Comparison of two numerical approaches for barrier and value of a simple pursuit-evasion game

P. Bernhard[1], S. Crepey[2] and A. Rapaport[3]

[1]Université de Nice/I3S CNRS, France,
[2]INRIA Sophia-Antipolis, France,
[3]INRA Montpellier, France.

Abstract : We investigate the barrier of a simple pursuit-evasion game for which we are able to compare two theoretical and numerical approaches. One is directly based on the capture time, and the second one, introduced by one of the authors, transforms the game in one of approach (or L_∞ criterion). This second approach gives both a new characterization of barriers and a new, potentially more robust, numerical method for the determination of barriers. We provide a detailed analytical solution of the various problems thus raised, and use it as a benchmark for the numerical method.

1 Introduction

We revisit a well known one-dimensional second-order servomechanism problem, proposed by Bernhard in [7], with a new approach that transforms the game in one of approach (or L_∞ criterion). This simple pursuit-evasion game allows us to compare the traditional approach with this new one, both on theoretical and numerical points of view.

We present numerical methods for the computation of the value functions of the two versions of the game (the *game in time* and the *game in distance*), with a particular emphasis on the determination of the *barrier* of the pursuit-evasion game. Our methods use the theory of *viscosity solutions* for the Isaacs equation (see Barles [4] or Crandall, Ishii, Lions [12] for the state of the art), which is an alternative to the *viability* approach proposed by Cardaliaguet, Quincampoix, Saint-Pierre [9, 10] or the *minimax* solutions of Subbotin [20].

The first method is based on a finite difference approximation of the discounted capture time function, involving viscosity lower-envelope solutions of the Isaacs equation (cf. the work of Bardi, Bottacin, Falcone [1]). The associated numerical scheme computes an approximation by discrete stochastic games, introduced by Pourtallier, Tidball [18] following the work of Kushner [16].

Nevertheless, when a barrier occurs in the capture-evasion game splitting the state space into capture and evasion areas, a detection of infinite value of the capture time function is required in order to characterize this manifold. (See Bernhard [8] for a state of the art description of barriers of differential games). From a numerical point of view, this previous method does not seem to be well suited for an accurate detection, since the barrier sought appears as the boundary of the set where the discounted value function is strictly less than one, a level it reaches with zero slope.

The second approach considers an approximation of the minimum oriented distance from the target, involving viscosity upper-envelope solutions of a variational inequality (see Rapaport [19]). The oriented distance from the target needs to be known, which may require a numerical computation for an arbitrary target. Nevertheless, for many games (such as the ones studied by Isaacs [15]), the target is given by a simple analytic expression and then its oriented distance is an also an analytic function, easy to compute. The numerical scheme computes a monotone sequence of continuous solutions for a sequence of perturbated Hamiltonians, using again approximation by discrete stochastic games (see Crepey [13]). The barrier for the game in time is then determined by the zero level set of the value function for the game in distance, an intrinsically robust determination, as the gradient is not zero there. Moreover, this gradient also measures sensitivity of the barrier location with respect to the target.

Finally, we illustrate these methods with numerical experimentations, using the analytical solutions to benchmark the numerical results we obtain.

2 Presentation of the game

Consider a one-dimensional second-order plant :

$$\ddot{y} = \beta\, v, \qquad |v| \leq 1\,,$$

where the objective is to keep y as close as possible to a set point z subject to an unknown drift :

$$\dot{z} = \alpha\, u, \qquad |u| \leq 1\,.$$

More precisely, for a given positive number γ, we are looking for a (state feedback) control law $v^*()$ that guarantees $|y(t) - z(t)| \leq \gamma$ for all $t \geq 0$ whatever is the disturbance $u(.)$.

Considering the state vector :

$$x := \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y - z \\ \dot{y} \end{pmatrix},$$

this problem can be formulated as a pursuit-evasion game, whose dynamics are :

$$\begin{cases} x(0) &= x_0\,, \\ \dot{x}(t) &= f(x(t), u(t), v(t)) := \begin{pmatrix} x_2(t) - \alpha u(t) \\ \beta v(t) \end{pmatrix}, \qquad |u(t)| \leq 1, \quad |v(t)| \leq 1\,, \end{cases}$$

with the target set :

$$\mathcal{T} := \{x \in \mathrm{I\!R}^2 \mid |x_1| \geq \gamma\}\,.$$

(the player u is the "pursuer" and the player v the "evader"). The usual way to study the existence of such a control law v^* is to study the *game in time* (cf. Isaacs [15]) :

$$V(x_0) = \sup_{\psi[]} \inf_{u(\cdot)} t^c(x_0, u, v)\,,$$

where $t^c(x_0, u, v) = \inf\{t \geq 0 \,|\, x(t) \in \mathcal{T}\}$ is the capture time, $(v(\cdot) = \psi[u(\cdot)], u(\cdot))$ are admissible controls, and $\psi[]$ belongs to a set of strategies defined below. This game has been investigated in detail by Masle [17] and Bernhard in [7]. A particular emphasis is made on the existence and the characterization of the barrier, that splits the state space between initial positions for which there exists a strategy for the player v avoiding a termination in finite time from its complementary.

Alternatively, we study another criterion related to the *game in distance* :

$$W(x_0) = \sup_{\psi[]} \inf_{u(\cdot)} \left[\inf_t d^o(x(t), \mathcal{T})\right],$$

where (u, v) belong to the same sets of strategies as for the previous game and d^o is the oriented distance function :

$$d^o(x, \mathcal{T}) = \begin{cases} d(x, \mathcal{T}) & \text{if } x \notin \mathcal{T}\,, \\ -d(x, \partial\mathcal{T}) & \text{otherwise}\,. \end{cases}$$

Here, $d^o(x,\mathcal{T}) = \gamma - |x_1|$, and we shall propose a new analytical resolution of this game. The barrier of the game in time is then determined by the set of points $\mathcal{B} = \{x \,|\, W(x) = 0\}$. Although this criterion does not provide any information on the capture time, it characterizes the sensitivity with respect to the target, which is of complementary interest compared with the traditional approach.

3 Analytical solutions

3.1 Preliminaries

We shall define more precisely for which class of strategies the value functions V and W defined above should be considered :

Definition 1 (VREK STRATEGIES) *Let $\mathcal{U}$, $\mathcal{V}$ be the sets of measurable functions from* $\mathbb{R}^+$ *to* $[-1,1]$ *or* open-loop *controls.* $u(\cdot)$ *is sought among* $\mathcal{U}$ *and* $\psi[\,]$ *among the* non-anticipative VREK strategies *:*

$$\{\ \psi[\,]\ : u \in \mathcal{U} \longmapsto \psi[u] \in \mathcal{V}\ \}$$

such that : $\forall u \in \mathcal{U}$, $[\forall t \le t',\ u(t) = u'(t)] \Longrightarrow [\forall t < t',\ \psi[u](t) = \psi[u'](t)]$.

Similarly, we can consider strategies for a reversed order of the players : $v(\cdot)$ *is then sought among* $\mathcal{V}$ *and* $\phi[\,]$ *among the* non-anticipative VREK strategies *:*

$$\{\ \phi[\,]\ : v \in \mathcal{V} \longmapsto \phi[v] \in \mathcal{U}\ \}$$

such that : $\forall v \in \mathcal{V}$, $[\forall t \le t',\ v(t) = v'(t)] \Longrightarrow [\forall t < t',\ \phi[v](t) = \phi[v'](t)]$.

