credit default swaps defined by $\delta_i(s) = \theta_i(s)e^{-r(T-s)}$ for $0 \le s \le T$ and $i = 1,\ldots,n$. Let us remark that the replication price at time *t*, is provided by $V_t = E^Q\left[Me^{-r(T-t)} \left| H_t\right.\right]$[4].

While the use of the representation theorem guarantees that, in our framework, any basket default swap can be perfectly hedged with respect to default risks, it does not provide a practical way to construct hedging strategies. As is the case with interest rate or equity derivatives, exhibiting hedging strategies involves some Markovian assumptions (see Subsection 2.3 and Section 4).

## 2 Homogeneous Markovian contagion models

### 2.1 Intensity specification

In the contagion approach, one starts from a specification of the risk-neutral pre-default default intensities $\alpha_1^Q,\ldots,\alpha_n^Q$[5]. In the previous section framework, the risk-neutral default intensities depend upon the complete history of defaults. More simplistically, it is often assumed that they depend only upon the current credit status, i.e. the default indicators; thus $\alpha_i^Q(t),\ i \in \{1,\ldots,n\}$ is a deterministic function of $N_1(t),\ldots,N_n(t)$. In this paper, we will further remain in this Markovian framework, i.e. the pre-default intensities will take the form $\alpha_i^Q\left(t, N_1(t),\ldots,N_n(t)\right)$[6]. Popular examples are the models of Kusuoka (1999), Jarrow and Yu (2001), Yu (2007), where the intensities are affine functions of the default indicators. The connexion between contagion models and Markov chains is described in the book of Lando (2004) and was further discussed in Herbertsson (2007a).

Another practical issue is related to name heterogeneity. Modelling all possible interactions amongst names leads to a huge number of contagion parameters and high dimensional

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[4] Let us notice that $M = E^Q\left[M \left| H_t\right.\right] + \sum_{i=1}^{n}\int_t^T \theta_i(s)\left(dN_i(s) - \alpha_i^Q(s)ds\right)$. As a consequence, we readily get $M = V_t e^{r(T-t)} + \sum_{i=1}^{n}\int_t^T \theta_i(s)\left(dN_i(s) - \alpha_i^Q(s)ds\right)$ which provides the time $t$ replication price of $M$.

[5] After default of name $i$, the intensity is equal to zero: $\alpha_i^Q(t) = 0$ on $\{N_i(t) = 1\}$.

[6] This Markovian assumption may be questionable, since the contagion effect of a default event may vanish as time goes by. The Hawkes process, that was used in the credit field by Giesecke and Goldberg (2006), Errais *et al*. (2007), provides such an example of a more complex time dependence. Other specifications with the same aim are discussed in Lopatin and Misirpashaev (2007).

problems, thus to numerical issues. For this practical purpose, we will further restrict to models where all the names share the same risk-neutral intensity[7]. This can be viewed as a reasonable assumption for CDO tranches on large indices, although this is obviously an issue with equity tranches for which idiosyncratic risk is an important feature. Since pre-default risk-neutral default intensities, $\alpha_1^Q,\ldots,\alpha_n^Q$ are equal, we will further denote these individual intensities by $\alpha_\bullet^Q$.

For further tractability, we will further rely on a strong name homogeneity assumption, that individual default intensities only depend upon the number of defaults. Let us denote by $N(t)=\sum_{i=1}^{n} N_i(t)$ the number of defaults at time $t$ within the pool of assets. Intensities thus take the form $\alpha_\bullet^Q\left(t, N(t)\right)$. This is related to mean-field approaches (see Frey and Backhaus (2007a)). As for parametric specifications, we can think of some additive effects, i.e. the pre-default name intensities take the form $\alpha_\bullet^Q(t)=\alpha+\beta N(t)$ for some constants $\alpha,\beta$ as mentioned in Frey and Backhaus (2007a), corresponding to the "linear counterparty risk model"[8], or multiplicative effects in the spirit of Davis and Lo (2001), i.e. the pre-default intensities take the form $\alpha_\bullet^Q(t)=\alpha\times\beta^{N(t)}$. Of course, we could think of a non-parametric model. Later on, we provide a calibration procedure of such unconstrained intensities onto market inputs.

For simplicity, we will further assume a constant recovery rate equal to $R$ and a constant exposure among the underlying names. The aggregate fractional loss at time $t$ is given by: $L(t)=\left(1-R\right)\dfrac{N(t)}{n}$. As a consequence of the no simultaneous defaults assumption, the intensity of $L(t)$ or of $N(t)$ is simply the sum of the individual default intensities and is itself only a function of the number of defaults process. Let us denote by $\lambda\left(t, N(t)\right)$ the risk-neutral loss intensity. It is related to the individual risk-intensities by:

$$\lambda(t, N(t))=\left(n-N(t)\right)\times\alpha_\bullet^Q\left(t, N(t)\right).$$

We are thus typically in a bottom-up approach, where one starts with the specification of name intensities and thus derives the dynamics of the aggregate loss.

## 2.2 Risk-neutral pricing

Let us remark that in a Markovian homogeneous contagion model, the process $N(t)$ is a Markov chain (under the risk-neutral probability $Q$), and more precisely a pure birth process, according to Karlin and Taylor (1975) terminology[9], since only single defaults can occur. The generator of the chain, $\Lambda(t)$ is quite simple:

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[7] This means that the pre-default intensities have the same functional dependence to the default indicators.

[8] Ding *et al*. (2006) consider the case where the intensity of the loss process is linear in the number of defaults. Then, the loss distribution is negative binomial.

[9] According to Feller's terminology, we should speak of a pure death process. Since, we later refer to Karlin and Taylor (1975), we will use that latter terminology.

$$\Lambda(t)=\begin{pmatrix} -\lambda(t,0) & \lambda(t,0) & 0 & 0 & 0 & 0 & 0 \\ 0 & -\lambda(t,1) & \lambda(t,1) & 0 & & & 0 \\ 0 & & \bullet & \bullet & & & 0 \\ 0 & & & & \bullet & & 0 \\ 0 & & & & & \bullet & 0 \\ 0 & & & & & -\lambda(t,n-1) & \lambda(t,n-1) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Such a simple model of the number of defaults dynamics was considered by Schönbucher (2006) where it is called the "one-step representation of the loss distribution". Our paper is a bottom-up view of the previous model, where the risk-neutral prices can actually be viewed as replicating prices. As an example of this approach, let us consider the replication price of a European payoff with payment date $T$, such as a "zero-coupon tranchelet", paying $1_{\{N(T)=k\}}$ at time $T$ for some $k\in\{0,1,\dots,n\}$. Let us denote by $V\left(t,N(t)\right)=e^{-r(T-t)}Q\left(N(T)=k\middle|N(t)\right)$ the time $t$ replication price and by $V(t,\bullet)$ the price vector whose components are $V(t,0),V(t,1),\dots,V(t,n)$ for $0\le t\le T$. We can thus relate the price vector $V(t,\bullet)$ to the terminal payoff, using the transition matrix:

$$V(t,\bullet)=e^{-r(T-t)}Q(t,T)V(T,\bullet),$$

where $V(T,N(T))=\delta_k\left(N(T)\right)$ and where the transition matrix between dates $t$ and $T$ is given by $Q\left(t,T\right)=\exp\left(\int_t^T\Lambda(s)ds\right)$[10].

These ideas have been put in practice by van der Voort (2006), Herbertsson and Rootzén (2006), Arnsdorf and Halperin (2007), de Koch and Kraft (2007), Epple *et al.* (2007), Herbertsson (2007a) and Lopatin and Misirpashaev (2007). These papers focus on the pricing of credit derivatives, while our concern here is the feasibility and implementation of replicating strategies.

### 2.3 Computation of credit deltas

A nice feature of homogeneous contagion models is that the credit deltas, i.e. the holdings in the instantaneous defaults swaps are the same for all (the non-defaulted) names, which results in a dramatic dimensionality reduction.

Let us consider a European[11] type payoff and denote its replication price at time $t$, $V(t,\bullet)$. In order to compute the credit deltas, let us remark that:

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[10] Since $e^{-rt}\times V\left(t,N(t)\right)$ is a $\left(Q,H_t\right)$ martingale and using Ito-Doeblin's formula, it can be seen that $V$ solves for the backward Kolmogorov equations:

$$\frac{\partial V\left(t,N(t)\right)}{\partial t}+\lambda\left(t,N(t)\right)\times\left(V\left(t,N(t)+1\right)-V\left(t,N(t)\right)\right)=rV\left(t,N(t)\right).$$

$$dV\left(t,N(t)\right)=\frac{\partial V\left(t,N(t)\right)}{\partial t}dt+\left(V\left(t,N(t)+1\right)-V\left(t,N(t)\right)\right)dN(t).$$

$V\left(t,N(t)+1\right)-V\left(t,N(t)\right)$ is associated with the jump in the price process when a default occurs in the credit portfolio, i.e. $dN(t)=1$. Thanks to the name homogeneity, $dN(t)=\sum_{i=1}^{n-N(t)} dN_i(t)$ [12] and using:

$$\frac{\partial V\left(t,N(t)\right)}{\partial t}+\lambda\left(t,N(t)\right)\times\left(V\left(t,N(t)+1\right)-V\left(t,N(t)\right)\right)=rV\left(t,N(t)\right),$$

we end up with:

$$dV\left(t,N(t)\right)=rV\left(t,N(t)\right)dt+\sum_{i=1}^{n-N(t)}\left(V\left(t,N(t)+1\right)-V\left(t,N(t)\right)\right)\times\left(dN_i(t)-\alpha_{\bullet}^{Q}\left(t,N(t)\right)dt\right).$$

As a consequence the credit deltas with respect to the individual instantaneous default swaps are equal to:

$$\delta_i(t)=e^{-r(T-t)}\left(V\left(t,N(t)+1\right)-V\left(t,N(t)\right)\right)\times\left(1-N_i(t)\right),$$

for $0\le t\le T$ and $i=1,\dots,n$.

Let us denote by $V_I(t,k)=e^{-r(T-t)}E^Q\left[1-\frac{N(T)}{n}\middle|N(t)=k\right]$ the time $t$ price of the equally weighted portfolio involving defaultable discount bonds and set $\delta_I\left(t,N(t)\right)=\frac{V\left(t,N(t)+1\right)-V\left(t,N(t)\right)}{V_I\left(t,N(t)+1\right)-V_I\left(t,N(t)\right)}$. It can readily be seen that:

$$dV\left(t,N(t)\right)=r\times\left(V\left(t,N(t)\right)-\delta_I\left(t,N(t)\right)V_I\left(t,N(t)\right)\right)dt+\delta_I\left(t,N(t)\right)dV_I\left(t,N(t)\right).$$

As a consequence, we can perfectly hedge a European type payoff, say a zero-coupon CDO tranche, using only the index portfolio and the risk-free asset[13]. The hedge ratio, with respect to the index portfolio is actually equal to $\delta_I\left(t,N(t)\right)=\frac{V\left(t,N(t)+1\right)-V\left(t,N(t)\right)}{V_I\left(t,N(t)+1\right)-V_I\left(t,N(t)\right)}$. The previous hedging strategy is feasible provided that $V_I\left(t,N(t)+1\right)\neq V_I\left(t,N(t)\right)$. The usual case corresponds to some positive dependence, thus $\alpha_{\bullet}^{Q}\left(t,0\right)\le\alpha_{\bullet}^{Q}\left(t,1\right)\le\cdots\le\alpha_{\bullet}^{Q}\left(t,n-1\right)$. Therefore $V_I\left(t,N(t)+1\right)<V_I\left(t,N(t)\right)$ [14]. The decrease in the index portfolio value is the

[11] At this stage, for notational simplicity, we assume that there are no intermediate payments. This corresponds for instance to the case of zero-coupon CDO tranches with up-front premiums. The more general case is considered in Section 4.