These classes of strategies are well suited to characterize the value functions in terms of *viscosity solutions* (see Crandall, Ishii, Lions [12] and Barles [4]), for which we recall the definition :

Definition 2 (VISCOSITY SOLUTIONS) *Consider a first order partial differential equation on a open domain Ω :*

$$\begin{array}{c} H(x, V(x), \nabla V(x)) = 0, \quad x \in \Omega \\ (\textit{possibly with a boundary condition } V(x) = K,\ x \in \partial\Omega) \end{array} \qquad (1)$$

*Let $D^+V(x)$ (*resp. *$D^-V(x)$) denote the Fréchet super-(*resp. *sub-)differential of the locally bounded function V at x,* i.e. *the set of formal gradients p, such that*

$$V(y) \le V(x) + < p, y - x > + \theta(y - x)||y - x||$$

$$(\textit{resp. } V(y) \ge V(x) + < p, y - x > - \theta(y - x)||y - x||)$$

for some continuous function θ, null at 0.

i) A subsolution *(resp.* supersolution*) of H on Ω is a* u.s.c. *(resp.* l.s.c.*) locally bounded function V* s.t. *$H(x, V(x), D^+V(x)) \ge 0$ (resp. $H(x, V(x), D^-V(x)) \le 0$) on Ω.*

ii) A Dirichlet subsolution *(resp.* Dirichlet supersolution*) of (1) on $\overline{\Omega}$ must satisfy also $V \le K$ (*resp. *$V \ge K$) on $\partial\Omega$.*

iii) If a subsolution *(resp.* supersolution*) of H on Ω satisfies at least $H(x, V(x), D^+V(x)) \ge 0$ (resp. $H(x, V(x), D^-V(x)) \le 0$), wherever it fails to satisfy $V \le K$ (resp. $V \ge K$) on $\partial\Omega$, we shall call it a* subsolution *(resp.* supersolution*) of (1) on $\overline{\Omega}$.*

iv) A (resp. Dirichlet) viscosity solution *means a function that is both a (resp. Dirichlet) sub- and a super-solution.*

v) The viscosity upper (resp. lower) envelope solution on Ω (resp. on $\overline{\Omega}$) means the largest viscosity sub-solution on Ω (resp. the smallest Dirichlet viscosity super-solution on $\overline{\Omega}$).

Alternatively, we shall also consider classes of *feedback strategies* :

Definition 3 (FEEDBACK STRATEGIES) *$\Phi \subset \{\phi : (t,x) \mapsto \phi(t,x) \in [-1,1]\}$ and $\Psi \subset \{\psi : (t,x) \mapsto \psi(t,x) \in [-1,1]\}$ are admissible classes of* feedback strategies *if :*

i) Open-loops are admissible : $\mathcal{U} \subset \Phi$ and $\mathcal{V} \subset \Psi$.

ii) Φ and Ψ are closed by concatenation (i.e. *switching from one strategy in the set to another one, at an intermediate instant of time, is allowed).*

iii) $\forall(\phi,\psi) \in \Phi \times \Psi$, $\forall x_0$, there exists an unique solution of $\dot{x} = f(x, \phi(.,x), \psi(.,x))$ over $\mathbb{R}^+$, leading to measurable controls : $u(.) = \phi(.,x(.)) \in \mathcal{U}$ and $v(.) = \psi(.,x(.)) \in \mathcal{V}$.

These properties do not uniquely define the pair (Φ,Ψ) but it is clear that such classes exist and are sub-classes of VREK non-anticipative strategies.

3.2 Game in time

We sketch here the analysis of [17] and [7], according to the classical Isaacs-Breakwell theory. From dimensional analysis, it is easy to see that the only meaningful parameter in that game is the ratio

$$p = \frac{\beta\gamma}{\alpha^2}\,.$$

First we find the usable part of the capture set (*i.e.* the subset of the boundary of the target such that $\sup_v \inf_u < \nu(x), f(x,u,v) > \leq 0$ where $\nu(x)$ is the outer normal to $\mathcal{T}$ at x) that , here made up of two symmetric pieces: $\{x_1 = \gamma, x_2 > -\alpha\}$ and $\{x_1 = -\gamma, x_2 < \alpha\}$. The boundary of the usable part (BUP) is thus made up of the two points $(x_1 = \varepsilon\gamma, x_2 = -\varepsilon\alpha)$ for $\varepsilon = \pm 1$. From the BUP, we attempt to construct a natural barrier. The semi-permeable normal is $(\nu_1 = -\varepsilon, \nu_2 = 0)$. Given the Hamiltonian of the game of kind,

$$H = \nu_1(x_2 - \alpha u) + \nu_2 \beta v\,,$$

we see on the one hand that the semi-permeable controls are $u = \operatorname{sign}\nu_1$ and $v = \operatorname{sign}\nu_2$, and on the other hand that the adjoint equations give

$$\begin{aligned}\dot{\nu}_1 &= 0\,,\\ \dot{\nu}_2 &= -\nu_1\,.\end{aligned}$$

Initialized with the proposed semi-permeable ν's on the BUP, this yields two parabola with the controls $u = v = -\varepsilon$: (we call t_1 the final time)

$$\begin{aligned}x_1(t) &= \varepsilon[\gamma - \frac{\beta}{2}(t_1 - t)^2]\,,\\ x_2(t) &= \varepsilon[-\alpha + \beta(t_1 - t)]\,.\end{aligned}$$

These intersect the "other edge" of the game space, *i.e.* the straight line $x_1 = -\varepsilon\gamma$, at $x_2 = \varepsilon(-\alpha + 2\sqrt{\beta\gamma})$. We must now distinguish two cases depending on whether these points are in the usable part or the non usable part.

The simple case is when this intersection happens in the non usable part, which is the case if $p > 1$. In that case the two parabola together with the pieces of (non usable) capture set boundary that join them (the thick lines in figure 2) indeed form a barrier, separating an escape zone "inside" from the capture zone outside.

Indeed that composite curve is a barrier. At all the points where it is smooth, the semipermeability condition holds (or, on the capture set boundary, a stronger inequality for the evader). At its points of non differentiability, the two intersections of the parabola with the opposite capture sets, the evader may play according to the parabola's dictum, *i.e.* $v = \varepsilon$. This insures that the state remains inside the escape zone, since $\dot{x}_1$ has the desired sign whatever the controls are.

Outside that region, we can construct a complete field of trajectories, that happen to be parabola translated from the previous ones parallel to the x_1 axis. It is a simple matter to check that they define a value function

$$V(x) = \frac{1}{\beta}\left[\alpha + \varepsilon x_2 - \sqrt{(\alpha + \varepsilon x_2)^2 - 2\beta(\gamma - \varepsilon x_1)}\right],$$

with $\varepsilon = 1$ in the upper region and $\varepsilon = -1$ in the lower region. Inside the escape zone, of course $V = +\infty$. (We should emphasize that the value function computed here is Isaacs', not the function V of the next paragraphs which is its Kruskov transform.)

In the case $p < 1$, the two parabola intersect each other inside the game space, delineating what we shall call the *lens*. This lens is *not* an escape zone however : the corners "leak". Following the classical analysis of intersection of barriers, we have an intersection with incoming trajectories that cross it. Therefore the composite surface is *not* a barrier.

As a matter of fact, the lens is the *intersection* of the proposed safety zones defined by each parabola. Therefore, to stay in it, the state should cross none of the parabola, what the pursuer cannot enforce since the required controls are +1 for one of the parabola, -1 for the other one. Upon reaching such a corner, the pursuer can keep its optimal control according to the incoming parabola, and the state necessarily leaves the "lens".