[12] The last $N(t)$ names have defaulted.

[13] As above, in order to ease the exposition, we neglect at this stage actual payoff features such as premium payments, amortization schemes, and so on. This is detailed in Section 4.

[14] In the case where $\alpha_{\bullet}^{Q}\left(t,0\right)=\alpha_{\bullet}^{Q}\left(t,1\right)=\cdots=\alpha_{\bullet}^{Q}\left(t,n-1\right)$, there are no contagion effects and default dates are independent. We still have $V_I\left(t,N(t)+1\right)<V_I\left(t,N(t)\right)$ since $V_I\left(t,N(t)\right)$ is linear in $k$.

consequence of a direct default effect (one name defaults) and an indirect effect related to a positive shift in the credit spreads associated with the non-defaulted names.

The idea of building a hedging strategy based on the change in value at default times was introduced in Arvanitis and Laurent (1999). The rigorous construction of a dynamic hedging strategy in a univariate case can be found in Blanchet-Scalliet and Jeanblanc (2004). Our result can be seen as a natural extension to the multivariate case, provided that we deal with Markovian homogeneous models: we simply need to deal with the number of defaults $N(t)$ and the index portfolio $V_I\left(t, N(t)\right)$ instead of a single default indicator $N_i(t)$ and the corresponding defaultable discount bond price.

Though this is not further needed in the computation of dynamic hedging strategies, we can actually build a bridge between the above Markov chain approach for the aggregate loss and well-known models involving credit migrations (see Appendix A).

## 3 Calibration of loss intensities

Another nice feature of the homogeneous Markovian contagion model is that the loss dynamics or equivalently the default intensities can be determined from market inputs such as CDO tranche premiums. Since the risk neutral dynamics are unambiguously derived from market inputs, so will be for dynamic hedging strategies of CDO tranches. This greatly facilitates empirical studies, since the replicating figures do not depend upon unobserved and difficult to calibrate parameters.

The construction of the implied Markov chain for the aggregate loss parallels the one made by Dupire (1994) to construct a local volatility model from call option prices. The local dynamics are derived thanks to the forward Kolmogorov equations. The main difference is the use of Markov chains instead of diffusion processes.

The calibration procedure depends on the available inputs. For a complete set of CDO tranche premiums or equivalently for a complete set of number of default distributions, Schönbucher (2006) provided the construction of the loss intensities. For the paper to be self-contained, we detail and comment this in the Appendix B. Lopatin and Misirpashaev (2007) also detail the similarities between the Dupire's approach and the building of the one step Markov chain of Schönbucher (2006).

In practical applications, we think that it is more appropriate to use a discrete set of loss distributions corresponding to liquid CDO tranche maturities. In the examples below, we will calibrate the loss intensities given a single calibration date $T$. For simplicity, we will be given the number of defaults probabilities $p(T,k)$, $k = 0,1,\dots,n$[15]. As for the computation of the latter quantities from quoted CDO tranche premiums, we refer to Krekel and Partenheimer (2006), Galiani *et al.* (2006), Meyer-Dautrich and Wagner (2007), Walker (2007a) and Torresetti *et al.* (2007). Practical issues related to the calibration inputs are also discussed in van der Voort (2006).

[15] Clearly, this involves more information that one could directly access through the quotes of liquid CDO tranches, especially with respect to small and large number of defaults.

For the sake of calibration on real market quotes, we have to put some restrictions on the previous model specifications. Now and in the sequel, we assume that the loss intensities are time homogeneous: the intensities do not depend on time but only on the number of realized defaults. We further denote by $\lambda_k = \lambda(t,k)$ for $0 \le t \le T$, the loss intensity for $k = 0,1,\ldots,n-1$ [16]. Our procedure is quite similar to Epple *et al.* (2007). For the paper to be self-contained, it is detailed in the Appendix C. Extensions to the calibration on several maturities are detailed in the Appendix D. Regarding the assumption of no simultaneous defaults, we also refer to Putyatin *et al*. (2005), Brigo *et al*. (2007), Walker (2007b). Allowing for multiple defaults could actually ease the calibration onto senior CDO tranche quotes.

Alternative calibrating approaches can be found in Herbertsson (2007a) or in Arnsdorf and Halperin (2007). In Herbertsson (2007a), the name intensities $\alpha_{\cdot}^{Q}\left(t, N(t)\right)$ are time homogeneous, piecewise linear in the number of defaults (the node points are given by standard detachment points) and they are fitted to spread quotes by a least square numerical procedure. Arnsdorf and Halperin (2007) propose a piecewise constant parameterization of name intensities (which are referred to as "contagion factors") in time. When intensities are piecewise linear in the number of defaults too, they use a "multi-dimensional solver" to calibrate onto the observed tranche prices[17]. In the same vein, Frey and Backhaus (2007a, 2007b) introduce a parametric form for the function $\lambda(t,k)$, a variant of the "convex counterparty risk model", and fit the parameters to some tranche spreads. Lopatin and Misirpashaev (2007) express the loss intensity $\lambda(t,k)$ as a polynomial function of an auxiliary variable involving the number of defaults.

# 4 Computation of credit deltas through a recombining tree

## 4.1 Building up a tree

We now address the computation of CDO tranche deltas with respect to the credit default swap index of the same maturity. As for the hedging instrument, the premium is set at the inception of the deal and remains fixed. Dealing with the credit default swap index at current market conditions would have been another possible choice. This would have led to a change of the hedging instrument at every step, due to changes in the par spread and to accrued coupon effects. We do not take either into account roll dates every six months and trade the same index series up to maturity. The former choice involves the same hedging instrument throughout the trading period[18]. Switching from one hedging instrument to another could be dealt with very easily in our framework and closer to market practice but we thought that using the same underlying across the tree would simplify the exposition.

The (fractional) loss at time $t$ is given by $L(t) = (1-R)\dfrac{N(t)}{n}$. Let us consider a tranche with attachment point $a$ and detachment point $b$, $0 \le a \le b \le 1$. Up to some minor adjustment for the premium leg (see below), the credit default swap index is a $[0,1]$ tranche. We denote by

[16] Therefore, the pre-default name intensity is such that $\alpha_{\cdot}^{Q}\left(t, N(t)\right) = \dfrac{\lambda_{N(t)}}{n - N(t)}$. Let us recall that $\lambda(t,n) = 0$.

[17] In both approaches, there are as many unknown parameters as available market quotes.

[18] Actually, the credit deltas at inception are the same whatever the choice.

$O(N(t))$ the outstanding nominal on a tranche. It is equal to $b-a$ if $L(t)<a$, to $b-L(t)$ if $a \le L(t) < b$ and to $0$ if $L(t) \ge b$.

Let us recall that, for a European type payoff the price vector fulfils $V(t,\bullet) = e^{-r(t'-t)} Q(t,t') V(t',\bullet)$ for $0 \le t \le t' \le T$. The transition matrix can be expressed as $Q(t,t') = \exp\left( \int_t^{t'} \Lambda(s) ds \right)$ where $\Lambda(t)$ is the generator matrix associated with the number of defaults process. In the time homogeneous framework discussed in the previous section, the generator matrix does not depend on time.

For practical implementation, we will be given a set of node dates $t_0 = 0, \ldots, t_i, \ldots, t_{n_s} = T$. For simplicity, we will further consider a constant time step $\Delta = t_1 - t_0 = \cdots = t_i - t_{i-1} = \cdots$; this assumption can easily be relaxed. The most simple discrete time approximation one can think of is $Q(t_i, t_{i+1}) = Id + \Lambda(t_i) \times (t_{i+1} - t_i)$, which leads to $Q\left(N(t_{i+1}) = k+1 \middle| N(t_i) = k\right) = \lambda_k \Delta$ and $Q\left(N(t_{i+1}) = k \middle| N(t_i) = k\right) = 1 - \lambda_k \Delta$. For large $\lambda_k$, the transition probabilities can become negative. Thus, we will rather use $Q\left(N(t_{i+1}) = k+1 \middle| N(t_i) = k\right) = 1 - e^{-\lambda_k \Delta}$ and $Q\left(N(t_{i+1}) = k \middle| N(t_i) = k\right) = e^{-\lambda_k \Delta}$.

Under the previous approximation the number of defaults process can be described through a recombining tree as in van der Voort (2006). One could clearly think of more sophisticated continuous Markov chain techniques[19], but we think that the tree implementation is quite intuitive from a financial point of view. Convergence of the discrete time Markov chain to its continuous limit is a rather standard issue and will not be detailed here.

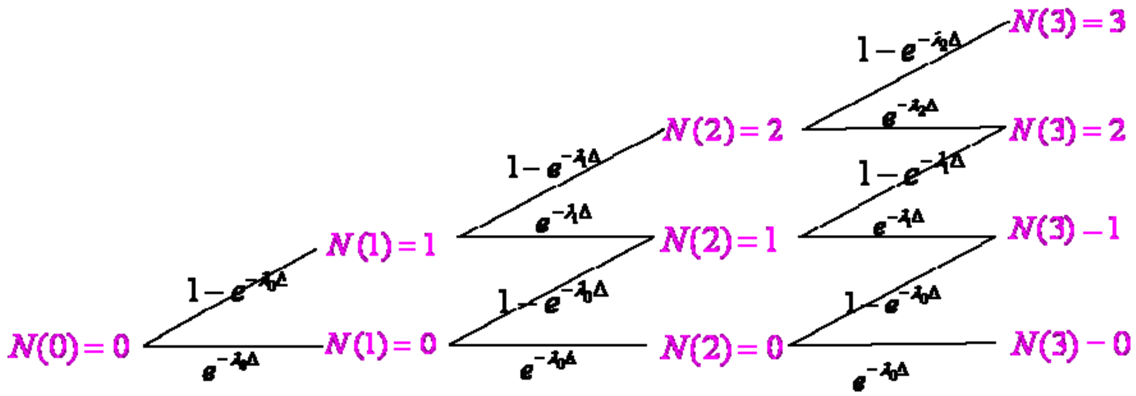

Figure 1. Number of defaults tree

**4.2 Computation of hedge ratios for CDO tranches**

[19] For such approaches, we refer to Herbertsson (2007a) and Moler and Van Loan (2003) regarding the numerical issues. However, we found that the tree approach led to efficient implementation. Clearly, the time step must be kept under control for large intensities.

Let us denote by $d(i,k)$ the value at time $t_i$ when $N(t_i)=k$ of the default payment leg of the CDO tranche[20]. The default payment at time $t_{i+1}$ is equal to $O(N(t_i))-O(N(t_{i+1}))$. Thus, $d(i,k)$ is given by the following recurrence equation:

$$d(i,k)=e^{-r\Delta}\left(\left(1-e^{-\lambda_k\Delta}\right)\times\left(d(i+1,k+1)+O(k)-O(k+1)\right)+e^{-\lambda_k\Delta}d(i+1,k)\right).$$

Let us now deal with a (unitary) premium leg. We denote the regular premium payment dates by $T_1,\ldots,T_p$ and for simplicity we assume that: $\{T_1,\ldots,T_p\}\subset\{t_0,\ldots,t_{n_s}\}$. Let us consider some date $t_{i+1}$ and set $l$ such that $T_l<t_{i+1}\leq T_{l+1}$. Whatever $t_{i+1}$, there is an accrued premium payment of $\left(O(N(t_i))-O(N(t_{i+1}))\right)\times\left(t_{i+1}-T_l\right)$. if $t_{i+1}=T_{l+1}$, i.e. $t_{i+1}$ is a regular premium payment date, there is an extra premium cash-flow at time $t_{i+1}$ of $O(N(T_{l+1}))\times\left(T_{l+1}-T_l\right)$. Thus, if $t_{i+1}$ is a regular premium payment date, the total premium payment is equal to $O(N(t_i))\times\left(T_{l+1}-T_l\right)$.

Let us denote by $r(i,k)$ the value at time $t_i$ when $N(t_i)=k$ of the unitary premium leg[21]. If $t_{i+1}\in\{T_1,\ldots,T_p\}$, $r(i,k)$ is provided by:

$$r(i,k)=e^{-r\Delta}\left(O(k)\times\left(T_{l+1}-T_l\right)+\left(1-e^{-\lambda_k\Delta}\right)\times r(i+1,k+1)+e^{-\lambda_k\Delta}r(i+1,k)\right)$$

If $t_{i+1}\notin\{T_1,\ldots,T_p\}$, then:

$$r(i,k)=e^{-r\Delta}\left(\left(1-e^{-\lambda_k\Delta}\right)\times\left(r(i+1,k+1)+\left(O(k)-O(k+1)\right)\times\left(t_{i+1}-T_l\right)\right)+e^{-\lambda_k\Delta}r(i+1,k)\right).$$

The CDO tranche premium is equal to $s=\dfrac{d(0,0)}{r(0,0)}$. The value of the CDO tranche (buy protection case) at time $t_i$ when $N(t_i)=k$ is given by $V_{CDO}(i,k)=d(i,k)-sr(i,k)$. The equity tranche needs to be dealt with slightly differently since its spread is set to $s=500\text{bp}$. However, the value of the CDO equity tranche is still given by $d(i,k)-sr(i,k)$.

As for the credit default swap index, we will denote by $r_{IS}(i,k)$ and $d_{IS}(i,k)$ the values of the premium and default legs. The credit default swap index spread at time $t_i$ when $N(t_i)=k$ is given by $s_{IS}(i,k)\times r_{IS}(i,k)=d_{IS}(i,k)$ [22]. The value of the credit default swap index, bought at inception, at node $(i,k)$ is given by $V_{IS}(i,k)=d_{IS}(i,k)-s_{IS}(0,0)\times r_{IS}(i,k)$. The default leg of the credit default swap index is computed as a standard default leg of a $[0,100\%]$ CDO tranche. Thus, in the recursion equation giving $d_{IS}(i,k)$ we write the outstanding nominal for

[20] We consider the value of the default leg immediately after $t_i$. Thus, we do not consider a possible default payment at $t_i$ in the calculation of $d(i,k)$.

[21] As for the default leg, we consider the value of the premium leg immediately after $t_i$. Thus, we do not take into account a possible premium payment at $t_i$ in the calculation of $r(i,k)$ either.

[22] This is an approximation of the index spread since, according to market rules, the first premium payment is reduced.

$k$ defaults as $O(k) = 1 - \frac{k(1-R)}{n}$, where $R$ is the recovery rate and $n$ the number of names. According to standard market rules, the premium leg of the credit default swap index needs a slight adaptation since the premium payments are based only upon the number of non-defaulted names and do not take into account recovery rates. As a consequence, the outstanding nominal to be used in the recursion equations providing $r_{IS}(i,k)$ is such that $O(k) = 1 - \frac{k}{n}$.

As usual in binomial trees, $\delta(i,k)$ is the ratio of the difference of the option value (at time $t_{i+1}$) in the upper state ($k+1$ defaults) and lower state ($k$ defaults) and the corresponding difference for the underlying asset. In our case, both the CDO tranche and the credit default swap index are "dividend-baring". For instance, when the number of defaults switches for $k$ to $k+1$, the default leg of the CDO tranche is associated with a default payment of $O(k) - O(k+1)$. Similarly, given the above discussion, when the number of defaults switches for $k$ to $k+1$, the premium leg of the CDO tranche is associated with an accrued premium payment of $-s \times 1_{t_{i+1} \notin \{T_1,\ldots,T_p\}} \left( O(k) - O(k+1) \right) \times \left( t_{i+1} - T_l \right)$[23]. Thus, when a default occurs the change in value of the CDO tranche is the outcome of a capital gain of $V_{CDO}\left(i+1,k+1\right) - V_{CDO}\left(i+1,k\right)$ and of a cash-flow of $\left( O(k) - O(k+1) \right) \times \left( 1 - s \times 1_{t_{i+1} \notin \{T_1,\ldots,T_p\}} \times \left( t_{i+1} - T_l \right) \right)$.

The credit delta of the CDO tranche at node $(i,k)$ with respect to the credit default swap index is thus given by:

$$\delta(i,k) = \frac{V_{CDO}\left(i+1,k+1\right) - V_{CDO}\left(i+1,k\right) + \left( O(k) - O(k+1) \right) \times \left( 1 - s \times 1_{t_{i+1} \notin \{T_1,\ldots,T_p\}} \times \left( t_{i+1} - T_l \right) \right)}{V_{IS}\left(i+1,k+1\right) - V_{IS}\left(i+1,k\right) + \frac{1-R}{n} - \frac{1}{n} \times s_{IS}(0,0) \times 1_{t_{i+1} \notin \{T_1,\ldots,T_p\}} \times \left( t_{i+1} - T_l \right)}.$$

Let us remark that using the previous credit deltas leads to a perfect replication of a CDO tranche within the tree, which is feasible since the approximating discrete market is complete.

We also remark that we can easily compute credit deltas with respect to the credit default swap index traded at current market conditions by using $s_{IS}(i,k)$ instead of $s_{IS}(0,0)$ when computing $V_{IS}$ at time $t_{i+1}$ and in the $\delta(i,k)$ expression.

### 4.3 Model calibrated on a loss distribution associated with a Gaussian copula

In this numerical illustration, the loss intensities $\lambda_k$ are computed from a loss distribution generated from a one factor Gaussian copula. The correlation parameter is equal to $\rho = 30\%$, the credit spreads are all equal to 20 basis points per annum, the recovery rate is such that

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[23] If $t_{i+1} \in \{T_1,\ldots,T_p\}$, the premium payment is the same whether the number of defaults is equal to $k$ or $k+1$. So, it does not appear in the computation of the credit delta.

$R = 40\%$ and the maturity is $T = 5$ years. The number of names is $n = 125$. Figure 2 shows the number of defaults distribution.

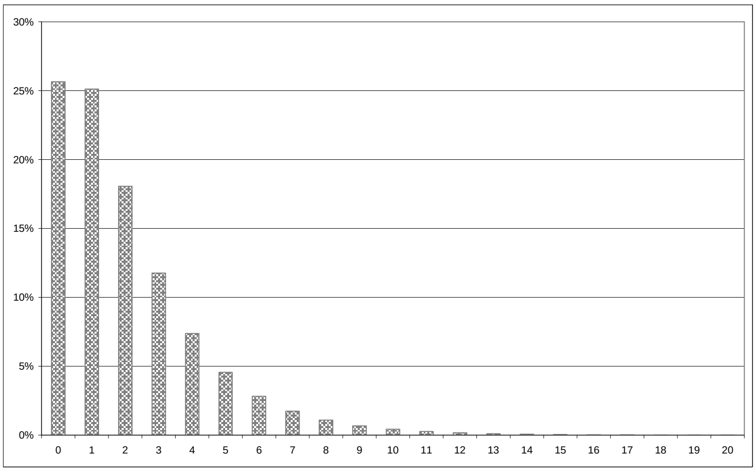


Figure 2. Number of defaults distribution. Number of defaults on the $x-$ axis.

Loss intensities $\lambda_k$ are calibrated as previously discussed up to $k = 49$ defaults (see Table 1).

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **0** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** |
| 0.27 | 0.41 | 0.57 | 0.75 | 0.94 | 1.15 | 1.36 | 1.59 | 1.82 | 2.05 |
| **10** | **11** | **12** | **13** | **14** | **15** | **16** | **17** | **18** | **19** |
| 2.29 | 2.54 | 2.79 | 3.04 | 3.29 | 3.55 | 3.80 | 4.06 | 4.32 | 4.58 |
| **20** | **21** | **22** | **23** | **24** | **25** | **26** | **27** | **28** | **29** |
| 4.84 | 5.10 | 5.35 | 5.61 | 5.87 | 6.12 | 6.38 | 6.63 | 6.88 | 7.13 |
| **30** | **31** | **32** | **33** | **34** | **35** | **36** | **37** | **38** | **39** |
| 7.37 | 7.62 | 7.86 | 8.10 | 8.34 | 8.57 | 8.80 | 9.03 | 9.25 | 9.47 |
| **40** | **41** | **42** | **43** | **44** | **45** | **46** | **47** | **48** | **49** |
| 9.69 | 9.91 | 10.12 | 10.32 | 10.53 | 10.72 | 10.92 | 11.11 | 11.30 | 11.48 |

Table 1. $\lambda_k, k = 0, \ldots, n_{\max} - 1$

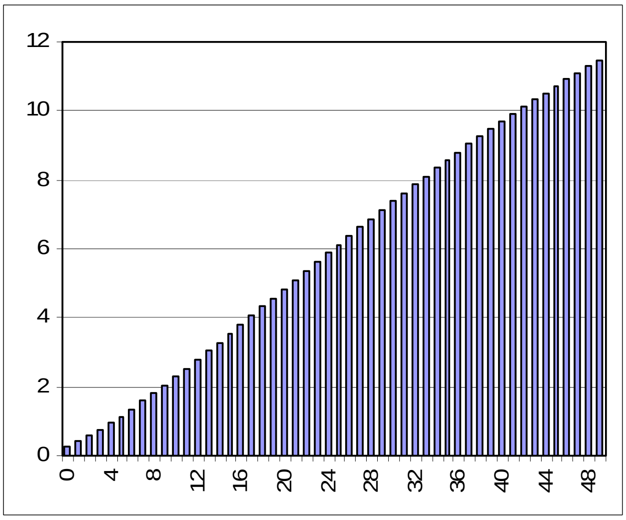


Figure 3. $\lambda_k, k = 0, \ldots, 49$

As can be seen from Figure 3, the loss intensity $\lambda_k$ changes almost linearly with respect to the number of defaults. Under the Gaussian copula assumption, the default probabilities $p(5, k)$

are insignificant for $k > 49$[24]. To avoid numerical difficulties, we computed the corresponding $\lambda_k$ by linear extrapolation. We checked that various choices of loss intensities for high number of defaults had no effect on the computation of deltas[25]. Let us also remark that such rather linear behaviour of loss intensities can be found in Lopatin and Misirpashaev (2007). Our results can also be related to the analysis of Ding *et al.* (2007): the Gaussian copula with flat correlation might be seen as a static version of a dynamical model

| | | Weeks | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | **0** | **14** | **28** | **42** | **56** | **70** | **84** |
| **Nb Defaults** | **0** | 20 | 19 | 19 | 18 | 18 | 17 | 17 |
| | **1** | 0 | 31 | 30 | 29 | 28 | 27 | 26 |
| | **2** | 0 | 46 | 45 | 43 | 41 | 40 | 38 |
| | **3** | 0 | 64 | 62 | 59 | 57 | 54 | 52 |
| | **4** | 0 | 84 | 81 | 77 | 74 | 71 | 68 |
| | **5** | 0 | 106 | 102 | 97 | 93 | 89 | 85 |
| | **6** | 0 | 130 | 125 | 119 | 114 | 109 | 104 |
| | **7** | 0 | 156 | 149 | 142 | 136 | 130 | 123 |
| | **8** | 0 | 184 | 175 | 167 | 159 | 152 | 144 |
| | **9** | 0 | 212 | 202 | 193 | 184 | 175 | 166 |
| | **10** | 0 | 242 | 231 | 220 | 209 | 199 | 189 |
| | **11** | 0 | 273 | 260 | 248 | 236 | 224 | 213 |
| | **12** | 0 | 305 | 291 | 277 | 263 | 250 | 238 |
| | **13** | 0 | 338 | 322 | 306 | 291 | 277 | 263 |
| | **14** | 0 | 372 | 354 | 337 | 320 | 304 | 289 |
| | **15** | 0 | 407 | 387 | 368 | 350 | 332 | 315 |

Table 2. $s_{IS}(i,k)$ in basis points per annum

Table 2 shows the dynamics of the credit default swap index spreads $s_{IS}(i,k)$ along the nodes of the tree. The continuously compounded default free rate is $r = 3\%$ and the time step is $\Delta = \frac{1}{365}$. It can be seen that default arrivals are associated with rather large jumps of credit spreads. For instance, if a (first) default occurs after a quarter, the credit default swap index spread jumps from 19 bps to 31 bps. An extra default by this time leads to an index spread of 46 bps (see Table 2).