In that case there is no escape zone. But the complete solution in terms of singularities of Isaacs'equation is extremely involved. A private communication of John Breakwell suggested that the number of commutations of the optimal controls from +1 to -1 and conversely can be arbitrarily large, depending on the initial state and the value of p.

3.3 Game in distance

In Rapaport [19], it is proved that the value function W for the game in distance is the viscosity upper-envelope solution of the following variational inequality (under technical assumptions that guarantee W to be u.s.c.) :

$$H(x, W(x), \nabla W(x)) = \min\left[d^o(x,\mathcal{T}) - W(x)\,,\, \min_u \max_v \nabla W(x).f(x,u,v)\right] = 0\,. \qquad (2)$$

Unfortunately, the technical assumptions proposed in [19] in a general framework are not fulfilled in this game. Nevertheless, we show here, thanks to analytical considerations, that the value function W is a continuous viscosity solution of (2).

When $W(x) < d^o(x,\mathcal{T})$, the characteristic fields of the considered game are obtained for $u^*(x) = \operatorname{sign}\partial_1 W(x)$ and $v^*(x) = \operatorname{sign}\partial_2 W(x)$:

$$\left\{\begin{array}{lcl} x_1(t) & = & \epsilon\beta t^2/2 + (x_1(0) - \epsilon\alpha)t + x_1(0) \\ x_2(t) & = & \epsilon\beta t + x_2(0) \end{array}\right. \qquad \text{for } \epsilon = \pm 1. \qquad (3)$$

A necessary condition for t_1 to minimize $t \mapsto \gamma - |x_1(t)|$ is to have $\dot{x}_1(t_1) = 0$, which gives :

$$\gamma - x_1(t_1) = \gamma + \epsilon x_1(0) - \frac{(x_2(0) - \epsilon\alpha)^2}{2\beta}.$$

This leads us to consider the following candidate Z solution of the variational inequality :

Definition 4

$$Z(x) = \begin{cases} \min(\,\gamma + x_1\,,\, P^+(x)\,) & \textit{when } x_2 \geq \alpha\,, \\ \min(\,\gamma - x_1\,,\, P^-(x)\,) & \textit{when } x_2 \leq -\alpha\,, \\ \min(\,\gamma - \alpha^2/\beta\,,\, P^+(x)\,,\, P^-(x)\,) & \textit{when } |x_2| \leq \alpha\,, \end{cases}$$

with

$$\begin{cases} P^+(x) &=& \gamma - x_1 - \dfrac{(x_2+\alpha)^2}{2\beta}\,, \\ P^-(x) &=& \gamma + x_1 - \dfrac{(x_2-\alpha)^2}{2\beta}\,. \end{cases}$$

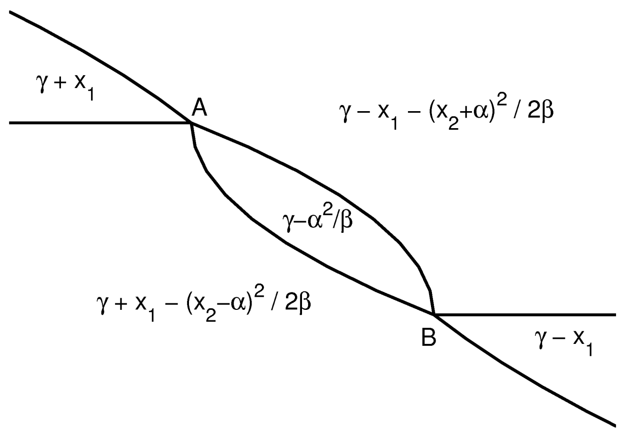

Figure 1: *Different areas defining the function Z.*

Remarks 1
1. Z is maximal and constant equal to $\gamma - \alpha^2/\beta$ inside the "lens" delimited by two arcs of parabola :

$$\mathcal{L} := \{x \,|\, P^-(x),\, P^+(x) \geq \gamma - \alpha^2/\beta\} \,\cap\, \{|x_2| \leq \alpha\}.$$

The constant value inside the lens $\mathcal{L}$ is equal to the common value kept by the three functions $\gamma - |x_1|$, $P^-(x)$ and $P^+(x)$ at points x such that $|x_2| \leq \alpha$ and where they are equal, which are exactly the two points $A = (-\alpha^2/\beta, \alpha)$ and $B = (\alpha^2/\beta, -\alpha)$ (see Figure 1).

2. The set of points where the function Z is null is :

   i) void if $\gamma - \alpha^2/\beta < 0$,

   ii) otherwise equal to

$$\{P^+(x) = 0, x_1 \geq -\gamma, x_2 \geq -\alpha\} \,\cup\, \{P^-(x) = 0, x_1 \leq \gamma, x_2 \leq \alpha\}$$
$$\cup\, \{-\gamma\} \times [\alpha, 2\sqrt{\beta\gamma} - \alpha] \,\cup\, \{\gamma\} \times [-\alpha, \alpha - 2\sqrt{\beta\gamma}]$$

   (see Figure 2).

We recognize in this last expression exactly the barrier found by Bernhard [7] for the game in time.

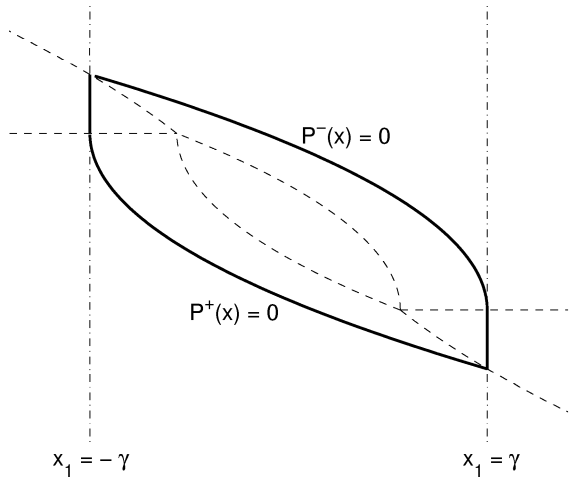

Figure 2: *The set of points x where $Z(x) = 0$ (when $\gamma - \alpha^2/\beta > 0$).*

Proposition 1 *Z is a continuous viscosity solution of (2).*

Proof Z is clearly continuous, nowhere above $d^o(.,\mathcal{T})$. Notice that requiring Z to be a continuous viscosity solution of (2) is then equivalent to :

$$\begin{cases} \min\limits_u \max\limits_v p.f(x,u,v) \geq 0, \ \forall p \in D^+Z(x)\,, \\ Z(x) = d^o(x,\mathcal{T}) \text{ or } \min\limits_u \max\limits_v p.f(x,u,v) \leq 0, \ \forall p \in D^-Z(x)\,. \end{cases}$$

Direct computation shows that Z satisfies the variational inequality (2) at its differentiable points. At non differentiable points, using non smooth calculus rules (see for instance Clarke [11]), we have :

i) for x such that $P^+(x) = \gamma + x_1$ and $x_2 \geq \alpha$,

$$D^+Z(x) = \left\{ -\lambda \begin{pmatrix} 1 \\ (x_2+\alpha)/\beta \end{pmatrix} + (1-\lambda) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}_{\lambda \in [0,1]}, \qquad D^-Z(x) = \emptyset.$$