The credit deltas with respect to the credit default swap index $\delta(i,k)$ have been computed for the $[0,3\%]$, $[3,6\%]$ and $[6,9\%]$ CDO tranches (see Tables 3, 6 and 7). As for the equity tranche, it can be seen that the credit deltas are positive and decrease up to zero. This is not surprising given that a buy protection equity tranche involves a short put position over the aggregate loss with a 3% strike. This is associated with positive deltas, negative gammas and thus decreasing deltas. When the number of defaults is above 6, the equity tranche is exhausted and the deltas obviously are equal to zero.

[24] $\sum_{k\geq 50} p(5,k) \simeq 2\times 10^{-9}$, $p(5,50) \simeq 6.1\times 10^{-10}$, $p(5,125) \simeq 2\times 10^{-33}$

[25] Let us stress that this applies for the Gaussian copula case since the loss distribution has thin tails. For the market case example, we proceeded differently.

| | | OutStanding Nominal | Weeks | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | 0 | 14 | 28 | 42 | 56 | 70 | 84 |
| Nb Defaults | 0 | 3.00% | 0.958 | 0.984 | 1.007 | 1.027 | 1.044 | 1.057 | 1.068 |
| | 1 | 2.52% | 0.000 | 0.736 | 0.780 | 0.822 | 0.862 | 0.900 | 0.935 |
| | 2 | 2.04% | 0.000 | 0.438 | 0.483 | 0.530 | 0.580 | 0.633 | 0.687 |
| | 3 | 1.56% | 0.000 | 0.208 | 0.235 | 0.266 | 0.303 | 0.344 | 0.391 |
| | 4 | 1.08% | 0.000 | 0.085 | 0.095 | 0.108 | 0.124 | 0.143 | 0.167 |
| | 5 | 0.60% | 0.000 | 0.031 | 0.034 | 0.038 | 0.042 | 0.047 | 0.054 |
| | 6 | 0.12% | 0.000 | 0.005 | 0.005 | 0.006 | 0.006 | 0.007 | 0.008 |
| | 7 | 0.00% | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

Table 3. $\delta(i,k)$ for the $\left[0,3\%\right]$ equity tranche

The credit deltas $\delta(i,k)$ can be decomposed into a default leg delta $\delta_d(i,k)$ and a premium leg delta $\delta_r(i,k)$ as follows: $\delta(i,k)=\delta_d(i,k)-s\delta_r(i,k)$ with:

$$\delta_d(i,k)=\frac{d\left(i+1,k+1\right)-d\left(i+1,k\right)+O(k)-O(k+1)}{V_{IS}\left(i+1,k+1\right)-V_{IS}\left(i+1,k\right)+\dfrac{1-R}{n}-\dfrac{1}{n}\times s_{IS}(0,0)\times 1_{t_{i+1}\notin\left\{T_1,\dots,T_p\right\}}\times\left(t_{i+1}-T_l\right)},$$

and:

$$\delta_r(i,k)=\frac{r\left(i+1,k+1\right)-r\left(i+1,k\right)+\left(O(k)-O(k+1)\right)1_{t_{i+1}\notin\left\{T_1,\dots,T_p\right\}}\times\left(t_{i+1}-T_l\right)}{V_{IS}\left(i+1,k+1\right)-V_{IS}\left(i+1,k\right)+\dfrac{1-R}{n}-\dfrac{1}{n}\times s_{IS}(0,0)\times 1_{t_{i+1}\notin\left\{T_1,\dots,T_p\right\}}\times\left(t_{i+1}-T_l\right)}.$$

Tables 4 and 5 detail the credit deltas associated with the default and premium legs of the equity tranche. As can be seen from Table 3, credit deltas for the equity tranche may be slightly above one when no default has occurred. Table 5 shows that this is due to the amortization scheme of the premium leg which is associated with significant negative deltas. Let us recall that premium payments are based on the outstanding nominal. Arrival of defaults thus reduces the commitment to pay. Furthermore, the increase in credit spreads due to contagion effects involves a decrease in the expected outstanding nominal. When considering the default leg only, we are led to credit deltas that actually remain within the standard 0%-100% range. The default leg of the equity tranche with respect to the credit default swap index is initially equal to 81.4%. Let us also remark that credit deltas of the default leg gradually increase with time which is consistent with a decrease in time value.

| | | OutStanding Nominal | Weeks | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | 0 | 14 | 28 | 42 | 56 | 70 | 84 |
| Nb Defaults | 0 | 3.00% | 0.810 | 0.839 | 0.865 | 0.889 | 0.911 | 0.929 | 0.946 |
| | 1 | 2.52% | 0 | 0.613 | 0.657 | 0.701 | 0.743 | 0.785 | 0.823 |
| | 2 | 2.04% | 0 | 0.343 | 0.386 | 0.432 | 0.483 | 0.536 | 0.591 |
| | 3 | 1.56% | 0 | 0.142 | 0.167 | 0.197 | 0.231 | 0.271 | 0.318 |
| | 4 | 1.08% | 0 | 0.046 | 0.055 | 0.066 | 0.080 | 0.097 | 0.119 |
| | 5 | 0.60% | 0 | 0.014 | 0.015 | 0.018 | 0.021 | 0.025 | 0.031 |
| | 6 | 0.12% | 0 | 0.002 | 0.002 | 0.002 | 0.003 | 0.003 | 0.004 |
| | 7 | 0.00% | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table 4. $\delta_d(i,k)$ for the $\left[0,3\%\right]$ equity tranche

| | | OutStanding Nominal | Weeks | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | 0 | 14 | 28 | 42 | 56 | 70 | 84 |
| Nb Defaults | 0 | 3.00% | -0.150 | -0.147 | -0.143 | -0.139 | -0.134 | -0.129 | -0.123 |
| | 1 | 2.52% | 0 | -0.127 | -0.126 | -0.124 | -0.121 | -0.118 | -0.114 |
| | 2 | 2.04% | 0 | -0.099 | -0.100 | -0.101 | -0.101 | -0.101 | -0.099 |
| | 3 | 1.56% | 0 | -0.067 | -0.070 | -0.072 | -0.074 | -0.076 | -0.077 |
| | 4 | 1.08% | 0 | -0.039 | -0.042 | -0.044 | -0.046 | -0.048 | -0.050 |
| | 5 | 0.60% | 0 | -0.018 | -0.019 | -0.021 | -0.022 | -0.023 | -0.024 |
| | 6 | 0.12% | 0 | -0.003 | -0.003 | -0.003 | -0.004 | -0.004 | -0.004 |
| | 7 | 0.00% | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table 5. $s\delta_r(i,k)$ for the $[0,3\%]$ equity tranche

This previous decomposition is useless for the $[3,6\%]$ and $[6,9\%]$ tranches since the impact of the CDO tranche premium leg becomes negligible.

| | | OutStanding Nominal | Weeks | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | 0 | 14 | 28 | 42 | 56 | 70 | 84 |
| Nb Defaults | 0 | 3.00% | 0.162 | 0.139 | 0.118 | 0.097 | 0.078 | 0.061 | 0.046 |
| | 1 | 3.00% | 0 | 0.325 | 0.296 | 0.265 | 0.232 | 0.198 | 0.164 |
| | 2 | 3.00% | 0 | 0.492 | 0.484 | 0.468 | 0.444 | 0.413 | 0.374 |
| | 3 | 3.00% | 0 | 0.516 | 0.546 | 0.570 | 0.584 | 0.588 | 0.580 |
| | 4 | 3.00% | 0 | 0.399 | 0.451 | 0.505 | 0.556 | 0.604 | 0.645 |
| | 5 | 3.00% | 0 | 0.242 | 0.289 | 0.344 | 0.405 | 0.471 | 0.540 |
| | 6 | 3.00% | 0 | 0.126 | 0.156 | 0.193 | 0.238 | 0.293 | 0.359 |
| | 7 | 2.64% | 0 | 0.061 | 0.075 | 0.093 | 0.118 | 0.150 | 0.193 |
| | 8 | 2.16% | 0 | 0.032 | 0.037 | 0.044 | 0.054 | 0.068 | 0.089 |
| | 9 | 1.68% | 0 | 0.019 | 0.021 | 0.023 | 0.027 | 0.032 | 0.039 |
| | 10 | 1.20% | 0 | 0.012 | 0.012 | 0.013 | 0.015 | 0.016 | 0.018 |
| | 11 | 0.72% | 0 | 0.006 | 0.007 | 0.007 | 0.008 | 0.008 | 0.009 |
| | 12 | 0.24% | 0 | 0.002 | 0.002 | 0.002 | 0.002 | 0.002 | 0.003 |
| | 13 | 0.00% | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table 6. $\delta(i,k)$ for the $[3,6\%]$ tranche

At inception, the credit delta of the junior mezzanine tranche is equal to 16.2% whilst it is only equal to 1.7% for the $[6,9\%]$ tranche which is deeper out of the money (see Tables 6 and 7). The $[3,6\%]$ and $[6,9\%]$ CDO tranches involve a call spread position over the aggregate loss. As a consequence the credit deltas are positive and firstly increase (positive gamma effect) and then decrease (negative gamma) up to zero as soon as the tranche is fully amortized.

Given the recovery rate assumption of 40%, the outstanding nominal of the $[3,6\%]$ is equal to 3% for six defaults and to 2.64% for seven defaults. One might thus think that at the sixth default the $[3,6\%]$ should behave almost like an equity tranche. However, as can be seen from Table 6, the credit delta is much lower, 12.6% instead of 84% for the default leg of the equity tranche. This is due to dramatic shifts in credit spreads from 19 bps to 127 bps (see Table 2) when moving from the no-defaults to the six defaults state. In the latter case, the expected loss on the tranche is much larger, which is consistent with smaller deltas given the call spread payoff.

Let us remark that the sum of the default leg cash-flows of the CDO tranches is equal to the default leg cash-flows of the credit default swap index. On the other hand, apart from the equity tranche, the premium effects are quite small. The sum of the credit deltas of the default leg of the equity tranche and of the $[3,6\%]$ and $[6,9\%]$ tranches is actually close to one when the number of defaults is equal to 0 or 1. For larger number of defaults, one has to take into account the credit deltas of the most senior tranches that gradually increase.

| | | OutStanding Nominal | Weeks | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | 0 | 14 | 28 | 42 | 56 | 70 | 84 |
| Nb Defaults | 0 | 3.00% | 0.018 | 0.012 | 0.008 | 0.006 | 0.003 | 0.002 | 0.001 |
| | 1 | 3.00% | 0 | 0.050 | 0.037 | 0.026 | 0.018 | 0.012 | 0.007 |
| | 2 | 3.00% | 0 | 0.134 | 0.108 | 0.084 | 0.063 | 0.045 | 0.030 |
| | 3 | 3.00% | 0 | 0.256 | 0.226 | 0.193 | 0.158 | 0.124 | 0.092 |
| | 4 | 3.00% | 0 | 0.365 | 0.350 | 0.326 | 0.292 | 0.252 | 0.207 |
| | 5 | 3.00% | 0 | 0.399 | 0.416 | 0.421 | 0.413 | 0.391 | 0.354 |
| | 6 | 3.00% | 0 | 0.344 | 0.389 | 0.428 | 0.458 | 0.474 | 0.473 |
| | 7 | 3.00% | 0 | 0.242 | 0.294 | 0.349 | 0.406 | 0.459 | 0.502 |
| | 8 | 3.00% | 0 | 0.144 | 0.185 | 0.236 | 0.296 | 0.363 | 0.433 |
| | 9 | 3.00% | 0 | 0.077 | 0.103 | 0.137 | 0.182 | 0.240 | 0.310 |
| | 10 | 3.00% | 0 | 0.043 | 0.055 | 0.074 | 0.100 | 0.137 | 0.189 |
| | 11 | 3.00% | 0 | 0.028 | 0.034 | 0.042 | 0.054 | 0.074 | 0.103 |
| | 12 | 3.00% | 0 | 0.023 | 0.025 | 0.029 | 0.034 | 0.042 | 0.056 |
| | 13 | 2.76% | 0 | 0.019 | 0.020 | 0.022 | 0.024 | 0.027 | 0.033 |
| | 14 | 2.28% | 0 | 0.014 | 0.015 | 0.016 | 0.017 | 0.019 | 0.021 |
| | 15 | 1.80% | 0 | 0.011 | 0.011 | 0.012 | 0.013 | 0.013 | 0.014 |
| | 16 | 1.32% | 0 | 0.007 | 0.008 | 0.008 | 0.009 | 0.009 | 0.010 |
| | 17 | 0.84% | 0 | 0.004 | 0.005 | 0.005 | 0.005 | 0.006 | 0.006 |
| | 18 | 0.36% | 0 | 0.002 | 0.002 | 0.002 | 0.002 | 0.002 | 0.002 |
| | 19 | 0.00% | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table 7. $\delta(i,k)$ for the $[6,9\%]$ tranche

#### 4.4 Sensitivity of hedging strategies to the recovery rate assumption

The previous deltas have been computed under the assumption that the recovery rate was equal to 40% which is a standard but somehow arbitrary assumption. We further investigate the dependence of the dynamic hedging strategy with respect to the choice of recovery rate. Of course, changing only the recovery rate and not the number of defaults distribution would lead to a change in the expected losses of the CDO tranches and of the CDO premiums. For our robustness study to be meaningful, we will modify recovery rates but keep the loss surface (or equivalently the CDO tranche premiums) unchanged. This implies a change in the number of defaults distribution. The procedure is detailed in Appendix E.

| | Recovery Rates | | | | | |
|---|---|---|---|---|---|---|
| Tranches | 10% | 20% | 30% | 40% | 50% | 60% |
| [0-3%] | 0.9924 | 0.9774 | 0.9680 | 0.9585 | 0.9418 | 0.9321 |
| [3-6%] | 0.1545 | 0.1605 | 0.1607 | 0.1618 | 0.1659 | 0.1668 |
| [6-9%] | 0.0169 | 0.0171 | 0.0174 | 0.0175 | 0.0177 | 0.0179 |

Table 8. $\delta(0,0)$ for different recovery rates

Table 8 shows the credit deltas at the initial date for various CDO tranches under different recovery assumptions. Fortunately, the recovery rate assumption has a very small effect on the computed credit deltas. Table 9 shows the dynamic credit deltas of the equity tranche when the recovery rate is shifted from $R = 40\%$ to $R^* = 30\%$. This should be compared with the figures in Table 3 exhibiting the credit deltas under a 40% recovery rate assumption. Up to one default, the credit deltas are fairly close. As the number of defaults increase, the credit deltas gradually depart one from the other, which is not surprising given that the amortization scheme now differs.

| | | OutStanding Nominal | Weeks | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | 0 | 14 | 28 | 42 | 56 | 70 | 84 |
| Nb Defaults | 0 | 3.00% | 0.968 | 0.991 | 1.011 | 1.029 | 1.044 | 1.056 | 1.066 |
| | 1 | 2.44% | 0.000 | 0.731 | 0.771 | 0.809 | 0.847 | 0.883 | 0.916 |
| | 2 | 1.88% | 0.000 | 0.417 | 0.456 | 0.498 | 0.542 | 0.589 | 0.638 |
| | 3 | 1.32% | 0.000 | 0.181 | 0.202 | 0.227 | 0.255 | 0.288 | 0.325 |
| | 4 | 0.76% | 0.000 | 0.062 | 0.069 | 0.077 | 0.087 | 0.098 | 0.113 |
| | 5 | 0.20% | 0.000 | 0.012 | 0.012 | 0.013 | 0.015 | 0.016 | 0.019 |
| | 6 | 0.00% | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

Table 9. $\delta^*(i,k)$ for the $[0,3\%]$ equity tranche, $R^* = 30\%$

### 4.5 Dependence of hedging strategies upon the correlation parameter

Let us recall that the recombining tree is calibrated on a loss distribution over a given time horizon. The shape of the loss distribution depends critically upon the correlation parameter which was set up to now to $\rho = 30\%$. Decreasing the dependence between default events leads to a thinner right-tail of the loss distribution and smaller contagion effects. We detail here the effects of varying the correlation parameter on the hedging strategies.

For simplicity, we firstly focus the analysis on the default leg of the equity tranche and shift the correlation parameter from 30% to 10%. It can be seen from Tables 4 and 10 that the credit deltas are much higher in the latter case. After 14 weeks, prior to the first default, the credit delta is equal to 84% for a 30% correlation and to 97% when the correlation parameter is equal to 10%.

| | | OutStanding Nominal | Weeks | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | 0 | 14 | 28 | 42 | 56 | 70 | 84 |
| Nb Defaults | 0 | 3.00% | 0.963 | 0.968 | 0.973 | 0.977 | 0.980 | 0.983 | 0.985 |
| | 1 | 2.52% | 0 | 0.928 | 0.939 | 0.948 | 0.957 | 0.965 | 0.971 |
| | 2 | 2.04% | 0 | 0.831 | 0.852 | 0.872 | 0.891 | 0.908 | 0.924 |
| | 3 | 1.56% | 0 | 0.652 | 0.681 | 0.711 | 0.742 | 0.772 | 0.801 |
| | 4 | 1.08% | 0 | 0.405 | 0.434 | 0.464 | 0.497 | 0.531 | 0.568 |
| | 5 | 0.60% | 0 | 0.171 | 0.186 | 0.203 | 0.223 | 0.244 | 0.269 |
| | 6 | 0.12% | 0 | 0.028 | 0.030 | 0.033 | 0.037 | 0.041 | 0.046 |
| | 7 | 0.00% | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table 10. $\delta_d(i,k)$ for the $[0,3\%]$ equity tranche, $\rho = 10\%$

To further investigate how changes in correlation levels alter credit deltas, we computed the market value of the default leg of the equity tranche at a 14 weeks horizon as a function of the number of defaults under different correlation assumptions (see Figure 5). The market value of the default leg, on the $y$ – axis, is computed as the sum of expected discounted cash-flows posterior to this 14 weeks horizon date and the accumulated defaults cash-flows paid before. We also plotted the accumulated losses which represents the intrinsic value of the equity tranche default leg. Unsurprisingly, we recognize some typical concave patterns associated with a short put option payoff.

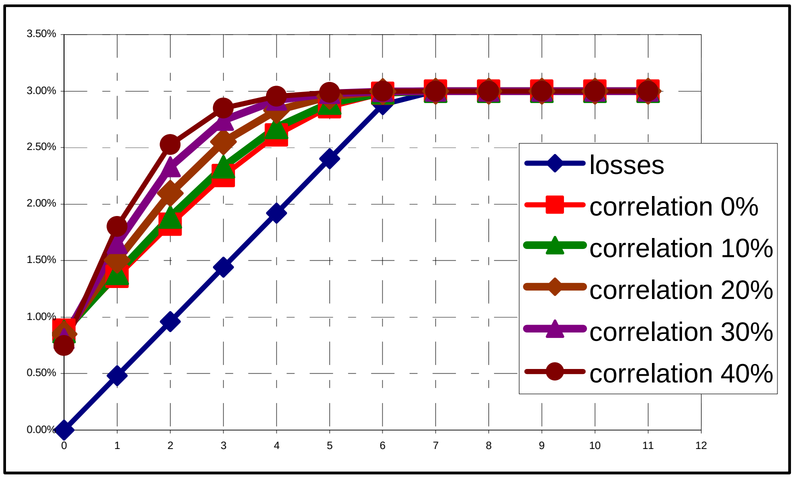


Figure 5. Market value of equity default leg under different correlation assumptions. Number of defaults on the $x$ – axis

As can be seen from Figure 5, prior to the first default, the value of the default leg of the equity tranche decreases as the correlation parameter increases from 0% to 40%[26]. However, after the first default the ordering of default leg values is reversed. This can be easily understood since larger correlations are associated with larger jumps in credit spreads at default arrivals and thus larger changes in the expected discounted cash-flows associated with the default leg of the equity tranche[27].

Therefore, varying the correlation parameter is associated with two opposite mechanisms:

- The first one is related to a typical negative vanna effect. Increasing correlation lowers loss "volatility" and leads to smaller expected losses on the equity tranche. In a standard option pricing framework, this should lead to an increase in the credit delta of the short put position on the loss.
- This is superseded by the shifts due to contagion effects. Increasing correlation is associated with bigger contagion effects and thus larger jumps in credit spreads at the arrival of defaults. This, in turn leads to a larger jump in the market value of the credit index default swap. Let us recall that the default leg of the equity tranche exhibit a concave payoff and thus a negative gamma. As a consequence the credit delta, i.e. the

[26] See Burtschell *et al.* (2005) for a formal proof of this well-known result.

[27] Let us remark that the larger the correlation the larger the change in market value of the default leg of the equity tranche at the arrival of the first default. This is not inconsistent with the previous results showing a decrease in credit deltas when the correlation parameter increases. The credit delta is the ratio of the change in value in the equity tranche and of the change in value in the credit default swap index. For a larger correlation parameter, the change in value in the credit default swap index is also larger due to magnified contagion effects.

ratio between the change in value of the option and the change in value of the underlying, decreases.

Let us also notice that for the 10% correlation example, the decrease in the credit delta when shifting from the no defaults case to the single default case is less pronounced than in the 30% correlation example. At the first default, the credit delta is still equal to 93% in the low correlation case and has dropped to 61% in the high correlation case. In other words, we have a smaller gamma at inception in the former case, but the gamma is ultimately larger after a few defaults since the deltas have to decrease to zero.

### 4.6 Taking into account a base correlation structure

Up to now, the probabilities of number of defaults were computed thanks to a Gaussian copula. In this example, we use a steep upward sloping base correlation curve for the iTraxx, typical of June 2007, as an input to derive the distribution of the probabilities of number of defaults (see Table 11). The maturity is still equal to 5 years, the recovery rate to 40% and the credit spreads to 20 bps. The default-free rate is now equal to 4%.

| | | | | |
|---|---|---|---|---|
| 3% | 6% | 9% | 12% | 22% |
| 16% | 24% | 30% | 35% | 50% |

Table 11. base correlation with respect to attachment points

Rather than spline interpolation, we used a parametric model to fit the market quotes and compute the probabilities of the number of defaults. This produces arbitrage free and smooth distributions that ease the calculation of the loss intensities[28]. Figure 6 shows the number of defaults distribution. This is rather different from the Gaussian copula case both for small and large losses. For instance, the probability of no defaults dropped from 25.6% to 19.5% while the probability of a single default rose from 25.1% to 36.5%. Let us stress that these figures are for illustrative purpose. The market does not provide direct information on first losses and thus the shape of the left tail of the loss distribution is a controversial issue. As for the right-tail, we have $\sum_{k\geq 50} p(5,k) \simeq 1.4\times 10^{-3}$ and $p(5,50) \simeq 3.3\times 10^{-6}$, $p(5,125) \simeq 1.38\times 10^{-3}$. The probabilities of large number of defaults, compared with the Gaussian copula case are much larger. The probability of the names defaulting altogether is also quite large, corresponding to some kind of Armageddon risk. Once again these figures need to be considered with caution, corresponding to high senior and super-senior tranche premiums and disputable assumptions about the probability of all names defaulting.