So $\min\limits_u \max\limits_v p.f(x,u,v) = (1-2\lambda)x_2 - |1-2\lambda|\alpha + \lambda(x_2+\alpha) \geq 0, \quad \forall p \in D^+Z(x)$.
(By symmetry, we have the same inequalities at points x such that $P^-(x) = \gamma - x_1$ and $x_2 \leq -\alpha$)

ii) for x such that $P^+(x) = \gamma - \alpha^2/\beta$ and $|x_2| \leq \alpha$,

$$D^+Z(x) = \left\{ -\lambda \begin{pmatrix} 1 \\ (x_2+\alpha)/\beta \end{pmatrix} + (1-\lambda) \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}_{\lambda \in [0,1]}, \qquad D^-Z(x) = \emptyset.$$

So $\min\limits_u \max\limits_v p.f(x,u,v) = -\lambda(x_2+\alpha) + \lambda(x_2+\alpha) = 0, \quad \forall p \in D^+Z(x)$.
(By symmetry, we have the same inequalities at points x such that $P^-(x) = \gamma - \alpha^2/\beta$ and $|x_2| \leq \alpha$)

iii) for A,

$$D^+Z(A) = \left\{ -\lambda_1 \begin{pmatrix} 1 \\ 2\alpha/\beta \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (1-\lambda_1-\lambda_2) \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}_{\substack{\lambda_1,\lambda_2 \geq 0, \\ \lambda_1+\lambda_2 \leq 1}}, \qquad D^-Z(x) = \emptyset.$$

So $\min_u \max_v p.f(A,u,v) = (\lambda_2-\lambda_1)\alpha - |\lambda_2-\lambda_1|\alpha + 2\lambda_1\alpha \geq 0, \quad \forall p \in D^+Z(A)$.
(By symmetry, we have the same inequalities at point B)

Z is then a viscosity solution of the variational inequality (2). ∎

Proposition 2 *Z is the value function with feedback strategies (for the game in distance).*

Proof Take a number c less or equal than $\gamma-\alpha^2/\beta$, then $(\gamma-c) \geq \alpha^2/\beta$ and so, according to section 3.2, there exists a barrier $\mathcal{B}_c$ for the game in time with the target $\mathcal{T}_c := \{x \in \mathbb{R}^2 \mid |x_1| \geq \gamma - c\}$. We notice also that the candidate Z is such that the manifold $\{x \mid Z(x) = c\}$ coincide exactly with the barrier $\mathcal{B}_c$ determined in the previous section. As the exterior of $\mathcal{B}_c$ is a guaranteed capture zone, the value function $W_{\gamma-c}(x_0)$ for the game in distance (with the target $\mathcal{T}_c$) is necessarily non positive, as soon as x_0 is such that $Z(x_0) > c$. Similarly, the interior of $\mathcal{B}_c$ is a guaranteed evasion zone, so $W_{\gamma-c}(x_0)$ is non negative as soon as x_0 is such that $Z(x) < c$. Remark also that the $W(x_0) = W_{\gamma-c}(x_0) + c$, $\forall x_0 \in \mathbb{R}^2$.

Consider now polar coordinates (r,θ) in the plane. For any θ, there exists $r > 0$ such that $re^{i\theta} \in \mathcal{B}_c$. A point $se^{i\theta}$ belongs to the exterior (resp. the interior) of $\mathcal{B}_c$ as soon as $s > r$ (resp. $s < r$) so $W(se^{i\theta}) \leq c = Z(re^{i\theta})$ (resp. $W(se^{i\theta}) \geq c = Z(re^{i\theta})$). This can be achieved for any c such that $c \leq \gamma - \alpha^2/\beta$, *i.e.* outside the lens :

$$\mathcal{L} := \{x \mid P^-(x),\, P^+(x) \geq \gamma - \alpha^2/\beta\} \,\cap\, \{|x_2| \leq \alpha\},$$

So we have :

$$x_0 = re^{i\theta} \notin \mathcal{L} \Rightarrow \begin{cases} s > r \Rightarrow W(se^{i\theta}) \leq Z(x_0) \\ s < r \Rightarrow W(re^{i\theta}) \geq Z(x_0) \end{cases}$$

Z being a continuous function, we conclude that $W(x_0) = Z(x_0)$, $\forall x_0 \notin \mathcal{L}$. We deduce also that $W(x_0) \geq \gamma - \alpha^2/\beta$, $\forall x_0 \in \mathcal{L}$. Note also that the reasoning above could be done with the "lower" value W^- (instead of the "upper" value W) :

$$W^-(x_0) = \inf_{\phi[]} \sup_{v \in \mathcal{V}} \left[\inf_{t \geq 0} d^o(x(t), \mathcal{T})\right] \leq W(x_0)$$

i.e. we have $W = W^- = Z$ on $\mathbb{R}^2 \setminus \mathcal{L}$ and $W^- \geq \gamma - \alpha^2/\beta$ on $\mathcal{L}$.

Consider the state space divided into the three domains :

$$\begin{array}{rcl} \mathcal{S} & := & \{\, x \mid Z(x) = P^+(x) \text{ or } Z(x) = \gamma - x_1 \,\} \\ \mathcal{I} & := & \{\, x \mid Z(x) = P^-(x) \text{ or } Z(x) = \gamma + x_1 \,\} \\ \mathring{\mathcal{L}} & := & \mathbb{R}^2 \setminus (\mathcal{S} \bigcup \mathcal{I}) \end{array}$$

and the following feedback strategies :

$$\tilde{u}(x) = \left| \begin{array}{rl} 1 & \text{if } x \in \mathcal{I} \\ -1 & \text{otherwise} \end{array} \right. \qquad \text{and} \quad \tilde{v}(x) = \tilde{u}(x).$$

From section 3.2 and the result above, it is clear that the pair $(\tilde{u},\tilde{v})$ realizes $W(x_0)$ for any initial condition x_0 outside the lens and give a minimal oriented distance larger or equal to $\gamma - \alpha^2/\beta$ for $x_0 \in \mathcal{L}$.

Consider now $x_0 \in \mathcal{L}$ and the pair $(\tilde{u}, v)$ for an arbitrary open-loop control $v \in \mathcal{V}$. Remember that inside the lens $x_2 \geq -\alpha$, so let $l = \inf_{t\geq 0}\{x_2(t) + \alpha\}$. Inside the lens, the dynamics in x_1 is : $\dot{x}_1 = x_2 + \alpha > l$.

If $l > 0$, the trajectory leaves $\mathcal{L}$ in finite time, let say at t_e. But then we have :

$$\forall \epsilon > 0,\ \exists \eta > 0 \text{ s.t. } \ |x_1(t)| \leq \alpha^2/\beta - \epsilon \implies \dot{x}_1(t) \geq \eta, \quad \forall t \geq t_e,$$

(if the trajectory reaches $\mathcal{S}$) and we conclude that $\inf_{t \geq t_e}\{\gamma - |x_1(t)|\} \leq \gamma - \alpha^2/\beta$ (by symmetry, we have the same inequality if the trajectory reaches $\mathcal{I}$).