[28] We also computed the number of defaults distribution using entropic calibration. Although we could still compute loss intensities, the pattern with respect to the number of defaults was not monotonic. Such oscillations of the loss intensities can also be found in Cont and Savescu (2007): depending on market inputs, direct calibration onto CDO tranche quotes can lead to shaky figures.

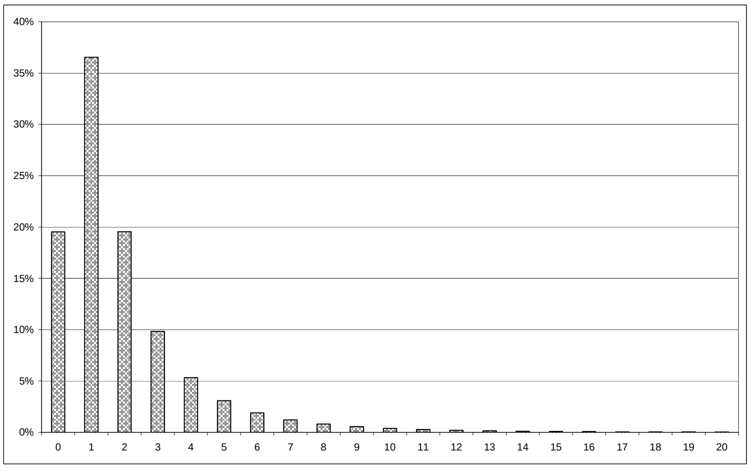


Figure 6. Number of defaults distribution. Number of defaults on the $x-$ axis.

Figure 7 shows the loss intensities calibrated onto market inputs compared with the loss intensities based on Gaussian copula inputs up to 39 defaults[29]. As can be seen, the loss intensity increases much quickly with the number of defaults as compared with the Gaussian copula approach. The average relative change in the loss intensities is equal to 19% when it is only equal to 10% when computed under the Gaussian copula assumption. Unsurprisingly, a steep base correlation curve is associated with fatter upper tails of the loss distribution and magnified contagion effects.

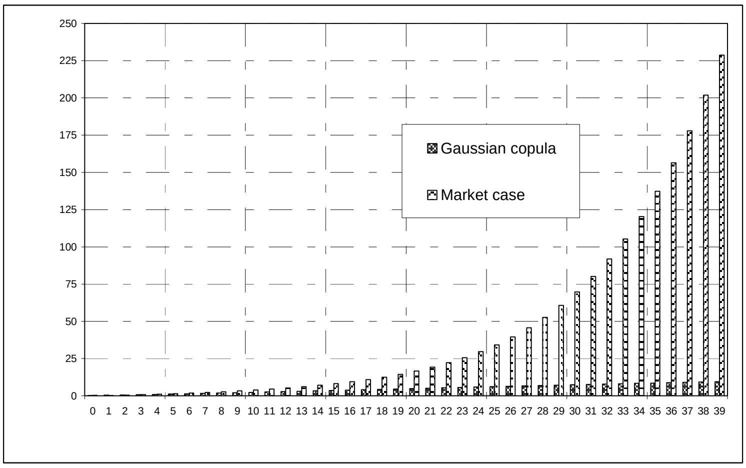


Figure 7. Loss intensities for the Gaussian copula and market case examples. Number of defaults on the $x-$ axis.

Table 12 shows the dynamics of the credit default swap index spreads $s_{IS}(i,k)$ along the nodes of the tree. As for tree implementation, the time step is still $\Delta = \frac{1}{365}$. Table 12 confirms the previous figure with much bigger contagion effects than in the Gaussian copula case. However, we notice that when going from the no default state to a single default at a 14 week horizon, credit spreads jump from 19 bps to 31 bps as in the Gaussian copula case. A further default leads to an index spread of 95 bps to be compared with only 46 bps in the Gaussian copula case. As mentioned above, this detailed pattern has to be considered with

[29] Contrary to the Gaussian copula example, we computed the complete set of loss intensities using the procedure described in subsection 3.2.

caution, since it involves the probability of 0, 1 and 2 defaults which are not directly observed in the market. After a few defaults, credit spreads become so large, that it is likely that most of the names will default by the 5 year time horizon.

| | | Weeks | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | **0** | **14** | **28** | **42** | **56** | **70** | **84** |
| **Nb Defaults** | **0** | 20 | 19 | 18 | 18 | 17 | 16 | 16 |
| | **1** | 0 | 31 | 28 | 25 | 23 | 21 | 20 |
| | **2** | 0 | 95 | 80 | 67 | 57 | 49 | 43 |
| | **3** | 0 | 269 | 225 | 185 | 150 | 121 | 98 |
| | **4** | 0 | 592 | 515 | 437 | 361 | 290 | 228 |
| | **5** | 0 | 1022 | 934 | 834 | 723 | 607 | 490 |
| | **6** | 0 | 1466 | 1395 | 1305 | 1193 | 1059 | 905 |
| | **7** | 0 | 1870 | 1825 | 1764 | 1680 | 1567 | 1420 |
| | **8** | 0 | 2243 | 2214 | 2177 | 2126 | 2052 | 1945 |
| | **9** | 0 | 2623 | 2597 | 2568 | 2534 | 2488 | 2423 |
| | **10** | 0 | 3035 | 3003 | 2971 | 2939 | 2903 | 2859 |
| | **11** | 0 | 3491 | 3450 | 3410 | 3371 | 3331 | 3290 |
| | **12** | 0 | 4001 | 3947 | 3896 | 3845 | 3795 | 3747 |
| | **13** | 0 | 4570 | 4501 | 4434 | 4369 | 4306 | 4245 |
| | **14** | 0 | 5206 | 5117 | 5031 | 4948 | 4868 | 4790 |
| | **15** | 0 | 5915 | 5801 | 5691 | 5586 | 5484 | 5386 |

Table 12. $s_{IS}(i,k)$ in basis points per annum

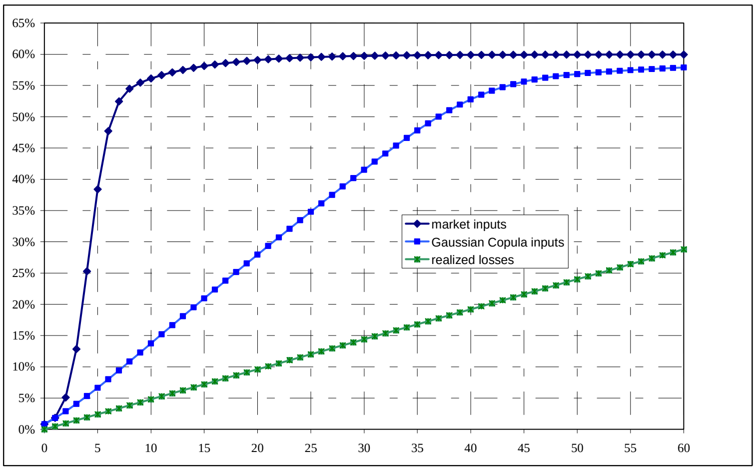


Figure 8. Expected loss on the credit portfolio after 14 weeks over a five year horizon ($y$ – axis) with respect to the number of defaults ($x$ – axis).

Figure 8 allows to further investigate the credit dynamics as deduced from market inputs. We plotted the conditional (with respect to the number of defaults) expected loss $E\left[L(T)\middle|N(t)\right]$ for $T=5$ years and $t=14$ weeks for the previous market inputs and for the 30% flat correlation Gaussian copula case. The conditional expected loss is expressed as a percentage of the nominal of the portfolio[30]. We also plotted the realized (or accumulated) losses on the portfolio. The expected losses are greater than the accumulated losses due to positive contagion effects. There are some dramatic differences between the Gaussian copula and the market inputs examples. In the Gaussian copula case, the expected loss is almost linear with

[30] Thus, given a recovery rate of 40%, the maximum expected loss is equal to 60%

respect to the number of defaults in a wide range (say up to 35 defaults). The pattern is quite different when using market inputs with huge non linearity effects. This shows large contagion effects after a few defaults as can also be seen from Table 12 and Figure 7. This rather explosive behaviour was also observed by Herbertsson (2007b), Tables 3 and 4. In Lopatin and Misirpashaev (2007), the contagion effects are also magnified when using market data, compared with Gaussian copula inputs, but only occur after a larger number of defaults than in our example.

| | | OutStanding Nominal | Weeks | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | **0** | **14** | **28** | **42** | **56** | **70** | **84** |
| **Nb Defaults** | **0** | 3.00% | 0.645 | 0.731 | 0.814 | 0.890 | 0.953 | 1.003 | 1.038 |
| | **1** | 2.52% | 0.000 | 0.329 | 0.402 | 0.488 | 0.584 | 0.684 | 0.777 |
| | **2** | 2.04% | 0.000 | 0.091 | 0.115 | 0.149 | 0.197 | 0.264 | 0.351 |
| | **3** | 1.56% | 0.000 | 0.023 | 0.028 | 0.035 | 0.045 | 0.062 | 0.090 |
| | **4** | 1.08% | 0.000 | 0.008 | 0.008 | 0.009 | 0.011 | 0.013 | 0.018 |
| | **5** | 0.60% | 0.000 | 0.004 | 0.004 | 0.003 | 0.003 | 0.003 | 0.004 |
| | 6 | 0.12% | 0.000 | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 |
| | 7 | 0.00% | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

Table 13. $\delta(i,k)$ for the $[0,3\%]$ equity tranche

| | | OutStanding Nominal | Weeks | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | **0** | **14** | **28** | **42** | **56** | **70** | **84** |
| **Nb Defaults** | **0** | 3.00% | 0.546 | 0.622 | 0.697 | 0.767 | 0.826 | 0.874 | 0.911 |
| | **1** | 2.52% | 0 | 0.283 | 0.349 | 0.427 | 0.516 | 0.608 | 0.695 |
| | **2** | 2.04% | 0 | 0.073 | 0.095 | 0.125 | 0.169 | 0.229 | 0.310 |
| | **3** | 1.56% | 0 | 0.016 | 0.020 | 0.026 | 0.035 | 0.050 | 0.074 |
| | **4** | 1.08% | 0 | 0.004 | 0.005 | 0.005 | 0.007 | 0.009 | 0.012 |
| | **5** | 0.60% | 0 | 0.002 | 0.002 | 0.002 | 0.002 | 0.002 | 0.002 |
| | **6** | 0.12% | 0 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| | **7** | 0.00% | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table 14. $\delta_d(i,k)$ for the $[0,3\%]$ equity tranche

Table 13 shows the dynamic deltas associated with the equity tranche. Table 14 focuses on the deltas of the default leg of the equity tranche[31]. We also notice that the credit deltas drop quite quickly to zero with the occurrence of defaults. This is not surprising given the surge in credit spreads and dependencies after the first default (see Figure 8): after only a few defaults the equity tranche is virtually exhausted.

It can be seen that the equity tranche deltas are much lower when taking into account a steep upward base correlation curve: for instance, at inception, the delta of the default leg is equal to 54.6% (see Table 14) while it was equal to 81% with a 30% flat correlation structure (see Table 4). Such a decrease in the credit delta is not related to a spread effect, since at 14 weeks the credit spreads of the index are the same in the no default and the single default cases. As a consequence, the change in value of the underlying credit default swap index when shifting to the first default is the same in the Gaussian copula and market inputs examples. The decrease

[31] As for the Gaussian copula example, we can see that the premium leg of the equity tranche significantly contributes to the total credit delta. We also found that the premium leg of the credit index default swap had some visible effect on the credit deltas after some defaults, when credit deltas are small.

in the credit delta is associated with a smaller value of the numerator in the delta computation (see Subsection 4.2) when using market inputs instead of Gaussian copula inputs. Let us recall that the numerator in the delta computation is the change of value of the equity tranche when shifting the number of defaults. Given the discussion in Subsection 4.5 about the dependence of credit deltas with respect to correlation parameters, the stated decrease in the credit delta of the equity tranche may look paradoxical: indeed the base correlation for the equity tranche in our market example is equal to 16% to be compared with 30% in the Gaussian copula example. As a consequence, one might wrongly conclude to an increase in the credit deltas when using market inputs.