If $l \leq 0$ and the trajectory does not leave the lens in finite time, it converges asymptotically inside $\mathcal{L}$ towards the corner point B and we have :

$$\inf_{t \geq 0}\{\gamma - |x_1(t)|\} \leq \max_{\xi \in \partial\mathcal{L}} Z(\xi) = Z(x_0), \quad \forall v \in \mathcal{V}.$$

So, in any case, we obtain :

$$Z(x_0) = \gamma - \alpha^2/\beta \geq \sup_{v \in \mathcal{V}} \left[\inf_{t \geq 0} d^o(x(t), \mathcal{T})\right] \geq W^-(x_0), \quad \forall x_0 \in \mathcal{L}.$$

We conclude then that $Z(x_0) = W(x_0) = W^-(x_0)$, $\forall x_0 \in \mathbb{R}^2$ (and that the game admits a saddle point for VREK strategies). We conclude also that the feedback strategies $(\tilde{u}, \tilde{v}) \in \Phi \times \Psi$ are optimal. ∎

Remark 2 W is necessarily non positive (resp. non negative) at capture points (resp. evasion points). So a barrier separates points x_c where $W(x_c) \leq 0$ from points x_e where $W(x_e) \geq 0$. Conversely, a point x belongs to a capture area (resp. evasion area) if $W(x) < 0$ (resp. $W(x) > 0$). So an hyper surface separating points where W is strictly negative from points where W is strictly positive is a barrier.

In this game, W is continuous and equal to Z (the value with feedback strategies). So the barrier of the capture-evasion game with feedback strategies is nothing else than the zero level set of the function Z. Existence condition and determination of the barrier are both derived explicitly from Z, and the analysis is independent of the ratio p (introduced in the previous section). In addition, we have proved that there does not exist a barrier when $p < 1$.

4 Numerical methods

We shall study numerical approximations of the value functions V and W on a given subset of the state space $\mathcal{E} = \mathbb{R}^2$. For the game in time, the domain of definition of the value function V is then $\Omega = \mathcal{E} \setminus \mathcal{T}$ (for the game in distance, we shall simply say that $\Omega = \mathcal{E}$).

4.1 Preliminaries

Definition 5 (KRUSKOV TRANSFORMATION) *Let U denote the Kruskov transform of the value function of the game, where by definition Kruskov transform is : $\phi(\xi) = 1 - \exp(-\xi)$.*

We recall how U is related to the discounted version of the game :

Proposition 3 *U is the value function (in the same meaning) of the discounted differential game, where by discounted game we mean the game with the same dynamics as before and the criterion t^c or $\inf_t d^o$ replaced by $\phi(t^c)$ or $\inf_t \phi(d^o)$, by monotonicity of ϕ.*

From now on, V and W will refer to the value functions of the discounted games, which have the numerical advantage to be bounded from above by 1.

Definition 6 (DISCOUNTED HAMILTONIANS) *To the discounted games, we associate the following Hamiltonians :*

i) for the game in time :

$$H(x,s,p) = \min_u \max_v < p, f(x,u,v) > +1 - s\,, \tag{4}$$

ii) for the game in distance :

$$H(x,s,p) = \min \left[\phi(d^o(x,\mathcal{T})) - s, \min_u \max_v < p, f(x,u,v) >\right]. \tag{5}$$

We introduce a finite difference scheme to approximate values of both differential games (game in time or game in distance). This scheme is nothing else than a classical *upwind* finite difference scheme for first-order p.d.e., adapted by Kushner to optimal control problems [16] and later to differential games by Pourtallier, Tidball [18]. This scheme can also be interpreted as approximation by discrete stochastic games: this leads to proofs of the convergence results, which are alternative to the ones used for standard p.d.e. (see Kushner [16] for the one player case and Crepey [13] for two players games).

Definition 7 (STARRED MESH) *A* starred mesh *of step h on $\mathcal{E}$ is given by :*

1. *A discrete set of nodes $\mathcal{E}^h \subseteq \mathcal{E}$.*
2. *A local triangulation of the space around each node.*

The last point means that about each node $x \in \mathcal{E}^h$ we choose a finite set of r-simplices, or *cells*, with edges linking the nodes of $\mathcal{E}^h$ (a typical instance is the square mesh we effectively use in the algorithms). These simplices must meet at x, and fit together to cover the space about x just once. Roughly said about each node x a finite sequence of boxes is constructed on the nodes of $\mathcal{E}^h$, so that these boxes meet at x and partition the space around x. The set of vertices of cells at x (included) will be noted $\mathcal{W}^h(x)$.

More precisely, we shall consider families $(\mathcal{E}^h)_{h>0}$ of starred meshes that fill $\mathcal{E}$, in the sense that the union of the nodes of all $\mathcal{E}^h$ $(h > 0)$ is dense in $\mathcal{E}$. Moreover we shall assume the non degeneracy of these families :

Assumption 1 (NON DEGENERACY) *There exist $\underline{\pi} \in (\pi/2, \pi)$, positive functions $\underline{\delta}(h)$ and $\overline{\delta}(h)$ going to 0 with h, such that for every $h > 0, x \in \mathcal{E}^h$ and $y \in \mathcal{W}^h(x)$:*

i) $\underline{\delta}(h) \leq ||y - x|| \leq \overline{\delta}(h)$.

ii) The angle between an edge $\overrightarrow{xy}$ of a cell at x and the opposite face is less or equal than $\underline{\pi}$.

Remark 3 The classical square mesh satisfies all these requirements.

Proposition 4 *For every $(x,u,v) \in (\mathcal{E}^h \times \mathcal{U} \times \mathcal{V})$, there is a unique family of nonnegative $f^y(x,u,v)$ $(y \in \mathcal{W}^h\{x\})$ s.t.:*

$$f(x,u,v) \quad = \sum_{y \in \mathcal{W}^h(x)} f^y(x,u,v)(y-x),$$

and $f^y(x,u,v) = 0$ if $y \in \mathcal{W}^h(x)$ does not belong to the intersection of the cells at x which meet $x + \mathbb{R}^+_\star f(x,u,v)$.

Proof It is a decomposition of a vector on a base. ▮

Let then $\Delta t(x,u,v)$ be a notation for $\left(\sum_{z\in\mathcal{W}^h(x)} f^z(x,u,v)\right)^{-1}$ when $f(x,u,v)\neq 0$, $+\infty$ otherwise.

We shall need the concept of *weak limit* introduced by Barles, Perthame [5] :

Definition 8 (WEAK LIMITS) *For any family of functions V_h on $\mathcal{E}^h$ $(h>0)$, we introduce lower and upper weak limits $\underline{V},\overline{V}:\mathcal{E}\to\mathbb{R}$ when h tends to 0 :*

$$\underline{V}(x)=\liminf_{\substack{x^h\in\mathcal{E}^h,x^h\to x\\ h\to 0}} V_h(x^h)\leq \limsup_{\substack{x^h\in\mathcal{E}^h,x^h\to x\\ h\to 0}} V_h(x^h)=\overline{V}(x). \tag{6}$$

We also need, in order to deal with discontinuities, the following definition from Bardi, Bottacin, Falcone (Definition 2.2 in [1]) :

Definition 9 (DOUBLE CONVERGENCE) *We call the doubly indexed family $(V_{\epsilon,h})_{\epsilon,h>0}$* doubly convergent *towards V at $x\in\mathcal{E}$, where $V_{\epsilon,h}$, V are real functions on $\mathcal{E}^h$, and write*

$$V(x)=\lim_{\substack{x^h\in\mathcal{E}^h,x^h\to x\\ h(\epsilon)\to 0}} V_{\epsilon,h}(x^h),$$

if for any $\gamma>0$ there exists a function $\overline{h}:(0,+\infty)\to(0,+\infty)$, and $\overline{\epsilon}>0$, such that

$$|V_{\epsilon,h}(x^h)-V(x)|\leq\gamma,$$

for all $\epsilon\leq\overline{\epsilon}$, $h\leq\overline{h}(\epsilon)$, and $x^h\in\mathcal{E}^h$ s.t. $||x-x^h||\leq\overline{h}(\epsilon)$.

For the next sections, we shall need the following assumption :

Assumption 2

a. f is continuous, Lipschitz continuous w.r.t. x, uniformly in (u,v) and has linear growth.

b. There exists constants $\underline{f},\overline{f}$ such that

$$0<\underline{f}\leq f(x,u,v)\leq\overline{f}<+\infty,\quad\forall(x,u,v)$$

c. There exists a constant $\underline{d^o}$ such that

$$-\infty<\underline{d^o}\leq d^0(x,\mathcal{T}),\quad\forall x$$

(Note that the function $d^0(.,\mathcal{T})$ is Lipschitz continuous, whatever is the target set $\mathcal{T}$).

4.2 Game in time

Definition 10 (DISCRETE STOCHASTIC GAME) *On the discrete space $\mathcal{E}^h$ we define a stochastic game (cf. Filar, Raghavan [14]), composed of the following elements.*

i) A discrete target : $\mathcal{T}^h=\mathcal{E}^h\cap\mathcal{T}$ and domain : $\Omega^h=\mathcal{E}^h\setminus\mathcal{T}^h$.

ii) Transition probabilities :

$$p(x,y \mid u,v) \quad = \quad \begin{cases} f^y(x,u,v)\Delta t(x,u,v) \ \textit{if}\ x \in \Omega^h, y \in \mathcal{W}^h(x), \\ 1 \ \textit{if}\ y = x \in \mathcal{T}^h, \\ 0 \ \textit{otherwise.} \end{cases}$$

For the particular case when $f(x,u,v) = 0$, we take :

$$p(x,y \mid u,v) \quad = \quad \begin{cases} 1 \ \textit{if}\ y = x, \\ 0 \ \textit{otherwise.} \end{cases}$$

iii) Instantaneous reward and discount factor :

$$k(x,u,v) = \begin{cases} \phi[\Delta t(x,u,v)] & \textit{when}\ x \in \Omega^h\,, \\ 0 & \textit{when}\ x \in \mathcal{T}^h\,, \end{cases}$$

$$\beta(x,u,v) = \begin{cases} \exp[-\Delta t(x,u,v)] & \textit{when}\ x \in \Omega^h\,, \\ 0 & \textit{when}\ x \in \mathcal{T}^h\,. \end{cases}$$

Classically, the value V_h of the discrete stochastic game so defined satisfies the following discrete averaged dynamic programming equation, known as Shapley equation:

$$V_h = T_h V_h, \tag{7}$$

where by definition T_h is the following non-linear operator from the metric complete space of all bounded real sequences $\mathrm{IR}_b^{\mathcal{E}^h}$ into itself:

$$[T_h V_h](x) = \min_u \max_v \{k(x,u,v) + \beta(x,u,v) E_x^{u,v} V_h\}\,. \tag{8}$$

Here $E_x^{u,v}V_h$ means the expected value of V_h viewed as a functional on the Markov random field (7). $E_x^{u,v}V^h(x) = \sum_{y\in\mathcal{W}^h(x)} p(x,y \mid u,v)V_h(y)$. In particular for $x \in \mathcal{T}^h$ (7) gives:

$$V_h(x) = 0$$

The following proposition is drawn from Prop. 3.1 by Pourtallier, Tidball [18].

Proposition 5 *Under assumptions 1 and 2a-2b, T_h is contractive from $\mathrm{IR}_b^{\mathcal{E}^h}$ to itself, so that Shapley equation (7) admits a unique solution V_h.*

Now, we relate this Shapley solution with the viscosity Dirichlet lower envelope solution of the Isaacs equation on $\overline{\Omega}$:

$$\begin{cases} H(x,V(x),\nabla V(x)) = 0, & x \in \Omega \\ V(x) = 0, & x \in \partial\Omega \end{cases} \tag{9}$$

Denoting $\underline{V}$ and $\overline{V}$ the weak limits when $h \to 0$ of V_h, solutions of Shapley fixed point equations (7), we have the fundamental result :

Proposition 6 *Under assumptions 1 and 2a-2b, $\overline{V}$ (resp. $\underline{V}$) is a viscosity subsolution (*resp. *supersolution) of Isaacs equation (9) on $\overline{\Omega}$.*

Proof See Pourtallier, Tidball [18] or Crepey [13] for application to differential games, following ideas of Barles, Souganidis [6]. ∎

When the discounted VREK value function V is continuous, it can be inferred that this scheme converges towards V *i.e.* $\lim_{h\to 0, x^h \to x} V_h(x^h) = V(x)$, under regularity assumption on the boundary target, fulfilled for the present game (see [3]). But when the value function turns out to be discontinuous, which is the case when a barrier occurs, we have to consider a double approximating scheme, adding a dilatation of the target according to the ideas introduced in Bardi, Bottacin, Falcone [1] :

Definition 11 (Double approximating scheme) *For $\epsilon > 0$, define $\mathcal{T}_\epsilon = \{x \in \mathcal{E} \mid \; d(x, \mathcal{T}) \leq \epsilon\}$, while Ω_ϵ is $\mathcal{E} \setminus \mathcal{T}_\epsilon$.*
Let $V_{\epsilon,h}$ be the value function of the stochastic game with $\Omega := \Omega_\epsilon$, and $\mathcal{T} := \mathcal{T}_\epsilon$.

Following Th. 2.5 in Bardi, Bottacin, Falcone [1], we have :

Proposition 7 *Under assumptions 1 and 2a-2b, $V_{\epsilon,h}$ converges doubly towards the viscosity lower Dirichlet envelope solution of (9) on $\overline{\Omega}$.*

Proof See the work of Bardi, Bottacin, Falcone [1] or Crepey [13]. ∎

Remarks 4
1. For pursuit-evasion problems, Bardi, Bottacin, Falcone have shown that the viscosity lower Dirichlet envelope solution of the Isaacs equation on $\overline{\Omega}$ is the value function for Friedman-like strategies, as well as the limit when ϵ tends towards zero of the VREK values for the target $\mathcal{T}_\epsilon$ [1]. For capture-time problems, proving that it is also the VREK value function is still an open problem, except in the case where the VREK value function is continuous.
2. The dependence $h(\epsilon)$ between the sequences $h \to 0$ and $\epsilon \to 0$ required to guarantee the practical convergence of the scheme is also an open problem.
3. In our pursuit-evasion game in time, the dynamics does not satisfy the assumption 2b but the numerical experiments (described in section 5) confronted to the analytical study of section 3.2 suggest that the scheme Nevertheless converges towards the value of the game.