The stated figures can be fully understood from the dynamics of correlation which is embedded in the model. When using market inputs and when considering the pricing of an equity tranche after a single default, the further contagion effects are much larger than when using Gaussian copula inputs (see Figure 8). Since larger contagion effects are associated with bigger dependencies between default dates, it is also associated with smaller values of equity tranches and thus with smaller deltas.

Let us further examine the credit deltas of the different tranches at inception. These are compared with the "sticky credit deltas" as computed by market participants under the previous base correlation structure assumption (see Table 15). These sticky deltas are computed by bumping the credit curves and computing the changes in present value of the tranches and of the credit default swap index. Once the credit curves are bumped, the moneyness varies, which is taken into account by using an updated base correlation when calculating the CDO tranches, thus the term "sticky". The delta is the ratio of the change in present value of the tranche and of the credit default swap index divided by the tranche's nominal. For example, a credit delta of an equity tranche previously equal to one would now lead to a figure of 33.33.

| | [0-3%] | [3-6%] | [6-9%] | [9-12%] | [12-22%] |
|---|---|---|---|---|---|
| market deltas | 27 | 4.5 | 1.25 | 0.6 | 0.25 |
| model deltas | 21.5 | 4.63 | 1.63 | 0.9 | NA |

Table 15. market and model deltas at inception

First of all we can see that the outlines are roughly the same, which is already noticeable since the two approaches are completely different. Then, we can remark that the model deltas are smaller for the equity tranche as compared with the market deltas, while there are larger for the other tranches. This is not surprising given the above discussion about the dynamic correlation effects. We actually believe that the sticky delta approach does not properly account for the shifts in correlation associated with the arrival of defaults[32].

Next, we thought that it was insightful to compare the previous table and the results provided by Arnsdorf and Halperin (2007), Figure 7 (see Table 16).

| | [0-3%] | [3-6%] | [6-9%] | [9-12%] | [12-22%] |
|---|---|---|---|---|---|
| market deltas | 26.5 | 4.5 | 1.25 | 0.65 | 0.25 |
| model deltas | 21.9 | 4.81 | 1.64 | 0.79 | 0.38 |

[32] Or with parallel shifts in the CDS spreads. The summer 2007 crisis is a good example of such effects with large increase of credit spreads and simultaneously large increases of correlation. Such inconsistencies are not surprising since the Gaussian copula fails to properly account for dynamic effects.

Table 16. market and model deltas as in Arnsdorf and Halperin (2007).

The market conditions are slightly different since the computations were done in March 2007, thus the maturity is slightly smaller than five years. The market deltas are quoted deltas provided by major trading firms. We can see that these are quite close to the previous market deltas since the computation methodology involving Gaussian copula and base correlation is quite standard. The models deltas (corresponding to “model B” in Arnsdorf and Halperin (2007)) have a quite different meaning from ours: there are related to credit spread deltas rather that then default risk deltas and are not related to a dynamical replicating strategy. However, it is noteworthy that these model deltas are similar to ours. Though this is not a formal proof, it appears from Figure 5, that (systemic) gammas are rather small prior to the first default. If we could view a shock on the credit spreads as a small shock on the expected loss while a default event induces a larger shock (but not so large given the risk diversification at the index level) on the expected loss, the similarity between the different model deltas are not so surprising. As above, model deltas are lower for the equity tranche and larger for the other tranches.

## Conclusion

The lack of internally consistent methods to hedge CDO tranches has paved the way to a variety of local hedging approaches that do not guarantee the full replication of tranche payoffs. Such incompleteness of the market may not look as such a practical issue as far as trade margins are high and holding periods short. However, we think that there might be a growing concern from investment banks about the long term credit risk management of trading books as the market matures.

A homogeneous Markovian contagion model can be implemented as a recombining binomial tree and thus provides a strikingly easy way to compute dynamic replicating strategies of CDO tranches. While such models have recently been considered for the pricing of exotic basket credit derivatives, our main concern here is to provide a rigorous framework to the hedging issue.

We do not aim at providing a definitive answer to the thorny issue of hedging CDO tranches. For this purpose, we would also need to tackle name heterogeneity, possible non Markovian effects in the dynamics of credit spreads, non deterministic intensities between two default dates, the occurrence of multiple defaults, … A fully comprehensive approach to the hedging of CDO tranches is likely to be quite cumbersome both on economic and numerical grounds.

However, from a practical perspective, we think that our approach might be useful to assess the default exposure of CDO tranches by quantifying the credit contagion effects in a reasonable way. We also found some noticeable similarities between credit spread deltas as computed under the standard base correlation methodology and the default risk deltas as computed from our recombining tree. A closer look at the discrepancies between the two approaches suggests some inconsistency in the market approach as far as the dynamics of the correlation is involved. Taking into account such dynamic effects lowers credit deltas of the equity tranche and therefore increases the credit deltas of the senior tranches. From a risk management perspective, understanding how credit deltas are related to base correlation curves requires a coupling of standard vanna analysis and the study of contagion and dynamic dependence effects.

## Appendix A: dynamics of defaultable discount bonds and credit spreads

Let us derive the dynamics of a (digital) defaultable discount bond associated with name $i \in \{1,\dots,n\}$ and maturity $T$. The corresponding payoff at time $T$ is equal to $1_{\{\tau_i > T\}} = 1 - N_i(T)$. Let us now consider a portfolio of the previously defined defaultable bonds with holdings equal to $\frac{1}{n}$ for all names. The portfolio payoff is equal to $V_I\left(T, N(T)\right) = 1 - \frac{N(T)}{n}$. The replication price at time $t$ given that $N(t) = k$ of such a portfolio is equal to $V_I(t,k) = e^{-r(T-t)} E^Q\left[1 - \frac{N(T)}{n} \middle| N(t) = k\right]$. Since the names are exchangeable, the $n-k$ non defaulted names have the same price which is thus $\frac{V_I(t,k)}{n-k}$. Thus the price time $t$ of the defaultable discount bond, $B_i\left(t,T\right)$ is given by:

$$B_i\left(t,T\right) = \left(1 - N_i(t)\right) \times \frac{V_I\left(t, N(t)\right)}{n - N(t)},\ V_I\left(t,\bullet\right) = e^{-r(T-t)} Q\left(t,T\right) V_I\left(T,\bullet\right)$$

where the pre-default intensity of $\tau_i$ is equal to $\alpha_{\bullet}^{Q}\left(t, N(t)\right) = \frac{\lambda\left(t, N(t)\right)}{n - N(t)}$. When $N(t) = n$, $\alpha_{\bullet}^{Q}\left(t, N(t)\right) = 0$ and $B_i\left(t,T\right) = 0$. Let us remark that the defaultable discount bond price follows a Markov chain with $n+1$ states $\left\{N(t) = 0, N_i(t) = 0\right\}, \dots, \left\{N(t) = n-1, N_i(t) = 0\right\}$ and $\left\{N_i(t) = 1\right\}$. The generator matrix, $\Lambda(t)$, is equal to:

$$\begin{pmatrix}
-\lambda(t,0) & \left((n-1)/n\right)\lambda(t,0) & 0 & 0 & 0 & 0 & \lambda(t,0)/n \\
0 & -\lambda(t,1) & \left((n-2)/(n-1)\right)\lambda(t,1) & 0 & & & \lambda(t,1)/(n-1) \\
0 & 0 & \bullet & \bullet & & & \bullet \\
0 & & & & \bullet & & \bullet \\
0 & & & & & & \bullet \\
0 & & & & & -\lambda(t,n-1) & \lambda(t,n-1) \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

Thus, the dynamics of the defaultable bond prices can be viewed as a special case of the one studied by Jarrow, Lando and Turnbull (1997) though the economic interpretation of the states slightly differs.

## Appendix B: Calibration equations on a complete set of number of defaults probabilities

While the pricing and thus the hedging involves a backward procedure, calibration is associated with forward Kolmogorov differential equations. We show here a non-parametric fitting procedure of a possibly non time homogeneous pure birth process onto a complete set of marginal distributions of number of defaults. This is quite similar to the one described in Schönbucher (2006), though the purpose is somehow different since the aim of the previous paper is to construct arbitrage-free, consistent with some complete loss surface, Markovian models of aggregate losses, possibly in incomplete markets, without detailing the feasibility and implementation of replication strategies.

We will further denote the marginal number of defaults probabilities by $p(t,k)=Q\left(N(t)=k\right)$ for $0\le t\le T$, $k=0,1,\dots,n$.

In the case of a pure birth process, the forward Kolmogorov equations can be written as:

$$\frac{dp(t,k)}{dt}=\lambda(t,k-1)\,p(t,k-1)-\lambda(t,k)\,p(t,k)\text{, for } k=1,\dots,n\text{, } \frac{dp(t,0)}{dt}=-\lambda(t,0)\,p(t,0)\,.$$

Since the space state is finite, there are no regularity issues and these equations admit a unique solution (see below for practical implementation). We refer to Karlin and Taylor (1975) for more details about the forward equations in the case of a pure birth process. These forward equations can be used to compute the loss intensity dynamics $t\in\left[0,T\right]\to\lambda(t,N(t))$, thanks to:

$$\lambda(t,0)=-\frac{1}{p(t,0)}\frac{dp(t,0)}{dt},\quad \lambda(t,k)=\frac{1}{p(t,k)}\left[\lambda(t,k-1)\,p(t,k-1)-\frac{dp(t,k)}{dt}\right]\quad\text{for}\quad k=1,\dots,n\,,$$

and $0\le t\le T$. Let us remark that we can also write:

$$\lambda(t,k)=-\frac{1}{p(t,k)}\frac{d\sum_{m=0}^{k}p(t,m)}{dt}=-\frac{1}{Q\left(N(t)=k\right)}\frac{dQ\left(N(t)\le k\right)}{dt}\,.$$

Eventually, the name intensities are provided by: $\alpha_{\cdot}^{Q}\left(t,N(t)\right)=\dfrac{\lambda(t,N(t))}{n-N(t)}$. This shows that, under the assumption of no simultaneous defaults, we can fully recover the loss intensities from the marginal distributions of the number of defaults. However, despite its simplicity, the previous approach (the inference of the $\lambda(t,k)$ from the default probabilities $p(t,m)$) involves some theoretical and practical issues.

As for the theoretical issues, we should deal with the assumption of no simultaneous defaults. We show below that, under standard no arbitrage requirements, (pseudo)-loss intensities might still be computed but that they may fail to reconstruct the input number of defaults distributions. Whatever the model, the marginal number of defaults probabilities must fulfil: $0\le p(t,m)\le 1$, $\forall\left(t,m\right)\in\left[0,T\right]\times\left\{0,1,\dots,n-1\right\}$, $\sum_{m=0}^{n}p(t,m)=1$, $\forall t\in\left[0,T\right]$ and since $N(t)$ is non decreasing, $\sum_{m=0}^{k}p(t,m)\ge\sum_{m=0}^{k}p(t',m)$, $\forall k\in\left\{0,1,\dots,n\right\}$, $\forall t,t'\in\left[0,T\right]$ and $t\le t'$. This implies that the $\lambda(t,k)$, as computed from the above equation, are non-negative. Moreover, since $\sum_{m=0}^{n}p(t,m)=1$, $\dfrac{d\sum_{m=0}^{n}p(t,m)}{dt}=0$, thus $\lambda(t,n)=0$, i.e. $\left\{N(t)=n\right\}$ is absorbing. In other words, standard no-arbitrage constraints on the probabilities of the number of defaults guarantee the existence of non-negative (pseudo)-loss intensities with the required boundary conditions. However, concluding that this (pseudo)-loss intensities may fail to reconstruct the input number of defaults distributions. The no simultaneous defaults assumption implies particularly that $\frac{dp(t,m)}{dt}=0$ for $t=0$ and $m>1$. If this constraint is not fulfilled by market inputs, we will not be able to reconstruct the input $p\left(t,m\right)$ from the (pseudo) -loss intensities.

On practical grounds, the computation of the $p(t,m)$ usually involves some arbitrary smoothing procedure and hazardous extrapolations for small time horizons.

For these reasons, we think that it is more appropriate and reasonable to calibrate the Markov chain of aggregate losses on a discrete set of meaningful market inputs corresponding to liquid maturities.

## Appendix C: calibration of time homogeneous loss intensities

Solving for the forward equations provides $p(T,0)=e^{-\lambda_0 T}$ and $p(T,k)=\lambda_{k-1}\int_0^T e^{-\lambda_k(T-s)}p(s,k-1)ds$ for $1\leq k\leq n-1$ (see Karlin and Taylor (1975) for more details). The previous equations can be used to determine $\lambda_0,\ldots,\lambda_{n-1}$ iteratively, even if our calibration inputs are the defaults probabilities at the single date $T$.

Assume for the moment that the intensities $\lambda_0,\ldots,\lambda_{n-1}$ are known, positive and distinct[33]. To solve the forward equations, we assume that the default probabilities can be written as $p(t,k)=\sum_{i=0}^{k}a_{k,i}e^{-\lambda_i t}$ for $0\leq t\leq T$ and $k=0,\ldots,n-1$[34]. Set $a_{0,0}=1$, the recurrence equations $a_{k,i}=\frac{\lambda_{k-1}}{\lambda_k-\lambda_i}a_{k-1,i}$ for $i=0,1,\ldots,k-1$, $k=1,\ldots,n-1$ and $a_{k,k}=-\sum_{i=0}^{k-1}a_{k,i}$. Then, we check easily that, if satisfied, these equations provide some solutions of the forward PDE. Since it is well-known that these solutions are unique, it means we have obtained explicitly the solution of the forward PDE, knowing the intensities $(\lambda_k)_{k=1,\ldots,n}$.

Therefore, using $p(0,k)=0$ and $\lambda_0=-\ln(p(T,0))/T$, we can compute iteratively $\lambda_1,\ldots,\lambda_{n-1}$ by solving the univariate non linear implicit equations $p(T,k)=\sum_{i=0}^{k}a_{k,i}e^{-\lambda_i T}$, or equivalently

[33] Due to the last assumption, the described calibration approach is not highly regarded by numerical analysts (see Moler and Van Loan (2003) for a discussion). However, it is well suited in our case studies.

[34] Since $\lambda_n=0$, $p(t,n)$ takes a slightly different form. Its detailed expression is useless here since we only need to deal with $p(t,0),\ldots p(t,n-1)$ to calibrate $\lambda_0,\ldots,\lambda_{n-1}$. Let us also remark that $p(t,n)$ can equally be recovered from $p(t,n)=\lambda_{n-1}\int_0^t p(s,n-1)ds$ or from $\sum_{k=0}^{n}p(t,k)=1$.

$$\sum_{i=0}^{k-1} a_{k-1,i} e^{-\lambda_i T} \times \left( \frac{1-e^{-(\lambda_k-\lambda_i)T}}{\lambda_k-\lambda_i} \right) = \frac{p(T,k)}{\lambda_{k-1}},\ k=1,\ldots,n-1 .$$

It can be seen easily that for any $k \in \{0,\ldots,n-1\}$, $p(T,k)$ is a decreasing function of $\lambda_k$, taking value $\lambda_{k-1}\int_0^T p(s,k-1)ds$ for $\lambda_k = 0$ and with a limit equal to zero as $\lambda_k$ tends to infinity. In other words, the previous $\lambda_k$ equations have a unique solution provided that: $p(T,k) < \lambda_{k-1} \times \left( \sum_{i=0}^{k-1} a_{k-1,i} \times \left( \frac{1-e^{-\lambda_i T}}{\lambda_i} \right) \right)$ for $k=1,\ldots,n-1$. Note that, in practice, all the intensities $\lambda_k$ will be different (almost surely). Thus, starting from the $T-$default probabilities only, we have found the explicit solutions of the forward equations and the intensities $(\lambda_k)_{k=1,\ldots,n}$ that would be consistent with these probabilities.

It is possible to extend this calibration procedure to fit simultaneously several maturities (for instance the usual tenors of credit indices), i.e. to fit the default probabilities $p(T_j,k)$ for $j=1,\ldots,J$ and $k=0,\ldots,n.$ Some details of a bootstrap procedure are provided in the Appendix D.

## Appendix D: multi-maturity calibration procedure

Now, the calibration set is the distribution of the number of defaults $p(T_j,k)$ at several time horizons $T_1,\ldots,T_p$. The intensities $\lambda(t,k)$ will be assumed piecewise constant in time: $\lambda(t,k) = \lambda_k^{(j)}$ for all integer $k$ and all $t \in ]T_{j-1},T_j]$, for every $j=1,\ldots,p$ (we have set $T_0 = 0$).

The general solution of the forward equations is $p(t,0) = e^{-\int_0^t \lambda(s,0)ds}$ and

$$p(t,k) = e^{-\int_0^t \lambda(u,k)du} \int_0^t \lambda(s,k-1) e^{\int_0^s \lambda(u,k)du} p(s,k-1)ds ,$$

for all time $t$ and $1 \le k \le n-1$.

The previous equations can be used to determine the intensities $\lambda_k^{(j)}$ iteratively, by starting with the shorter maturities. As previously, to solve the forward equations, we assume that the default probabilities can be written as $p(t,k) = \sum_{i=0}^{k} a_{k,i}^{(j)} \exp(-\lambda_i^{(j)}(t-T_{j-1}))$ for $T_{j-1} \le t \le T_j$, $k=0,\ldots,n-1$ and $j=1,\ldots,p$. Here, it is sufficient to set the recurrence equations:

$$a_{0,0}^{(j)} = \exp(-\sum_{l=1}^{j-1} \lambda_0^{(l)}(T_l - T_{l-1})),$$

$$a_{k,i}^{(j)} = \frac{\lambda_{k-1}^{(j)}}{\lambda_k^{(j)} - \lambda_i^{(j)}} a_{k-1,i}^{(j)}, \text{ and } a_{k,k}^{(j)} = p(T_{j-1},k) - \sum_{i=0}^{k-1} a_{k,i}^{(j)},$$

for $i=0,1,\ldots,k-1$, $k=1,\ldots,n-1$ and $j=1,\ldots,p$. Then, we can check that, if satisfied, these equations provide the solution of the forward PDE, knowing the intensities $(\lambda_k^{(j)})_{k=1,\ldots,n;\,j=1,\ldots,p}$.

Therefore, using $p(0,k)=0$ and $\lambda_0^{(j)}=[\ln(p(T_{j-1},0))-\ln(p(T_j,0))]/(T_j-T_{j-1})$, we can compute iteratively the model default intensities by solving the univariate non linear implicit equations

$$\sum_{i=0}^{k-1}\frac{\lambda_{k-1}^{(j)}a_{k-1,i}^{(j)}}{\lambda_k^{(j)}-\lambda_i^{(j)}}[\mathrm{e}^{-\lambda_i^{(j)}(T_j-T_{j-1})}-\mathrm{e}^{-\lambda_k^{(j)}(T_j-T_{j-1})}]+p(T_{j-1},k)\,\mathrm{e}^{-\lambda_k^{(j)}(T_j-T_{j-1})}=p(T_j,k)$$

for all $k=1,\ldots,n-1$ and $j=1,\ldots,p$.

Since, for any $k\in\{0,\ldots,n-1\}$, $p(T_j,k)$ is a decreasing function of $\lambda_k^{(j)}$, the previous $\lambda_k^{(j)}$ equations have a unique solution provided that

$$p(T_j,k)<\sum_{i=0}^{k-1}\frac{\lambda_{k-1}^{(j)}a_{k-1,i}^{(j)}}{\lambda_k^{(j)}-\lambda_i^{(j)}}[\mathrm{e}^{-\lambda_i^{(j)}(T_j-T_{j-1})}-1]+p(T_{j-1},k).$$

Thus, starting from a set of default probabilities for $p$ different time horizons, we have found the explicit solutions of the forward equations and the intensities $(\lambda_k)_{k=1,\ldots,n}$ that would be consistent with these probabilities.

## Appendix E: tree computations for different recovery rates

Given a recovery rate of $R$, the (fractional) loss at time $t$ on the credit portfolio is such that $L(t)=(1-R)\dfrac{N(t)}{n}$. The mapping $(t,\tilde{k})\in[0,T]\times[0,1]\to EL(t,\tilde{k})=E^Q\left[\min\left(\tilde{k},L(t)\right)\right]$ is known as the "loss surface". We readily relate the loss surface to the number of defaults distributions: $EL(t,\tilde{k})=\sum_{m=1}^{n}\min\left(\tilde{k},\dfrac{m(1-R)}{n}\right)p(t,m)$. Conversely, we can compute the probabilities of number of defaults from the $EL(t,\tilde{k})$ (see below). Figure 4 plots the expected loss $EL(T,\tilde{k})$ for $T=5Y$, $R=40\%$. The $p(T,m)$ are computed as above from a Gaussian copula dependence structure.

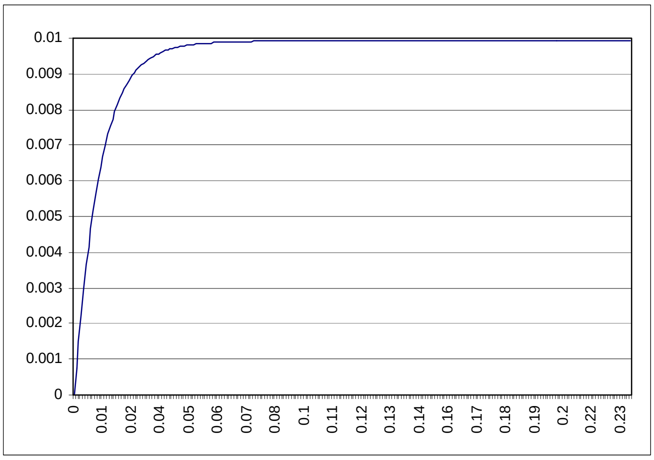


Figure 4. $EL(T,\tilde{k})$, $0\le\tilde{k}\le1$, $R=40\%$

Let us change the recovery rate from $R$ to $R^*$. Then, it can be quickly checked that the new probabilities of number of defaults are given by:

$$p^*(t,k)=\frac{n}{R^*-1}\times\left(EL\left(t,\frac{(k-1)\times(1-R^*)}{n}\right)-2EL\left(t,\frac{k\times(1-R^*)}{n}\right)+EL\left(t,\frac{(k+1)\times(1-R^*)}{n}\right)\right),$$

for $k=1,\ldots,n-1$ and $p^*(t,n)=\frac{n}{1-R^*}\times\left(EL\left(t,1-R^*\right)-EL\left(t,\frac{n-1}{n}\times\left(1-R^*\right)\right)\right)$. Eventually, $p^*(t,0)$ is obtained from $\sum_{k=0}^{n}p^*(t,k)=1$. Once we have obtained a new set of probabilities of number of defaults, we calibrated some new loss intensities $\lambda_k^*$, reconstructed a tree and recomputed some dynamic hedging strategies $\delta^*\left(i,k\right)$.

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