4.3 Game in distance

Following Rapaport [19], for a given positive number ϵ, we consider the ϵ-game :

$$W^\epsilon(x_0) = \sup_{\psi} \inf_{u(\cdot),t} \left\{ d^o(x(t), \mathcal{T}) + \int_0^t \epsilon \, d\tau \right\}.$$

Proposition 8 *Under assumptions 2a and 2c, W^ϵ is a non-increasing sequence of bounded continuous functions, unique viscosity solutions of the variational inequalities :*

$$\min \Big[d^o(x, \mathcal{T}) - W^\epsilon(x) \, , \, \min_u \max_v \nabla W^\epsilon(x).f(x,u,v) + \epsilon \Big] = 0, \quad \forall x \in \mathcal{E}.$$

Proof See Rapaport [19]. ∎

The Hamiltonian associated to the discounted version of this ϵ-game is then :

$$H^\epsilon(x,s,p) = \min \Big[\phi(d^o(x, \mathcal{T})) - s \, , \, \min_u \max_v < p, f(x,u,v) > + \epsilon(1-s) \Big] \, . \tag{10}$$

The scheme described in previous section to compute numerically a continuous value can be adapted here to approximate W^ϵ. More precisely, the dynamic programming for an appropriate approximation W_h^ϵ of W^ϵ on a grid $\mathcal{E}_h$ yields the Shapley-like equation :

$$W_h^\epsilon = T_h^\epsilon W_h^\epsilon \, , \tag{11}$$

with

$$\begin{aligned} [T_h^\epsilon W_h^\epsilon](x) = \min \Big[\phi[d^o(x, \mathcal{T})] - W_h^\epsilon(x) \, , \, \min_u \max_v k(x,u,v) + \beta(x,u,v) E_x^{u,v} W_h^\epsilon \Big] \, , \\ k(x,u,v) = \phi(\epsilon \Delta t(x,u,v)) \quad \text{and} \quad \beta(x,u,v) = exp(-\epsilon \Delta t(x,u,v)) \end{aligned} \tag{12}$$

(remind that there is no boundary condition for this game : $\mathcal{T}_h = \emptyset$ in the definition of transition probabilities for this version of the game).

T_h^ϵ is a contractive operator on $\mathbb{R}_b^{\mathcal{E}^h}$, as T_h defined through Shapley equation used to be for the game in time (noticing that whatever are three real numbers a, b, c, we have $|\min(a,b) - \min(a,c)| \leq |b - c|$). So the fixed point equation (11) defines a unique W_h^ϵ.

Theorem 1 *Under assumptions 1 and 2a-2b-2c, W_h^ϵ converges doubly to the viscosity upper-envelope solution of (2) on Ω.*

Proof See Crepey [13]. ▮

Remark 5
1. It is still an open problem to know if the viscosity upper-envelope solution coincides with the VREK value function of this game. For general sufficient conditions ensuring such coincidence, see Rapaport [19].
2. In the minimum distance game, f and d^0 do not fulfill assumptions 2b and 2c but numerical experiments confronted to the analytical solutions of section 3.3 suggest that the scheme converges towards the value of the game.

5 Algorithms

In order to approximate the capture time or the minimum oriented distance, we are led to solve the Shapley equation (8) (with a target dilated by ϵ) or (11). But these equations are infinite algebraic systems, since an infinite number of nodes are needed to cover the whole state space $\mathcal{E}$. So, their numerical resolutions require to localize a bounded window of interest. Classically, on the border of the discretized domain, the probabilities of transitions that would lead the state outside the domain have been chosen equal to zero.

Moreover, we need also to discretize the control sets into U_f, V_f. We use a rough discretization as usual for such problems, without prejudice on the quality of the results (Indeed, most of the optimal controls are *bang bang* or median in this example).

In the following experiments, we have used the set of parameters $(\alpha, \beta, \gamma) = (3, 2, 5)$, a window of 20×20 centered at the origin, a grid $\mathcal{E}_b^h$ of about 10^5 nodes in this window and sets of discretized controls of 5 values (experiments with more values have been made without any significant improvement on the precision of the results).

5.1 Game in time

A first possible algorithm to solve the fixed point equation (7) (with Ω replaced by Ω_ϵ for small $\epsilon > 0$) is the Shapley one *i.e.* iterations on the values :

$$V_h^{n+1}(x) = \min_{u \in U_f} \max_{v \in V_f} \{k(x,u,v) + \beta(x,u,v) E_x^{u,v} V_h^n\}, \ x \in \mathcal{E}_b^h \, .$$

This is a gradient method, as remarked by Filar, Raghavan [14]. Therefore its convergence is quite slow, and consequently it is not the algorithm that we shall use in practice.

Another possible algorithm is the Hoffman-Karp one, making iterations on the policies, which consists in solving the linear systems ($n \in \mathbb{N}$) :

$$V_h^n(x) = k(x, u^{n-1}(x), v^{n-1}(x)) + \beta(x, u^{n-1}(x), v^{n-1}(x)) E_x^{u^{n-1}(x), v^{n-1}(x)} V_h^n, \ x \in \mathcal{E}_b^h \, , \qquad (13)$$

where $(u^{n-1}(x), v^{n-1}(x)) \in U_f \times V_f$ mini-maximizes $\{k(x,u,v) + \beta(x,u,v) E_x^{u,v} V_h^{n-1}\}$, and V_h^0 is arbitrary in $\mathbb{R}_b^{\mathcal{E}^h}$.

It is of Newton-Raphson type (see Filar, Raghavan [14]), converging much faster than the Shapley one, although its convergence is not proved in general. It is the one we shall use in practice (The linear systems (13) have been solved iteratively using a Picard method).

Figure 3 shows the value $\widehat{V}$ obtained for small values of ϵ and h, after an hundred of Newton-Raphson iterations, which was the required amount of iterations to obtain the stabilization of the algorithm. On figure 3, the results are presented in terms of level curves. Curves of level less than 0.9 are represented in light color, while those of greater level are darker. The lens that can be seen on this figure is the area $\{x \,|\, \widehat{V}(x) \geq 0.9\}$, therefore it approximates the evasion zone. The existence and the general shape of this evasion zone are consistent with the analytical results obtained for this game by Bernhard ([7] and section 3.2).

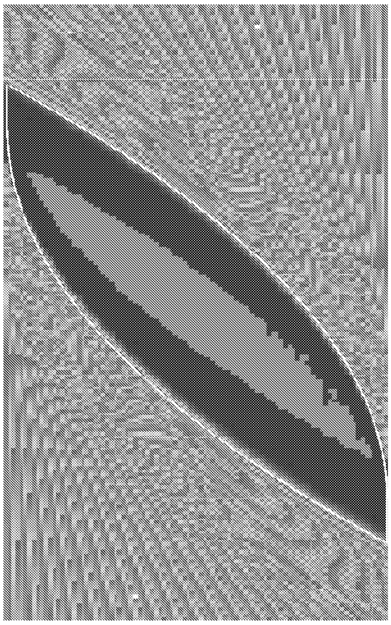

Figure 3: Iso-values for the game in time
(the superposed white curve is the exact barrier)

5.2 Game in distance

Let $(W_h^n)_{n\in\mathbb{N}}$ be the sequence $W_h^n = T_h^\epsilon W_h^{n-1}$ $(n \in \mathbb{N}^*)$, where W_h^0 is arbitrary in $\mathbb{R}_b^{\mathcal{E}^h}$. By Picard fixed point theorem, W_h^ϵ is the uniform limit of W_h^n when $n \to \infty$. But as before we prefer to use a Newton-Raphson algorithm on the policies adapted from Hoffman- Karp, *i.e.* we solve iteratively the linear systems :

$$W_h^n(x) = \min[\ \phi[d(x,\mathcal{T})] - W_h^n(x)\,,\\ k(x,u^{n-1}(x),v^{n-1}(x)) + \beta(x,u^{n-1}(x),v^{n-1}(x))E_x^{u^{n-1}(x),v^{n-1}(x)}W_h^n\ ], \qquad x \in \mathcal{E}_b^h \tag{14}$$

where $(u^{n-1}(x), v^{n-1}(x)) \in U_f \times V_f$ mini-maximizes $k(x,u,v) + \beta(x,u,v)E_x^{u,v}W_h^{n-1}$ and $W_h^0 = \phi(d(\cdot,\mathcal{T}))$ on $\mathcal{E}^h$.

Figure 4 shows discounted value $\widehat{W}$ obtained for small positive ϵ and h : The curves of negative level are represented in light color, while those of positive level are darker. These numerical results are consistent with the theoretical study of section 3.3, except in the anti-first diagonal corners of the window, where the edge effects are important. Looking at the figure, we recognize the optimal fields (3), and their separation along the abstract target $\mathcal{T}^\star := \{x \,|\, W(x) = d^o(x, \mathcal{T})\}$. The *oppidum* that should split the optimal fields is clearly visible (it corresponds to the inner lens whose upper border is well drawn). $\widehat{W}$ is roughly constant at its maximum value in this approximate *oppidum*, as expected.

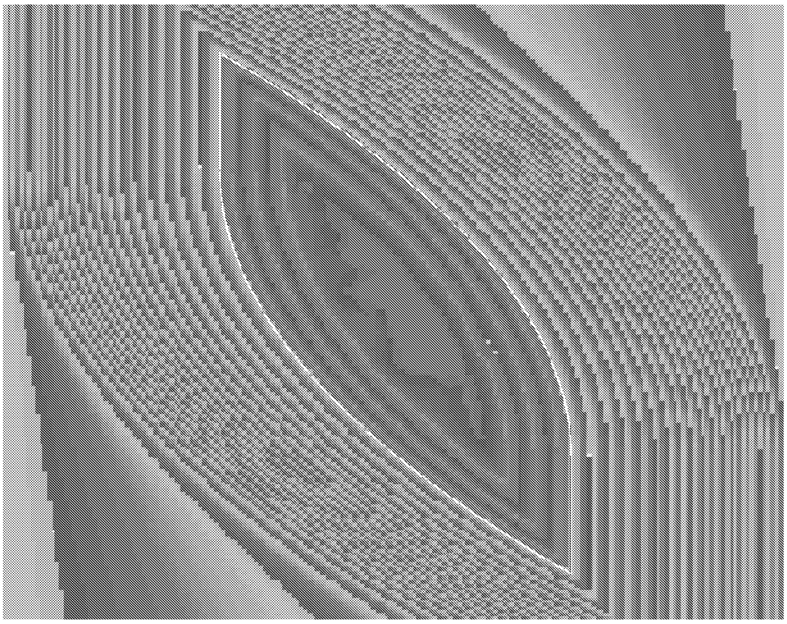

Figure 4: Iso-values for the game in distance
(the superposed white curve is the exact barrier)

5.3 Comparison

To compare the two methods, we have chosen the following criterion of convergence :
1. For any fix control, the linear systems (13) and (14) have been solved using Picard iterations up to a relative error (between two iterations) less or equal to 10^{-5}.
2. Then, iterations on the controls have been achieved until a relative error of 10^{-4} on the fix points V_h, W_h has been reached.
We have experimented approximatively the same amount of iterations for both methods.

Figures 3 and 4 allow one to compare the time and distance approaches, as far as the determination of the barrier of the capture-evasion game is concerned. As already mentioned, if we consider level sets of $\widehat{V}$ less than $1 - \mu$ (for small μ), these domains depend strongly on the arbitrary value of μ, so they are numerically very sensitive. Indeed, we can see on figure 3 that the level curves are very sparse inside the dark lens.

On the opposite, consider once again the figure 4 illustrating the approach in distance. This time, the level curves are very close to each other about the border of the lens $\{x \mid \widehat{W}(x) < 0\}$. Indeed, there is no reason why W should be flat about the level curve 0. Therefore it is not a surprise that the lens $\{x \mid W(x) < 0\}$ be less numerically sensitive than $\{x \mid V(x) < 1\}$.

6 Conclusion

Viscosity solutions of Isaacs equation provide two complementary viewpoints on the solution of our capture-evasion game. The p.d.e. equation allows one to investigate the capture time, while the variational inequality is an efficient way to investigate the barrier.

Both approaches allow us to construct candidate solutions, and lead to one same numerical approximation scheme, doubly indexed by two parameters (grid mesh h and dilatation ϵ) due to the possible discontinuities of the value function.

Numerical experimentations performed on this analytical example suggest that the scheme should obtain in general good results for both time and distance approaches (although this requires be checked on further examples...). The current major drawback is that we do not know how to choose ϵ when h goes towards 0, at least theoretically (In this respect, the viability framework appears to be more satisfactory from a numerical point of view, see [9, 10]). Mixing with other approaches could be a fruitful future task : for instance using construction techniques from Isaacs-Breakwell theory and completing the results obtained by numerical investigations.

References

[1] M. Bardi, S. Bottacin, and M. Falcone. *Convergence of Discrete Schemes for Discontinuous Value Functions of Pursuit-Evasion Games*, pages 273–304. New Trends in Dynamic Games and Applications. 5th International Symposium in Differential Games, Birkhaüser, 1995.

[2] M. Bardi and I. Capuzzo-Dolcetta. *Optimal Control and Viscosity solutions of Hamilton-Jacobi-Bellman Equations.* Birkhaüser, 1997.

[3] M. Bardi and P. Soravia. A comparison result for Hamilton-Jacobi equations and applications to some differential games lacking controllability. *Funkc. Ekvacioj*, Ser. Int. 37, No.1, 19–43, 1994.

[4] G. Barles. *Solutions de viscosité des équations de Hamilton-Jacobi.* Springer-Verlag, 1994.

[5] G. Barles and B. Perthame. Discontinuous solutions of deterministic optimal stopping time problems. *Mathematical Modeling and Numerical Analysis*, 21: 557–579, 1987.

[6] G. Barles and P.E. Souganidis. Convergence of approximation schemes for fully nonlinear second order equations. *Asymptotic Anal.*, (4): 271–283, 1991.

[7] P. Bernhard. Contribution à l'étude des jeux différentiels à deux joueurs, somme nulle et information parfaite. Thèse d'état, August 1978.

[8] P. Bernhard. Differential games lecture notes on the Isaacs-Breakwell theory. *Summer school on game theory, Cagliari*, 1993.

[9] P. Cardaliaguet, M. Quincampoix, and P. Saint-Pierre. Some algorithms for a game with two players and one target. *Mathematical Modeling and Numerical Analysis*, 28: 441–461, 1994.

[10] P. Cardaliaguet, M. Quincampoix, and P. Saint-Pierre. Numerical methods for optimal control and differential games. Technical Report 9510, CEREMADE, 1995.

[11] F.H. Clarke. *Optimization and Nonsmooth Analysis*. Wiley, 1983.

[12] M. Crandall, H. Ishii, and P.-L. Lions. User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc.*, 1992.

[13] S. Crepey. Stochastic games approximation schemes for differential games with barriers. PhD in preparation, 1998.

[14] J. Filar and T. Raghavan. Algorithms for stochastic games - a survey. *Math. Methods Oper. Res.*, (35): 437–472, 1991.

[15] R. Isaacs. *Differential games*. Wiley, 1965.

[16] H. Kushner. *Probability methods for approximations in stochastic control and for elliptic equations*. Academic Press, 1977.

[17] J-F. Masle. Problèmes qualitatif et quantitatif liés en jeux différentiels. Thèse de docteur ingénieur, Université Paris 9, 1976.

[18] O. Pourtallier and M. Tidball. Approximation of the value function for a class of differential games with target. *Research Report INRIA 2942*, 1996.

[19] A. Rapaport. Characterization of barriers of differential games. *Journal of Optimization, Theory and Applications*, 97(1): 151–180, April 1998.

[20] A.I. Subbotin. Generalized Solutions of First-Order PDEs, the Dynamical Optimization Perspective. Birkhäuser 1995.